Topological Modular Forms

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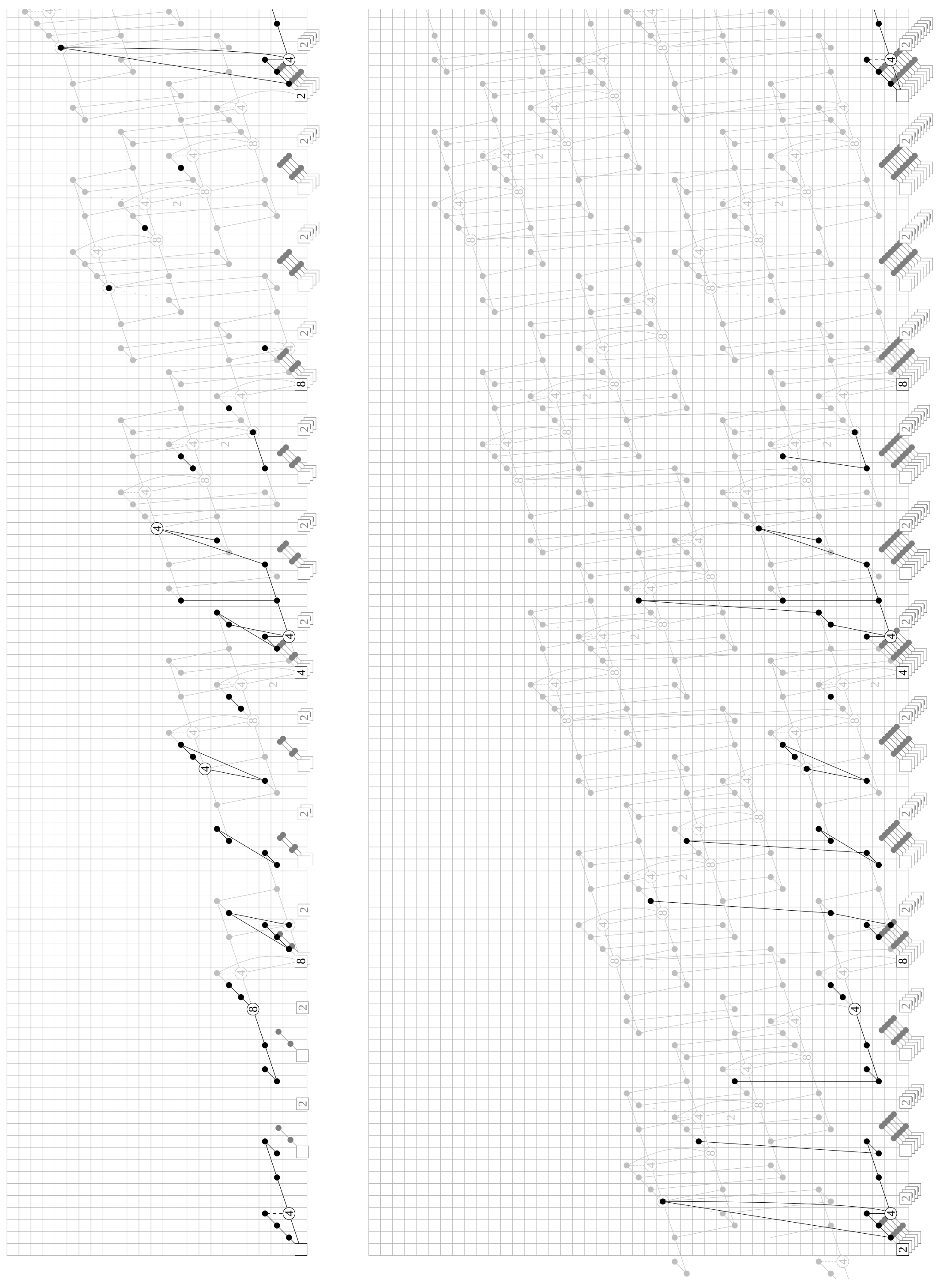

Preface

This is a book about the theory of topological modular forms. It is also a record of the efforts of a group of graduate students to learn that theory at the 2007 Talbot Workshop, and so a book born of and steeped in the Talbot vision.

In the fall of 2003, Mike Hopkins taught a course at MIT about *tmf*. Our generation of Cambridge algebraic topologists, having survived and thrived in Haynes Miller's Kan seminar, found in Mike's class our next, and really our last common, mathematical crucible. The course hacked through the theory of algebraic modular forms, formal groups, multiplicative stable homotopy theory, stacks, even more stacks, moduli stacks of elliptic curves, Bousfield localization, Morava K- and E-theory, the arithmetic and Hasse squares, André-Quillen cohomology, obstruction theory for moduli of associative and commutative ring spectra—by this point we were having dreams, or maybe nightmares, about the spiral exact sequence.

In the middle of the course, we all hopped over to Münster for a week-long workshop on *tmf* with lectures by Mike, Haynes, Matt Ando, Charles Rezk, and Paul Goerss. We spent the late afternoons alternately steaming and freezing the knowledge in at a spa off Steinfurter Strasse, and there sketched a vision of what would become the Talbot Workshops: a gathering for graduate students, focused on a single topic of contemporary research interest, lectured by graduate students and guided by a single faculty mentor, having talks in the morning and in the evening and every afternoon free for discussion and outdoor activities, with participants sleeping and lecturing and cooking together under the same roof. We pitched it to Mike and Haynes and they promptly wrote us a check (signed "our NSF grant"). Talbot was born.

Three years later, in 2007, we decided to bring Talbot home with a workshop on *tmf*, mentored by Mike Hopkins. Mike stopped by Staples on his way to the workshop and picked up a big red "That was easy" button. Throughout the workshop, whenever he or anyone else completed a particularly epic spectral sequence computation or stacky decomposition, he'd hit the button and a scratchy electronic voice would remind us, "That was easy!" It became the workshop joke (for much of it was evidently not easy) and mantra (for shifting perspective, whether to multiplicative stable homotopy or to stacky language or to a suitable localization, did make the intractable seem possible).

This book is a record and expansion of the material covered in the Talbot 2007 workshop. Though the authors of the various chapters have brought their own expositional perspectives to bear (particularly heroically in the case of Mark Behrens), the contemporary material in this book is due to Mike Hopkins, Haynes Miller, and Paul Goerss, with contributions by Mark Mahowald, Matt Ando, and Charles Rezk.

