A Combinatorial Problem in Geometry

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Introduction.

Our present problem has been suggested by Miss Esther Klein in connection with the following proposition.

From 5 points of the plane of which no three lie on the same straight line it is always possible to select 4 points determining a convex quadrilateral.

We present E. Klein's proof here because later on we are going to make use of it. If the least convex polygon which encloses the points is a quadrilateral or a pentagon the theorem is trivial. Let therefore the enclosing polygon be a triangle ABC. Then the two remaining points D and E are inside ABC. Two of the given points (say A and C) must lie on the same side of the connecting straight line $\overline{DE}$. Then it is clear that $AEDC$ is a convex quadrilateral.

Miss Klein suggested the following more general problem. *Can we find for a given n a number $N(n)$ such that from any set containing at least N points it is possible to select n points forming a convex polygon?*

There are two particular questions: (1) does the number N corresponding to n exist? (2) If so, how is the least $N(n)$ determined as a function of n? (We denote the least N by $N_0(n)$.)

We give two proofs that the first question is to be answered in the affirmative. Both of them will give definite values for $N(n)$ and the first one can be generalised to any number of dimensions. Thus we obtain a certain preliminary answer to the second question. But the answer is not final for we generally get in this way a number N which is too large. Mr. E. Makai proved that $N_0(5) = 9$, and from our second demonstration, we obtain $N(5) = 21$ (from the first a number of the order 2^{10000}).

Thus it is to be seen, that our estimate lies pretty far from the true limit $N_0(n)$. It is notable that $N(3) = 3 = 2 + 1$, $N_0(4) = 5 = 2^2+1$, $N_0(5) = 9 = 2^3 + 1$.

We might conjecture therefore that $N_0(n) = 2^{n-2} + 1$, but the limits given by our proofs are much larger.

It is desirable to extend the usual definition of convex polygon to include the cases where three or more consecutive points lie on a straight line.

First proof.

The basis of the first proof is a combinatorial theorem of Ramsey [1]). In the introduction it was proved that from 5 points it is always possible to select 4 forming a convex quadrangle. Now it can be easily proved by induction that n points determine a convex polygon if and only if any 4 points of them form a convex quadrilateral.

Denote the given points by the numbers $1, 2, 3, \ldots, N$, then any k-gon of the set of points is represented by a set of k of these numbers, or as we shall say, by a k-combination. Let us now suppose each n-gon to be concave, then from what we observed above we can divide the 4-combinations into two classes (i. e. into „convex" and „concave" quadrilaterals) such that every 5-combination shall contain at least one „convex" combination and each n-combination at least one concave one. (We regard one combination as contained in another, if each element of the first is also an element of the second.)

From Ramsey's theorem, it follows that this is impossible for a sufficiently large N.

Ramsey's theorem can be stated as follows:

Let k, l, i be given positive integers, $k \geqq i$; $l \geqq i$. Suppose that there exist two classes, α and β, of i-combinations of m elements such that each k-combination shall contain at least one combination from class α und each l-combination shall contain at least one combination from class β. Then for sufficiently great $m < m_i(k, l)$ this is not possible. Ramsey enunciated his theorem in a slightly different form.

In other words: if the members of α had been determined as above at our discretion and $m \geqq m_i(k, l)$, then there must be at least one l-combination with every combination of order i belonging to class α.

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[1]) F. P. Ramsey, Collected papers. On a problem of formal logic, 82—111. Recently Skolem also proved Ramsey's theorem [Fundamenta Math. **20** (1933), 254—261].

We give here a new proof of Ramsey's theorem, which differs entirely from the previous ones and gives for $m_i(k, l)$ slightly smaller limits.

a) If $i = 1$, the theorem holds for every k and l. For if we select out of m some determined elements (combinations of order 1) as the class α, so that every k-gon (this shorter denomination will be given to the combination of order k) must contain at least one of the α elements, there are at most $(k-1)$ elements which do not belong to the class α. Then there must be at least $(m-k+1)$ elements of α. If $(m-k+1) \geqq l$, then there must be an l-gon of the α elements and thus

$$m \leqq k + l - 2$$

which is evidently false for sufficiently great m.

