Bachelor thesis

Knot theory and the Alexander polynomial

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Abstract

An introduction to knot theory, the Alexander polynomial and an application in physics is presented.

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1 Abstract

An introduction to the theory of knots is given. Properties of knots are discussed and certain knot invariants are obtained. The emphasis lies on a particular type of knot invariant: a knot polynomial, namely the Alexander polynomial. The Alexander polynomial is a Laurent polynomial which can be assigned to a knot. It turns out that this assignment only depends on the equivalence class of the knot, i.e. on the knot type. For this reason we speak of a knot invariant. Otherwise formulated, if two knots have different Alexander polynomials they cannot be equivalent. Therefore knot invariants such as the Alexander polynomial can be used to distinguish between knot types. The aim is to understand the construction of the Alexander polynomial, its geometrical meaning and to analyse some of its properties. Then a connection is made between knots and physical particles who live in 2+1 dimensional space time. These particles are called anyons and they have some interesting features. A model of such particles is specified by three rules. By the consideration of a certain equation resulting from an anyonic model, a knot polynomial called the Jones polynomial is obtained.

2 Populair wetenschappelijke samenvatting

Deze bachelorscriptie geeft een introductie in de wereld van knopen. Vrijwel iedereen heeft in zijn of haar leven wel eens te maken met knopen. Zo hebben de meesten geleerd om een knoop in de schoenveters te leggen om ervoor te zorgen dat de schoenen goed blijven zitten. Ook zullen velen het ontwarren van bijvoorbeeld de iPod-oortjes herkennen nadat deze voor de zoveelste keer in elkaar zijn geknoopt. Dit zijn allemaal voorbeelden van knopen in het dagelijks leven. Echter, als iemand mij op dit moment zou vragen om een dubbele paalsteek in een stuk touw te leggen, zou ik diegene het antwoord schuldig moeten blijven. Maar waar gaat de knopentheorie in wiskundige en natuurkundige zin dan wel over?

Om te beginnen is het in de wiskunde van belang om objecten zoals knopen precies te definiëren. We komen al een heel stuk verder door de twee uiteinden van een stuk touw met een knoop erin, aan elkaar te verbinden. Een wiskundige knoop heeft dus geen begin- of eindpunt. De triviale knoop, ook wel de onknoop genoemd, is een cirkel. Alle andere soorten knopen zijn in feite vervormingen van de cirkel. Meer precies: manieren waarop de cirkel kan knopen in de drie-dimensionale ruimte. De feitelijke definitie van een knoop is iets te wiskundig om hier te noemen.

Nu is het heel goed mogelijk dat twee knopen op het eerste gezicht niet op elkaar lijken, maar na een beetje trekken en buigen toch één en dezelfde knoop blijken. We zeggen wel dat twee knopen equivalent zijn als zij door middel van trekken, buigen en duwen in elkaar over zijn te voeren. Handelingen zoals knippen en plakken zijn niet toegestaan. Dit soort manipulaties met knopen worden in wiskundige termen beschreven. De hamvraag in de knopentheorie is of, gegeven twee knopen, zij equivalent zijn of niet.

In de praktijk blijkt het vaak erg lastig om knopen met veel kruisingen en lusjes te manipuleren zoals op bovenstaande manier beschreven staat. Daarom worden er eigenschappen aan knopen toegekend. Vervolgens gaat men op zoek naar knoopinvarianten. Dit zijn eigenschappen van knopen die hetzelfde zijn voor equivalente knopen. Met behulp van deze invarianten is het mogelijk knopen te classificeren. Als een knoopinvariant de vorm van een polynoom aanneemt, spreekt men van een knooppolynoom. Het Alexanderpolynoom is een voorbeeld van een dergelijk knooppolynoom. In deze scriptie heb ik enkele eigenschappen en de constructie van dit polynoom beschreven.

De knopentheorie kent veel toepassingen in de natuurkunde. In deze scriptie heb ik één toepassing, in de vorm van anyonen, besproken. Anyonen zijn fysische deeltjes met bijzondere eigenschappen. Natuurkundigen gingen er lange tijd van uit dat alle verschillende deeltjes in twee hoofdsoorten konden worden onderverdeeld: de fermionen en de bosonen. Een bekend voorbeeld van een fermion is een elektron en het foton is een typisch boson. Op theoretische gronden is in de 20^e eeuw het bestaan van andere soorten deeltjes geopperd, te weten anyonen.

Anyonen leven, in tegenstelling tot fermionen en bosonen, in een twee-dimensionale wereld. Het wel of niet bestaan van deze deeltjes is enigszins controversieel. Niet iedere fysicus is ervan overtuigd dat deze theoretische deeltjes daadwerkelijk in de natuur voorkomen. Toekomstige experimenten zullen deze vete moeten beslechten. Desalniettemin zijn anyonen het bestuderen waard. Zo beschrijf ik in mijn scriptie een manier om een knooppolynoom te verkrijgen na het analyseren van een bepaald systeem dat uit anyonen bestaat.

In conclusie kan ik stellen dat mijn scriptie slechts een kleine introductie geeft in een populair onderzoeksgebied dat sinds zijn bestaan enorm in omvang is gegroeid. In het bijzonder heb ik het Alexanderpolynoom uitgelicht omdat dat het eerste en wellicht best begrepen knooppolynoom is. De knopentheorie is echter nog lang niet af. Er zijn nog veel open vragen en ik kan iedereen van harte aanbevelen zelf een beginnetje in de wereld van knopen te maken door in de bijvoorbeeld zeer begrijpelijke boeken [1] en [15] te lezen.

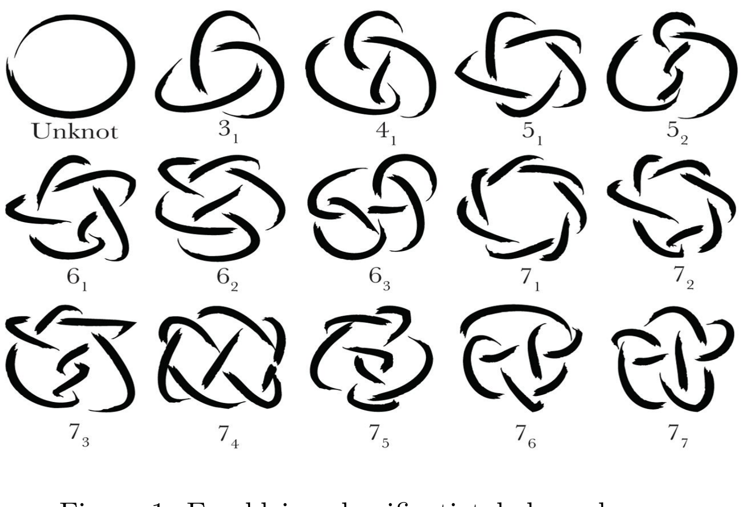

Figure 1: Een kleine classificatietabel van knopen

3 Introduction

In this paper an introduction to knot theory is given. Knot theory is a mathematical theory that investigates properties of knots. But what precisely is a knot? Most people think of knots as being tied up ropes which are used by sailors. Or they refer to the knot in their shoelaces. If one now ties both ends of the shoelace or rope together, one gets the knot that mathematicians study. One of the main questions in knot theory is whether or not two given knots are the same. Though this question intuitively may seem very simple, it requires hard work to overcome many technical difficulties and to provide an answer. There exists an algorithm that solves the question [11]. Unfortunately it may take a very long time before the algorithm can distinct two knots. This is because the algorithms complexity is unknown.

To distinguish between two knots, one assigns properties to knots in order to obtain knot invariants. A knot invariant is a quantity that is the same for equivalent knots; in the first part of this research the precise equivalence relation is described. That part also discusses a few examples of knot invariants. The aim of this thesis is to examine the so-called Alexander polynomial. That is; its mathematical definition, its properties and its geometrical meaning. The Alexander polynomial turns out to be a knot invariant and is good in distinguishing knots with not to many crossings. Two different methods that lead to this Laurent polynomial are proposed. In order to fully understand the Alexander polynomial, tools from different areas of mathematics are needed. They mainly come from algebraic and geometric topology. The reader is assumed to have a knowledge of undergraduate level in mathematics. The second part of the thesis is devoted to the Alexander polynomial.

Knot theory naturally lends itself to make graphs and diagrams. In order to prevent this paper to become a picture book, they are only used when they improve clarity and when it is necessary. Though these pictures often offer a helping hand, it is premature to make knot theory out to be a straightforward and simple part of mathematics. As said, the tools used to deal with knots come from many branches of mathematics and often they are very complex. Although knot theory exists for over a hundred years, there are still many unsolved questions. A lot of these questions can be understood with very little background information. This is one of the motivations for this thesis. To understand where the difficulties in approaching knot theory lie.

Another motivation for this thesis can be found in the applications of knot theory in physics. As will be seen later, knots have something to do with physical particles with strange statistics. It turns out that these particles live in 2+1 dimensional space time and that they behave completely different than the well-known bosons and fermions. The connection between these particles and knots will be studied in the third and last part of this research.

Finally I want to thank my supervisors. Eric Opdam for the intensive and fruitful sessions without which this thesis could never have been realized in its present shape. Also the discussions about topics which are not directly related to knot theory have been very insightful. Furthermore I am grateful to Jan de Boer for his explanation about anyons and pointing me in the good direction.

4 Knot equivalence

4.1 Ambient isotopy

Knot theory originally has a topological flavour. It is intended to deal with the possible ways a circle can lie in three-dimensional Euclidean space. There exists a generalization of a mathematical knot, namely a link. We will first give the formal definition of the link.

Definition 4.1 A *link* of n components is a subset of $\mathbb{R}^3$, or its one-point compactification which is homeomorphic with S^3, that consists of n disjoint, piecewise linear simple closed curves. A *knot* is a link that consists of one component.

The simple closed curves have to be piecewise linear because we want to exclude certain links which are hard to deal with. We restrict our attention to links and knots that are built up of a finite number of straight line segments. Such knots and links are called piecewise linear. There exists a more comprehensive definition of a knot in which a knot is defined as an embedding of S^1 in $\mathbb{R}^3$ (or S^3). For a moment we will work with this definition. The justification of this step will be given later on.

It may occur that two different embeddings represent the same knot if we allow for some manipulations on the knot. The knot can be considered as a very elastic rope that can be pulled tighter or may be stretched. Actions as splitting, cutting and pasting are forbidden. Now recall the following definition from topology.

Definition 4.2 Let f, g: X $\rightarrow$ Y be homeomorphisms between topological spaces X and Y. Then f is *isotopic* to g if there exists a homotopy H: X x I $\rightarrow$ Y such that H(x,0) = f, H(x,1) = g and H(x,t) is a homeomorphism for all t$\in$I.

Where I denotes the unit interval [0,1]. So an isotopy is nothing more than a continuous family of homeomorphisms. A naive attempt to define equivalence between two knots would be that the two embeddings defining the knots must be isotopic to each other. Note that an embedding is a homeomorphism between a topological space and its image. But such a notion of isotopy turns out to be incorrect because it allows for the possibility that knotted segments shrink to a point by "pulling the string", thus simplifying the knot type.

It is clear that additional requirements have to be placed on an isotopy as described above in order to satisfy our geometrical intuition concerning the fact whether two knots are equivalent or not. This leads to the following definition.

Definition 4.3 Two knots (K_1 and K_2) are said to be *equivalent* if there is an *ambient isotopy* (in the piecewise-linear category) between the two embeddings (i_1 and i_2) of S^1 in S^3. That is: there exists a piecewise linear map F: S^3 x I $\rightarrow$ S^3 such that $F(x,0) = id_{S^3}$, $F(i_1(S^1),1) = i_2$ and F(x,t) is a homeomorphism from S^3 to itself for all t$\in$I.

Lemma 4.4 *The relation "i_1 is ambient isotopic with i_2" is an equivalence relation on the set of embeddings of S^1 in S^3.*

A knot can be given an orientation by specifying a direction to travel around the knot. It follows from the definition of equivalence between knots that an ambient isotopy preserves the orientation of a knot.

A knot is called *tame* if every point on the knot has a neighborhood which is isotopic to the standard arc-ball pair. The standard arc-ball pair is a ball with an arc from the north pole to the south pole of the ball. A knot which is not tame is called wild. An example of a wild knot is the Alexander horned sphere. It turns out that a tame knot is equivalent to a piecewise linear knot as defined in 4.1. That is, every tame knot can be represented by a piecewise linear knot up to ambient isotopy. This justifies working with the alternative definition of a knot earlier.

Considering the above, it is most convenient to work in the piecewise linear category. This is called combinatorial knot theory. In this approach to knot theory not only the knot itself is represented by a piecewise linear curve, but all spaces are simplicial complexes and all maps are simplicial maps. For example, the ambient space S^3 is represented by a three-dimensional simplicial complex homeomorphic to S^3. The embedded knot is an one-dimensional simplicial subcomplex. There are other approaches to knot theory. Some authors prefer to work in the differentiable or topological category. It is known that for *tame knots* all these approaches are equivalent and all spaces and maps can be equipped with compatible piecewise-linear structures. However, the proof of this statement is far beyond the scope of this thesis. Nevertheless we will sometimes tacitly use these fundamental facts and work in these other categories whenever this turns out to be more convenient.

4.2 Reidemeister moves

Now a start has been made, it is time to make things a little more geometrical. For each knot a knot diagram can be given. A knot diagram is a projection of the knot on $\mathbb{R}^2$ in which the crossings are marked. The precise projection will now be specified.

A piecewise linear knot consists of straight line segments. These straight line segments can be changed by the aid of delta moves. A delta move can be explained as follows. If a planar triangle intersects a link or a knot in one edge of the triangle, this part of the link is deleted and it is replaced by the two other parts of the triangle. The straight line segments of a knot can be arranged in such a way that each line segment projects to a line segment in $\mathbb{R}^2$. It is required that two projected line segments intersect in at most one point. No three points of the knot project to the same point and furthermore vertices of line segments of the knot do not project to the same point as any other point of the knot.

A knot diagram then is such a projection of a piecewise linear knot together with the information where one strand goes under or over another. It is like a shadow of a knot. Of course, it is also possible to give a link diagram. An orientation in the knot diagram is indicated by drawing arrows. Furthermore, if a link is oriented, one can define a linking number. For a crossing there are two possibilities and to each of them a sign is given in the following way.

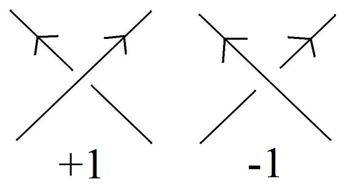

Figure 2: Signs

The linking number is then defined as follows.

Definition 4.5 The *linking number* of a link L with two components L_1 and L_2 is denoted as $lk(L_1, L_2)$ and is defined as half the sum of the signs in a diagram for the link L.

Notice that this definition implies that the linking number is symmetric in its argument. Given one particular knot, the reverse knot is the same subset of S^3 but with orientation altered. The reflected or mirrored knot is the knot one obtains if the over-crossings are changed in under-crossings and the other way around. In some situations the operations reversing and reflecting change the knot, in other situations this is not the case. If a knot is equivalent to its reflected image, the knot is called amphicheiral. The knot and link diagram may be very useful in order to determine if two knots or links are equivalent. It points out to be that there are three kind of manipulations of a diagram that can be carried out in order to transform knots and links into others that are equivalent to it. These maneuvers are called Reidemeister moves and they are pictured below.

