Short Introduction to Fractional Calculus

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Abstract

In the past few years fractional calculus appeared as an important tool to deal with anomalous diffusion processes. An anomalous diffusion process can be visualized as an ant in a labyrinth where the average square of the distance covered by the ant is $\langle x^2(t) \rangle \propto t^{2\mu}$ where μ is a phenomenological constant; for $\mu = 1/2$ we have the ordinary diffusion processes. A more physical approach of anomalous diffusion processes has several applications in many field such as diffusion in porous media or long range correlation of DNA sequence. In this short report we shall present an introductory view of the the mathematical aspects of fractional calculus and its basic foundation.

1 Elementary properties of fractional derivatives

The concept of derivative is traditionally associated to an integer; given a function, we can derive it one, two, three times and so on. It can be have an interest to investigate the possibility to derive a real number of times a function. The main idea is to examine the properties of the ordinary derivative and see where and how it is possible to generalize the concepts. As often happen there is not only a way to do that; we are going to use the most intuitive and, in a certain sense, less rigorous way. Let us consider the general properties of the derivative D_t^n for $n \in \mathcal{N}$, where n is an integer. This operator is, in fact, defined to have the following properties, all of which we would like the fractional derivative to share. The first property of interest is that of association

$$D_t^n [C f(t)] = C D_t^n [f(t)] \tag{1}$$

where C is a constant. The second property we would like to incorporate into the fractional calculus is the distributive law

$$D_t^n [f(t) \pm g(t)] = D_t^n [f(t)] \pm D_t^n [g(t)] \,. \tag{2}$$

The final property is that the operator obeys Leibniz rule for taking the derivative of the product of two functions

$$\begin{aligned} D_t^n [f(t)\, g(t)] &= \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} D_t^{n-k} [f(t)]\, D_t^k [g(t)] \\ &= \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} D_t^{n-k} [g(t)]\, D_t^k [f(t)] \end{aligned} \tag{3}$$

where $\begin{pmatrix} n \\ k \end{pmatrix} = \frac{n!}{k!(n-k)!}$ is the binomial coefficient. The above properties are certainly retained for the nth derivative of a monomial t^m with $m \in \mathcal{N}$, so that

$$D_t^n [t^m] = m(m-1)(m-2)\cdots(m-n+1)\, t^{m-n} = \frac{m!}{(m-n)!} t^{m-n} \tag{4}$$

for $m > n$. Properties (1) and (2) establish that the operator D_t^n is linear and (4) enables us to compute the $n-th$ derivative of an analytic function expressed in terms of a Taylor's series.

We now extend these considerations to fractional derivatives. Looking at Eq. (4) the most easy thing would be to replace the integer numbers with real numbers. The main difficulty is how to replace the factorial function that is defined for integer numbers. Fortunately it is exists a special function, the gamma function, that has this property. The gamma function is defined as:

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt. \tag{5}$$

The integral (5) is defined for $z > 0$ (or $Re[z] > 0$ if z is a complex number) and can be checked by elementary integration that for z integer this function coincide with the factorial; more precisely it holds: $\Gamma(n+1) = n!$ We are ready now to define a real-indexed derivative, or more generally, a complex-indexed derivative D_t^α with $\alpha \in \mathcal{R}$ (or $\alpha \in \mathcal{C}$), of a monomial t^β, as

$$\frac{d^\alpha}{dt^\alpha}\left[t^\beta\right] \equiv D_t^\alpha\left[t^\beta\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} t^{\beta-\alpha} \tag{6}$$

where $\beta + 1 \neq 0, -1, \cdots, -n$. It can be proved, via Eq. (3) known as Leibniz rule, that this equation can be generalized to fractional derivatives as

$$\begin{aligned} D_t^\alpha\left[f(t)\,g(t)\right] &= \sum_{k=0}^{\infty} \left(\begin{array}{c} \alpha \\ k \end{array}\right) D_t^{\alpha-k}\left[f(t)\right] D_t^k\left[g(t)\right] \\ &= \sum_{k=0}^{\infty} \left(\begin{array}{c} \alpha \\ k \end{array}\right) D_t^{\alpha-k}\left[g(t)\right] D_t^k\left[f(t)\right] \end{aligned} \tag{7}$$

where we use the binomial coefficient

$$\left(\begin{array}{c} \alpha \\ k \end{array}\right) = \frac{\Gamma(\alpha+1)}{\Gamma(k+1)\,\Gamma(\alpha+1-k)}.$$

and since α is not integer the upper limit of the sum in (7) is infinite. If one of the functions in the product is a constant, say $g(t) = C$, then (7) reduces to

$$\begin{aligned} D_t^\alpha\left[f(t)\,C\right] &= \sum_{k=0}^{\infty} \left(\begin{array}{c} \alpha \\ k \end{array}\right) D_t^{\alpha-k}\left[f(t)\right] D_t^k\left[C\right] \\ &= D_t^\alpha\left[f(t)\right] C \end{aligned} \tag{8}$$

since only the $k = 0$ term survives in the series because the integer derivatives of the constant vanish. Thus, property (1) is retained by the generalized Leibniz rule (7).

