Radial Motion of Two Mutually Attracting Particles

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A pair of masses or opposite-sign charges released from rest will move directly toward each other under the action of the inverse-distance-squared force of attraction between them. An exact expression for the separation distance as a function of time can only be found by numerically inverting the solution of a differential equation. A simpler, approximate formula can be obtained by combining dimensional analysis, Kepler's third law, and the familiar quadratic dependence of distance on time for a mass falling near Earth's surface. These exact and approximate results are applied to several interesting examples: the flight time and maximum altitude attained by an object fired straight upward from Earth's surface; the time required for an asteroid of known starting position and speed to cross Earth's orbit if it is bearing toward the Sun; and the collision time of two oppositely charged particles starting from rest.

Problem Statement

Suppose that the first particle, whose motion is of primary interest, has mass m and charge q, while the second particle has mass M and charge $-Q$. To ensure that the electrical force between them is not repulsive, assume that $Qq \geq 0$. (Note, however, that q can be positive, negative, or zero. In the last case, the force of attraction is solely gravitational.)

We will analyze the problem in the center-of-mass frame of reference and assume that the two particles constitute a bound, isolated system. In other words, the total mechanical energy is negative relative to the usual potential reference at infinite separation, and the particles always have equal and opposite linear momenta. Consequently, there is some maximum distance r_0 between the two particles at which point they are both instantaneously at rest. Define the time of that event as $t = 0$. It corresponds to the initial separation between the particles if they start from rest; it represents their turning points if the particles start out moving away from one another; and it represents a point in the extrapolated past if they are initially moving toward each other.

To summarize, without loss of generality we suppose that two particles 1 and 2 are released from rest at time $t = 0$ when they are a center-to-center distance r_0 apart. Define the coordinate origin to be at the stationary center of mass of the system. The two particles attract and begin to move toward each other. Their radial separation and relative speed at any later time t (before they collide with each other at time t_c at the origin if they are of point size) are r and υ, as described in Fig. 1.

The attractive force between the two particles is the sum of the gravitational and electrostatic forces, with magnitude $F = \gamma / r^2$, where $\gamma \equiv kQq + GMm$. (Here k is the Coulomb constant and G is the universal gravitational constant.) In practice, one of the two forces will usually dominate and we can neglect the other, but it is just as easy to be general and include both. Newton's second law describing the magnitude of the relative acceleration $a \equiv a_1 + a_2$ of the particles toward each other is[1]

$$\mu a = \frac{\gamma}{r^2}, \quad (1)$$

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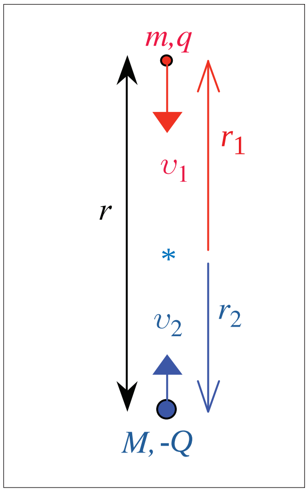

Fig. 1. Particle 1 of mass *m* and charge *q*, and particle 2 of mass *M* and charge –*Q* moving radially toward each other. The position and speed of the first particle relative to the center of mass (indicated by the asterisk) are r_1 and υ_1, while those of the second particle are r_2 and υ_2. Consequently the separation and relative speed of the two particles are $r \equiv r_1 + r_2$ and $\upsilon \equiv \upsilon_1 + \upsilon_2$.

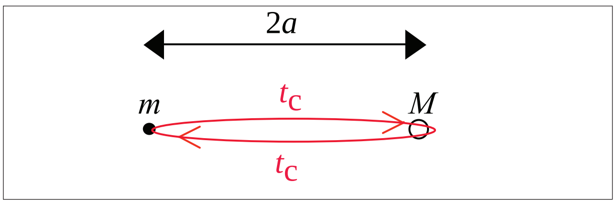

Fig. 2. A highly elliptical two-dimensional orbit of a planet of mass *m* shown at the initial instant when it is located at aphelion with respect to a sun of mass *M*.

where the reduced mass of the system is $\mu \equiv mM/(m+M)$, which simplifies to $\mu \approx m$ for the common case of $M >> m$. The signs of the kinematic variables have been chosen so that r, υ, and a are all positive for $0 < t < t_c$. Consequently $\upsilon = -dr/dt$ since r decreases with t during this time interval, while $a = d\upsilon/dt = -d^2r/dt^2$ because υ increases.