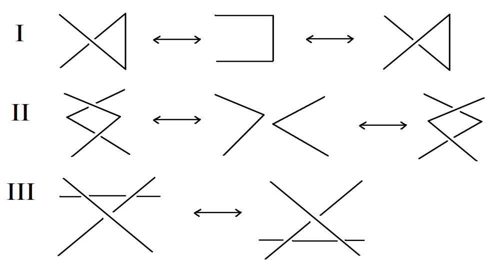

Figure 3: Reidemeister moves

An equivalence relation generated by the second and third Reidemeister move is called a regular isotopy. A regular isotopy, unlike an ambient isotopy, conserves the winding number of a knot diagram. Kurt Reidemeister, a German mathematician, proved that two equivalent knots are related by a sequence of Reidemeister moves. He proved this with the aid of the earlier described delta moves.

Reidemeister's result seems very useful. However, there is no bound on the number of moves that takes one knot into another. In this way it is very hard to prove that two knots are not equivalent for we can not forever go on checking Reidemeister moves. In the next section is explained how to deal with this problem. It is also quite hard to write down all Reidemeister moves if two equivalent knots are given. For example, try do so for the following knot, which is in fact the unknot in disguise.

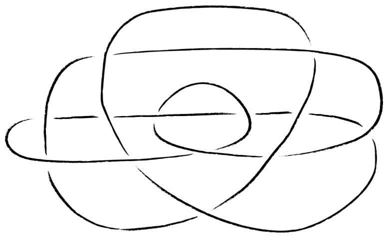

Figure 4: An unknot in disguise

4.3 Knot and link invariants

As mentioned in the previous section, it is necessary to search for better ways for detecting equivalence between knots and links. In order to do so, we have to assign new properties to links and knots and see if they match for equivalent ones. This is a quest for invariants. So as a start, let us study the behavior of the linking number under the Reidemeister moves. Obviously, move I does not affect the number of crossings between two link components; it only changes the number of self-crossings.Therefore the linking number does not change. Moves II and III do change the number of crossings but in either case there is a plus one and a minus one which cancel each other out. We conclude that the linking number is a link invariant; it is not changed by ambient isotopy.

There is an important theorem concerning the linking number. It deals with homology theory. As is known from algebraic topology, the fundamental group is a powerful tool to measure holes of codimension two of a topological space. Due to the fact that the fundamental group is defined in terms of homotopy classes of loops, one would not expect that it is very useful to detect holes of higher codimension. There exists a theory of homotopy groups of higher order but they have the disadvantage that they are very difficult to compute. On that account a new theory has been developed, called homology theory.

In short the difference between the first homology group and the fundamental group is that in homology one looks at 'sums' of paths, which are called 1-chains, instead of loops. A closed 1-chain is called a 1-cycle and two 1-cycles are said to be homologous if they differ by a boundary of a 2-dimensional chain. This equivalence relation is called homology and the set of homology classes of closed 1-chains of a topological space X is called the first homology group $H_1(X)$. It is easy to extend this definition to the higher homology groups $H_k(K)$.

There are different types of homology theories. All these theories must satisfy a list of axioms. One of the most important properties of homology theories is that they are invariant for homotopy: two homotopic maps f,g X: $\rightarrow$ Y, with X and Y being topological spaces, define the same map $H_k(f) = H_k(g) : H_k(X) \rightarrow H_k(Y)$ in any homology theory. It follows that two spaces which are homotopy equivalent have the same homology groups in any homology theory. Two theories are especially important in the present context: simplicial and singular homology. Though the two theories are certainly different, for most of the theorems that will follows it is indifferently which homology theory is being used. This should sound familiar to the reader. It is presumed that some basis results from algebraic topology such as homology groups and the Mayer-Vietoris sequence are known.

Back to knots. It is not hard to imagine that every knot in S^3 has a regular neighbourhood N that is a solid torus. Note that the knot itself is not a solid torus because it has no interior points. The exterior X of a knot is defined as the closure of the complement of N in S^3, $X = \overline{S^3 \setminus N}$. Observe that X is homotopy equivalent to the complement of the knot in S^3. The theorem says something about the first homology group of the exterior X of a knot in S^3. Moreover it relates the homology class of a simple closed curve c in X to the linking number of c and the knot K. Its proof uses the Mayer-Vietoris sequence.

Theorem 4.6 *Let K be an oriented knot and X its exterior. Then $H_1(X)$ is isomorphic to $\mathbb{Z}$ and its generator is a simple closed curve μ in ∂N that bounds a disc in N that meets K at one point. If c is an oriented simple closed curve in X, then the homology class $[c] \in H_1(X)$ is lk(c,k).*

Now a link invariant has been found, one is maybe tempted to think that the classification of links is within reach. Nothing is further from the truth. Take a look at the link below, called the Whitehead link. The mirror image of the Whitehead link is the logo of the International Mathematical Olympiad.

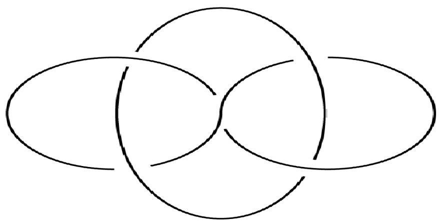

Figure 5: The Whitehead Link

Its linking number is zero, just as for the unlink. An unlink consists of two unknots situated in S^3 in such a way that there are no crossings in the link diagram. However, the Whitehead link is definitely not equivalent to the unlink. But for now, we are not able to tell the difference between the Whitehead link and the unlink. Apparently a search must be made with the aim to find other invariants. In the next chapter a new and very useful invariant will be introduced.

5 The Jones polynomial

The history of knot theory goes back to the nineteenth century. During a long period of time, no big breakthroughs had been made. That all changed in the year 1984 when Vaughan Jones introduced the Jones polynomial. It was a real milestone and in 1990 Jones received the Fields medal for his discovery. The Jones polynomial is a Laurent polynomial assigned to a link and turns out to be a link invariant. Some work must be done before giving the definition of the polynomial.

5.1 The Kauffman bracket

We would like the Jones polynomial to satisfy some requirements. Therefore we first introduce the *Kauffman bracket* for a link L, denoted as $<L>$. As a first rule we demand that the Kauffman bracket of the unknot equals one: $<O> = 1$. Now consider the following diagram.

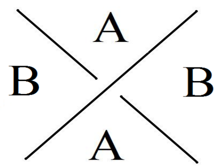

Figure 6: The Whitehead Link

If one travels, in arbitrary direction, on the upper strand the area before the crossing on the right is denoted as A. The area on the left is marked as B. If the strands are cut open in the centre of the crossing and are pasted in such a way that the areas denoted with A are connected, a simpler figure with an A in front is obtained. Similarly this holds if the B areas are joined. If a link is simplified in accordance with the above procedure, a linear combination of the simpler figures is obtained in the following Kauffman bracket notation.

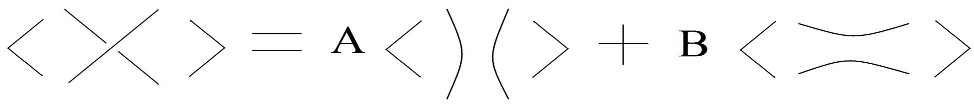

Figure 7: Rule 2

In addition, as a third rule is required that the adding of an unknot behaves like this.

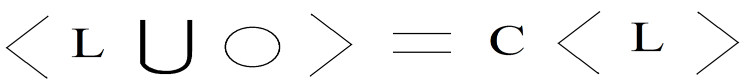

Figure 8: Rule 3

In the latter picture the L stands for a link, the circle represents the unknot and the U stands for the union of both. We must get on with the task to find a relation between the variables A, B and C. Let us explore the behavior of the Kauffman bracket under the Reidemeister moves. After all, we are looking for a knot invariant. It is advantageous to start with Reidemeister move II.

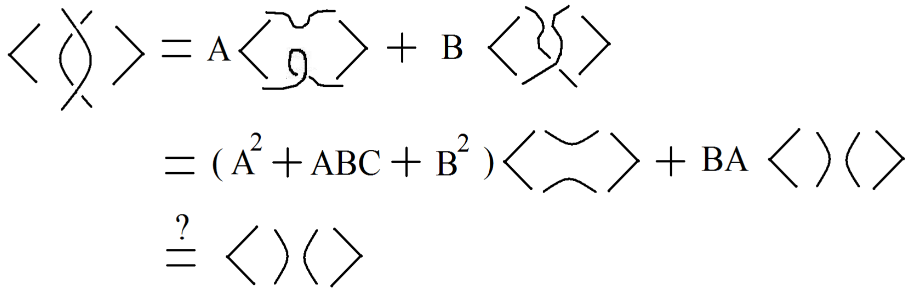

Figure 9: The effect of Reidemeister move II

Where the second equality is obtained by repeated use of the bracket rules. It follows that the next relations must apply in order for the equality to hold: $B = A^{-1}$ and $C = -(A^2 + A^{-2})$. The third Reidemeister move leaves the Kauffman bracket unchanged, as a quick check will tell. Only an inspection of the first Reidemeister move remains.

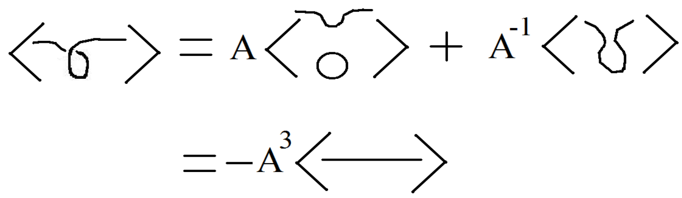

Figure 10: The effect of Reidemeister move I

The second equality is obtained by applying bracket rule three on the first term. The first Reidemeister move seems to throw grit into the machine. To solve this problem an orientation must be given to a link. Then a sign is associated to each crossing in the knot diagram alike has been done to crossings in a link diagram. This gives rise to another definition.

Definition 5.1 The *writhe* of a link L, denoted as w(L), is the sum of all signs.

Note that the writhe of a knot is certainly not a link invariant as Reidemeister move I takes care of one crossing more or less which changes the writhe. So this Reidemeister move causes an extra plus or minus one for the writhe. The writhe of a link is however unaffected by a regular isotopy as one can easily verify. Here comes the definition of the Kauffman bracket.

Definition 5.2 The *Kauffman polynomial* for an oriented link L is a Laurent polynomial in the variable A with integer coefficients and is defined by $P_L(A) = (-A)^{-3w(L)}<L>$.

It remains to check if this is a knot invariant.

Theorem 5.3 *The Kauffman polynomial is a link invariant.*

Proof. Because the writhe and the Kauffman bracket are unchanged by a Reidemeister move II or III, it remains to show that a Reidemeister move I does not affect $P_L(A)$. By symmetry it is enough to check this for the first variant of a Reidemeister move I as in figure 2.
We then find that $w(L') = w(L) + 1$. For $P_{L'}(A)$ we find: $P_{L'}(A) = (-A)^{-3w(L')}<L'> = (-A)^{-3(w(L)+1)}(-A^3<L>) = (-A)^{-3w(L)}<L> = P_L(A)$. So the Kauffman polynomial is invariant under all Reidemeister moves and therefore it is a link invariant. □

Definition 5.4 The *Jones polynomial* V(L) of an oriented link L is defined by $V(L) = (P_L(A))_{A^{-1}=t^{\frac{1}{4}}} = ((-A)^{-3w(L)}<L>)_{A^{-1}=t^{\frac{1}{4}}} \in \mathbb{Z}[t^{-\frac{1}{2}}, t^{\frac{1}{2}}]$.

Because of theorem 5.3, the Jones polynomial is a link invariant (and thus a knot invariant) as well. Furthermore it is well-defined and $V(O) = 1$. In the definition of the Jones polynomial it is stated that $V(L)$ lies in the ring $\mathbb{Z}[t^{-\frac{1}{2}}, t^{\frac{1}{2}}]$, but why is this the case? Because if the variable A in the Kauffman polynomial is replaced by $t^{-\frac{1}{4}}$, it could be possible that factors of t with powers like $\frac{1}{4}$ occur in the Jones polynomial. We come to the following lemma.

Lemma 5.5 *The Jones polynomial lies in the ring* $\mathbb{Z}[t^{-\frac{1}{2}}, t^{\frac{1}{2}}]$.

Proof. The lemma is proved by induction on the number of crossings in a diagram for an oriented link. If an oriented link has zero crossings and consists of l components then by the first and second rule for the Kauffman bracket we have $<L> = (-(A^2 + A^{-2}))^{l-1}<O> = (-(A^2+A^{-2}))^{l-1}$. Then $V(L) = ((-A)^{-3w(L)}<L>)_{A^{-1}=t^{\frac{1}{4}}} = ((-A)^0(-(A^2+A^{-2}))^{l-1}_{A^{-1}=t^{\frac{1}{4}}} = (-(t^{\frac{1}{2}} + t^{-\frac{1}{2}}))^{l-1} \in \mathbb{Z}[t^{-\frac{1}{2}}, t^{\frac{1}{2}}]$.
Now assume the claim holds for n crossings in the diagram. Then for $n+1$ crossings we find $<L_{n+1}> = A<L_n> + A^{-1}<L_n>$, where L_{n+1} represents a link with $n+1$ crossings in its link diagram and L_n a link with n crossings. So $V(L_{n+1}) = ((-A)^{-3w(L_{n+1})}<L_{n+1}>)_{A^{-1}=t^{\frac{1}{4}}} = ((-A)^{-3w(L_{n+1})+1}<L_n> - (A)^{-3w(L_{n+1})-1}<L_n>)_{A^{-1}=t^{\frac{1}{4}}}$.
The writhe of an oriented link with $n+1$ crossings differs by one with the writhe of an oriented link with n crossings. Without loss of generality it is assumed that $w(L_{n+1}) = w(L_n) + 1$. It follows that $V(L_{n+1}) = ((-A)^{-3w(L_n)+2}<L_n> + (-A)^{-3w(L_n)}<L_n>)_{A^{-1}=t^{\frac{1}{4}}} = (A^2(-A)^{-3w(L_n)}<L_n> + (-A)^{-3w(L_n)}<L_n>)_{A^{-1}=t^{\frac{1}{4}}}$. And the last expression lies in the ring $\mathbb{Z}[t^{-\frac{1}{2}}, t^{\frac{1}{2}}]$ because of the induction hypothesis and the fact that $A^2 = t^{-\frac{1}{2}}$. □

5.2 A skein relation

In the previous section a first knot polynomial has been found. We know that two equivalent knots have the same Jones polynomial. In other words, if two knots have different Jones polynomials the knots are not equivalent. However, if two knots have the same Jones polynomial this does not imply that the knots are equivalent. For example the Kinoshita-Terasaka knot and the Conway knot are two non-equivalent knots with the same Jones polynomial. It was found that every knot with nine or less crossings in its knot diagram, has a distinct Jones polynomial (see [1]).

There is a relation which the Jones polynomial satisfies and this relation provides us a different method for obtaining the Jones polynomial. Such a relation for obtaining knot polynomials is called a skein relation and it will appear another time in this thesis. We consider three oriented links: L_+, L_- and L_0 which are identical except in the neighborhood of some point where they appear as below

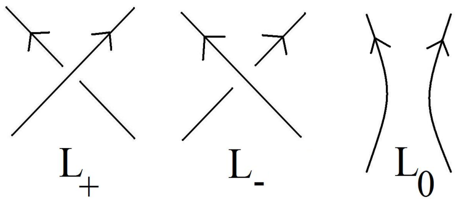

Figure 11: Three oriented links

Then we have the following proposition.

Proposition 5.6 *The Jones polynomial is a function* $V\colon$ {*Oriented links in* S^3} $\to \mathbb{Z}[t^{-\frac{1}{2}}, t^{\frac{1}{2}}]$, *such that* $t^{-1}V(L_+) - tV(L_-) + (t^{-\frac{1}{2}} - t^{\frac{1}{2}})V(L_0) = 0$.