Other properties of ordinary derivative that hold for the fractional derivative can be found using Eq. (7); in particular:

$$\begin{aligned} D_t^\alpha\left[h(t)+g(t)\right] &= \sum_{k=0}^{\infty} \left(\begin{array}{c} \alpha \\ k \end{array}\right) D_t^{\alpha-k}\left[t^0\right] D_t^k\left[h(t)+g(t)\right] \\ &= \sum_{k=0}^{\infty} \left(\begin{array}{c} \alpha \\ k \end{array}\right) D_t^{\alpha-k}\left[t^0\right] D_t^k\left[h(t)\right] + \\ &\quad \sum_{k=0}^{\infty} \left(\begin{array}{c} \alpha \\ k \end{array}\right) D_t^{\alpha-k}\left[t^0\right] D_t^k\left[g(t)\right] \\ &= D_t^\alpha\left[h(t)\right] + D_t^\alpha\left[g(t)\right] \end{aligned} \tag{9}$$

and, in similar way, we can show that:

$$D_t^\alpha \left[f\left(at\right)\right] = a^\alpha D_x^\alpha \left[f\left(x\right)\right] \ , \, x = at. \tag{10}$$

Thus, we see that the associative property is also true for the fractional derivative D_t^α. Further, Eqs. (8) and (9), taken together, establish that the fractional derivative is a linear operator.

Now let us consider the case where the index of the monomial is negative integer valued: $\beta + 1 = 0, -1, \cdots, -n$ and we operate with the ordinary integer derivative. Consider the monomial function $f\left(t\right) = t^{-m}$ with m a positive definite integer, from which we obtain

$$D_t^n \left[t^{-m}\right] = \left(-1\right)^n m\left(m+1\right)\cdots\left(m+n-1\right) t^{-(m+n)}$$

or using the properties of gamma functions

$$D_t^n \left[t^{-m}\right] = \left(-1\right)^n \frac{\Gamma\left(m+n\right)}{\Gamma\left(m\right)} t^{-(m+n)} \tag{11}$$

with $n \in \mathcal{N}$. If we restrict ourselves to real indices, then again proceeding by analogy we write for $0 < \alpha < 1$,

$$D_t^\alpha \left[t^{-m}\right] = \left(-1\right)^\alpha \frac{\Gamma\left(m+\alpha\right)}{\Gamma\left(m\right)} t^{-(m+\alpha)}, \tag{12}$$

but we have to change the definition of the gamma functions when the argument in the numerator is a negative integer. This new definition transforms real functions into complex functions and vice versa, because there is the complex factor $\left(-1\right)^\alpha = e^{i\alpha\pi}$. We shall have occasion to use (12).

2 Constant functions

We define $\mathcal{A}\left(\alpha\right)$ to be the set of constant functions under the real indexed derivative D_t^α and $\mathcal{C}\left(\alpha\right)$ is the generic constant of index α. So, for example, we consider the two functions: $f\left(t\right) = t^{-1/2}$ and $f\left(t\right) = C$ and use the derivative of the monomial (6) to obtain:

$$D_t^{1/2} \left[t^{-1/2}\right] = \frac{\Gamma\left(1/2\right)}{\Gamma\left(0\right)} t^{-1} = 0 \tag{13}$$

since $\Gamma\left(0\right) = \infty$. Thus, a particular function is effectively a "constant" with regard to a certain fractional derivative. In the second example

$$D_t^{1/2} \left[C\right] = C \frac{\Gamma\left(1\right)}{\Gamma\left(1/2\right)} t^{-1/2} = \frac{C}{\sqrt{\pi t}} \tag{14}$$

