A Capacitor Paradox

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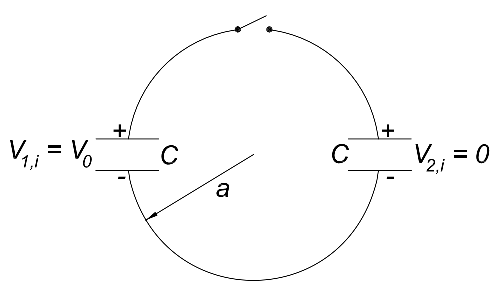
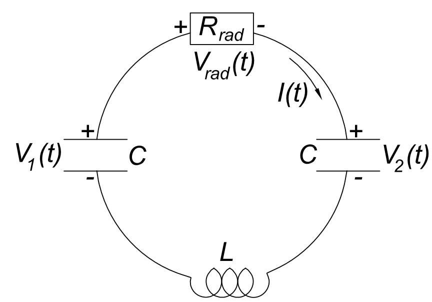
1 Problem

Two capacitors of equal capacitance C are connected in parallel by wires of negligible resistance and a switch, as shown in the lefthand figure below. Initially the switch is open, one capacitor is charged to voltage V_0, and charge $Q_0 = CV_0$, while the other is uncharged. At time $t = 0$ the switch is closed. If there were no damping (dissipative) mechanism, the circuit would then oscillate forever, at a frequency dependent on the self inductance $L \approx \mu_0 \ln a/b$ of the loop of radius a of wire of radius $b \ll a$ and the total capacitance $C_{\rm tot} \approx C/2$, namely $f = \omega/2\pi \approx 1/2\pi\sqrt{LC_{\rm tot}} \approx 1/2\pi\sqrt{(\mu_0 C/2)\ln(a/b)} \approx 200$ Hz$/\sqrt{C\ln(a/b)}$ for C in farards. However, even in a circuit with zero Ohmic resistance, damping occurs due to the radiation of the oscillating charges, and eventually a static charge distribution results, with charge $Q_{\rm i}/2$ and voltage $V_{\rm i}/2$, on each capacitor.





The "paradox" is that the final stored energy is $U_{\rm f} = 2(CV_{\rm f}^2/2) = CV_0^2/4 = U_{\rm i}/2$, where $U_{\rm i} = CV_0^2/2$ is the initial stored energy.[1] Hence, half the initial energy is "missing" in the final state.

Where is the "missing" energy?[2]

2 Solution

This problem is (in the view of this author) meant to illustrate the limitations of "ordinary" circuit analysis,[3] and has been discussed many times, including [1]-[38]. A substantial fraction of these papers argue that "ordinary" circuit analysis suffices for a practical understanding of the two-capacitor problem, remarking that if the circuit contains a large enough

[1]If the two capacitances were unequal, more than half of the initial energy would go "missing". Better energy efficiency while charging a capacitor can be obtained using nonlinear circuit elements, as in sec. 9.1 of http://www.ti.com/lit/ds/symlink/lm2664.pdf.

[2]This problem can also be posed for a single capacitor that is initially charged with $\pm Q$ on its plates, and then discharged by "shorting" its terminals with a wire. This can be dangerous, so "don't try this at home". That is, a spark generally occurs during the discharge, which is a clue that the physics here can be intricate. The experiment discussed in sec. 2.3 below is for the single-capacitor version of the "paradox".

[3]Another example that illustrates the limitations of "ordinary" circuit analysis is [39].

Ohmic resistance, the associated Joule heating accounts for essentially all of the "missing" energy.[4]

Recall that in Poynting's view [40], the energy the energy that is transferred from one capacitor to the other passes through the intervening space, not down the connecting wires. In the present example, some of the energy in the electrostatic field of the initially charged capacitor "escapes" from the circuit in the form of electromagnetic radiation.[5] Hence, we should examine the possibility that radiation carries away a significant fraction of the "missing" energy.[6]

2.1 Ordinary Circuit Analysis of the Two-Capacitor Problem

If the quantity labeled $R_{\rm rad}$ in the circuit diagram on p. 1 were an ordinary resistor of value R, then the circuit equation would be,

$$-V_1 + V_2 + L\dot{I} + IR = 0, \qquad \frac{Q}{C} + \frac{L\ddot{Q}}{2} + \frac{R\dot{Q}}{2} = 0, \tag{1}$$

where $L \approx \mu_0 \ln(a/b)$ is the self inductance of the circuit, $Q \equiv Q_2 - Q_1$ and we note that $I = \dot{Q}_2 = -\dot{Q}_1 = \dot{Q}/2$. The initial conditions are $Q(0) = -Q_1(0) = -CV_0$ and $\dot{Q}(0) = 0$. Use of a trial solution of the form $e^{i\omega t}$ leads to

$$\omega = \frac{iR}{2L} \pm \omega_0, \qquad \omega_0 \equiv \sqrt{\frac{2}{LC} - \frac{R^2}{4L^2}}, \tag{2}$$

so the (real) solution that obeys the initial conditions can be written as

$$Q(t) = -CV_0\, e^{-Rt/2L}\left(\cos\omega_0 t + \frac{R}{2L\omega_0}\sin\omega_0 t\right), \qquad I(t) = \frac{\dot{Q}}{2} = \frac{V_0}{L\omega_0}\, e^{-Rt/2L}\sin\omega_0 t. \tag{3}$$

