

Intersection Theory with Applications to the Computation of Gromov-Witten Invariants

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*To my family
to my wife, Thuong Nghi
and to our son, Minh Khang*

Contents

List of Algorithms	iii
List of Figures	v
List of Tables	vii
Introduction	1
1. A Review of Intersection Theory	5
1.1. Chow Rings	5
1.1.1. Cycles and Rational Equivalence	5
1.1.2. Pushforwards and Pullbacks	6
1.1.3. Intersection Products	8
1.1.4. Examples	9
1.2. Chern and Segre Classes	13
1.2.1. Splitting Principle	14
1.3. Chern Characters and Todd Classes	16
1.4. Hirzebruch-Riemann-Roch Theorem	18
1.5. Excess Intersection Formula	20
2. Schubert Calculus	23
2.1. Grassmannians	23
2.2. Littlewood-Richardson Rule	28
2.3. Fano Schemes	29
2.4. Linear Subspaces on Hypersurfaces	31
2.5. Universal Fano Schemes	33
2.6. Lines on Complete Intersection Calabi-Yau Threefolds	35
3. Enumerative Geometry of Conics	39
3.1. Enumeration of Conics in \mathbb{P}^3	39
3.2. Conics on Quintic Hypersurfaces in \mathbb{P}^4	42
3.3. Conics on Complete Intersection Calabi-Yau Threefolds	43
4. Toric Intersection Theory	45
4.1. Cones and Affine Toric Varieties	45
4.2. Fans and Toric Varieties	46
4.3. Chow Rings of Toric Varieties	48
4.4. Todd classes of Toric Varieties	50
5. Equivariant Intersection Theory and Bott's Formula	53
5.1. Basic Notions	53
5.1.1. Equivariant Chow Rings	53

5.1.2.	Pullback and Pushforward Homomorphisms	55
5.1.3.	Equivariant vector bundles and Chern Classes	56
5.2.	Localization Theorem and Bott's Formula	58
5.3.	Schubert Calculus, Revisited	61
5.3.1.	Euler Characteristic of Grassmannians	63
5.3.2.	Fano Schemes, Revisited	63
5.3.3.	Linear Subspaces on Hypersurfaces, Revisited	65
6.	Gromov-Witten Invariants	67
6.1.	Moduli Spaces of Stable Maps	67
6.2.	Virtual Fundamental Classes	68
6.3.	Defining Gromov-Witten Invariants	69
6.4.	Gromov-Witten Invariants for Projective Spaces	70
7.	Rational Curves on Calabi-Yau Threefolds	75
7.1.	Localization of $\overline{M}_{0,0}(\mathbb{P}^r, d)$	75
7.2.	Lines on Hypersurfaces, Revisited	81
7.3.	Multiple Covers	82
7.4.	Rational Curves on Quintic Hypersurfaces in \mathbb{P}^4	83
7.5.	Rational Curves on Complete Intersections	86
7.6.	Future work	91
A.	A Sage Package for Intersection Theory and Enumerative Geometry	93
A.1.	Code	93
A.2.	Examples	94
B.	A Singular library for Intersection Theory and Enumerative Geometry	105
B.1.	Examples	106
	Bibliography	119

List of Algorithms

1.4.1.The Hirzebruch-Riemann-Roch formula	18
1.5.1.The excess intersection formula	21
2.1.1.Degree on Grassmannians	26
2.2.1.The Littlewood-Richardson coefficients	28
2.2.2.The Littlewood-Richardson rule	29
2.3.1.The degree of Fano schemes	30
2.3.2.Dual to Algorithm 2.3.1	30
2.4.1.Linear subspaces on hypersurfaces	32
5.3.1.Bott's formula for Grassmannians	61
6.4.1.Gromov-Witten invariants for projective spaces	72
7.4.1.The Gromov-Witten invariants N_d	84
7.5.1.The Gromov-Witten invariants $N_d^{(d_1, \dots, d_k)}$	88

List of Figures

2.1.	27 lines on a general cubic surface.	32
4.1.	The cone $\sigma = \text{Cone}((0, 1), (1, 0))$ in the plane.	46
4.2.	The fan Δ for \mathbb{P}^2	47
4.3.	The fan Δ_2 for Hirzebruch surface \mathcal{H}_2	48

List of Tables

2.1. Lines on hypersurfaces	33
2.2. Lines on complete intersection Calabi-Yau threefolds	37
3.1. Some numbers concerning conics in \mathbb{P}^3	41
3.2. Conics on complete intersection Calabi-Yau threefolds	44
5.1. Timings for linear subspaces on hypersurfaces.	66
6.1. Some Kontsevich numbers.	70
6.2. The number of rational curves of degree d meeting a lines and b points in \mathbb{P}^3 .	73
6.3. The number of lines meeting a planes, b lines and c points in \mathbb{P}^4	74
6.4. The number of conics meeting a planes, b lines and c points in \mathbb{P}^4	74
7.1. Some numbers n_d	86
7.2. Some instanton numbers $n_d^{(d_1, \dots, d_k)}$	91
A.1. The top degree monomial values of a Pfaffian Calabi-Yau threefold.	100
A.2. Some Gromov-Witten invariants for lines.	102

Introduction

Modern computer algebra methods allow us to study explicit examples in algebraic geometry. In this way, algebraic geometry becomes accessible to experiments. One of the leading computer algebra systems for applications in algebraic geometry is SINGULAR [21], developed under the direction of Wolfram Decker, Gert-Martin Greuel, Gerhard Pfister, and Hans Schönemann at the Technische Universität Kaiserslautern. Recently, an open-source mathematics software system which is called SAGE [99] has been developing. It combines the power of many existing open-source computer algebra systems and packages into a common interface which is based on the programming language PYTHON¹. For polynomial computations, SAGE relies on SINGULAR. One of the advantages of SAGE is that it provides interfaces to many computer algebra systems. This allows us to rely not only on tools for polynomial computations but also, for example, on combinatorial methods.

The main topic of this thesis is computational intersection theory. Let us introduce the history of this field. The first package for computations in intersection theory on smooth projective varieties is `Schubert` [68] in MAPLE which was written by Sheldon Katz and Stein A. Strømme in 1992, and revised and updated to new versions of MAPLE by Jan-Magnus Økland. However, `Schubert` is no longer actively supported. The second package is `Schubert2` [48] in MACAULAY2 which has been developed by Daniel R. Grayson, Michael E. Stillman, Stein A. Strømme, David Eisenbud, and Charley Crissman. `Schubert2` deals with abstract varieties, vector bundles on abstract varieties, morphisms between abstract varieties. An abstract variety is not given by equations. Instead, it is given by a nonnegative integer which is its dimension, and a graded ring which is its Chow ring. A detailed description of `Schubert2` can be found in [48].

In this thesis, we developed a SAGE package `Schubert3` and a SINGULAR library `schubert.lib` which both provide the key functionality necessary for computations in intersection theory and enumerative geometry. In particular, we focus on the classical problems in Schubert calculus and the enumerative geometry of conics. We briefly treat some problems in toric intersection theory. We have also implemented tools for computations in equivariant intersection theory. These tools are based on the localization theorem and Bott's formula. This work gives us an alternative method for solving the problems in Schubert calculus via Bott's formula.

Another focus of this thesis is related to the computation of Gromov-Witten invariants. For this, we deal with moduli spaces of stable maps, which were introduced by Kontsevich [75], and the graphs corresponding to the fixed point components of a torus action on the moduli spaces of stable maps. The basic ideas are as follows.

In [15], the string theorists Candelas, de la Ossa, Green and Parkes predicted the numbers n_d of degree d rational curves on a general quintic hypersurface in 4-dimensional projective space. By similar methods, Libgober and Teitelbaum [82] predicted the numbers of rational curves on the remaining general Calabi-Yau threefolds in projective spaces. The computation of the physicists is based on arguments from topological quantum field theory and mirror

¹www.python.org

symmetry. These predictions went far beyond anything algebraic geometry could prove at the time and became a challenge for mathematicians to understand mirror symmetry and to find a mathematically rigorous proof of the physical predictions. The process of creating a rigorous mathematical foundation for mirror symmetry is still far from being finished. For important aspects concerning this question, we refer to Morrison [87], Givental [51, 52], and Lian, Liu, and Yau [83]. For an introduction to the algebro-geometric aspects of mirror symmetry, see Cox and Katz [16]. To verify the physical predictions by algebro-geometric methods, we have to find a suitable moduli space of rational curves separately for every degree d , and express the locus of rational curves on a general Calabi-Yau threefold as a certain zero-dimensional cycle class on the moduli space. We then need in particular to evaluate the degree of a given zero-dimensional cycle class. This is possible, in principle, whenever the Chow ring of the moduli space is known. Unfortunately, in general it is quite difficult to describe the Chow ring of a moduli space. Alternatively, Bott's formula allows us to express the degree of a certain zero-dimensional cycle class on a moduli space endowed with a torus action in terms of local contributions supported on the components of the fixed point locus. The local contributions are of course much simpler than the whole space. By these methods, algebraic geometers checked the results of physicists up to degree 4: $n_1 = 2875$ is classically known, $n_2 = 609250$ has been shown by Katz [64], $n_3 = 317206375$ by Ellingsrud and Strømme [30, 31], $n_4 = 242467530000$ by Kontsevich [75]. In Chapter 7, we present a strategy for computing Gromov-Witten invariants using the localization theorem and Bott's formula. This method is based on the work of Kontsevich [75]. As an insightful example, the numbers of rational curves on general complete intersection Calabi-Yau threefolds in projective spaces are computed up to degree 6. The results are all in agreement with predictions made from mirror symmetry in [15, 82].

This thesis is organized in the following way:

- Chapter 1 introduces some of the basic notions of intersection theory such as Chow rings, Segre and Chern classes, Chern characters, Todd classes, the Hirzebruch-Riemann-Roch theorem, and the excess intersection formula. The topics we cover in this chapter are selected to consider some classical enumerative problems.
- Chapter 2 explains the classical algorithms in Schubert calculus for computing the intersection numbers of cycle classes on Grassmannians. In this chapter, we also present an effective method for computing Littlewood-Richardson coefficients and the degree of the Fano schemes of linear subspaces on general hypersurfaces. In particular, we review the problem of counting linear subspaces on a general hypersurface. As a well-known example, we recompute that there are $n_1 = 2875$ lines on a general quintic hypersurface in \mathbb{P}^4 .
- Chapter 3 presents further examples in enumerative geometry such as the problem of counting conics on a general quintic hypersurface in \mathbb{P}^4 , that is how to compute the number $n_2 = 609250$. Other enumerative problems concerning conics in \mathbb{P}^3 are also reviewed in this chapter. The key idea for solving these problems is a result describing the Chow rings of parameter spaces for conics.
- Chapter 4 presents some basic notions of toric geometry such as cones and fans. We also describe the Chow rings of toric varieties associated to fans and present some examples in the toric setting.

- Chapter 5 describes some basic notions of equivariant intersection theory. The most important results of this theory are the localization theorem and Bott's formula. As an application, we present an alternative method for solving the problems in Schubert calculus already discussed in Chapter 2.
- Chapter 6 focuses on intersection theory over moduli spaces of stable maps. In particular, we explain how to define Gromov-Witten invariants via algebraic geometry. Then we present some numerical results on Gromov-Witten invariants for projective spaces.
- Chapter 7 presents an important technique for computing Gromov-Witten invariants, that is the localization of moduli spaces of stable maps. This technique allows us to compute Gromov-Witten invariants via Bott's formula. In particular, we revisit some examples of the previous chapters and verify many results in enumerative geometry.

Furthermore, there are two appendices in this thesis:

- Appendix A introduces the SAGE package `Schubert3`.
- Appendix B introduces the SINGULAR library `schubert.lib`.

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Chapter 1.

A Review of Intersection Theory

The aim of this chapter is to quickly introduce some basic notions and properties taken from *intersection theory in algebraic geometry*. The details will not be fully worked out and we shall not provide the proofs, which one may find in the books of William Fulton [35, 36], the lecture notes of Andreas Gathmann [41], the book of Hartshorne [59], and the forthcoming book on intersection theory of David Eisenbud and Joe Harris [29]. Note that in sections where our presentation does not follow [36], the assumptions may be more restrictive than actually needed (see the introduction to [29, Chapter 5]).

Blanket assumptions

Throughout this thesis, we work over the field of complex numbers. By a scheme we shall mean a scheme of finite type over \mathbb{C} . A variety will be an integral scheme, and a subvariety of a scheme will be a closed subscheme which is a variety. By a point on a scheme, we will always mean a closed point. All morphisms of schemes are morphisms over \mathbb{C} . All vector spaces and projective spaces are defined over \mathbb{C} . For convenience, we always write \mathbb{P}^n instead of $\mathbb{P}_{\mathbb{C}}^n$.

1.1. Chow Rings

One of the requirements in intersection theory is to construct the rings of cycle classes on smooth quasiprojective varieties, commonly known as the Chow rings. In particular, we have to find a way to define the intersection product of cycle classes that should be the multiplication of Chow rings.

1.1.1. Cycles and Rational Equivalence

Let X be a scheme. A k -cycle on X is a finite formal sum $\sum n_i[V_i]$, where the V_i are k -dimensional subvarieties of X , and the $n_i \in \mathbb{Z}$. The group of k -cycles on X , denoted $Z_k(X)$, is the free abelian group on the k -dimensional subvarieties of X . The graded group

$$Z_*(X) = \bigoplus_{k=0}^{\dim X} Z_k(X)$$

is called the *group of cycles* on X .

For any $(k + 1)$ -dimensional subvariety W of X , the field of rational functions on W is denoted by $R(W)$. The non-zero elements of this field form the multiplicative group $R(W)^*$, and for any $f \in R(W)^*$, we define a k -cycle $[\operatorname{div}(f)]$ on X by

$$[\operatorname{div}(f)] = \sum \operatorname{ord}_V(f)[V].$$

Here the sum runs over all k -dimensional subvarieties V of W , and ord_V is the order function on $R(W)^*$ defined by

$$\text{ord}_V(f) = l_{\mathcal{O}_{W,V}}(\mathcal{O}_{W,V}/(f)),$$

where $l_{\mathcal{O}_{W,V}}(\mathcal{O}_{W,V}/(f))$ is the length of $\mathcal{O}_{W,V}/(f)$ as an $\mathcal{O}_{W,V}$ -module.

We say that a k -cycle c is *rationally equivalent to zero*, written $c \sim 0$, if there are a finite number of $(k+1)$ -dimensional subvarieties W_i of X , and $f_i \in R(W_i)^*$ such that

$$c = \sum [\text{div}(f_i)].$$

Since $[\text{div}(f^{-1})] = -[\text{div}(f)]$, the k -cycles rationally equivalent to zero form a subgroup $\text{Rat}_k(X)$ of $Z_k(X)$. The group of k -cycles modulo rational equivalence on X is the quotient group

$$A_k(X) = Z_k(X) / \text{Rat}_k(X).$$

The graded group

$$A_*(X) = \bigoplus_{k=0}^{\dim X} A_k(X)$$

is called the *Chow group* of X .

A *cycle* (respectively *cycle class*) on X is an element of $Z_*(X)$ (respectively $A_*(X)$). If α is a cycle class in $A_*(X)$, and k is an integer, we denote by $\{\alpha\}_k$ the component of α in $A_k(X)$.

Let X be a scheme, Y be a closed subscheme of X , and Y_1, \dots, Y_m be the irreducible components of Y . The *geometric multiplicity* m_i of Y_i in Y is defined to be the length of the local ring of Y along Y_i as an \mathcal{O}_{Y,Y_i} -module:

$$m_i = l_{\mathcal{O}_{Y,Y_i}}(\mathcal{O}_{Y,Y_i}).$$

We define the *fundamental cycle* of Y by

$$[Y] = \sum_{i=1}^m m_i [Y_i] \in Z_*(X).$$

By abuse of notation, we also write $[Y]$ for the corresponding cycle class in $A_*(X)$. If Y is purely k -dimensional, that is, $\dim(Y_i) = k$ for all i , then $[Y] \in Z_k(X)$.

If $n = \dim X$, then we have $A_n(X) \cong \mathbb{Z} \cdot [X]$. If $k < 0$ or $k > n$, then we have $A_k(X) = 0$.

For each k , we set

$$A^k(X) = A_{n-k}(X),$$

and

$$A^*(X) = \bigoplus_{k=0}^n A^k(X).$$

Let $1 \in A^0(X)$ be the cycle class $[X] \in A_n(X)$. Throughout this thesis we write $A(X)$ instead of $A^*(X)$ or $A_*(X)$ whenever the grading does not play a role.

1.1.2. Pushforwards and Pullbacks

We first recall the basic definitions of some properties of morphisms in algebraic geometry (see [59, Chapter 2, Section 4] for more details).

Definition 1.1.1. Let X, Y be two schemes and $f : X \rightarrow Y$ be a morphism from X to Y .

- (i) We say that f is *proper* if it is separated, of finite type, and universally closed.
- (ii) We say that f is *flat* if locally the induced maps $f_x : \mathcal{O}_{Y,f(x)} \rightarrow \mathcal{O}_{X,x}$ for all $x \in X$ are flat maps of rings.
- (iii) We say that f has *relative dimension* n if $\dim V = \dim W + n$ for all subvarieties W of Y and irreducible components V of $f^{-1}(W)$.

Let $f : X \rightarrow Y$ be a proper morphism. For any k -dimensional subvariety V of X , we set $W = f(V)$ and define

$$f_*[V] = \begin{cases} [R(V) : R(W)][W] & \text{if } \dim(W) = k, \\ 0 & \text{otherwise,} \end{cases}$$

where $[R(V) : R(W)]$ denotes the degree of the field extension. This extends linearly to a homomorphism of groups

$$f_* : Z_k(X) \rightarrow Z_k(Y).$$

Theorem 1.1.2. *Let $f : X \rightarrow Y$ be a proper morphism and let c be a k -cycle on X which is rationally equivalent to zero. Then $f_*(c)$ is a k -cycle which is rationally equivalent to zero on Y .*

Proof. See [36, Theorem 1.4]. □

This theorem implies that there is an induced homomorphism of Chow groups

$$f_* : A_*(X) \rightarrow A_*(Y),$$

which is called the *pushforward homomorphism* associated to f .

Definition 1.1.3. Let X be a complete scheme, that is, X is proper over $\text{Spec}(\mathbb{C})$, and let c be a zero-cycle on X . The *degree* of c , denoted $\int_X c$, is defined by

$$\int_X c = p_*(c),$$

where p is the structure morphism from X to $\text{Spec}(\mathbb{C})$, and

$$A_0(\text{Spec}(\mathbb{C})) \cong \mathbb{Z} \cdot [\text{Spec}(\mathbb{C})]$$

is identified with \mathbb{Z} . If $c = \sum_i n_i [P_i]$ for some points $P_i \in X$, then

$$\int_X c = \sum_i n_i.$$

By Theorem 1.1.2, this is well-defined for any zero-dimensional cycle class. We then extend the degree homomorphism to all of $A_*(X)$,

$$\int_X : A_*(X) \rightarrow \mathbb{Z}$$

by defining

$$\int_X \alpha = 0 \text{ if } \alpha \in A_k(X), k > 0.$$

As a special case of functoriality, for any morphism $f : X \rightarrow Y$ of complete schemes and any $\alpha \in A_*(X)$, we have

$$\int_X \alpha = \int_Y f_*(\alpha).$$

Let $f : X \rightarrow Y$ be a flat morphism of relative dimension n and W be a k -dimensional subvariety of Y . Then $f^{-1}(W)$ is a subvariety of pure dimension $k + n$ and we set

$$f^*[W] = [f^{-1}(W)].$$

This extends linearly to a homomorphism

$$f^* : Z_k(Y) \longrightarrow Z_{k+n}(X).$$

Theorem 1.1.4. *Let $f : X \rightarrow Y$ be a flat morphism of relative dimension n and c be a k -cycle on Y which is rationally equivalent to zero. Then $f^*(c)$ is a $(k + n)$ -cycle which is rationally equivalent to zero in $Z_{k+n}(X)$.*

Proof. See [36, Theorem 1.7]. □

This theorem implies that there is an induced homomorphism of Chow groups

$$f^* : A_k(Y) \longrightarrow A_{k+n}(X).$$

Therefore there is a homomorphism of Chow groups

$$f^* : A_*(Y) \longrightarrow A_*(X).$$

This homomorphism is called the *pullback homomorphism* associated to f .

1.1.3. Intersection Products

In this subsection, we shall see that the cycles modulo rational equivalence on a smooth quasiprojective variety form a graded ring.

Definition 1.1.5. Let X be a variety.

- (i) We say that two irreducible subschemes A and B of X are *dimensionally transverse* if for every component C of $A \cap B$, we have

$$\text{codim}(C) = \text{codim}(A) + \text{codim}(B).$$

- (ii) We say that two subvarieties A and B of X are *transverse* at a point p if A, B , and X are all smooth at p , and the tangent spaces to A and B at p together span the tangent space to X .
- (iii) We say that two subvarieties A and B of X intersect transversely if they are transverse at each point $p \in A \cap B$.
- (iv) We say that two subvarieties A and B of X are *generically transverse* if they are transverse at a general point of each component C of $A \cap B$.
- (v) We say that two cycles $\sum m_i A_i$ and $\sum n_j B_j$ are *dimensionally* (respectively *generically*) *transverse* if A_i, B_j are dimensionally (respectively generically) transverse for every i, j .

Generic transversality implies dimensional transversality, but the reverse is wrong in general. More precisely, we have the following statement.

Theorem 1.1.6. *Two subvarieties A and B of a variety X are generically transverse if and only if they are dimensionally transverse and each irreducible component of $A \cap B$ is reduced and contains a smooth point of X .*

Proof. See [29, Proposition 5.2]. □

An important result of this section is as follows:

Theorem 1.1.7. *If X is a smooth quasiprojective variety, then there is a unique product structure on $A^*(X)$ that satisfies*

$$[A] \cdot [B] = [A \cap B]$$

whenever A and B are subvarieties that are generically transverse. This structure makes $A^(X)$ into an associative, commutative ring with unit $[X]$. This ring is graded by codimension.*

Proof. See [29, Theorem 5.3]. □

Definition 1.1.8. Let X be a smooth quasiprojective variety. Then the *Chow ring* of X is the ring described in Theorem 1.1.7.

The key idea of the proof is to find a way to compare an arbitrary pair of cycles with a pair of cycles that are generically transverse. For this, we need the following lemma.

Lemma 1.1.9 (Moving Lemma). *Let X be a smooth quasiprojective variety.*

(a) *For every $\alpha, \beta \in A(X)$, there are cycles $A = \sum m_i A_i$ and $B = \sum n_j B_j$ in $Z(X)$ that represent α and β such that A_i intersects B_j generically transversely for each i and j .*

(b) *The cycle class*

$$\sum_{i,j} m_i n_j [A_i \cap B_j]$$

in $A(X)$ is independent of the choice of such A, B .

Proof. A proof of this lemma can be found in [29, Section 5.6]. □

1.1.4. Examples

Theorem 1.1.10. *The Chow ring of \mathbb{P}^n is*

$$A(\mathbb{P}^n) = \mathbb{Z}[h]/(h^{n+1}),$$

where h is the cycle class of a hyperplane. In addition, the cycle class of a subvariety of codimension r and degree d is dh^r .

Proof. A proof of this theorem can be found in [29, Section 1.2.2]. □

We illustrate the Chow ring of \mathbb{P}^2 by Schubert3 as follows:

```

sage: P = ProjectiveSpace(2); P
A projective space of dimension 2
sage: P.hyperplane()
h
sage: R = P.chow_ring(); R
Univariate Quotient Polynomial Ring in h over
Rational Field with modulus h^3
sage: R.gen() == P.hyperplane()
True
sage: I = R.modulus(); I
h^3

```

Here is the implementation of the Chow ring of \mathbb{P}^2 in `schubert.lib`:

```

> variety P = projectiveSpace(2);
> P;
A variety of dimension 2
> P.dimension;
2
> def r = P.baseRing;
> setring r;
> ChowRing(P);
// characteristic : 0
// number of vars : 1
//      block   1 : ordering wp
//                  : names   h
//                  : weights  1
//      block   2 : ordering C
// quotient ring from ideal ...
> P.relations;
_[1]=h3

```

Remark 1.1.11. In `Schubert3` and `schubert.lib`, an abstract variety is represented by a nonnegative integer which is its dimension and a graded ring which is its Chow ring. For example, the n -dimensional projective space \mathbb{P}^n is represented by its dimension and the graded ring $\mathbb{Q}[h]/(h^{n+1})$. Note that we have to work with rational coefficients rather than integer ones.

As an immediate consequence of Theorem 1.1.10, we get a general form of Bézout's theorem.

Theorem 1.1.12 (Bézout's theorem). *If $Z_1, \dots, Z_k \subset \mathbb{P}^n$ are subvarieties of codimensions r_1, \dots, r_k with $\sum r_i \leq n$ and the Z_i are pairwise generically transverse, then*

$$\deg(Z_1 \cap \dots \cap Z_k) = \prod \deg(Z_i).$$

In particular, two subvarieties $X, Y \subset \mathbb{P}^n$ having complementary dimension and intersecting transversely will intersect in exactly $\deg(X) \deg(Y)$ points.

Proof. By induction, we only need to show the statement in case $k = 2$. From the assumption, for each i , we have

$$[Z_i] = \deg(Z_i)h^{r_i} \in A^*(\mathbb{P}^n),$$

where h is the cycle class of a hyperplane in \mathbb{P}^n . Since Z_1 and Z_2 are generically transverse, in particular, $\text{codim}(Z_1 \cap Z_2) = r_1 + r_2$, we have

$$[Z_1 \cap Z_2] = \deg(Z_1 \cap Z_2)h^{r_1+r_2}.$$

On the other hand, we have

$$[Z_1 \cap Z_2] = [Z_1] \cdot [Z_2] = \deg(Z_1)h^{r_1} \deg(Z_2)h^{r_2} = \deg(Z_1) \deg(Z_2)h^{r_1+r_2}.$$

This implies that

$$\deg(Z_1 \cap Z_2) = \deg(Z_1) \deg(Z_2).$$

By Theorem 1.1.6, if two subvarieties $X, Y \subset \mathbb{P}^n$ have complementary dimension and intersect transversely, then they are generically transverse. This implies that X and Y intersect in exactly $\deg(X \cap Y) = \deg(X) \deg(Y)$ points. \square

We recall that the degree of an n -dimensional subvariety $X \subset \mathbb{P}^N$ is the number of intersection points of X with the intersection of n general hyperplanes in \mathbb{P}^N . For example, the degree of a conic in \mathbb{P}^2 is two, because a general line meets a conic in two points.

One can use Bézout's theorem to compute the degree of a Veronese variety. Indeed, let

$$\nu = \nu_{n,d} : \mathbb{P}^n \longrightarrow \mathbb{P}^{\binom{n+d}{n}-1}$$

be the Veronese embedding given by

$$[x_0 : \cdots : x_n] \longmapsto [\cdots : x^I : \cdots],$$

where x^I ranges over all monomials of degree d in $n+1$ variables. The image $V_{n,d} = \nu_{n,d}(\mathbb{P}^n) \subset \mathbb{P}^N$ of the Veronese embedding is called the d -th Veronese variety of \mathbb{P}^n .

Proposition 1.1.13. *The degree of the d -th Veronese variety of \mathbb{P}^n is d^n .*

Proof. The degree of $V_{n,d}$ is the number of intersection points of $V_{n,d}$ with the intersection of n general hyperplanes $H_1, \dots, H_n \subset \mathbb{P}^{\binom{n+d}{n}-1}$. Since ν is injective, this is the cardinality of the intersection $\nu^{-1}(H_1) \cap \cdots \cap \nu^{-1}(H_n) \subset \mathbb{P}^n$. Since the preimages of the hyperplanes H_i are the hypersurfaces of degree d in \mathbb{P}^n , by Bézout's theorem, this cardinality is d^n . \square

Theorem 1.1.14. *The Chow ring of $\mathbb{P}^n \times \mathbb{P}^m$ is given by the formula*

$$A(\mathbb{P}^n \times \mathbb{P}^m) \cong A(\mathbb{P}^n) \otimes_{\mathbb{Z}} A(\mathbb{P}^m).$$

Equivalently, if $\alpha, \beta \in A^1(\mathbb{P}^n \times \mathbb{P}^m)$ denote the pullbacks of the hyperplane classes on \mathbb{P}^n and \mathbb{P}^m via the projection maps, then

$$A(\mathbb{P}^n \times \mathbb{P}^m) \cong \mathbb{Z}[\alpha, \beta]/(\alpha^{n+1}, \beta^{m+1}).$$

Moreover, the cycle class of a hypersurface defined by a bihomogeneous form of bidegree (d, e) on $\mathbb{P}^n \times \mathbb{P}^m$ is $d\alpha + e\beta$.

Proof. See [29, Theorem 1.26]. \square

We illustrate the Chow ring of $\mathbb{P}^3 \times \mathbb{P}^5$ by Schubert3 as follows:

```

sage: P3 = ProjectiveSpace(3,'h'); P3
A projective space of dimension 3
sage: P5 = ProjectiveSpace(5,'H'); P5
A projective space of dimension 5
sage: W = ProductVariety(P3,P5); W
A product variety of A projective space of dimension
3 and A projective space of dimension 5
sage: R = W.chow_ring(); R
Quotient of Multivariate Polynomial Ring in h, H over
Rational Field by the ideal (h^4, H^6)
sage: I = R.defining_ideal(); I
Ideal (h^4, H^6) of Multivariate Polynomial Ring in
h, H over Rational Field
sage: I.gens()
[h^4, H^6]

```

Similarly, we illustrate the Chow ring of $\mathbb{P}^3 \times \mathbb{P}^5$ by `schubert.lib` as follows:

```

> variety P3 = projectiveSpace(3,"h");
> variety P5 = projectiveSpace(5,"H");
> variety W = productVariety(P3,P5);
> def r = W.baseRing;
> setring r;
> W.dimension;
8
> ChowRing(W);
// characteristic : 0
// number of vars : 2
//      block   1 : ordering wp
//                : names     h
//                : weights   1
//      block   2 : ordering wp
//                : names     H
//                : weights   1
//      block   3 : ordering C
// quotient ring from ideal ...
> W.relations;
_[1]=h4
_[2]=H6

```

Remark 1.1.15. In `Schubert3` and `schubert.lib`, a product variety of two abstract varieties X and Y is represented as an abstract variety by a nonnegative integer which is equal to $\dim(X) + \dim(Y)$, and a graded ring which is the tensor product of two graded rings $A(X)$ and $A(Y)$.

As a consequence of Theorem 1.1.14, we obtain the degree formula of a Segre variety. The *Segre variety* $S_{n,m}$ is by definition the image of the product $\mathbb{P}^n \times \mathbb{P}^m$ in $\mathbb{P}^{(n+1)(m+1)-1}$ under the Segre embedding

$$\sigma_{n,m} : ([x_0, \dots, x_n], [y_0, \dots, y_m]) \longmapsto [\dots, x_i y_j, \dots].$$

Proposition 1.1.16. *The degree of the Segre variety $S_{n,m}$ is the binomial coefficient $\binom{n+m}{n}$.*

Proof. The degree of $S_{n,m}$ is the number of points in which $S_{n,m}$ meets the intersection of $n+m$ hypersurfaces in $\mathbb{P}^{(n+1)(m+1)-1}$. Since $\sigma_{n,m}$ is an embedding, we may compute this number by pulling back these hypersurfaces to $\mathbb{P}^n \times \mathbb{P}^m$ and computing in the Chow ring of $\mathbb{P}^n \times \mathbb{P}^m$. Thus $\deg S_{n,m} = \deg(\alpha + \beta)^{n+m}$, which gives us the desired formula because

$$(\alpha + \beta)^{n+m} = \binom{n+m}{n} \alpha^n \beta^m$$

in $\mathbb{Z}[\alpha, \beta]/(\alpha^{n+1}, \beta^{m+1})$. □

1.2. Chern and Segre Classes

For convenience, a variety is always assumed to be smooth and quasiprojective in this section.

Let E be a vector bundle of rank r on a variety X , and let \mathcal{E} be the locally free sheaf of sections of E . The projective bundle $\mathbb{P}(E)$ of E on X is defined to be

$$p : \mathbb{P}(E) = \text{Proj}(\text{Sym}(\mathcal{E}^\vee)) \longrightarrow X,$$

Note that the projection morphism $p : \mathbb{P}(E) \rightarrow X$ is proper and flat. Let ζ be the cycle class of the divisor associated to the line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$. By [36, Theorem 3.3(b)], the cycle class ζ^r has a unique presentation of the form

$$\zeta^r = - \sum_{k=1}^r p^*(\alpha_k) \zeta^{r-k},$$

where $\alpha_k \in A^k(X)$. Then the k -th Chern class of E is defined to be the cycle class

$$c_k(E) := \alpha_k \in A^k(X).$$

For convenience, we define $c_0(E) = 1$ and $c_k(E) = 0$ if $k < 0$ or $k > r$. The total Chern class of E is defined to be the sum

$$c(E) = 1 + c_1(E) + c_2(E) + \cdots + c_r(E).$$

The Segre classes $s_1(E), \dots, s_r(E)$ of E are defined by the recursion formulas

$$s_k(E) + s_{k-1}(E)c_1(E) + \cdots + s_1(E)c_{k-1}(E) + c_k(E) = 0$$

for all $k = 1, \dots, r$. For convenience, we define $s_0(E) = 1$ and $s_k(E) = 0$ if $k < 0$ or $k > r$. More precisely, we have

$$s_1(E) = -c_1(E), s_2(E) = c_1(E)^2 - c_2(E), \dots$$

The total Segre class of E is defined to be the sum

$$s(E) = 1 + s_1(E) + s_2(E) + \cdots + s_r(E).$$

The total Chern class and the total Segre class of a vector bundle E on a variety X are inverses of each other in the Chow ring of X . This means that we have the following identity

$$c(E)s(E) = 1 \in A(X).$$

Here are the important properties of Chern classes:

Proposition 1.2.1 (Whitney sum formula). *If E and F are two vector bundles, then*

$$c(E \oplus F) = c(E)c(F).$$

Equivalently, we have

$$c_k(E \oplus F) = \sum_{i=0}^k c_i(E)c_{k-i}(F).$$

Proof. See [29, Theorem 7.5]. □

Proposition 1.2.2 (Pullback formula). *Let X and Y be two projective varieties. If $f : X \rightarrow Y$ is a flat morphism of relative dimension, say, n and E is a vector bundle on Y , then*

$$c_k(f^*E) = f^*c_k(E)$$

for all k .

Proof. See [29, Proposition 7.12]. □

1.2.1. Splitting Principle

The splitting principle is a useful technique for reducing questions concerning vector bundles to questions concerning line bundles.

Let E be a vector bundle of rank r on a variety X . The splitting principle says that there exists a variety Y and a flat morphism $f : Y \rightarrow X$ such that

- (i) the pullback homomorphism $f^* : A(X) \rightarrow A(Y)$ is injective,
- (ii) the pullback bundle f^*E has a filtration by the vector bundles

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{r-1} \subset E_r = f^*E,$$

such that the quotients E_i/E_{i-1} are line bundles on Y .

The injectivity of the pullback homomorphism f^* shows that any identity concerning Chern classes which holds in $A(Y)$ (and is invariant under pullback homomorphisms) also holds in $A(X)$. Moreover, these identities are easier to understand for direct sums of line bundles than for arbitrary vector bundles, so they should be understood in $A(Y)$ and then pushed down to $A(X)$.

Using the splitting principle, we can regard the Chern classes of E as the elementary symmetric polynomials in r variables α_i for all $i = 1, \dots, r$, which are called the *Chern roots* of E . More precisely, we have

$$\begin{aligned} c_0(E) &= 1, \\ c_1(E) &= \sum_{1 \leq i \leq r} \alpha_i, \\ c_2(E) &= \sum_{1 \leq i < j \leq r} \alpha_i \alpha_j, \\ &\vdots \\ c_r(E) &= \alpha_1 \alpha_2 \cdots \alpha_r. \end{aligned}$$

Using the splitting principle and the Chern roots, we have the following statements.

Proposition 1.2.3. *Let E and F be two vector bundles with Chern roots $(\alpha_i)_i$ and $(\beta_j)_j$, respectively. Then we have the following statements:*

(i) E^\vee has Chern roots $(-\alpha_i)_i$. Hence $c_k(E^\vee) = (-1)^k c_k(E)$ for all k .

(ii) $E \otimes F$ has Chern roots $(\alpha_i + \beta_j)_{i,j}$.