where we see that a constant is not "constant" with regard to fractional derivatives. These two examples demonstrate that there are functions that, under real-indexed derivatives, are additive constants and additive constants that, under real-indexed derivatives, are functions. This functions, that behave as constant under fractional derivative, can destroy the composition property of the index of derivation. In fact, let $f(t)$ be a function having a power series representation and assume that there exists derivatives $D_t^{\mu}[f(t)]$, $D_t^{\nu}[f(t)]$ and $D_t^{\alpha}[f(t)]$ with $\alpha = \mu + \nu$; if $f(t)$ does not contain function that are constant for the derivative operator D_t^{μ} and D_t^{ν} then

$$D_t^{\alpha}[f(t)] = D_t^{\mu+\nu}[f(t)] = D_t^{\mu}[D_t^{\nu}[f(t)]] = D_t^{\nu}[D_t^{\mu}[f(t)]]. \tag{15}$$

In conclusion of this section we are going to check the intuitive idea that D^{-1} is the integration operator; examine the $\mathcal{C}(-n)$ constant and in particular the $\mathcal{C}(-1)$ constant. We first examine the operator D_t^{-1} applied to a monomial

$$D_t^{-1}\left[t^{\beta}\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1+1)} t^{\beta+1} = \frac{t^{\beta+1}}{\beta+1} \tag{16}$$

from which we see that effectively this is the integral operator. Now again using the linearity property of the operator we know that we can take a sum of infinitesimals to obtain the standard definition of the integral and therefore in general we can write

$$D_t^{-1}[f(t)] = \int_0^t f(\tau)\, d\tau. \tag{17}$$

3 Application to integral calculus

In the general situation the fractional derivative of a function is a series. However, there are some cases where it is possible to express the result in terms of elementary functions. It is not our purpose here to provide an exhaustive list of the fractional derivative of functions, but it may be useful to see how such expressions are constructed from the definitions provided. An example is given by the function $f(t) = [a+bt]^{\mu-1}$, where applying the fractional derivative (9) we obtain

$$D_t^{\mu}\left[(a+bt)^{\mu-1}\right] = \sum_{k=0}^{\infty} \binom{\mu}{k} D_t^{\mu-k}\left[t^0\right] D_t^k \left[(a+bt)^{\mu-1}\right] \tag{18}$$

so that in terms of the fractional derivative of a constant and the integer k derivative of the function we have

$$\begin{aligned} D_t^{\mu}\left[(a+bt)^{\mu-1}\right] = \textstyle\sum_{k=0}^{\infty} \binom{\mu}{k} \frac{t^{k-\mu}}{\Gamma(k-\mu+1)} (\mu-1)(\mu-2)\cdots \\ \cdots(\mu-k-1)\, b^k (a+bt)^{\mu-1-k}. \end{aligned} \tag{19}$$

The multiplicative factors in (19) may be expressed in terms of gamma functions

$$\begin{aligned}
\Gamma(\mu) &= \Gamma(\mu-1+1) = (\mu-1)\Gamma(\mu-1) \\
&= (\mu-1)(\mu-2)\cdots(\mu-k-1)\Gamma(\mu-k)
\end{aligned}$$

so that

$$D_t^{\mu}\left[(a+bt)^{\mu-1}\right] = b^{\mu}\sum_{k=0}^{\infty}\left(\begin{array}{c}\mu\\ k\end{array}\right)\frac{(bt)^{k-\mu}}{\Gamma(k-\mu+1)}\frac{\Gamma(\mu)}{\Gamma(\mu-k)}(a+bt)^{\mu-1-k}. \quad (20)$$

This expression may be further simplified by using the gamma function relation

$$\Gamma(\mu-k)\Gamma(k-\mu+1) = \frac{\pi}{\sin\pi(\mu-k)}$$

to obtain, using $\sin\pi(\mu-k) = (-1)^k\sin\pi\mu$,

$$D_t^{\mu}\left[(a+bt)^{\mu-1}\right] = \Gamma(\mu)\frac{[a+bt]^{\mu-1}\sin\pi\mu}{\pi t^{\mu}}\sum_{k=0}^{\infty}\left(\begin{array}{c}\mu\\ k\end{array}\right)\frac{(-bt)^k}{(a+bt)^k}. \qquad (21)$$

We sum the series using the binomial relation

$$\sum_{k=0}^{\infty}\left(\begin{array}{c}\mu\\ k\end{array}\right)z^k = (1+z)^{\mu}$$

where $z = -bt(a+bt)^{-1}$, to obtain

$$\begin{aligned}
D_t^{\mu}\left[(a+bt)^{\mu-1}\right] &= \Gamma(\mu)\frac{[a+bt]^{\mu-1}\sin\pi\mu}{\pi t^{\mu}}\left[1-\frac{bt}{a+bt}\right]^{\mu} \\
&= \frac{a^{\mu}\sin\pi\mu}{\pi t^{\mu}}\frac{\Gamma(\mu)}{(a+bt)} \qquad (22)
\end{aligned}$$