Acknowledgments

We thank the participants of Talbot 2007 for their dedication and enthusiasm during the workshop: Ricardo Andrade, Vigleik Angeltveit, Tilman Bauer, Mark Behrens, Thomas Bitoun, Andrew Blumberg, Ulrik Buchholtz, Scott Carnahan, John Duncan, Matthew Gelvin, Teena Gerhardt, Veronique Godin, Owen Gwilliam, Henning Hohnhold, Valentina Joukhovitski, Jacob Lurie, Carl Mautner, Justin Noel, Corbett Redden, Nick Rozenblyum, and Samuel Wüthrich. Special thanks to Corbett, Carl, Henning, Tilman, Jacob, and Vigleik for writing up their talks from the workshop. And super-special thanks to Mark for his enormous effort writing up his perspective on the construction of *tmf*, and for extensive support and assistance throughout the development of the book. Our appreciation and thanks also to Tilman, to Mark, and to Niko Naumann for substantial contributions to and help with the glossary, and to Paul Goerss, Charles Rezk, and David Gepner for suggestions and corrections.

Our appreciation goes to Nora Ganter, who in 2003 ran a student seminar on *tmf*—it was our first sustained exposure to the subject; Nora also prepared the original literature list for the Talbot 2007 program. We also acknowledge the students who wrote up the 'Course notes for elliptic cohomology' based on Mike Hopkins' 1995 course, and the students who wrote up 'Complex oriented cohomology theories and the language of stacks' based on Mike's 1999 course—both documents, distributed through the topology underground, were helpful for us in the years leading up to and at Talbot 2007.

The book would not have happened without Talbot, and the Talbot workshops would not have happened without the support of MIT and the NSF. Christopher Stark at the NSF was instrumental in us securing the first Talbot grant, DMS-0512714, which funded the workshops from 2005 til 2008. MIT provided the facilities for the first workshop, at the university retreat Talbot House, and has provided continual logistical, administrative, and technical support for the workshops ever since. We'd like to thank the second and third generations of Talbot organizers, Owen Gwilliam, Sheel Ganatra, and Hiro Lee Tanaka, and Saul Glasman, Gijs Heuts, and Dylan Wilson, for carrying on the tradition.

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Our greatest debt is to Haynes Miller and Mike Hopkins. Haynes was and has been a source of wisdom and insight and support for us through the years, and it is with fondness and appreciation that we acknowledge his crucial role in our mathematical upbringing and thus in the possibility of this book. We are singularly grateful to our advisor, Mike, for inspiring us and guiding us, all these years, with his characteristic energy and brilliance, humor and care. Thank you.

Introduction

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1. Elliptic cohomology

A ring-valued cohomology theory E is *complex orientable* if there is an 'orientation class' $x \in E^2(\mathbb{CP}^\infty)$ whose restriction along the inclusion $S^2 \cong \mathbb{CP}^1 \hookrightarrow \mathbb{CP}^\infty$ is the element 1 in $E^0S^0 \cong E^2\mathbb{CP}^1$. The existence of such an orientation class implies, by the collapse of the Atiyah–Hirzebruch spectral sequence, that

$$E^*(\mathbb{CP}^\infty) \cong E^*[[x]].$$

The class x is a universal characteristic class for line bundles in E-cohomology; it is the E-theoretic analogue of the first Chern class. The space $\mathbb{CP}^\infty$ represents the functor

$$X \mapsto \{\text{isomorphism classes of line bundles on } X\},$$

and the tensor product of line bundles induces a multiplication map $\mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$. Applying E^* produces a ring map

$$E^*[[x]] \cong E^*(\mathbb{CP}^\infty) \to E^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) \cong E^*[[x_1, x_2]];$$

the image of x under this map is a formula for the E-theoretic first Chern class of a tensor product of line bundles in terms of the first Chern classes of the two factors. That ring map $E^*[[x]] \to E^*[[x_1, x_2]]$ is a (1-dimensional, commutative) formal group law—that is, a commutative group structure on the formal completion $\hat{\mathbb{A}}^1$ at the origin of the affine line $\mathbb{A}^1$ over the ring E^*.

A formal group often arises as the completion of a group scheme at its identity element; the dimension of the formal group is the dimension of the original group scheme. There are three kinds of 1-dimensional group schemes:

(1) the additive group $\mathbb{G}_a = \mathbb{A}^1$ with multiplication determined by the map $\mathbb{Z}[x] \to \mathbb{Z}[x_1, x_2]$ sending x to $x_1 + x_2$,
(2) the multiplicative group $\mathbb{G}_m = \mathbb{A}^1 \backslash \{0\}$ with multiplication determined by the map $\mathbb{Z}[x^{\pm 1}] \to \mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}]$ sending x to $x_1 x_2$, and
(3) elliptic curves (of which there are many isomorphism classes).

Ordinary cohomology is complex orientable, and its associated formal group is the formal completion of the additive formal group. Topological K-theory is also complex orientable, and its formal group is the formal completion of the multiplicative formal group. This situation naturally leads one to search for 'elliptic' cohomology theories whose formal groups are the formal completions of elliptic curves. These elliptic cohomology theories should, ideally, be functorial for morphisms of elliptic curves.

Complex bordism MU is complex orientable and the resulting formal group law is the universal formal group law; this means that ring maps from MU_* to R are in natural bijective correspondence with formal group laws over R. Given a commutative ring R and a map $MU_* \to R$ that classifies a formal group law over R, the functor

$$X \mapsto MU_*(X) \otimes_{MU_*} R$$

is a homology theory if and only if the corresponding map from $\mathrm{Spec}(R)$ to the moduli stack $\mathcal{M}_{FG}$ of formal groups is flat. There is a map

$$\mathcal{M}_{ell} \to \mathcal{M}_{FG}$$

from the moduli stack of elliptic curves to that of formal groups, sending an elliptic curve to its completion at the identity; this map is flat. Any flat map $\mathrm{Spec}(R) \to \mathcal{M}_{ell}$ therefore provides a flat map $\mathrm{Spec}(R) \to \mathcal{M}_{FG}$ and thus a homology theory, or equivalently, a cohomology theory (a priori only defined on finite CW-complexes). In other words, to any affine scheme with a flat map to the moduli stack of elliptic curves, there is a functorially associated cohomology theory.