(iii) $\text{Sym}^d E$ has Chern roots $(\alpha_{i_1} + \cdots + \alpha_{i_d})_{i_1 \leq \cdots \leq i_d}$.

(iv) $\wedge^d E$ has Chern roots $(\alpha_{i_1} + \cdots + \alpha_{i_d})_{i_1 < \cdots < i_d}$.

Here we denote by $\text{Sym}^d E$ the d -th symmetric power of E , and $\wedge^d E$ the d -th exterior power of E .

Proof. See [36, Remark 3.2.3 and Example 3.2.6]. □

Remark 1.2.4. Proposition 1.2.3 shows how to implement the standard operations on vector bundles.

Example 1.2.5. Let E be a vector bundle of rank 2 on a variety X of dimension 4. We want to compute the Chern classes of $\text{Sym}^3 E$ in terms of the Chern classes of E . If α_1 and α_2 are the Chern roots of E , then $\text{Sym}^3 E$ has the Chern roots $3\alpha_1, 2\alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2, 3\alpha_2$. Thus we have

$$\begin{aligned} c_1(\text{Sym}^3 E) &= 3\alpha_1 + 2\alpha_1 + \alpha_2 + \alpha_1 + 2\alpha_2 + 3\alpha_2 \\ &= 6(\alpha_1 + \alpha_2) \\ &= 6c_1(E). \end{aligned}$$

Similarly, we have

$$\begin{aligned} c_2(\text{Sym}^3 E) &= 11c_1(E)^2 + 10c_2(E), \\ c_3(\text{Sym}^3 E) &= 6c_1(E)^3 + 30c_1(E)c_2(E), \\ c_4(\text{Sym}^3 E) &= 18c_1(E)^2c_2(E) + 9c_2(E)^2. \end{aligned}$$

This can be made with `Schubert3` as follows:

```
sage: X = Base(4, bundles=[[2, 'c'], [3, 'd']]); X
A variety of dimension 4
sage: E = X.bundle(1); E
A vector bundle of rank 2 on A variety of dimension 4
sage: E.chern_class(1), E.chern_class(2)
(c1, c2)
sage: S = E.symmetric_power(3); S
A vector bundle of rank 4 on A variety of dimension 4
sage: S.chern_class(1)
6*c1
sage: S.chern_class(2)
11*c1^2 + 10*c2
sage: S.chern_class(3)
6*c1^3 + 30*c1*c2
sage: S.chern_class(4)
18*c1^2*c2 + 9*c2^2
```

We also illustrate this calculation by `schubert.lib` in Example B.1.1.

Example 1.2.6. Let X be a variety of dimension 2 and E, F be two vector bundles on X with $\text{rank } E = 2, \text{rank } F = 3$. We want to compute the Chern classes of the vector bundle $E \otimes F$ in terms of the Chern classes of E and F . If α_1, α_2 are the Chern roots of E , and $\beta_1, \beta_2, \beta_3$ are the Chern roots of F , then $E \otimes F$ has the Chern roots $\alpha_1 + \beta_1, \alpha_1 + \beta_2, \alpha_1 + \beta_3, \alpha_2 + \beta_1, \alpha_2 + \beta_2, \alpha_2 + \beta_3$. Thus we have

$$\begin{aligned} c_1(E \otimes F) &= 3(\alpha_1 + \alpha_2) + 2(\beta_1 + \beta_2 + \beta_3) \\ &= 3c_1(E) + 2c_1(F). \end{aligned}$$

Similarly, we have

$$c_2(E \otimes F) = 3c_1(E)^2 + 3c_2(E) + 5c_1(E)c_1(F) + c_1(F)^2 + 2c_2(F).$$

This can be made with `Schubert3` as follows:

```
sage: X = Base(2,bundles=[[2,'c'],[3,'d']]); X
A variety of dimension 2
sage: E = X.bundle(1); E
A vector bundle of rank 2 on A variety of dimension 2
sage: E.chern_class(1),E.chern_class(2)
(c1, c2)
sage: F = X.bundle(2); F
A vector bundle of rank 3 on A variety of dimension 2
sage: F.chern_class(1),F.chern_class(2),F.chern_class(3)
(d1, d2, d3)
sage: S = E & F; S #the tensor product of E and F
A vector bundle of rank 6 on A variety of dimension 2
sage: S.chern_class(1)
3*c1 + 2*d1
sage: S.chern_class(2)
3*c1^2 + 3*c2 + 5*c1*d1 + d1^2 + 2*d2
```

We also illustrate this calculation by `schubert.lib` in Example B.1.2.

1.3. Chern Characters and Todd Classes

Let E be a vector bundle (locally free sheaf) of rank r on a smooth projective variety X . For convenience, we write $A(X)_{\mathbb{Q}}$ instead of $A(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. Using the notion of Chern roots, we can define formally the *Chern character* and the *Todd class* of E as follows:

$$\text{ch}(E) = \sum_{i=1}^r \exp(\alpha_i) \in A(X)_{\mathbb{Q}},$$

and

$$\text{td}(E) = \prod_{i=1}^r \frac{\alpha_i}{1 - \exp(-\alpha_i)} \in A(X)_{\mathbb{Q}},$$

where $\alpha_1, \dots, \alpha_r$ are the Chern roots of E and the expressions in the α_i are understood as formal power series, that is

$$\exp(\alpha_i) = 1 + \alpha_i + \frac{1}{2}\alpha_i^2 + \frac{1}{6}\alpha_i^3 + \dots,$$

and

$$\frac{\alpha_i}{1 - \exp(-\alpha_i)} = 1 + \frac{1}{2}\alpha_i + \frac{1}{12}\alpha_i^2 + \dots.$$

Here are some basic properties of Chern characters and Todd classes:

1. If E and F are two vector bundles, then we have the following formulas:
 - (i) $\text{ch}(E \otimes F) = \text{ch}(E) \cdot \text{ch}(F)$,
 - (ii) $\text{ch}(E \oplus F) = \text{ch}(E) + \text{ch}(F)$,
 - (iii) $\text{td}(E \oplus F) = \text{td}(E) \cdot \text{td}(F)$.
2. The Chern character and the Chern classes of E are related by the following formal identity

$$\begin{aligned} \text{ch}(E) &= r + c_1 + \frac{1}{2}(c_1^2 - 2c_2) + \frac{1}{6}(c_1^3 - 3c_1c_2 + 3c_3) + \dots \\ &= \sum_{k \geq 0} \frac{p_k}{k!}, \end{aligned}$$

where $c_i = c_i(E)$ and p_k is the determinant of matrix

$$\begin{pmatrix} c_1 & 1 & 0 & \dots & 0 \\ 2c_2 & c_1 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 1 \\ kc_k & c_{k-1} & \dots & \dots & c_1 \end{pmatrix}.$$

This is useful for the implementation of vector bundles in `Schubert3` and `schubert.lib`.

3. *Borel-Serre formula*: The Chern character and the Todd class of E are related by the following formal identity:

$$\sum_{i=0}^r (-1)^i \text{ch}(\Lambda^i E^\vee) = c_r(E) \cdot \text{td}(E)^{-1},$$

where r is the rank of E . This immediately follows using the splitting principle (see [36, Example 3.2.5]). This formula gives us an effective method for computing the Todd class of a vector bundle.

Example 1.3.1. Let X be a smooth projective variety of dimension 3 and E be a vector bundle of rank 2 on X . Then we have

$$\text{ch}(E) = 2 + c_1 + \frac{1}{2}(c_1^2 - 2c_2) + \frac{1}{6}(c_1^3 - 3c_1c_2),$$

and

$$\text{td}(E) = 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}c_1c_2,$$

where $c_i = c_i(E)$ be i -th Chern class of E .

We illustrate this by `Schubert3` as follows:

```

sage: X = Base(3,bundles=[[2,'c'],[3,'d']]); X
A variety of dimension 3
sage: E = X.bundle(1); E
A vector bundle of rank 2 on A variety of dimension 3
sage: E.chern_class(1),E.chern_class(2)
(c1, c2)
sage: E.chern_character()
1/6*c1^3 - 1/2*c1*c2 + 1/2*c1^2 - c2 + c1 + 2
sage: E.todd_class()
1/24*c1*c2 + 1/12*c1^2 + 1/12*c2 + 1/2*c1 + 1

```

1.4. Hirzebruch-Riemann-Roch Theorem

We now state the Hirzebruch-Riemann-Roch theorem.

Theorem 1.4.1 (Hirzebruch-Riemann-Roch). *Let E be a vector bundle on a smooth projective variety X . Then we have the following formula:*

$$\chi(X, E) = \int_X \text{ch}(E) \cdot \text{td}(X),$$

where $\chi(X, E)$ is the Euler characteristic of E on X , $\text{td}(X)$ is the Todd class of the tangent bundle on X , and $\int_X \alpha$ is the degree of the cycle class α in $A(X)_{\mathbb{Q}}$.

Proof. See [57, Chapter 4]. □

For more general versions of the Hirzebruch-Riemann-Roch theorem, we refer to [36, Chapter 15].

The formula in Theorem 1.4.1 gives us an effective method for computing the Euler characteristic of a vector bundle E on a smooth projective variety X in terms of the Chern character of E and the Todd class of the tangent bundle T_X on X .

Algorithm 1.4.1 The Hirzebruch-Riemann-Roch formula

Input: A vector bundle E on a smooth projective variety X .

Output: The Euler characteristic $\chi(X, E)$ of E on X .

- 1: Compute the Chern character $\text{ch}(E)$ of E .
 - 2: Compute the Todd class $\text{td}(X)$ of the tangent bundle on X .
 - 3: **Return** the degree of the zero-dimensional part of $\text{ch}(E) \cdot \text{td}(X)$ in $A(X)_{\mathbb{Q}}$.
-

Example 1.4.2. In this example, we use the Hirzebruch-Riemann-Roch formula for computing the Euler characteristic of the Horrocks-Mumford bundle.

The Horrocks-Mumford bundle F is an indecomposable rank 2 vector bundle on \mathbb{P}^4 . It was introduced by Horrocks and Mumford in [58]. The total Chern class of F is

$$c(F) = 1 + 5h + 10h^2,$$

where h is the hyperplane class of \mathbb{P}^4 .

Here is how the construction of the Horrocks-Mumford bundle can be made with `Schubert3`:

```

sage: P = ProjectiveSpace(4); P
A projective space of dimension 4
sage: h = P.hyperplane()
sage: F = VectorBundle(P, 2, chern_class = 1 + 5*h + 10*h^2); F
A vector bundle of rank 2 on A projective space of dimension 4
sage: F.rank()
2
sage: F.total_chern_class()
10*h^2 + 5*h + 1

```

Alternatively, we can use the representation of the Horrocks-Mumford bundle as the cohomology of a monad of the form (see [89, Chapter 2, Section 3.3] for more details)

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^4}(-1)^5 \longrightarrow (\wedge^2 \Omega_{\mathbb{P}^4} \otimes \mathcal{O}_{\mathbb{P}^4}(2))^2 \longrightarrow \mathcal{O}_{\mathbb{P}^4}^5 \longrightarrow 0,$$

where $\Omega_{\mathbb{P}^4}$ is the cotangent bundle on \mathbb{P}^4 . In this way, the Horrocks-Mumford bundle can be constructed as follows:

```

sage: A = P.cotangent_bundle().wedge(2) & P.O(2)
sage: B = A^2 - P.O(0)^5 - P.O(-1)^5
sage: C = B & P.O(-2)
sage: F = C.dual(); F
A vector bundle of rank 2 on A projective space of dimension 4
sage: F.total_chern_class()
10*h^2 + 5*h + 1

```

The Euler characteristic $\chi(\mathbb{P}^4, F)$ of the Horrocks-Mumford bundle F on \mathbb{P}^4 is computed as follows:

```

sage: F.chi()
2

```

Moreover, we can compute the Euler characteristic of the twists of the Horrocks-Mumford bundle

$$\chi(\mathbb{P}^4, F \otimes \mathcal{O}_{\mathbb{P}^4}(n)),$$

where n is an integer.

```

sage: def twists_Horrocks_Mumford_bundle(n):
.....:     P = ProjectiveSpace(4)
.....:     h = P.hyperplane()
.....:     c = 1 + 5*h + 10*h^2
.....:     F = VectorBundle(P, 2, chern_class = c)
.....:     L = P.O(n)
.....:     E = F & L
.....:     return E.chi()
.....:
sage: twists_Horrocks_Mumford_bundle(0)
2
sage: twists_Horrocks_Mumford_bundle(1)
35

```

```

sage: twists_Horrocks_Mumford_bundle(2)
100
sage: twists_Horrocks_Mumford_bundle(3)
210
sage: twists_Horrocks_Mumford_bundle(4)
380

```

1.5. Excess Intersection Formula

Let us consider a simple example.

Example 1.5.1. [65, Chapter 8] Consider two conics $C_1, C_2 \subset \mathbb{P}^2$ which have a line in common, for instance, $C_1 = Z(x_0x_1), C_2 = Z(x_0x_2)$. In this case, $C_1 \cap C_2$ has two components, the point $p = (1, 0, 0)$ and the line $L = Z(x_0)$. In the general case, if the intersection is transverse, Bézout's Theorem says that there are four intersection points of C_1 and C_2 counting multiplicity. However even if the intersection is not transverse, we can still compute

$$\int_{\mathbb{P}^2} [C_1][C_2] = \int_{\mathbb{P}^2} (2h)(2h) = \int_{\mathbb{P}^2} 4h^2 = 4,$$

where h is the hyperplane class of \mathbb{P}^2 . We want to find a way to make sense of the answer 4 in terms of the geometry of $C_1 \cap C_2 = L \cup \{p\}$. It is clear that the point p is a transverse intersection, so $\int_{\mathbb{P}^2} [p] = 1$. Thus we need to find a way to make sense of the intersection number of C_1 and C_2 along L such that $\int_{\mathbb{P}^2} [L] = 3$.

The excess intersection formula gives us an effective method for defining the cycle class of connected components where the intersection is not transverse. We first need the following notations.

Let X be a smooth projective variety of dimension n . We denote by T_X the tangent bundle on X . If $Z \subset X$ is a smooth subvariety, we denote by $T_X|_Z$ the restriction of T_X to Z . We claim that T_Z is a subbundle of $T_X|_Z$. The quotient $N_{Z/X} = (T_X|_Z) - T_Z$ is called the normal bundle of Z in X . If $i : Z \hookrightarrow X$ is the inclusion morphism, then we have

$$\text{ch}(N_{Z/X}) = \text{ch}(T_X|_Z) - \text{ch}(T_Z) = i^*(\text{ch}(T_X)) - \text{ch}(T_Z).$$

Definition 1.5.2. Let X_1, \dots, X_k be subvarieties of a smooth projective variety X . Suppose that Z is a connected component of $X_1 \cap \dots \cap X_k$, and set

$$m = \dim(Z) - \text{codim}(X_1) - \dots - \text{codim}(X_k).$$

We write $(X_1 \cap \dots \cap X_k)^Z \in A_m(Z)$ for the part of $X_1 \cap \dots \cap X_k$ supported on Z and call it the *equivalence class of Z for the intersection $X_1 \cap \dots \cap X_k$* . Let N_i be the restriction of $N_{X_i/X}$ to Z . By the arguments as in [29, Section 15.2], we have

$$(X_1 \cap \dots \cap X_k)^Z = \left\{ \frac{c(N_1) \dots c(N_k)}{c(N_{Z/X})} \right\}_m \in A_m(Z).$$

Theorem 1.5.3 (Excess intersection formula). *Let X_1, \dots, X_k be subvarieties of a smooth projective variety X . Suppose that $X_1 \cap \dots \cap X_k = \cup_j Z_j$ is a disjoint union of smooth varieties Z_j . Write $i_j : Z_j \hookrightarrow X$ for the inclusion morphism. Then we have*

$$\prod_{i=1}^k [X_k] = \sum_j (i_j)_* ((X_1 \cap \dots \cap X_k)^{Z_j}) \in A(X).$$

Proof. See [29, Section 15.2]. □

We give an effective algorithm (see Algorithm 1.5.1) for computing the degree of $(X_1 \cap \cdots \cap X_k)^Z$ in the case $X = \mathbb{P}^n$ and the X_i are smooth hypersurfaces.

Algorithm 1.5.1 The excess intersection formula

Input: X_1, \dots, X_k are k smooth hypersurfaces in \mathbb{P}^n , and Z is a connected component of $X_1 \cap \cdots \cap X_k$.

Output: The degree of the equivalence class of Z for the intersection $X_1 \cap \cdots \cap X_k$ in $A(Z)$.

- 1: Compute the vector bundle $N = N_1 \oplus \cdots \oplus N_k$, where N_i is the restriction of $N_{X_i/X}$ to Z .
 - 2: Compute the vector bundle $E = N - N_{Z/\mathbb{P}^n}$.
 - 3: Compute the top Chern class of E in $A(Z)$.
 - 4: **Return** the degree of this top Chern class.
-

As an example, we illustrate how to compute the intersection number of C_1 and C_2 along L as in Example 1.5.1 by `Schubert3` as follows:

```
sage: P1 = ProjectiveSpace(1); P2 = ProjectiveSpace(2)
sage: h = P1.hyperplane(); H = P2.hyperplane()
sage: f = Morphism(P1,P2,[h])
sage: B = VectorBundle(P2,2,chern_class=(1+2*H)^2)
sage: E = f.excess_bundle(B)
sage: P1.integral(E.top_chern_class())
3
```

Example 1.5.4. [36, Example 9.1.4]

1. Let Q_1, Q_2 , and Q_3 be three quadratic hypersurfaces in \mathbb{P}^3 . Suppose that

$$Q_1 \cap Q_2 \cap Q_3 = L \cup \{p_1, \dots, p_k\},$$

where L is a line, and p_1, \dots, p_k are points. In order to determine the number of the points p_i , we need to compute the intersection number of Q_1, Q_2 , and Q_3 along L , that is

$$\int_L (Q_1 \cap Q_2 \cap Q_3)^L.$$

```
sage: P1 = ProjectiveSpace(1,'h'); P3 = ProjectiveSpace(3,'H')
sage: h = P1.hyperplane(); H = P3.hyperplane()
sage: f = Morphism(P1,P3,[h])
sage: B = VectorBundle(P3,3,chern_class=(1+2*H)^3)
sage: E = f.excess_bundle(B)
sage: P1.integral(E.top_chern_class())
4
```

We conclude that there are $2^3 - 4 = 4$ points of intersection outside the line.

2. Let Q_1, Q_2, Q_3 , and Q_4 be four quadratic hypersurfaces in \mathbb{P}^4 . Suppose that

$$Q_1 \cap Q_2 \cap Q_3 \cap Q_4 = C \cup \{p_1, \dots, p_k\},$$

where C is a smooth rational curve of degree 4 and p_1, \dots, p_k are points. In order to determine the number of the points p_i , we need to compute the intersection number of Q_1, Q_2, Q_3 , and Q_4 along C , that is

$$\int_C (Q_1 \cap Q_2 \cap Q_3 \cap Q_4)^C.$$

```
sage: P1 = ProjectiveSpace(1, 'h'); P4 = ProjectiveSpace(4, 'H')
sage: h = P1.hyperplane(); H = P4.hyperplane()
sage: f = Morphism(P1, P4, [4*h])
sage: B = VectorBundle(P4, 4, chern_class=(1+2*H)^4)
sage: E = f.excess_bundle(B)
sage: P1.integral(E.top_chern_class())
14
```

We conclude that there are $2^4 - 14 = 2$ points of intersection outside the curve.

Example 1.5.5. [36, Example 9.1.8] We consider the following question: How many smooth conics in \mathbb{P}^2 are tangent to each of five general lines? The plane conics are parametrized by \mathbb{P}^5 . The conics tangent to a given line form a quadric in \mathbb{P}^5 . Thus the cycle of conics which are tangent to 5 general lines is the intersection of 5 quadratic hypersurfaces H_1, \dots, H_5 in \mathbb{P}^5 . However, the Veronese surface Z of double lines ($Z \cong \mathbb{P}^2$) is a connected component of $H_1 \cap \dots \cap H_5$. The equivalence class of Z for $H_1 \cap \dots \cap H_5$ has degree

$$\int_Z (H_1 \cap \dots \cap H_5)^Z = 31.$$

We illustrate this calculation by `Schubert3` as follows:

```
sage: P2 = ProjectiveSpace(2, 'h'); P5 = ProjectiveSpace(5, 'H')
sage: h = P2.hyperplane(); H = P5.hyperplane()
sage: f = Morphism(P2, P5, [2*h])
sage: B = VectorBundle(P5, 5, chern_class=(1+2*H)^5)
sage: E = f.excess_bundle(B)
sage: P2.integral(E.top_chern_class())
31
```

We conclude that the number of smooth conics tangent to five general lines is $32 - 31 = 1$.

Chapter 2.

Schubert Calculus

A classical problem in projective geometry is to compute the number of points, lines, planes, or higher dimensional linear subspaces satisfying given geometric conditions. In the nineteenth century, Hermann Schubert [97] collected many results involving this problem and introduced deep ideas of his own. In 1972, Kleiman and Laksov [71] gave a modern introduction to Schubert calculus. Fulton [36] also reviewed this subject in modern language. Nowadays, there are several computer algebra systems developing the computation of Schubert calculus. However, there are only a few articles describing implementation details. The algorithms used by `Schubert` in `MAPLE` and by `Schubert2` in `MACAULAY2` are described in [46].

In this chapter, we review the basic notions involving Grassmannians such as Schubert cycles and Schubert classes. We then present the classical formulas of Pieri and Giambelli and describe the Chow ring of a Grassmannian. We also present an effective method for computing Littlewood-Richardson coefficients and the degree of a Fano scheme. We also illustrate a number of classical examples, and show how the computation can be done by `Schubert3`. Most of the examples in this chapter will be reconsidered using `schubert.lib` in Appendix B.

2.1. Grassmannians

Let $G(k, n)$ be the Grassmannian of k -dimensional linear subspaces in an n -dimensional vector space V . The tautological subbundle S on $G(k, n)$ is the vector bundle of rank k whose fiber at $W \in G(k, n)$ is the vector subspace $W \subset V$ itself. The tautological quotient bundle Q on $G(k, n)$ is the vector bundle of rank $n - k$ whose fiber at $W \in G(k, n)$ is the quotient vector space V/W . The tangent bundle T on $G(k, n)$ is isomorphic to $\text{Hom}(S, Q) \cong S^\vee \otimes Q$.

Let \mathcal{V} be a *flag* in V , that is, a strictly increasing sequence of linear subspaces

$$0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = V,$$

where $\dim V_i = i$. For any sequence $a = (a_1, \dots, a_k)$ of integers with

$$n - k \geq a_1 \geq a_2 \geq \cdots \geq a_k \geq 0,$$

we define the *Schubert cycle* by

$$\Sigma_a(\mathcal{V}) = \{W \in G(k, n) : \dim(V_{n-k+i-a_i} \cap W) \geq i, i = 1, \dots, k\}.$$

One can show that this is an irreducible subvariety of $G(k, n)$ of codimension

$$|a| = \sum_{i=1}^k a_i,$$

and its cycle class $[\Sigma_a(\mathcal{V})]$ does not depend on the choice of flag. We then define the *Schubert class* to be the cycle class $\sigma_a := [\Sigma_a(\mathcal{V})]$.

Proposition 2.1.1. *The number of Schubert classes on the Grassmannian $G(k, n)$ is equal to the binomial coefficient $\binom{n}{k}$.*

Proof. A Schubert class on $G(k, n)$ is given by a sequence $a = (a_1, a_2, \dots, a_k)$ of integers such that $n - k \geq a_1 \geq a_2 \geq \dots \geq a_k \geq 0$. Indeed, such a sequence is a partition of an integer less than or equal to $k(n - k)$ in which there are at most k parts and in which no part is larger than $n - k$. There is a bijection between k -subsets of $\{1, 2, \dots, n\}$ and such sequences, given by

$$b \mapsto a = (n - k - b_1 + 1, n - k - b_2 + 2, \dots, n - b_k),$$

where $b = (b_1, b_2, \dots, b_k)$ with $1 \leq b_1 < b_2 < \dots < b_k \leq n$. Hence the number of Schubert classes on the Grassmannian $G(k, n)$ is equal to the binomial coefficient $\binom{n}{k}$. \square

To shorten the notation, we write Σ_a in place of $\Sigma_a(\mathcal{V})$, write $\Sigma_{a_1, \dots, a_s}, \sigma_{a_1, \dots, a_s}$ whenever $a = (a_1, \dots, a_s, 0, \dots, 0)$ and $\Sigma_{p^i}, \sigma_{p^i}$ whenever $a = (p, \dots, p, 0, \dots, 0)$ with i the first components equal to p . Then the cycle classes $\sigma_i, i = 1, \dots, n - k$ and $\sigma_{1^i}, i = 1, \dots, k$ are called *special Schubert classes*.

The special Schubert classes are intimately connected with the tautological bundles on $G(k, n)$, and both $\{\sigma_1, \sigma_2, \dots, \sigma_{n-k}\}$ and $\{\sigma_1, \sigma_{1^2}, \dots, \sigma_{1^k}\}$ are minimal generating sets for the Chow ring of $G(k, n)$. More precisely, we have the following statements.

Theorem 2.1.2. *The Chern classes of S and Q are as follows:*

$$c_i(S) = (-1)^i \sigma_{1^i}, \quad i = 1, \dots, k$$

and

$$c_i(Q) = \sigma_i, \quad i = 1, \dots, n - k.$$

Proof. See [84, Proposition 3.5.5] and [29, Section 7.3.5]. \square

By [29, Corollary 3.5], the Schubert classes form a free \mathbb{Z} -basis for $A(G(k, n))$. The multiplication is determined by the following formulae.

Theorem 2.1.3 (Duality formula). *If $|a| + |b| = k(n - k)$, we have*

$$\sigma_a \cdot \sigma_b = \begin{cases} \sigma_{(n-k)^k} & \text{if } a_i + b_{k-i} = n - k \text{ for all } i, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, both $(\sigma_{n-k})^k$ and $(\sigma_{1^k})^{n-k}$ are equal to the class of a point in the Chow ring of $G(k, n)$.

Proof. See [4, Theorem 4.3] and [29, Corollary 3.2 and Proposition 3.4]. \square

Theorem 2.1.4 (Pieri formula). *For any Schubert class $\sigma_a \in A^*(G(k, n))$ and any integer i with $0 \leq i \leq n - k$, we have*

$$\sigma_a \cdot \sigma_i = \sum_c \sigma_c,$$

where the sum is over all c with $n - k \geq c_1 \geq a_1 \geq c_2 \geq \dots \geq c_k \geq a_k \geq 0$, and $|c| = |a| + i$.

Proof. See [29, Proposition 3.7]. □

Theorem 2.1.5 (Giambelli formula). *For any $a = (a_1, \dots, a_k)$ with $n - k \geq a_1 \geq a_2 \geq \dots \geq a_k \geq 0$, we have*

$$\sigma_a = \det(\sigma_{a_i+j-i})_{1 \leq i, j \leq k},$$

where $\sigma_0 = 1$ and $\sigma_m = 0$ whenever $m < 0$ or $m > n - k$.

Proof. A proof of this theorem can be found in [53, Chapter 1, Section 5]. □

Pieri's formula shows how to determine the product of an arbitrary Schubert class and a special Schubert class. Giambelli's formula shows how to express an arbitrary Schubert class in terms of special ones. Therefore, both formulas give us an effective way to determine the product of two arbitrary Schubert classes.

Example 2.1.6. In case $G(2, 4)$, there are exactly 6 Schubert classes:

$$\{1, \sigma_1, \sigma_2, \sigma_{1,1}, \sigma_{2,1}, \sigma_{2,2}\}.$$

```
sage: G = Grassmannian(2,4)
sage: G.schubert_classes()
[[], [1], [2], [1, 1], [2, 1], [2, 2]]
sage: len(_)
6
```

By Pieri's formula, we have

$$\begin{aligned} \sigma_1 \cdot \sigma_1 &= \sigma_2 + \sigma_{1,1}, \\ \sigma_1 \cdot \sigma_2 &= \sigma_1 \cdot \sigma_{1,1} = \sigma_{2,1}, \\ \sigma_1 \cdot \sigma_{2,1} &= \sigma_2 \cdot \sigma_2 = \sigma_{2,2}. \end{aligned}$$

By Giambelli's formula, we have

$$\begin{aligned} \sigma_{1,1} &= \begin{vmatrix} \sigma_1 & \sigma_2 \\ 1 & \sigma_1 \end{vmatrix} = \sigma_1^2 - \sigma_2, \\ \sigma_{2,1} &= \begin{vmatrix} \sigma_2 & 0 \\ 1 & \sigma_1 \end{vmatrix} = \sigma_1 \sigma_2, \\ \sigma_{2,2} &= \begin{vmatrix} \sigma_2 & 0 \\ \sigma_1 & \sigma_2 \end{vmatrix} = \sigma_2^2. \end{aligned}$$

Theorem 2.1.7. *The Chow ring of the Grassmannian $G(k, n)$ has the form*

$$A(G(k, n)) \cong \frac{\mathbb{Z}[\sigma_1, \dots, \sigma_{n-k}]}{I},$$

where the ideal I is generated by the $n - k$ polynomials $f_m \in \mathbb{Z}[\sigma_1, \dots, \sigma_{n-k}]$, $m = 1, \dots, n - k$. The polynomials f_m are defined by Giambelli's formula as follows:

$$f_m = \sigma_{1^{k+m}} = \det(\sigma_{1+j-i})_{1 \leq i, j \leq k+m}.$$

Moreover, this Chow ring is graded with $\deg(\sigma_i) = i$.

Proof. By duality, there is an isomorphism between the Chow rings of $G(k, n)$ and $G(n-k, n)$, which sends $c_i(Q)$ to $c_i(S)$ with Q is the tautological quotient bundle on $G(k, n)$ and S is the tautological subbundle on $G(n-k, n)$. By [29, Theorem 7.17], the Chow ring of $G(n-k, n)$ has the form

$$A(G(n-k, n)) \cong \frac{\mathbb{Z}[c_1(S), \dots, c_{n-k}(S)]}{I},$$

where the ideal I is generated by the terms of total degree $k+1, \dots, n$ in the power series expansion

$$\frac{1}{1 + c_1(S) + \dots + c_{n-k}(S)} = 1 - (c_1(S) + \dots + c_{n-k}(S)) + \dots \in \mathbb{Z}[[c_1(S), \dots, c_{n-k}(S)]].$$

It is easy to see that these terms are $(-1)^{k+m} \det(c_{1+j-i}(S))_{1 \leq i, j \leq k+m}$ for all $m = 1, \dots, n-k$. Note that $c_0(S) = 1$ and $c_l(S) = 0$ whenever $l < 0$ or $l > n-k$. Hence the Chow ring of $G(k, n)$ has the desired form. \square

Since the Chow ring of the Grassmannian $G(k, n)$ is the quotient of a polynomial ring in the $n-k$ variables $\sigma_1, \dots, \sigma_{n-k}$ modulo an ideal generated by the $n-k$ polynomials as in Theorem 2.1.7, we can implement the computation of the intersection numbers of cycle classes on Grassmannians in any computer algebra system that supports the computation of Gröbner bases.

Algorithm 2.1.1 Degree on Grassmannians

Input: A zero-dimensional cycle class α on the Grassmannian $G(k, n)$, represented by a homogeneous polynomial of degree $k(n-k)$ in $\mathbb{Z}[\sigma_1, \dots, \sigma_{n-k}]$.

Output: The degree of the cycle class α in the Chow ring of $G(k, n)$, denoted by

$$\int_{G(k, n)} \alpha.$$

- 1: Use Giambelli's formula to compute the ideal I generated by the polynomials f_m for all $m = 1, \dots, n-k$ as in Theorem 2.1.7.
 - 2: Use Gröbner bases to compute the remainder, say f , of α modulo I .
 - 3: **Return** the leading coefficient of f .
-

Example 2.1.8. In order to compute the number of lines meeting four general lines in \mathbb{P}^3 , we need to compute the integration $\int_{G(2,4)} \sigma_1^4$. By Giambelli's formula, we have

$$\sigma_{1,1,1} = \begin{vmatrix} \sigma_1 & \sigma_2 & 0 \\ 1 & \sigma_1 & \sigma_2 \\ 0 & 1 & \sigma_1 \end{vmatrix} = \sigma_1^3 - 2\sigma_1\sigma_2,$$

and

$$\sigma_{1,1,1,1} = \begin{vmatrix} \sigma_1 & \sigma_2 & 0 & 0 \\ 1 & \sigma_1 & \sigma_2 & 0 \\ 0 & 1 & \sigma_1 & \sigma_2 \\ 0 & 0 & 1 & \sigma_1 \end{vmatrix} = \sigma_1^4 - 3\sigma_1^2\sigma_2 + \sigma_2^2.$$

Thus the Chow ring of $G(2, 4)$ has the form

$$A^*(G(2, 4)) \cong \frac{\mathbb{Z}[\sigma_1, \sigma_2]}{(\sigma_1^3 - 2\sigma_1\sigma_2, \sigma_1^4 - 3\sigma_1^2\sigma_2 + \sigma_2^2)}.$$

In this Chow ring, we have $\sigma_1^4 = 2\sigma_2^2$. This implies that

$$\int_{G(2,4)} \sigma_1^4 = 2.$$

We illustrate this by `Schubert3` as follows:

```
sage: G = Grassmannian(2,4)
sage: s1 = G.schubert_class([1])
sage: G.integral(s1^4)
2
```

We conclude that there are precisely two lines meeting four general lines in \mathbb{P}^3 .

Example 2.1.9. [29, Section 2.3.4] Let $C \subset \mathbb{P}^3$ be a smooth, nondegenerate curve of degree d and genus g . One can show that the locus $S_1(C)$ of secant lines to C is a cycle of codimension 2 in $G(2, 4)$ and its class is

$$[S_1(C)] = \left(\binom{d-1}{2} - g \right) \sigma_2 + \binom{d}{2} \sigma_{1,1}.$$

Taking $d = 3$ and $g = 0$, we claim that the number of common secant lines to two general twisted cubic curves in \mathbb{P}^3 is equal to the degree of class $(\sigma_2 + 3\sigma_{1,1})^2$ in the Chow ring of $G(2, 4)$,

$$\int_{G(2,4)} (\sigma_2 + 3\sigma_{1,1})^2.$$

This number can be computed by `Schubert3` as follows:

```
sage: G = Grassmannian(2,4)
sage: s2 = G.schubert_class([2])
sage: s11 = G.schubert_class([1,1])
sage: G.integral((s2+3*s11)^2)
10
```

We conclude that there are precisely 10 secant lines to two general twisted cubic curves in \mathbb{P}^3 .

Example 2.1.10. In [61], Huber, Sottile, and Sturmfels considered the following problem: Given n general linear subspaces V_1, \dots, V_n in \mathbb{C}^{k+m} with $\dim V_i = m+1-a_i$ and $a_1 + \dots + a_n = mk$, how many k -dimensional linear subspaces in \mathbb{C}^{k+m} meet all V_i nontrivially? We can solve this problem as follows: Let $G(k, k+m)$ be the Grassmannian of k -dimensional linear subspaces in \mathbb{C}^{k+m} . The locus of k -dimensional linear subspaces in \mathbb{C}^{k+m} meeting V_i nontrivially is a cycle of codimension a_i and its cycle class in the Chow ring of $G(k, k+m)$ is the special Schubert class σ_{a_i} . Thus the number of k -dimensional linear subspaces in \mathbb{C}^{k+m} meeting all V_i nontrivially is equal to the degree of the zero-dimensional cycle class $\sigma_{a_1} \cdot \sigma_{a_2} \cdot \dots \cdot \sigma_{a_n}$ in the Chow ring $A^*(G(k, k+m))$,

$$\int_{G(k,k+m)} \sigma_{a_1} \cdot \sigma_{a_2} \cdot \dots \cdot \sigma_{a_n}. \quad (2.1)$$

Since $a_1 + \cdots + a_n = mk$, this number is finite and can be computed by Algorithm 2.1.1. In the special case $a_1 = \cdots = a_n = 1$, there is an explicit formula for the degree of the Grassmannian in the Plücker embedding,

$$\deg(G(k, k+m)) = \int_{G(k, k+m)} \sigma_1^{km} = (km)! \frac{\prod_{l=0}^{k-1} l!}{\prod_{l=0}^{k-1} (m+l)!}.$$

Also in that paper, the authors developed the Pieri homotopy algorithm for solving problems from classical Schubert calculus. Afterwards, this algorithm was improved by Huber and Verschelde in [62]. A different approach to the Pieri homotopy algorithm was given by Li, Wang and Wu in [81]. In Chapter 5, we will present another method for solving this problem, using equivariant intersection theory and Bott's formula.