Proof. The second rule that defines the Kauffman bracket reads

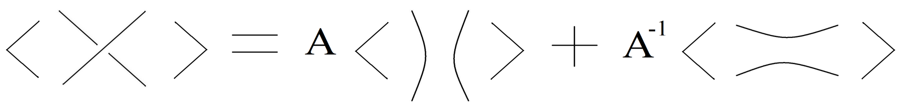

Figure 12: Second bracket rule

Or in equivalent notation

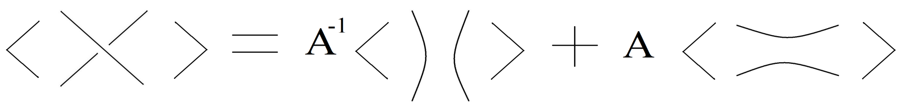

Figure 13: Second bracket rule in equivalent form

Now multiplying the first equation with A and the second with A^{-1} and then subtracting the second of the first gives

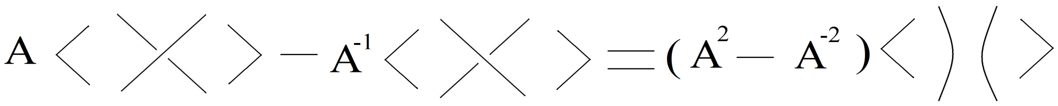

Figure 14: New expression

Because the links L_+, L_- and L_0 are identical besides from the indicated point, it follows that $w(L_+) - 1 = w(L_O) = w(L_-) + 1$. If we now multiply the third equation with $(-A)^{-3w(L_O))}$ we get: $-A^4V(L_+) + A^{-4}V(L_-) = (A^2 - A^{-2})V(L_O)$.
Replace A^{-2} by $t^{\frac{1}{2}}$ to obtain the wanted expression. □

There is another useful proposition that compares the Jones polynomial of a link with the Jones polynomial of the mirror image of the link. It is an indicator for amphicheirality.

Proposition 5.7 *Let L be an oriented link and let L_* denote its mirror image. Then* $V(L)_{t=t^{-1}} = V(L_*)$

Proof. It is clear that $w(L_*) = -w(L)$. Furthermore we have that $<L_*> = (<L>)_{A=A^{-1}}$ because of the rules that determine the Kauffman bracket and the fact that all are changed for a mirror image of a link. The statement then follows by the definition of the Jones polynomial. □

The proposition tells us that if a knot is amphicheiral, its Jones polynomial must be palindromic.

6 Seifert surfaces

The Jones polynomial turns out to be very good in distinguishing knots. An advantage of this knot invariant is that it can tell the difference between a knot and its mirror image. It must be said that Jones originally discovered his polynomial by studying certain operator algebras and not by means of the Kauffman bracket as has been done in the last chapter. The coming of the Jones polynomial put earlier work in knot theory in a new light. The fact is, another knot polynomial had been found in the beginning of the nineteenth century [3]. It is nowadays called the Alexander polynomial and even though it had been found much earlier than the Jones polynomial, its importance was not recognized until Jones came up with his polynomial. A significant part of this thesis is devoted to the Alexander polynomial, its properties and ways to acquire this knot invariant. However, a lot of preliminary work has to be done. That is what this chapter is all about.

6.1 Knot factorisation

It is possible to add two oriented knots in a natural way. Intuitively, one cuts the two knots at one place and glues the remaining parts together, such that orientations match, to form a new knot . In a figure

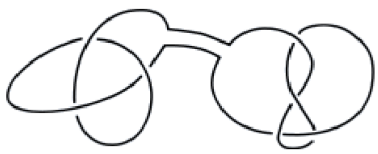

Figure 15: The adding of two knots

Just as there are prime numbers in the set of natural numbers, there are some kind of irreducible knots within the set of oriented knots.

Definition 6.1 A nontrivial knot K is said to be a *prime knot* if $K = K_1 + K_2$ implies that either K_1 or K_2 is the unknot.

It will now be shown that for every oriented link in S^3 there exists a surface , called a Seifert surface, such that the link is the boundary of the surface. First the definition of a Seifert surface will be given.

Definition 6.2 Let L be an oriented link in S^3. A *Seifert surface* for L is an oriented, connected and compact surface in S^3 whose boundary is L, with induced orientation that corresponds to the orientation of L.

A surface is non-orientable if it contains a subset that is homeomorphic with the Möbius strip. The following theorem provides an algorithm for the construction of a Seifert surface if an oriented link is given.

Theorem 6.3 *Any oriented link in S^3 has a Seifert surface.*

Proof. Let D be the link diagram of the oriented link. Now at each crossing in the diagram there are two incoming and two outcoming strands. Cut each crossing open and connect the incoming strand that goes to the right with the outgoing strand that goes to the left. Do likewise for the other strands. A modified diagram D_* is obtained. D_* is a set of non-intersecting oriented circles, called Seifert circuits. So D_* is the boundary of a set of oriented disjoint discs. The discs inherit an orientation of the oriented circles. We continue by connecting the discs with half twisted strips at the places where the crossings used to be. The strips are twisted in such a way that they correspond to the type of crossing there was before. It may occur that the resulting surface is not connected. In that case the surface can be made connected by removing small discs from the larger discs and by inserting long tubes. □

It is important to notice that there could exist different Seifert surfaces for one and the same link in S^3. In practice it is not convenient to deal with very exotic surfaces. What we are searching for is the most economical Seifert surface. Therefore the following property is established.

Definition 6.4 The *genus* g(K) of a knot is defined by
$g(K) = \min.\{\text{genus(F): F is a Seifert surface for K}\}$

Recall that the (topological) genus of an orientable surface equals the number of handles the surface has. For example the disc $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leq 1, z = 0\}$ has genus zero. Furthermore this disc is a Seifert surface for the unknot. It is easy to check that a knot K is the unknot if and only if it has genus zero. The genus of a knot is also a knot invariant since an ambient isotopy does not change the genus of a Seifert surface. The genus is a topological invariant. The following very important theorem states that the genus is an additive function. The proof is quite long and technical.

Theorem 6.5 *Let K_1 and K_2 be knots. Then $g(K_1 + K_2) = g(K_1) + g(K_2)$.*

Proof. First the inequality $g(K_1 + K_2) \leq g(K_1) + g(K_2)$ will be proved. Suppose we have two knots K_1 and K_2 with minimal Seifert surfaces F_1 and F_2 situated far apart in S^3. Then each Seifert surface is a connected surface with non-empty boundary by definition. It will be shown now that $F_1 \cup F_2$ does not separate S^3.
Let $U_1 = S^3 \setminus F_1$, $U_2 = S^3 \setminus F_2$. Then $U_1 \cup U_2 = S^3$ and $U_1 \cap U_2 = S^3 \setminus (F_1 \cup F_2)$. We have to show that $U_1 \cap U_2$ is connected. From elementary homology theory we know that $H_0(X) \cong \mathbb{Z}^{\#p.c.c(X)}$, where $\#p.c.c(X)$ means the number of path-connected components of X. So we have to show that $H_0(U_1 \cap U_2) \cong \mathbb{Z}$. First it will be proved that U_1 is path-connected. This will be done by investigation of the Mayer-Vietoris exact sequence for F_1. The sequence goes as follows
$H_1(F_1 \cup U_1; \mathbb{Z}) \to H_0(F_1 \cap U_1; \mathbb{Z}) \to H_0(F_1; \mathbb{Z}) \oplus H_0(U_1; \mathbb{Z}) \to H_0(U_1 \cup F_1; \mathbb{Z})$.
Now $U_1 \cup F_1 = S^3$ and $U_1 \cap F_1 = \partial F_1 = K_1$. Therefore we have the sequence
$H_1(S^3; \mathbb{Z}) \to H_0(K_1; \mathbb{Z}) \to H_0(F_1; \mathbb{Z}) \oplus H_0(U_1; \mathbb{Z}) \to H_0(S^3; \mathbb{Z})$.
Inspection of the homology groups yields the following. By the Hurewicz theorem: $H_1(S^3; \mathbb{Z}) = \pi_1(S^3, s_0)/[\pi_1(S^3, s_0), \pi_1(S^3, s_0)] = (0)$. $H_0(K_1; \mathbb{Z}) \cong \mathbb{Z}$ because a knot, being an embedding of the circle, is path-connected. $H_0(F_1; \mathbb{Z}) \cong \mathbb{Z}$ because F_1 path-connected. For the same reason also $H_0(S^3; \mathbb{Z}) \cong \mathbb{Z}$. Then the sequence reduces to
$(0) \xrightarrow{\alpha} \mathbb{Z} \xrightarrow{\beta} \mathbb{Z} \oplus \mathbb{Z}^a \xrightarrow{\gamma} \mathbb{Z}$, with $a \in \mathbb{Z}$ the number to be determined.

Exactness of the sequence yields that β is injective and $ker(\gamma) = im(\beta)$. So $\mathbb{Z}^a = \mathbb{Z} \Rightarrow a = 1$. Thus $H_0(U_1;\mathbb{Z}) \cong \mathbb{Z}$ and therefore U_1 is path-connected. A similar argument holds for U_2 so U_2 is also path-connected.
The Mayer-Vietoris sequence for $U_1 \cap U_2$ is given by
$H_1(S^3;\mathbb{Z}) \cong (0) \xrightarrow{\alpha} H_0(U_1 \cap U_2;\mathbb{Z}) \cong \mathbb{Z}^b \xrightarrow{\beta} H_0(U_1;\mathbb{Z}) \oplus H_0(U_2;\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\gamma} H_0(S^3;\mathbb{Z}) \cong \mathbb{Z}$.
Because the sequence is exact $ker(\gamma) = im(\beta) \Rightarrow im(\beta) \cong \mathbb{Z} \Rightarrow b = 1 \Rightarrow H_0(U_1 \cap U_2;\mathbb{Z}) \cong \mathbb{Z} \Rightarrow U_1 \cap U_2$ is path-connected.
Now we can choose an arc α from K_1 to K_2 that never crosses $F_1 \cup F_2$ such that it cuts the 2-sphere which separates K_1 from K_2 once. The arc α can be extended to a thin strip that connects F_1 and F_2 and preserves the orientations of the Seifert surfaces. The constructed surface is a Seifert surface for $K_1 + K_2$. The genus of the new surface is the sum of the genera of the surfaces F_1 and $F_2 \Rightarrow g(K_1 + K_2) \leq g(K_1) + g(K_2)$. The last statement follows from the fact that we at least have an upper bound for the genus of the knot by the constructed Seifert surface.

Now the other inequality will be proved. Suppose F is a Seifert surface for $K_1 + K_2$ with minimal genus. Let Σ be a 2-sphere that cuts $K_1 + K_2$ transversely in two points $\{1\}$ and $\{2\}$. This is another example of the earlier mentioned ball-arc pair. To visualize the situation it is helpful to draw a picture. The 2-sphere Σ separates the knot $K_1 + K_2$ into two arcs α_1 and α_2. If β is an arc on Σ that unites $\{1\}$ and $\{2\}$ then $\alpha_1 \cup \beta$ and $\alpha_2 \cup \beta$ are copies of K_1 and K_2.
F and Σ are surfaces which are piecewise-linear embedded in S^3. Because we are working in the PL-category, S^3 can be triangulated. Therefore F and Σ are sub-complexes of a certain triangulation of S^3. The sphere Σ can be oriented in such a way that its position is transversely with respect to F. So Σ cuts F transversely in the points $\Sigma \cap F$. This means that the tangent spaces of Σ and F in a point in $\Sigma \cap F$ span the tangent spaces of S^3 in that point.
Now without loss of generality we may assume that $\Sigma \cap F$ is a 1-dimensional manifold which consists of a finite collection of simple closed curves and an arc β which connects the points $\{1\}$ and $\{2\}$. The surface F crosses the 2-sphere Σ only a finite number of times and each time it crosses Σ, this yields a simple closed curve. The 2-dimensional variant of the Schönflies theorem states that every simple closed curve in a plane separates the plane into two regions one of which is homeomorphic with the inside of the standard circle. The other region is homeomorphic with the outside of the standard circle in the plane. If the plane is replaced by the 2-sphere then every simple closed curve in a 2-sphere separates the 2-sphere into two discs. In our case only one of the two regions in Σ contains the arc β. We now choose a simple closed curve c of $\Sigma \cap F$ such that it is *innermost* on $\Sigma \setminus \beta$. This means that the curve c bounds a disc D in Σ the interior of which does not contain F.
The disc D can be used to do surgery on the Seifert surface F. The aim is to examine the consequences for the genus of the Seifert surface. A new surface $\widehat{F}$ can be created from F by removing a small annular neighborhood of c and replacing it by two copies of D. A copy is placed on either side of D. This closes the new surface $\widehat{F}$. If the simple closed curve c did not separate F, the new created surface $\widehat{F}$ would be a new Seifert surface for $K_1 + K_2$ with a lower genus than F has since the surgery has the effect of removing a hole. This contradicts our assumption that F was a Seifert surface with minimal genus.
We can conclude that c separates F and that $\widehat{F}$ is disconnected by construction. One of the components of $\widehat{F}$ contains the knot $K_1 + K_2$. This component is a Seifert surface for the knot

with the same genus as F. However, the number of simple closed curves in the intersection of this component and Σ is less than the number in $\Sigma \cap F$ for the simple closed curve c has been removed. Because the number of simple closed curves in $\Sigma \cap F$ was finite, we can repeat this process until we obtain a surface F' which is a Seifert surface for $K_1 + K_2$ with the same genus as F and which intersects Σ only in β.
Then the 2-sphere Σ separates our new constructed surface F' in two surfaces which are Seifert surfaces for K_1 and K_2. Because we started with a Seifert surface with minimal genus for $K_1 + K_2$, it follows that $g(K_1) + g(K_2) \leq g(K_1 + K_2)$. Together with the first part of the proof this proves the theorem. □

This theorem has many consequences. A few are named here.

Corollary 6.6 *Let K_1 and K_2 be knots. If $K_1 + K_2$ is the unknot, then K_1 and K_2 are the unknot.*

This just means that no nontrivial knot has an additive inverse.

Corollary 6.7 *There are infinitely many distinct knots.*

Corollary 6.8 *A knot of genus one is prime.*

Corollary 6.9 *Every knot can be expressed as a finite sum of prime knots.*

The last corollary can be proved by using induction on the genus. The corollary can even be made stronger in the sense that every knot has a unique expression as a finite sum of prime knots. No proof of this statement will be given here, but see for instance [18]. The theorem implies that in our quest to classify knots, it is convenient to restrict attention to prime knots.

7 The Alexander polynomial

In the last chapter the concept of Seifert surfaces has been introduced. We will soon find out that they are crucial in the construction of the Alexander polynomial. Other ingredients result particular from homology theory. This is the original approach that also will be taken during this chapter. The Alexander polynomial is then described as being an invariant of a homology module. It does not matter much whether one uses singular or simplicial homology along the way. They can proven to be equivalent to each other. However, for the reason that we consider knots and links as subsets of S^3, which has a triangulation, it is convenient to use the simpler variant of both, viz. simplicial homology.

There is another way to describe a particular normalized version of the Alexander polynomial, namely by use of a skein relation just as has been done with the Jones polynomial. However, such an approach does not contain the true essence of the knot polynomial and is not much insightful. It just provides an easy method to compute the polynomial. In the following section some necessary results from module theory are introduced.