The energy dissipated by the resistor R is

$$\begin{aligned} \Delta U &= \int_0^\infty I^2 R\, dt = \frac{V_0^2 R}{L^2\omega_0^2}\int_0^\infty e^{-Rt/L}\sin^2\omega_0 t\, dt = \frac{V_0^2 R}{L^2\omega_0^2}\frac{2\omega_0^2}{(R/L)(R^2/L^2 + 4\omega_0^2)} \\ &= \frac{CV_0^2}{4} = \frac{U_{\rm i}}{2}, \end{aligned} \tag{4}$$

[4] The YouTube video https://www.youtube.com/watch?v=cmverrUVOQA adds a motor + mechanical load to the circuit, so that the "missing" energy can be "seen" as the mechanical work done after the switch is closed, thereby avoiding the need to consider radiation or even Joule heating, which concepts the video author finds too abstract. Yet, this author gets it right that a dissipative mechanism is required for the circuit to end up with only half of the initial stored field energy.

[5] As noted in [13], when a capacitor is discharged near a radio, the latter detects a burst of noise at any frequency, associated with the initial "switching" transients that last a few nsec. See also the caption of Fig. 3 of [7]. For additional commentary on this phenomenon, see [41].

[6] Some authors [12, 17, 18, 20, 21, 30, 31, 32, 34] have argued that the two-capacitor problem is analogous to the "two-tank problem," in which water is transfered from one tank to another via a connecting pipe (although this "plumbing analogy" was objected to already in [13]). If the water were frictionless, the eventual "missing" potential energy (*i.e.*, gravitational-field energy) would be radiated away by gravitational waves. Since this is a very weak process, the frictionless water would oscillate from one tank to the other for a very long time, before eventually coming to equilibrium with each tank half full. In practice, the friction (viscosity) of water is large enough that there would be no observable oscillation of the water (*i.e.*, overdamped "oscillation").

using Dwight 861.10.[7] Thus, if the voltage drop associated with the dissipative mechanism has the form IR for a constant R, the dissipated energy equals the "missing" energy $U_{\rm i}/2 = CV_0^2/4$. It does not, however, follow that this demonstrates R to be purely an Ohmic resistance.

Indeed, for low Ohmic resistance, the current in the circuit would perform a damped oscillation with nominal angular frequency $\omega_0 \approx \sqrt{2/LC}$, and the associated electric and magnetic dipole radiation would have power well described by $P_{\rm rm}(t) = I^2(t) R_{\rm rad}$ where $R_{\rm rad}$ is a constant with dimensions of electrical resistance.

2.2 Model Calculation of Magnetic Dipole Radiation

We assume that the wires form a circle of radius a and we neglect charge accumulation in the wires compared to that on the capacitor plates. In this approximation the current in the wires is spatially uniform, and the total electric dipole moment of the system (with symmetrically arrayed capacitors) is constant. Then, electric dipole radiation does not exist, and magnetic dipole radiation dominates.

The "radiation resistance" of this circuit causes a voltage drop $V_{\rm rad}$ within the circuit that can be identified as

$$V_{\rm rad}(t) = \frac{P_{\rm rad}(t)}{I(t)} = I(t)\frac{P_{\rm rad}(t)}{I^2(t)} \equiv I(t) R_{\rm rad}\,, \tag{5}$$

where $P_{\rm rad}$ is the radiated power, $I(t)$ is the current in the wire, and the radiation resistance is $R_{\rm rad} = P/I^2$ The latter is constant in the further approximation that the damping time is large compared to the period of oscillation of the current, *i.e.*, $\ddot I \approx -\omega_0^2 I \approx 2I/LC$.