2.2. Littlewood-Richardson Rule

As a generalization of Pieri's formula, we have a product formula for two arbitrary Schubert classes $\sigma_a, \sigma_b \in A^*(G(k, n))$,

$$\sigma_a \cdot \sigma_b = \sum_{|c|=|a|+|b|} N_{a,b}^c \sigma_c \in A^{|a|+|b|}(G(k, n)),$$

where the $N_{a,b}^c$ are given by the *Littlewood-Richardson rule*. These coefficients appear in many combinatorial subjects and can be computed by many combinatorial methods. We here present a geometric method for computing the Littlewood-Richardson coefficients $N_{a,b}^c$.

Let $a = (a_1, a_2, \dots, a_k)$ be a partition with $n - k \geq a_1 \geq a_2 \geq \cdots \geq a_k \geq 0$. We set $a^\vee = (n - k - a_k, \dots, n - k - a_1)$. By the duality formula, we have $\sigma_a \cdot \sigma_{a^\vee}$ is the class of a point in the Chow ring $A^*(G(k, n))$. This means that $\int_{G(k, n)} \sigma_a \cdot \sigma_{a^\vee} = 1$. Hence

$$N_{a,b}^c = \int_{G(k, n)} \sigma_a \cdot \sigma_b \cdot \sigma_{c^\vee}.$$

This formula gives us an effective method (see Algorithm 2.2.1) for computing the Littlewood-Richardson coefficients.

Algorithm 2.2.1 The Littlewood-Richardson coefficients

Input: Three partitions a, b, c in which there are at most k parts and in which no part is larger than $n - k$.

Output: $N_{a,b}^c$.

1: **if** $|a| + |b| \neq |c|$ **then**

2: $N_{a,b}^c = 0$.

3: **end if**

4: **if** $|a| + |b| = |c|$ **then**

5: Set $c^\vee = (n - k - c_k, \dots, n - k - c_1)$.

6: Use Giambelli's formula to compute the Schubert classes $\sigma_a, \sigma_b, \sigma_{c^\vee}$ on the Grassmannian $G(k, n)$ in terms of the special Schubert classes $\sigma_1, \dots, \sigma_{n-k}$.

7: Use Algorithm 2.1.1 to return the degree of cycle class $\sigma_a \cdot \sigma_b \cdot \sigma_{c^\vee}$ in the Chow ring of $G(k, n)$.

8: **end if**

Algorithm 2.2.2 The Littlewood–Richardson rule

Input: Two partitions a, b in which there are at most k parts and in which no part is larger than $n - k$.

Output: $\sigma_a \cdot \sigma_b = \sum_{|c|=|a|+|b|} N_{a,b}^c \sigma_c$.

1: List all partitions c with $|c| = |a| + |b|$ and set

$$L = \{c \mid |c| = |a| + |b|\}.$$

2: Use Algorithm 2.2.1 to compute $N_{a,b}^c$ for all $c \in L$.

3: **Return** list of $(N_{a,b}^c, c)$ for all $c \in L$.

Example 2.2.1. In case $G(3, 6)$, we have

$$\sigma_{2,1} \cdot \sigma_{2,1} = \sigma_{3,3} + 2\sigma_{3,2,1} + \sigma_{2,2,2}.$$

We illustrate this by `Schubert3` as follows:

```
sage: G = Grassmannian(3,6); G
A Grassmannian of all 3-planes in 6-space
sage: G.Littlewood_Richardson_coefficient([2,1],[2,1],[3,3])
1
sage: G.Littlewood_Richardson_coefficient([2,1],[2,1],[3,2,1])
2
sage: G.Littlewood_Richardson_coefficient([2,1],[2,1],[2,2,2])
1
sage: G.Littlewood_Richardson_rule([2,1],[2,1])
[(1, [3, 3]), (2, [3, 2, 1]), (1, [2, 2, 2])]
```

2.3. Fano Schemes

Let $X \subset \mathbb{P}^n$ be a general hypersurface of degree d . Let $F_k(X)$ be the set of k -planes on X . One can show that $F_k(X)$ is a subscheme of the Grassmannian $G(k+1, n+1)$. This subscheme is called the *Fano scheme of k -planes* on X . The following proposition gives us the expected dimension of the Fano schemes.

Proposition 2.3.1. *Let $X \subset \mathbb{P}^n$ be a general hypersurface of degree d . Suppose that $d \neq 2$ or $n \geq 2k + 1$. Then we have*

1. $F_k(X)$ is empty, if $\phi(k, d, n) < 0$,
2. $F_k(X)$ is smooth of expected dimension $\phi(k, d, n)$, if $\phi(k, d, n) \geq 0$.

Here we denote $\phi(k, d, n) = (k+1)(n-k) - \binom{d+k}{d}$.

Proof. See [77, Theorem 0.1]. □

The following theorem describes the cycle class $[F_k(X)]$ in the Chow ring $A^*(G(k+1, n+1))$.

Theorem 2.3.2. *Let S be the tautological subbundle on $G(k+1, n+1)$ and $\text{Sym}^d S^\vee$ be the d -th symmetric power of the dual of S . Then*

$$[F_k(X)] = c_{\text{top}}(\text{Sym}^d S^\vee) \in A^*(G(k+1, n+1)),$$

where $c_{\text{top}}(E)$ is the top Chern class of the vector bundle E .

Proof. See [84, Section 3.5]. □

This implies that the degree of $[F_k(X)]$ is equal to the degree of the top Chern class of $\text{Sym}^d S^\vee$ in the Chow ring of $G(k+1, n+1)$. Thus we have

$$\deg(F_k(X)) = \int_{G(k+1, n+1)} c_{\text{top}}(\text{Sym}^d S^\vee) \cdot \sigma_1^{\phi(k, d, n)}.$$

Algorithm 2.3.1 The degree of Fano schemes

Input: $k, d, n \in \mathbb{N}$ with $d \neq 2$ (or $n \geq 2k+1$) and $\phi(k, d, n) \geq 0$.

Output: The degree of the Fano scheme of k -planes on a general hypersurface of degree d in \mathbb{P}^n .

- 1: Consider $G = G(k+1, n+1)$ with its tautological subbundle S .
 - 2: Use the splitting principle (see Section 1.2.1) and Giambelli's formula to compute the cycle class $c_{\text{top}}(\text{Sym}^d S^\vee)$. This should be represented by a polynomial in $n-k$ variables $\sigma_1, \dots, \sigma_{n-k}$ with coefficients in \mathbb{Z} .
 - 3: Use Algorithm 2.1.1 to return the degree of the cycle class $c_{\text{top}}(\text{Sym}^d S^\vee) \cdot \sigma_1^{\phi(k, d, n)}$ in the Chow ring of $G(k+1, n+1)$.
-

By duality, we also have

$$\deg(F_k(X)) = \int_{G(n-k, n+1)} c_{\text{top}}(\text{Sym}^d Q) \cdot \sigma_1^{\phi(k, d, n)}.$$

Algorithm 2.3.2 Dual to Algorithm 2.3.1

Input: $k, d, n \in \mathbb{N}$ with $d \neq 2$ (or $n \geq 2k+1$) and $\phi(k, d, n) \geq 0$.

Output: The degree of the Fano scheme of k -planes on a general hypersurface of degree d in \mathbb{P}^n .

- 1: Consider $G = G(n-k, n+1)$ with its tautological quotient bundle Q .
 - 2: Use the splitting principle to compute the cycle class $c_{\text{top}}(\text{Sym}^d Q)$. This should be represented by a polynomial in $k+1$ variables $\sigma_1, \dots, \sigma_{k+1}$ with coefficients in \mathbb{Z} .
 - 3: Use Algorithm 2.1.1 to return the degree of the cycle class $c_{\text{top}}(\text{Sym}^d Q) \cdot \sigma_1^{\phi(k, d, n)}$ in the Chow ring of $G(n-k, n+1)$.
-

Example 2.3.3. Let X be a general cubic hypersurface in \mathbb{P}^4 , then the expected dimension of $F_1(X)$ is two. In order to compute the degree of $F_1(X)$, we first consider the Grassmannian $G(2, 5)$ with the tautological subbundle S . Then we compute the cycle class

$$c_{\text{top}}(\text{Sym}^3 S^\vee) = 105\sigma_1^4 - 226\sigma_1^2\sigma_2 + 43\sigma_2^2 + 112\sigma_1\sigma_3 \in A^*(G(2, 5)).$$

By duality, if we consider the Grassmannian $G(3, 5)$ with the tautological quotient bundle Q , then we can compute the cycle class

$$c_{\text{top}}(\text{Sym}^3 Q) = 18\sigma_1^2\sigma_2 + 9\sigma_2^2 \in A^*(G(3, 5)).$$

Thus the degree of $F_1(X)$ is as follows:

$$\deg(F_1(X)) = \int_{G(2,5)} c_{\text{top}}(\text{Sym}^3 S^\vee) \cdot \sigma_1^2 = \int_{G(3,5)} c_{\text{top}}(\text{Sym}^3 Q) \cdot \sigma_1^2 = 45.$$

We illustrate this calculation by `Schubert3` as follows:

```
sage: G = Grassmannian(2,5)
sage: S = G.tautological_subbundle().dual()
sage: B = S.symmetric_power(3)
sage: B.top_chern_class()
105*q1^4 - 226*q1^2*q2 + 43*q2^2 + 112*q1*q3
sage: s1 = G.schubert_class([1])
sage: G.integral(B.top_chern_class()*s1^2)
45
sage: G = Grassmannian(3,5)
sage: Q = G.tautological_quotient_bundle()
sage: B = Q.symmetric_power(3)
sage: B.top_chern_class()
18*q1^2*q2 + 9*q2^2
sage: s1 = G.schubert_class([1])
sage: G.integral(B.top_chern_class()*s1^2)
45
```

2.4. Linear Subspaces on Hypersurfaces

In this section we describe the solution of the following problem.

Question 2.4.1. *Let $X \subset \mathbb{P}^n$ be a general hypersurface of degree d . How many k -planes in \mathbb{P}^n are there on X ?*

In many cases, the answers are infinite. However, if $d, k, n \in \mathbb{N}$ satisfy $d \neq 2$ (or $n \geq 2k + 1$) and

$$\binom{d+k}{k} = (k+1)(n-k),$$

then the answers are finite.

Example 2.4.2. Let $X \subset \mathbb{P}^3$ be a general cubic surface. How many lines in \mathbb{P}^3 does X contain? The answer is exactly 27 lines (see Figure 2.1). If we replace X by a general quintic hypersurface in \mathbb{P}^4 , then there will be exactly 2875 lines on X .

Let $X \subset \mathbb{P}^n$ be a general hypersurface of degree d . In order to count the k -planes on X , we consider the Fano scheme $F_k(X)$ as a subscheme of $G(k+1, n+1)$, which is a parameter space for the k -planes in \mathbb{P}^n .

In case $d \neq 2$ or $n \geq 2k + 1$ and $\phi(k, d, n) = 0$, by Theorem 2.3.1, we have $F_k(X)$ is zero-dimensional. This implies that the number of k -planes on X is finite and equal to the degree of $[F_k(X)]$ in $A(G(k+1, n+1))$. We present an effective algorithm (see Algorithm 2.4.1) for counting k -planes on a general hypersurface of degree d in \mathbb{P}^n .

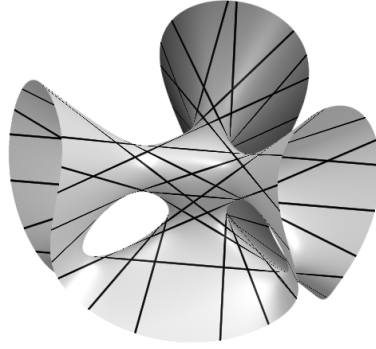


Figure 2.1.: 27 lines on a general cubic surface.

Algorithm 2.4.1 Linear subspaces on hypersurfaces

Input: $k, d, n \in \mathbb{N}$ with $d \neq 2$ (or $n \geq 2k + 1$) and $\phi(k, d, n) = 0$.

Output: The number of k -planes on a general hypersurface of degree d in \mathbb{P}^n .

- 1: Consider the Grassmannian $G(k + 1, n + 1)$ (respectively $G(n - k, n + 1)$) with the tautological subbundle S (respectively the tautological quotient bundle Q).
 - 2: Use Algorithm 2.3.1 (respectively Algorithm 2.3.2) to return the degree of the Fano scheme of k -planes on a general hypersurface of degree d in \mathbb{P}^n .
-

Example 2.4.3. [29, Chapter 8] In this example, we compute the number of lines on a general hypersurface of degree $d = 2n - 3 \geq 3$ in \mathbb{P}^n .

This number is the degree of the top Chern class of the d -th symmetric power of the tautological quotient bundle on $G(n - 1, n + 1)$.

We illustrate this calculation by `Schubert3` as follows:

```
sage: def lines_on_hypersurfaces(n):
.....:     d = 2*n-3
.....:     G = Grassmannian(n-1,n+1)
.....:     Q = G.tautological_quotient_bundle()
.....:     B = Q.symmetric_power(d)
.....:     return G.integral(B.top_chern_class())
.....:
sage: for n in range(2,11):
.....:     lines_on_hypersurfaces(n)
.....:
1
27
2875
698005
305093061
210480374951
210776836330775
289139638632755625
520764738758073845321
```

We collect the computations above in Table 2.1.

More generally, if $k, d, n \in \mathbb{N}$ satisfy $d \geq 3$ and

$$\binom{d+k}{k} = (k+1)(n-k),$$

then we can return the number of k -planes on a general hypersurface of degree d in \mathbb{P}^n .

```
sage: def linear_subspaces_on_hypersurfaces(k,d,n):
.....:     G = Grassmannian(n-k,n+1)
.....:     Q = G.tautological_quotient_bundle()
.....:     B = Q.symmetric_power(d)
.....:     return G.integral(B.top_chern_class())
.....:
sage: linear_subspaces_on_hypersurfaces(2,4,7)
3297280
sage: linear_subspaces_on_hypersurfaces(3,3,8)
321489
```

This means that there are 3297280 2-planes on a general hypersurface of degree 4 in \mathbb{P}^4 and 321489 3-planes on a general hypersurface of degree 3 in \mathbb{P}^8 .

2.5. Universal Fano Schemes

In this section, we present a different approach to the problem counting lines on a general cubic surface in \mathbb{P}^3 .

It is easy to see that there is a one-to-one correspondence

$$\{\text{hypersurfaces of degree } d \text{ in } \mathbb{P}^n\} \longleftrightarrow \{\text{points in } \mathbb{P}^{\binom{n+d}{d}-1}\}.$$

Therefore, a hypersurface X of degree d in \mathbb{P}^n will be considered as a point in $\mathbb{P}^{\binom{n+d}{d}-1}$.

Definition 2.5.1. The incidence correspondence

$$F_{n,d} = \{(X, L) \in \mathbb{P}^{\binom{n+d}{d}-1} \times G(2, n+1) \mid L \subset X\}$$

is called the *universal Fano scheme* of lines on hypersurfaces of degree d in \mathbb{P}^n .

n	Number of lines
2	1
3	27
4	2875
5	698005
6	305093061
7	210480374951
8	210776836330775
9	289139638632755625
10	520764738758073845321

Table 2.1.: Lines on hypersurfaces

We work with the product variety $W := \mathbb{P}^{\binom{n+d}{d}-1} \times G(2, n+1)$ and write p_1, p_2 for its canonical projections

$$\begin{array}{ccc} & \mathbb{P}^{\binom{n+d}{d}-1} \times G(2, n+1) & \\ p_1 \swarrow & & \searrow p_2 \\ \mathbb{P}^{\binom{n+d}{d}-1} & & G(2, n+1). \end{array}$$

We consider the vector bundle

$$E := p_2^* \text{Sym}^d S^\vee \otimes p_1^* \mathcal{O}_{\mathbb{P}^{\binom{n+d}{d}-1}}(1),$$

where S is the tautological subbundle on $G(2, n+1)$. The following theorem describe the cycle class of $F_{n,d}$ in the Chow ring of W .

Theorem 2.5.2. *The universal Fano scheme $F_{n,d}$ of lines on hypersurfaces of degree d in \mathbb{P}^n is reduced and of codimension $d+1$ in the product variety W . In addition, it is the zero locus of a section of E and its cycle class is*

$$[F_{n,d}] = c_{d+1}(E) \in A^{d+1}(W).$$

Proof. See [29, Theorem 8.37]. □

Example 2.5.3. [29, Corollary 8.39] In case $n = d = 3$, we consider the vector bundle

$$E := p_2^* \text{Sym}^3 S^\vee \otimes p_1^* \mathcal{O}_{\mathbb{P}^{19}}(1)$$

on the product variety $\mathbb{P}^{19} \times G(2, 4)$, where S is the tautological subbundle on $G(2, 4)$. Then the cycle class of $F_{3,3}$ is

$$\begin{aligned} [F_{3,3}] &= c_4(E) \\ &= h^4 + 6h^3\sigma_1 + 21h^2\sigma_1^2 - 10h^2\sigma_2 + 42h\sigma_1\sigma_2 + 27\sigma_2^2. \end{aligned}$$

We here use the symbol h for the pullback to $\mathbb{P}^{19} \times G(2, 4)$ of the hyperplane class on \mathbb{P}^{19} and the symbols σ_i for the pullbacks to $\mathbb{P}^{19} \times G(2, 4)$ of the corresponding Schubert classes on $G(2, 4)$.

We illustrate this calculation by `Schubert3` as follows:

```
sage: P = ProjectiveSpace(19)
sage: G = Grassmannian(2,4)
sage: W = ProductVariety(P,G)
sage: p1 = W.projection_morphism(1)
sage: p2 = W.projection_morphism(2)
sage: S = G.tautological_subbundle().dual()
sage: B = S.symmetric_power(3)
sage: h = P.hyperplane()
sage: L = P.O(1)
sage: E = p2.pullback(B) & p1.pullback(L)
sage: W.chow_ring()(E.top_chern_class())
h^4 + 6*h^3*q1 + 21*h^2*q1^2 - 10*h^2*q2 + 42*h*q1*q2 + 27*q2^2
```

In this case we claim that the number of lines on a general cubic surface in \mathbb{P}^3 is equal to the degree of $[F_{3,3}] \cdot h^{19}$ in $A^*(\mathbb{P}^{19} \times G(2, 4))$

$$\int_{\mathbb{P}^{19} \times G(2,4)} [F_{3,3}] \cdot h^{19} = \int_{\mathbb{P}^{19} \times G(2,4)} c_4(E) \cdot h^{19} = 27.$$

This number can be computed as follows:

```
sage: h = p1.pullback(h)
sage: W.integral(E.top_chern_class()*h^19)
27
```

Moreover, if C is the set of lines on a general pencil of cubic surfaces in \mathbb{P}^3 , then C will be a cycle of dimension 1 on the product variety $\mathbb{P}^{19} \times G(2, 4)$ and its cycle class will be $[F_{3,3}] \cdot h^{18}$. We may consider C as a curve on $\mathbb{P}^{19} \times G(2, 4)$, then the degree of C is 42. Indeed, the degree of C is equal to the degree of $[F_{3,3}] \cdot h^{18} \cdot \sigma_1$ in $A^*(\mathbb{P}^{19} \times G(2, 4))$

$$\int_{\mathbb{P}^{19} \times G(2,4)} [F_{3,3}] \cdot h^{18} \cdot \sigma_1 = \int_{\mathbb{P}^{19} \times G(2,4)} c_4(E) \cdot h^{18} \cdot \sigma_1 = 42.$$

This number can be computed as follows:

```
sage: sigma1 = G.schubert_class([1])
sage: sigma1 = p2.pullback(sigma1)
sage: W.integral(E.top_chern_class()*h^18*sigma1)
42
```

2.6. Lines on Complete Intersection Calabi-Yau Threefolds

Let

$$X = \bigcap_{i=1}^k X_i \subset \mathbb{P}^{k+3}$$

be a smooth complete intersection threefold, where X_i is a smooth hypersurface of degree $d_i \geq 2$ for all $i = 1, \dots, k$. In this case, we will say that X has type (d_1, \dots, d_k) . If the canonical bundle K_X on X is trivial, then X is called *Calabi-Yau*. By the adjunction formula [36, Example 3.2.12], we have

$$K_X \cong \mathcal{O} \left(\sum_{i=1}^k d_i - (k+4) \right).$$

Suppose that X is Calabi-Yau. Then we obtain

$$\sum_{i=1}^k d_i = k+4.$$

Without loss of generality, we assume that $d_1 \geq \dots \geq d_k \geq 2$. Thus we have

$$2k \leq \sum_{i=1}^k d_i = k+4,$$

so $k \leq 4$. Therefore there are precisely five possibilities.

1. If $k = 1$, then $d_1 = 5$. In this case, X is a quintic hypersurface in \mathbb{P}^4 .
2. If $k = 2$, then there will be two possibilities as follows:
 - a) $d_1 = 4, d_2 = 2$. Then X is a complete intersection of type $(4, 2)$ in \mathbb{P}^5 .
 - b) $d_1 = d_2 = 3$. Then X is a complete intersection of type $(3, 3)$ in \mathbb{P}^5 .
3. If $k = 3$, then $d_1 = 3, d_2 = d_3 = 2$. In this case, X is a complete intersection of type $(3, 2, 2)$ in \mathbb{P}^6 .
4. If $k = 4$, then $d_1 = d_2 = d_3 = d_4 = 2$. In this case, X is a complete intersection of type $(2, 2, 2, 2)$ in \mathbb{P}^7 .

We have already shown in Example 2.4.3 that there are 2875 lines on a general quintic hypersurface in \mathbb{P}^4 . Similarly, we can show that the numbers of lines on general complete intersection Calabi-Yau threefolds are finite.

Let X be a general complete intersection of type $(4, 2)$ in \mathbb{P}^5 . We may consider X as the intersection of two general hypersurfaces, say X_1 and X_2 , with $\deg(X_1) = 4$ and $\deg(X_2) = 2$. By Theorem 2.3.2, the cycle class of lines on X is

$$[F_1(X_1)] \cdot [F_1(X_2)] = c_5(\text{Sym}^4(S^\vee)) \cdot c_3(\text{Sym}^2(S^\vee)) \in A(G(2, 6)).$$

Hence the number $n_1^{(4,2)}$ of lines on X is equal to the degree of this cycle class, that is,

$$n_1^{(4,2)} = \int_{G(2,6)} c_5(\text{Sym}^4(S^\vee)) \cdot c_3(\text{Sym}^2(S^\vee)).$$

In the same way, the number $n_1^{(3,3)}$ of lines on a general complete intersection of type $(3, 3)$ in \mathbb{P}^5 is

$$n_1^{(3,3)} = \int_{G(2,6)} c_4(\text{Sym}^3(S^\vee))^2.$$

The number $n_1^{(3,2,2)}$ of lines on a general complete intersection of type $(3, 2, 2)$ in \mathbb{P}^6 is

$$n_1^{(3,2,2)} = \int_{G(2,7)} c_4(\text{Sym}^3(S^\vee)) \cdot c_3(\text{Sym}^2(S^\vee))^2.$$

The number $n_1^{(2,2,2,2)}$ of lines on a general complete intersection of type $(2, 2, 2, 2)$ in \mathbb{P}^7 is

$$n_1^{(2,2,2,2)} = \int_{G(2,8)} c_3(\text{Sym}^2(S^\vee))^4.$$

Here is how to compute these numbers via `Schubert3`:

```
sage: def lines(D):
.....:     G = Grassmannian(2, len(D)+4)
.....:     S = G.tautological_subbundle().dual()
.....:     B = [S.symmetric_power(d) for d in D]
.....:     c = prod(V.top_chern_class() for V in B)
.....:     return G.integral(c)
.....:
sage: D = [[5], [4, 2], [3, 3], [3, 2, 2], [2, 2, 2, 2]]
sage: [lines(d) for d in D]
[2875, 1280, 1053, 720, 512]
```


We also illustrate this calculation by `schubert.lib` in Example B.1.10.

Theorem 2.6.1 (Compare with Table 1.3 in [66]). *The numbers of lines on general complete intersection Calabi-Yau threefolds are given by Table 2.2.*

Type	Number of lines
(5)	2875
(4, 2)	1280
(3, 3)	1053
(3, 2, 2)	720
(2, 2, 2, 2)	512

Table 2.2.: Lines on complete intersection Calabi-Yau threefolds

In Section 3.3, we will present a same problem counting conics on general complete intersection Calabi-Yau threefolds. This problem will be revisited more generally for higher degree rational curves in Chapter 7.

Chapter 3.

Enumerative Geometry of Conics

In this chapter, we review some enumerative problems concerning conics in projective spaces. We first recall the Chow ring of a projective bundle. Let E be a vector bundle of rank r on a smooth projective variety X , and let $\mathbb{P}(E)$ be the projective bundle of E on X . If $p : \mathbb{P}(E) \rightarrow X$ is the projection morphism, then the pullback homomorphism $p^* : A(X) \rightarrow A(\mathbb{P}(E))$ is an injection of graded rings. Furthermore, we have

$$A(\mathbb{P}(E)) \cong A(X)[\zeta]/(\zeta^r + c_1(E)\zeta^{r-1} + \cdots + c_{r-1}(E)\zeta + c_r(E)),$$

where $\zeta = c_1(\mathcal{O}_{\mathbb{P}(E)}(1)) \in A^1(\mathbb{P}(E))$. This is clear from [36, Theorem 3.3(b)] and the definition of the Chern classes of E (see Section 1.2).

3.1. Enumeration of Conics in \mathbb{P}^3

In this section, we describe the answer to the following questions (see [29, Section 11.8] and [2, Section 3.4]):

Question 3.1.1. *How many conics are there in \mathbb{P}^3 meeting 8 general lines? How many conics are there in \mathbb{P}^3 contained in a general cubic surface and meeting a general line? How many conics are there in \mathbb{P}^3 contained in a general cubic surface and whose supporting planes pass through a general point?*

A key step for solving these problems is the choice of a parameter space for the conics in \mathbb{P}^3 . Let $G(3, 4)$ be the Grassmannian of 2-planes in \mathbb{P}^3 , with tautological subbundle S . By the arguments in [29, Section 11.8], the parameter space for conics in \mathbb{P}^3 may be identified with the projective bundle $\mathbb{P}(\text{Sym}^2(S^\vee))$ over $G(3, 4)$.

We write P for the subvariety of conics whose supporting planes pass through a general point, L for the subvariety of conics meeting a general line, and C for the subvariety of conics contained in a general cubic surface. By [2, Lemma 3.4.1], the cycle classes $[P], [L], [C] \in A(\mathbb{P}(\text{Sym}^2(S^\vee)))$ are given by

$$\begin{aligned} [P] &= c_1(\mathcal{O}_{G(3,4)}(1)), \\ [L] &= 2c_1(\mathcal{O}_{G(3,4)}(1)) + c_1(\mathcal{O}_{\mathbb{P}(\text{Sym}^2(S^\vee))}(1)), \\ [C] &= c_7(A), \end{aligned}$$

where A is the vector bundle

$$A = \text{Sym}^3 S^\vee - S^\vee \otimes \mathcal{O}_{\mathbb{P}(\text{Sym}^2(S^\vee))}(-1).$$

This implies that the number of conics meeting 8 general lines in \mathbb{P}^3 is equal to

$$\int_{\mathbb{P}(\text{Sym}^2(S^\vee))} [L]^8.$$

This number can be computed by Schubert3 as follows:

```
sage: G = Grassmannian(3,4)
sage: S = G.tautological_subbundle().dual()
sage: B = S.symmetric_power(2)
sage: PB = ProjectiveBundle(B)
sage: Q = G.tautological_quotient_bundle()
sage: P = Q.chern_class(1)
sage: Z = PB.tautological_line_bundle().dual()
sage: z = Z.chern_class(1)
sage: PB.integral((2*P + z)^8)
92
```

Theorem 3.1.2. *There are 92 conics in \mathbb{P}^3 meeting 8 general lines, and each of them is smooth.*

Proof. See [29, Theorem 11.30]. □

Similarly, there are 34 conics whose supporting planes pass through a general point and which meet 7 general lines. This number is equal to

$$\int_{\mathbb{P}(\mathrm{Sym}^2(S^\vee))} [L]^7 \cdot [P],$$

and can be computed as follows:

```
sage: PB.integral((2*P + z)^7*P)
34
```

There are 8 conics whose supporting planes pass through 2 general points and which meet 6 general lines. This number is equal to

$$\int_{\mathbb{P}(\mathrm{Sym}^2(S^\vee))} [L]^6 \cdot [P]^2,$$

and can be computed as follows:

```
sage: PB.integral((2*P + z)^6*P^2)
8
```

There is precisely one conic whose supporting plane passes through 3 general points and which meets 5 general lines. This number is equal to

$$\int_{\mathbb{P}(\mathrm{Sym}^2(S^\vee))} [L]^5 \cdot [P]^3,$$

and can be computed as follows:

```
sage: PB.integral((2*P + z)^5*P^3)
1
```

The number of conics in \mathbb{P}^3 contained in a general cubic surface and meeting a general line is equal to

$$\int_{\mathbb{P}(\mathrm{Sym}^2(S^\vee))} [C] \cdot [L].$$

This number can be computed as follows:

```
sage: V = PB.0(-1) & S
sage: A = S.symmetric_power(3) - V
sage: C = A.top_chern_class()
sage: PB.integral(C*(2*P + z))
81
```

The number of conics in \mathbb{P}^3 whose supporting planes pass through a general point and which are contained in a general cubic surface is equal to

$$\int_{\mathbb{P}(\mathrm{Sym}^2(S^\vee))} [C] \cdot [P].$$

This number can be computed as follows:

```
sage: PB.integral(C*P)
27
```

We summarize the computations above in Table 3.1.

α	$\int_{\mathbb{P}(\mathrm{Sym}^2(S^\vee))} \alpha$
$[C] \cdot [L]$	81
$[C] \cdot [P]$	27
$[L]^8$	92
$[L]^7 \cdot [P]$	34
$[L]^6 \cdot [P]^2$	8
$[L]^5 \cdot [P]^3$	1

Table 3.1.: Some numbers concerning conics in \mathbb{P}^3

The parameter space for conics in \mathbb{P}^3 can also be identified with the projective bundle of the vector bundle $E = \mathrm{Sym}^2(S^\vee \otimes Q^\vee)$ on $G(3, 4)$, where S and Q are the tautological bundles on $G(3, 4)$ (see [36, Example 3.2.22]). With this description, the total Segre class of E can be computed as follows:

```
sage: G = Grassmannian(3,4)
sage: S = G.tautological_subbundle()
sage: Q = G.tautological_quotient_bundle()
sage: B = S.dual() & Q.dual()
sage: E = B.symmetric_power(2)
sage: E.total_segre_class()
92*q1^3 + 34*q1^2 + 8*q1 + 1
```

This means that the total Segre class of E is

$$s(E) = 1 + 8c_1(Q) + 34c_1(Q)^2 + 92c_1(Q)^3.$$

Furthermore, the cycle class $[L] \in A(\mathbb{P}(E))$ is given by

$$[L] = c_1(\mathcal{O}_{\mathbb{P}(E)}(1)).$$

By [29, Definition 12.1], we have

$$\int_{\mathbb{P}(E)} [L]^8 = \int_{G(3,4)} p_*(c_1(\mathcal{O}_{\mathbb{P}(E)}(1))^8) = \int_{G(3,4)} s_3(E) = 92,$$

where $p : \mathbb{P}(E) \rightarrow G(3, 4)$ is the projection morphism. Similarly, we obtain

$$\int_{\mathbb{P}(E)} [L]^7 \cdot [P] = \int_{G(3,4)} s_2(E) \cdot c_1(Q) = 34,$$

$$\int_{\mathbb{P}(E)} [L]^6 \cdot [P]^2 = \int_{G(3,4)} s_1(E) \cdot c_1(Q)^2 = 8,$$

$$\int_{\mathbb{P}(E)} [L]^5 \cdot [P]^3 = \int_{G(3,4)} s_0(E) \cdot c_1(Q)^3 = 1.$$

These numbers agree with Table 3.1.

3.2. Conics on Quintic Hypersurfaces in \mathbb{P}^4

Let X be a general quintic hypersurface in \mathbb{P}^4 .

Question 3.2.1. *How many rational curves of degree d are there on X ?*

This is a difficult question in enumerative geometry and still unresolved in general. In case $d = 1$, we have seen in Example 2.4.3 that there are 2875 lines on X . In this section, we will set up the computation of the number of degree 2 rational curves (conics) on X . This number was computed by Sheldon Katz [64] in 1985. Here is how the computation can be made with `Schubert3`.

Let $G(3, 5)$ be the Grassmannian of planes in \mathbb{P}^4 , with tautological subbundle S . The parameter space for conics in \mathbb{P}^4 may be identified with the projective bundle $\mathbb{P}(\text{Sym}^2(S^\vee))$ over $G(3, 5)$.

```
sage: G = Grassmannian(3,5)
sage: S = G.tautological_subbundle().dual()
sage: B = S.symmetric_power(2)
sage: PB = ProjectiveBundle(B)
```

The cycle class of conics on X is the top Chern class of the vector bundle

$$A = \text{Sym}^5(S^\vee) - \text{Sym}^3(S^\vee) \otimes \mathcal{O}_{\mathbb{P}(\text{Sym}^2(S^\vee))}(-1).$$

```
sage: V = PB.0(-1) & S.symmetric_power(3)
sage: A = S.symmetric_power(5) - V
sage: C = A.top_chern_class()
```

The number of conics on X is equal to the degree of this cycle class in the Chow ring $A^*(\mathbb{P}(\mathrm{Sym}^2(S^\vee)))$,

$$\int_{\mathbb{P}(\mathrm{Sym}^2(S^\vee))} c_{11}(A).$$

```
sage: PB.integral(C)
609250
```

We also illustrate this calculation by `schubert.lib` in Example B.1.14. If $p : \mathbb{P}(\mathrm{Sym}^2(S^\vee)) \rightarrow G(3, 5)$ is the projection morphism, then we have

$$\int_{\mathbb{P}(\mathrm{Sym}^2(S^\vee))} c_{11}(A) = \int_{G(3,5)} p_*(c_{11}(A)).$$

This means that the number of conics on X is also equal to the degree of $p_*(c_{11}(A))$ in the Chow ring $A^*(G(3, 5))$.

```
sage: p = PB.projection_morphism()
sage: G.integral(p.pushforward(C))
609250
```

Theorem 3.2.2. *There are 609250 conics on a general quintic hypersurface in \mathbb{P}^4 .*

Proof. See [64, Theorem 3.1]. □

3.3. Conics on Complete Intersection Calabi-Yau Threefolds

We have already shown that there are precisely five types of complete intersection Calabi-Yau threefolds (see Section 2.6). In this section, by the same arguments as in Section 3.2, we present how to compute the numbers of conics on these complete intersections via `Schubert3`.

Let X be a general complete intersection threefold of type (d_1, \dots, d_k) in \mathbb{P}^{k+3} . Then X is the intersection of k general hypersurfaces, say X_1, \dots, X_k , with $\deg(X_i) = d_i$. The parameter space for conics in \mathbb{P}^{k+3} may be identified with the projective bundle $\mathbb{P}(\mathrm{Sym}^2(S^\vee))$, where S is the tautological subbundle on the Grassmannian $G(3, k+4)$. For each i , the cycle class of conics on X_i is the top Chern class of the vector bundle

$$A_i = \mathrm{Sym}^{d_i}(S^\vee) - \mathrm{Sym}^{d_i-2}(S^\vee) \otimes \mathcal{O}_{\mathbb{P}(\mathrm{Sym}^2(S^\vee))}(-1).$$

Thus the cycle class of conics on X is the product of the top Chern classes of the A_i . This implies that the number of conics on X is equal to the degree of this cycle class. For example, the number $n_2^{(4,2)}$ of conics on a general complete intersection of type $(4, 2)$ in \mathbb{P}^5 is

$$n_2^{(4,2)} = \int_{\mathbb{P}(\mathrm{Sym}^2(S^\vee))} c_{\mathrm{top}}(A_1) \cdot c_{\mathrm{top}}(A_2),$$

where

$$\begin{aligned} A_1 &= \mathrm{Sym}^4(S^\vee) - \mathrm{Sym}^2(S^\vee) \otimes \mathcal{O}_{\mathbb{P}(\mathrm{Sym}^2(S^\vee))}(-1), \\ A_2 &= \mathrm{Sym}^2(S^\vee) - \mathcal{O}_{\mathbb{P}(\mathrm{Sym}^2(S^\vee))}(-1), \end{aligned}$$

and S is the tautological subbundle on the Grassmannian $G(3, 6)$. We also have similar formulas for the numbers $n_2^{(3,3)}$, $n_2^{(3,2,2)}$, and $n_2^{(2,2,2,2)}$ of conics on the general complete intersections of types $(3, 3)$, $(3, 2, 2)$, and $(2, 2, 2, 2)$ respectively.