7.1 The Seifert matrix

It is well known in group theory that if a group-ring R acts on a certain group, the group can be considered as a module M over R. This situation will also be encountered in the construction of the Alexander polynomial. Given the module M, a matrix called the presentation matrix can be associated to it. This is done in the following way.

Let R be a commutative ring with identity and M a module over R. M can be regarded as a vector space with the scalar elements lying in a ring R instead of a field. Not all modules have a basis. If we have a basis for the module then M is called free if every element of M can be expressed in a unique way as a linear combination of basis elements. This is well defined because a commutative ring always satisfies the invariant basis number condition. This means that all free modules over the ring R have a unique rank. Let E and F be two such free modules over R with finite bases. Then the exact sequence $F \xrightarrow{\alpha} E \xrightarrow{\phi} M \to 0$ is called a *finite presentation* for M. The exactness of the sequence implies that ϕ is surjective. Furthermore if $f_1, ..., f_n$ is a basis for F and $e_1, ..., e_m$ a basis for E, then the map α can be represented by a $m \times n$ matrix A in such a way that $\alpha f_i = \sum_j A_{ji} e_j$. A is called a *presentation matrix* for M. The images of $e_1, ..., e_m$ are generators for M and the images of $f_1, ..., f_n$ act as relations between the generators.

Theorem 7.1 *If A and A_1 are two representation matrices for a module M over a commutative ring R, then these two matrices are related by a sequence of matrix moves of the following form and their inverses.*
(i) Permutation of rows and columns
(ii) Replacement of the matrix A by $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$
(iii) Addition of an extra column of zeros to the matrix A
(iv) Addition of a scalar multiple of a row (or column) to another row (or column)

Proof. Because A and A_1 are presentation matrices for M, they correspond to maps α and α_1 in the following finite presentation

$$\begin{tikzcd} F \arrow[r, "\alpha"] \arrow[d, "\gamma"] & E \arrow[r, "\phi"] \arrow[d, "\beta"] & M \arrow[r] \arrow[d, "1"] & 0 \\ F_1 \arrow[r, "\alpha_1"'] & E_1 \arrow[r, "\phi_1"'] & M \arrow[r] & 0 \end{tikzcd}$$

Because the module E has a free basis we can pick a basis element e_i and fix $\phi(e_i)$. We can choose such a $\phi(e_i)$ because there are no relations amongst the generators because E is a free module. Now we pick $e_i^{'} \in E_1$ such that $\phi_1(e_i^{'}) = \phi(e_1)$. This is possible for the reason that ϕ_1 is surjective and E_1 is a free module. Then we construct β such that $\beta(e_i) = e_i^{'}$. It follows that β is linear and that $\phi_1\beta = \phi$.

In a similar way we can construct γ such that $\beta\alpha = \alpha_1\gamma$. The maps β and γ are linear so they can be represented by matrices B and C respectively with respect to the given bases. Then $BA = A_1C$. Now by means of symmetry the roles of α, α_1, ϕ and ϕ_1 could be interchanged in order to produce linear maps γ_1 and β_1 with accessory matrices C_1 and B_1. Then the following relation holds $B_1A_1 = AC_1$.

Now we define that two matrices are equivalent, denoted by " $\sim$ ", if they differ by the matrix moves named above. Then the following matrices are equivalent
$A \sim \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} A & B_1 \\ 0 & I \end{pmatrix} \sim \begin{pmatrix} A & B_1 & 0 \\ 0 & I & 0 \end{pmatrix} \sim \begin{pmatrix} A & B_1 & B_1A_1 \\ 0 & I & A_1 \end{pmatrix} \sim \begin{pmatrix} A & B_1 & 0 \\ 0 & I & A_1 \end{pmatrix} \sim$
$\begin{pmatrix} A & B_1 & 0 & 0 \\ 0 & I & A_1 & 0 \end{pmatrix} \sim \begin{pmatrix} A & B_1 & 0 & B_1B \\ 0 & I & A_1 & B \end{pmatrix}$.
Because $\phi_1\beta = \phi$ and $\phi\beta_1 = \phi_1$ we have that $\phi\beta_1\beta = \phi$. The exactness of the sequence implies that $im(\beta_1\beta - id_E) \subseteq im(\alpha)$. The fact that E is free can be used to construct a linear map $\delta : E \to F$ such that $\alpha\delta = \beta_1\beta - id_E$. Then again a matrix D can be associated to δ, satisfying $AD = B_1B - I$. It follows that

$$A \sim \begin{pmatrix} A & B_1 & 0 & B_1B \\ 0 & I & A_1 & B \end{pmatrix} \sim \begin{pmatrix} A & B_1 & 0 & I \\ 0 & I & A_1 & B \end{pmatrix} \sim \begin{pmatrix} A_1 & B & 0 & I \\ 0 & I & A & B_1 \end{pmatrix} \sim A_1$$

Where in the last step a symmetrical argument is used. □

Recall that the minor M_{ij} of a matrix element A_{ij} of a matrix A is given by the determinant of the matrix A that remains after removing all elements in the same row and column as A_{ij} A $k \times k$ minor of a $m \times n$ matrix A is the determinant of the $k \times k$ matrix obtained from A after removing $m - k$ arbitrary rows and $n - k$ arbitrary columns.

Definition 7.2. Let M be a module over a commutative ring R with an $m \times n$ presentation matrix A. Then the r^{th} *elementary ideal* ε_r of M is the ideal of R generated by the $(m - r + 1) \times (m - r + 1)$ minors of A.

The properties of the determinant together with theorem 7.1 make sure that different presentation matrices for one and the same module have the same elementary ideals. Furthermore, if we are dealing with a square matrix, say an $n \times n$-matrix, then ε_1 is the principal ideal generated by the determinant of the matrix. In that case the n^{th} elementary ideal is the ideal generated by all elements of the $n \times n$-matrix.

It is time to make things more concrete. Therefore we consider a connected compact, orientable surface F with non-empty boundary. If this surface is now piecewise-linearly contained in S^3, so the "hollow handles" of F may be twisted or linked, we come to the following theorem.

Theorem 7.3 *Let F be a surface as just mentioned. Then $H_1(S^3 \setminus F; \mathbb{Z}) \cong H_1(F; \mathbb{Z})$ and there exists a unique non-singular bilinear form β: $H_1(S^3 \setminus F; \mathbb{Z}) \times H_1(F; \mathbb{Z}) \to \mathbb{Z}$ such that for two oriented simple closed curves c in $S^3 \setminus F$ and d in F, $\beta([c], [d]) = lk(c, d)$.*

Here the 1-cycles $[c]$ and $[d]$ are the homology classes of c and d. The proof that the two homology groups are isomorphic makes use of the Mayer-Vietoris sequence. The bilinear form β is defined by means of bases $\{[f_i]\}$ and $\{[e_i]\}$ for $H_1(S^3 \setminus F; \mathbb{Z})$ and $H_1(F; \mathbb{Z})$ respectively. In fact, β only relies on the linking number and hence it is independent of the chosen bases. The homology groups are considered to be $\mathbb{Z}$-modules. The map β is bilinear, which means that it is linear in each of its components if the other component is held fixed. Being non-singular means that the matrix which represents β is invertible.

It is important to note that a Seifert surface F for a link L meets the assumptions of theorem 7.3. A regular neighbourhood N of L is then a disjoint union of solid tori. The exterior X of a link is the closure of the complement of N in S^3, $X = \overline{S^3 \setminus N}$. This in agreement with the definition of the exterior of a knot in section 4.3. Now $X \cap F$ is a copy of F with a small neighbourhood of $\partial F = L$ removed. This can be seen intuitively by drawing a picture. We will now treat $X \cap F$ as just being F. Then the following embeddings can be constructed $i^{\pm} : F \to S^3 \setminus F$, $x \mapsto x \times \pm 1$. It follows that $i^-(F) \cup i^+(F) = F \times \{-1, 1\}$. The last expression is contained as a subset in the regular neighbourhood $F \times [-1, 1]$ of F in X. For an oriented simple closed curve c, we denote $c^{\pm} = i^{\pm} c$.

Definition 7.4 Let F be a Seifert surface for an oriented link L. Then the bilinear form
$\alpha : H_1(F; \mathbb{Z}) \times H_1(F; \mathbb{Z}) \to \mathbb{Z}$, $([c], [d]) \mapsto lk(c^-, d)$
for two simple closed curves c and d in F is called the *Seifert form* for F.

Definition 7.5 Let α be a Seifert form associated to a Seifert surface F. Let $\{[f_i]\}$ be a basis for $H_1(F; \mathbb{Z})$. The matrix A which represents α is called the *Seifert matrix*. We have $A_{ij} = \alpha([f_i], [f_j]) = lk(f_i^-, f_j)$.

First of all the map α is bilinear and well defined by theorem 7.3. Further, the following holds $lk(c^-, d) = lk(c, d^+)$ because we can slide the curves with respect to the second coordinate of $F \times [-1, 1]$ without changing the shape of the curves. In the next section it will be made clear that this Seifert matrix A can be used to obtain a presentation matrix for a certain module. After this has been done, the Alexander polynomial is within reach.

7.2 The Alexander polynomial

In this section the Alexander polynomial in all its glory will be presented. All the pieces introduced earlier will fall into place. With a view to obtain the knot invariant, a particular space has to be studied. Now the construction of this space will first be discussed.

Let F be a Seifert surface for an oriented link L and let X be its exterior. We can cut X along F to get a space Y. The cut edge splits into two edges which both are a copy of F. The copies are called F_- and F_+. The compact space Y is homeomorphic with $X \setminus (F \times (-1, 1))$. Because F_- and F_+ are copies of F we have a homeomorphism $\phi : F_- \to F \to F_+$. Let $\sim_\phi$ denote the equivalence relation generated by ϕ: $x \sim_\phi y \iff \phi(x) = \phi(y)$. Then $X = Y/\sim_\phi$. Geometrically we say that the space X can be recovered from the space Y by gluing F_- and F_+ back together.

The next step is to take countably many copies of Y, denoted by the set $\{Y_i : i \in \mathbb{Z}\}$, and glue them together in the following way. The spaces Y_i can be thought of as puzzle pieces and the collar F_- of Y_i is attached to the collar F_+ of Y_{i+1}. This is done for all $i \in \mathbb{Z}$. The resulting space is called X_∞. More formally: let $h_i : Y \to Y_i$ be a homeomorphism. The subsets $h_i F_-$ of Y_i and $h_{i+1}F_+$ of Y_{i+1} can be identified with the aid of the homeomorphism $h_{i+1}\phi h_i^{-1}$, this is done for all i. Then X_∞ is the disjoint union of the $Y_i's$ divided by the equivalence relation generated by $h_{i+1}\phi h_i^{-1}$ for all i. This is explained graphically in the picture below.

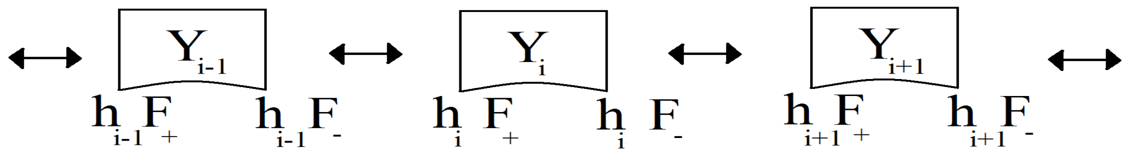

Figure 16: Construction of X_∞

The space X_∞ is found to be a covering space of X and in fact it is the universal cover of X. This will be shown later on.

Now on X_∞ there exists a self-homeomorphism $t : X_\infty \to X_\infty$. This homeomorphism maps Y_i to Y_{i+1} and can be seen as a translation of the puzzle pieces making up X_∞. The pieces are translated one unit to the right. So $t|_{Y_i} = h_{i+1}h_i^{-1}$. It is also possible to apply t twice or several times. Furthermore t^{-1} also is a self-homeomorphism which can be seen as a translation to the left. The infinite cyclic group generated by t, $<t>$, acts on X_∞ by homeomorphisms. This induces an action of a homology automorphism on $H_1(X_\infty; \mathbb{Z})$. Namely, homology groups are functorial. If we have a continuous map $f : X \to Y$, this induces a homomorphism, $f_\star$, between the free abelian groups generated by singular k-simplices of X and Y. Therefore we have a homomorphism between the k-th homology groups of the spaces X and Y for every $k \in \mathbb{Z}_{\geq 0}$. In the case that the f is a self-homeomorphism of a space X, this induces an automorphism of the homology group of X. In this context it is common to denote the induced automorphism also by t instead of $t_\star$.

It is a well-known fact in group- and module theory that the ring $\mathbb{Z}$ acts on any abelian group. This is because any abelian group is a $\mathbb{Z}$-module and a module is an action on an abelian group by definition. Because the first homology group of a topological space is equal to the fundamental group of the space with a certain base point, divided by its commutator subgroup, the first homology group is abelian. Hence, $\mathbb{Z}$ also acts on $H_1(X_\infty; \mathbb{Z})$. This implies that the group-ring $\mathbb{Z}<t>$ acts on $H_1(X_\infty;\mathbb{Z})$. The group-ring $\mathbb{Z}<t>$ consists of elements of the form $\sum_i a_i g_i$ with $a_i \in \mathbb{Z}$ and $g_i \in <t>$ and $n_i \neq 0$ for finitely many i. Therefore the group-ring $\mathbb{Z}<t>$ can be identified with the ring $\mathbb{Z}[t^{-1}, t]$. By means of the action just described, $H_1(X_\infty;\mathbb{Z})$ can be considered as a module over the ring $\mathbb{Z}[t^{-1}, t]$. For this module, a finite presentation with presentation matrix will be found.

Theorem 7.6 *Let L be an oriented link in S^3 with Seifert surface F. Let α be the Seifert form associated to F and let A be a matrix for α with respect to a basis for $H_1(F;\mathbb{Z})$. Then the expression $tA - A^\tau$ is a presentation matrix for the $\mathbb{Z}[t^{-1}, t]$-module $H_1(X_\infty;\mathbb{Z})$.*

Here A^τ represents the transposed matrix of A. The term tA just stands for the matrix A with every element It follows directly from the theorem that the module $H_1(X_\infty;\mathbb{Z})$ has a square presentation matrix. A proof of 7.6 may be found in [18]. Now finally the definition of the Alexander polynomial can be given.

Definition 7.7 The r^{th} *Alexander ideal* of an oriented link L is the r^{th} elementary ideal of the $\mathbb{Z}[t^{-1}, t]$-module $H_1(X_\infty;\mathbb{Z})$. The r^{th} *Alexander polynomial* of L is a generator of the smallest principal ideal of $\mathbb{Z}[t^{-1}, t]$ that contains the r^{th} Alexander ideal.

In this thesis only the first Alexander polynomial will be discussed. It is referred to as *the* Alexander polynomial and is indicated as $\Delta_L(t)$. The definition of the Alexander polynomials seems quite complex. Besides that, there is the disadvantage that the ring $\mathbb{Z}[t^{-1}, t]$ is not a principal ideal domain, which has nice properties. On top the invertible elements of this ring, called the units, are $\pm t^{\pm n}$ with $n \in \mathbb{N}$. Hence, the Alexander polynomial is defined only up to multiplication by such a unit.

The calculation of the Alexander polynomial of an oriented link L amounts to the determination of the Seifert matrix A corresponding with a Seifert form associated, in turn, with an Seifert surface F for L. If this A can be determined, theorem 7.6 says that a presentation matrix for the module $H_1(X_\infty;\mathbb{Z})$ can be given. From that matrix the elementary ideals can be calculated and ultimately the Alexander polynomial. Because the presentation matrix is always square the first elementary ideal is the ideal generated by the determinant of the presentation matrix. Thus the first elementary ideal is principal and the Alexander polynomial is given by

$$\Delta_L(t) \doteq det(tA - A^\tau) \tag{1}$$

Here the symbol $\doteq$ means that the polynomial is equal to the expression on the right side of the equation up to a multiplication by a unit of the ring $\mathbb{Z}[t^{-1}, t]$.