We can use the previous result to apply fractional derivative with respect to a parameter to produce the possibility of new transformations. Consider, for example, the integral

$$I(a,b) = \int_0^{\infty} t^{\alpha}\left(a+bt^{\beta}\right)^{\gamma-1}dt \qquad (23)$$

which is convergent for $\alpha > -1$ and $(\alpha+1)/\beta+\gamma < 1$ (with $a, b \neq 0$). We rewrite (23) in terms of parametric derivatives

$$\begin{aligned}
I(a,b) &= b^{\gamma-1}\int_0^{\infty} t^{\alpha}\left(a/b+t^{\beta}\right)^{\gamma-1}dt \\
&= b^{\gamma-1}D_{\frac{a}{b}}^{-\gamma}\left[D_{\frac{a}{b}}^{\gamma}\int_0^{\infty} t^{\alpha}\left(a/b+t^{\beta}\right)^{\gamma-1}dt\right] \qquad (24)
\end{aligned}$$

in order to simplify its evaluation. We can use (22) in terms of the parametric fractional derivative to obtain from (24)

$$I\left(a,b\right)=b^{\gamma-1}\frac{\sin\pi\gamma}{\pi}\Gamma\left(\gamma\right)D_{\lambda}^{-\gamma}\left[\lambda^{-\gamma}\int_{0}^{\infty}t^{\alpha+\gamma\beta}\left(\lambda+t^{\beta}\right)^{-1}dt\right]\tag{25a}$$

where $\lambda=a/b$. Making the further substitution $z=t^{\beta}$ in (25a) we have:

$$I\left(a,b\right)=b^{\gamma-1}\frac{\sin\pi\gamma}{\pi}\Gamma\left(\gamma\right)D_{\lambda}^{-\gamma}\left[\lambda^{-\gamma}\beta^{-1}\int_{0}^{\infty}\frac{z^{\frac{\alpha+1}{\beta}+\gamma-1}}{z+\lambda}dz\right]\tag{26}$$

and using the calculus of residues to evaluate the simple pole in the remaining integral we have

$$I\left(a,b\right)=b^{\gamma-1}\frac{\Gamma\left(\gamma\right)\sin\pi\gamma}{\pi}\frac{\pi}{\sin\left[\left(\frac{\alpha+1}{\beta}+\gamma\right)\pi\right]}D_{\lambda}^{-\gamma}\left[\lambda^{\frac{\alpha+1}{\beta}-1}\right]$$

so that we finally obtain

$$I\left(a,b\right)=b^{\gamma-1}\frac{\Gamma\left(\gamma\right)\sin\pi\gamma}{\beta\sin\left[\left(\frac{\alpha+1}{\beta}+\gamma\right)\pi\right]}\frac{\Gamma\left(\gamma\right)\Gamma\left(\frac{\alpha+1}{\beta}\right)}{\Gamma\left(\frac{\alpha+1}{\beta}+\gamma\right)}\left(\frac{a}{b}\right)^{\frac{\alpha+1}{\beta}+\gamma-1}.\tag{27}$$

4 The generalized exponential function

We now turn our attention to the fractional derivative of the exponential function e^{t}, which when expressed in terms of an infinite series, yields

$$D_{t}^{\mu}\left[e^{t}\right]=D_{t}^{\mu}\left[\sum_{k=0}^{\infty}\frac{t^{k}}{k!}\right]=\sum_{k=0}^{\infty}\frac{t^{k-\mu}}{\Gamma\left(k+1-\mu\right)}\equiv E_{\mu}^{t}\tag{28}$$

where we define the generalized exponential function, E_{μ}^{t}, by the series.

Let us consider, for example the situation when the real-valued index in (28) is a negative integer $\mu=-1,-2,\cdots$. Starting from the definition (28) we have for $\mu=-1$

$$E_{-1}^{t}=D_{t}^{-1}\left[e^{t}\right]=\sum_{k=0}^{\infty}\frac{t^{k+1}}{\Gamma\left(k+2\right)}$$

so that reindexing the series we have

$$E_{-1}^{t}=\sum_{j=1}^{\infty}\frac{t^{j}}{\Gamma\left(j+1\right)}=e^{t}-1.\tag{29}$$

Of course, we can also write the negatively indexed generalized exponential as the first-order integral

$$E_{-1}^{t}=D_{t}^{-1}\left[e^{t}\right]=\int_{0}^{t}e^{\tau}d\tau=e^{t}-1. \tag{30}$$

as it was expected because of Eq. (17).