These numbers can be computed by `Schubert3` as follows:

```

sage: def conics(D):
.....:     G = Grassmannian(3,len(D)+4)
.....:     S = G.tautological_subbundle().dual()
.....:     B = S.symmetric_power(2)
.....:     P = ProjectiveBundle(B)
.....:     c = 1
.....:     for d in D:
.....:         V = P.O(-1) & S.symmetric_power(d-2)
.....:         A = S.symmetric_power(d) - V
.....:         c = c*A.top_chern_class()
.....:     return P.integral(c)
.....:
sage: D = [[5],[4,2],[3,3],[3,2,2],[2,2,2,2]]
sage: [conics(d) for d in D]
[609250, 92288, 52812, 22428, 9728]

```

This calculation will be done by `schubert.lib` in Example B.1.14.

Theorem 3.3.1 (Compare with Table 1 in [82]). *The numbers of conics on general complete intersection Calabi-Yau threefolds are given by Table 3.2.*

Type	Number of conics
(5)	609250
(4, 2)	92288
(3, 3)	52812
(3, 2, 2)	22428
(2, 2, 2, 2)	9728

Table 3.2.: Conics on complete intersection Calabi-Yau threefolds

Chapter 4.

Toric Intersection Theory

In this chapter, we describe the Chow ring of a toric variety and present the implementation for computing in the toric setting.

We first review some of the basic notions in toric geometry such as cones and fans.

4.1. Cones and Affine Toric Varieties

We start with the general definition of an affine toric variety.

Definition 4.1.1. An *affine toric variety* is an affine variety X such that

1. a torus $\mathbb{T} \cong (\mathbb{C}^*)^n$ is a Zariski open subset of X ,
2. the natural action of \mathbb{T} on itself extends to X .

The most basic examples of affine toric varieties are $(\mathbb{C}^*)^n$ and \mathbb{C}^n .

Let N and M be the lattices of rank n which are dual to each other. Then $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ and $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$ will be a pair of dual vector spaces of dimension n over \mathbb{R} .

Definition 4.1.2. A *rational polyhedral cone* $\sigma \subset N_{\mathbb{R}}$ is a cone generated by finitely many elements u_1, \dots, u_r of N , that is

$$\sigma = \text{Cone}(u_1, \dots, u_r) := \left\{ \sum_{i=1}^r a_i u_i \mid a_i \geq 0 \text{ for all } i \right\} \subset N_{\mathbb{R}}.$$

Also set $\text{Cone}(\emptyset) = \{0\}$.

For the rest of this chapter, we will always assume that $\sigma \subset N_{\mathbb{R}}$ is a rational polyhedral cone.

A rational polyhedral cone is called *strongly convex* if $\sigma \cap (-\sigma) = \{0\}$.

The dimension of σ is defined to be the dimension of the smallest subspace containing σ . The dual of σ is defined to be

$$\sigma^{\vee} := \{m \in M_{\mathbb{R}} \mid \langle m, u \rangle \geq 0 \text{ for all } u \in \sigma\}.$$

A *face* of σ is the intersection of σ with the zero locus of an element of σ^{\vee} . A *ray* ρ is an one dimensional face of σ which comes with a minimal generator $u_{\rho} \in \rho \cap N$.

Here are some important strongly convex rational polyhedral cones.

1. σ is *smooth* if its minimal generators form part of a \mathbb{Z} -basis of N .
2. σ is *simplicial* if its minimal generators are linearly independent over \mathbb{R} .

For each cone σ , we define a semigroup

$$S_\sigma := \sigma^\vee \cap M.$$

By Gordan's lemma [19, Proposition 1.2.27], this semigroup is finitely generated. In this case, we have the following theorem.

Theorem 4.1.3. *Let $\sigma \subset N_{\mathbb{R}}$ be a rational polyhedral cone with semigroup $S_\sigma = \sigma^\vee \cap M$. Then*

$$X_\sigma := \text{Spec}(\mathbb{C}[S_\sigma])$$

is an affine toric variety. Furthermore, $\dim X_\sigma = n$ if and only if σ is strongly convex.

Proof. See [19, Theorem 1.2.18]. □

Example 4.1.4.

1. Let $\sigma = \{0\}$, then $S_\sigma = M$ and

$$X_\sigma = \text{Spec}(\mathbb{C}[S_\sigma]) = \text{Spec}(\mathbb{C}[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]) = (\mathbb{C}^*)^n.$$

2. Let $\sigma = \text{Cone}((0, 1), (1, 0))$ (see Figure 4.1), then σ^\vee is generated by the same vectors and

$$X_\sigma = \text{Spec}(\mathbb{C}[S_\sigma]) = \text{Spec}(\mathbb{C}[x_1, x_2]) = \mathbb{C}^2.$$

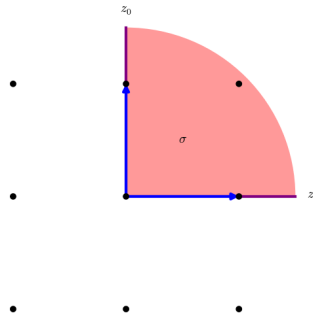


Figure 4.1.: The cone $\sigma = \text{Cone}((0, 1), (1, 0))$ in the plane.

4.2. Fans and Toric Varieties

In this section we construct the toric variety X_Δ associated to a fan Δ . Here is the general definitions of a toric variety and a fan.

Definition 4.2.1. A *toric variety* is a variety X containing a torus as a Zariski open subset such that the action of the torus extends to an action on X .

Definition 4.2.2. A *fan* Δ in $N_{\mathbb{R}}$ is a set of cones such that the following conditions hold:

1. $\sigma \in \Delta$ is a strongly convex rational polyhedral cone,

2. for any $\sigma \in \Delta$, each face of σ is also an element of Δ ,
3. the intersection of two cones of Δ is also an element of Δ .

The *support* of Δ is

$$|\Delta| = \bigcup_{\sigma \in \Delta} \sigma \subseteq N_{\mathbb{R}}.$$

$\Delta(r)$ is the set of r -dimensional cones of Δ . Furthermore, we say that

1. Δ is *smooth* if every cone σ in Δ is smooth,
2. Δ is *simplicial* if every cone σ in Δ is simplicial,
3. Δ is *complete* if its support $|\Delta|$ is all of $N_{\mathbb{R}}$.

Let Δ be a fan and $\sigma, \sigma' \in \Delta$ with $\sigma \cap \sigma' = \tau$. Then we have two affine toric varieties X_{σ} and $X_{\sigma'}$, both containing the affine toric variety X_{τ} as an open subset. Hence we can glue the affine toric varieties along their common faces and obtain an abstract variety X_{Δ} associated to the fan Δ .

Example 4.2.3. Consider the fan Δ in $N_{\mathbb{R}}$ (see Figure 4.2), where $N = \mathbb{Z}^2$ has standard basis e_1, e_2 . The fan Δ has three 2-dimensional cones $\sigma_0 = \text{Cone}(e_1, e_2)$, $\sigma_1 = \text{Cone}(e_1, -e_1 - e_2)$, and $\sigma_2 = \text{Cone}(-e_1 - e_2, e_2)$, together with the three rays $\tau_{ij} = \sigma_i \cap \sigma_j$ for $i \neq j$, and the origin. The toric variety $X_{\Delta} \cong \mathbb{P}^2$ is covered by the affine parts

$$X_{\sigma_0} \cong \text{Spec}(\mathbb{C}[x, y]), X_{\sigma_1} \cong \text{Spec}(\mathbb{C}[xy^{-1}, y^{-1}]), X_{\sigma_2} \cong \text{Spec}(\mathbb{C}[x^{-1}, x^{-1}y]).$$

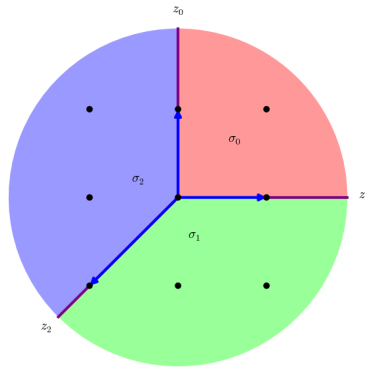


Figure 4.2.: The fan Δ for \mathbb{P}^2 .

More generally, let $N_{\mathbb{R}} = \mathbb{R}^n$, where $N = \mathbb{Z}^n$ has standard basis e_1, \dots, e_n . Set

$$e_0 = -e_1 - \dots - e_n,$$

and let Δ be the fan in $N_{\mathbb{R}}$ consisting of the cones generated by all proper subsets of $\{e_0, e_1, \dots, e_n\}$. This fan gives a toric variety $X_{\Delta} \cong \mathbb{P}^n$.

Example 4.2.4. Let $r \in \mathbb{N}$ and consider the fan Δ_r in $N_{\mathbb{R}} = \mathbb{R}^2$ consisting of the four cones

$$\begin{aligned}\sigma_0 &= \text{Cone}((1, 0), (0, 1)), \\ \sigma_1 &= \text{Cone}((1, 0), (0, -1)), \\ \sigma_2 &= \text{Cone}((0, -1), (-1, r)), \\ \sigma_3 &= \text{Cone}((-1, r), (0, 1)),\end{aligned}$$

together with all of their faces. The corresponding toric variety X_{Δ_r} is called the *Hirzebruch surface* \mathcal{H}_r (see Figure 4.3 for the case when $r = 2$).

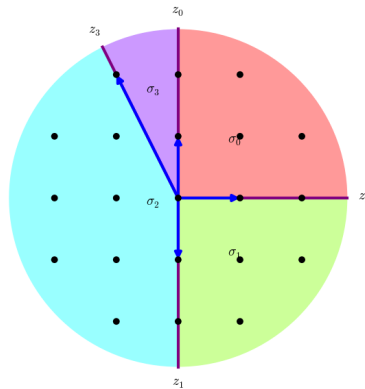


Figure 4.3.: The fan Δ_2 for Hirzebruch surface \mathcal{H}_2 .

4.3. Chow Rings of Toric Varieties

Let Δ be a complete simplicial fan in $N_{\mathbb{R}}$ and assume that $\Delta(1) = \{\tau_1, \dots, \tau_r\}$. Also let u_i be the minimal generator of τ_i and introduce a variable x_i for each τ_i . In the ring $\mathbb{Q}[x_1, \dots, x_r]$, let \mathcal{I} be the monomial ideal as follows:

$$\mathcal{I} = \langle x_{i_1} \dots x_{i_s} \mid i_j \text{ are distinct and } \tau_{i_1} + \dots + \tau_{i_s} \text{ is not a cone of } \Delta \rangle.$$

This ideal is called the *Stanley-Reisner ideal*. Also let \mathcal{J} be the ideal generated by the linear forms

$$\sum_{i=1}^r \langle m, u_i \rangle x_i,$$

where m runs over some basis for M . Then the *rational Chow ring* of the toric variety X_{Δ} associated to the fan Δ has the form

$$A^*(X_{\Delta})_{\mathbb{Q}} \cong \frac{\mathbb{Q}[x_1, \dots, x_r]}{\mathcal{I} + \mathcal{J}}.$$

Moreover, if Δ is smooth, then the *Chow ring* of X_{Δ} has the form

$$A^*(X_{\Delta}) \cong \frac{\mathbb{Z}[x_1, \dots, x_r]}{\mathcal{I} + \mathcal{J}}.$$

Note that these rings are graded with $\deg(x_i) = 1$ for all i .

Example 4.3.1. As a toric variety, \mathbb{P}^n comes from the fan with ray generators $u_i = e_i$ for $i = 1, \dots, n$ and $u_{n+1} = -e_1 - \dots - e_n$. It is easy to check that the Stanley-Reisner ideal

$$\mathcal{I} = \langle x_1 \dots x_{n+1} \rangle.$$

Using the basis e_1, \dots, e_n for M , we have

$$\mathcal{J} = \langle x_1 - x_{n+1}, \dots, x_n - x_{n+1} \rangle.$$

Thus the Chow ring of \mathbb{P}^n has the form

$$A^*(\mathbb{P}^n) \cong \frac{\mathbb{Z}[x_1, \dots, x_{n+1}]}{\langle x_1 \dots x_{n+1}, x_1 - x_{n+1}, \dots, x_n - x_{n+1} \rangle} \cong \frac{\mathbb{Z}[x_{n+1}]}{\langle x_{n+1}^{n+1} \rangle}.$$

This agrees with Theorem 1.1.10.

Here is the implementation of \mathbb{P}^n as a toric variety in `Schubert3`.

```
sage: def ToricProjectiveSpace(n):
.....:     cones = []
.....:     V = IntegerVectors(1,n).list()
.....:     for i in range(n+1):
.....:         e = [tuple(a) for a in V] + [tuple([-1]*n)]
.....:         e.pop(-i-1)
.....:         cones.append(Cone(e))
.....:     F = Fan(cones)
.....:     return ToricVariety(F)
.....:
sage: P2 = ToricProjectiveSpace(2); P2
A toric variety of dimension 2 corresponding to Rational
polyhedral fan in 2-d lattice N
sage: R = P2.chow_ring(); R
Quotient of Multivariate Polynomial Ring in x1, x2, x3 over
Rational Field by the ideal (x1*x2*x3, x2 - x3, x1 - x3)
sage: I = R.defining_ideal()
sage: I.groebner_basis()
[x3^3, x1 - x3, x2 - x3]
```

Example 4.3.2. The Hirzebruch surface \mathcal{H}_r comes from the fan as in Example 4.2.4. In this case, we have four ray generators $u_1 = e_1, u_2 = e_2, u_3 = -e_2, u_4 = -e_1 + re_2$. Then the Stanley-Reisner ideal

$$\mathcal{I} = \langle x_1x_4, x_2x_3 \rangle.$$

Using the basis e_1, e_2 for M , we have two linear relations for \mathcal{J} as follows

$$\mathcal{J} = \langle x_1 - x_4, x_2 - x_3 + rx_4 \rangle.$$

Thus the Chow ring of Hirzebruch surface \mathcal{H}_r has the form

$$A^*(\mathcal{H}_r) \cong \frac{\mathbb{Z}[x_1, x_2, x_3, x_4]}{\langle x_1x_4, x_2x_3, x_1 - x_4, x_2 - x_3 + rx_4 \rangle}.$$

The two linear relations from \mathcal{J} can be used to eliminate x_1, x_2 . This implies that

$$A^*(\mathcal{H}_r) \cong \frac{\mathbb{Z}[x_3, x_4]}{\langle x_4^2, x_3^2 - rx_3x_4 \rangle}.$$

Here is the implementation of Hirzebruch surface \mathcal{H}_r and its Chow ring in `Schubert3`.

```

sage: def HirzebruchSurface(r):
.....:     C1 = Cone([(0,1),(1,0)])
.....:     C2 = Cone([(1,0),(0,-1)])
.....:     C3 = Cone([(0,-1),(-1,r)])
.....:     C4 = Cone([(-1,r),(0,1)])
.....:     F = Fan([C1,C2,C3,C4])
.....:     return ToricVariety(F)
.....:
sage: H2 = HirzebruchSurface(2); H2
A toric variety of dimension 2 corresponding to Rational
polyhedral fan in 2-d lattice N
sage: R = H2.chow_ring(); R
Quotient of Multivariate Polynomial Ring in x1, x2, x3, x4
over Rational Field by the ideal
(x1*x2, x3*x4, x3 - x4, x1 - x2 + 2*x4)
sage: I = R.defining_ideal()
sage: I.groebner_basis()
[x2^2 - 2*x2*x4, x4^2, x1 - x2 + 2*x4, x3 - x4]

```

4.4. Todd classes of Toric Varieties

Let Δ be a smooth complete fan in $N_{\mathbb{R}}$. Let $X = X_{\Delta}$ be the toric variety associated to Δ and let T_X be the tangent bundle on X . Assume that the Chow ring of X_{Δ} has the form as in Section 4.3. Then we have the following statements:

1. The total Chern class of T_X has the form

$$c(T_X) = \prod_{i=1}^r (1 + x_i) \in A^*(X_{\Delta}).$$

In particular, we have

$$c_1(T_X) = \sum_{i=1}^r x_i \in A^*(X_{\Delta}).$$

2. The Todd class of T_X has the form

$$\text{td}(T_X) = \prod_{i=1}^r \frac{x_i}{1 - \exp(-x_i)} \in A^*(X_{\Delta})_{\mathbb{Q}}.$$

Note that the x_i are as in Section 4.3.

Remark 4.4.1.

1. For the proof of these statements, we refer to [19, Section 13.1].
2. We can use the `ToricVarieties` package in `Sage`, written by Braun and Novoseltsev, for computations in toric geometry. However, with this package, one can only compute the Todd class of a toric variety of small dimension (up to four). We can overcome this with the `Schubert3` package.

Example 4.4.2. In this example, we compute the Euler characteristic of a line bundle $\mathcal{L} = \mathcal{O}_{\mathcal{H}_2}(D)$ on the Hirzebruch surface \mathcal{H}_2 with $D = -5x_2 + 3x_3 \in A^*(\mathcal{H}_2)$ and x_2, x_3 are as in Example 4.3.2. By the Hirzebruch-Riemann-Roch formula, we have

$$\chi(\mathcal{H}_2, \mathcal{L}) = \int_{\mathcal{H}_2} \text{ch}(\mathcal{L}) \cdot \text{td}(\mathcal{H}_2).$$

We illustrate this by `Schubert3` as follows:

```
sage: H2 = HirzebruchSurface(2)
sage: D = H2.divisor([0,-5,3,0]); D
-5*x2 + 3*x3
sage: L = H2.0(D)
sage: tdX = H2.todd_class()
sage: chL = L.chern_character()
sage: H2.integral(tdX*chL)
4
```

We conclude that $\chi(\mathcal{H}_2, \mathcal{L}) = 4$.

Example 4.4.3. In this example, we compute the Todd class of \mathbb{P}^5 as a toric variety.

```
sage: P5 = ToricProjectiveSpace(5)
sage: P5.todd_class()
x5^5 + 137/60*x5^4 + 15/4*x5^3 + 17/4*x5^2 + 3*x5 + 1
```

This means that the Todd class of \mathbb{P}^5 is

$$\text{td}(\mathbb{P}^5) = x_5^5 + \frac{137}{60}x_5^4 + \frac{15}{4}x_5^3 + \frac{17}{4}x_5^2 + 3x_5 + 1,$$

where x_5 is defined as in Example 4.3.1.

Chapter 5.

Equivariant Intersection Theory and Bott's Formula

In this chapter, we introduce basic notions such as Chow rings, vector bundles, and Chern classes in the equivariant setting. We will also describe the localization theorem and Bott's formula. Finally, we will present an alternative method for solving the enumerative problems discussed in Chapter 2. All the theoretical results in this chapter can be found in [2, 26, 27].

A variety is always assumed to be smooth and quasiprojective. If G is a linear algebraic group such as a *split torus* $G = T = (\mathbb{C}^*)^g$, a G -variety is a variety X endowed with an action $G \times X \rightarrow X$.

5.1. Basic Notions

In this section, we introduce the equivariant Chow ring of a G -variety X . The notions of pullback and pushforward homomorphisms will be reviewed in the equivariant setting.

Let G be a linear algebraic group of dimension g , and let X be a G -variety of dimension n . For any $i \geq 0$, we may choose a representation V of G with, say, $\dim V = l$ together with a dense open subset $U \subset V$ on which G acts freely, where the codimension of $V \setminus U$ in V is larger than $n - i$, and such that the principal bundle quotient $U \rightarrow U/G$ exists in the category of schemes (see [26, Lemma 9]). The diagonal action on $X \times U$ is then also free, which implies that under mild assumptions, a principal bundle quotient $X \times U \rightarrow (X \times U)/G$ exists in the category of schemes (see [26, Proposition 23]). One possible assumption here is that G is special. This applies in our examples: we will only consider split torus actions. Therefore, in what follows, we will tacitly assume that the scheme $(X \times U)/G$ exists and denote it by X_G .

5.1.1. Equivariant Chow Rings

With notation as above, the i -th G -equivariant Chow group of X is defined by

$$A_i^G(X) := A_{i+l-g}(X_G),$$

where A_* stands for the ordinary Chow group.

Theorem 5.1.1. *The group $A_i^G(X)$ is independent from all choices made above.*

Proof. See [26, Definition-Proposition 1]. □

The G -equivariant Chow group of X is defined to be

$$A_*^G(X) = \bigoplus A_i^G(X).$$

Since we assume that X and, thus, X_G are smooth, $A_*^G(X)$ inherits an intersection product from the ordinary Chow groups. This endows $A_*^G(X)$ with the structure of a graded ring, called the G -equivariant Chow ring of X . For convenience, we take the grading given by codimension, writing

$$A_G^i(X) = A_{n-i}^G(X)$$

and

$$A_G^*(X) = \bigoplus A_G^i(X).$$

Example 5.1.2. [2, Section 1.5] Let $G = \mathbb{C}^*$ be a torus of dimension 1 and $X = pt$ a point. Consider the diagonal representation of G on $V = \mathbb{C}^l$ for some l , and set $U = V \setminus \{0\} = \mathbb{C}^l \setminus \{0\}$. In this case, it is clear that G acts freely on U and X_G is just $U/G = \mathbb{P}^{l-1}$. For each $i \geq 0$, picking $l > i$, we have

$$A_{\mathbb{C}^*}^i(pt) = A_{-i}^{\mathbb{C}^*}(pt) = A_{-i+l-1}(\mathbb{P}^{l-1}) = A^i(\mathbb{P}^{l-1}) = \mathbb{Z} \cdot h^i,$$

where h denotes the hyperplane class of \mathbb{P}^{l-1} . Letting l go to infinity, we see that $A_{\mathbb{C}^*}^*(pt)$ is a polynomial ring, say

$$A_{\mathbb{C}^*}^*(pt) = \bigoplus_{i=0}^{\infty} \mathbb{Z} \cdot \lambda^i = \mathbb{Z}[\lambda].$$

More generally, if $G = T = (\mathbb{C}^*)^g$ is a split torus of dimension g , then we may choose

$$U = \prod_{j=1}^g (\mathbb{C}^{l_j} \setminus \{0\}) \subset \prod_{j=1}^g (\mathbb{C}^{l_j}).$$

The quotient U/T is a product of projective spaces,

$$U/T = \prod_{j=1}^g \mathbb{P}^{l_j-1}.$$

Thus the T -equivariant Chow ring of a point,

$$R_T := A_T^*(pt) = \mathbb{Z}[\lambda_1, \dots, \lambda_g],$$

is a polynomial ring with integer coefficients in g variables $\lambda_1, \dots, \lambda_g$.

Example 5.1.3. [26, Section 3.3] Consider the action of $G = T = \mathbb{C}^*$ on $X = \mathbb{P}^n$ given in coordinates by

$$t \cdot [x_0 : \dots : x_n] = [t^{a_0}x_0 : \dots : t^{a_n}x_n],$$

where $t \in T$, and where the $a_i \in \mathbb{N}$ are fixed weights. Like in the previous example, we may choose $U = \mathbb{C}^l \setminus \{0\}$. Then we have $U/T = \mathbb{P}^{l-1}$. In this case, one can show that $\mathbb{P}_T^n \rightarrow \mathbb{P}^{l-1}$ is the projective bundle of the vector bundle

$$E = \mathcal{O}(a_0) \oplus \dots \oplus \mathcal{O}(a_n)$$

on \mathbb{P}^{l-1} . The Chow ring of this projective bundle is given by

$$A^*(\mathbb{P}_T^n) = A^*(\mathbb{P}^{l-1})[\zeta]/\langle f(\zeta, h) \rangle,$$

where h is the hyperplane class of \mathbb{P}^{l-1} , $\zeta = c_1(\mathcal{O}_{\mathbb{P}(E)}(1))$ is the equivariant hyperplane class, and

$$\begin{aligned} f(\zeta, h) &= \zeta^{n+1} + c_1(E)\zeta^n + \cdots + c_n(E)\zeta + c_{n+1}(E) \\ &= \prod_{i=0}^n (\zeta + a_i h). \end{aligned}$$

Letting l go to infinity, we see that the \mathbb{C}^* -equivariant Chow ring of \mathbb{P}^n is

$$A_{\mathbb{C}^*}^*(\mathbb{P}^n) = \mathbb{Z}[\zeta, \lambda] / \langle \prod_{i=0}^n (\zeta + a_i \lambda) \rangle.$$

5.1.2. Pullback and Pushforward Homomorphisms

Let X, Y be G -varieties. A morphism $f : X \rightarrow Y$ is called G -equivariant if it satisfies the condition $f(g \cdot x) = g \cdot f(x)$ for all $g \in G$ and $x \in X$. Throughout this section, all morphisms of G -varieties are assumed to be G -equivariant.

Given a morphism $f : X \rightarrow Y$ of G -varieties, the morphism $f \times id : X \times U \rightarrow Y \times U$ induces a morphism $f_G : X_G \rightarrow Y_G$ such that the diagram

$$\begin{array}{ccc} X \times U & \xrightarrow{f \times id} & Y \times U \\ \downarrow & & \downarrow \\ X_G & \xrightarrow{f_G} & Y_G \end{array}$$

commutes. In addition, one may deduce that if $f : X \rightarrow Y$ is proper, or flat of relative dimension k , or an embedding, then $f_G : X_G \rightarrow Y_G$ will have the same property (see [26, Proposition 2]).

If $f : X \rightarrow Y$ is a proper morphism, then the pushforward homomorphism

$$f_* : A_i^G(X) \longrightarrow A_i^G(Y)$$

is given by

$$f_{G*} : A_{i+l-g}(X_G) \longrightarrow A_{i+l-g}(Y_G).$$

If $f : X \rightarrow Y$ is a flat morphism of relative dimension k , then the pullback homomorphism

$$f^* : A_i^G(Y) \longrightarrow A_{i+k}^G(X)$$

is given by

$$f_G^* : A_{i+l-g}(Y_G) \longrightarrow A_{i+k+l-g}(X_G).$$

Theorem 5.1.4. *The maps f_* and f^* are independent of the choices made.*

Proof. See [2, Proposition 1.6.1]. □

5.1.3. Equivariant vector bundles and Chern Classes

Let X be a G -variety. A G -equivariant vector bundle is a vector bundle E on X such that the action of G on X lifts to an action of G on E which is linear on fibers. More concretely, for each $g \in G$ and $x \in X$, the map $g : E_x \rightarrow E_{g \cdot x}$ is linear.

By [26, Lemma 1], E_G is a vector bundle over X_G . The G -equivariant Chern classes $c_i^G(E) \in A_G^*(X)$ are defined to be the Chern classes $c_i(E_G)$. If E has rank r , then the top Chern class $c_r^G(E)$ is called the G -equivariant Euler class of E and denoted

$$e^G(E) \in A_G^*(X).$$

Note that a G -equivariant vector bundle over a point is a representation of G (see also [26, Section 3.2]). Our primary interest is when $G = T = (\mathbb{C}^*)^g$ is a split torus and $X = pt$ is a point. In this case, let $M(T)$ be the character group of the torus T . There is a group isomorphism

$$\psi : M(T) \longrightarrow A_T^1(pt) \quad , \quad \rho_i \longmapsto \lambda_i,$$

where ρ_i is the character of T defined by $\rho_i(t_1, \dots, t_g) = t_i$. This induces a ring isomorphism $\text{Sym}(M(T)) \simeq R_T$. We call $\psi(\rho)$ the *weight* of ρ . In particular, λ_i is the weight of ρ_i .

Example 5.1.5. [16, Example 9.1.1.1] The diagonal action of $G = T = (\mathbb{C}^*)^g$ on \mathbb{C}^g gives a T -equivariant vector bundle E over a point. Given $i \geq 0$, choose

$$U = \prod_{j=1}^g (\mathbb{C}^{l_j} \setminus \{0\}) \subset \prod_{j=1}^g (\mathbb{C}^{l_j}),$$

as in Example 5.1.2. Then

$$E_T \cong \mathcal{O}(h_1) \oplus \dots \oplus \mathcal{O}(h_g),$$

where the h_j are the pullbacks of the hyperplane classes on the \mathbb{P}^{l_j-1} to the product $\prod_{j=1}^g \mathbb{P}^{l_j-1}$. It follows from the definition of the Chern classes that

$$c_k^T(E) = c_k(E_T) = s_k(\lambda_1, \dots, \lambda_g) \in R_T \cong \mathbb{Z}[\lambda_1, \dots, \lambda_g],$$

where s_k is the k -th elementary symmetric function. In particular, the T -equivariant Euler class of E is

$$e^T(E) = \lambda_1 \cdots \lambda_g \in R_T.$$

Example 5.1.6. Consider the diagonal action of $T = (\mathbb{C}^*)^4$ on \mathbb{C}^4 given in coordinates by

$$(t_1, t_2, t_3, t_4) \cdot (x_1, x_2, x_3, x_4) = (t_1 x_1, t_2 x_2, t_3 x_3, t_4 x_4).$$

This induces an action of T on the Grassmannian $G(2, 4)$ with the isolated fixed points L_I corresponding to coordinate 2-planes in \mathbb{C}^4 . Each L_I is indexed by the 2-subset I of the set $\{1, 2, 3, 4\}$ so that L_I is defined by the equations $x_j = 0$ for $j \notin I$. Let S be the tautological subbundle on $G(2, 4)$. At each L_I , the restriction of the action of T on the fiber $S|_{L_I}$ gives a representation of T with characters ρ_i for $i \in I$. This representation gives a T -equivariant vector bundle $S|_{L_I}$ of rank 2 over a point L_I . If $I = \{i_1, i_2\}$, then we have

$$c_1^T(S|_{L_I}) = \lambda_{i_1} + \lambda_{i_2},$$

$$c_2^T(S|_{L_I}) = \lambda_{i_1} \cdot \lambda_{i_2},$$

where λ_{i_1} and λ_{i_2} are the weights of ρ_{i_1} and ρ_{i_2} respectively.

Remark 5.1.7 (The implementation of equivariant vector bundles and Chern classes). In `Schubert3`, a T -equivariant vector bundle over a point is represented by a list of weights. The command `EquivariantVectorBundle` makes a class in `Schubert3` which represents such a T -equivariant vector bundle.

Let E be a T -equivariant vector bundle over a point. Suppose that $\lambda_1, \dots, \lambda_r$ are the weights of E . The rank of E is represented by the number of the weights of E , that is r . The k -th T -equivariant Chern class $c_k^T(E)$ is represented by $s_k(\lambda_1, \dots, \lambda_r)$, where s_k is the k -th elementary symmetric function in r variables. The T -equivariant Euler class $e^T(E)$ is represented by the product $\lambda_1 \cdots \lambda_r$. The dual of E is represented by a T -equivariant vector bundle over a point with weights $-\lambda_1, \dots, -\lambda_r$. If F is a T -equivariant vector bundle over a point with weights $\gamma_1, \dots, \gamma_s$, then the tensor product of E and F is represented by a T -equivariant vector bundle over a point with weights $\lambda_i + \gamma_j$ for all i and j . The d -th symmetric power of E is represented by a T -equivariant vector bundle over a point with weights

$$\left\{ \sum_{i=1}^r v_i \lambda_i \mid v_i \in \mathbb{N}, \sum_{i=1}^r v_i = d \right\}.$$

We illustrate these functionalities by `Schubert3` as follows:

```
sage: R.<h1,h2,h3,h4> = QQ[]
sage: w = [h1,h2,h3,h4]
sage: E = EquivariantVectorBundle(w); E
A T-equivariant vector bundle of rank 4 over a point
sage: E.weights()
[h1, h2, h3, h4]
sage: E.rank() == len(E.weights())
True
sage: E.chern_class(1)
h1 + h2 + h3 + h4
sage: E.euler_class()
h1*h2*h3*h4
sage: F = E.dual(); F
A T-equivariant vector bundle of rank 4 over a point
sage: F.weights()
[-h1, -h2, -h3, -h4]
sage: F.rank() == E.rank()
True
sage: T = E & F; T #the tensor product of E and F
A T-equivariant vector bundle of rank 16 over a point
sage: T.weights()
[0, h1 - h2, h1 - h3, h1 - h4, -h1 + h2, 0, h2 - h3, h2 - h4,
-h1 + h3, -h2 + h3, 0, h3 - h4, -h1 + h4, -h2 + h4, -h3 + h4, 0]
sage: S = E.symmetric_power(2); S
A T-equivariant vector bundle of rank 10 over a point
sage: S.weights()
[2*h1, h1 + h2, h1 + h3, h1 + h4, 2*h2, h2 + h3, h2 + h4, 2*h3,
h3 + h4, 2*h4]
```

5.2. Localization Theorem and Bott's Formula

In this section, we focus on the action of a split torus $T = (\mathbb{C}^*)^g$ on a smooth projective variety X .

Lemma 5.2.1. *If T acts trivially on X , then we have*

$$A_T^*(X) = A^*(X) \otimes R_T.$$

Proof. See [27, Proposition 2]. □

Given an action of T on X , we denote the fixed point locus by X^T and write

$$X^T = \bigcup_j Z_j,$$

where the Z_j are the components of X^T . Since X is assumed to be smooth, X^T and, hence, the Z_j are also smooth (see [63]). For each j , let

$$i_j : Z_j \hookrightarrow X$$

be the inclusion. Then i_j induces the pushforward homomorphism

$$i_{j*} : A_T^*(Z_j) \longrightarrow A_T^*(X),$$

and the pullback homomorphism

$$i_j^* : A_T^*(X) \longrightarrow A_T^*(Z_j).$$

Since the action of T on each Z_j is trivial, we have

$$A_T^*(Z_j) \simeq A^*(Z_j) \otimes R_T.$$

We have shown in Example 5.1.2 that the T -equivariant Chow ring of a point is

$$R_T = \mathbb{Z}[\lambda_1, \dots, \lambda_g].$$

In addition, via the pullback homomorphism of the map to a point, we may consider $A_T^*(Z_j)$ and $A_T^*(X)$ as R_T -modules. Set $R_T^* = R_T \setminus \{0\}$. We then get the localization of the modules $A_T^*(Z_j), A_T^*(X)$ with respect to R_T^* , denoted by

$$A_T^*(Z_j) \otimes Q_T, A_T^*(X) \otimes Q_T,$$

where $Q_T = \mathbb{Q}(\lambda_1, \dots, \lambda_g)$ is the field of fractions of R_T .

For each j , we denote by N_j the normal bundle of Z_j in X . In [27, Proposition 4], Edidin and Graham proved that $e^T(N_j)$ is an invertible element in $A_T^*(Z_j) \otimes Q_T$. More concretely, we have the following theorem.

Theorem 5.2.2 (Localization theorem, compare with Proposition 9.1.2 in [16]). *There is an isomorphism*

$$\varphi : A_T^*(X) \otimes Q_T \longrightarrow \bigoplus_j A_T^*(Z_j) \otimes Q_T,$$

given by

$$\alpha \mapsto \left(\frac{i_j^*(\alpha)}{e^T(N_j)} \right)_j.$$

In addition, the inverse

$$\varphi^{-1} : \bigoplus_j A_T^*(Z_j) \otimes Q_T \longrightarrow A_T^*(X) \otimes Q_T,$$

is given by

$$(\alpha_j)_j \mapsto \sum_j i_{j*}(\alpha_j).$$

In particular, for any $\alpha \in A_T^*(X) \otimes Q_T$, we have

$$\alpha = \sum_j i_{j*} \left(\frac{i_j^*(\alpha)}{e^T(N_j)} \right).$$

Proof. By [27, Theorem 1], there is an isomorphism

$$A^*(X^T) \otimes Q_T \cong A_T^*(X) \otimes Q_T.$$

This implies that, for any $\alpha \in A_T^*(X) \otimes Q_T$, we can write

$$\alpha = \sum_j i_{j*}(\alpha_j),$$

where $\alpha_j \in A_T^*(Z_j) \otimes Q_T$. Therefore, $i_j^*(\alpha) = i_j^* i_{j*}(\alpha_j)$. By the equivariant self-intersection formula (see [2, Section 1.6.4]), we obtain $i_j^*(\alpha) = e^T(N_j)\alpha_j$. Hence

$$\alpha_j = \frac{i_j^*(\alpha)}{e^T(N_j)}$$

as desired. □

Since we assume that X is projective, the trivial morphism from X to a point $\pi_X : X \rightarrow pt$ induces a pushforward homomorphism $\pi_{X*} : A_*^T(X) \rightarrow R_T$ which is given by the degree of zero cycles in $A_0^T(X)$ and is zero on $A_i^T(X)$ for $i \neq 0$. Tensorizing by Q_T , we define the *equivariant degree* $\int_{X^T} \alpha$ to be the image of α under π_{X*} :

$$\begin{aligned} \pi_{X*} : A_T^*(X) \otimes Q_T &\longrightarrow Q_T \\ \alpha &\longmapsto \int_{X^T} \alpha. \end{aligned}$$

Corollary 5.2.3 (Degree formula). *Let $\alpha \in A_T^*(X) \otimes Q_T$. Then we have*

$$\int_{X^T} \alpha = \sum_j \int_{(Z_j)^T} \frac{i_j^*(\alpha)}{e^T(N_j)} \in Q_T.$$

Proof. See [27, Corollary 1]. □

In many applications, we want to compute $\int_X \alpha$, where $\alpha \in A_0(X)$ is a zero-dimensional cycle class on X . In order to do this, we consider the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\pi_X} & pt \\ i_X \downarrow & & \downarrow i_{pt} \\ X_T & \xrightarrow{\pi_X^T} & U/T. \end{array}$$

Suppose that $\alpha = i_X^*(\beta)$ for $\beta \in A_0^T(X)$. Then we have

$$\int_X \alpha = \int_X i_X^*(\beta) = i_{pt}^* \int_{X_T} \beta.$$

By the degree formula, we have the following theorem.