We can now succinctly state the problem: Find the distance r between the particles as a function of time t. An exact solution for the inverse relationship $t(r)$ has been published in several places[2–6] and is reviewed in the appendix. The purpose of the present paper is to develop an approximate formula using ideas that would be more appropriate in an introductory course. Not only is the resulting expression for $t(r)$ simpler, but it is analytically invertible to give $r(t)$, in contrast to the exact expression that is not.

Three steps are used to develop the approximate formula. First, dimensional analysis quickly gives a good estimate for the collision time t_c of two point particles that begin at rest. Second, Kepler's third law determines the missing numerical prefactor that dimensional analysis cannot give. Third, the familiar kinematic result for an object falling in Earth's surface gravity motivates the final expression. Instructors need not follow all three steps. For example, one could skip directly to the second step to find the exact collision time. Alternatively, one could omit the second step and simply state without proof the small numerical correction to the formula for the collision time found in the first step. A third possibility would be to skip the third step and restrict consideration to the one-dimensional collision of two point particles starting from rest.

Step One—Approximate Solution for the Collision Time

Equation (1) implicitly relates the initial separation r_0 to the collision time t_c in terms of the quantity γ/μ that has SI units of m^3/s^2. To convert this quantity into one with units of time, we must divide it by the cube of r_0 and take the reciprocal square root. Therefore, we immediately predict that

$$t_c \approx \sqrt{\frac{\mu r_0^3}{\gamma}}, \tag{2}$$

assuming both particles have negligible size relative to their initial separation. (In contrast, an example problem is solved later in this article where the second particle is taken to be the Earth and its radius is accounted for.) As shown below, this estimate for t_c is only 10% smaller than the exact answer and so it is an effective illustration of the power of dimensional reasoning.[7] For the case of two uncharged particles (so that $\gamma = GMm$), when the second particle is much more massive than the first (so that $\mu \approx m$), then Eq. (2) becomes

$$t_c \approx \sqrt{\frac{r_0^3}{GM}}. \tag{3}$$

Step Two—Numerical Prefactor for the Collision Time

The elliptical orbit of a planet (of mass m) about a sun (of mass $M >> m$) with semi-major axis a has a period squared of

$$\tau^2 = \frac{4\pi^2}{GM}a^3 \tag{4}$$

according to Kepler's third law, as can be derived at an introductory level using conservation laws and elementary properties of ellipses.[8] Now consider an elliptical orbit in the limit that the eccentricity $e \to 1$. Then the sun (located at one of the elliptical foci) moves to an end of the highly elongated orbit, as illustrated in a letter to the editor[9] by a sequence of orbits for which e is increased from 0 to 1. To make contact with the present paper, let us suppose that the planet starts at aphelion, where its radial velocity is zero. In that case, as Fig. 2 makes clear, the planet and sun are initially separated by the major axis so that $r_0 = 2a$. To complete one orbit, the planet first has to move to the sun's position, which by definition takes the collision time t_c, and then return to aphelion in the same time. Thus, the orbital period is $\tau = 2t_c$. Substituting this result and $a = r_0/2$ into Eq. (4) gives

$$t_c = \frac{\pi}{2}\sqrt{\frac{r_0^3}{2GM}}. \tag{5}$$

Consequently the required numerical prefactor in front of Eq. (3) is $2^{-3/2}\pi \approx 1.11$. Likewise correcting the more general expression in Eq. (2) leads to the exact result for two point particles,

$$t_c = \frac{\pi}{2}\sqrt{\frac{\mu r_0^3}{2\gamma}}. \tag{6}$$

Noting that the relative separation between the two particles decreases by r_0 in a time of t_c, it proves convenient in the subsequent analysis to recast the position and time variables in dimensionless form. Define the normalized radial separation as

$$R \equiv \frac{r}{r_0} \tag{7}$$

and the normalized time as

$$T \equiv \frac{t}{t_c}. \tag{8}$$

Note that $R = 1$ at the initial point $T = 0$, and that point particles would reach the origin $R = 0$ at $T = 1$.