Now let us consider the fractional derivative of the negative exponential function, e^{-t}. We do this by considering the fractional derivative

$$D_{t}^{\mu}\left[e^{at}\right]=D_{t}^{\mu}\left[\sum_{k=0}^{\infty}\frac{(at)^{k}}{k!}\right]=a^{\mu}\sum_{k=0}^{\infty}\frac{(at)^{k-\mu}}{\Gamma(k+1-\mu)}\equiv a^{\mu}E_{\mu}^{at} \tag{31}$$

where a is an arbitrary constant. If we choose $a=-1$ we can use (31) to write

$$D_{t}^{\mu}\left[e^{-t}\right]=(-1)^{\mu}E_{\mu}^{-t}=e^{i\pi\mu}E_{\mu}^{-t} \tag{32}$$

which we can further use to define another generalized exponential function

$$^{*}E_{\mu}^{-t}\equiv e^{i\pi\mu}E_{\mu}^{-t}. \tag{33}$$

In series form we write this new generalized exponential function as

$$D_{t}^{\mu}\left[e^{-t}\right]=^{*}E_{\mu}^{-t}=\sum_{k=0}^{\infty}\frac{(-1)^{k}t^{k-\mu}}{\Gamma(k+1-\mu)}. \tag{34}$$

Both (33) and (34) make it abundantly clear that the function $^{*}E_{\mu}^{-t}$ is not E_{μ}^{t} calculated with $-t$; the new function differs from the old by the phase factor $e^{i\pi\mu}$. Using the property of Eq. (10) we have:

$$D_{t}^{\mu}\left[e^{-t}\right]=(-1)^{\mu}D_{x}^{\mu}\left[e^{x}\right]_{x=-t}=e^{i\pi\mu}E_{\mu}^{-t} \tag{35}$$

just as we obtained in (32) and here E_{μ}^{-t} is a function in the complex field. For real functions it is convenient to define $^{*}E_{\mu}^{-t}$ as E_{μ}^{t} calculated with $-t$, but in order to do this we need to define the generalized exponential as

$$E_{\mu}^{t}=|t|^{-\mu}\sum_{k=0}^{\infty}\frac{t^{k}}{\Gamma(k+1-\mu)} \tag{36}$$

where it is possible to evaluate this function for both positive and negative values of the independent variable. In general, however, when we are dealing with complex functions we use the first definition of the generalized exponential given by (28).

For completeness we define the inverse of the generalized exponential function as the generalized logarithm, which is to say the function that satisfies the relation

$$\ln_{\mu}E_{\mu}^{t}=t. \tag{37}$$

5 Generalized trigonometric functions

Now that we have a generalization of the complex exponential function, it should, of course, be possible to construct a generalization of the Euler relation, that being,

$$E_\mu^{it} = \cos_\mu t + i \sin_\mu t. \tag{38}$$

From the real part of (38) we obtain the equation for the generalized cosine function

$$\cos_\mu t = \frac{1}{2}\left(E_\mu^{it} + E_\mu^{-it}\right) \tag{39}$$

and from the imaginary part of (38) we obtain the equation for the generalized sine function

$$\sin_\mu t = \frac{1}{2i}\left(E_\mu^{it} - E_\mu^{-it}\right). \tag{40}$$

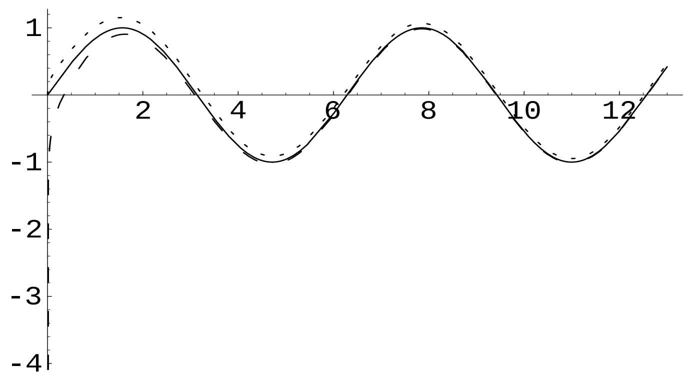

Figure 1: *The dashed line is* $\sin_\alpha x$ *with* $\alpha = 0.3$*, the dotted line is* $\sin_\alpha x$ *with* $\alpha = -0.3$ *and the continuous line is the ordinary trigonometric function* $\sin x$*. Clearly after a period the three function assume the same values.*