Theorem 5.2.4. *Let $\alpha \in A_0(X)$ be a zero-dimensional cycle class on X . Suppose that $\alpha = i_X^*(\beta)$ for $\beta \in A_0^T(X)$. Then we have*

$$\int_X \alpha = \sum_j \int_{(Z_j)_T} \frac{i_j^*(\beta)}{e^T(N_j)}.$$

Proof. See [27, Proposition 5]. □

Bott's formula shows that we can compute the degree of a zero-dimensional cycle class on X in terms of local contributions supported on the components of the fixed point locus of a torus action on X .

As an interesting example, we have the following statement.

Proposition 5.2.5. *Suppose that the action of T on X has a finite number m of fixed points. Then we have $\chi(X) = m$, where $\chi(X)$ is the Euler characteristic of X .*

Proof. By the Hirzebruch-Riemann-Roch theorem (see Theorem 1.4.1) and Bott's formula, we have

$$\chi(X) = \chi(X, \mathcal{O}_X) = \int_X c_{\text{top}}(T_X) = \sum_j \int_{Z_j} \frac{e^T(N_j)}{e^T(N_j)} = \sum_j 1 = m.$$

□

Let us describe Bott's formula in case where the fixed point locus X^T is finite. For each point $Z_j \in X^T$, assume that the action of T on $E|_{Z_j}$ has characters $\chi_j^1, \dots, \chi_j^s$, where each χ_j^i is a linear combination of the basic characters ρ_1, \dots, ρ_g defined in Section 5.1.3. Write $\chi_j^i = a_j^i(\rho_1, \dots, \rho_g)$ for this linear combination. Since ρ_i has weight λ_i , the weight of χ_j^i is $a_j^i(\lambda)$, where $a_j^i(\lambda) \in R_T$ denotes the linear combination of $\lambda_1, \dots, \lambda_g$ obtained by replacing λ_i by ρ_i in $a_j^i(\rho_1, \dots, \rho_g)$. Thus the T -action on $E|_{Z_j}$ has weights $a_j^i(\lambda)$. This implies that

$$i_j^*(c_k^T(E)) = c_k^T(E|_{Z_j}) = s_k(a_j^1(\lambda), \dots, a_j^s(\lambda)) \in R_T,$$

where s_k is the k -th elementary symmetric function. In addition, since Z_j is just a point, the normal bundle N_j of Z_j in X is the tangent space to X at Z_j . If the action of T on the tangent space to X at Z_j has weights $b_j^1(\lambda), \dots, b_j^n(\lambda)$, where $n = \dim X$, then

$$e^T(N_j) = \prod_{k=1}^n b_j^k(\lambda) \in R_T.$$

We can now apply i_{pt}^* to the right hand side of the degree formula. Keeping the notation as above, we obtain the following theorem.

Theorem 5.2.6 (Bott formula). *Suppose that T acts on X with isolated fixed points Z_j and that E is a T -equivariant vector bundle on X . If $P(c_k(E))$ is a polynomial in the Chern classes of E , then we have*

$$\int_X P(c_k(E)) = \sum_j \frac{P(s_k(a_j^1(\lambda), \dots, a_j^s(\lambda)))}{\prod_{k=1}^n b_j^k(\lambda)}.$$

Proof. See [16, Proposition 9.1.5] or [2, Section 2.6]. □

5.3. Schubert Calculus, Revisited

The formula in Theorem 5.2.6 gives us an alternative algorithm (see Algorithm 5.3.1) for computing the degree of a zero-dimensional cycle class in terms of the fixed points under a torus action on a Grassmannian.

Algorithm 5.3.1 Bott's formula for Grassmannians

Input: A zero-dimensional cycle class on the Grassmannian $G(k, n)$ which is represented by a homogeneous polynomial P of degree $k(n - k)$ in $A(G(k, n))$.

Output: The degree of this cycle class.

- 1: Choose a suitable torus action on $G(k, n)$.
 - 2: Compute the fixed point locus F of this action.
 - 3: Set $r = 0$.
 - 4: **for** p in F **do**
 - 5: Compute the T -equivariant Chern classes of the tautological quotient bundle on $G(k, n)$ at p . Store the results as a tuple, say s .
 - 6: Compute the T -equivariant Euler class t of the tangent bundle on $G(k, n)$ at p .
 - 7: Set $r = r + \frac{P(s)}{t}$.
 - 8: **end for**
 - 9: **Return** r .
-

Remark 5.3.1. In the first step of Algorithm 5.3.1, choosing a suitable torus action means that we have to choose a torus action such that the fixed point locus is finite. For example, choose the diagonal action of $T = (\mathbb{C}^*)^n$ on \mathbb{C}^n given in coordinates by

$$(t_1, \dots, t_n) \cdot (x_1, \dots, x_n) = (t_1 x_1, \dots, t_n x_n).$$

With this choice, we have an induced action of T on the Grassmannian $G(k, n)$ with the isolated fixed points L_I corresponding to the coordinate k -planes in \mathbb{C}^n . Each L_I is indexed by the k -subset $I \subset \{1, \dots, n\}$ such that L_I is defined by the equations $x_j = 0$ for all $j \notin I$. For example, in the case of $G(2, 4)$, the fixed point locus F is

$$F = \{\langle x_1, x_2 \rangle, \langle x_1, x_3 \rangle, \langle x_1, x_4 \rangle, \langle x_2, x_3 \rangle, \langle x_2, x_4 \rangle, \langle x_3, x_4 \rangle\},$$

where $\langle x_i, x_j \rangle$ represents the line given by $x_i = x_j = 0$.

We illustrate this by `Schubert3` as follows:

```
sage: G = Grassmannian(2,4)
sage: F = G.fixed_points(); F
[[1, 2], [1, 3], [1, 4], [2, 3], [2, 4], [3, 4]]
```

In `Schubert3`, the function `fixed_points` of the class `Grassmannian` $G(k, n)$ returns a list of the k -subsets of $\{1, \dots, n\}$.

Algorithm 5.3.1 has been implemented in `Schubert3` as a functionality (`bott`) of the class `Grassmannian`. We have checked the examples in Chapter 2, the results are all as desired (see Appendix A).

Example 5.3.2. We recompute the number of lines meeting 4 general lines in \mathbb{P}^3 as follows:

```
sage: G = Grassmannian(2,4)
sage: sigma1 = G.schubert_class([1])
sage: G.bott(sigma1^4)
2
```

Remark 5.3.3 (The implementation of Bott's formula for Grassmannians). In order to shorten the computation time, we replace the variables λ_j as in Section 5.1.3 by the integers j (the sum of rational functions on the right hand side of Bott's formula does not, in fact, depend on the λ_j).

Let L_I be a fixed point of the action of $T = (\mathbb{C}^*)^n$ on the Grassmannian $G(k, n)$, represented by a k -subset I of $\{1, \dots, n\}$. Then the tautological subbundle S on $G(k, n)$ at L_I is represented by a T -equivariant vector bundle over a point with weights which are the elements of the k -subset I . The tautological quotient bundle Q on $G(k, n)$ at L_I is represented by a T -equivariant vector bundle over a point with weights which are the elements of $\{1, \dots, n\} \setminus I$. The tangent bundle on $G(k, n)$ at L_I is represented by a T -equivariant vector bundle over a point which is the tensor product of S^\vee and Q at L_I .

We illustrate these functionalities by `Schubert3` as follows:

```
sage: G = Grassmannian(2,4)
sage: F = G.fixed_points()
sage: I = F[0]; I
{1, 2}
sage: S = G.equivariant_subbundle(I); S
A T-equivariant vector bundle of rank 2 over a point
sage: Q = G.equivariant_quotient_bundle(I); Q
A T-equivariant vector bundle of rank 2 over a point
sage: Q.weights()
[3, 4]
sage: B = S.dual() & Q
sage: T = G.equivariant_tangent_bundle(I); T
A T-equivariant vector bundle of rank 4 over a point
sage: B.weights() == T.weights()
True
sage: E = Q.symmetric_power(3); E
A T-equivariant vector bundle of rank 4 over a point
sage: E.weights()
[6, 7, 8, 9]
sage: E.bernstein_class(1)
30
sage: E.euler_class()
3024
```

5.3.1. Euler Characteristic of Grassmannians

It is clear that the number of fixed points is equal to the binomial coefficient $\binom{n}{k}$. Therefore, we have the following statement.

Proposition 5.3.4. *The Euler characteristic of the Grassmannian $G(k, n)$ is equal to the binomial coefficient $\binom{n}{k}$.*

Proof. Choose the torus action on $G(k, n)$ as above. Then the number of fixed points is finite and equal to the number of ways to choose k elements from a set of n elements. Hence, by Proposition 5.2.5, the Euler characteristic of $G(k, n)$ is equal to the binomial coefficient $\binom{n}{k}$. \square

Propositions 2.1.1 and 5.3.4 show that the Euler characteristic and the number of Schubert classes on $G(k, n)$ coincide, both are equal to the binomial coefficient $\binom{n}{k}$.

5.3.2. Fano Schemes, Revisited

We have already shown how to compute the degree of a Fano scheme of k -planes on a general hypersurface of degree d in \mathbb{P}^n via Schubert calculus (see Algorithms 2.3.1 and 2.3.2). In this section, using Bott's formula for Grassmannians, we give an explicit formula for this.

Theorem 5.3.5. *Let $k, d, n \in \mathbb{N}$ satisfy $d \neq 2$ (or $n \geq 2k + 1$) and $\phi(k, d, n) = (k + 1)(n - k) - \binom{d+k}{d} \geq 0$, and let $X \subset \mathbb{P}^n$ be a general hypersurface of degree d . Choose any set of integers a_1, \dots, a_{n+1} such that $a_i \neq a_j$ for $i \neq j$. Then the degree of the Fano scheme $F_k(X)$ of k -planes on X is given by the following formula:*

$$\deg(F_k(X)) = (-1)^{\phi(k, d, n)} \sum_{I \subset \{1, \dots, n+1\}, |I|=k+1} \frac{\left(\prod_{v_i \in \mathbb{N}, \sum_{i \in I} v_i = d} \left(\sum_{i \in I} v_i a_i \right) \right) \left(\sum_{j \notin I} a_j \right)^{\phi(k, d, n)}}{\prod_{i \in I} \prod_{j \notin I} (a_i - a_j)},$$

where the sum runs over all the $(k + 1)$ -subsets of $\{1, \dots, n + 1\}$.

Proof. Consider the diagonal action of $T = (\mathbb{C}^*)^{n+1}$ on \mathbb{C}^{n+1} given in coordinates by

$$(t_1, \dots, t_{n+1}) \cdot (x_1, \dots, x_{n+1}) = (t_1 x_1, \dots, t_{n+1} x_{n+1}).$$

This induces an action of T on the Grassmannian $G(k + 1, n + 1)$ with $\binom{n+1}{k+1}$ isolated fixed points L_I corresponding to the $\binom{n+1}{k+1}$ coordinate k -planes in \mathbb{P}^n . Each fixed point L_I is indexed by a $(k + 1)$ -subset I of the set $\{1, \dots, n + 1\}$. Let S and Q be the tautological subbundle and quotient bundle on $G(k + 1, n + 1)$. The key idea is that, at each L_I , the T -action on the fibers $S|_{L_I}$ and $Q|_{L_I}$ have characters ρ_i for $i \in I$ and ρ_j for $j \notin I$ respectively. Since the tangent bundle on the Grassmannian is isomorphic to $S^\vee \otimes Q$, the characters of the torus action on the tangent space at L_I are

$$\{\rho_j - \rho_i \mid i \in I, j \notin I\}.$$

The normal bundle N_{L_I} of L_I in $G(k + 1, n + 1)$ is just the tangent space of $G(k + 1, n + 1)$ at L_I , hence

$$e^T(N_{L_I}) = \prod_{i \in I} \prod_{j \notin I} (\lambda_j - \lambda_i) = (-1)^{(k+1)(n-k)} \prod_{i \in I} \prod_{j \notin I} (\lambda_i - \lambda_j) \in R_T,$$

where λ_i is the weight of ρ_i defined in Section 5.1.3, and $R_T = \mathbb{Z}[\lambda_1, \dots, \lambda_{n+1}]$ is the T -equivariant Chow ring of a point.

We also need to compute $i_{L_I}^*(e^T(\text{Sym}^d S^\vee))$ and $i_{L_I}^*(c_1^T(Q))$, where $i_{L_I} : L_I \hookrightarrow G(k+1, n+1)$ is an inclusion. Since the characters of the torus action on $S^\vee|_{L_I}$ are $-\rho_i$ for $i \in I$, the torus action on $\text{Sym}^d S^\vee|_{L_I}$ has characters

$$\left\{ \sum_{i \in I} v_i(-\rho_i) \mid v_i \in \mathbb{N}, \sum_{i \in I} v_i = d \right\}.$$

Hence

$$\begin{aligned} i_{L_I}^*(e^T(\text{Sym}^d S^\vee)) &= e^T(\text{Sym}^d S^\vee|_{L_I}) \\ &= \prod_{v_i \in \mathbb{N}, \sum_{i \in I} v_i = d} \left(\sum_{i \in I} v_i(-\lambda_i) \right) \\ &= (-1)^{\binom{d+k}{d}} \prod_{v_i \in \mathbb{N}, \sum_{i \in I} v_i = d} \left(\sum_{i \in I} v_i \lambda_i \right). \end{aligned}$$

Since the characters of the torus action on $Q|_{L_I}$ are ρ_j for $j \notin I$, we have

$$i_{L_I}^*(c_1^T(Q)) = c_1^T(Q|_{L_I}) = \sum_{j \notin I} \lambda_j.$$

Theorems 2.3.2 and 5.2.6 imply that

$$\begin{aligned} \deg(F_k(X)) &= \int_{G(k+1, n+1)} c_{\text{top}}(\text{Sym}^d S^\vee) \cdot c_1(Q)^{\phi(k, d, n)} \\ &= \sum_{L_I} \frac{i_{L_I}^*(e^T(\text{Sym}^d S^\vee)) i_{L_I}^*(c_1^T(Q))}{e^T(N_{L_I})} \\ &= (-1)^{\phi(k, d, n)} \sum_{I \subset \{1, \dots, n+1\}, |I|=k+1} \frac{\left(\prod_{v_i \in \mathbb{N}, \sum_{i \in I} v_i = d} \left(\sum_{i \in I} v_i \lambda_i \right) \right) \left(\sum_{j \notin I} \lambda_j \right)^{\phi(k, d, n)}}{\prod_{i \in I} \prod_{j \notin I} (\lambda_i - \lambda_j)}. \end{aligned}$$

Replacing λ_i by a_i , we get the desired formula. □

We illustrate the formula in Theorem 5.3.5 by directly using SAGE:

```
sage: def degree_fano_scheme(k, d, n):
.....:     phi = (k+1)*(n-k)-binomial(d+k, d)
.....:     if phi < 0:
.....:         raise ValueError('Fano scheme is empty')
.....:     S = Set([i for i in range(1, n+2)])
.....:     F = S.subsets(k+1).list()
.....:     V = IntegerVectors(d, k+1)
.....:     result = 0
```

```

.....:     for p in F:
.....:         q = S - p
.....:         s = prod(sum(v[i]*p[i] for i in range(k+1)) for v in V)
.....:         t = prod(i - j for i in p for j in q)
.....:         result = result + (s*(-sum(q)**phi)/t
.....:     return result
.....:
sage: degree_fano_scheme(1,3,4)
45

```

In Schubert3, the computation is done as follows:

```

sage: def Fano_schemes_via_Bott_formula(k,d,n):
.....:     phi = (k+1)*(n-k)-binomial(d+k,d)
.....:     if phi < 0:
.....:         raise ValueError('The Fano scheme is empty')
.....:     G = Grassmannian(n-k,n+1)
.....:     F = G.fixed_points()
.....:     result = 0
.....:     for p in F:
.....:         Q = G.equivariant_quotient_bundle(p)
.....:         q1 = Q.chern_class(1)
.....:         B = Q.symmetric_power(d)
.....:         s = B.euler_class()
.....:         T = G.equivariant_tangent_bundle(p)
.....:         t = T.euler_class()
.....:         result = result + (s*q1**phi)/t
.....:     return result
.....:
sage: Fano_schemes_via_Bott_formula(1,3,4)
45

```

5.3.3. Linear Subspaces on Hypersurfaces, Revisited

By Proposition 2.3.1, if $k, d, n \in \mathbb{N}$ satisfy $d \neq 2$ (or $n \geq 2k + 1$) and $\phi(k, d, n) = 0$, then the Fano scheme $F_k(X)$ of k -planes on a general hypersurface X of degree d in \mathbb{P}^n is zero-dimensional. In this case, the degree of $F_k(X)$ is equal to the number of k -planes on X . As a special case of Theorem 5.3.5, we have the following corollary.

Corollary 5.3.6. *Let $k, d, n \in \mathbb{N}$ satisfy $d \neq 2$ (or $n \geq 2k + 1$) and $\phi(k, d, n) = 0$, and let $X \subset \mathbb{P}^n$ be a general hypersurface of degree d . Choose any set of integers a_1, \dots, a_{n+1} such that $a_i \neq a_j$ for $i \neq j$. Then we have the following formula:*

$$\text{number of } k\text{-planes on } X = \sum_{I \subset \{1, \dots, n+1\}, |I|=k+1} \frac{\prod_{v_i \in \mathbb{N}, \sum_{i \in I} v_i = d} \left(\sum_{i \in I} v_i a_i \right)}{\prod_{i \in I} \prod_{j \notin I} (a_i - a_j)},$$

where the sum runs over all the $(k + 1)$ -subsets of $\{1, \dots, n + 1\}$.

Proof. This is clear from Theorem 5.3.5. □

Remark 5.3.7. The explicit formulas in Theorem 5.3.5 and Corollary 5.3.6 lead to a much better performance than Schubert calculus (see Algorithms 2.3.1, 2.3.2, and 2.4.1). We give timings on a computer with 4GB RAM and an Intel(R) Core(TM) i3 CPU 530 2.93GHz processor. The results (in seconds) are collected in Table 5.1.

k	d	n	Schubert calculus Schubert2	Bott's formula Schubert3	The formula in Corollary 5.3.6
1	17	10	0.31	0.13	0.05
1	27	15	1.97	0.21	0.10
1	37	20	8.80	0.37	0.20
1	47	25	34.08	0.68	0.50
2	5	9	0.29	0.22	0.10
2	7	14	6.54	0.90	0.70

Table 5.1.: Timings for linear subspaces on hypersurfaces.

Chapter 6.

Gromov-Witten Invariants

Gromov-Witten invariants have their origin in physics. There are several ways to define Gromov-Witten invariants using algebraic geometry or symplectic geometry. For details of the definitions and properties of Gromov-Witten invariants, we refer to [16, Chapter 7].

Roughly speaking, Gromov-Witten invariants are the intersection numbers of cycle classes in the rational Chow rings of moduli spaces of stable maps subject to certain geometric conditions on the curves. In order to define Gromov-Witten invariants in algebraic geometry, first of all we need to recall some basic facts about moduli spaces of stable maps and virtual fundamental classes.

6.1. Moduli Spaces of Stable Maps

In [74], Kontsevich introduced the notion of moduli spaces of stable maps. In this section, we define and state some results about these moduli spaces. We only concentrate on the case of rational curves (genus zero), but some results are still true in the general case. An introduction to this subject was given by Fulton and Pandharipande in [39].

Let X be a smooth projective variety and fix a cycle class $\beta \in A_1(X)$.

Definition 6.1.1. An n -pointed stable map $(C; p_1, \dots, p_n; f)$ is a morphism $f : C \rightarrow X$ satisfying the following properties:

1. C is a compact connected curve with at most nodes as singularities.
2. p_1, \dots, p_n are distinct smooth points of C , called the *marked points* of the curve.
3. *Stability:* If C_i is a component of C such that $C_i \simeq \mathbb{P}^1$ and f is constant on C_i , then C_i contains at least 3 special (nodal or marked) points.

We say that $(C; p_1, \dots, p_n; f)$ is isomorphic to $(C'; p'_1, \dots, p'_n; f')$ if there exists an isomorphism $\tau : C \rightarrow C'$ with $f' \circ \tau = f$ and $\tau(p_i) = p'_i$ for all i . The condition of stability means that the group of automorphisms of $(C; p_1, \dots, p_n; f)$ is finite (see [75, Definition 1.1]). We denote the set of all n -pointed stable maps $(C; p_1, \dots, p_n; f)$ satisfying $f_*([C]) = \beta$ by $\overline{M}_{0,n}(X, \beta)$. The subset of $\overline{M}_{0,n}(X, \beta)$ of all smooth stable maps (that is, stable maps such that C is smooth) is denoted $M_{0,n}(X, \beta)$.

Theorem 6.1.2. *The moduli space $\overline{M}_{0,n}(X, \beta)$ has a structure of a projective scheme.*

Proof. See [39, Theorem 1]. □

Definition 6.1.3. Let X be a smooth projective variety with tangent bundle T_X . We say that X is *convex* if for every morphism $f : \mathbb{P}^1 \rightarrow X$,

$$H^1(\mathbb{P}^1, f^*(T_X)) = 0.$$

The following theorem concerns the convex case.

Theorem 6.1.4. *Let X be a convex variety and $\beta \in A_1(X)$. Then $\overline{M}_{0,n}(X, \beta)$ is a normal projective variety of pure dimension*

$$\dim(X) + \int_{\beta} c_1(T_X) + n - 3.$$

Proof. See [39, Theorem 2]. □

For each marked point p_i , there is a natural morphism

$$\begin{aligned} e_i : \overline{M}_{0,n}(X, \beta) &\longrightarrow X \\ (C; p_1, \dots, p_n; f) &\longmapsto f(p_i) \end{aligned}$$

which is called the *evaluation map* at p_i . These morphisms play an important role when it comes to relating the geometry of $\overline{M}_{0,n}(X, \beta)$ to the geometry of X .

In case $X = \mathbb{P}^r$, we can write $\beta = d\ell$, where ℓ is the cycle class of a line. The moduli space $\overline{M}_{0,n}(\mathbb{P}^r, d\ell)$ is an orbifold which is the quotient of a smooth variety (see [16, Section 7.1.1]). By Theorem 6.1.4, $\overline{M}_{0,n}(\mathbb{P}^r, d\ell)$ is also a normal projective variety of dimension $rd + r + d + n - 3$. We will write $\overline{M}_{0,n}(\mathbb{P}^r, d)$ in place of $\overline{M}_{0,n}(\mathbb{P}^r, d\ell)$. For more details on the construction and properties of $\overline{M}_{0,n}(\mathbb{P}^r, d)$, we refer to [73, Chapter 2].

6.2. Virtual Fundamental Classes

For details of the construction and properties of virtual fundamental classes, we refer to [16, Section 7.1.4]. The virtual fundamental class $[\overline{M}_{0,n}(X, \beta)]^{\text{virt}}$ is an element of the Chow group of $\overline{M}_{0,n}(X, \beta)$. In many cases, we have

$$[\overline{M}_{0,n}(X, \beta)]^{\text{virt}} = [\overline{M}_{0,n}(X, \beta)].$$

This holds, for instance, in case X is a projective space, and more generally a convex variety.

Example 6.2.1. In this example, we construct the virtual fundamental class for the quintic hypersurface in \mathbb{P}^4 . Let $X \subset \mathbb{P}^4$ be a smooth quintic hypersurface and let d be a positive integer. The inclusion $X \subset \mathbb{P}^4$ induces a natural embedding

$$i : \overline{M}_{0,0}(X, d\ell) \longrightarrow \overline{M}_{0,0}(\mathbb{P}^4, d),$$

where $\ell \in A_1(X)$ is the cycle class of a line. Let \mathcal{V}_d be the rank $5d + 1$ vector bundle on the moduli space $\overline{M}_{0,0}(\mathbb{P}^4, d)$ whose fiber over a stable map $f : C \rightarrow \mathbb{P}^4$ is $H^0(C, f^*\mathcal{O}_{\mathbb{P}^4}(5))$. One can show that $\overline{M}_{0,0}(X, d\ell)$ is the zero locus of a general section s of \mathcal{V}_d . Moreover, we have

$$[\overline{M}_{0,0}(X, d\ell)]^{\text{virt}} = s^*[C_Z Y],$$

where $Z = \overline{M}_{0,0}(X, d\ell)$, $Y = \overline{M}_{0,0}(\mathbb{P}^4, d)$, and $C_Z Y$ is the normal cone to Z in Y . For details of the definition and properties of normal cones, we refer to [35, Section 2.5].

6.3. Defining Gromov-Witten Invariants

Definition 6.3.1. Let X be a smooth projective variety and let $\beta \in A_1(X)$ be a cycle class. The *Gromov-Witten invariant* of β associated with the cycle classes $\gamma_1, \dots, \gamma_n \in A^*(X)$ is the rational number defined by

$$I_{n,\beta}(\gamma_1, \dots, \gamma_n) := \int_{\overline{M}_{0,n}(X,\beta)} e_1^*(\gamma_1) \cup \dots \cup e_n^*(\gamma_n) \cup [\overline{M}_{0,n}(X,\beta)]^{\text{virt}},$$

where \cup is the cup product. Note that we have to use the cup product since the moduli space $\overline{M}_{0,n}(X,\beta)$ is a singular variety. For the definition of cup product, we refer to [73, Section 4.1.2].

In case $n = 0$, we denote the Gromov-Witten invariant of the cycle class $\beta \in A_1(X)$ by

$$I_\beta := \int_{\overline{M}_{0,0}(X,\beta)} [\overline{M}_{0,0}(X,\beta)]^{\text{virt}},$$

that is the degree of the virtual fundamental class $[\overline{M}_{0,0}(X,\beta)]^{\text{virt}}$.

Example 6.3.2. Let $X \subset \mathbb{P}^4$ be a smooth quintic hypersurface. We have shown in Example 6.2.1 that the virtual fundamental class $[\overline{M}_{0,0}(X,d\ell)]^{\text{virt}}$ is a cycle class on $\overline{M}_{0,0}(X,d\ell)$. By Definition 6.3.1, we have the Gromov-Witten invariant

$$I_{d\ell} = \int_{\overline{M}_{0,0}(X,d\ell)} [\overline{M}_{0,0}(X,d\ell)]^{\text{virt}}.$$

By [16, Lemma 7.1.5], we have

$$i_*([\overline{M}_{0,0}(X,d\ell)]^{\text{virt}}) = c_{5d+1}(\mathcal{V}_d) \in A^*(\overline{M}_{0,0}(\mathbb{P}^4,d)).$$

This will lead to a nice formula of the form

$$I_{d\ell} = \int_{\overline{M}_{0,0}(\mathbb{P}^4,d)} c_{5d+1}(\mathcal{V}_d).$$

In Chapter 7, we will present how to compute this Gromov-Witten invariant using the localization of $\overline{M}_{0,0}(\mathbb{P}^4,d)$ and Bott's formula.

We recall some properties of Gromov-Witten invariants. Their proofs can be found in [39, Section 7].

1. **Degree Axiom.** In the case when $\sum \text{codim } \gamma_i \neq \dim(\overline{M}_{0,n}(X,\beta))$, we have

$$I_{n,\beta}(\gamma_1, \dots, \gamma_n) = 0.$$

2. **Point Mapping Axiom.** In the case when $\beta = 0$, we have

$$I_{n,0}(\gamma_1, \dots, \gamma_n) = \begin{cases} \int_X \gamma_1 \cdot \gamma_2 \cdot \gamma_3 & \text{if } n = 3, \\ 0 & \text{otherwise.} \end{cases}$$

3. **Fundamental Class Axiom.** In the case when $\gamma_1 = 1$, we have

$$I_{n,\beta}(1, \gamma_2, \dots, \gamma_n) = \begin{cases} \int_X \gamma_2 \cdot \gamma_3 & \text{if } n = 3 \text{ and } \beta = 0, \\ 0 & \text{otherwise.} \end{cases}$$

4. **Divisor Axiom.** In the case when $\beta \neq 0$ and $\text{codim}(\gamma_1) = 1$, we have

$$I_{n,\beta}(\gamma_1, \gamma_2, \dots, \gamma_n) = \left(\int_X \beta \cdot \gamma_1 \right) I_{n-1,\beta}(\gamma_2, \dots, \gamma_n).$$

6.4. Gromov-Witten Invariants for Projective Spaces

The following theorem relates the Gromov-Witten invariants to enumerative geometry.

Theorem 6.4.1. *Let $\ell \in A_1(\mathbb{P}^r)$ be the cycle class of a line, $d \geq 1$ be an integer, and $\gamma_1, \dots, \gamma_n \in A^*(\mathbb{P}^r)$ be homogeneous cycle classes of codimension at least 2, with*

$$\sum_{i=1}^n \text{codim } \gamma_i = \dim(\overline{M}_{0,n}(\mathbb{P}^r, d)).$$

Then for general subvarieties $\Gamma_1, \dots, \Gamma_n \subset \mathbb{P}^r$ with $[\Gamma_i] = \gamma_i$, the Gromov-Witten invariant $I_{n,d\ell}(\gamma_1, \dots, \gamma_n)$ is the number of rational curves of degree d meeting $\Gamma_1, \dots, \Gamma_n$.

Proof. See [73, Proposition 4.1.5]. □

Example 6.4.2. In particular, for \mathbb{P}^2 , we write

$$K_d := I_{3d-1,d\ell}(\underbrace{pt, \dots, pt}_{3d-1}),$$

where pt is the cycle class of a point in \mathbb{P}^2 . This is the number of rational curves of degree d passing through $3d - 1$ general points. These invariants are called the *Kontsevich numbers* and can be computed by Theorem 6.4.3.

Theorem 6.4.3 (Kontsevich). *Let K_d be the number of rational irreducible plane curves of degree d through $3d - 1$ general points in \mathbb{P}^2 . Then we have the recursion formula*

$$K_d = \sum_{\substack{d_1+d_2=d \\ d_1 \geq 1, d_2 \geq 1}} K_{d_1} K_{d_2} \left(d_1^2 d_2^2 \binom{3d-4}{3d_1-2} - d_1^3 d_2 \binom{3d-4}{3d_1-1} \right)$$

for all $d > 1$ and the initial value is $K_1 = 1$.

Proof. See [73, Section 3.3 or 5.4]. □

Some Kontsevich numbers K_d with $1 \leq d \leq 10$ are listed in Table 6.1.

d	K_d
1	1
2	1
3	12
4	620
5	87304
6	26312976
7	14616808192
8	13525751027392
9	19385778269260800
10	40739017561997799680

Table 6.1.: Some Kontsevich numbers.

An important result is that it is possible to reduce the computation of any Gromov-Witten invariant $I_{n,d\ell}(\gamma_1, \dots, \gamma_n)$ to the single one $I_{2,\ell}(pt, pt) = 1$. This means that we can give a recursive algorithm for computing the Gromov-Witten invariants for \mathbb{P}^r .

Theorem 6.4.4 (Reconstruction theorem for projective spaces). *All the Gromov-Witten invariants for \mathbb{P}^r can be computed recursively, and the only necessary initial value is $I_{2,\ell}(pt, pt) = 1$, the number of lines through two general points.*

Proof. An algorithmic sketch of the proof of this theorem can be found in [73, Section 4.4.1]. For more details of the proof, we refer to [76]. \square

Remark 6.4.5 (Gromov-Witten invariants for projective spaces). In `Schubert3`, the computation of Gromov-Witten invariants for projective spaces is implemented in the function `Gromov_Witten_invariant` of the class `ProjectiveSpace`. It computes the Gromov-Witten invariants $I_{n,d\ell}(\gamma_1, \dots, \gamma_n)$ corresponding to the number of rational curves of degree d meeting n general linear subspaces $\Gamma_1, \dots, \Gamma_n$ with $\gamma_i = [\Gamma_i]$ in a projective space. The input of this functionality is a positive integer which is the degree of rational curves and a list of the codimensions of the Γ_i . More concretely, we translate the proof of the reconstruction theorem for projective spaces (see [73, Section 4.4.1]) into a recursive algorithm (see Algorithm 6.4.1).

Example 6.4.6. Let $h \in A^1(\mathbb{P}^3)$ be the cycle class of a hyperplane in \mathbb{P}^3 and $\ell \in A_1(\mathbb{P}^3)$ be the cycle class of a line in \mathbb{P}^3 . The Gromov-Witten invariant of ℓ associated with the classes $\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = h^2$,

$$I_{4,\ell}(h^2, h^2, h^2, h^2),$$

is equal to the number of lines meeting four general lines in \mathbb{P}^3 . This number was computed in Chapter 2 via Schubert calculus and Chapter 5 via Bott's formula for Grassmannians.

```
sage: P = ProjectiveSpace(3)
sage: P.Gromov_Witten_invariant(1, [2,2,2,2])
2
```

Example 6.4.7. In this example, we reconsider the example from Section 3.1, which computes the number of conics in \mathbb{P}^3 meeting 8 general lines. This number is equal to the Gromov-Witten invariant of cycle class 2ℓ associated with 8 classes h^2 ,

$$I_{8,2\ell}(\underbrace{h^2, \dots, h^2}_8),$$

where h, ℓ as in the previous example. We illustrate how to compute this number as a Gromov-Witten invariant by `Schubert3` as follows:

```
sage: P = ProjectiveSpace(3)
sage: P.Gromov_Witten_invariant(2, [2]*8)
92
```

Example 6.4.8. In \mathbb{P}^3 , the Gromov-Witten invariant

$$I_{9,3\ell}(\underbrace{h^2, \dots, h^2}_6, \underbrace{h^3, \dots, h^3}_3),$$

is equal to the number of twisted cubics meeting 6 lines and 3 points in general position. This number can be computed as follows:

Algorithm 6.4.1 Gromov-Witten invariants for projective spaces

- Input:**
1. An integer $r \geq 2$, that is the dimension of the projective space \mathbb{P}^r .
 2. A list of the integers $c = [c_1, \dots, c_n]$, where $c_i = \text{codim}(\gamma_i)$.
 3. An integer $d \geq 0$ which is the degree of a stable map to \mathbb{P}^r .
 4. We may assume that $n \geq 3, d \geq 1, c_i \geq 2$ for all i , and

$$\sum_{i=1}^n c_i = rd + r + d + n - 3.$$

Otherwise, the respective Gromov-Witten invariant is either 0 or 1 by the degree, point mapping, fundamental class, and divisor axioms.

Output: The Gromov-Witten invariant $I_d(\gamma_1, \dots, \gamma_n)$, written as $GW(r, d, c)$ for short. This is the number of rational curves of degree d meeting n general linear subspaces $\Gamma_1, \dots, \Gamma_n$ in \mathbb{P}^r with $\gamma_i = [\Gamma_i]$ and $\text{codim}(\gamma_i) = c_i$.