A problem in the definition of the Alexander polynomial is whether or not the action of the infinite cyclic group on X_∞ is uniquely determined by the oriented link. This is necessary to check for if it is not true, the Alexander polynomial cannot be a link invariant. If the statement is true, then we have found our link invariant. Namely, it implies that the first homology group $H_1(X_\infty;\mathbb{Z})$ is a link invariant. Furthermore it has been said before that theorem 7.1 together with the properties of the determinant function comprise the fact that the elementary ideals of a module are invariants. They are independent of the chosen presentation matrix. It can thus be concluded that the Alexander polynomial is a link invariant. The question about the action on X_∞ being dependent on the oriented link or not will be answered in section 7.4 which deals with covering spaces. The next section contains a few properties of the Alexander polynomial. From these properties the effectiveness of the Alexander polynomial in distinguishing knots can be deduced.

7.3 Properties

In chapter 5 the Jones polynomial has been introduced. This knot invariant is quite good in distinguishing knots. That means that it can tell apart prime knots with a relative high number of crossings in their knot diagram. All the prime knots with nine or less crossings in their knot diagram have a distinct Jones polynomial [1]. The same holds for the Alexander polynomial for prime knots with eight or less crossings in their knot diagram [18]. Now this may cause one to think that the Alexander polynomial is useless compared with the Jones polynomial. Wrongly thought; the Alexander polynomial is completely different from the Jones polynomial. It has some nice properties and provides utile information about knots. A few properties will now be explored.

Theorem 7.8 *Let L be an oriented link and K be an oriented knot, then*
(i) $\Delta_L(t) \doteq \Delta_L(t^{-1})$
(ii) $\Delta_K(1) = \pm 1$.

Proof. (i) Let A be an $n \times n$ Seifert matrix for L. Then we have $\Delta_L(t) \doteq det(tA - A^\tau) = det((tA - A^\tau)^\tau) = det(tA^\tau - A) = det(-t(t^{-1}A - A^\tau)) = (-t)^n det(t^{-1}A - A^\tau) \doteq \Delta_L(t^{-1})$
(ii) Let F be a Seifert surface of K with genus g. Because F has one boundary component, we have that $H_1(F;\mathbb{Z}) = \oplus_{2g}\mathbb{Z}$ generated by $\{[f_i]\}$. Here the f_i are simple closed curves as shown below.

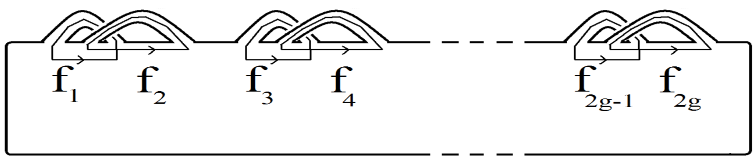

Figure 17: Generators of $H_1(F;\mathbb{Z})$

Now we can calculate the presentation matrix. $\Delta_K(1) = \pm det(A - A^\tau)$. By definition 7.5 the matrix elements are given by $(A - A^\tau)_{ij} = lk(f_i^-, f_j) - lk(f_i^+, f_j)$. By looking at the picture for the Seifert surface we see that the diagonal elements of $A - A^\tau$ are zero. In fact only the super- and subdiagonal entries are non zero. By inspection and recalling the definition of the linking number we see that the matrix consists of g blocks of the form $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The determinant of this matrix is 1. So $\Delta_K(1) = \pm 1$. □

Corollary 7.9 *For a knot K we have $\Delta_K(t) \doteq a_0 + a_1(t^{-1} + t) + a_2(t^{-2} + t^2) + ...,$, where $a_i \in \mathbb{N}$ and a_0 is odd.*

Proof. The Alexander polynomial of a knot can be multiplied with a positive power of t in order to get rid of all terms with negative powers of t. Thus $\Delta_K(t) = a_0 + a_1t + ... + a_nt^n$. Because of Theorem 7.8(i) we have that $\Delta_K(t) \doteq \Delta_K(t^{-1})$. Therefore the coefficients must obey the following relation: $a_r = \pm a_{n-r}$. The choice of the sign must be the same for all r for it is not allowed to multiply some terms of the Alexander polynomial with one unit and the other terms with the same unit with altered sign. Assume n is odd. Then $\Delta_K(1) = a_0 + a_1 + ... + a_n = 2(a_0 + a_1 + ... + a_{\frac{n-1}{2}})$. This implies that $\Delta_K(1)$ is even, contradicting Theorem 7.8(ii). Thus n is even. Now assume $a_r = -a_{n-r}$. Then $a_{\frac{n}{2}} = 0$ and $\Delta_K(1) = 0$ which contradicts Theorem 7.8(ii) again. Hence $a_r = a_{n-r}$. We may also conclude that $a_{\frac{n}{2}}$ is odd for if otherwise, $\Delta_K(1)$ is the sum of two even numbers and therefore even. Returning to the more general form of the Alexander polynomial we then find $\Delta_K(t) \doteq a_0 + a_1(t^{-1} + t) + a_2(t^{-2} + t^2) + ...,$. □

The shape of the Alexander polynomial for a knot as given in corollary 7.9 with the normalization $\Delta_K(1) = 1$ is the standard one. In tables of the Alexander polynomial for prime knots only the coefficients $a_0, a_1, ...$ are specified. The breadth of a Laurent polynomial in the variable t is the difference between the highest degree of t and the smallest degree of t that occur in the polynomial.

Proposition 7.10 *If K is a knot with genus g, then $2g \geq breadth\Delta_K(t)$.*

Proof. Let F be a Seifert surface for K with genus g and Seifert matrix A. Then $tA - A^\tau$ is a $2g \times 2g$ matrix by the same argument as in theorem 7.8(ii). Then the degree of the polynomial $det(tA - A^\tau)$ is at most $2g$. □

Proposition 7.10 says that the breadth of the Alexander polynomial for a knot gives a lower bound for the genus of the knot. Therefore the Alexander polynomial gives geometrical information about the knot. The proposition can easily be extended to a similar statement for links with n components. A link with n components has a Seifert surface with n boundary components. This is a surface which can be obtained by a surface with genus g with no boundary. One has to remove the interior of n disjoint discs to obtain the n boundary components. The genus of the constructed surface remains the same, g. The inequality then becomes: $2g + n - 1 \geq breadth\Delta_L(t)$. This is because a link with n boundary components has $2g + n - 1$ simple closed curves which are the generators for $H_1(F; \mathbb{Z})$, with F being a Seifert surface for the link. The latter can be seen by inspection of a surface similar to the one given in the proof of theorem 7.8 with n boundary components instead of one.

7.4 Covering spaces

In this section the formulation of the Alexander polynomial will be completed. This is done by showing that the action of the self-homeomorphism t on X_∞ is well defined, i.e: independent of the choice of the Seifert surface for the oriented link that is considered. In order to do so, an introduction in the theory of covering spaces is necessary.

First the definition of a covering map will be given. It will be assumed that B is a topological space which is Hausdorff, locally path-connected and path connected.

Definition 7.11 A continuous map $p : E \to B$ is called a *covering map* of the *covering space* E if
(i) E is locally path-connected and non-empty
(ii) $\forall b \in B$ there exists an open neighborhood V of b such that $p^{-1}(V)$ is a disjoint union of open sets in E. Each of these open sets must be homeomorphic with V by the map p.
E is said to be a *universal covering space* if it is simply-connected.

An easy example of a covering map is the map $p : \mathbb{R} \to S^1, x \mapsto (cos(2\pi x), sin(2\pi x))$. The 'line' $\mathbb{R}$ can be wrapped around the circle like a helix. The full original of every open subset of the circle corresponds to an infinite collection of open subsets of $\mathbb{R}$. There is a nice connection between the theory of covering spaces and classical Galois theory. This connection is best described by Grothendieck. Classical Galois theory makes an one-to-one correspondence between Galois extensions of a field and subgroups of the Galois group. In the theory of covering spaces there exists a connection between the path-connected covering spaces of a path-connected space X and the subgroups of the fundamental group of X. Grothendieck formalizes this collective structure with the aid of what he calls a Galoiscategory. We will however not go into further detail about this beautiful Galois connection. Instead some technical machinery will be discussed in order to come back on the Alexander polynomial.

Definition 7.12 Let X be a topological space and $p : E \to B$ be a covering map. Let $h : X \to B$ be a continuous map. A continuous map $\tilde{h} : X \to E$ such that $h = p\tilde{h}$ is called a *lift* of h.

Lemma 7.13 *Let X, E be topological spaces. Let $F : X \times I \to E$ be a continuous map. We write $f_t(x) = F(x, t)$. Let $\tilde{f}_0 : X \to E$ be a lift of a map f_0. Then there exists a unique lift $\tilde{F} : X \times I \to E$ of F such that $\tilde{F}(x, 0) = \tilde{f}_0(x)$ for all $x \in X$.*

This lemma means that if we have a collection of continuous maps $f_t(x)$ and $f_0(x)$ is a lift, we can lift all these continuous maps $f_t(x)$ to $\tilde{f}_t(x)$. There follow two important consequences.

Corollary 7.14 *Let $p : E \to B$ be a covering map. Let $e_o \in E$ be a point with $p(e_0) = b_0$. Then we have the following two properties*
(i) Path lifting property. Let γ be a path in B with base point $b_0 \in B$. Then there exists a unique path $\tilde{\gamma}$ in E with starting point e_0 which lifts γ, i.e: $\gamma = p\tilde{\gamma}$
(ii) Homotopy lifting property. Let γ_0 and γ_1 be paths in B with starting point b_0 and let $F : I \times I \to B$ be a homotopy from γ_0 to γ_1 relative $\{0\}$. Then there is a unique lifting $\tilde{F} : I \times I \to E$ of F with $\tilde{F}(0, t) = e_0$ for all $t \in I$.

Proof. (i) This statement follows directly from lemma 7.13 with the space X being a point.
(ii) We apply lemma 7.13 with X being the unit interval I. We have the homotopy $F : I \times I \to B$ and write $\gamma_t(x) = F(x, t)$. Because of (i) there exists a unique lift $\tilde{\gamma}_0 : I \to E$ of γ_0 with starting point e_0. From lemma 7.13 it follows that there exists a unique lift $\tilde{F} I \times I \to E$ such that $\tilde{F}(x, o) = \tilde{\gamma}_0(x)$. Then the map defined by $t \mapsto \tilde{F}(0, t)$ is a path in E with starting point e_0 which is a lift of the constant path in B with value b_0. Because of the unicity of (i) we have that $\tilde{F}(0, t) = e_0$ for all $t \in I$. $\square$

The question arises in which cases it is possible to lift a certain map if a covering map is given. The next theorem gives the answer.

Theorem 7.15 *Let $p : E \to B$ be a covering map with base points $e_0 \in E$ and $b_0 \in B$ such that $p(e_0) = b_0$. Let X be a path-connected, locally path-connected topological space with base point x_0. Let $f : (X, x_0) \to (B, b_0)$ be a continuous map. Then there exists a continuous $g : (X, x_0) \to (E, e_0) \iff f_\star(\pi_1(X, x_0)) \subseteq p_\star\pi_1(E, e_0)$.*

Proof. " $\Rightarrow$ " If g exists then $(p \circ g)_\star = p_\star \circ g_\star = f_\star$ and it follows that $f_\star(\pi_1(X, x_0)) = p_\star \circ g_\star(\pi_1(X, x_0)) \subseteq p_\star\pi_1(E, e_0)$.
" $\Leftarrow$ " Assume that $f_\star(\pi_1(X, x_0)) \subseteq p_\star\pi_1(E, e_0)$. For $x_1 \in X$ choose a path $\alpha : I \to X$ with $\alpha(0) = x_0, \alpha(1) = x_1$. Then $f(\alpha)$ is a path in E with $f(\alpha(0)) = f(x_0) = b_0$ and $f(\alpha(1)) = f(x_1)$. Now $f(0) = b_0 = p(e_0)$ so because of corollary 7.14(i) there exists a lift $\widehat{f(\alpha)} : I \to E$ with $\widehat{f(\alpha(0))} = e_0$. If there exists a continuous $g : X \to E$ with $g(x_0) = e_0$ then the following holds $g(x) = \widehat{f(\alpha(1))} = \widehat{f(x_1)}$. This is due to the fact that the lift is unique by corollary 5.14(i) because $g(\alpha)$ is a lift of $f(\alpha)$.
Define $g(x) = \widehat{f(\alpha(1))}$. We have to investigate if this function is well-defined. Let β be an other path in X from x_0 to x_1. Then $f_\star([\alpha \star \bar{\beta}])$ is the image of a homotopy class of a loop with basis point x_0. Here $\bar{\beta}$ denotes the inverse path. It follows that $f_\star([\alpha \star \bar{\beta}]) \in f_\star(\pi_1(X, x_0)) \subseteq p_\star(\pi_1(E, e_0))$. So $f_\star(\pi_1(X, x_0))$ is a loop in B with basis point b_0. Then there is a loop $\gamma : I \to E$ with $\gamma(0) = e_0 = \gamma(1)$ such that $p(\gamma)$ is homotopic with $f(\alpha \star \bar{\beta})$ relative $\{0, 1\}$. The homotopy from $p(\gamma)$ to $f(\alpha \star \bar{\beta})$ can be lifted relative $\{0, 1\}$ in a unique way by the homotopy lifting property. In this way we obtain a loop $\widehat{\gamma} : I \to E$ with $\widehat{\gamma}(0) = \widehat{\gamma}(1) = e_0$ which is homotopic relative $\{0, 1\}$ with $\widehat{\alpha \star \bar{\beta}}$ such that $p(\widehat{\gamma}) = f(\alpha \star \bar{\beta})$. This implies that $g(1) = \widehat{f(\alpha(1))} = \widehat{\gamma(\frac{1}{2})} = \widehat{f(\beta(1))}$. Where the second equality comes from the fact that $p(\widehat{\gamma(t)}) = f(\alpha(2t)) \iff \widehat{f(t)} = \widehat{f(\alpha(2t))}$. We conclude that g is well defined.
It remains to check continuity of g. Take $x_1 \in X$. Let U be an open neighborhood of $f(x_1) \in B$ which has a lift $\widehat{U} \subset E$ which contains $g(x_1)$ such that $p : \widehat{U} \to U$ is a homeomorphism. Such an U exists because p is a covering map. Take a path-connected $V \subset X$ of x_1 such that $f(V) \subset U$. This is possible because X is locally path-connected. So the requirement of X being locally path-connected really cannot be missed.
For paths from x_0 to $x_2 \in V$ we can take the composition of the path from x_0 to x_1 (α) with a path η from x_1 to x_2. There exists such an η because x_1 and x_2 lie in the same path-connected subset. Then $f(\alpha) \star f(\eta)$ has a lift in B: $\widehat{\widehat{f(\alpha)} \star \widehat{f(\alpha)}}$. We have that $\widehat{f(\eta)} = p^{-1}(f(\eta))$ and $p^{-1} : U \to \widehat{U}$ is the inverse of p which exists for the reason that p is a homemorphism locally. Now $g(V) \subset \widehat{U}$ for $f(V) \subset U$, thus $p^{-1}(f(V)) = g(V) \subset \widehat{U}$ because p is homeomorphism locally. Furthermore $g|V = p^{-1}(f)$ which implies that g is continuous at $x_2 \in V$ due to the fact that f and p^{-1} are continuous. Because $x_2 \in V$ was arbitrary it follows that g is continuous on V. Therefore g is continuous because continuity is local on the domain. $\square$

The subgroup $p_\star(\pi_1(E, e_0))$ of $\pi_1(B, b_0)$ is called the *group of covering.*

Proposition 7.16 *Let $p : (E, e_0) \to (B, b_0)$ and $p' : (E', e'_0) \to (B, b_0)$ be two covering maps of B with the same group of covering. Then p and p' differ by a homeomorphism: there exists a homeomorphism $h : (E', e'_0) \to (E, e_0)$ such that $ph = p'$.*

A proof of the last proposition is fairly easy ([18]) and relies on theorem 7.15. In section 7.2 it was said that the space X_∞ is a universal cover of the exterior X of a link. The covering map is $p : X_\infty \to X$. This map sends the infinite many copies of the Seifert surface F, for each $i \in \mathbb{Z}$ this includes F_- and F_+, to $F \subset X$. Remember that we regarded $F \cap X$ as being F, so indeed $F \subset X$. What remains in X_∞ are just copies of $X \setminus F$, therefore the rest of X_∞ is just projected on $X \setminus F$ by p.