Step Three—Approximate Relationship Between the Separation and Time

Suppose that a particle of mass m is dropped near Earth's surface (so that r_0 is approximately equal to Earth's radius r_E) and falls a small distance $\Delta r << r_0$ to a new radial position $r = r_0 - \Delta r$ in a time t. Elementary kinematics tells us that

$$\Delta r = \frac{1}{2}gt^2, \tag{9}$$

where $g = GM/r_E^2 = 9.8$ N/kg is Earth's surface gravitational field strength (with M the mass of the Earth). Use Eqs. (7) and (8) to re-express the fall distance and time in normalized form as

$$R = 1 - X \quad \text{where} \quad X \equiv \frac{\Delta r}{r_0} << 1 \tag{10}$$

and

$$T = \frac{t}{t_c} \quad \text{where} \quad t_c = \frac{\pi}{2}\sqrt{\frac{r_0}{2g}} \tag{11}$$

from Eq. (5). Using these variables, Eq. (9) can be rewritten as

$$X = \frac{1}{n}T^2 \quad \text{where} \quad n \equiv \left(\frac{4}{\pi}\right)^2 \approx 1.62. \tag{12}$$

Multiplying this equation for X by n and adding and subtracting unity gives

$$1 - (1 - nX) = T^2. \tag{13}$$

We want to extend this result to hold for values of X that approach 1. The expression in parentheses looks like the first two terms in the binomial expansion of $(1 - X)^n = R^n$, using Eq. (10) to relate X to R. Making this identification in Eq. (13) leads to the following hypothesized expressions relating the normalized separation and time,

$$1 - R^n \approx T^2 \quad \Rightarrow \quad T(R) \approx \sqrt{1 - R^n}$$
$$\text{or} \quad R(T) \approx \left(1 - T^2\right)^{1/n}. \tag{14}$$

Note that these expressions fit the two endpoints $R = 1$ at $T = 0$, and $R = 0$ at $T = 1$.

Equation (14) is plotted in red in Fig. 3. For comparison, the exact result from Eq. (30) in the appendix is graphed in blue. The two curves overlap almost perfectly across their whole range! The agreement can be slightly improved by rounding off n to the value 1.6, which is more convenient to work with than the exact irrational value; in that case, the mean discrepancy[10] between the exact and approximate values of $T(R)$ is about a quarter of a percent.

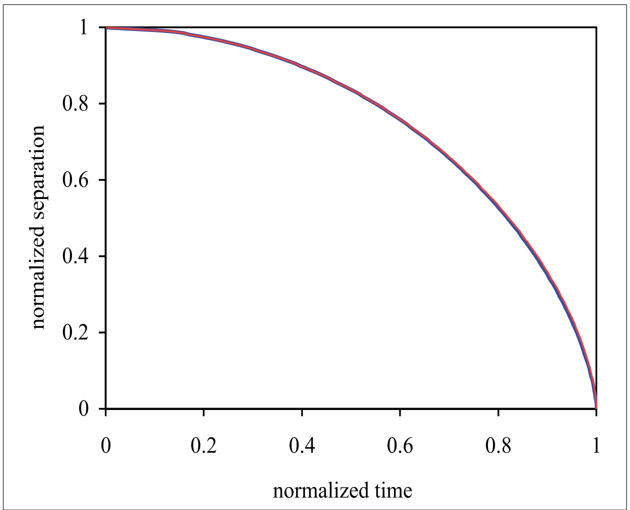

Fig. 3. Approximate (red) and exact (blue) plots of the separation of the two particles as a function of time in dimensionless form.

Three Example Problems

1. Suppose a particle is launched upward from the surface of the Earth with a speed υ_{orbit} equal to that of a low-Earth-orbit satellite. What is the maximum height that it reaches above Earth's surface and how long does it take to get there?[11]

The speed of orbit of a satellite of mass m just above the surface of the Earth (ignoring its atmosphere and topography) is obtained from Newton's second law as

$$\frac{GMm}{r_E^2} = m\frac{\upsilon_{\text{orbit}}^2}{r_E} \quad \Rightarrow \qquad (15)$$
$$\upsilon_{\text{orbit}} = \sqrt{g r_E} = 7.9 \text{ km/s},$$

where M and r_E are Earth's mass and radius, respectively. Since the velocity on the way up is equal and opposite to the subsequent velocity back down from the peak height, we can substitute $r = r_E$, $\upsilon = \upsilon_{\text{orbit}}$, and $\gamma/\mu = GM = g r_E^2$ into Eq. (25) to obtain the maximum radial position $r_0 = 2r_E$, i.e., the particle attains a maximum height of one Earth radius (6370 km) above the surface. It then falls back to the ground, at which point $R = r_E/r_0 = 1/2$. Substituting this value into Eq. (30), one then finds that the fall (or equivalently the rise) time is

$$t = \sqrt{\frac{r_E}{g}}\left(1+\frac{\pi}{2}\right) = 34.5 \text{ min} \qquad (16)$$

since $T = t/t_c$, where $t_c = \pi\sqrt{r_E/g}$ according to Eq. (5). For comparison, Eq. (14) predicts a time of

$$t = \pi\sqrt{\frac{r_E}{g}\left(1-2^{-1.6}\right)}, \qquad (17)$$

using the rounded-off value of n. The numerical values of Eqs. (16) and (17) disagree by less than 1 s.