- 1: **if** $r = 2$ **then**
 - 2: The result will be the Kontsevich number K_d . Use the recursion formula in Theorem 6.4.3 to compute this number.
 - 3: **end if**
 - 4: **if** $r > 2$ **then**
 - 5: Set $l = [c_1, \dots, c_{n-2}, c_{n-1} + c_n - 1]$ and result := $d \cdot GW(r, d, l)$.
 - 6: Set $S = \{(s_0, s_1) \mid s_0 \cup s_1 = [c_1, \dots, c_{n-2}], s_0 \cap s_1 = \emptyset\}$.
 - 7: **for** (s_0, s_1) in S **do**
 - 8: **for** i from 1 to d **do**
 - 9: Set $j = d - i$.
 - 10: Set $e = ri + r + i + l(s_0) - \sum(s_0) - c_{n-1} - c_n + 1$, where $l(s_0)$ the length of the list s_0 and $\sum(s_0)$ the sum of elements in the list s_0 .
 - 11: Set $f = ri + r + i + l(s_0) - \sum(s_0) - c_n$.
 - 12: **if** $0 \leq e \leq r$ **then**
 - 13: Set $l_1 = [c_n - 1, c_{n-1}, s_0, e]$.
 - 14: Set $l_2 = [c_{n-2}, s_1, r - e]$.
 - 15: result = result + $d \cdot GW(r, i, l_1) \cdot GW(r, j, l_2)$.
 - 16: **end if**
 - 17: **if** $0 \leq f \leq r$ **then**
 - 18: Set $l_1 = [c_n - 1, s_0, f]$.
 - 19: Set $l_2 = [c_{n-1}, c_{n-2}, s_1, r - f]$.
 - 20: result = result - $d \cdot GW(r, i, l_1) \cdot GW(r, j, l_2)$.
 - 21: **end if**
 - 22: **end for**
 - 23: **end for**
 - 24: **Return** result
 - 25: **end if**
-

```
sage: P = ProjectiveSpace(3)
sage: P.Gromov_Witten_invariant(3, [2]*6+[3]*3)
190
```

More generally, we denote by $N_d(a, b)$ the number of rational curves of degree d meeting a general lines and b general points in \mathbb{P}^3 . By Theorem 6.4.1, we have

$$N_d(a, b) = I_{a+b, dl}(\underbrace{h^2, \dots, h^2}_a, \underbrace{h^3, \dots, h^3}_b).$$

Again by the degree axiom, we may assume that $a + 2b = 4d$ (otherwise, the respective Gromov-Witten invariants are zero).

In case $d = 1$, we have the following possibilities:

$$\begin{array}{c|ccc} a & 0 & 2 & 4 \\ \hline b & 2 & 1 & 0 \end{array}$$

In case $d = 2$, we have the following possibilities:

$$\begin{array}{c|ccccc} a & 0 & 2 & 4 & 6 & 8 \\ \hline b & 4 & 3 & 2 & 1 & 0 \end{array}$$

In case $d = 3$, we have the following possibilities:

$$\begin{array}{c|cccccc} a & 0 & 2 & 4 & 6 & 8 & 10 & 12 \\ \hline b & 6 & 5 & 4 & 3 & 2 & 1 & 0 \end{array}$$

We have computed these Gromov-Witten invariants using the `Schubert3` package (see Example A.2.18). We collect the results in Table 6.2.

d	a	b	$N_d(a, b)$	d	a	b	$N_d(a, b)$
1	0	2	1	3	0	6	1
1	2	1	1	3	2	5	5
1	4	0	2	3	4	4	30
2	0	4	0	3	6	3	190
2	2	3	1	3	8	2	1312
2	4	2	4	3	10	1	9864
2	6	1	18	3	12	0	80160
2	8	0	92				

Table 6.2.: The number of rational curves of degree d meeting a lines and b points in \mathbb{P}^3

Example 6.4.9. The number of conics in \mathbb{P}^4 meeting 11 general 2-planes $\gamma_1 \dots \gamma_{11} \subset \mathbb{P}^4$ is the Gromov-Witten invariant

$$I_{11, 2\ell}(\underbrace{h^2, \dots, h^2}_{11}),$$

where h is the cycle class of a hyperplane in \mathbb{P}^4 . This number can be computed as follows:

```
sage: P = ProjectiveSpace(4)
sage: P.Gromov_Witten_invariant(2, [2]*11)
6620
```

More generally, we denote by $N_d(a, b, c)$ the number of rational curves of degree d meeting a planes, b lines, and c points in general position in \mathbb{P}^4 . By Theorem 6.4.1, we have

$$N_d(a, b, c) = I_{a+b+c, d}(\underbrace{h^2, \dots, h^2}_a, \underbrace{h^3, \dots, h^3}_b, \underbrace{h^4, \dots, h^4}_c).$$

Like in the previous example, we also have the following tables (see Tables 6.3 and 6.4) which list all the numbers $N_d(a, b, c)$ in the cases where $d = 1$ and $d = 2$ (see Example A.2.19).

a	b	c	$N_1(a, b, c)$	a	b	c	$N_1(a, b, c)$
0	0	2	1	3	0	1	1
0	3	0	1	4	1	0	3
1	1	1	1	6	0	0	5
2	2	0	2				

Table 6.3.: The number of lines meeting a planes, b lines and c points in \mathbb{P}^4

a	b	c	$N_2(a, b, c)$	a	b	c	$N_2(a, b, c)$
0	1	3	0	4	2	1	21
2	0	3	1	3	4	0	36
1	2	2	1	6	1	1	67
0	4	1	2	5	3	0	132
3	1	2	4	8	0	1	219
2	3	1	6	7	2	0	473
1	5	0	10	9	1	0	1734
5	0	2	11	11	0	0	6620

Table 6.4.: The number of conics meeting a planes, b lines and c points in \mathbb{P}^4

Chapter 7.

Rational Curves on Calabi-Yau Threefolds

In this chapter, we present an important technique for computing the Gromov-Witten invariants, namely, the localization of moduli spaces of stable maps. This technique allows us to compute the Gromov-Witten invariants via Bott's formula. As an insightful example, the numbers related to the problem of counting rational curves on general Calabi-Yau threefolds in projective spaces are computed up to degree 6. The results are all in agreement with predictions made from mirror symmetry.

7.1. Localization of $\overline{M}_{0,0}(\mathbb{P}^r, d)$

The natural action of $T = (\mathbb{C}^*)^{r+1}$ on \mathbb{P}^r induces an action of T on $\overline{M}_{0,0}(\mathbb{P}^r, d)$ by the composition of the action with stable maps. The fixed point loci and their equivariant normal bundles were determined for $\overline{M}_{0,0}(\mathbb{P}^r, d)$ by Kontsevich in [75].

We denote the set of fixed points by $\overline{M}_{0,0}(\mathbb{P}^r, d)^T$. A fixed point of the T -action on $\overline{M}_{0,0}(\mathbb{P}^r, d)$ corresponds to a stable map (C, f) where each irreducible component C_i of C is either mapped to a fixed point of \mathbb{P}^r or a multiple cover of a coordinate line. Each node of C and each ramification point of f is also mapped to a fixed point of \mathbb{P}^r . Thus each fixed point (C, f) of the T -action on $\overline{M}_{0,0}(\mathbb{P}^r, d)$ can be associated with a graph Γ . Let q_0, \dots, q_r be the fixed points of \mathbb{P}^r under the torus action.

1. The vertices of Γ are in one-to-one correspondence with the connected components C_i of $f^{-1}(\{q_0, \dots, q_r\})$, where each C_i is either a point or a non-empty union of irreducible components of C .
2. The edges of Γ correspond to irreducible components C_e of C which are mapped onto some coordinate line l_e in \mathbb{P}^r .

The graph Γ has the following labels: Associate to each vertex v the number i_v defined by $f(C_v) = q_{i_v}$. Associate to each edge e the degree d_e of the map $f|_{C_e}$. The connected components of $\overline{M}_{0,0}(\mathbb{P}^r, d)^T$ are naturally labelled by equivalence classes of connected graphs Γ . Furthermore, the following conditions must be satisfied:

1. If an edge e connects v and v' , then $i_v \neq i_{v'}$, and l_e is the coordinate line joining q_{i_v} and $q_{i_{v'}}$.
2. $\sum_e d_e = d$.

We denote the number of edges connected to v by $\text{val}(v)$. The stable maps associated with the fixed graph Γ define a subspace

$$\overline{M}_\Gamma \subset \overline{M}_{0,0}(\mathbb{P}^r, d).$$

Fix $(C, f) \in \overline{M}_\Gamma$. For each vertex v such that the component C_v of C is one dimensional, C_v has $\text{val}(v)$ special points. The data consisting of C_v plus these $\text{val}(v)$ points forms a stable curve, giving an element of $\overline{M}_{0, \text{val}(v)}$. Using the data of Γ , we can construct a map

$$\psi_\Gamma : \prod_{v: \dim C_v=1} \overline{M}_{0, \text{val}(v)} \longrightarrow \overline{M}_\Gamma.$$

See [16, Section 9.2.1] for more details of this construction. Note that $\text{val}(v)$ can be defined for all vertices v and that $\dim C_v = 1$ if and only if C_v contains a component of C contracted by f if and only if $\text{val}(v) \geq 3$. We define

$$F_\Gamma = \prod_{v: \dim C_v=1} \overline{M}_{0, \text{val}(v)}.$$

If there are no contracted components, then we let F_Γ be a point. The map ψ_Γ is finite. There is a finite group of automorphisms A_Γ acting on F_Γ such that the quotient space is \overline{M}_Γ . The group A_Γ fits into an exact sequence

$$0 \longrightarrow \prod_e \mathbb{Z}/d_e \mathbb{Z} \longrightarrow A_\Gamma \longrightarrow \text{Aut}(\Gamma) \longrightarrow 0, \quad (7.1)$$

where $\text{Aut}(\Gamma)$ is the group of automorphisms of Γ which preserve the labels.

Since we would like to apply the localization theorem and Bott's formula to the moduli spaces $\overline{M}_{0,0}(\mathbb{P}^r, d)$, we actually need another version of Corollary 5.2.3 for these moduli spaces. Such a formula can be found in the following corollary.

Corollary 7.1.1 (Degree formula for orbifolds). *Let X be an orbifold with a T -action which is the quotient of a smooth variety. If $\alpha \in A_T^*(X) \otimes Q_T$, then*

$$\int_{X_T} \alpha = \sum_j \int_{(Z_j)_T} \frac{i_j^*(\alpha)}{a_j e^T(N_j)},$$

where a_j is the order of the group H occurring in a local chart at the generic point of Z_j .

Proof. See [16, Corollary 9.1.4]. □

We can also generalize Theorems 5.2.4 and 5.2.6 to the moduli spaces. This is done by replacing Corollary 5.2.3 by Corollary 7.1.1 in the arguments of Section 5.2. When Corollary 7.1.1 is applied to the moduli spaces $\overline{M}_{0,0}(\mathbb{P}^r, d)$, the number a_j appearing in the denominator of the term corresponding to the component \overline{M}_Γ is just the order a_Γ of the group A_Γ .

Example 7.1.2. Let us describe the fixed point components of the natural action of T on $\overline{M}_{0,0}(\mathbb{P}^r, 1)$. The possible graphs are of the following type:

$$\Gamma_{i,j} : \begin{array}{c} \bullet \text{---} \overset{1}{\text{---}} \text{---} \bullet \\ i \qquad \qquad \qquad j \end{array}$$

In these labelled graphs, we have added the labels $i, j \in \{0, 1, \dots, r\}$ with $i \neq j$ below the vertices and the degree 1 above the edge. For each graph, it is easy to see that $|\text{Aut}(\Gamma_{i,j})| = 2$, hence we have $a_{\Gamma_{i,j}} = 2$. For example, in the case of $\overline{M}_{0,0}(\mathbb{P}^4, 1)$, there are 20 graphs.

Here is the implementation enumerating these graphs in `Schubert3`:


```

sage: P = ProjectiveSpace(4)
sage: M = ModuliSpace(P,1); M
A moduli space of dimension 6
sage: F = M.fixed_points(); len(F)
20
sage: K = F.keys(); len(K)
20
sage: G = K[0]; G
A graph on 2 vertices and 1 edges
sage: G.vertices()
[(0, 1, (0, 1, 1)), (1, 1, (1, 0, 1))]
sage: G.edges()
[(0, 1, 1)]
sage: F[G]
2

```

Here is the implementation enumerating these graphs in `schubert.lib`:

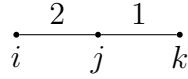
```

> ring r = 0,x,dp;
> variety P = projectiveSpace(4);
> stack M = moduliSpace(P,1);
> M;
A moduli space of dimension 6
> list F = fixedPoints(M);
> size(F);
20
> F[1];
[1]:
  A graph with 2 vertices and 1 edges
[2]:
  2
> graph G = F[1][1];
> G;
A graph with 2 vertices and 1 edges
> size(G.vertices);
2
> size(G.edges);
1

```

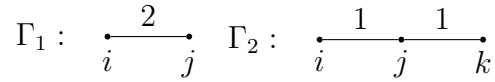
Remark 7.1.3. In `Schubert3` and `schubert.lib`, a graph is represented by a list of vertices and a list of edges. For later use we store some additional information in the graph: A vertex of a graph is represented by a list of integers. The first element of such a list is the label of the vertex, the second one is the number of edges containing the vertex, and the remaining data describing the edges which contain the vertex as indicated in Example 7.1.4. An edge of a graph is represented by a list of length 3. The first two entries are the labels of the two vertices of the edge, the last entry is the degree of the irreducible component corresponding to the edge.

Example 7.1.4. Consider the following graph:



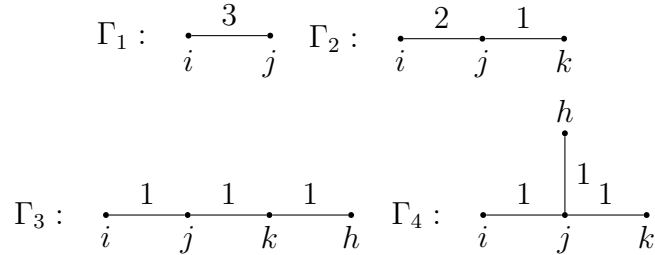
This graph has 3 vertices $(i, 1, (i, j, 2))$, $(j, 2, ((j, i, 2), (j, k, 1)))$, and $(k, 1, (k, j, 1))$, and 2 edges $(i, j, 2)$ and $(j, k, 1)$.

Example 7.1.5. Let us describe the fixed point components of the natural action of T on $\overline{M}_{0,0}(\mathbb{P}^r, 2)$. The possible graphs are of the following types:



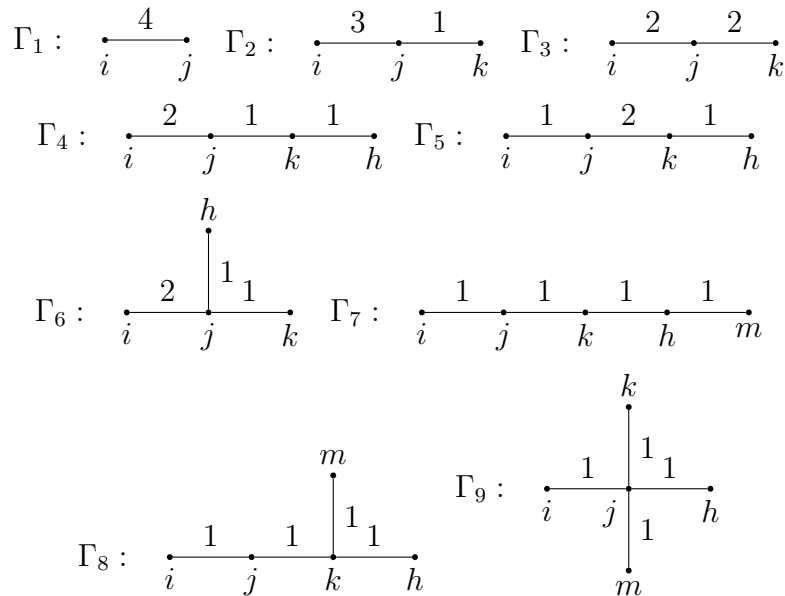
Note that $i, j, k \in \{0, 1, \dots, r\}$, and the vertices of an edge must have different labels. For each type, it is easy to see that $|\text{Aut}(\Gamma_i)| = 2$, hence $a_{\Gamma_1} = 4$ and $a_{\Gamma_2} = 2$. For instance, if $r = 1$, then there will be 4 graphs. If $r = 4$, then there will be 100 graphs.

Example 7.1.6. In the case of $\overline{M}_{0,0}(\mathbb{P}^r, 3)$, the possible graphs have the following types:



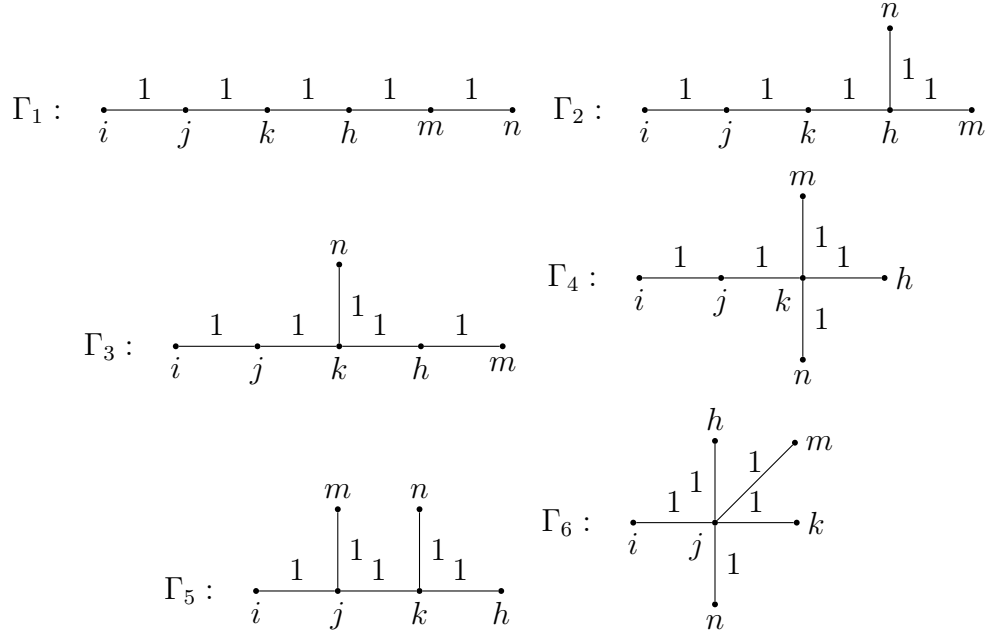
Note that $i, j, k, h \in \{0, 1, \dots, r\}$, and the vertices of an edge must have different labels. It is easy to see that $|\text{Aut}(\Gamma_1)| = 2$, $|\text{Aut}(\Gamma_2)| = 1$, $|\text{Aut}(\Gamma_3)| = 2$, and $|\text{Aut}(\Gamma_4)| = 6$. Hence $a_{\Gamma_1} = 6$, $a_{\Gamma_2} = a_{\Gamma_3} = 2$, and $a_{\Gamma_4} = 6$.

Example 7.1.7. In the case of $\overline{M}_{0,0}(\mathbb{P}^r, 4)$, the possible graphs have the following types:



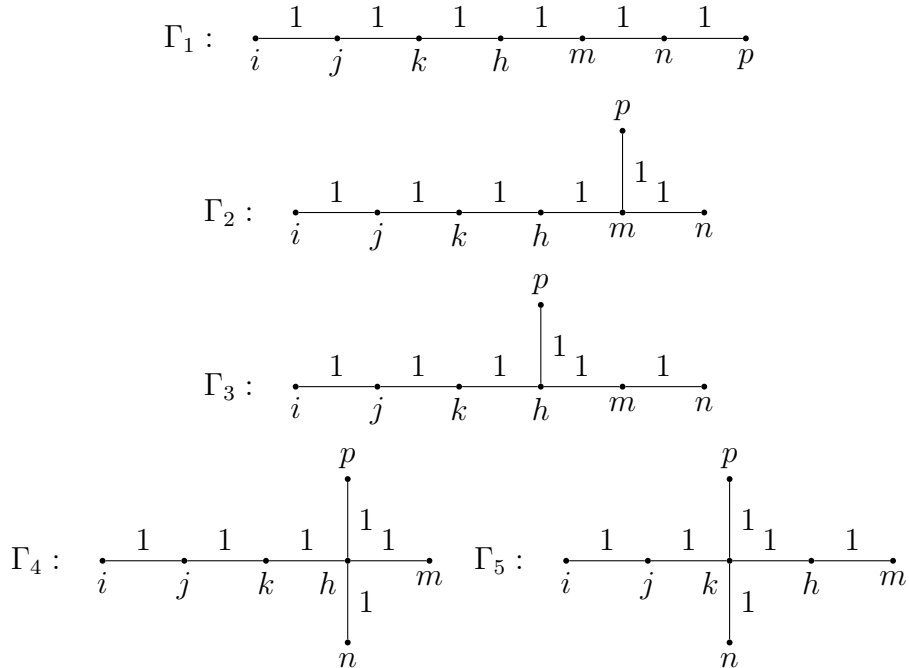
Note that $i, j, k, h, m \in \{0, 1, \dots, r\}$, and the vertices of an edge must have different labels. For these graphs, we have $|\text{Aut}(\Gamma_1)| = |\text{Aut}(\Gamma_3)| = |\text{Aut}(\Gamma_5)| = |\text{Aut}(\Gamma_6)| = |\text{Aut}(\Gamma_7)| = |\text{Aut}(\Gamma_8)| = 2$, $|\text{Aut}(\Gamma_2)| = |\text{Aut}(\Gamma_4)| = 1$, and $|\text{Aut}(\Gamma_9)| = 24$. Hence $a_{\Gamma_1} = a_{\Gamma_3} = 8$, $a_{\Gamma_2} = 3$, $a_{\Gamma_4} = a_{\Gamma_7} = a_{\Gamma_8} = 2$, $a_{\Gamma_5} = a_{\Gamma_6} = 4$, and $a_{\Gamma_9} = 24$.

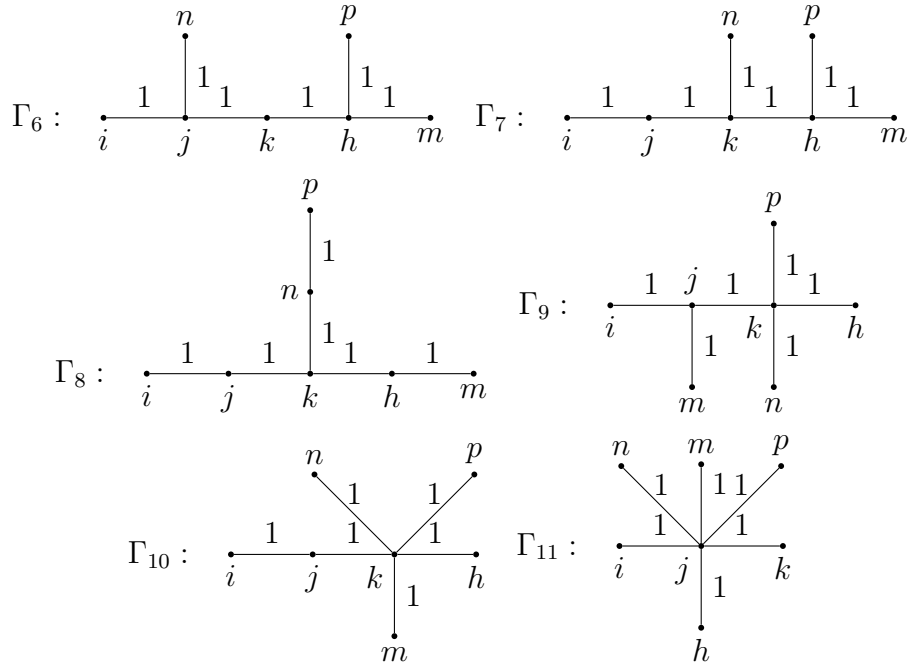
Example 7.1.8. In the case of $\overline{M}_{0,0}(\mathbb{P}^r, 5)$, in addition to the analogues of the graphs above, we have the following graphs:



Note that $i, j, k, h, m, n \in \{0, 1, \dots, r\}$, and the vertices of an edge must have different labels. For these graphs, we have $a_{\Gamma_1} = a_{\Gamma_2} = a_{\Gamma_3} = 2$, $a_{\Gamma_4} = 6$, $a_{\Gamma_5} = 8$, and $a_{\Gamma_6} = 120$.

Example 7.1.9. In the case of $\overline{M}_{0,0}(\mathbb{P}^r, 6)$, in addition to the analogues of the graphs above, we have the following graphs:





Note that $i, j, k, h, m, n, p \in \{0, 1, \dots, r\}$, and the vertices of an edge must have different labels. For these graphs, we have $a_{\Gamma_1} = a_{\Gamma_2} = a_{\Gamma_7} = 2, a_{\Gamma_3} = 1, a_{\Gamma_4} = a_{\Gamma_8} = 6, a_{\Gamma_5} = 4, a_{\Gamma_6} = 8, a_{\Gamma_9} = 12, a_{\Gamma_{10}} = 24,$ and $a_{\Gamma_{11}} = 720$.

In order to apply Bott's formula to $\overline{M}_{0,0}(\mathbb{P}^r, d)$, we need a formula for computing the T -equivariant Euler class of the normal bundle N_Γ of \overline{M}_Γ . We define a flag F of a graph to be a pair (v, e) such that v is a vertex of e . Put $i(F) = v$ and let $j(F)$ be the other vertex of e . Set

$$\omega_F = \frac{\lambda_{i(F)} - \lambda_{j(F)}}{d_e}.$$

This corresponds to the weight of a torus action on the tangent space of the component C_e of C at the point p_F lying over i_v . If v is a vertex with $\text{val}(v) = 1$, then we denote by $F(v)$ the unique flag containing v . If $\text{val}(v) = n \geq 2$, then we denote by $F_1(v), F_2(v), \dots, F_n(v)$ the n flags containing v . We also denote by $v_1(e)$ and $v_2(e)$ the two vertices of an edge e .

Theorem 7.1.10. *The T -equivariant Euler class of the normal bundle N_Γ is a product of contributions from the vertices and edges. More precisely, we have*

$$e^T(N_\Gamma) = e_\Gamma^v e_\Gamma^e,$$

where e_Γ^v and e_Γ^e are defined by the following formulas:

$$e_\Gamma^v = \prod_v \left(\left(\prod_{j \neq i_v} (\lambda_{i_v} - \lambda_j) \right)^{1-\text{val}(v)} \left(\sum_i \omega_{F_i(v)}^{-1} \right)^{3-\text{val}(v)} \prod_i \omega_{F_i(v)} \right),$$

$$e_\Gamma^e = \prod_e \left(\frac{(-1)^{d_e} (d_e!)^2 (\lambda_{i_{v_1(e)}} - \lambda_{i_{v_2(e)}})^{2d_e}}{d_e^{2d_e}} \prod_{a,b \in \mathbb{N}, a+b=d_e, k \neq i_{v_j(e)}} \left(\frac{a\lambda_{i_{v_1(e)}} + b\lambda_{i_{v_2(e)}}}{d_e} - \lambda_k \right) \right).$$

Proof. Complete proofs can be found in [75, Sections 3.3.3 and 3.3.4] and [16, Theorem 9.2.1]. \square

The formula in Theorem 7.1.10 gives an effective algorithm for computing $e^T(N_\Gamma)$. We will present some examples in the next sections.

7.2. Lines on Hypersurfaces, Revisited

In this part, we reconsider the problem of counting lines on a general hypersurface of degree $d = 2r - 3$ in \mathbb{P}^r . This problem was considered in Chapter 2 using classical Schubert calculus. In this section, we use the localization of $\overline{M}_{0,0}(\mathbb{P}^r, 1)$ and Bott's formula for solving the problem. That is, given a general hypersurface of degree $d = 2r - 3$ in \mathbb{P}^r and the cycle class ℓ of a line, we compute the Gromov-Witten invariant

$$I_\ell^{(r)} = \int_{\overline{M}_{0,0}(X,\ell)} [\overline{M}_{0,0}(X, \ell)]^{\text{virt}}.$$

By arguments similar to those in Example 6.3.2, we have the following formula:

$$I_\ell^{(r)} = \int_{\overline{M}_{0,0}(\mathbb{P}^r, 1)} c_{2r-2}(\mathcal{E}_r),$$

where \mathcal{E}_r is the vector bundle on $\overline{M}_{0,0}(\mathbb{P}^r, 1)$ whose fiber at a stable map $f : C \rightarrow \mathbb{P}^r$ is $H^0(C, f^* \mathcal{O}_{\mathbb{P}^r}(2r - 3))$.

By Example 7.1.2, there are $r(r+1)$ graphs $\Gamma_{i,j}$ corresponding to the fixed point components of the natural action of T on $\overline{M}_{0,0}(\mathbb{P}^r, 1)$. By arguments similar to those in the paper of Kontsevich (see [75], Example 2.2) and using Corollary 7.1.1, we get:

$$I_\ell^{(r)} = \int_{\overline{M}_{0,0}(\mathbb{P}^r, 1)} c_{2r-2}(\mathcal{E}_r) = \sum_{\Gamma_{i,j}} \frac{c_{2r-2}^T(\mathcal{E}_r|_{\Gamma_{i,j}})}{a_{\Gamma_{i,j}} e^T(N_{\Gamma_{i,j}})},$$

where $c_{2r-2}^T(\mathcal{E}_r|_{\Gamma_{i,j}})$ is defined by the following formula:

$$c_{2r-2}^T(\mathcal{E}_r|_{\Gamma_{i,j}}) = \prod_e \left(\prod_{a,b \in \mathbb{N}, a+b=(2r-3)d_e} \frac{a\lambda_{i_{v_1(e)}} + b\lambda_{i_{v_2(e)}}}{d_e} \right) \prod_v ((2r-3)\lambda_{i_v})^{1-\text{val}(v)}. \quad (7.2)$$

Since each graph $\Gamma_{i,j}$ has two vertices and one edge, we have $d_e = 1$, $i_{v_1(e)} = i$, $i_{v_2(e)} = j$, and $\text{val}(v) = 1$. Thus we have

$$c_{2r-2}^T(\mathcal{E}_r|_{\Gamma_{i,j}}) = \prod_{a,b \in \mathbb{N}, a+b=2r-3} (a\lambda_i + b\lambda_j).$$

For computing $e^T(N_{\Gamma_{i,j}})$, we use the formulas coming from Theorem 7.1.10. More precisely, we have

$$e_{\Gamma_{i,j}}^v = \frac{1}{(\lambda_i - \lambda_j)(\lambda_j - \lambda_i)},$$

$$e_{\Gamma_{i,j}}^e = -(\lambda_i - \lambda_j)^2 \prod_{k \neq i,j} (\lambda_i - \lambda_k)(\lambda_j - \lambda_k).$$

Thus

$$e^T(N_{\Gamma_{i,j}}) = e_{\Gamma_{i,j}}^v e_{\Gamma_{i,j}}^e = \prod_{k \neq i,j} (\lambda_i - \lambda_k)(\lambda_j - \lambda_k).$$

Since $a_{\Gamma_{i,j}} = 2$ for all $\Gamma_{i,j}$, we have

$$I_\ell^{(r)} = \sum_{0 \leq i, j \leq r, i \neq j} \frac{\prod_{a,b \in \mathbb{N}, a+b=2r-3} (a\lambda_i + b\lambda_j)}{2 \prod_{k \neq i,j} (\lambda_i - \lambda_k)(\lambda_j - \lambda_k)} = \sum_{0 \leq i < j \leq r} \frac{\prod_{a,b \in \mathbb{N}, a+b=2r-3} (a\lambda_i + b\lambda_j)}{\prod_{k \neq i,j} (\lambda_i - \lambda_k)(\lambda_j - \lambda_k)}. \quad (7.3)$$

By [73, Example 2.1.15], the moduli space $\overline{M}_{0,0}(\mathbb{P}^r, 1)$ is isomorphic to the Grassmannian $G(2, r+1)$ of lines in \mathbb{P}^r . In this case, using arguments similar to those in [16, Example 7.1.3.1], we construct the virtual fundamental class $[\overline{M}_{0,0}(X, \ell)]^{\text{virt}}$ as follows. Let S be the tautological subbundle on the Grassmannian $G(2, r+1)$. The fiber S_ℓ at a line ℓ is the 2-dimensional subspace of \mathbb{C}^{r+1} whose projectivization is ℓ . An equation for X gives a section s of the vector bundle $\text{Sym}^{2r-3} S^\vee$. Then $\overline{M}_{0,0}(X, \ell)$ is the zero locus of s . This contribution produces the cycle class

$$s^*[C_Z Y] \in A_0(\overline{M}_{0,0}(X, \ell)),$$

where $C_Z Y$ is the normal cone of $Z = \overline{M}_{0,0}(X, \ell) = Z(s) \subset Y = G(2, r+1)$. By arguments similar to those in [16, Example 7.1.5.1], this class is the virtual fundamental class $[\overline{M}_{0,0}(X, \ell)]^{\text{virt}}$. Then [16, Lemma 7.1.5] implies that

$$i_*([\overline{M}_{0,0}(X, \ell)]^{\text{virt}}) = c_{\text{top}}(\text{Sym}^{2r-3} S^\vee) \in A^*(G(2, r+1)),$$

where $i : \overline{M}_{0,0}(X, \ell) \hookrightarrow G(2, r+1)$ is an embedding. This will lead to the following formula:

$$I_\ell^{(r)} = \int_{G(2, r+1)} c_{\text{top}}(\text{Sym}^{2r-3} S^\vee). \quad (7.4)$$

In summary, we have the following theorem.

Theorem 7.2.1. *Let $X \subset \mathbb{P}^r$ be a general hypersurface of degree $2r-3$. The number of lines on X is given by*

$$\sum_{0 \leq i < j \leq r} \frac{\prod_{a, b \in \mathbb{N}, a+b=2r-3} (a\lambda_i + b\lambda_j)}{\prod_{k \neq i, j} (\lambda_i - \lambda_k)(\lambda_j - \lambda_k)}.$$

Proof. In Chapter 2, we have already shown that the number of lines on X is finite and can be computed by the following formula:

$$\text{number of lines on } X = \int_{G(2, r+1)} c_{\text{top}}(\text{Sym}^{2r-3} S^\vee).$$

The induced action on $\overline{M}_{0,0}(\mathbb{P}^r, 1)$ gives us formula (7.3). By (7.4), we get the desired formula. \square

Note that this theorem is a special case of Corollary 5.3.6. Furthermore, we can replace the variables λ_i in the formula above by certain numbers since they cancel in the final result, which is the number of lines on X .

We will illustrate the calculation of these numbers by `Schubert3` in Example A.2.15 and by `schubert.lib` in Example B.1.9.

7.3. Multiple Covers

Let $C \simeq \mathbb{P}^1$ be a smooth rational curve in a Calabi-Yau threefold V with the normal bundle $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. Then the contribution of degree d multiple covers of C to the Gromov-Witten invariant $I_{d[C]}$ is

$$M_d := \int_{\overline{M}_{0,0}(\mathbb{P}^1, d)} c_{2d-2}(\mathcal{F}_d),$$

where \mathcal{F}_d is a vector bundle of rank $2d - 2$ whose fiber at each stable map $f : C_0 \rightarrow C$ is equal to

$$H^1(C_0, f^*(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1))).$$

By Bott's formula (see Corollary 7.1.1), we have

$$M_d = \int_{\overline{M}_{0,0}(\mathbb{P}^1, d)} c_{2d-2}(\mathcal{F}_d) = \sum_{\Gamma} \frac{c_{2d-2}^T(\mathcal{F}_d|_{\Gamma})}{a_{\Gamma} e^T(N_{\Gamma})},$$

where $c_{2d-2}^T(\mathcal{F}_d|_{\Gamma})$ is defined by the following formula:

$$c_{2d-2}^T(\mathcal{F}_d|_{\Gamma}) = \left(\prod_e \left(\prod_{a, b \in \mathbb{N}^*, -a-b=-d_e} \frac{-a\lambda_{i_{v_1}(e)} - b\lambda_{i_{v_2}(e)}}{d_e} \right) \prod_v (-\lambda_{i_v})^{\text{val}(v)-1} \right)^2. \quad (7.5)$$

This formula was given by Kontsevich in [75]. One can show that $M_d = 1/d^3$ (see [16, Theorem 7.4.4 or Theorem 9.2.3]). We present here the computation of M_d using the localization of $\overline{M}_{0,0}(\mathbb{P}^1, d)$ and Bott's formula in the cases of low degree.