Suppose then that $i > 1$.

b) The theorem is trivial, if k or l equals i. If, for example, $k = i$, then it is sufficient to choose $m = l$.

For $k = 1$ means that all i-gons are α combinations and thus in virtue of $m = l$ there is one polygon (i. e. the l-gon formed of all the elements), whose i-gons are all α-combinations.

The argument for $l = i$ runs similarly.

c) Suppose finally that $k > i$; and suppose that the theorem holds for $(i-1)$ and every k and l, further for i, k, $l-1$ and i, $k-1$, l. We shall prove that it will hold for i, k, l also and in virtue of (a) and (b) we may say that the theorem is proved for all i, k, l.

Suppose then that we are able to carry out the division of the i-polygons mentioned above. Further let k' be so great that if in every l-gon of k' elements there is at least one β combination, then there is one $(k-1)$-gon all of whose i-gons are β combinations. This choice of k' is always possible in virtue of the induction-hypothesis, we have only to choose $k' = m_i(k-1, l)$.

Similarly we choose l' so great that if each k-gon of l' elements contains at least one α combination, then there is one $(l-1)$-gon all of whose i-gons are α combinations.

We then take m larger than k' and l'; and let

$$(a_1, a_2, \ldots, a_{k'}) \equiv A$$

be an arbitrary k'-gon of the first $(n-1)$ elements. By hypothesis each l-gon contains at least one β combination, hence owing to the choice of k', A contains one $(k-1)$-gon $(a_{m_1}, a_{m_2}, \ldots, a_{m_{k-1}})$ whose i-gons all belong to the class β. Since in $(a_{m_1}, \ldots, a_{m_{k-1}}, n)$

there is at least one α combination, it is clear that this must be one of the i-gons

$$(a_{p_1}, a_{p_2}, \ldots, a_{p_{i-1}}, n) \equiv B.$$

In just the same way we may prove by replacing the roles of k and l by k' and l' and of α by β, that if

$$(b_1, b_2, \ldots, b_{l'}) \equiv A'$$

is an arbitrary l'-gon of the first $(n-1)$ elements, then among the i-gons

$$(b_{r_1}, b_{r_2}, \ldots, b_{r_{i-1}}, n) \equiv B'$$

there must be a β combination.

Thus we can divide the $(i-1)$-gons of the first $(n-1)$ elements into classes α' and β' so that each k'-gon A shall contain at least one α' combination B and each l'-gon A' at least one β' combination B'. But, by the induction-hypotheses this is impossible for $m \geqq m_{i-1}(k'l') + 1$.

By following the induction, it is easy to obtain for $m_i(k, l)$ the following functional equation;

$$m_i(k, l) = m_{i-1}\left[m_i(k-1, l),\ m_i(k, l-1)\right] + 1. \qquad (1)$$

By this recurrence-formula and the initial values

$$\left.\begin{aligned} m_1(k, l) &= k + l - 1 \\ m_i(i, l) = l,\ \ m_i(k, i) &= k \end{aligned}\right\} \qquad (2)$$

obtained from (a) and (b) we can calculate every $m_i(k, l)$.

We obtain e. g. easily

$$m_2(k+1, l+1) = \binom{k+l}{k}. \qquad (3)$$

The function mentioned in the introduction has the form

$$N(k) = m(5, k). \qquad (4)$$

Finally, for the special case $i = 2$, we give a graphotheoretic formulation of Ramsey's theorem and present a very simple proof of it.

THEOREM: *In an arbitrary graph let the maximum number* of *independent points* [2]) *be* k; *if the number of points is* $N \geqq m(k, l)$ *then there exists in our graph a complete graph* [3]) *of order* l.

[2]) Two points are said to be independent if they are not connected; k points are independent if every pair is independent.

[3]) A complete graph is one in which every pair of points is connected.

Proof. For $l = 1$, the theorem is trivial for any k, since the maximum number of independent points is k and if the number of points is $(k+1)$, there must be an edge (complete graph of order 1).