The main theorem of Goerss–Hopkins–Miller is that this functor (that is, presheaf)

$$\{\text{flat maps from affine schemes to } \mathcal{M}_{ell}\} \to \{\text{multiplicative cohomology theories}\},$$

when restricted to maps that are étale, lifts to a sheaf

$$\mathcal{O}^{\mathrm{top}}: \{\text{étale maps to } \mathcal{M}_{ell}\} \to \{E_\infty\text{-ring spectra}\}.$$

(Here the subscript 'top' refers to it being a kind of 'topological', rather than discrete, structure sheaf.) The value of this sheaf on $\mathcal{M}_{ell}$ itself, that is the E_∞-ring spectrum of global sections, is the periodic version of the spectrum of topological modular forms:

$$TMF := \mathcal{O}^{\mathrm{top}}(\mathcal{M}_{ell}) = \Gamma(\mathcal{M}_{ell}, \mathcal{O}^{\mathrm{top}}).$$

The spectrum TMF owes its name to the fact that its ring of homotopy groups is rationally isomorphic to the ring

$$\mathbb{Z}[c_4, c_6, \Delta^{\pm 1}]/(c_4^3 - c_6^2 - 1728\Delta) \;\cong\; \bigoplus_{n \geq 0} \Gamma\big(\mathcal{M}_{ell}, \omega^{\otimes n}\big)$$

of weakly holomorphic integral modular forms. Here, the elements c_4, c_6, and Δ have degrees 8, 12, and 24 respectively, and ω is the bundle of invariant differentials (the restriction to $\mathcal{M}_{ell}$ of the (vertical) cotangent bundle of the universal elliptic curve $\mathcal{E} \to \mathcal{M}_{ell}$). That ring of modular forms is periodic with period 24, and the periodicity is given by multiplication by the discriminant Δ. The discriminant is not an element in the homotopy groups of TMF, but its twenty-fourth power $\Delta^{24} \in \pi_{24^2}(TMF)$ is, and, as a result, $\pi_*(TMF)$ has a periodicity of order $24^2 = 576$.

One would like an analogous E_∞-ring spectrum whose homotopy groups are rationally isomorphic to the subring

$$\mathbb{Z}[c_4, c_6, \Delta]/(c_4^3 - c_6^2 - 1728\Delta)$$

of integral modular forms. For that, one observes that the sheaf $\mathcal{O}^{\mathrm{top}}$ is defined not only on the moduli stack of elliptic curves, but also on the Deligne–Mumford compactification $\overline{\mathcal{M}}_{ell}$ of the moduli stack—this compactification is the moduli stack of elliptic curves possibly with nodal singularities. The spectrum of global sections over $\overline{\mathcal{M}}_{ell}$ is denoted

$$Tmf := \mathcal{O}^{\mathrm{top}}(\overline{\mathcal{M}}_{ell}) = \Gamma(\overline{\mathcal{M}}_{ell}, \mathcal{O}^{\mathrm{top}}).$$

The element $\Delta^{24} \in \pi_{24^2}(Tmf)$ is no longer invertible in the homotopy ring, and so the spectrum Tmf is not periodic. This spectrum is not connective either, and the mixed capitalization reflects its intermediate state between the periodic version TMF and the connective version tmf, described below, of topological modular forms.

In positive degrees, the homotopy groups of Tmf are rationally isomorphic to the ring $\mathbb{Z}[c_4, c_6, \Delta]/(c_4^3 - c_6^2 - 1728\Delta)$. The homotopy groups $\pi_{-1}, \ldots, \pi_{-20}$ are all zero, and the remaining negative homotopy groups are given by:

$$\pi_{-n}(Tmf) \;\cong\; \left[\pi_{n-21}(Tmf)\right]_{\textit{torsion-free}} \oplus \left[\pi_{n-22}(Tmf)\right]_{\textit{torsion}}.$$

This structure in the homotopy groups is a kind of Serre duality reflecting the properness (compactness) of the moduli stack $\overline{\mathcal{M}}_{ell}$.

If we take the (-1)-connected cover of the spectrum Tmf, that is, if we kill all its negative homotopy groups, then we get

$$tmf := Tmf\langle 0\rangle,$$

the connective version of the spectrum topological modular forms. This spectrum is now, as desired, a topological refinement of the classical ring of integral modular forms. Note that one can recover TMF from either of the other versions by inverting the element Δ^{24} in the 576^{th} homotopy group:

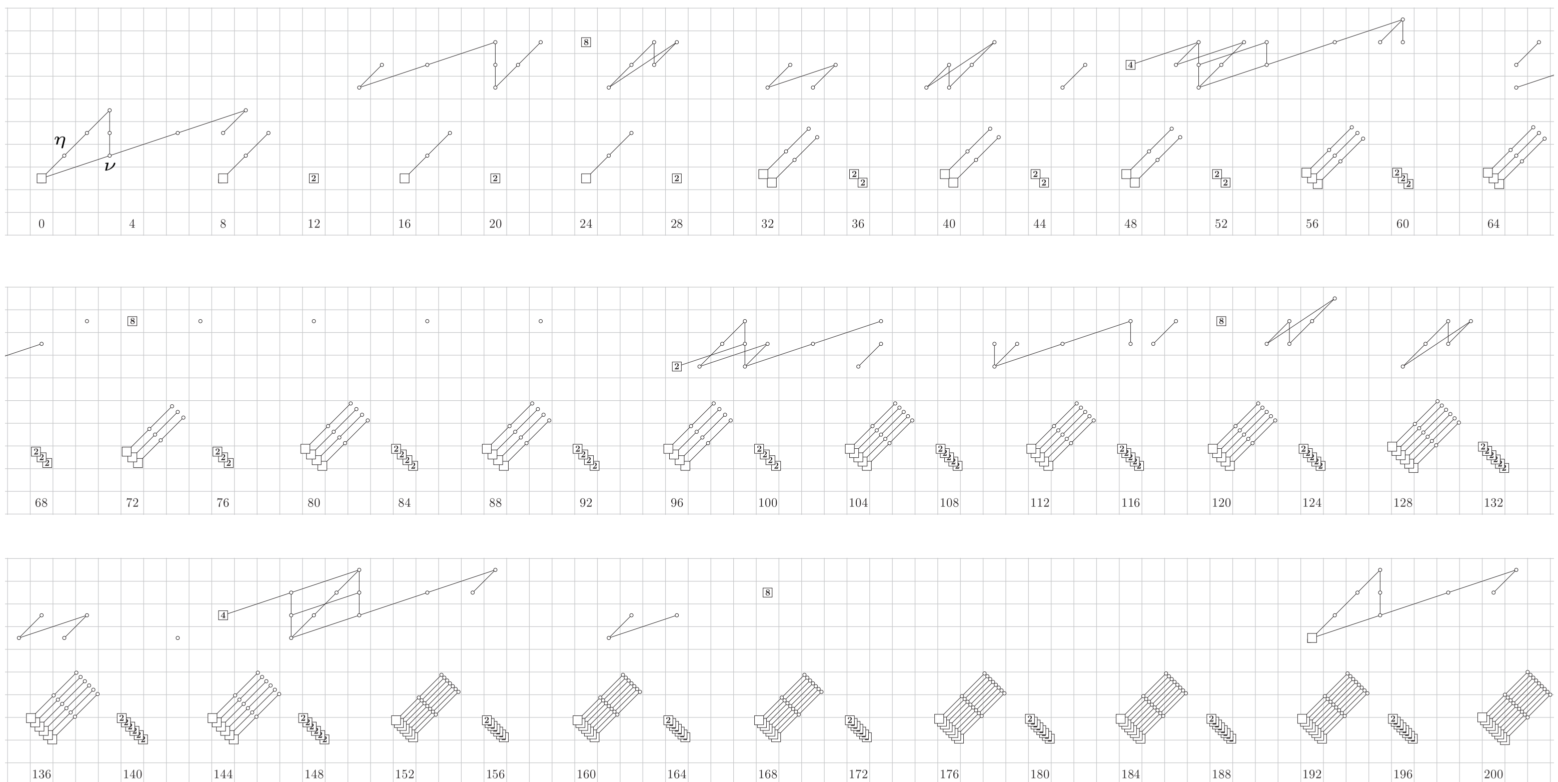
$$TMF = tmf[\Delta^{-24}] = Tmf[\Delta^{-24}].$$

There is another moduli stack worth mentioning here, the stack $\overline{\mathcal{M}}_{ell}^{+}$ of elliptic curves with possibly nodal or cuspidal singularities. There does not seem to be an extension of $\mathcal{O}^{\mathrm{top}}$ to that stack. However, if there were one, then a formal computation, namely an elliptic spectral sequence for that hypothetical sheaf, shows that the global sections of the sheaf over $\overline{\mathcal{M}}_{ell}^{+}$ would be the spectrum tmf. That hypothetical spectral sequence is the picture that appears before the preface. It is also, more concretely, the Adams–Novikov spectral sequence for the spectrum tmf.

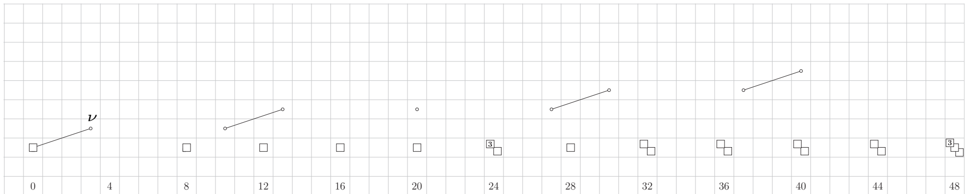
So far, we have only mentioned the connection between tmf and modular forms. The connection of tmf to the stable homotopy groups of spheres is equally strong and the unit map from the sphere spectrum to tmf detects an astounding amount of the 2- and 3-primary parts of the homotopy $\pi_*(\mathbb{S})$ of the sphere.

The homotopy groups of tmf are as follows at the prime 2:

The homotopy groups of *tmf* at the prime 2.



and as follows at the prime 3:



Here, a square indicates a copy of $\mathbb{Z}$ and a dot indicates a copy of $\mathbb{Z}/p$. A little number n drawn in a square indicates that the copy of $\mathbb{Z}$ in $\pi_*(tmf)$ maps onto an index n subgroup of the corresponding $\mathbb{Z}$ in the ring of modular forms. A vertical line between two dots indicates an additive extension, and a slanted line indicates the multiplicative action of the generator $\eta \in \pi_1(tmf)$ or $\nu \in \pi_3(tmf)$. The y-coordinate, although vaguely reminiscent of the filtration degree in the Adams spectral sequence, has no meaning in the above charts.

Note that, at the prime 2, the pattern on the top of the chart (that is, above the expanding ko pattern on the base) repeats with a periodicity of $192 = 8 \cdot 24$. A similar periodicity (not visible in the above chart) happens at the prime 3, with period $72 = 3 \cdot 24$. Over $\mathbb{Z}$, taking the least common multiple of these two periodicities results in a periodicity of $24 \cdot 24 = 576$.

2. A brief history of *tmf*

In the sixties, Conner and Floyd proved that complex K-theory is determined by complex cobordism: if X is a space, then its K-homology can be described as $K_*(X) \cong MU_*(X) \otimes_{MU_*} K_*$, where K_* is a module over the complex cobordism ring of the point via the Todd genus map $MU_* \to K_*$. Following this observation, it was natural to look for other homology theories that could be obtained from complex cobordism by a similar tensor product construction. By Quillen's theorem (1969), MU_* is the base ring over which the universal formal group law is defined; ring maps $MU_* \to R$ thus classify formal groups laws over R.

Given such a map, there is no guarantee in general that the functor $X \mapsto MU_*(X) \otimes_{MU_*} R$ will be a homology theory. If R is a flat MU_*-module, then long exact sequences remains exact after tensoring with R and so the functor in question does indeed define a new homology theory. However, the condition of being flat over MU_* is quite restrictive. Landweber's theorem (1976) showed that, because arbitrary MU_*-modules do not occur as the MU-homology of spaces, the flatness condition can be greatly relaxed. A more general condition, Landweber exactness, suffices to ensure that the functor $MU_*(-)\otimes_{MU_*} R$ satisfies the axioms of a homology theory. Shortly after the announcement of Landweber's result, Morava applied that theorem to the formal groups of certain elliptic curves and constructed the first elliptic cohomology theories (though the term 'elliptic cohomology' was coined only much later).