1. If $d = 2$, then there are 4 fixed point components of the natural torus action on $\overline{M}_{0,0}(\mathbb{P}^1, 2)$ as in Example 7.1.5. Thus we have

$$\begin{aligned} M_2 &= \int_{\overline{M}_{0,0}(\mathbb{P}^1, 2)} c_2(\mathcal{F}_2) = \sum_{i=1}^4 \frac{c_2^T(\mathcal{F}_2|_{\Gamma_i})}{a_{\Gamma_i} e^T(N_{\Gamma_i})} \\ &= \frac{-(\lambda_0 + \lambda_1)^2}{16(\lambda_0 - \lambda_1)^2} + \frac{-(\lambda_0 + \lambda_1)^2}{16(\lambda_0 - \lambda_1)^2} + \frac{\lambda_0^2}{4(\lambda_0 - \lambda_1)^2} + \frac{\lambda_1^2}{4(\lambda_0 - \lambda_1)^2} = \frac{1}{8}. \end{aligned}$$

2. If $d = 3$, then there are 8 fixed point components of the natural torus action on $\overline{M}_{0,0}(\mathbb{P}^1, 3)$ as in Example 7.1.6. Thus we have $M_3 = \frac{1}{27}$.
3. If $d = 4$, then there are 18 fixed point components of the natural torus action on $\overline{M}_{0,0}(\mathbb{P}^1, 4)$ as in Example 7.1.7. Thus we have $M_4 = \frac{1}{64}$.
4. If $d = 5$, then there are 42 fixed point components of the natural torus action on $\overline{M}_{0,0}(\mathbb{P}^1, 5)$ as in Example 7.1.8. Thus we have $M_5 = \frac{1}{125}$.
5. If $d = 6$, there are 110 fixed point components of the natural torus action on $\overline{M}_{0,0}(\mathbb{P}^1, 6)$ as in Example 7.1.9. Hence $M_6 = \frac{1}{216}$.

We will illustrate the calculation of these numbers by `Schubert3` in Example A.2.16 and by `schubert.lib` in Example B.1.16.

7.4. Rational Curves on Quintic Hypersurfaces in \mathbb{P}^4

Let us reconsider Question 3.2.1. Let $X \subset \mathbb{P}^4$ be a general quintic hypersurface and d be a positive integer. The question is: how many rational curves of degree d are there on X ? This is an important question in enumerative geometry. It has already been mentioned that one can predict the numbers of rational curves of degree d on X via topological quantum field theory and mirror symmetry. These predictions from physics cannot be easily translated into the mathematical language. By algebro-geometric methods, these predictions have been checked

mathematically up to degree 4. We here present how to use the localization of $\overline{M}_{0,0}(\mathbb{P}^4, d)$ and Bott's formula for verifying these predictions up to degree 6.

First of all, we need to compute the Gromov-Witten invariant of the cycle class $d\ell \in A_1(X)$, that is

$$N_d = \int_{\overline{M}_{0,0}(X, d\ell)} [\overline{M}_{0,0}(X, d\ell)]^{\text{virt}}.$$

By Example 6.3.2, we have

$$N_d = \int_{\overline{M}_{0,0}(\mathbb{P}^4, d)} c_{5d+1}(\mathcal{V}_d), \quad (7.6)$$

where \mathcal{V}_d is the vector bundle on $\overline{M}_{0,0}(\mathbb{P}^4, d)$ whose fiber at a stable map $f : C \rightarrow \mathbb{P}^4$ is $H^0(C, f^*\mathcal{O}_{\mathbb{P}^4}(5))$. By arguments as in the paper of Kontsevich (see [75, Example 2.2]) and using Corollary 7.1.1, we have

$$N_d = \int_{\overline{M}_{0,0}(\mathbb{P}^4, d)} c_{5d+1}(\mathcal{V}_d) = \sum_{\Gamma} \frac{c_{5d+1}^T(\mathcal{V}_d|_{\Gamma})}{a_{\Gamma} e^T(N_{\Gamma})},$$

where the sum runs over all graphs corresponding to the fixed point components of a torus action on $\overline{M}_{0,0}(\mathbb{P}^4, d)$. Given a graph Γ , the T -equivariant Chern class $c_{5d+1}^T(\mathcal{V}_d|_{\Gamma})$ is computed by the following formula:

$$c_{5d+1}^T(\mathcal{V}_d|_{\Gamma}) = \prod_e \left(\prod_{a, b \in \mathbb{N}, a+b=5d_e} \frac{a\lambda_{i_{v_1}(e)} + b\lambda_{i_{v_2}(e)}}{d_e} \right) \prod_v (5\lambda_{i_v})^{1-\text{val}(v)}, \quad (7.7)$$

where the products run over all the edges and vertices of Γ . The number a_{Γ} is the order of the group A_{Γ} as in (7.1). We sketch an algorithm (see Algorithm 7.4.1) for computing the Gromov-Witten invariants N_d .

Algorithm 7.4.1 The Gromov-Witten invariants N_d

Input: A positive integer d .

Output: The Gromov-Witten invariants N_d defined by (7.6).

- 1: The natural action of $T = (\mathbb{C}^*)^5$ on \mathbb{P}^4 induces a T -action on the moduli space $\overline{M}_{0,0}(\mathbb{P}^4, d)$. Enumerate all the possible graphs Γ corresponding to the fixed point components of this action.
 - 2: Set $r = 0$.
 - 3: **for** each graph Γ **do**
 - 4: Use formula (7.7) to compute the T -equivariant Chern class $c_{5d+1}^T(\mathcal{V}_d|_{\Gamma})$ of the contribution bundle \mathcal{V}_d at Γ .
 - 5: Use the formulas in Theorem 7.1.10 to compute the T -equivariant Euler class $e^T(N_{\Gamma})$ of the normal bundle N_{Γ} .
 - 6: Use the exact sequence (7.1) to compute the order a_{Γ} of the group A_{Γ} .
 - 7: Set $r = r + \frac{c_{5d+1}^T(\mathcal{V}_d|_{\Gamma})}{a_{\Gamma} e^T(N_{\Gamma})}$.
 - 8: **end for**
 - 9: **Return** r .
-

Remark 7.4.1. In order to shorten the computation time, we replace λ_i by 10^i . This is possible since the final results do not depend on the λ_i .

Here is the implementation of Algorithm 7.4.1 in `Schubert3` (see Example B.1.17 for the implementation in `schubert.lib`):


```

sage: def rational_curves_on_quintic_threefold(d):
.....:     P = ProjectiveSpace(4)
.....:     M = ModuliSpace(P,d)
.....:     F = M.fixed_points()
.....:     result = 0
.....:     for p in F.keys():
.....:         s = M.contribution_bundle(p)
.....:         t = M.equivariant_normal_bundle(p)
.....:         result = result + s/(F[p]*t)
.....:     return result
.....:
sage: for d in range(1,7):
.....:     rational_curves_on_quintic_threefold(d)
.....:
2875
4876875/8
8564575000/27
15517926796875/64
229305888887648
248249742157695375

```

In `Schubert3`, a moduli space of stable maps is represented by a projective space and a positive integer which is the degree of stable maps. The command `ModuliSpace` makes a class which represents such a moduli space. The function `contribution_bundle` in the class `ModuliSpace` is implemented using the formulas (7.2), (7.5), (7.7), and (7.10). The function `equivariant_normal_bundle` is implemented using the formulas in Theorem 7.1.10.

In order to better understand the relation between N_d and the numbers of rational curves of degree d on X , we define *instanton numbers* \tilde{n}_d for the quintic hypersurface in \mathbb{P}^4 such that the equation

$$N_d = \sum_{k|d} \frac{\tilde{n}_{d/k}}{k^3} \tag{7.8}$$

is automatically true. By [16, Example 7.4.4.1], if $1 \leq d \leq 9$, then \tilde{n}_d is equal to the number n_d of rational curves of degree d on X . Unfortunately, this is not true in general. For example, if the Clemens conjecture [16, Conjecture 7.4.3] is true for all $d \leq 10$, then

$$\tilde{n}_{10} = 6 \times 17601000 + n_{10}.$$

See [16, Section 7.4.4] for further discussions on this topic.

From the computations above, we obtain

$$\begin{aligned}
n_1 &= N_1 = \mathbf{2875}, \\
n_2 &= N_2 - \frac{n_1}{8} = \frac{4876875}{8} - \frac{2875}{8} = \mathbf{609250}, \\
n_3 &= N_3 - \frac{n_1}{27} = \frac{8564575000}{27} - \frac{2875}{27} = \mathbf{317206375}, \\
n_4 &= N_4 - \frac{n_2}{8} - \frac{n_1}{64} = \frac{15517926796875}{64} - \frac{609250}{8} - \frac{2875}{64} = \mathbf{242467530000}, \\
n_5 &= N_5 - \frac{n_1}{125} = 229305888887648 - \frac{2875}{125} = \mathbf{229305888887625}, \\
n_6 &= N_6 - \frac{n_3}{8} - \frac{n_2}{27} - \frac{n_1}{216} = 248249742157695375 - \frac{317206375}{8} - \frac{609250}{27} - \frac{2875}{216} \\
&= \mathbf{248249742118022000}.
\end{aligned}$$

We collect these numbers in Table 7.1.

d	n_d
1	2875
2	609250
3	317206375
4	242467530000
5	229305888887625
6	248249742118022000

Table 7.1.: Some numbers n_d .

The numbers with $5 \leq d \leq 6$ have not been obtained before by the localization theorem and Bott's formula. They are all in agreement with predictions made from mirror symmetry in [15] up to degree 6. For a different method for computing these numbers, we refer to the work of Gathmann [43, Example 2.5.8].

7.5. Rational Curves on Complete Intersections

Using the method applied to the quintic hypersurfaces in \mathbb{P}^4 , we can also deal with any complete intersection of hypersurfaces. In Section 2.6, we have already shown that there are five complete intersection Calabi-Yau threefolds of type (d_1, \dots, d_k) in \mathbb{P}^{k+3} . In this section, we present how to compute the numbers of rational curves of degree d on general complete intersection Calabi-Yau threefolds using the localization of moduli spaces of stable maps and Bott's formula. Like in the previous section, we also sketch an algorithm for computing the Gromov-Witten invariants corresponding to the geometric numbers of rational curves of degree d on general complete intersection Calabi-Yau threefolds.

Let X be a general complete intersection Calabi-Yau threefold of type (d_1, \dots, d_k) in \mathbb{P}^{k+3} and let d be a positive integer. We need to compute the Gromov-Witten invariant of the cycle class $d\ell \in A_1(X)$, that is

$$N_d^{(d_1, \dots, d_k)} = \int_{\overline{M}_{0,0}(X, d\ell)} [\overline{M}_{0,0}(X, d\ell)]^{\text{virt}}. \quad (7.9)$$

By arguments similar to those in [16, Example 7.1.5.1], we construct the virtual fundamental class $[\overline{M}_{0,0}(X, d\ell)]^{\text{virt}}$ as follows. Let $\mathcal{V}_d^{(d_1, \dots, d_k)}$ be the vector bundle on the moduli space $\overline{M}_{0,0}(\mathbb{P}^{k+3}, d)$ whose fiber over a stable map $f : C \rightarrow \mathbb{P}^{k+3}$ is

$$H^0(C, f^*(\mathcal{O}_{\mathbb{P}^{k+3}}(d_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^{k+3}}(d_k))).$$

In this case, $\overline{M}_{0,0}(X, d\ell)$ is the zero locus of a general section s of $\mathcal{V}_d^{(d_1, \dots, d_k)}$. Moreover, we have

$$[\overline{M}_{0,0}(X, d\ell)]^{\text{virt}} = s^*[C_Z Y],$$

where $Z = \overline{M}_{0,0}(X, d\ell)$, $Y = \overline{M}_{0,0}(\mathbb{P}^{k+3}, d)$, and $C_Z Y$ is the normal cone to Z in Y .

By [16, Lemma 7.1.5], we obtain

$$i_*([\overline{M}_{0,0}(X, d\ell)]^{\text{virt}}) = c_{(k+4)d+k}(\mathcal{V}_d^{(d_1, \dots, d_k)}) \in A^*(\overline{M}_{0,0}(\mathbb{P}^{k+3}, d)),$$

where $i : \overline{M}_{0,0}(X, d\ell) \hookrightarrow \overline{M}_{0,0}(\mathbb{P}^{k+3}, d)$ is an embedding. This implies the following formula:

$$N_d^{(d_1, \dots, d_k)} = \int_{\overline{M}_{0,0}(\mathbb{P}^{k+3}, d)} c_{(k+4)d+k}(\mathcal{V}_d^{(d_1, \dots, d_k)}).$$

In other words, we have

$$\mathcal{V}_d^{(d_1, \dots, d_k)} = \bigoplus_{i=1}^k \mathcal{V}_d^{(d_i)},$$

where $\mathcal{V}_d^{(d_i)}$ is the vector bundle on the moduli space $\overline{M}_{0,0}(\mathbb{P}^{k+3}, d)$ whose fiber over a stable map $f : C \rightarrow \mathbb{P}^{k+3}$ is $H^0(C, f^*\mathcal{O}_{\mathbb{P}^{k+3}}(d_i))$. Note that $\mathcal{V}_d^{(5)}$ is \mathcal{V}_d in the quintic case. Therefore, we have the following formula:

$$N_d^{(d_1, \dots, d_k)} = \int_{\overline{M}_{0,0}(\mathbb{P}^{k+3}, d)} \prod_{i=1}^k c_{d_i d+1}(\mathcal{V}_d^{(d_i)}).$$

Using Corollary 7.1.1, we have

$$N_d^{(d_1, \dots, d_k)} = \sum_{\Gamma} \frac{\prod_{i=1}^k c_{d_i d+1}^T(\mathcal{V}_d^{(d_i)}|_{\Gamma})}{a_{\Gamma} e^T(N_{\Gamma})},$$

where the sum runs over all graphs corresponding to the fixed point components of a torus action on $\overline{M}_{0,0}(\mathbb{P}^{k+3}, d)$. For each graph Γ and each i , the T -equivariant Chern class $c_{d_i d+1}^T(\mathcal{V}_d^{(d_i)}|_{\Gamma})$ is computed by the following formula:

$$c_{d_i d+1}^T(\mathcal{V}_d^{(d_i)}|_{\Gamma}) = \prod_e \left(\prod_{a, b \in \mathbb{N}, a+b=d_i d_e} \frac{a\lambda_{i_{v_1(e)}} + b\lambda_{i_{v_2(e)}}}{d_e} \right) \prod_v (d_i \lambda_{i_v})^{1-\text{val}(v)}, \quad (7.10)$$

where the products run over all the edges and vertices of Γ .

We sketch an algorithm (see Algorithm 7.5.1) for computing the Gromov-Witten invariants $N_d^{(d_1, \dots, d_k)}$.

We have implemented Algorithm 7.5.1 in `Schubert3` and `schubert.lib`. Here is the implementation in `Schubert3` (see Example B.1.17 for the implementation in `schubert.lib`):

Algorithm 7.5.1 The Gromov-Witten invariants $N_d^{(d_1, \dots, d_k)}$

Input: A positive integer d and a list of the integers d_1, \dots, d_k with $d_i \geq 2$ for all i .

Output: The Gromov-Witten invariants $N_d^{(d_1, \dots, d_k)}$ defined by (7.9).

- 1: The natural action of $T = (\mathbb{C}^*)^{k+4}$ on \mathbb{P}^{k+3} induces a T -action on the moduli space $\overline{M}_{0,0}(\mathbb{P}^{k+3}, d)$. Enumerate all the possible graphs Γ corresponding to the fixed point components of this action.
 - 2: Set $r = 0$.
 - 3: **for** each graph Γ **do**
 - 4: **for** i from 1 to k **do**
 - 5: Use formula (7.10) to compute the equivariant Chern class $c_{d_i d+1}^T(\mathcal{V}_d^{(d_i)}|_\Gamma)$ of the contribution bundle $\mathcal{V}_d^{(d_i)}$ at Γ .
 - 6: **end for**
 - 7: Set $s = \prod_{i=1}^k c_{d_i d+1}^T(\mathcal{V}_d^{(d_i)}|_\Gamma)$.
 - 8: Use the formulas in Theorem 7.1.10 to compute the equivariant Euler class $e^T(N_\Gamma)$ of the normal bundle N_Γ .
 - 9: Use the exact sequence (7.1) to compute the order a_Γ of the group A_Γ .
 - 10: Set $r = r + \frac{s}{a_\Gamma e^T(N_\Gamma)}$.
 - 11: **end for**
 - 12: **Return** r .
-

```

sage: def rational_curves(d,D):
.....:     P = ProjectiveSpace(len(D)+3)
.....:     M = ModuliSpace(P,d)
.....:     F = M.fixed_points()
.....:     r = 0
.....:     for p in F.keys():
.....:         s = prod(M.contribution_bundle(p,k) for k in D)
.....:         t = M.equivariant_normal_bundle(p)
.....:         r = r + s/(F[p]*t)
.....:     return r
.....:
sage: T = [[4,2],[3,3],[3,2,2],[2,2,2,2]]

```

As for the quintic hypersurface in \mathbb{P}^4 , we may define *instanton numbers* $\tilde{n}_d^{(d_1, \dots, d_k)}$ for complete intersection Calabi-Yau threefolds of type (d_1, \dots, d_k) in \mathbb{P}^{k+3} such that the equation

$$N_d^{(d_1, \dots, d_k)} = \sum_{k|d} \frac{\tilde{n}_{d/k}^{(d_1, \dots, d_k)}}{k^3} \quad (7.11)$$

is automatically true. In the following examples, the instanton numbers agree with the numbers of rational curves. We therefore use the notation $n_d^{(d_1, \dots, d_k)}$ also for the instanton numbers.

For lines, corresponding to the 4 remaining types of complete intersection Calabi-Yau threefolds as indicated above, we get the following numbers:

```

sage: [rational_curves(1,t) for t in T]
[1280, 1053, 720, 512]

```

This gives the instanton numbers $n_1^{(4,2)}$, $n_1^{(3,3)}$, $n_1^{(3,2,2)}$, and $n_1^{(2,2,2,2)}$ for complete intersection Calabi-Yau threefolds of types $(4, 2)$, $(3, 3)$, $(3, 2, 2)$, and $(2, 2, 2, 2)$ respectively. These instanton numbers agree with the numbers of lines on general complete intersection Calabi-Yau threefolds computed in Section 2.6 (see Table 2.2).

For conics, we get the following numbers:

```
sage: [rational_curves(2,t) for t in T]
[92448, 423549/8, 22518, 9792]
```

Hence the instanton numbers $n_2^{(4,2)}$, $n_2^{(3,3)}$, $n_2^{(3,2,2)}$, and $n_2^{(2,2,2,2)}$ for complete intersection Calabi-Yau threefolds of types $(4, 2)$, $(3, 3)$, $(3, 2, 2)$, and $(2, 2, 2, 2)$, respectively, are as follows:

$$n_2^{(4,2)} = 92448 - \frac{1280}{8} = \mathbf{92288},$$

$$n_2^{(3,3)} = \frac{423549}{8} - \frac{1053}{8} = \mathbf{52812},$$

$$n_2^{(3,2,2)} = 22518 - \frac{720}{8} = \mathbf{22428},$$

$$n_2^{(2,2,2,2)} = 9792 - \frac{512}{8} = \mathbf{9728}.$$

These instanton numbers agree with the numbers of conics on general complete intersection Calabi-Yau threefolds computed in Section 3.3 (see Table 3.2).

For degree 3 rational curves, we get the following numbers:

```
sage: [rational_curves(3,t) for t in T]
[422690816/27, 6424365, 4834592/3, 11239424/27]
```

So the instanton numbers $n_3^{(4,2)}$, $n_3^{(3,3)}$, $n_3^{(3,2,2)}$, and $n_3^{(2,2,2,2)}$ for complete intersection Calabi-Yau threefolds of types $(4, 2)$, $(3, 3)$, $(3, 2, 2)$, $(2, 2, 2, 2)$, respectively, are as follows:

$$n_3^{(4,2)} = \frac{422690816}{27} - \frac{1280}{27} = \mathbf{15655168},$$

$$n_3^{(3,3)} = 6424365 - \frac{1053}{27} = \mathbf{6424326},$$

$$n_3^{(3,2,2)} = \frac{4834592}{3} - \frac{720}{27} = \mathbf{1611504},$$

$$n_3^{(2,2,2,2)} = \frac{11239424}{27} - \frac{512}{27} = \mathbf{416256}.$$

These instanton numbers agree with the numbers of twisted cubic curves on general complete intersection Calabi-Yau threefolds computed by Ellingsrud and Strømme [31, Theorem 1.1].

For degree 4 rational curves, we get the following numbers:

```
sage: [rational_curves(4,t) for t in T]
[3883914084, 72925120125/64, 672808059/4, 25705160]
```

Hence the instanton numbers $n_4^{(4,2)}$, $n_4^{(3,3)}$, $n_4^{(3,2,2)}$, and $n_4^{(2,2,2,2)}$ for complete intersection Calabi-Yau threefolds of types (4, 2), (3, 3), (3, 2, 2), and (2, 2, 2, 2), respectively, are as follows:

$$\begin{aligned} n_4^{(4,2)} &= 3883914084 - \frac{92288}{8} - \frac{1280}{64} = \mathbf{3883902528}, \\ n_4^{(3,3)} &= \frac{72925120125}{64} - \frac{52812}{8} - \frac{1053}{64} = \mathbf{1139448384}, \\ n_4^{(3,2,2)} &= \frac{672808059}{4} - \frac{22428}{8} - \frac{720}{64} = \mathbf{168199200}, \\ n_4^{(2,2,2,2)} &= 25705160 - \frac{9728}{8} - \frac{512}{64} = \mathbf{25703936}. \end{aligned}$$

For degree 5 rational curves, we get the following numbers:

```
sage: [rational_curves(5,t) for t in T]
[29773082054656/25, 31223486573928/125, 541923292944/25, 244747968512/125]
```

Hence the instanton numbers $n_5^{(4,2)}$, $n_5^{(3,3)}$, $n_5^{(3,2,2)}$, and $n_5^{(2,2,2,2)}$ for complete intersection Calabi-Yau threefolds of types (4, 2), (3, 3), (3, 2, 2), and (2, 2, 2, 2), respectively, are as follows:

$$\begin{aligned} n_5^{(4,2)} &= \frac{29773082054656}{25} - \frac{1280}{125} = \mathbf{1190923282176}, \\ n_5^{(3,3)} &= \frac{31223486573928}{125} - \frac{1053}{125} = \mathbf{249787892583}, \\ n_5^{(3,2,2)} &= \frac{541923292944}{25} - \frac{720}{125} = \mathbf{21676931712}, \\ n_5^{(2,2,2,2)} &= \frac{244747968512}{125} - \frac{512}{125} = \mathbf{1957983744}. \end{aligned}$$

For degree 6 rational curves, we get the following numbers:

```
sage: [rational_curves(6,t) for t in T]
[417874607302656, 501287722516269/8, 3195558106836, 511607926784/3]
```

Hence the instanton numbers $n_6^{(4,2)}$, $n_6^{(3,3)}$, $n_6^{(3,2,2)}$, and $n_6^{(2,2,2,2)}$ for complete intersection Calabi-Yau threefolds of types (4, 2), (3, 3), (3, 2, 2), and (2, 2, 2, 2), respectively, are as follows:

$$\begin{aligned} n_6^{(4,2)} &= 417874607302656 - \frac{15655168}{8} - \frac{92288}{27} - \frac{1280}{216} = \mathbf{417874605342336}, \\ n_6^{(3,3)} &= \frac{501287722516269}{8} - \frac{6424326}{8} - \frac{52812}{27} - \frac{1053}{216} = \mathbf{62660964509532}, \\ n_6^{(3,2,2)} &= 3195558106836 - \frac{1611504}{8} - \frac{22428}{27} - \frac{720}{216} = \mathbf{3195557904564}, \\ n_6^{(2,2,2,2)} &= \frac{511607926784}{3} - \frac{416256}{8} - \frac{9728}{27} - \frac{512}{216} = \mathbf{170535923200}. \end{aligned}$$

Summarizing the computations above, we obtain Table 7.2.

The numbers with $4 \leq d \leq 6$ have not been obtained before by algebro-geometric methods. They are all in agreement with predictions made from mirror symmetry in [82] up to degree 6.

In summary, we have the following theorem.

d	(4, 2)	(3, 3)	(3, 2, 2)	(2, 2, 2, 2)
1	1280	1053	720	512
2	92288	52812	22428	9728
3	15655168	6424326	1611504	416256
4	3883902528	1139448384	168199200	25703936
5	1190923282176	249787892583	21676931712	1957983744
6	417874605342336	62660964509532	3195557904564	170535923200

Table 7.2.: Some instanton numbers $n_d^{(d_1, \dots, d_k)}$.

Theorem 7.5.1. *Let d be an integer with $1 \leq d \leq 6$. The instanton numbers n_d , $n_d^{(4,2)}$, $n_d^{(3,3)}$, $n_d^{(3,3,2)}$, and $n_d^{(2,2,2,2)}$ for Calabi-Yau threefolds in projective spaces are given by Tables 7.1 and 7.2.*

Compare with [15, Table 4] and [82, Table 1] for the numbers obtained by mirror symmetry.

7.6. Future work

In this part, we describe some problems which are still open with respect to algebro-geometric methods. We hope to come back to these problems in the near future.

First of all, we recall the following problem which arose in this chapter:

Problem 7.6.1. *Let X be a general Calabi-Yau threefold which is a complete intersection in some projective space, and let $d \geq 1$ be an integer. How many rational curves of degree d are there on X ?*

We have computed these numbers up to degree 6 using purely algebro-geometric methods. So the next problem will be to find an effective method for computing these numbers for larger degree. For this, we need an effective way to enumerate all the graphs corresponding to the fixed point components of a torus action on a moduli space of stable maps. This comes down to enumerate all isomorphism classes of trees with a given number of edges.

Instead of just rational curves, we may as well consider curves of higher genus. Since a moduli space of higher genus stable maps is in general nonreduced, reducible, and singular, we need a virtual localization formula. Such a formula can be found in [45]. In this case, we also need to find a formula for computing the equivariant Euler class of the normal bundle of a moduli space of higher genus stable maps. This formula may be found in [45] as well.

Greene, Morrison, and Plesser [49] predicted the numbers of rational curves on some Calabi-Yau hypersurfaces of higher dimension. Katz [67] verified these numbers for lines and conics on Calabi-Yau hypersurfaces of dimension up to 10. Ellingsrud and Strømme [31] verified some numbers of degree 3 rational curves. We have already verified these numbers for lines using the localization theorem and Bott's formula (see Example A.2.17). By the same method, we hope that the numbers of higher degree and genus curves as above can be obtained.

In case X is a Pfaffian Calabi-Yau threefold, which is not a complete intersection, Rødland [94] predicted some numbers of rational curves on X using mirror symmetry methods. Afterwards, Tjøtta [100] computed the quantum cohomology of X using a generalization of Givental-Kim's quantum hyperplane principle. By this way, Tjøtta verified the mirror symmetry predictions of Rødland. So it is a natural question, whether we can use the method of this chapter and our calculation to verify these predictions.

Markwig [85] proved a number of important results related to tropical enumerative geometry. This work shows that there are many deep relations between enumerative geometry and tropical geometry. Recently, Böhm, Bringmann, Buchholz, and Markwig [11] proved a tropical generalization of mirror symmetry for elliptic curves. Their result is based on a method for computing tropical Hurwitz numbers of an elliptic curve. By the Correspondence Theorem (see [11, Theorem 2.13]), these numbers are equal to the classical Gromov-Witten invariants. This raises the question whether the proof of mirror symmetry can be generalized to higher dimensional Calabi-Yau varieties using the techniques of equivariant intersection theory. A first step would be to compute the Hurwitz numbers obtained via tropical geometry by algebro-geometric methods using `Schubert3` and `schubert.lib`.

Appendix A.

A Sage Package for Intersection Theory and Enumerative Geometry

The first package for computations in intersection theory and enumerative geometry is `Schubert` [68] in `MAPLE` which was written by Sheldon Katz and Stein A. Strømme in 1992, and had been revised and updated to new versions of `MAPLE` by Jan-Magnus Økland. However, it is no longer actively supported. The second package is `Schubert2` [48] in `MACAULAY2` which has been developed by Daniel R. Grayson, Michael E. Stillman, Stein A. Strømme, David Eisenbud and Charley Crissman.

Our package is `Schubert3` developed as a `SAGE` package in the `PYTHON` programming language. This package provides the key functionality necessary for computations in intersection theory and enumerative geometry. It deals with abstract varieties, vector bundles on abstract varieties, morphisms between abstract varieties, equivariant vector bundles on abstract varieties endowed with a torus action, moduli spaces of stable maps, and graphs.

A.1. Code

In `Schubert3`, an abstract variety is represented by a nonnegative integer which is its dimension, a list of variables which are the minimal generators of its Chow ring, a list of positive integers which are the degrees of the variables in the graded Chow ring, and a list of polynomials which give the relations between the variables which define the Chow ring as an affine ring. For example, the n -dimensional projective space \mathbb{P}^n is represented by its dimension n , by the variable h of degree 1 which is the class of a hyperplane, and by the relation $h^{n+1} = 0$. The Grassmannian $G(k, n)$ is represented by its dimension $k(n - k)$, by $n - k$ variables $\sigma_1, \dots, \sigma_{n-k}$ with $\deg(\sigma_i) = i$, and by $n - k$ relations $\det(\sigma_{1+b-a})_{1 \leq a, b \leq k+m} = 0$, for $m = 1, \dots, n - k$, where the σ_i are the special Schubert classes on $G(k, n)$ defined in Section 2.1. Note that $\sigma_0 = 1$ and $\sigma_i = 0$ whenever $i < 0$ or $i > n - k$. In some cases, the relations may not be known, but we must know their monomial values. This allows us to compute the degree of zero-dimensional cycle classes. Moreover, we also developed an alternative method for computing the degree of zero-dimensional cycle classes via Bott's formula. There are two ways to give a vector bundle E on an abstract variety in `Schubert3`. The first one is to represent E by a positive integer which is the rank of E , and an element of the Chow ring which is the total Chern class of E . The second one is to represent E by only an element of the Chow ring which is the Chern character of E . The main idea for the implementation of vector bundles is based on the splitting principle. A morphism between two abstract varieties is represented by its pullback homomorphism. In order to use Bott's formula for computing the degree of a zero-dimensional cycle class, we deal with T -equivariant vector bundles over a point and moduli spaces of stable maps. A T -equivariant vector bundle over a point is represented by a list of weights defined

in Section 5.1.3. A moduli space of stable maps is represented by a projective space and a positive integer which are the target space and the degree of the stable maps, respectively. We also deal with graphs which allow us to represent the fixed point components of a torus action on a moduli space of stable maps.

In order to use `Schubert3`, we first attach the file `Schubert3.py` to a running session using the `%attach` command in SAGE as follows:

```
sage: %attach Schubert3.py
```

This file is available from the author upon request or at

<https://github.com/hiepdang/Sage>.

The current version of this package is the result of many discussions on mathematical background and on implementation algorithms, and of many hours of coding. It should not be considered a complete, unchangeable, totally stable version. We will keep working on this project by fixing bugs, improving algorithms, and by adding functionalities.

A.2. Examples

Example A.2.1. Compute the total Chern class, Chern polynomial, Chern character, Todd class, total Segre class, and Segre polynomial of a vector bundle.

```
sage: X = Base(3,bundles=[[2,'c'],[3,'d']]); X
A variety of dimension 3
sage: E = X.bundle(1); E
A vector bundle of rank 2 on A variety of dimension 3
sage: E.total_chern_class()
c2 + c1 + 1
sage: E.bern_polynomial()
c2*t^2 + c1*t + 1
sage: E.bern_character()
1/6*c1^3 - 1/2*c1*c2 + 1/2*c1^2 - c2 + c1 + 2
sage: E.todd_class()
1/24*c1*c2 + 1/12*c1^2 + 1/12*c2 + 1/2*c1 + 1
sage: E.total_segre_class()
-c1^3 + 2*c1*c2 + c1^2 - c2 - c1 + 1
sage: E.segre_polynomial()
(-c1^3 + 2*c1*c2)*t^3 + (c1^2 - c2)*t^2 - c1*t + 1
```

Example A.2.2. Compute the number of lines meeting 4 general lines in \mathbb{P}^3 .

```
sage: G = Grassmannian(2,4)
sage: G.dim()
4
sage: G.degree()
2
sage: sigma1 = G.schubert_class([1])
sage: G.integral(sigma1^4)
```

```

2
sage: G.bott(sigma1^4)
2

```

Example A.2.3. Let X be a general quartic hypersurface in \mathbb{P}^4 . Then the Fano scheme $F_1(X)$ is a curve in $G(2, 5)$ and its degree is computed as follows:

```

sage: G = Grassmannian(2,5)
sage: S = G.tautological_subbundle().dual()
sage: B = S.symmetric_power(4)
sage: sigma1 = G.schubert_class([1])
sage: G.integral(B.top_chern_class()*sigma1)
320
sage: G.bott(B.top_chern_class()*sigma1)
320

```

Moreover, the Hilbert function of $F_1(X)$ is determined as follows:

```

sage: F = BundleSection(B)
sage: def hilbert_function(t):
.....:     L = F.0(t*sigma1)
.....:     return L.chi()
.....:
sage: hilbert_function(0)
- 800
sage: hilbert_function(1)
- 480
sage: hilbert_function(2)
- 160
sage: hilbert_function(3)
160

```

This implies that the Hilbert polynomial of $F_1(X)$ is $320t - 800$.

