A Dozen Proofs of Stirling's Formula

Steven R. Dunbar

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Wallis' Formula is the amazing limit

$$\lim_{n\to\infty}\left(\frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\ldots(2n)\cdot(2n)}{1\cdot 3\cdot 3\cdot 5\cdot 5\ldots(2n-1)\cdot(2n-1)\cdot(2n+1)}\right)=\frac{\pi}{2}.$$

1. One proof of Wallis' formula uses a recursion formula from integration by parts of powers of sine.
2. Another proof uses only basic algebra on the partial products, the Pythagorean Theorem, and πr^2 for the area of a circle.
3. A complex analysis proof uses the infinite product expansion for the sine function.

Gaussian Probability Integral

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-\frac{x^2}{2}} \, \mathrm{d}x = 1.$$

- First complete evaluation by Laplace in 1774.
- Polar coordinates, due to Poisson, popularized by Sturm.
- Volume integration by shells.
- Double Integral, change of variables, product of integrals, one is an $\arctan$.
- Interpolation between two integrals, one is an $\arctan$.
- Approximate e^{-x^2} with $(1 - x^2/n)^n$ on $[0, \sqrt{n}]$, change variables to sine functions, use Wallis formula.

Forms of Stirling's Formula

The easy ones in order of increasing precision:

1. $$\sqrt[n]{n!} \sim \frac{n}{\mathrm{e}}.$$
2. $$n! \approx \sqrt{2\pi} n^{n+1/2} \mathrm{e}^{-n}.$$
3. $$n! \sim \sqrt{2\pi} n^{n+1/2} \mathrm{e}^{-n}.$$

Forms of Stirling's Formula

Inequalities in order of increasing precision:

1. $$n! = \sqrt{2\pi} n^{n+1/2} \mathrm{e}^{-n} (1 + \epsilon_n), \quad |\epsilon_n| < \frac{A}{n}.$$

2. $$\sqrt{2\pi} n^{n+1/2} \mathrm{e}^{-n} < n! < \sqrt{2\pi} n^{n+1/2} \mathrm{e}^{-n+1/(12(n-1/2))}.$$

3. $$\sqrt{2\pi} n^{n+1/2} \mathrm{e}^{-n} < n! < \sqrt{2\pi} n^{n+1/2} \mathrm{e}^{-n+1/(12n)}.$$

4. $$\sqrt{2\pi} n^{n+1/2} \mathrm{e}^{-n+1/(12n+1)} < n! < \sqrt{2\pi} n^{n+1/2} \mathrm{e}^{-n+1/(12n)}.$$

5. $$\left| \frac{n!}{\sqrt{2\pi} n^{n+1/2} \mathrm{e}^{-n}} - 1 - \frac{1}{12n} \right| \le \frac{1}{288n^2} + \frac{1}{9940n^3}.$$

Forms of Stirling's Formula

Complex variable versions in order of increasing precision:

1. $$\Gamma(z+1) \sim \sqrt{2\pi z} z^z e^{-z} \left(1 + \frac{1}{12z} + \frac{1}{288z^2} + \dots\right).$$

2. $$\log(\Gamma(z+1)) \sim \frac{1}{2}\log(2\pi) + (z + \frac{1}{2})\log(z) - z + \frac{1}{2}\sum_{n=1}^{\infty} \frac{B_{2n}}{n(2n-1)} \frac{1}{z^{2n-1}}.$$

Overview of Proofs

Altogether

- 2 heuristic proofs
- 8 rigorous proofs
- 2 sketches of proofs using advanced complex variables

Sequence-oriented Proofs

The proof of

$$\sqrt[n]{n!} \sim \frac{n}{\mathrm{e}}$$

follows from easy estimations of the power series of the exponential. The $n!$ comes from the denominator in the power series.

A proof of

$$n! \sim \sqrt{2\pi} n^{n+1/2} \mathrm{e}^{-n}$$

follows from showing $x_n = \log\left(\frac{n!}{n^{n+1/2}\mathrm{e}^{-n}}\right)$ increases to a limit. Then use Wallis' Formula to evaluate the limit.

The disadvantage of this proof is that it requires the form of Stirling's Formula in order to create the sequence which is the main object of the proof.

Integral-oriented Proofs

The proof of

$$n! = \sqrt{2\pi} n^{n+1/2} \mathrm{e}^{-n}(1+\epsilon_n), \quad |\epsilon_n| < \frac{A}{n}$$

and

$$\sqrt{2\pi} n^{n+1/2} \mathrm{e}^{-n} < n! < \sqrt{2\pi} n^{n+1/2} \mathrm{e}^{-n+1/(12(n-1/2))}.$$

follow from converting $\log(n!)$ to $\sum_{k=1}^{n} \log(k)$. Then consider $\sum_{k=1}^{n} \log(k)$ as an approximation to $\int \log(t) \, \mathrm{d}t$ over some interval.

Integral-oriented Proofs

There are three ways to estimate the approximation:

1. Use the Euler-Maclaurin summation formula, which gives an explicit form for the error.
2. Break into integrals over unit intervals, then estimate the $\log(k) - \int_k^{k+1} \log(t)\, \mathrm{d}t$ approximation with the trapezoidal rule.
3. Break into integrals over unit-length intervals, then estimate the difference $\log(k) - \int_{k-1/2}^{k+1/2} \log(t)\, \mathrm{d}t$ with Taylor series.

In each case, the limit constant $\sqrt{2\pi}$ is evaluated with Wallis' formula.

Proofs using the Gamma Function

$$\Gamma(t+1) = \int_0^\infty x^t \mathrm{e}^{-x} \, \mathrm{d}x$$

The Gamma Function is the continuous representation of the factorial, so estimating the integral is natural.

Note that $x^t \mathrm{e}^{-x}$ has its maximum value at $x = t$. That is, most of the value of the Gamma Function comes from values of x near t.

Change variables, estimate, repeat!

Show that an integral resulting in the estimate approaches the Gaussian probability integral to get the asymptotic constant.

Proofs using Complex Variables

Many complex variables books give a proof using Gauss's Formula

$$\Gamma(z) = \lim_{n\to\infty} \frac{n^z n!}{z(z+1)(z+2)\dots(z+n)}$$

and a version of the Euler-Maclaurin summation formula.

Proofs using Complex Variables

$$\log(\Gamma(z+1)) \sim \frac{1}{2}\log(2\pi) + (z+\frac{1}{2})\log(z) - z + \frac{1}{2}\sum_{n=1}^{\infty} \frac{B_{2n}}{n(2n-1)} \frac{1}{z^{2n-1}}.$$

Derivation of the asymptotic limit for $\Gamma(z+1)$:

1. Start with the digamma function,
2. Expand the integrand in a power series,
3. Define the Bernoulli numbers B_n,
4. Use the definition of the Gamma function as the derivative of the logarithm of the digamma function,
5. Derive the asymptotic expansion.

Proofs using Probability Theory

$$n! \approx \sqrt{2\pi} n^{n+1/2} \mathrm{e}^{-n}.$$

A heuristic proof uses

1. On the one hand, the distribution of the sum of Poisson random variables.
2. On the other hand, the central limit theorem.
3. The constant is evaluated with a form of the Gaussian probability integral.

Proofs using Probability Theory

$$\lim_{n\to\infty} \frac{\sqrt{2\pi n}\, n^n \mathrm{e}^{-n}}{n!} = 1.$$

A rigorous proof uses the same general idea expressed with the product of characteristic functions of Poisson random variables, then estimating the resulting exponentials. The proof shows the characteristic function of the sum of Poisson random variables converges to the characteristic function of a normal random variable with corresponding variance.