In the mid-eighties, Ochanine introduced certain genera (that is homomorphisms out of a bordism ring) related to elliptic integrals, and Witten constructed a genus that took values in the ring of modular forms, provided the low-dimensional characteristic classes of the manifold vanish. Landweber–Ravenel–Stong made explicit the connection between elliptic genera, modular forms, and elliptic cohomology by identifying the target of the universal Ochanine elliptic genus with the coefficient ring of the homology theory $X \mapsto MU_*(X) \otimes_{MU_*} \mathbb{Z}[\frac{1}{2}][\delta, \epsilon, \Delta^{-1}]$ associated to the Jacobi quartic elliptic curve $y^2 = 1 - 2\delta x^2 + \epsilon x^4$ (here, Δ is the discriminant of the polynomial in x). Segal had also presented a picture of the relationship between elliptic cohomology and Witten's physics-inspired index theory on loop spaces. In hindsight, a natural question would have been whether there existed a form of elliptic cohomology that received Witten's genus, thus explaining its integrality and modularity properties. But at the time, the community's attention was on Witten's rigidity conjecture for elliptic genera (established by Bott and Taubes), and on finding a geometric interpretation for elliptic cohomology—a problem that remains open to this day, despite a tantalizing proposal by Segal and much subsequent work.

Around 1989, inspired in part by work of McClure and Baker on A_∞ structures and actions on spectra and by Ravenel's work on the odd primary Arf invariant, Hopkins and Miller showed that a certain profinite group known as the Morava stabilizer group acts by A_∞ automorphisms on the Lubin–Tate spectrum E_n (the representing spectrum for the Landweber exact homology theory associated to the universal deformation of a height n formal group law). Of special interest was the action of the binary tetrahedral group on the spectrum E_2 at the prime 2. The homotopy fixed point spectrum of this action was called EO_2, by analogy with the real K-theory spectrum KO being the homotopy fixed points of complex conjugation on the complex K-theory spectrum.

Mahowald recognized the homotopy of EO_2 as a periodic version of a hypothetical spectrum with mod two cohomology $A /\!/ A(2)$, the quotient of the Steenrod algebra by the submodule generated by Sq^1, Sq^2, and Sq^4. It seemed likely that there would be a corresponding connective spectrum eo_2 and indeed a bit later Hopkins and Mahowald produced such a spectrum; (in hindsight, that spectrum eo_2 is seen as the 2-completion of *tmf*). However, Davis–Mahowald (1982) had proved, by an intricate spectral sequence argument, that it is impossible to realize $A /\!/ A(2)$ as the cohomology of a spectrum. This conundrum was resolved only much later, when Mahowald found a missing differential around the 55^{th} stem of the Adams spectral sequence for the sphere, invalidating the earlier Davis–Mahowald argument.

In the meantime, computations of the cohomology of $MO\langle 8\rangle$ at the prime 2 revealed an $A /\!/ A(2)$ summand, suggesting the existence of a map of spectra from $MO\langle 8\rangle$ to eo_2. While attempting to construct a map $MO\langle 8\rangle \to EO_2$, Hopkins (1994) thought to view the binary tetrahedral group as the automorphism group of the supersingular elliptic curve at the prime 2; the idea of a sheaf of ring spectra over the moduli stack of elliptic curves quickly followed—the global sections of that sheaf, TMF, would then be an integral version of EO_2.

The language of stacks, initially brought to bear on complex cobordism and formal groups by Strickland, proved crucial for even formulating the question TMF would answer. In particular, the stacky perspective allowed a reformulation of Landweber's exactness criterion in a more conceptual and geometric way: $MU_* \to R$ is Landweber exact if and only if the corresponding map to the moduli stack of formal groups, $\mathrm{Spec}(R) \to \mathcal{M}_{FG}$, is flat. From this viewpoint, Landweber's theorem defined a presheaf of homology theories on the flat site of the moduli stack $\mathcal{M}_{FG}$ of formal groups. Restricting to those formal groups coming from elliptic curves then provided a presheaf of homology theories on the moduli stack of elliptic curves.

Hopkins and Miller conceived of the problem as lifting this presheaf of homology theories to a sheaf of spectra. In the 80s and early 90s, Dwyer, Kan, Smith, and Stover had developed an obstruction theory for rigidifying a diagram in a homotopy category (here a diagram of elliptic homology theories) to an honest diagram (here a sheaf of spectra). Hopkins and Miller adapted the Dwyer–Kan–Stover theory to treat the seemingly more difficult problem of rigidifying a diagram of multiplicative cohomology theories to a diagram of A_∞-ring spectra. The resulting multiplicative obstruction groups vanished, except at the prime 2—Hopkins addressed that last case by a direct construction in the category of $K(1)$-local E_∞-ring spectra. Altogether the resulting sheaf of spectra provided a universal elliptic cohomology theory, the spectrum *TMF* of global sections (and its connective version *tmf*). Subsequently, Goerss and Hopkins upgraded the A_∞ obstruction theory to an obstruction theory for E_∞-ring spectra, leading to the definitive theorem of Goerss–Hopkins–Miller: the presheaf of elliptic homology theories on the compactified moduli stack of elliptic curves lifts to a sheaf of E_∞-ring spectra.

Equipped finally with the spectrum *tmf*, Ando–Hopkins–Strickland (2001) established the connection to elliptic genera by constructing a *tmf*-orientation for almost complex manifolds with certain vanishing characteristic classes; specifically, they built a map of ring spectra from $MU\langle 6\rangle$ to *tmf*. This map was later refined by Ando–Hopkins–Rezk to a map of E_∞-ring spectra $MO\langle 8\rangle \to \mathit{tmf}$ that recovers Witten's genus at the level of homotopy groups. In the meantime, the source spectrum $MO\langle 8\rangle$ of that map had been optimistically rebranded as *MString*.

Later, an interpretation of *tmf* was given by Lurie (2009) using the theory of spectral algebraic geometry, based on work of Töen and Vezzosi. Lurie interpreted the stack $\mathcal{M}_{ell}$ with its sheaf $\mathcal{O}^{\mathrm{top}}$ as a stack not over commutative rings but over E_∞-ring spectra. Using Goerss–Hopkins–Miller obsturction theory and a spectral form of Artin's representability theorem, he identified that stack as classifying oriented elliptic curves over E_∞-ring spectra. Unlike the previous construction of *tmf* and of the sheaf $\mathcal{O}^{\mathrm{top}}$, this description specifies the sheaf and therefore the spectrum *tmf* up to a contractible space of choices.