Theorem 7.17 *Let L be an oriented link and denote $p : X_\infty \to X$ for the covering map. Then the covering space X_∞ is independent of the choice of the Seifert surface F for L. Furthermore the action of $<t>$ on X_∞ is independent of the choice of F.*

Proof. By the path lifting property a loop $\alpha : I \to X$ lifts to a path $\widehat{\alpha} : I \to X_\infty$. If $\widehat{\alpha}(0)$ and $\widehat{\alpha}(1)$ are in the same copy of Y then the loop α is lifted to a loop in X_∞. By the construction of X_∞ it is clear that $\widehat{\alpha}(0)$ and $\widehat{\alpha}(1)$ lie in the same copy of Y if and only if α intersects F zero times algebraically. Remember that the copies of F served as 'glue' in the construction of X_∞ so if α crosses F, its lift is moved from one copy of Y to an adjacent copy. By the definitions of the linking number this means that α lifts to a loop if and only if the sum of the linking numbers of α with each component of L is zero. Note that the last statement is independent of the choice of a Seifert surface F. So now we have a criterion for loops α lifting to loops in X_∞. But this means that we have fully described the group of covering. Namely, the group of covering, $p_\star(\pi_1(X_\infty))$, consists of homotopy classes, relative $\{0, 1\}$, of loops $\alpha : I \to X$ that lift to a loop in X_∞. This implies that the group of covering is independent of F. By proposition 7.16 the first result follows.
For the second statement it is important to remember that the infinite cyclic group $<t>$ acts on X_∞ by a sort of translation. Consider a path $\gamma : I \to X_\infty$ from a to ta. Because $p(a) = p(ta)$, $p\gamma$ is a loop in X which has linking number 1 with L. Now the other way around: if $\gamma' : I \to X$ is a loop with basis point a, then this loop lifts to a path from some a to ta by the path lifting property. Suppose that we have another Seifert surface F' for L which is used to construct another infinite cyclic cover X'_∞ with covering map $p' : X'_\infty \to X$. Let $h' : X_\infty \to X'_\infty$ be the homeomorphism such that $p'h' = p$, such a homeomorphism exists by proposition 7.16. Then due to the fact that $p'h'\gamma = p\gamma$, $h'\gamma$ lifts from the loop $p\gamma$ with respect to the covering p' to the path in X'_∞ from a point a to ta. Thus $th'(a) = h'(ta)$. This means that h' preserves an action by t. Therefore the action is independent of F. □

Theorem 7.17 and the construction of the Alexander polynomial for an oriented link L imply that the polynomial is a link invariant. To restate what this means: equivalent links have the same Alexander polynomial and if two links have different Alexander polynomial, they are certainly not equivalent. The knot polynomial turns out to be quite good in telling knots apart [18]. Prime knots with eight or less crossings in their knot diagram have a different Alexander polynomial. Moreover the polynomial has interesting features. It was seen that the Alexander polynomial of a knot can be used to give a lower bound for the genus of the knot. An important shortcoming of the polynomial is that it cannot tell the difference between a link, the reflected link and the mirror link. This can be seen by the fact that if A is a Seifert matrix for the link, $-A$ is a Seifert matrix for the reflected link and A^τ is a Seifert matrix for the mirror link. That means that the Alexander polynomial is the same, up to multiplication by a unit, for all three links. Another property is that there are infinitely many non-trivial knots with Alexander polynomial equal to one. We refer to [23] for an argument for this statement.

To obtain the Alexander polynomial it was necessary to pick an (arbitrary) Seifert surface for the given link. Then one has to determine a matrix for the corresponding Seifert form. By evaluating a simple determinant, the Alexander polynomial emerges. There are several other approaches to the Alexander polynomial. One of them is introduced in the following section.

7.5 A skein relation

A disadvantage of the Alexander polynomial is that we cannot add two polynomials. This is due to the fact that the Alexander polynomial is defined only up to multiplication by a unit $\pm t^n$. In contrary, two Jones polynomials can be added. Therefore it was possible to deduce a skein relation for the Jones polynomial. It follows that we cannot give a skein relation for the Alexander polynomial in its current state.

There exists a normalization of the Alexander polynomial which resolves the ambiguity concerning the multiplication by a unit. This normalized polynomial is called the Conway polynomial. For now, it requires too much work to give the formal definition of the Conway polynomial but the polynomial can be regarded as just being the Alexander polynomial with a small modification. The skein relation is given by the following relation.

Theorem 7.18 *Let L_+, L_- and L_0 be three oriented links which are identical except in the neighbourhood of a point where they appear as in figure 10. Then the Conway-normalized Alexander polynomial $\Delta_L(t) \in \mathbb{Z}[t^{-\frac{1}{2}}, t^{\frac{1}{2}}]$ is characterized by*
(i) $\Delta_{unknot}(t) = 1$,
(ii) $\Delta_{L_+} - \Delta_{L_-} = (t^{-\frac{1}{2}} - t^{\frac{1}{2}})\Delta_{L_0}$.

By repeated use of this relation, one is able to determine the Conway polynomial for an oriented link by use of a link diagram only. In the following chapter the Alexander polynomial will be abandoned. An application of knot theory in theoretical physics will be presented.

8 Anyons

In the previous chapters an elementary introduction to knot theory has been given. In particular the Alexander polynomial has been studied as a knot invariant. We now should have a pretty good idea what the variable 't' represents in the expression for the Alexander polynomial, namely a homeomorphism from the space X_∞ to itself. More geometrically it can be seen as a translation: the pieces making up X_∞ are slided one piece to the right. In contrary to the Alexander polynomial, there is not yet a good explanation for what exactly the variable 't' means in the expression for the Jones polynomial. This is because the Jones polynomial originally has been discovered in a field outside knot theory, viz. by a representation of the braid group, coming from a set of generators that satisfies the Temperley-Lieb algebra [14].

In this section a relative new application of knot theory in theoretical physics is presented. We study particles with strange statistics: so-called anyons. It turns out that the Jones polynomial arises from one specific anyonic model. This supplies an alternative way to look at the Jones polynomial. In order to describe these anyons, a very short introduction to yet another branch of mathematics, braid theory, will be given.

8.1 Braids

If one cuts every piecewise linear simple closed curve making up a link at one place and rearranges the obtained strands a bit, one gets a braid. This mathematical braid does not differ much from for example the braid in the hair of a girl. Here comes the definition, as given in [21], of a braid.

Definition 8.1 A *braid on n strings* is given by the following properties:
(i) n points P_1, P_2,...,P_n in $\mathbb{R}^3$ with the same z-coordinate, $z = a$. Furthermore the x-coordinate strictly increases if one goes from P_i to P_{i+1} along the line segment P_iP_{i+1}, for each $i \in \{1, 2, .., n-1\}$.
(ii) n points Q_1, Q_2, ..., Q_n in $\mathbb{R}^3$ with the same z-coordinate, $z = b$ with $a > b$. Furthermore the x-coordinate strictly increases if one goes from Q_i to Q_{i+1} along the line segment Q_iQ_{i+1}, for each $i \in \{1, 2, .., n-1\}$.
(iii) For all $j \in \{1, 2, .., n\}$ there is a finite piecewise linear curve that joins Q_j and P_{j_k}, with j_k a permutation of $1, 2, ..., n$. If one travels along this curve from Q_j to P_{j_k} the z-coordinate strictly increases.
(iv) No two distinct piecewise linear curves intersect.

These conditions seem rather restrictive but it turns out that they are necessary to avoid complicated and undesirable situations. For example the demand that the piecewise linear curves are finite, i.e: they are made up of a finite number of straight line segments, is needful to avoid ever smaller becoming strings. There are other ways to define a braid. For instance a braid can be defined as an embedding of a collection of disjoint strands, with initial and end points just as stated above, in $\mathbb{R}^3$ [16]. The individual strands are not permitted to knot. Two braids are said to be equivalent if they are ambient isotopic. That means that for two equivalent braids there is an ambient isotopy from the one embedding to the other.

Lemma 8.2 *The relation "braid one on n strings is ambient isotopic with braid two on n strings" is an equivalence relation on the set of braids on n strings.*

There is an operation on braids. If σ_1 and σ_2 are n-braids, then $\sigma_1\sigma_2$ is a n-braid. The product is constructed as follows: one takes σ_1 and attaches σ_2 on it in such a way that the end points of the strings of σ_1 are the initial points of σ_2. The composition of these n-braids is well-defined on the ambient isotopy classes. This is a fact we will not verify here. We then come to the following theorem.

Theorem 8.3 *Fix n initial and n end points in $\mathbb{R}^3$. The set of ambient isotopy classes of braids on n strings with respect to the fixed points, with an operation being the composition of n-braids as formulated above, forms a group. This group is called the Artin braid group and is denoted by B_n.*

The Artin braid group can be described completely by a set of generators and relations between the generators. The generators are σ_1, σ_2,...,σ_{n-1} and their inverses σ_1^{-1},σ_2^{-1},...,σ_{n-1}^{-1} and they are easy to picture.

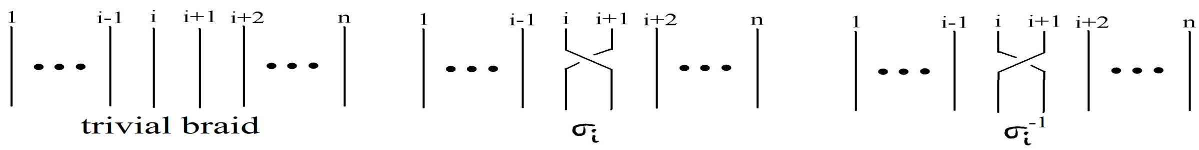

Figure 18: The trivial braid and the generators of the Artin-Braid group

The diagrams above are called braid diagrams, in these diagrams it is made clear how the strings cross. The leftmost braid is called the trivial braid. The relations, called Artin relations, among the generators are the following
(i) $\sigma_i\sigma_i^{-1} = 1$, for i = 1,2,...,n-1
(ii) $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$, for i = 1,2,...,n-1
(iii) $\sigma_i\sigma_j = \sigma_j\sigma_i$, for $|i-j| > 1$.

The second relation being the Yang-Baxter equation which plays an important role in statistical mechanics. In his paper [4], Emil Artin assumed that these generators and the relations define the group. He did not give a proof of this statement. A proof, which is quite hard, can be found in [21]. It is important to notice that the Artin braid group can be obtained by different means. The description in theorem 8.3 is a topological one (B_n^{top}). The characterization just given above is algebraic. The braid group on n strings, B_n^{alg}, is now defined as the free abelian group on the $\sigma_1, ..., \sigma_{n-1}$ and their inverses subject to the Artin relations. Another way to construct the braid group is by the use of configuration spaces.

The configuration space of n points is given by $C_n = \{D \subset \mathbb{C} | \#D = n\}$. So $D = \{z_1, z_2, ..., z_n | \forall i : z_i \in \mathbb{C}\ and\ z_i \neq z_j\ if\ i \neq j\}$. Consider now the following set $F_{\mathbb{C},n} = \mathbb{C}^n \setminus \cup_{i\neq j}\Delta_{ij}$, where Δ_{ij} denotes the set $\{z_1, z_2, .., z_n | \forall i : z_i \in \mathbb{C}\ and\ z_i = z_j\}$. An element of $F_{\mathbb{C},n}$ can be seen as a n-tuple with distinct coordinates. The symmetric group S_n acts freely on $F_{\mathbb{C},n}$ by permuting the coordinates of the n-tuple. This gives rise to an orbit space projection μ. It follows that the orbit space equals the configuration space, $C_n = S_n \setminus (\mathbb{C}^n \setminus \cup_{i\neq j}\Delta_{ij})$. Because the action is also fixed point free, μ is in fact a covering map. The braid group on n strings can now be defined by $\pi_1(S_n \setminus (\mathbb{C}^n \setminus \cup_{i\neq j}\Delta_{ij}), p_0) = B_n^{config}$, with $p_0 = (1, 2, ..., n)$.

It is not immediately clear what this fundamental group has to do with the topological and algebraic braid group. A very short explanation is given. An element in B_n^{config} is a loop that lifts uniquely to a path in $\mathbb{C}^n \setminus \cup_{i\neq j}\Delta_{ij}$, because μ is a covering map. This path consists of n coordinate functions $z_1(t), ..., z_n(t)$ that map n initial points at $t = 0$ to n end points at $t = 1$. The graph of these coordinate functions represents a geometrical braid. There are two theorems, one by Artin and one by Fox and Neuwirth, that, together, state that the braid groups obtained in various ways are isomorphic as groups: $B_n^{config} \cong B_n^{alg} \cong B_n^{top}$ ([4], [9]).

Next we want to say something about closed braids. One can take the closure of a braid by connecting the initial points to the end points with the aid of parallel strands. This is shown schematically below.

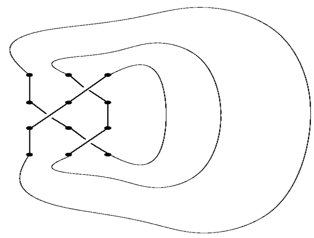

Figure 19: The closure of a braid

The closure of this braid in B_2 is ambient isotopic with the trefoil knot. By taking the closure of a braid one gets a knot or a link. The mathematician James Alexander, the same Alexander as the polynomial, proved in 1923 that every link in S^3 is ambient isotopic to a link obtained by a closed braid [2].

8.2 Anyons

All cards are now on the table. It is time to play the anyon game. From quantum statistical mechanics it has been known for a long period of time that in four-dimensional spacetime there are two classes in which every particle can be put depending on its nature. On the one hand there are fermions, particles that obey Fermi-Dirac statistics. On the other hand there are bosons, particles that obey Bose-Einstein statistics. In order to get a feel for the theory of anyons, it is important to understand why particles in four-dimensional spacetime come in two different types.