2. A new asteroid is discovered that is heading straight for the Sun. It is initially observed at a distance from the Sun of 3 AU with a speed of half of the speed υ_{esc} it would need to escape from the Sun at that point in its orbit. How much time will elapse until the asteroid crosses Earth's orbit?

Let $r_i = 4.5 \times 10^{11}$ m and $r_f = 1.5 \times 10^{11}$ m be the initial and final radial positions of the asteroid, respectively, and let r_0 denote the maximum distance from the Sun that the asteroid would attain if it were traveling directly away from the Sun with an initial speed of $\upsilon_{\text{esc}}/2$. Also let $R_i \equiv r_i/r_0$ and $R_f \equiv r_f/r_0$. The initial speed of the asteroid (of mass m) is obtained from energy conservation as

$$\frac{1}{2}m\upsilon_{\text{esc}}^2 - \frac{GMm}{r_i} = 0 \quad \Rightarrow \qquad (18)$$
$$\upsilon_i \equiv \frac{1}{2}\upsilon_{\text{esc}} = \sqrt{\frac{GM}{2r_i}} = 12.1 \text{ km/s},$$

where M is the mass of the Sun. Next we use Eq. (25) with $\gamma/\mu = GM$ to find the reciprocal of the asteroid's radial turning point,

$$\frac{1}{r_0} = \frac{1}{r_i} - \frac{\upsilon_i^2}{2GM} = \frac{3}{4r_i}, \qquad (19)$$

with the help of Eq. (18). Consequently, $R_i = 3/4$ and $R_f = 1/4$. Thus Eq. (30) gives

$$\Delta t \equiv t_f - t_i = \sqrt{\frac{r_0^3}{2GM}}\left[\cos^{-1}\sqrt{\frac{1}{4}} - \cos^{-1}\sqrt{\frac{3}{4}}\right] \qquad (20)$$
$$= \frac{2\pi}{3}\sqrt{\frac{2r_f^3}{GM}} = 173 \text{ days},$$

using Eqs. (5) and (8). In comparison, Eq. (14) implies that

$$\Delta t = 2\pi\sqrt{\frac{2r_f^3}{GM}}\left[\sqrt{1-0.25^{1.6}} - \sqrt{1-0.75^{1.6}}\right] \qquad (21)$$

and the quantity in square brackets is nearly equal to 1/3, in agreement with the result of Eq. (20). If such an asteroid really were discovered tomorrow, we would therefore have about half a year to prepare if the Moon or Earth were to be impacted.

3. An electron and a positron start out at rest a distance r_0 = 10 nm apart. How much time will it take for them to annihilate each other?

The two particles have equal and opposite charges of magnitude $q = Q = e$, where $e = 1.6 \times 10^{-19}$ C. Furthermore they have equal masses $m = M = 9.11 \times 10^{-31}$ kg so that the reduced mass is $\mu = m/2$. The electrostatic force completely overwhelms the gravitational force between them and thus $\gamma = ke^2$. Equation (6) now becomes

$$t_c = \frac{\pi}{4e}\sqrt{\frac{mr_0^3}{k}} = 49 \text{ fs.} \quad (22)$$

They disappear in less than a blink of an eye!

Appendix: *Exact Relationship Between the Separation and Time*

Substituting $a = -d^2r/dt^2$ into Eq. (1) results in a second-order differential equation for $r(t)$ that can be solved in two steps. First we find the relative speed υ of the particles as a function of their separation r; then we integrate that to determine the time T required for the particles to approach each other to within some distance R in dimensionless form.