We can then extend these definitions even further and construct the generalized tangent function as well

$$\tan_\mu t \equiv \frac{\sin_\mu t}{\cos_\mu t}. \tag{41}$$

We can also express the generalized sine and generalized cosine functions in series form using the series definition of the generalized exponential. The generalized cosine function is given by

$$
\begin{aligned}
\cos_\mu t &= \frac{1}{2}\left(E_\mu^{it} + E_\mu^{-it}\right) \\
&= \sum_{k=0}^{\infty} \frac{t^{k-\mu}}{\Gamma(k+1-\mu)} \frac{e^{i(k-\mu)\pi/2} + e^{-i(k-\mu)\pi/2}}{2} \\
&= \sum_{k=0}^{\infty} \frac{t^{k-\mu}}{\Gamma(k+1-\mu)} \cos\left[(k-\mu)\pi/2\right]
\end{aligned} \tag{42}
$$

and the generalized sine function is given by

$$
\begin{aligned}
\sin_\mu t &= \frac{1}{2i}\left(E_\mu^{it} - E_\mu^{-it}\right) \\
&= \sum_{k=0}^{\infty} \frac{t^{k-\mu}}{\Gamma(k+1-\mu)} \frac{e^{i(k-\mu)\pi/2} - e^{-i(k-\mu)\pi/2}}{2i} \\
&= \sum_{k=0}^{\infty} \frac{t^{k-\mu}}{\Gamma(k+1-\mu)} \sin\left[(k-\mu)\pi/2\right].
\end{aligned} \tag{43}
$$

From (42) and (43) we can see that for integer μ the generalized trigonometric series $\sin_\mu t$ and $\cos_\mu t$ become the ordinary trigonometric functions $\sin t$ and $\cos t$.

It is useful to study the derivatives of the generalized trigonometric functions in order to understand how these periodic functions differ from those in the standard form. Consider the first-order time derivative of the generalized cosine function

$$
\begin{aligned}
D_t\left[\cos_\mu t\right] &= \sum_{k=0}^{\infty} \frac{(k-\mu)\, t^{k-\mu-1}}{\Gamma(k+1-\mu)} \cos\left[(k-\mu)\pi/2\right] \\
&= \sum_{k=0}^{\infty} \frac{t^{k-\mu-1}}{\Gamma(k-\mu)} \cos\left[(k-\mu)\pi/2\right]
\end{aligned} \tag{44}
$$

where by reindexing the series, $k = j+1$, we can write

$$
D_t\left[\cos_\mu t\right] = \sum_{j=-1}^{\infty} \frac{t^{j-\mu}}{\Gamma(j+1-\mu)} \cos\left[(j+1-\mu)\pi/2\right]. \tag{45}
$$

Separating the $j = -1$ term from the series and using the trigonometric identity $\cos(j+1-\mu)\pi/2 = -\sin(j-\mu)\pi/2$ yields

$$
D_t\left[\cos_\mu t\right] = -\sin_\mu t + \frac{\cos(\mu\pi/2)}{\Gamma(-\mu)\, t^{\mu+1}} \tag{46}
$$

where we have used (43) to replace the series. We see that the formal relation resulting from the derivative of the generalized cosine differs from that of the derivative of the cosine by a term that decays as an inverse power law in the independent variable. Thus, as $t \to \infty$, the formal relation for the two derivatives approach one another:

$$\lim_{t \to \infty} D_t \left[\cos_\mu t\right] = -\sin t. \tag{47}$$

The inverse power-law form of the term in (46) is quite suggestive, since the memory in dynamical processes that make it impossible to join the microscopic and macroscopic descriptions of complex phenomena are exactly of this inverse power-law form.