Imagine we have n identical particles with a wave vector $|\psi\rangle$ in the complex Hilbert space describing the quantum state of the particles. Since the particles are identical, the physics does not change if we permute the particles. This is translated into the fact that a permutation causes an extra phase with modulus one to the wave vector: $P|\psi\rangle = e^{i\theta}|\psi\rangle$, for the reason that an extra phase does not change the physical properties of the system. It follows that we must have an one-dimensional representation of the symmetric group S_n on the complex Hilbert space. The representation is one-dimensional because $e^{i\theta} \in \mathbb{C}$, so we choose our base field to be $\mathbb{C}$. Then we are looking for homomorphisms ρ: $S_n \to GL(1, \mathbb{C}) \cong \mathbb{C}^*$. Due to the fact that all elements of S_n have finite order, $\rho(\sigma)$ is a root of unity. Because if an element of the symmetric group has order k, then $\rho(\sigma)^k = \rho(\sigma^k) = \rho(e) = 1$. This implies that $|\rho(\sigma)| = 1$ for all $\sigma \in S_n$. Furthermore every permutation on the n identical particles must be a homothety, i.e a permutation must be mapped to a scalar multiple of the identity. There are two such representations: the trivial representation and the sign representation. So we have $P|\psi\rangle = a|\psi\rangle$, with $a \in \mathbb{Z}$ and $|a| = 1$, so $a = \pm 1$. If $a = 1$ the wave vector does not change under a permutation of the particles, this particle type is called a boson. In the other case, if $a = -1$, the corresponding particles are called fermions.

The situation described above is the four-dimensional case. In three-dimensional spacetime (two spatial dimensions and one dimension of time) however, it is a different story. For instance, if we have two identical particles and interchange them a couple of times, the world lines of the particles form a 2-braid. So not only the initial and end state of the system are important, but also the manner in which the particles are interchanged. This in contrary to the four-dimensional situation. The role that S_n played before, is now taken over by the Artin-Braid group B_n. This is because the possible permutations on the n identical particles correspond one-to-one with the homotopy classes of braids. In fact the only requirement that is abandoned in the 2+1 dimensional case is that $\sigma_i^2 = 1$.

This subtle difference has far reaching consequences. For instance, we have many more one-dimensional representations of the braid group on the Hilbert space. The one-dimensional representations correspond with an extra phase $e^{i\theta}$ in front of the wave vector. Each $\theta \in [0, 2\pi)$ gives a different type of particle. These possible particles, except for $\theta = 0, \pi$, are called anyons and in the case of $\theta = 0$ we just get a boson, for $\theta = \pi$ we have a fermion. One must distinguish between so-called abelian and non-abelian anyons. Anyons that match with one-dimensional representations of the braid group are always abelian, because the phases in front of the wave vector commute. This does not have to be the case for higher dimensional representations. For instance, by permuting non-abelian anyons it is possible that the wave vector is multiplied by an unitary matrix. And matrices in general do not commute. Non-abelian anyons find their application in the construction of topological quantum computers. A good and understandable article about topological quantum computation can be found here [22].

8.3 Anyonic models

In the last section we saw that 2+1 dimensional space time permits the existence of indistinguishable particles, other than bosons and fermions, which are called anyons. Anyons obey other types of statistics than bosons and fermions do. They arised from a theoretical consideration of identical particles in three dimensional space time but do anyons really exist? It turns out they do. They appear for instance in a physical phenomenon that is called the fractional quantum Hall effect [17]. We shall not go into the details of this effect but it certainly justifies the investigation of other properties of anyons.

The following step is to examine how anyons behave in groups and how they interact. Just as for ordinary bosons and fermions, anyons come with an antiparticle. Together they may annihilate to form a vacuum state, a state with no particles in it. There is also the possibility that a couple of anyons form a new particle. Moreover, the opposite reaction, where a number of particles is created out of one particle, may occur as well. Finally the consequences of braiding, the interchanging of anyons, must be specified. The foregoing translates into the fact that an anyonic model is defined by three properties.

An anyonic model is described first of all by a list of particles, including the unique trivial particle (vacuum state) and a unique antiparticle for each particle. Such a list can be denoted by $L = \{a, b, c, ...\}$. On top, rules for fusion and splitting should be given. If two particles a and b are combined to fuse to an arbitrary particle c, this is indicated as follows: $a \times b = \sum_c N_{ab}^c c$, where $N_{ab}^c \in \mathbb{Z}_{\geq 0}$ represents the number of possible manners in which a and b can form c. Read backwards this gives the splitting of a particle c. In the case that $N_{ab}^c = 0$, a and b cannot fuse into c. If $N_{ab}^c = 1$, the fusion happens in precisely one way. When $N_{ab}^c > 1$ this gives rise to non-abelian anyons.

The fusion of particles must be associative and commutative, i.e $a \times (b \times c) = (a \times b) \times c$ and $a \times b = b \times a$. It follows that $N_{ab}^c = N_{ba}^c$. For the trivial particle, denoted by e, we have $a \times e = a$. Fusion of a particle a and its antiparticle $\bar{a}$ gives $a \times \bar{a} = e + \sum_{c \neq 1} N_{a\bar{a}}^c c$. Now one can construct a vector space, called the fusion space, by taking as a basis for the vector space the set $\{|(ab)c, \mu\rangle, \mu = 1, 2, ..., N_{ab}^c\}$. So to each label set $\{a, b, c\}$ a vector space is assigned. The basis vector $|(ab)c, \mu\rangle$, representing a fusion, and its dual $\langle (ab)c, \mu|$, representing a splitting, can be pictured as follows

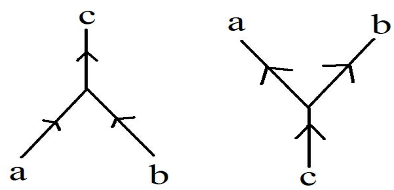

Figure 20: Fusion and splitting of particles

The fusion space is referred to as V_{ab}^c and we have $dim(V_{ab}^c) = N_{ab}^c$. The basis of the space is orthonormal $\langle (ab)c, \mu | (ab)c, \mu' \rangle = \delta_{\mu\mu'}$. Moreover one can consider a general fusion space where the outcome of a fusion between two particles a and b is not fixed. This space is the direct sum of all subspaces which do have a determined fusion outcome, so $dim(\bigoplus_c V_{ab}^c) = \sum_c N_{ab}^c$ and $V_{ab}^{c_1} \cap V_{ab}^{c_2} = e$ for distinct $c_1, c_2 \in L$. If $\sum_c N_{ab}^c > 1$, then the theory is called non-abelian. This means that the outcome of a fusion is not unique. When the outcome of a fusion is unique this implies that $\sum_c N_{ab}^c = 1$, i.e $\exists c_1$ for which $\bigoplus_c V_{ab}^c \cong V_{ab}^{c_1}$ and this representation space is irreducible because it is not the direct sum of two representations, other than the trivial decomposition.

A vector in the fusion space V_{ab}^c can be regarded as a linear combination of ways in which particles a and b can form another particle c. The dual space of V_{ab}^c, the splitting space, is $V_c^{ab} = Hom(V_{ab}^c)$ and consists of vectors that are linear combinations of ways in which a particle c can split in two particles a and b. A basis can be given by $\{\langle (ab)c, \mu|, \mu = 1, 2, ..., N^c_{ab}\}$. Similar expressions for the dimension hold for the splitting space. Furthermore one can verify that $\langle (ab)c, \mu|(ab)c', \mu'\rangle = \delta_{cc'}\delta_{\mu\mu'}$.

The fusion of two particles can be extended to a fusion of n particles. First two particles are fused into an intermediate particle, which in turn fuses with one of the remaining particles. This process continues until all original particles have been used. Schematically

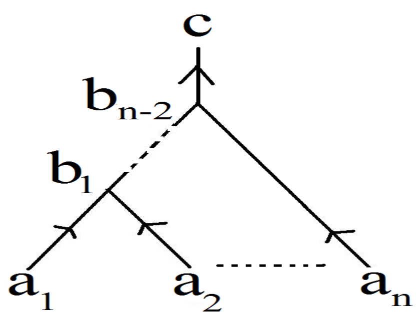

Figure 21: Fusion of n particles

In similar fashion as before a fusion space can be assigned to the label set $\{a_1, a_2, ..., a_n\}$. Because fusion is an associative action, this space can be regarded as the direct product of the tensor product of some subspaces. As a result of the fact that the order of fusion is not important, such a decomposition of the fusion space is unique up to isomorphism. In summary $V^c_{a_1a_2...a_n} = \bigoplus\limits_{b_1,...,b_{n-2}} (V^{b_1}_{a_1a_2} \otimes V^{b_2}_{b_1a_3} \otimes ... \otimes V^c_{b_{n-2}a_n}) \cong \bigoplus\limits_{b_1,...,b_{n-2}} (V^c_{a_1b_{n-2}} \otimes V^{b_{n-2}}_{a_2b_{n-3}} \otimes ... \otimes V^{b_1}_{a_{n-1}a_n})$. Where $b_1, b_2, ..., b_{n-2}$ stand for the intermediate particles as discussed. Note that only two of the many possible decompositions are named. The left one is obtained by repeatedly fusing the two left most particles. Likewise, the right one is obtained by repeatedly fusing the two right most particles. A basis for the new fusion space can be given by the sum of the product of the bases of the subspaces that appear in the decomposition, hence a consideration of the dimension yields $dim(V^c_{a_1a_2...a_n}) = \sum\limits_{b_1,...,b_{n-2}} N^{b_1}_{a_1a_2} \cdot N^{b_2}_{b_1a_3} \cdot ... \cdot N^c_{b_{n-2}a_n}$.

Now that the rules for fusion have been formulated, it remains to investigate the rules for braiding. This last defining property of an anyonic model consists of rules that specify what happens when two or more particles are exchanged. In case two particles are fused, the order is irrelevant because the fusion is commutative. Therefore we must have $V^c_{ab} \cong V^c_{ba}$. This implies that there exists a unitary transformation R, called the braid operator, between the two Hilbert spaces. In terms of the bases we have R: $|(ab)c, \mu\rangle \mapsto |(ba)c, \mu'\rangle$. For the reason that the operator R does not have physical consequences for the anyons, it can be seen as an intrinsic or innate freedom of the system. The transformations can be represented by unitary matrices. In general such transformations can be described as actions of the generators of the braid group on fusion spaces. Thus, geometrically the particles are just twined. Such an intertwining is called a R-move.

The same reasoning applies when three particles fuse. Three particles, if not interchanged, can fuse in two different ways: $(a \times b) \times c \to d$ and $a \times (b \times c) \to d$. For each of these possibilities there are many decompositions of the total fusion space V^d_{abc}, depending on the potential particle that may be formed. Because all these decompositions are isomorphic, they are related by unitary transformations called F-moves, in analogy with R-moves in the case of two particles. So $F : V^d_{abc} = \bigoplus_{e \in L} (V^e_{ab} \otimes V^d_{ce}) \to \bigoplus_{f \in L} (V^d_{af} \otimes V^f_{bc}) = V^d_{bca}$, $\sum_e |(ab)e, \mu\rangle |(ce)d, \nu\rangle \mapsto \sum_f |(af)d, \mu'\rangle |(bc)f, \nu'\rangle$.

With the specification of the R-, and F-moves our description of an anyonic model is almost complete. For sake of consistency there are two relations which the unitary transformations must satisfy. Consider the fusion of four particles. This process can proceed in five manners. The fusion spaces are related by F-moves as pictured in the following diagram

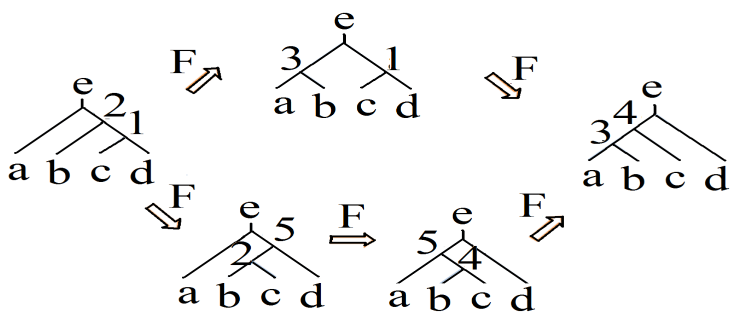

Figure 22: The hexagon diagram

The following rule is imposed on this diagram. If one goes from one pattern in the above diagram to another by a sequence of F-moves, this must yield the same answer irrespective of the path chosen. Therefore the consistency relation, called the pentagon relation, becomes $(F^e_{ab3})^4_1 (F^e_{1cd})^3_2 = \sum_5 (F^4_{bcd})^3_5 (F^e_{a5d})^4_2 (F^2_{abc})^5_1$. This should be read as follows: $(F^g_{abc})^f_e$ represents the move where particle e is removed and in stead f is formed out of b and c. Then a fuses with f to particle g. Graphically

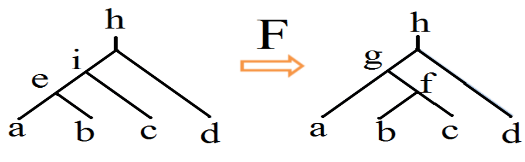

Figure 23: An F-move

There is one more equation that the R-, and F-moves must obey. It is called the hexagon equation and is given by $R^3_{ac} (F^d_{bac})^3_1 R^1_{ab} = \sum_2 (F^d_{bca})^3_2 R^d_{a2} (F^d_{abc})^2_1$. Here the convention is that R^c_{ab} represents a move from a space where particles a and b, in this order, fuse into c, to a space where b and a form c.

A lot of effort has been made to introduce the concept of an anyonic model. Now we can reap the fruits of our labor. In the next section a bridge is build to the Jones polynomial by regarding a specific anyonic model that originally has been described by the renowned mathematician and physicist Edward Witten [26].

8.4 The Jones polynomial

In stead of first investigating the properties of an uncomplicated anyonic model, this step is skipped by means of time. In this chapter a particular type of Chern-Simons theory will be studied in the light of anyonic models. Chern-Simons theory is a topological quantum field theory that is used in different branches of both mathematics and physics. In our case it is necessary to pick a certain Lie group, in this context also referred to as a gauge group, on which Chern-Simons theory is applied. By taking $SU(n)$ as a gauge group, several knot invariants can be calculated. It turns out, as we will see later, that for $n = 2$ the Jones polynomial appears. So we will study $SU(2)$.

First of all a list of particles must be given for our anyonic model. Let $k \in \mathbb{N}$ be a fixed number. Then the irreducible representations of $SU(2)$ of dimension k or lower, act as particles. This anyonic model is referred to as $SU(2)_k$. Just as in four-dimensional space time the trivial representation of S_n was associated with a boson and the sign representation with a fermion, the irreducible representations of a group again impersonate particles. A problem that one has to deal with is the fact that $SU(2)$ is not a finite group. Therefore it is quite hard to apply representation theory on it. Luckily, $SU(2)$ is a compact Lie group so the Peter-Weyl theorem holds for this group. That $SU(2)$ is compact can be seen by the map $\varphi : S^3 \to SU(2),\ (x, y) \in \mathbb{C}^2\ with\ |x|^2 + |y|^2 = 1 \mapsto \begin{pmatrix} x & y \\ -\bar{y} & \bar{x} \end{pmatrix}$ which establishes an isomorphism between S^3 and $SU(2)$ as manifolds.

Now because $SU(2)$ is compact we have $\mu_H(SU(2))<\infty$, with μ_H being the Haar-measure. Therefore we can normalize the Haar-integral, $vol(SU(2)) = \int_{G=SU(2)} d\mu_H(g) = 1$. Recall that a representation $\rho : G \to GL(V)$ can be characterized by a complex valued function χ, called the *character* of a representation, given by $\chi_\rho = Tr(\rho_s) = \sum_i a_{ii},\ with\ s \in G\ and$ $\rho_s = \rho(s) = (a_{ij}) \in GL(V)$. So $\chi : G \to F$ with F the base field of the vector space and in our case $F = \mathbb{C}$. If a character satisfies the relation $\chi(tst^{-1}) = \chi(s)\ \forall s, t \in G$ then χ is called a $class\ function$. Given a group G, a class function on G is a function $f : G \to \mathbb{C}$ with the property that $f(tst^{-1}) = f(s)\ \forall s, t \in G$. All characters of finite dimensional representations are class functions due to the fact that $Tr(st) = Tr(ts)$ if the products st and ts are well-defined. Given two class functions χ and η, a product can be defined by $(\chi, \eta) = \frac{1}{|G|} \sum_{g \in G} \chi(g)\overline{\eta(g)}$, where $|G|$ stands for the order of the group. There is a useful proposition called Schur's lemma which will be needed in order to prove an important theorem about the irreducibility of representations.