Example A.2.4. Using Bott's formula for Grassmannians, we illustrate how to compute the number of k -planes on a general hypersurface of degree d in \mathbb{P}^n by `Schubert3` as follows:

```

sage: def linear_subspaces_via_Bott_formula(k,d,n):
.....:     phi = (k+1)*(n-k)-binomial(d+k,d)
.....:     if phi <> 0:
.....:         raise ValueError('Invalid data')
.....:     G = Grassmannian(n-k,n+1)
.....:     F = G.fixed_points()
.....:     result = 0
.....:     for p in F:
.....:         Q = G.equivariant_quotient_bundle(p)
.....:         B = Q.symmetric_power(d)
.....:         s = B.euler_class()
.....:         T = G.equivariant_tangent_bundle(p)

```

```

.....:         t = T.euler_class()
.....:         result = result + s/t
.....:     return result
.....:
sage: linear_subspaces_via_Bott_formula(1,17,10)
520764738758073845321
sage: linear_subspaces_via_Bott_formula(1,27,15)
231191601420598135249236900564098773215
sage: linear_subspaces_via_Bott_formula(1,37,20)
4798492409653834563672780605191070760393640761817269985515
sage: linear_subspaces_via_Bott_formula(2,4,7)
3297280
sage: linear_subspaces_via_Bott_formula(2,5,9)
420760566875
sage: linear_subspaces_via_Bott_formula(2,7,14)
279101475496912988004267637
sage: linear_subspaces_via_Bott_formula(3,3,8)
321489

```

Example A.2.5. We illustrate the Porteous formula (see [29, Chapter 14] for more details) as follows:

```

sage: X = Base(3,bundles=[[3,'a'],[5,'b']]); X
A variety of dimension 3
sage: A = X.bundle(1); A
A vector bundle of rank 3 on A variety of dimension 3
sage: B = X.bundle(2); B
A vector bundle of rank 5 on A variety of dimension 3
sage: A.porteous(B,2)
-a1^3 + 2*a1*a2 - a3 + a1^2*b1 - a2*b1 - a1*b2 + b3

sage: P = ProjectiveSpace(4)
sage: h = P.hyperplane()
sage: L = P.O(1)
sage: E = VectorBundle(P, chern_character = 4)
sage: A = VectorBundle(P, chern_character = 2)
sage: B = E & L
sage: A.porteous(B,1)
4*h^3
sage: AA = VectorBundle(P, 2, chern_class = - h)
sage: AA.porteous(B,1)
15*h^3

```

The following example is to illustrate how to use the Porteous formula for computing the number of quintic surfaces in a general net that contains a line.

```

sage: G = Grassmannian(2,4)
sage: S = G.tautological_subbundle().dual()
sage: B = S.symmetric_power(5)

```

```

sage: L = VectorBundle(G, chern_character = 3)
sage: M2 = L.porteous(B,2)
sage: G.integral(M2)
1990
sage: G.bott(M2)
1990

```

Example A.2.6. We illustrate how to compute the number of conics tangent to five given conics (see [29, Chapter 15] for more details).

```

sage: P2 = ProjectiveSpace(2)
sage: P5 = ProjectiveSpace(5)
sage: h = P2.hyperplane()
sage: H = P5.hyperplane()
sage: f = Morphism(P2,P5,[2*h])
sage: B = Blowup(f)
sage: E = B.exceptional_divisor()
sage: B.integral1((6*H-2*E)^5)
3264

```

Example A.2.7. Compute the number of bitangents to a plane curve.

```

sage: P = ProjectiveSpace(2,'h')
sage: Q = ProjectiveSpace(2,'H')
sage: H = Q.hyperplane()
sage: L = Q.O(4*H)
sage: B = BundleSection(L)
sage: f = Morphism(B,P,[3*H])
sage: D = f.double_point(); D
26*H
sage: B.integral1(D)/2 - 3*4*(4-2)
28

```

Example A.2.8. Compute the Betti numbers of a product of Grassmannians.

```

sage: G = Grassmannian(2,5); H = Grassmannian(3,6)
sage: W = ProductVariety(G,H)
sage: W.betti()
[1, 2, 5, 9, 14, 19, 24, 26, 26, 24, 19, 14, 9, 5, 2, 1]

```

Example A.2.9. Compute the Todd class of \mathbb{P}^3 , that is the Todd class of the tangent bundle of \mathbb{P}^3 .

```

sage: P = ProjectiveSpace(3)
sage: P.todd_class()
h^3 + 11/6*h^2 + 2*h + 1
sage: T = P.tangent_bundle()
sage: T.todd_class() == P.todd_class()
True

```

Example A.2.10. Cubic hypersurfaces containing some \mathbb{P}^2 in \mathbb{P}^5 form a hypersurface in \mathbb{P}^{55} . Here is how to compute the degree of this hypersurface (see [103, Proposition 2.2.1] for more details).

```
sage: G = Grassmannian(3,6)
sage: S = G.tautological_subbundle().dual()
sage: B = S.symmetric_power(3)
sage: G.integral(B.chern_class(9))
3402
sage: G.bott(B.chern_class(9))
3402
```

More generally, hypersurfaces of degree d containing some \mathbb{P}^2 in \mathbb{P}^5 form a subvariety of codimension $\binom{d+2}{2} - 9$ and its degree is computed as follows:

```
sage: def degree(d):
.....:     G = Grassmannian(3,6)
.....:     S = G.tautological_subbundle().dual()
.....:     B = S.symmetric_power(d)
.....:     return G.integral(B.chern_class(9))
.....:
sage: degree(3)
3402
sage: degree(4)
8754732
sage: degree(5)
2547516517
sage: degree(6)
246775402104
```

Example A.2.11. [72, Table 2] The number of elliptic cubics on a general sextic hypersurface in \mathbb{P}^5 can be computed as follows:

```
sage: G = Grassmannian(3,6)
sage: S = G.tautological_subbundle().dual()
sage: B = S.symmetric_power(3)
sage: P = ProjectiveBundle(B)
sage: V = P.0(-1) & S.symmetric_power(3)
sage: A = S.symmetric_power(6) - V
sage: P.integral(A.top_chern_class())
2734099200
sage: p = P.projection_morphism()
sage: G.integral(p.pushforward(A.top_chern_class()))
2734099200
sage: G.bott(p.pushforward(A.top_chern_class()))
2734099200
```

Example A.2.12. Compute the number of conics in \mathbb{P}^4 meeting each of 11 general 2-planes $\gamma_1 \dots \gamma_{11} \subset \mathbb{P}^4$. This number was computed as a Gromov-Witten invariant in Example 6.4.9.

```

sage: G = Grassmannian(3,5)
sage: S = G.tautological_subbundle().dual()
sage: B = S.symmetric_power(2)
sage: P = ProjectiveBundle(B)
sage: zeta = P.zeta_class()
sage: sigma1 = G.schubert_class([1])
sage: P.integral((2*sigma1+zeta)^11)
6620
sage: p = P.projection_morphism()
sage: G.integral(p.pushforward((2*sigma1+zeta)^11))
6620
sage: G.bott(p.pushforward((2*sigma1+zeta)^11))
6620

```

Example A.2.13. In this example, we compute the top degree monomial values of a Grassmannian. There are two ways to do this using the Schubert calculus or Bott's formula.

```

sage: def top_monomial_values_via_schubert_calculus(G):
.....:     d = {}
.....:     for m in G.monomials(G.dim()):
.....:         d[m] = G.integral(m)
.....:     return d
.....:
sage: G = Grassmannian(2,4)
sage: top_monomial_values_via_schubert_calculus(G)
{q2^2: 1, q1^2*q2: 1, q1^4: 2}

sage: def top_monomial_values_via_bott_formula(G):
.....:     d = {}
.....:     for m in G.monomials(G.dim()):
.....:         r = 0
.....:         for p in G.fixed_points():
.....:             q = G.equivariant_variable(p)
.....:             T = G.equivariant_tangent_bundle(p)
.....:             t = T.euler_class()
.....:             r = r + m(q) / t
.....:         d[m] = r
.....:     return d
.....:
sage: G = Grassmannian(2,4)
sage: top_monomial_values_via_bott_formula(G)
{q2^2: 1, q1^2*q2: 1, q1^4: 2}

```

Example A.2.14. In this example, we compute the top degree monomial values of a Pfaffian Calabi-Yau threefold as the description in [100, Section 2]. The results are all in Table A.1.

```

sage: P = ProjectiveSpace(20); O1 = P.O(1)
sage: G = Grassmannian(4,7)
sage: Ub = G.tautological_subbundle()

```

h^6	14	$q_1 h^5$	28	$q_2 h^4$	28	$q_1^2 h^4$	56
$q_3 h^3$	14	$q_1 q_2 h^3$	56	$q_1^3 h^3$	112	$q_1 q_3 h^2$	28
$q_2^2 h^2$	59	$q_1^2 q_2 h^2$	112	$q_1^4 h^2$	224	$q_2 q_3 h$	37
$q_1^2 q_3 h$	56	$q_1 q_2^2 h$	118	$q_1^3 q_2 h$	224	$q_1^5 h$	448
q_3^2	41	$q_1 q_2 q_3$	74	$q_1^3 q_3$	112	q_2^3	117
$q_1^2 q_2^2$	236	$q_1^4 q_2$	448	q_1^6	896		

Table A.1.: The top degree monomial values of a Pfaffian Calabi-Yau threefold.

```

sage: W = ProductVariety(G,P)
sage: p1 = W.projection_morphism(1)
sage: p2 = W.projection_morphism(2)
sage: O1p = p2.pullback(O1); Ubq = p1.pullback(Ub)
sage: Wedge2Ubq = Ubq.wedge(2) & O1p
sage: T = VectorBundle(W,chern_character=21)
sage: B = T & O1p; A = B - Wedge2Ubq
sage: P9 = ProjectiveSpace(9)
sage: W1 = ProductVariety(G,P9)
sage: i = Morphism(W1,W,sb=W1.variables())
sage: A1 = i.pullback(A); E = BundleSection(A1)
sage: E.variables()
(q1, q2, q3, h)
sage: E.dim()
6
sage: E.monomial_values()
{h^6: 14, q1*h^5: 28, q2*h^4: 28, q1^2*h^4: 56,
 q3*h^3: 14, q1*q2*h^3: 56, q1^3*h^3: 112,
 q1*q3*h^2: 28, q2^2*h^2: 59, q1^2*q2*h^2: 112,
 q1^4*h^2: 224, q2*q3*h: 37, q1^2*q3*h: 56,
 q1*q2^2*h: 118, q1^3*q2*h: 224, q1^5*h: 448,
 q3^2: 41, q1*q2*q3: 74, q1^3*q3: 112, q2^3: 117,
 q1^2*q2^2: 236, q1^4*q2: 448, q1^6: 896}
sage: E.betti_numbers()
[1, 1, 2, 2, 2, 1, 1]

```

Example A.2.15. In this example, we illustrate the calculation of the Gromov-Witten invariants corresponding to the numbers of lines on general hypersurfaces of degree $2n - 3$ in \mathbb{P}^n by the localization of $\overline{M}_{0,0}(\mathbb{P}^n, 1)$ and Bott's formula.

```

sage: def lines_on_hypersurfaces(n):
.....:     P = ProjectiveSpace(n)
.....:     M = ModuliSpace(P,1)
.....:     F = M.fixed_points()
.....:     result = 0
.....:     for p in F.keys():
.....:         s = M.contribution_bundle(p)
.....:         t = M.equivariant_normal_bundle(p)

```



```

.....:         result = result + s/(F[p]*t)
.....:     return result
.....:
sage: for n in range(2,11):
.....:     lines_on_hypersurfaces(n)
.....:
1
27
2875
698005
305093061
210480374951
210776836330775
289139638632755625
520764738758073845321

```

These numbers agree with Table 2.1.

Example A.2.16. We illustrate the calculation of Section 7.3 as follows:

```

sage: def multiple_cover(d):
.....:     P = ProjectiveSpace(1)
.....:     M = ModuliSpace(P,d)
.....:     F = M.fixed_points()
.....:     r = 0
.....:     for p in F.keys():
.....:         s = M.contribution_bundle(p)
.....:         t = M.equivariant_normal_bundle(p)
.....:         r = r + s/(F[p]*t)
.....:     return r
.....:
sage: [multiple_cover(d) for d in range(1,7)]
[1, 1/8, 1/27, 1/64, 1/125, 1/216]

```

Example A.2.17. In this example, we illustrate how to compute the Gromov-Witten invariants $n_b^a(1)$ corresponding to the numbers of lines on general Calabi-Yau hypersurfaces of dimension k which meet 3 general linear subspaces L_a, L_b, L_c such that $\text{codim}(L_a) = a$, $\text{codim}(L_b) = b$, $\text{codim}(L_c) = c$, and $a + b + c = k$ (see [67, Section 3] for more details). The results are in Table A.2.

```

sage: def GWLines_via_Schubert_calculus(k,a,b):
.....:     G = Grassmannian(2,k+2)
.....:     S = G.tautological_subbundle().dual()
.....:     B = S.symmetric_power(k+2)
.....:     qa = G.schubert_class([a-1])
.....:     qb = G.schubert_class([b-1])
.....:     qc = G.schubert_class([k-a-b-1])
.....:     return G.integral(B.top_chern_class()*qa*qb*qc)
.....:
sage: GWLines_via_Schubert_calculus(3,1,1)
2875

```

We also illustrate this calculation by Bott's formula for Grassmannians as follows:

```

sage: def GWLines_via_Bott_formula(k,a,b):
.....:     G = Grassmannian(2,k+2)
.....:     F = G.fixed_points()
.....:     result = 0
.....:     for p in F:
.....:         S = G.equivariant_subbundle(p)
.....:         Q = G.equivariant_quotient_bundle(p)
.....:         D = S.dual()
.....:         B = D.symmetric_power(k+2)
.....:         qa = Q.chern_class(a-1)
.....:         qb = Q.chern_class(b-1)
.....:         qc = Q.chern_class(k-a-b-1)
.....:         s = B.euler_class()*qa*qb*qc
.....:         T = G.equivariant_tangent_bundle(p)
.....:         t = T.euler_class()
.....:         result = result + s/t
.....:     return result
.....:
sage: GWLines_via_Bott_formula(3,1,1)
2875

```

k	a	b	$n_b^a(1)$	k	a	b	$n_b^a(1)$
3	1	1	2875	9	1	1	69407571816
4	1	1	60480	9	1	2	307393401172
5	1	1	1009792	9	1	3	695221679878
5	1	2	1707797	9	1	4	905702054829
6	1	1	15984640	9	2	2	933207509234
6	1	2	37502976	9	2	3	1531516162891
6	2	2	59021312	9	3	3	1919344441597
7	1	1	253490796	10	1	1	1217507106816
7	1	2	763954092	10	1	2	6306655500288
7	1	3	1069047153	10	1	3	17225362851840
7	2	2	1579510449	10	1	4	28015971489792
8	1	1	4120776000	10	2	2	22314511245312
8	1	2	15274952000	10	2	3	44023827234816
8	1	3	27768048000	10	2	4	54814435872768
8	2	2	38922224000	10	3	3	65733143224320
8	2	3	51415320000				

Table A.2.: Some Gromov-Witten invariants for lines.

Example A.2.18. In this example, we compute the Gromov-Witten invariants for \mathbb{P}^3 as in Example 6.4.8.

```

sage: P = ProjectiveSpace(3)
sage: P.Gromov_Witten_invariant(1, [3]*2)
1
sage: P.Gromov_Witten_invariant(1, [2]*2+[3])
1
sage: P.Gromov_Witten_invariant(1, [2]*4)
2
sage: P.Gromov_Witten_invariant(2, [3]*4)
0
sage: P.Gromov_Witten_invariant(2, [2]*2+[3]*3)
1
sage: P.Gromov_Witten_invariant(2, [2]*4+[3]*2)
4
sage: P.Gromov_Witten_invariant(2, [2]*6+[3])
18
sage: P.Gromov_Witten_invariant(2, [2]*8)
92
sage: P.Gromov_Witten_invariant(3, [3]*6)
1
sage: P.Gromov_Witten_invariant(3, [2]*2+[3]*5)
5
sage: P.Gromov_Witten_invariant(3, [2]*4+[3]*4)
30
sage: P.Gromov_Witten_invariant(3, [2]*6+[3]*3)
190
sage: P.Gromov_Witten_invariant(3, [2]*8+[3]*2)
1312
sage: P.Gromov_Witten_invariant(3, [2]*10+[3])
9864
sage: P.Gromov_Witten_invariant(3, [2]*12)
80160

```

Example A.2.19. In this example, we compute the Gromov-Witten invariants for \mathbb{P}^4 as in Example 6.4.9.

```

sage: P = ProjectiveSpace(4)
sage: P.Gromov_Witten_invariant(1, [4]*2)
1
sage: P.Gromov_Witten_invariant(1, [3]*3)
1
sage: P.Gromov_Witten_invariant(1, [2,3,4])
1
sage: P.Gromov_Witten_invariant(1, [2]*2+[3]*2)
2
sage: P.Gromov_Witten_invariant(1, [2]*3+[4])
1
sage: P.Gromov_Witten_invariant(1, [2]*4+[3])
3

```

```

sage: P.Gromov_Witten_invariant(1, [2]*6)
5
sage: P.Gromov_Witten_invariant(2, [3]+[4]*3)
0
sage: P.Gromov_Witten_invariant(2, [2]*2+[4]*3)
1
sage: P.Gromov_Witten_invariant(2, [2]+[3]*2+[4]*2)
1
sage: P.Gromov_Witten_invariant(2, [3]*4+[4])
2
sage: P.Gromov_Witten_invariant(2, [2]*3+[3]+[4]*2)
4
sage: P.Gromov_Witten_invariant(2, [2]*2+[3]*3+[4])
6
sage: P.Gromov_Witten_invariant(2, [2]+[3]*5)
10
sage: P.Gromov_Witten_invariant(2, [2]*5+[4]*2)
11
sage: P.Gromov_Witten_invariant(2, [2]*4+[3]*2+[4])
21
sage: P.Gromov_Witten_invariant(2, [2]*3+[3]*4)
36
sage: P.Gromov_Witten_invariant(2, [2]*6+[3]+[4])
67
sage: P.Gromov_Witten_invariant(2, [2]*5+[3]*3)
132
sage: P.Gromov_Witten_invariant(2, [2]*8+[4])
219
sage: P.Gromov_Witten_invariant(2, [2]*7+[3]*2)
473
sage: P.Gromov_Witten_invariant(2, [2]*9+[3])
1734
sage: P.Gromov_Witten_invariant(2, [2]*11)
6620

```

Appendix B.

A Singular library for Intersection Theory and Enumerative Geometry

This appendix gives an outline of the SINGULAR library `schubert.lib`, and illustrate some examples in this thesis. In order to use `schubert.lib`, we first must run the LIB command for loading this library in SINGULAR. The file `schubert.lib` is available from the author upon request or at

<https://github.com/hiepdang/Singular>.

Like `Schubert3`, the current version of `schubert.lib` should not be considered a complete, unchangeable, totally stable version. We will keep working on this project in the near future.

Overview

Our goal when writing this library was to make functionality of `Schubert3` available to SINGULAR users. As before, we implemented new classes (variety, sheaf, stack, graph) and methods for computing with them. An abstract variety is represented by a nonnegative integer which is its dimension and a graded ring which is its Chow ring. A sheaf is represented by an abstract variety and a polynomial which is its Chern character. In particular, we also implemented the specific varieties such as projective spaces, Grassmannians, and projective bundles.

An important focus of this library is related to the computation of Gromov-Witten invariants. In particular, we implemented new tools for computations in equivariant intersection theory. These tools are based on the localization theorem and Bott's formula. In order to compute Gromov-Witten invariants, we deal with moduli spaces of stable maps and the graphs corresponding to the fixed point components of a torus action on these moduli spaces.

As an insightful example, the numbers of rational curves on general complete intersection Calabi-Yau threefolds in projective spaces are computed up to degree 6. The results are all in agreement with predictions made from mirror symmetry.

Keywords

Intersection theory, enumerative geometry, Gromov-Witten invariants, Schubert calculus, Bott's formula.

Procedures

<code>mod_init</code>	create new objects in this library
<code>makeVariety</code>	create a variety
<code>printVariety</code>	print procedure for a variety
<code>productVariety</code>	make the product of two varieties
<code>ChowRing</code>	create the Chow ring of a variety
<code>basis</code>	compute a basis for the Chow ring of a variety
<code>dualBasis</code>	compute a dual basis for the Chow ring of a variety
<code>intersectionMatrix</code>	compute the intersection matrix of a variety
<code>BettiNumber</code>	compute the i-th Betti number of a variety
<code>BettiNumbers</code>	compute the Betti numbers of a variety
<code>Grassmannian</code>	create a Grassmannian as a variety
<code>projectiveSpace</code>	create a projective space as a variety
<code>projectiveBundle</code>	create a projective bundle as a variety
<code>integral</code>	degree of a 0-cycle on a variety
<code>makeSheaf</code>	create a sheaf
<code>printSheaf</code>	print procedure for sheaves
<code>rankSheaf</code>	return the rank of a sheaf
<code>totalChernClass</code>	compute the total Chern class of a sheaf
<code>ChernClass</code>	compute the k-th Chern class of a sheaf
<code>topChernClass</code>	compute the top Chern class of a sheaf
<code>totalSegreClass</code>	compute the total Segre class of a sheaf
<code>dualSheaf</code>	make the dual of a sheaf
<code>tensorSheaf</code>	make the tensor of two sheaves
<code>symmetricPowerSheaf</code>	make the k-th symmetric power of a sheaf
<code>quotSheaf</code>	make the quotient of two sheaves
<code>addSheaf</code>	make the direct sum of two sheaves
<code>makeGraph</code>	create a graph from a list of vertices and a list of edges
<code>printGraph</code>	print procedure for graphs
<code>moduliSpace</code>	create a moduli space of stable maps as an algebraic stack
<code>printStack</code>	print procedure for stacks
<code>dimStack</code>	compute the dimension of a stack
<code>fixedPoints</code>	compute the list of graphs corresponding to the fixed point components of a torus action on the stack
<code>contributionBundle</code>	compute the contribution bundle on a stack at a graph
<code>normalBundle</code>	compute the normal bundle on a stack at a graph
<code>sumofquotients</code>	prepare a command for parallel computation
<code>part</code>	compute a homogeneous component of a polynomial
<code>parts</code>	compute the sum of homogeneous components of a polynomial
<code>logg</code>	compute Chern characters from total Chern classes
<code>expp</code>	compute total Chern classes from Chern characters
<code>SchubertClass</code>	compute the Schubert classes on a Grassmannian

B.1. Examples

Example B.1.1. The computation of Example 1.2.5 can be made with `schubert.lib` as follows:

```

> variety X;
> X.dimension = 4;
> ring r = 0,(c(1..2),d(1..3)),wp(1..2,1..3);
> ideal I;
> X.baseRing = r;
> X.relations = I;
> X;
A variety of dimension 4
> poly c = 1+c(1)+c(2);
> poly ch = 2+logg(c,4);
> sheaf E = makeSheaf(X,ch);
> E;
A sheaf of rank 2
> sheaf S = E^3; //the 3rd symmetric power of E
> S;
A sheaf of rank 4
> ChernClass(S,1);
6*c(1)
> ChernClass(S,2);
11*c(1)^2+10*c(2)
> ChernClass(S,3);
6*c(1)^3+30*c(1)*c(2)
> ChernClass(S,4);
18*c(1)^2*c(2)+9*c(2)^2

```

Example B.1.2. The computation of Example 1.2.6 can be made with `schubert.lib` as follows:

```

> variety X;
> X.dimension = 2;
> ring r = 0,(c(1..2),d(1..3)),wp(1..2,1..3);
> ideal I;
> X.baseRing = r;
> X.relations = I;
> X;
A variety of dimension 2
> poly c = 1+c(1)+c(2);
> poly ch1 = 2+logg(c,2);
> sheaf E = makeSheaf(X,ch1);
> E;
A sheaf of rank 2
> poly d = 1+d(1)+d(2)+d(3);
> poly ch2 = 3+logg(d,2);
> sheaf F = makeSheaf(X,ch2);
> F;
A sheaf of rank 3
> sheaf S = E*F; //the tensor of E and F
> S;
A sheaf of rank 6

```

```

> ChernClass(S,1);
3*c(1)+2*d(1)
> ChernClass(S,2);
3*c(1)^2+3*c(2)+5*c(1)*d(1)+d(1)^2+d(2)
> ChernClass(S,3);
0

```

Example B.1.3. Compute the number of lines intersecting 4 given lines in general position in \mathbb{P}^3 (see Example 2.1.8 for more details).

```

> variety G24 = Grassmannian(2,4);
> G24;
A variety of dimension 4
> def r = G24.baseRing;
> setring r;
> integral(G24,q(1)^4);
2

```

Example B.1.4. Compute the number of lines intersecting 6 given planes in general position in \mathbb{P}^4 .

```

> variety G25 = Grassmannian(2,5);
> def r = G25.baseRing;
> setring r;
> integral(G25,q(1)^6);
5

```

We can also solve a similar problem where we switch the roles of the line and the plane: Given 6 lines in general position in \mathbb{P}^4 , computing the number of planes intersect all of them.

```

> variety G35 = Grassmannian(3,5);
> def r = G35.baseRing;
> setring r;
> integral(G35,q(1)^6);
5

```

Example B.1.5. Compute the Schubert classes on Grassmannian $G(2,4)$ (see Example 2.1.6 for more details).

```

> variety G24 = Grassmannian(2,4);
> def r = G24.baseRing;
> setring r;
> list p;
> SchubertClass(p);
1
> list p = 1;
> SchubertClass(p);
q(1)
> list p = 2;
> SchubertClass(p);

```



```

q(2)
> list p = 1,1;
> SchubertClass(p);
q(1)^2-q(2)
> list p = 2,1;
> SchubertClass(p);
q(1)*q(2)
> list p = 2,2;
> SchubertClass(p);
q(2)^2

```

Example B.1.6. Compute the number of secant lines to two general twisted cubic curves in \mathbb{P}^3 (see Example 2.1.9 for more details).

```

> variety G24 = Grassmannian(2,4);
> def r = G24.baseRing;
> setring r;
> list p = 2;
> poly s2 = SchubertClass(p);
> list p = 1,1;
> poly s11 = SchubertClass(p);
> integral(G24, (s2+3*s11)^2);
10

```

Example B.1.7. Compute the Littlewood-Richardson coefficients in the intersection product $\sigma_{2,1} \cdot \sigma_{2,1}$ on the Grassmannian $G(3,6)$ (see Section 2.2 for more details).

```

> variety G36 = Grassmannian(3,6);
> def r = G36.baseRing;
> setring r;
> list a = 2,1;
> poly f = SchubertClass(a);
> list b1 = 3,3;
> poly g1 = SchubertClass(dualPartition(3,6,b1));
> integral(G36, f^2*g1);
1
> list b2 = 3,2,1;
> poly g2 = SchubertClass(dualPartition(3,6,b2));
> integral(G36, f^2*g2);
2
> list b3 = 2,2,2;
> poly g3 = SchubertClass(dualPartition(3,6,b3));
> integral(G36, f^2*g3);
1

```

Example B.1.8. Compute the number of lines on a general cubic hypersurface in \mathbb{P}^3 (see Section 2.4 for more details).

```

> variety G24 = Grassmannian(2,4);
> def r = G24.baseRing;
> setring r;
> sheaf S = makeSheaf(G24,subBundle);
> S; //tautological subbundle
A sheaf of rank 2
> sheaf B = dualSheaf(S)^3;
> B;
A sheaf of rank 4
> integral(G24,topChernClass(B));
27

```

Example B.1.9. Compute the number of lines on a general quintic hypersurface in \mathbb{P}^4 (see Section 2.4 for more details).

```

> variety G35 = Grassmannian(3,5);
> def r = G35.baseRing;
> setring r;
> sheaf Q = makeSheaf(G35,quotientBundle);
> Q; //tautological quotient bundle
A sheaf of rank 2
> sheaf B = Q^5;
> integral(G35,topChernClass(B));
2875

```

More general, we can write a procedure to compute the number of lines on a general hypersurface of degree $2n - 3$ in \mathbb{P}^n .

```

> proc linesHypersurfaces(int n)
{
  variety G = Grassmannian(n-1,n+1);
  def r = G.baseRing;
  setring r;
  sheaf Q = makeSheaf(G,quotientBundle);
  sheaf B = Q^(2*n-3);
  return (integral(G,topChernClass(B)));
}
> ring r = 0,x,dp;
> int n;
> for (n=2;n<=9;n++){linesHypersurfaces(n);}
1
27
2875
698005
305093061
210480374951
210776836330775
289139638632755625

```

These numbers can also be computed by the localization of $\overline{M}_{0,0}(\mathbb{P}^n, 1)$ and Bott's formula as follows:

```

> proc linesHypersurfaces(int n)
{
    variety P = projectiveSpace(n);
    stack M = moduliSpace(P,1);
    def F = fixedPoints(M);
    int i;
    number r;
    for (i=1;i<=size(F);i++)
    {
        number s = contributionBundle(M,F[i][1]);
        number t = normalBundle(M,F[i][1]);
        r = r + s/t;
        kill s,t;
    }
    return (r);
}
> ring r =0,x,dp;
> int n;
> for (n=2;n<=9,n++) {linesHypersurfaces(n);}
1
27
2875
698005
305093061
210480374951
210776836330775
289139638632755625

```

Example B.1.10. Compute the numbers of lines on the general complete intersection Calabi-Yau threefolds using Schubert calculus as in Section 2.6.

```

> proc lines(list D) //use Schubert calculus
{
    int i;
    variety G = Grassmannian(2,size(D)+4);
    def r = G.baseRing;
    setring r;
    sheaf S = makeSheaf(G,subBundle);
    poly c = 1;
    for (i=1;i<=size(D);i++)
    {
        sheaf B = dualSheaf(S)^D[i];
        c = c*topChernClass(B);
        kill B;
    }
    return (integral(G,c));
}

```

```

}
> ring r = 0,x,dp;
> lines(list(5));
2875
> lines(list(4,2));
1280
> lines(list(3,3));
1053
> lines(list(3,2,2));
720
> lines(list(2,2,2,2));
512

```

Compute the numbers of lines on the general complete intersection Calabi-Yau threefolds using Bott's formula as the description in Section 7.5.

```

> proc lines(list D) //use Bott's formula
{
    variety P = projectiveSpace(size(D)+3);
    stack M = moduliSpace(P,1);
    def F = fixedPoints(M);
    int i,j;
    number r;
    for (i=1;i<=size(F);i++)
    {
        number s = 1;
        for (j=1;j<=size(D);j++)
        {
            s = s*contributionBundle(M,F[i][1],list(D[j]));
        }
        number t = F[i][2]*normalBundle(M,F[i][1]);
        r = r + s/t;
        kill s,t;
    }
    return (r);
}
> ring r = 0,x,dp;
> lines(list(5));
2875
> lines(list(4,2));
1280
> lines(list(3,3));
1053
> lines(list(3,2,2));
720
> lines(list(2,2,2,2));
512

```

These numbers agree with Table 2.2.

Example B.1.11. The computation of Example 2.5.3 can be made with `schubert.lib` as follows:

```

> variety P = projectiveSpace(19);
> variety G = Grassmannian(2,4);
> variety W = productVariety(P,G);
> def r = W.baseRing;
> setring r;
> poly s = imap(G.baseRing,subBundle);
> sheaf S = makeSheaf(W,s);
> sheaf B = dualSheaf(S)^3;
> poly ch = 1+logg(1+h,19);
> sheaf L = makeSheaf(W,ch);
> sheaf E = B*L;
> topChernClass(E);
h^4+6*h^3*q(1)+21*h^2*q(1)^2-10*h^2*q(2)+42*h*q(1)*q(2)+27*q(2)^2
> integral(W,topChernClass(E)*h^19);
27
> integral(W,topChernClass(E)*h^18*q(1));
42

```

Example B.1.12. Compute the number of conics in \mathbb{P}^3 meeting 8 general lines. There are two ways to solve this problem (see Section 3.1 for more details).

```

> variety G = Grassmannian(3,4);
> def R = G.baseRing;
> setring R;
> sheaf S = makeSheaf(G,subBundle);
> sheaf B = dualSheaf(S)^2;
> variety PB = projectiveBundle(B);
> def P = PB.baseRing;
> setring P;
> poly L = 2*q(1)+z;
> integral(L^8);
92
> integral(L^7*q(1));
34
> integral(L^6*q(1)^2);
8
> integral(L^5*q(1)^3);
1

> variety G34 = Grassmannian(3,4);
> def R = G34.baseRing;
> setring R;
> sheaf S = makeSheaf(G34,subBundle);
> sheaf Q = makeSheaf(G34,quotientBundle);
> sheaf B = dualSheaf(S)*dualSheaf(Q);
> sheaf E = B^2;

```

```
> totalSegreClass(E);
92*q(1)^3+34*q(1)^2+8*q(1)+1
```

Example B.1.13. Compute the number of conics in \mathbb{P}^3 contained in a general cubic surface and meeting a general line (see Section 3.1 for more details).

```
> variety G = Grassmannian(3,4);
> def R = G.baseRing;
> setring R;
> sheaf S = makeSheaf(G,subBundle);
> sheaf B = dualSheaf(S)^2;
> sheaf B3 = dualSheaf(S)^3;
> variety PB = projectiveBundle(B);
> def P = PB.baseRing;
> setring P;
> poly L = 2*q(1)+z;
> sheaf Q = makeSheaf(PB,QuotientBundle);
> sheaf V = dualSheaf(Q)*dualSheaf(S);
> sheaf A = B3-V;
> poly C = topChernClass(A);
> integral(PB,C*L);
81
```

Compute the number of conics in \mathbb{P}^3 whose supporting planes pass through a general point and which are contained in a general cubic surface.

```
> integral(PB,C*q(1));
27
```

Example B.1.14. Compute the number of conics on a general quintic hypersurface in \mathbb{P}^4 (see Section 3.2 for more details).

```
> variety G = Grassmannian(3,5);
> def R = G.baseRing;
> setring R;
> sheaf S = makeSheaf(G,subBundle);
> sheaf B = dualSheaf(S)^2;
> sheaf B3 = dualSheaf(S)^3;
> sheaf B5 = dualSheaf(S)^5;
> variety PB = projectiveBundle(B);
> def P = PB.baseRing;
> setring P;
> sheaf Q = makeSheaf(PB,QuotientBundle);
> sheaf V = dualSheaf(Q)*B3;
> sheaf A = B5-V;
> integral(PB,topChernClass(A));
609250
```

More generally, we compute the numbers of conics on the general complete intersection Calabi-Yau threefolds as in Section 3.3.

```

> proc conics(list D)
{
  int i;
  variety G = Grassmannian(3,size(D)+4);
  def r = G.baseRing;
  setring r;
  sheaf S = makeSheaf(G,subBundle);
  sheaf B = dualSheaf(S)^2;
  variety PB = projectiveBundle(B);
  def P = PB.baseRing;
  setring P;
  sheaf Q = makeSheaf(PB,QuotientBundle);
  poly c = 1;
  for (i=1;i<=size(D);i++)
  {
    sheaf V = dualSheaf(Q)*(dualSheaf(S)^(D[i]-2));
    sheaf A = dualSheaf(S)^D[i] - V;
    c = c*topChernClass(A);
    kill V,A;
  }
  return (integral(PB,c));
}
> ring r = 0,x,dp;
> conics(list(5));
609250
> conics(list(4,2));
92288
> conics(list(3,3));
52812
> conics(list(3,2,2));
22428
> conics(list(2,2,2,2));
9728

```

Example B.1.15. Compute the number of elliptic cubics on a general sextic hypersurface in \mathbb{P}^5 (compare with [72, Table 2]).

```

> variety G = Grassmannian(3,6);
> def R = G.baseRing;
> setring R;
> sheaf S = makeSheaf(G,subBundle);
> sheaf B = dualSheaf(S)^3;
> variety PB = projectiveBundle(B);
> def P = PB.baseRing;
> setring P;
> sheaf Q = makeSheaf(PB,QuotientBundle);
> sheaf V = dualSheaf(Q)*B;
> sheaf A = dualSheaf(S)^6-V;

```

```
> integral(PB,topChernClass(A));
2734099200
```

Example B.1.16. We illustrate the computation of Section 7.3 as follows:

```
> proc multipleCover(int d)
{
    variety P = projectiveSpace(1);
    stack M = moduliSpace(P,d);
    def F = fixedPoints(M);
    int i;
    number r = 0;
    for (i=1;i<=size(F);i++)
    {
        number s = contributionBundle(M,F[i][1]);
        number t = F[i][2]*normalBundle(M,F[i][1]);
        r = r + s/t;
        kill s,t;
    }
    return (r);
}
> ring r = 0,x,dp;
> int d;
> for (d=1;d<=6;d++) {multipleCover(d);}
1
1/8
1/27
1/64
1/125
1/216
```

Example B.1.17. The computation of Sections 7.4 and 7.5 can be made with `schubert.lib` as follows:

```
> proc rationalCurve(int d, list #)
{
    int n,i,j;
    if (size(#) == 0) {n = 4; list l = 5;}
    else {n = size(#)+3; list l = #;}
    variety P = projectiveSpace(n);
    stack M = moduliSpace(P,d);
    def F = fixedPoints(M);
    number r = 0;
    for (i=1;i<=size(F);i++)
    {
        graph G = F[i][1];
        number s = 1;
        for (j=1;j<=size(l);j++)
        {
```



```

        s = s*contributionBundle(M,G,list(l[j]));
    }
    number t = F[i][2]*normalBundle(M,G);
    r = r + s/t;
    kill s,t,G;
}
return (r);
}
> ring r = 0,x,dp;
> for (int d=1;d<=6;d++) {rationalCurve(d);}
2875
4876875/8
8564575000/27
15517926796875/64
229305888887648
248249742157695375
> list D = list(4,2),list(3,3),list(3,2,2),list(2,2,2,2);
> int i;
> for (i=1;i<=4;i++) {rationalCurve(1,D[i]);}
1280
1053
720
512
> for (i=1;i<=4;i++) {rationalCurve(2,D[i]);}
92448
423549/8
22518
9792
> for (i=1;i<=4;i++) {rationalCurve(3,D[i]);}
422690816/27
6424365
4834592/3
11239424/27
> for (i=1;i<=4;i++) {rationalCurve(4,D[i]);}
3883914084
72925120125/64
672808059/4
25705160
> for (i=1;i<=4;i++) {rationalCurve(5,D[i]);}
29773082054656/25
31223486573928/125
541923292944/25
244747968512/125
> for (i=1;i<=4;i++) {rationalCurve(6,D[i]);}
417874607302656
501287722516269/8
3195558106836
511607926784/3

```

Example B.1.18. We illustrate the calculation of Example A.2.17 by `schubert.lib` as follows:

```
> proc gromovWittenLines(int k, int a, int b)
{
  variety G = Grassmannian(2,k+2);
  def r = G.baseRing;
  setring r;
  sheaf S = makeSheaf(G,subBundle);
  sheaf B = dualSheaf(S)^(k+2);
  poly qa = SchubertClass(list(a-1));
  poly qb = SchubertClass(list(b-1));
  poly qc = SchubertClass(list(k-a-b-1));
  return (integral(G,topChernClass(B)*qa*qb*qc));
}
> ring r = 0,x,dp;
> gromovWittenLines(3,1,1);
2875
> gromovWittenLines(4,1,1);
60480
> gromovWittenLines(5,1,1);
1009792
> gromovWittenLines(5,1,2);
1707797
```

Example B.1.19. Compute the Betti numbers of the product of two Grassmannians.

```
> variety G = Grassmannian(2,5,list("g"));
> variety H = Grassmannian(3,6,list("h"));
> variety W = productVariety(G,H);
> def r = W.baseRing;
> setring r;
> BettiNumbers(W);
1,2,5,9,14,19,24,26,26,24,19,14,9,5,2,1
```

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Declaration

I hereby declare that this thesis is my own work and effort, and that no other sources than those listed have been used.

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