I. Relative Speed as a Function of the Separation

In the center-of-mass reference frame, the two particles have equal and opposite linear momenta of magnitude[1] $m\upsilon_1 = M\upsilon_2 = \mu\upsilon \equiv p$, so that their total kinetic energy is

$$\frac{p^2}{2m} + \frac{p^2}{2M} = \frac{\mu^2\upsilon^2}{2}\left(\frac{1}{m}+\frac{1}{M}\right) = \frac{1}{2}\mu\upsilon^2. \quad (23)$$

The potential energy corresponding to the force of interaction in Eq. (1) is $-\gamma/r$. Conservation of mechanical energy for the system of two particles between the initial and an arbitrary final state therefore becomes

$$\frac{1}{2}\mu\upsilon^2 - \frac{\gamma}{r} = 0 - \frac{\gamma}{r_0}, \quad (24)$$

which can be rearranged as

$$\upsilon = \sqrt{\frac{2\gamma}{\mu}\left(\frac{1}{r} - \frac{1}{r_0}\right)}\,. \quad (25)$$

The square root is always real because $r \leq r_0$. Note that the speed becomes infinite as $r \to 0$, but in the real world that would be prevented by the finite sizes of the two particles.

II. Relationship Between the Separation and Time

Substituting $\upsilon = -dr/dt$ into Eq. (25), the variables r and t can be separated and rewritten in dimensionless form using Eqs. (6) through (8) to get

$$\frac{\pi}{2}\int_0^T dT = -\int_1^R \frac{dR}{\sqrt{1/R - 1}}\,. \quad (26)$$

The lower limits correspond to the initial condition $R = 1$ at $T = 0$. The time integral on the left-hand side is equal to T. One can solve the radial integral by making a change of variable. By trial and error, one is led to the trigonometric substitution $R = \cos^2\theta$, whose virtue is that it lends itself to the Pythagorean identity

$$\sec^2\theta - 1 = \tan^2\theta. \quad (27)$$

Substituting this identity and the differential $dR = -2\cos\theta\sin\theta\, d\theta$ into Eq. (26) leads to

$$\frac{\pi}{2}T = \int_0^\theta 2\cos^2\theta\, d\theta = \int_0^\theta (1+\cos 2\theta)\, d\theta, \quad (28)$$

using the cosine double-angle formula and noting that $\theta = 0$ when $R = 1$. Integrating gives

$$\frac{\pi}{2}T = \theta + \frac{\sin 2\theta}{2} = \theta + \cos\theta\sin\theta = \theta + \cos\theta\sqrt{1-\cos^2\theta} \quad (29)$$

with the help of the sine double-angle formula and another Pythagorean identity. Returning to the original variable R gives the final result

$$\frac{\pi}{2}T = \cos^{-1}\sqrt{R} + \sqrt{R(1-R)}. \quad (30)$$

This equation correctly predicts that $T = 0$ at the initial point $R = 1$, and that $T = 1$ at $R = 0$. We cannot analytically invert it to find $R(T)$, but by computing a table of values (as plotted in Fig. 3) we can find any desired result numerically, which points out the advantage of working with a "universal" (i.e., dimensionally normalized) form of the equation.

One can show that Eq. (30) becomes Eq. (12) to lowest order in X if Eq. (10) holds. This result follows by proving that each of the two terms on the right-hand side of Eq. (30) is approximately equal to $\sqrt{X}$ if $R = 1 - X$, where $X << 1$. For the first term, one demonstrates the inverse result $\sqrt{1-X} \approx \cos\sqrt{X}$ by Taylor expanding both sides up to linear powers in X.

Note added in proof: S.K. Foong ["From Moon-fall to motions under inverse square laws," *Eur. J. Phys.* **29**, 987–1003 (Sep. 2008)] has obtained other approximate expressions for $R(T)$.

References

1. By eliminating r_2 between the two equations $r \equiv r_1 + r_2$ and $mr_1 = Mr_2$, one can quickly show that $\mu r = mr_1$. Differentiating both sides with respect to time proves that $\mu v = mv_1$. Differentiating again gives $\mu a = ma_1$. Then Eq. (1) is equivalent to Newton's second law for particle 1. Alternatively one can establish that $\mu v = Mv_2$ and $\mu a = Ma_2$, identifying Eq. (1) as Newton's second law for particle 2. Note that $ma_1 = Ma_2$ in agreement with Newton's third law.
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10. Let the fractional discrepancy between the exact and approximate values of T be $f(R) \equiv (T_{\text{exact}} - T_{\text{approx}})/T_{\text{exact}}$. Then I define the mean discrepancy to be the square root of the integral of f^2 from $R = 0$ to $R = 1$, in accordance with the usual statistical idea of variance.
11. Problem adapted from B. Korsunsky, "Physics Challenge for Teachers and Students: The drill team rocks!" *Phys. Teach.* **45**, 568 (Dec. 2007).

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Carl Mungan *is a frequent contributor to* The Physics Teacher. *This article was inspired by a question by his colleague John Fontanella about the vertical motion of an object in Earth's gravity when its change in altitude is large.*
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