Let us now examine the derivative of the generalized sine function

$$\begin{aligned} D_t \left[\sin_\mu t\right] &= \sum_{k=0}^{\infty} \frac{(k-\mu)\, t^{k-\mu-1}}{\Gamma(k+1-\mu)} \sin\left[(k-\mu)\,\pi/2\right] \\ &= \sum_{k=0}^{\infty} \frac{t^{k-\mu-1}}{\Gamma(k-\mu)} \sin\left[(k-\mu)\,\pi/2\right] \end{aligned} \tag{48}$$

where by reindexing the series, $k = j+1$, we can write

$$D_t \left[\sin_\mu t\right] = \sum_{j=-1}^{\infty} \frac{t^{j-\mu}}{\Gamma(j+1-\mu)} \sin\left[(j+1-\mu)\,\pi/2\right]. \tag{49}$$

Separating the $j = -1$ term from the series and using the trigonometric identity $\sin(j+1-\mu)\,\pi/2 = \cos(j-\mu)\,\pi/2$ yields

$$D_t \left[\sin_\mu t\right] = \cos_\mu t - \frac{\sin(\mu\pi/2)}{\Gamma(-\mu)\, t^{\mu+1}} \tag{50}$$

where we have used (42) to replace the series. We see that the formal relation resulting from the derivative of the generalized sine differs from the derivative of the sine by a term that decays as an inverse power law in the independent variable, just as it did for the generalized cosine. Thus, as $t \to \infty$ the formal relations for the two derivatives approach one another:

$$\lim_{t \to \infty} D_t \left[\sin_\mu t\right] = D_t \left[\sin t\right] = \cos t. \tag{51}$$

Here again, the inverse power-law form of the term in (50) is quite suggestive.

We have examined what happens to a generalized trigonometric function when we take an ordinary derivative. Now let us examine what happens to an ordinary trigonometric function when we take a fractional derivative. Consider the fractional derivative of the sine function

$$\begin{aligned} D_t^\mu \left[\sin t\right] &= \frac{1}{2i}\left(D_t^\mu\left[e^{it}\right] - D_t^\mu\left[e^{-it}\right]\right) \\ &= \frac{1}{2i}\left(e^{i\mu\pi/2} E_\mu^{it} - e^{i\mu\pi/2} E_\mu^{it}\right) \end{aligned}$$

so that using the Euler relations for both the exponential and generalized exponential and combining terms we obtain

$$D_t^{\mu}\left[\sin t\right] = \sin\left(\mu\pi/2\right)\cos_{\mu} t + \cos\left(\mu\pi/2\right)\sin_{\mu} t. \tag{52}$$

Equation (52) is reminiscent of the trigonometric expansion of $\sin\left(t+\mu\pi/2\right)$. In the same way we take the fractional derivative of the cosine function

$$\begin{aligned} D_t^{\mu}\left[\cot t\right] &= \frac{1}{2}\left(D_t^{\mu}\left[e^{it}\right] + D_t^{\mu}\left[e^{-it}\right]\right) \\ &= \frac{1}{2}\left(e^{i\mu\pi/2}E_{\mu}^{it} + e^{-i\mu\pi/2}E_{\mu}^{-it}\right) \end{aligned}$$

so again using the Euler relations and combining terms we obtain

$$D_t^{\mu}\left[\cot t\right] = \cos\left(\mu\pi/2\right)\cos_{\mu} t - \sin\left(\mu\pi/2\right)\sin_{\mu} t. \tag{53}$$

Equation (53) is reminiscent of the trigonometric expansion of $\cos\left(t+\mu\pi/2\right)$.

A geometrical interpretation of the derivative relations in (52) and (53) can be obtained by introducing the rotation matrix

$$\mathbf{R} = \begin{pmatrix} \cos\left(\mu\pi/2\right) & \sin\left(\mu\pi/2\right) \\ -\sin\left(\mu\pi/2\right) & \cos\left(\mu\pi/2\right) \end{pmatrix}$$

and the vector

$$\mathbf{v}_{\mu} = \begin{pmatrix} \sin_{\mu} t \\ \cos_{\mu} t \end{pmatrix}$$

so that we can write

$$D_t^{\mu}\left[\mathbf{v}_0\right] = \mathbf{R}\mathbf{v}_{\mu}. \tag{54}$$

Both the above fractional derivatives are included in the rotation equation given by (54) since we know that the generalized functions reduce to their ordinary counterparts when $\mu = 0$

$$\mathbf{v}_0 = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}.$$

The existence of (54) allows us to infer that the vectors $\mathbf{v}_{\mu}$ and $\mathbf{w}$ have the same length, where

$$\mathbf{w} \equiv D_t^{\mu}\left[\mathbf{v}_0\right],$$

since the "length" of the rotation matrix is unity.

A similar kind of analysis can be done for the generalization of the hyperbolic sines and cosines.