3. Overview

Part I

Chapter 1: Elliptic genera and elliptic cohomology. One-dimensional formal group laws entered algebraic topology though complex orientations, in answering the question of which generalized cohomology theories E carry a theory of Chern classes for complex vector bundles. In any such theory, the E-cohomology of $\mathbb{CP}^\infty$ is isomorphic to $E^*[[c_1]]$, the E-cohomology ring of a point adjoined a formal power series generator in degree 2. The tensor product of line bundles defines a map $\mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$, which in turn defines a comultiplication on $E^*[[c_1]]$, i.e., a formal group law. Ordinary homology is an example of such a theory; the associated formal group is the additive formal group, since the first Chern class of the tensor product of line bundles is the sum of the respective Chern classes, $c_1(L \otimes L') = c_1(L) + c_1(L')$. Complex K-theory is another example of such a theory; the associated formal group is the multiplicative formal group.

Complex cobordism also admits a theory of Chern classes, hence a formal group. Quillen's theorem is that this is the universal formal group. In other words, the formal group of complex cobordism defines a natural isomorphism of MU^* with the Lazard ring, the classifying ring for formal groups. Thus, a one-dimensional formal group over a ring R is essentially equivalent to a complex genus, that is, a ring homomorphism $MU^* \to R$. One important example of such a genus is the Todd genus, a map $MU^* \to K^*$. The Todd genus occurs in the Hirzebruch–Riemann–Roch theorem, which calculates the index of the Dolbeault operator in terms of the Chern character. It also determines the K-theory of a finite space X from its complex cobordism groups, via the Conner–Floyd theorem: $K^*(X) \cong MU^*(X) \otimes_{MU^*} K^*$.

Elliptic curves form a natural source of formal groups, and hence complex genera. An example of such is Euler's formal group law over $\mathbb{Z}[\frac{1}{2}, \delta, \epsilon]$ associated to Jacobi's quartic elliptic curve; the corresponding elliptic cohomology theory is given on finite spaces by $X \mapsto MU^*(X) \otimes_{MU^*} \mathbb{Z}[\frac{1}{2}, \delta, \epsilon]$. Witten defined a genus $MSpin \to \mathbb{Z}[[q]]$ (not an elliptic genus in the above strict sense, because not a map out of MU^*) which lands in the ring of modular forms, provided the characteristic class $\frac{p_1}{2}$ vanishes. He also gave an index theory interpretation, at a physical level of rigor, in terms of Dirac operators on loop spaces. It was later shown, by Ando–Hopkins–Rezk that the Witten genus can be lifted to a map of ring spectra $MString \to tmf$. The theory of topological modular forms can therefore be seen as a solution to the problem of finding a kind of elliptic cohomology that is connected to the Witten genus in the same way that the Todd genus is to K-theory.

Chapter 2: Elliptic curves and modular forms. An elliptic curve is a non-singular curve in the projective plane defined by a Weierstrass equation:

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

Elliptic curves can also be presented abstractly, as pointed genus one curves. They are equipped with a group structure, where one declares the sum of three points to be zero if they are collinear in $\mathbb{P}^2$. The bundle of Kähler differentials on an elliptic curve, denoted ω, has a one-dimensional space of global sections.

When working over a field, one-dimensional group varieties can be classified into additive groups, multiplicative groups, and elliptic curves. However, when working over an arbitrary ring, the object defined by a Weierstrass equation will be typically a combination of those three cases. The general fibers will typically be elliptic curves, some fibers will be nodal (multiplicative groups), and some cuspidal (additive groups).

By a 'Weierstrass curve' we mean a curve defined by a Weierstrass equation—no smoothness requirement. An *integral modular form* can then be defined, abstractly, to be a law that associates to every (family of) Weierstrass curves a section of $\omega^{\otimes n}$, in a way compatible with base change. Integral modular forms form a graded ring, graded by the power of ω. Here is a concrete presentation of that ring:

$$\mathbb{Z}[c_4, c_6, \Delta]/(c_4^3 - c_6^2 - 1728\Delta).$$

In the context of modular forms, the degree is usually called *weight*: the generators c_4, c_6, and Δ have weight 4, 6, and 12, respectively. As we'll see later, those weights correspond to the degrees 8, 12, and 24 in the homotopy groups of tmf.

Chapter 3: The moduli stack of elliptic curves. We next describe the geometry of the moduli stack of elliptic curves over fields of prime characteristic, and over the integers. At large primes, the stack $\mathcal{M}_{ell}$ looks rather like the way it does over $\mathbb{C}$: general elliptic curves have an automorphism group of order two, and there are two special curves with automorphism groups of orders four and six. That picture needs to be modified when dealing with small primes. At the prime $p = 3$ (respectively $p = 2$), there is only one special 'orbifold point', and the automorphism group of the corresponding elliptic curve has order 12 (respectively 24). The automorphism group of that curve is given by $\mathbb{Z}/4 \ltimes \mathbb{Z}/3$ at the prime 3, and by $\mathbb{Z}/3 \ltimes Q_8$ (also known as the binary tetrahedral group) at the prime 2.

In characteristic p, there is an important dichotomy between *ordinary* and *supersingular* elliptic curves. An elliptic curve C is ordinary if its group of p-torsion points has p connected components, and supersingular if the group of p-torsion points is connected. This dichotomy is also reflected in the structure of the multiplication-by-p map, which is purely inseparable in the supersingular case, and the composite of an inseparable map with a degree p covering in the case of an ordinary elliptic curve. The supersingular elliptic curves form a zero-dimensional substack of $(\mathcal{M}_{ell})_{\mathbb{F}_p}$—the stack of elliptic curves in characteristic p—whose cardinality grows roughly linearly in p. If one counts supersingular curves with a multiplicity equal to the inverse of the order of their automorphism group, then there are exactly $(p-1)/24$ of them.

The stratification of $(\mathcal{M}_{ell})_{\mathbb{F}_p}$ into ordinary and supersingular alliptic curves is intimately connected to the stratification of the moduli stack of formal groups by the *height* of the formal group. A formal group has height n if the first non-zero coefficient of the multiplication-by-p map is that of x^{p^n}. The ordinary elliptic curves are the ones whose associated formal group has height one, and the supersingular elliptic curves are the ones whose associated formal group has height two. Higher heights cannot occur among elliptic curves.