Now suppose the theorem proved for $(l-1)$ with any k. Then at least $\frac{N-k}{k}$ edges start from one of the independent points. Hence if

$$\frac{N-k}{k} \geqq m(k, l-1),$$

i. e.,

$$N \geqq k \cdot m(k, l-1) + k, \tag{5}$$

then, out of the end points of these edges we may select, in virtue of our induction hypothesis, a complete graph whose order is at least $(l-1)$. As the points of this graph are connected with the same point, they form together a complete graph of order l.

Second proof.

The foundation of the second proof of our main theorem is formed partly by geometrical and partly by combinatorial considerations. We start from some similar problems and we shall see, that the numerical limits are more accurate then in the previous proof; they are in some respects exact.

Let us consider the first quarter of the plane, whose points are determined by coordinates (x, y). We choose n points with monotonously increasing abscissae [4]).

Theorem: *It is always possible to choose at least $\sqrt{n}$ points with increasing abscissae and either monotonously increasing or monotonously decreasing ordinates.* If two ordinates are equal, the case may equally be regarded as increasing or decreasing.

Let us denote by $f(n, n)$ the minimum number of the points out of which we can select n monotonously increasing or decreasing ordinates.

We assert that

$$f(n+1, n+1) = f(n, n) + 2n - 1. \tag{6}$$

Let us select n monotonously increasing or decreasing points out of the $f(n, n)$. Let us replace the last point by one of the $(2n-1)$ new points. Then we shall have once more $f(n, n)$ points, out

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[4]) The same problem was considered independently by Richard Rado.

of which we can select as before n monotonous points. Now we replace the last point by one of the new ones and so on. Thus we obtain $2n$ points each an endpoint of a monotonous set. Suppose that among them $(n+1)$ are end points of monotonously increasing sets. Then if $y_l \geqq y_k$ for $l > k$ we add P_l to the monotonously increasing set of P_k and thus, with it, we shall have an increasing set of $(n+1)$ points. If $y_k \geqq y_l$ for every $k < l$, then the $(n+1)$ decreasing end-points themselves give the monotonous set of $(n+1)$ members. If between the $2n$ points there are at least $(n+1)$ end-points of monotonous decreasing sets, the proof will run in just the same way.

But it may happen that, out of the $2n$ points, just n are the end-points of increasing sets, and n the end-points of decreasing sets. Then by the same reasoning, the end-points of the decreasing sets necessarily increase. But after the last end-point P there is no point, for its ordinate would be greater or smaller than that of P. If it is greater, then together with the n end-points it forms a monotonously increasing $(n+1)$ set and if it is smaller, with the n points belonging to P, it forms a decreasing set of $(n+1)$ members. But by the same reasoning the last of the n increasing end-points Q ought to be also an extreme one and that is evidently impossible. Thus we may deduce by induction

$$f(n+1, n+1) = n^2 + 1. \tag{7}$$

Similarly let $f(i, k)$ denote the minimum number of points out of which it is impossible to select either i monotonously increasing or k monotonously decreasing points. We have then

$$f(i, k) = (i-1)(k-1) + 1. \tag{8}$$

The proof is similar to the previous one.

It is not difficult to see, that this limit is exact i. e. we can give $(i-1)(k-1)$ points such that it is impossible to select out of them the desired number of monotonously increasing or decreasing ordinates.

We solve now a similar problem:

$P_1, P_2, \ldots$ are given points on a straight line. Let $f_1(i, k)$ denote the minimum number of points such that proceeding from left to right we shall be able to select either i points so that the distances of two neighbouring points monotonously increase or k points so that the same distances monotonously decrease. We assert that

$$f_1(i, k) = f_1(i-1, k) + f_1(i, k-1) - 1. \tag{9}$$

Let the point C bisect the distance $\overline{AB}$ (A and B being the first and the last points). If the total number of points is $f_1(i-1, k) + f_1(i, k-1) - 1$, then either the number of points in the first half is at least $f_1(i-1, k)$, or else there are in the second half at least $f_1(i, k-1)$ points. If in the first half there are $f_1(i-1, k)$ points then either there are among them k points whose distances, from left to right, monotonously decrease and then the equation for $f_1(i, k)$ is fulfilled, or there must be $(i-1)$ points with increasing distances. By adding the point B, we have i points with monotonely increasing distances. If in the second interval there are $f_1(i, k-1)$ points, the proof runs in the same way. (The case, in which two distances are the same, may be classed into either the increasing or the decreasing sets.)