6 Certain fractional integrals

We now consider how to construct the definite integrals of certain functions using the properties of the fractional derivatives discussed in the previous sections using negative values of the $\mu-$index and the series representations of the generalized functions. Consider an integral of the form

$$I_\alpha(t) = \int_0^t \tau^\alpha e^{-\tau} d\tau \tag{55}$$

where $\alpha > -1$. Using the fractional calculus formalism we write

$$I_\alpha(t) = D_t^{-1}\left[t^\alpha e^{-t}\right] \tag{56}$$

and applying the generalized Leibniz rule (7) to (56) we obtain

$$\begin{aligned} I_\alpha(t) &= \sum_{k=0}^{\infty} \begin{pmatrix} -1 \\ k \end{pmatrix} D_t^{-k-1}\left[t^\alpha\right] D_t^k\left[e^{-t}\right] \\ &= \Gamma(\alpha+1) \sum_{k=0}^{\infty} (-1)^k \frac{(-1)^k t^{k+\alpha+1}}{\Gamma(k+\alpha+2)} e^{-t}. \end{aligned} \tag{57}$$

Introducing the series definition of the generalized exponential into (57) we have

$$I_\alpha(t) = \Gamma(\alpha+1)\, e^{-t} E_{-(\alpha+1)}^t \tag{58}$$

for the integral (55). We can extend (58) to $\alpha < -1$ with $\alpha \neq -1, -2, \cdots$. Equation (58) can be used to determine the integral for the gamma function

$$\lim_{t\to\infty} I_\alpha(t) = \Gamma(\alpha+1) \tag{59}$$

since $E_{-(\alpha+1)}^t \to e^t$ as $t \to \infty$. In general we use the same logic to obtain

$$\int_a^t \tau^\alpha e^{\pm\tau} d\tau = \Gamma(\alpha+1)\, e^{\pm t}\ {}^*E_{-(\alpha+1)}^{\mp t} + c \tag{60}$$

where c is a constant dependent on the lower limit of the integral. We can now use (60) to again obtain the derivative of the generalized exponential.

We now consider the integrals of trigonometric functions of monomials such as

$$I_\alpha(t) = \int_0^t \cos(\tau^\alpha)\, d\tau. \tag{61}$$

that appear in several field of physics. For example for $\alpha = 2$, the integral (61) is basically the Fresnel cosine integral $C(\tau)$, well known in optics. Making the change of variables $y = \tau^\alpha$ so that $dy = \alpha\tau^{\alpha-1}d\tau$ and (61) becomes

$$I_\alpha(t) = \frac{1}{\alpha}\int_0^t dy y^{\frac{1}{\alpha}-1}\cos y = \mathrm{Re}\left[\frac{1}{\alpha}\int_0^t dy y^{\frac{1}{\alpha}-1} e^{-iy}\right] \tag{62}$$

and using (60) we obtain

$$\int_0^t \cos(\tau^\alpha)\, d\tau = \mathrm{Re}\left[\frac{\Gamma(1/\alpha)}{\alpha i^{1/\alpha}} e^{-iy}\ E_{-1/\alpha}^{iy}\right]_0^\infty = \frac{\Gamma(1/\alpha)}{\alpha}\cos(\pi/2\alpha)\,. \tag{63}$$

In a similar way we obtain for the integral of the sine function

$$\int_0^t \sin(\tau^\alpha)\, d\tau = \mathrm{Im}\left[\frac{\Gamma(1/\alpha)}{\alpha i^{1/\alpha}} e^{-iy}\ E_{-1/\alpha}^{iy}\right]_0^\infty = \frac{\Gamma(1/\alpha)}{\alpha}\sin(\pi/2\alpha) \tag{64}$$

where we have used the property that for "$t = \infty$" the generalized exponential function becomes the ordinary exponential function.

7 Concluding Remarks

This short review showed as the fractional calculus is a very helpful tool to perform calculation specifically dealing with power law[1, 2]. Despite to the mathematical examples presented, many physical application can be faced[3]; let us, for example, consider the ordinary diffusion equation, $\frac{\partial}{\partial t}P(x,t) = \nabla^2 P(x,t)$ that leads to gaussian processes. A possible generalization of the diffusion equation is the fractional diffusion equation $\frac{\partial}{\partial t}P(x,t) = \nabla^\alpha P(x,t)$ where the second derivative is substituted by a non integer order of derivation α. The solution of this new diffusion equation (fractional diffusion equation) leads to Levy processes that are considered as possible source of anomalous diffusion processes. Finally a brief and absolutely not exhaustive bibliography is reported.

References

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