Chapter 4: The Landweber exact functor theorem. The next main result is that there this a presheaf Ell of homology theories on the (affines of the) flat site of the moduli stack of elliptic curves—the category whose objects are flat maps $\mathrm{Spec}(R) \to \mathcal{M}_{ell}$. That presheaf is defined as follows. Given an elliptic curve C over a ring R, classified by a flat map $\mathrm{Spec}(R) \to \mathcal{M}_{ell}$, the associated formal group $\widehat{C}$ corresponds to a map $MP_0 \to R$, where $MP_* = \bigoplus_{n\in\mathbb{Z}} MU_{*+2n}$ is the periodic version of complex cobordism. Ell^C is then defined by

$$Ell^C(X) := MP_*(X) \underset{MP_0}{\otimes} R.$$

We claim that for every elliptic curve C whose classifying map $\mathrm{Spec}(R) \to \mathcal{M}_{ell}$ is flat, the functor Ell^C is a homology theory, i.e., satisfies the exactness axiom. An example of an elliptic curve whose classifying map is flat, and which therefore admits an associated elliptic homology theory, is the universal smooth Weierstrass curve.

The proof is a combination of several ingredients. The main one is the Landweber exact functor theorem, which provides an algebraic criterion (Landweber exactness, which is weaker than flatness) on a ring map $MP_0 \to R$, that ensures the functor $X \mapsto MP_*(X) \otimes_{MP_0} R$ satisfies exactness. The other ingredients, due to Hopkins and Miller, relate the geometry of $\mathcal{M}_{ell}$ and $\mathcal{M}_{FG}$ to the Landweber exactness criterion. These results are the following: (1) A formal group law $MP_0 \to R$ over R is Landweber exact if and only if the corresponding map $\mathrm{Spec}(R) \to \mathcal{M}_{FG}$ is flat; together with Landweber exactness, this gives a presheaf of homology theories on the flat site of the moduli stack of formal groups $\mathcal{M}_{FG}$. (2) The map of stacks, $\mathcal{M}_{ell} \to \mathcal{M}_{FG}$ defined by sending an elliptic to its associated formal group, is flat.

Chapter 5: Sheaves in homotopy theory. By the above construction, using the Landweber Exact Functor Theorem, we have a presheaf $\mathcal{O}^{\mathrm{hom}}$ of homology theories (previously called *Ell*) on the moduli stack of elliptic curves. One might try to define a single 'universal elliptic homology theory' as the limit $\lim_{U \in \mathfrak{U}} \mathcal{O}^{\mathrm{hom}}(U)$, where $\mathfrak{U}$ is an affine cover of the moduli stack. However, the category of homology theories does not admit limits. If, though, we can rigidify the presheaf $\mathcal{O}^{\mathrm{hom}}$ of homology theories to a presheaf $\mathcal{O}^{\mathrm{top}}$ of spectra, then we can use instead a homotopy limit construction in the category of spectra. The main theorem is that there does indeed exist such a presheaf, in fact a sheaf, of spectra.

THEOREM (Goerss–Hopkins–Miller). *There exists a sheaf $\mathcal{O}^{\mathrm{top}}$ of E_∞ ring spectra on $(\mathcal{M}_{ell})_{\acute{e}t}$, the étale site of the moduli stack of elliptic curves (whose objects are étale maps to $\mathcal{M}_{ell}$), such that the associated presheaf of homology theories, when restricted to those maps whose domain is affine, is the presheaf $\mathcal{O}^{\mathrm{hom}}$ built using the Landweber Exact Functor Theorem.*

In this theorem, it is essential that the sheaf $\mathcal{O}^{\mathrm{top}}$ is a functor to a point-set-level, not homotopy, category of E_∞ ring spectra. Moreover, the functor is defined on *all* étale maps $\mathcal{N} \to \mathcal{M}_{ell}$, not just those where $\mathcal{N}$ is affine; (in fact, $\mathcal{N}$ can be itself a stack, as long as the map to $\mathcal{M}_{ell}$ is étale). Given a cover $\mathfrak{N} = \{\mathcal{N}_i \to \mathcal{N}\}$ of an object $\mathcal{N}$, we can assemble the n-fold 'intersections' $\mathcal{N}_{ij} := \mathcal{N}_i \times_{\mathcal{N}} \mathcal{N}_j$, $\mathcal{N}_{ijk} := \mathcal{N}_i \times_{\mathcal{N}} \mathcal{N}_j \times_{\mathcal{N}} \mathcal{N}_k$, etc., into a simplicial object

$$\mathfrak{N}_\bullet \;=\; \Big[\coprod \mathcal{N}_i \leftleftarrows \coprod \mathcal{N}_{ij} \;\substack{\leftarrow\\\leftarrow\\\leftarrow}\; \coprod \mathcal{N}_{ijk} \;\substack{\leftarrow\\\leftarrow\\\leftarrow\\\leftarrow}\; \cdots \Big].$$

The sheaf condition is that the natural map from the totalization (homotopy limit) of the cosimplicial spectrum

$$\mathcal{O}^{\mathrm{top}}(\mathfrak{N}_\bullet) \;=\; \Big[\mathcal{O}^{\mathrm{top}}(\coprod \mathcal{N}_i) \rightrightarrows \mathcal{O}^{\mathrm{top}}(\coprod \mathcal{N}_{ij}) \;\substack{\to\\\to\\\to}\; \mathcal{O}^{\mathrm{top}}(\coprod \mathcal{N}_{ijk}) \cdots \Big]$$

to $\mathcal{O}^{\mathrm{top}}(\mathcal{N})$ is an equivalence.