It is possible to prove that this limit is exact. If the limits $f_1(i-1, k)$ and $f_1(i, k-1)$ are exact (i. e. if it is possible to give $[f_1(i-1, k) - 1]$ points so that there are no $(i-1)$ increasing nor k decreasing distances) then the limit $f_1(i, k)$ is exact too. For if we choose e. g. $[f_1(i-1, k) - 1]$ points in the $0 \ldots 1$ interval, and $[f_1(i, k-1) - 1]$ points in the $2 \ldots 3$ interval, then we have $[f_1(i, k) - 1]$ points out of which it is equally impossible to select i points with monotonously increasing and k points with decreasing distances.

We now tackle the problem of the convex n-gon. If there are n given points, there is always a straight line which is neither parallel nor perpendicular to any join of two points. Let this straight line be e. Now we regard the configuration $A_1A_2A_3A_4 \ldots$ as convex, if the gradients of the lines A_1A_2, A_2A_3, $\ldots$ decrease monotonously, and as concave if they increase monotonously. Let $f_2(i, k)$ denote the minimum number of the points such that from them we may pick out either i sided convex or k-sided concave configurations. We assert that

$$f_2(i, k) = f_2(i-1, k) + f_2(i, k-1) - 1. \tag{10}$$

We consider the first $f_2(i-1, k)$ points. If out of them there can be taken a concave configuration of k points then the equation for $f_2(i, k)$ is fulfilled. If not, then there is a convex configuration of $(i-1)$ points. The last point of this convex configuration we replace by another point. Then we have once more either k concave points and then the assertion holds, or $(i-1)$ convex ones. We go on replacing the last point, until we have made use of all points. Thus we obtain $f_2(i, k-1)$ points, each of which is an end-point of a convex configuration of $(i-1)$

elements. Among them, there are either i convex points and then our assertion is proved, or $(k-1)$ concave ones. Let the first of them be A_1, the second A_2. A_1 is the end-point of a convex configuration of $(i-1)$ points. Let the neighbour of A_1 in this configuration be B. If the gradient of BA_1 is greater than that of A_1A_2, then A_2 together with the $(i-1)$ points form a convex configuration; if the gradient is smaller, then B together with $A_1A_2\dots$ form a concave k-configuration. This proves our assertion.

The deduction of the recurrence formula may start from the statement: $f_2(3, n) = f_2(n, 3) = n$ (by definition). Thus we easily obtain

$$f_2(k, k) = \binom{2k-4}{k-2} + 1. \tag{11}$$

As before we may easily prove that the limit given by (11) is exact, i. e. it is possible to give $\binom{2k-4}{k-2}$ points such, that they contain neither convex nor concave k points.

Since by connection of the first and last points, every set of k convex or concave points determines a convex k-gon it is evident that $\left[\binom{2k-4}{k-2} + 1\right]$ points always contain a convex k-gon.

And as in every convex $(2k-1)$ polygon there is always either a convex or a concave configuration of k points, it is evident that it is possible to give $\binom{2k-4}{k-2}$ points, so that out of them no convex $(2k-1)$ polygon can be selected. Thus the limit is also estimated from below.

Professor D. König's lemma [5]) of infinity also gives a proof of the theorem that if k is a definite number and n sufficiently great, the n points always contain a convex k-gon. But we thus obtain a pure existence-proof, which allows no estimation of the number n. The proof depends on the statement that if M is an infinite set of points we may select out of it another convex infinite set of points.

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[5]) D. König, Über eine Schlußweise aus dem Endlichen ins Unendliche [Acta Szeged **3** (1927), 121—130].