To estimate the radiated power we note that the magnetic moment m of the circuit is (in Gaussian units)

$$m(t) = \frac{\pi a^2 I(t)}{c}\,, \tag{6}$$

where c is the speed of light. According to the Larmor formula [42], the radiated power is

$$P_{\rm rad} = \frac{2\ddot m^2}{3c^3} = \frac{2\pi^2 a^4 \ddot I^2}{3c^5} \approx \frac{2\pi^2 a^4 \omega_0^4 I^2}{3c^5}\,. \tag{7}$$

The radiation resistance is

$$R_{\rm rad} = \frac{P_{\rm rad}}{I^2} \approx \frac{2\pi^2}{3c}\left(\frac{a\omega_0}{c}\right)^4 = \frac{2^5\pi^6}{3c}\left(\frac{a}{\lambda}\right)^4 \approx 3\times 10^5 \left(\frac{a}{\lambda}\right)^4\ \Omega, \tag{8}$$

noting that $\omega_0 = 2\pi c/\lambda$, and $1/c$ in Gaussian units equals 30 Ω.

While this radiation resistance appears large at first glance, in practice a/λ (the ratio of the size of the circuit compared to the wavelength of the radiation) will be quite small, and the circuit would oscillate a very long time before the "missing" energy $CV_0^2/4$ would be radiated away.

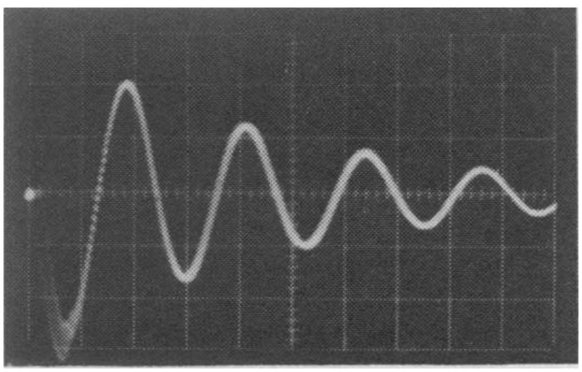
[7] In [1], the self inductance was ignored, so the resulting circuit equation $\dot Q = -2Q/RC$ has the solution $Q(t) = -CV_0\, e^{-2t/RC}$, with $I(t) = \dot Q/2 = (V_0/R)\, e^{-2t/RC}$. Then, the total energy dissipated by the resistor R is $\Delta U = \int_0^\infty I^2 R\, dt = (V_0^2/R)\int_0^\infty e^{-4t/RC}\, dt = CV_0^2/4 = U_{\rm i}/2$.

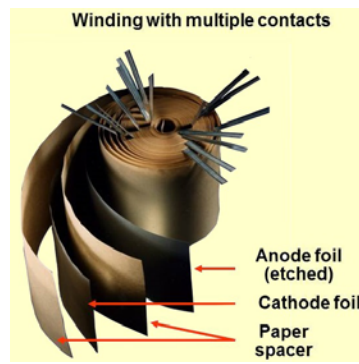
2.3 An Experiment

Hence, it is useful to consider the only experimental data in the literature related to the two-capacitor problem, in Fig. 3 of [7], shown on left on the next page, where the current trace has 10 μs per horizontal (time) division. This experiment was on the "short circuit" discharge of a single capacitor with $C = 11.5\ \mu$F, where the observed frequency of the damped oscillations was $f = 41$ kHz ($\lambda = 7.3 \times 10^5$ cm), and the damping time was observed to be approximately two periods, $\tau \approx 2/f$. Considering the equivalent circuit to be a series R-L-C circuit, where the charge on the capacitor varies as $Q = Q_0\, e^{i\omega t}$, the (complex) angular frequency ω is

$$\omega = \frac{1}{\sqrt{LC}}\sqrt{1 - \frac{R^2 C}{4L}} + \frac{iR}{2L} \approx \frac{1}{\sqrt{LC}} + \frac{iR}{2L} = 2\pi f + \frac{i}{\tau}, \tag{9}$$

where the approximation holds for small resistance R, as holds for this example. Then the observed frequency implies that the self inductance of the circuit was $L = 1/4\pi^2 f^2 C = 1.3\ \mu$H (consistent with the circuit being a loop of 2-cm radius made of 24-gauge wire), and the observed damping time implies that the effective resistance was $R = fL = 0.05\ \Omega$.





The wires in the circuit were stated to be "very short," such that it is implausible that the Ohmic resistance of the circuit was 0.05 Ω (for example, the resistance of 2000 feet of 24-gauge wire is 0.05 Ω). However, the *conventional capacitor contained a "rolled up" sandwich of foil and dielectric*, for which the equivalent series resistance of the thin foil was very plausibly close to the observed 50 mΩ.[8] In contrast, the radiation resistance (8) is only $1.7 \times 10^{-17}\ \Omega$ for $a = 2$ cm and $\lambda = 7.3 \times 10^5$ cm.

This supports the view in many of the discussions of the two-capacitor problem [1]-[38] that the Ohmic resistance of the circuit dissipates the vast majority of the "missing" energy (unless, of course, the electrical circuit is used to drive a nonelectrical load that dissipates the energy, as in footnote 2).

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