Proposition 8.4 (Schur's lemma) *Let V,W be finite dimensional complex vector spaces which are irreducible representation spaces. If $V \cong W$ then $Hom_G(V,W)$ is one-dimensional. If $V \ncong W$ then $Hom_G(V,W)$ is zero-dimensional.*

The proposition implies that in the case of V and W being isomorphic finite-dimensional complex vector spaces, there exists an isomorphism $f : V \to W$ such that for all $g \in GL(V)$ and $v \in V$ we have that $f(gv) = gf(v)$. Then this isomorphism is the generator of $Hom_G(V,W)$.

Theorem 8.5 *Let V, W be n-dimensional complex vector spaces with $V \ncong W$. Let χ_V be a character of the representation space V and χ_W a character of the representation space W. Then*
(i) V is irreducible $\Leftrightarrow (\chi_V, \chi_V) = 1$
(ii) V, W are irreducible $\Leftrightarrow (\chi_V, \chi_W) = 0$.

Proof. (i) " $\Rightarrow$ " Let ρ be an n-dimensional irreducible representation of G with character χ_V. Then $\rho_s = (a_{ij}(s))$ and $\chi_V(s) = \sum_i a_{ii}(s)$, for $s \in G$. Define $<\phi, \psi> = \frac{1}{|G|}\sum_{s\in G}\phi(s)\psi(s^{-1})$ and define $\widehat{\psi(s)} = \overline{\psi(s^{-1})}$. Then $(\phi,\psi) = \frac{1}{|G|}\sum_{s\in G}\phi(s)\widehat{\psi(s^{-1})} = <\phi, \widehat{\psi}>$. Now because ρ is n-dimensional, ρ_s has finite order and so do the eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ of ρ_s. Therefore the absolute value of each eigenvalue equals one and we have $\overline{\lambda_i} = \lambda_i^{-1}$ for each i. Given a character χ of ρ then $\overline{\chi(s)} = \overline{Tr(\rho_s)} = \sum_i \overline{\lambda_i} = \sum_i \lambda_i^{-1} = Tr(\rho_s^{-1}) = Tr(\rho_{s^{-1}}) = \chi(s^{-1})$. It follows that $\widehat{\psi} = \psi$ and $(\phi,\psi) = <\phi,\psi>$ for characters ψ. Therefore $(\chi_V,\chi_V) = <\chi_V,\chi_V> = \sum_{i,j} <a_{ii}, a_{jj}>$. Now it follows from Schur's lemma, see [24], that $<a_{ii}, a_{jj}> = \frac{\delta_{ij}}{n}$. Thus $(\chi_V,\chi_V) = \frac{1}{n}\sum_{i,j}\delta_{ij} = \dfrac{n}{n} = 1$.
(ii) " $\Rightarrow$ " The argument for (ii) is analogous to (i). The only difference is that we now have $(\chi_V, \chi_W) = <\chi_V, \chi_W> = 0$, which also follows from Schur's lemma. □

For the other implications we refer to [24]. Next, we define an equivalence relation on the set of representations of a group G on a vector space V by saying that ρ_1 is equivalent to ρ_2 if they have the same character. It follows from a theorem that equivalent representations are isomorphic. We now come the Peter-Weyl theorem.

Theorem 8.6 (Peter-Weyl) *Let G be a compact group not necessarily abelian. Then*
(i) All irreducible representations of G are finite dimensional.
(ii) The characters $\{\chi_S\}$, where S runs through a full system of representatives of equivalence classes of irreducible representations of G, form an orthonormal basis for $L^2(G, \mu_H)$.

Here $L^2(G,\mu_G)$ represents the infinite dimensional Hilbert space of square integrable functions φ with $\int_G |\varphi(g)|^2 d\mu_H < \infty$. The Peter-Weyl theorem asserts us that the irreducible representations of $SU(2)$ can indeed act as particles, as will be seen later. However, it does not provide a way to obtain these representations.

Consider the following space $P_n = \{binary\ polynomials\ of\ homogeneous\ degree\ n\ in\ two\ complex\ variables\ x\ and\ y\}$. So an element of P_n can be written like $f(x,y) = \sum_{i=0}^{n} a_n x^{n-i} y^i$. A basis for this space is $\{x^n, x^{n-1}y, x^{n-2}y^2, ..., xy^{n-1}, y^n\}$. Thus $dim(P_n) = n+1$. The group $SU(2)$ acts on this space by $\pi : SU(2) \times P_n \to P_n$, $\pi(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, f(x,y)) = f((x,y)\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = f(ax+cy, bx+dy)$. In this context we will also denote π as π_U, with $U \in SU(2)$ if U is given. This gives rise to a homomorphism $\alpha_n : SU(2) \to Aut(P_n)$, $U \mapsto \pi_U$. The map α_n is a homomorphism because $(x\ y)(U_1U_2) = ((x\ y)U_1)U_2$, so $\alpha_n(U_1)\alpha_n(U_2) = \pi_{U_1}\pi_{U_2} = \pi_{U_1U_2} = \alpha_n(U_1U_2)$. As a consequence α_n is a representation of $SU(2)$.

The $\alpha_n' s$ turn out to be irreducible and they are the only irreducible representations [7]. The argument goes as follows. Every character associated with a representation is a class function. One can show that the product, (χ, χ), of such a character with itself is one. This amounts to the calculation of an integral. By theorem 8.5 the representation is irreducible. Furthermore it can be shown that the characters of the representations α_n form an orthonormal basis for $L^2(G, \mu_H)$. Therefore there are no other irreducible representations. The last statement can also be seen with an argument from the absurd. Take an irreducible representation other than α_n for all n. This representation must be finite dimensional by the Peter-Weyl theorem. A calculation will show that the product of the associated character with itself gives zero. This contradicts the assumption by theorem 8.5.

By now we have concreted a list of particles for the anyonic model $SU(2)_k$. The next step is to specify the fusion rules. However, there is a natural way to formulate the fusion rules. Namely, if one takes the tensor product of two irreducible representation spaces of $SU(2)$, there is a decomposition of the obtained space as the direct sum of irreducible representations. This is called the Clebsch-Gordan decomposition and is given by $V_n \otimes V_m = \bigoplus_{j=|n-m|}^{n+m} V_j$, with the step size of j being two. Furthermore one requires that $j + n + m < k$ with $k \in \mathbb{N}$ the specified number for the anyonic model $SU(2)_k$, for every representation has to k-dimensional or lower. Translated into the fusion of two particles a and b we have: $a \times b = \sum_{j=|a-b|}^{k} j$, again with step size two and $j + a + b < k$.

The braiding rules also follow from theory [8]. They correspond to definite phases and are given by: $R_{ab}^c = (-1)^{\frac{a+b-c}{2}} A^{-\frac{(a(a+2)+b(b+2)-c(c+2))}{2}}$ and for the F-move we have $(F_{abc}^d)_e^f = \begin{bmatrix} a & b & f \\ c & d & e \end{bmatrix}$. The variable A is the same one that appears in the derivation of the Jones polynomial. The matrix in square brackets denotes the 6j-symbol. The description of our anyonic model is now complete.

To obtain the Jones polynomial, we analyse a particular equation and try to solve it. As said, the irreducible representations of $SU(2)$ act as anyons. In addition these anyons may intertwine in 2+1 dimensional space time while satisfying the braiding rules. Their word lines form braids and by closing these braids a link is constructed. Therefore we can say that every component of such a link can be associated with a state in two-dimensional Hilbert space, because that is the space where the anyons live in. Considering this, take a look at the following states

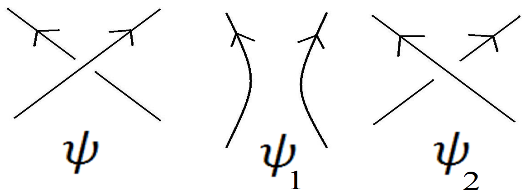

Figure 24: The states

The equation that is to be analysed is the following

$$\alpha\psi + \beta\psi_1 + \gamma\psi_2 = 0 \tag{2}$$

with $\alpha, \beta, \gamma \in \mathbb{C}$. This equation has a non-trivial solution because there always exists a relation of linear dependence between three vectors in a two-dimensional vector space. It can be seen that ψ, ψ_1 and ψ_2 obey another equation that relates them to each other [20]. It turns out they only differ by the braid operator R. We have: $\psi_1 = R\psi$ and $\psi_2 = R^2\psi$. Remember that this R-move is a unitary transformation acting, in this case, in a two-dimensional space. We can therefore also regard R as a unitary 2×2 matrix. It is a two-dimensional representation. The Cayley-Hamilton theorem implies that R satisfies its own characteristic equation: $R^2 - Tr(R)R + det(R)R = 0$. Multiplying by ψ reduces the equation in

$$\psi_2 - \psi_1 Tr(R) + det(R)\psi = 0 \tag{3}$$

The last equation is precisely the one wherewith we started. The problem then reduces to the determination of the eigenvalues of R. This is first done by Moore and Seiberg in [20] and also, in the light of our approach, by Witten in [26]. That R has two distinct eigenvalues is not very trivial to deduce. Moore and Seiberg originally derived these eigenvalues because they wanted to prove a conjecture of Verlinde stated in [25]. We will not go into the detail of this but for those interested Verlinde conjectured that 'the modular transformation S(0) of one-loop vacuum characters diagonalizes the fusion rules' [20].

The eigenvalues, in the case of an anyonic model $SU(n)_k$, are given by

$$\lambda_{\pm} = \pm exp(\frac{i\pi(1 \mp n)}{n(n+k)}) \tag{4}$$

Now α, β and γ can be derived. Yet one has to make a few corrections due to technical issues depending on the framing of the knot. The framing of a knot is a way of looking at the knot by determining to what extent the knot is linked with the ribbon $I \times S^1$. The corrections are taken along in the calculation below. Eventually the following values are obtained

$$\alpha = det(R) = \lambda_+\lambda_- = -exp(\frac{2\pi i}{n(n+k)}) \tag{5}$$

$$\beta = -exp(\frac{-2\pi i(n^2-1)}{2n(n+k)})Tr(R) = exp(\frac{\pi i(2+n-n^2)}{n(n+k)}) - exp(\frac{\pi i(2-n-n^2)}{n(n+k)}) \tag{6}$$

$$\gamma = exp(\frac{-2\pi i(n^2-1)}{n(n+k)}) \tag{7}$$

The numbers are filled in equation (1) and then (1) is multiplied by a factor $exp(\frac{\pi i(n^2-2)}{n(n+k)})$. We substitute $t = exp(\frac{2\pi i}{n+k})$ to get

$$t^{-\frac{n}{2}}V(L_+) - t^{\frac{n}{2}}V(L_-) + (t^{-\frac{1}{2}} - t^{\frac{1}{2}})V(L_0) = 0 \tag{8}$$

Here L_-, L_0 and L_+ stand for the links one gets by closing the braids ψ, ψ_1 and ψ_2. This equation reduces to the skein relation for the Jones polynomial for $n = 2$.

The conclusion is that the variable 't' in the Jones polynomial is a sort of a permutation of anyons in a system of anyons. In the construction of the Jones polynomial, crossings appearing in the knot diagram are cut open and pasted back together in two ways. This matches with the permutation of two anyons in a certain anyonic model. However, because R is a two-dimensional representation, the anyons may be non-abelian. This means that a permutation of the anyons could change the particles and sometimes also the way they move. In special cases, the wave vector just gains a complex phase.

9 Conclusion

This thesis can be divided in three parts. The first part consists of an introduction into the world of knots and links. After this introduction the Alexander polynomial is accentuated. That is the second part of this thesis. The last part is devoted to an application of knot theory in theoretical physics, in the form of anyons. The last part also comprises an introduction in braid theory and a connection to the Jones polynomial.

The key question in knot theory is to tell whether two given knots are equivalent or not. By looking at the knot diagram it is possible to relate equivalent knots by a sequence of Reidemeister moves. Since this is a painful process, better ways in distinguishing knots had to be found. The hunt for knot invariants was opened.

The Jones polynomial turned out to be a very useful invariant. It is easy to determine, it tells knots from their mirror images apart and their exists a skein relation for this Laurent polynomial. Furthermore prime knots with nine or less crossings in their knot diagram have distinct Jones polynomials [1]. However, a full understanding of the Jones polynomial misses. The unknown meaning of the variable 't' makes the Jones polynomial not very intuitive. An important unsolved question is whether there exists a non-trivial knot which has a Jones polynomial of one.

A second knot polynomial is given in the second part: the Alexander polynomial. This knot invariant has a long history and there are several ways to approach it. The approach in terms of homology theory was taken in this thesis. The Alexander polynomial is mainly build upon results coming from geometric topology and algebraic topology. This resulted not only in a quite complex definition for the polynomial but also in nice properties. One of the most important consequences is that the breadth of the Alexander polynomial for a knot gives a lower bound for the genus of the knot. Further, the Alexander polynomial can tell prime knots with knot diagrams up to eight crossings apart. Compared with the Jones polynomial, the geometrical meaning of the variable 't' in the Alexander polynomial is clear. It is a self-homeomorphism of the space X_∞. The variable can be seen as a translation function of the topological spaces making up X_∞.

The Alexander polynomial also has a downside. Unlike the Jones polynomial it cannot differentiate between a knot and its mirror knot. There turn out to be infinitely many non-trivial knots with Alexander polynomial equal to one [23]. Furthermore the Alexander polynomial is only defined up to multiplication with a unit $\pm t^n$. This gives rise to a normalized version of the Alexander polynomial called the Conway polynomial. The latter has the advantage that there can be deduced a skein relation for it which makes computations much easier.

The last part of the thesis deals with braids, anyons and anyonic models. A braid can be seen as an embedding of a collection of disjoint strands in three-dimensional Euclidean space. Such a braid can be closed to obtain a link. The connection with physics now goes as follows. It points out to be that in 2+1 dimensional space time there are particles other than bosons and fermions, called anyons. These particles may intertwine in time and their world lines form a braid. The effect of anyons has been measured in experiments [17].

An anyonic model is a described by three properties: a list of particles, fusion rules and braiding rules. The penultimate rules specify what happens when two anyons fuse into a new particle. The last property describes the effect of the braiding of two or more anyons. By considering a particular type of anyonic model certain knot invariants can be derived [26]. The model that is investigated goes under the name of $SU(2)_k$. After solving an equation coming from this model the skein relation for the Jones polynomial appeared.

In conclusion it can be said that knot theory is a very promising field. It has its roots in mathematics but it has also many important applications in mathematical and theoretical physics. The interplay between these different fields through knot theory is fascinating. New discoveries follow each other rapidly.

Knot theory has the advantage that it naturally lend itself to make pictures. This is a great help to simplify matters. However one may never forget that the technical machinery to deal with knot theory is very complex. There are many subtleties that may be overlooked but which require a lot of work.

The last remark is also expressed by the fact that there are still many unsolved questions in knot theory, see for instance [1]. Questions that every layman can understand, like Fermat's last theorem, but which turn out to be extremely hard to solve. In these questions lies a great challenge for the future generation of mathematicians and physicists.

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