Now consider a cover $\mathfrak{N} = \{\mathcal{N}_i \to \mathcal{M}_{ell}\}$ of $\mathcal{M}_{ell}$ by affine schemes. The above cosimplicial spectrum has an assoicated tower of fibrations

$$\ldots \to \mathrm{Tot}^2\, \mathcal{O}^{\mathrm{top}}(\mathfrak{N}_\bullet) \to \mathrm{Tot}^1\, \mathcal{O}^{\mathrm{top}}(\mathfrak{N}_\bullet) \to \mathrm{Tot}^0\, \mathcal{O}^{\mathrm{top}}(\mathfrak{N}_\bullet)$$

whose inverse limit is $\mathrm{Tot}\, \mathcal{O}^{\mathrm{top}}(\mathfrak{N}_\bullet) = \mathcal{O}^{\mathrm{top}}(\mathcal{M}_{ell}) = TMF$. The spectral sequence associated to this tower has as E^2 page the Cech cohomology $\check{H}^q_{\mathfrak{N}}(\mathcal{M}_{ell}, \pi_p \mathcal{O}^{\mathrm{top}})$ of $\mathfrak{N}$ with coefficients in $\pi_p \mathcal{O}^{\mathrm{top}}$. Since $\mathfrak{N}$ is a cover by affines, the Cech cohomology of that cover is the same as the sheaf cohomology of $\mathcal{M}_{ell}$ with coefficients in the sheafification $\pi_p^\dagger \mathcal{O}^{\mathrm{top}}$ of $\pi_p \mathcal{O}^{\mathrm{top}}$; (that sheafification happens to agree with $\pi_p \mathcal{O}^{\mathrm{top}}$ on maps to $\mathcal{M}_{ell}$ whose domain is affine). Altogether, we get a spectral sequence, the so-called descent spectral sequence, that converges to the homotopy groups of the spectrum of global sections:

$$E^2_{pq} = H^q(\mathcal{M}_{ell}, \pi_p^\dagger \mathcal{O}^{\mathrm{top}}) \;\Rightarrow\; \pi_{p-q}\, TMF.$$

<u>Chapter 6</u>: Bousfield localization and the Hasse square. We would like a sheaf of spectra $\mathcal{O}^{\mathrm{top}}$ on the moduli stack of elliptic curves $\mathcal{M}_{ell}$. As we will see, this moduli stack is built out of its p-completions $\mathcal{O}^{\mathrm{top}}_p$ and its rationalization. The p-completion $\mathcal{O}^{\mathrm{top}}_p$ is in turn built from certain localizations of $\mathcal{O}^{\mathrm{top}}$ with respect to the first and second Morava K-theories.

Localizing a spectrum X at a spectrum E is a means of systematically ignoring the part of X that is invisible to E. A spectrum A is called E-acyclic if $A \wedge X$ is contractible. A spectrum B is called E-local if there are no nontrivial maps from an E-acyclic spectrum into B. Finally, a spectrum Y is an E-localization of X if it is E-local and there is a map $X \to Y$ that is an equivalence after smashing with E. This localization is denoted $L_E X$ or X_E.

The localization $L_p X := L_{M(\mathbb{Z}/p)} X$ of a spectrum X at the mod p Moore spectrum is the p-completion of X (when X is connective); we denote this localization map $\eta_p : X \to L_p X$. The localization $L_{\mathbb{Q}} X := L_{H\mathbb{Q}} X$ at the rational Eilenberg–MacLane spectrum is the rationalization of X; we denote this localization map $\eta_{\mathbb{Q}} : X \to L_{\mathbb{Q}} X$.

Any spectrum X can be reconstructed from its p-completion and rationalization by means of the 'Sullivan arithmetic square', which is the following homotopy pullback square:

$$\begin{tikzcd}
X \arrow[r, "\prod \eta_p"] \arrow[d, "\eta_{\mathbb{Q}}"'] & \prod_p L_p X \arrow[d, "\eta_{\mathbb{Q}}"] \\
L_{\mathbb{Q}} X \arrow[r, "L_{\mathbb{Q}}(\prod \eta_p)"] & L_{\mathbb{Q}} \left(\prod_p L_p X\right).
\end{tikzcd}$$

The above pullback square is a special case of the localization square

$$\begin{tikzcd}
L_{E \vee F} X \arrow[r, "\eta_E"] \arrow[d, "\eta_F"'] & L_E X \arrow[d, "\eta_F"] \\
L_F X \arrow[r, "L_F(\eta_E)"] & L_F L_E X,
\end{tikzcd}$$

which is a homotopy pullback square if one assumes that $E_*(L_F X) = 0$.

An application of this localization square gives the so-called 'chromatic fracture square':

$$\begin{tikzcd}
L_{K(1) \vee K(2)} X \arrow[r, "\eta_{K(2)}"] \arrow[d, "\eta_{K(1)}"'] & L_{K(2)} X \arrow[d, "\eta_{K(1)}"] \\
L_{K(1)} X \arrow[r, "L_{K(1)}(\eta_{K(2)})"] & L_{K(1)} L_{K(2)} X.
\end{tikzcd}$$

Here $K(1)$ and $K(2)$ are the first and second Morava K-theory spectra.

When the spectrum in question is an elliptic spectrum, the above square simplifies into the 'Hasse square': for any elliptic spectrum E, there is a pullback square

$$\begin{array}{ccc} L_pE & \longrightarrow & L_{K(2)}E \\ \downarrow & & \downarrow \\ L_{K(1)}E & \longrightarrow & L_{K(1)}L_{K(2)}E. \end{array}$$

By means of the arithmetic square, the construction of the sheaf $\mathcal{O}^{\mathrm{top}}$ is reduced to the construction of its p-completions, of its rationalization, and of the comparison map between the rationalization and the rationalization of the product of the p-completions. In turn, the construction of the p-completion $\mathcal{O}_p^{\mathrm{top}}$ of the sheaf $\mathcal{O}^{\mathrm{top}}$ is reduced to the construction of the corresponding $K(1)$- and $K(2)$-localizations and of a comparison map between the $K(1)$-localization and the $K(1)$-localization of the $K(2)$-localization.

Chapter 7: The local structure of the moduli stack of formal groups. By Landweber's theorem, flat maps $\mathrm{Spec}(R) \to \mathcal{M}_{FG}$ to the moduli stack of one-dimensional formal groups give rise to even-periodic homology theories:

$$X \mapsto MP_*(X) \otimes_{MP_0} R.$$

Here, MP is periodic complex bordism, $MP_0 = MU_* \cong \mathbb{Z}[u_1, u_2, \ldots]$ is the Lazard ring, and the choice of a formal group endows R with the structure of algebra over that ring. We wish to understand the geometry of $\mathcal{M}_{FG}$ with an eye towards constructing such flat maps.

The geometric points of $\mathcal{M}_{FG}$ have a simple description. If k is a separably closed field of characteristic $p > 0$, then formal groups over k are classified by their height, where again a formal group has height n if the first non-trivial term of its p-series (the multiplication-by-p map) is the one involving x^{p^n}. Given a formal group of height n, classified by $\mathrm{Spec}(k) \to \mathcal{M}_{FG}$, one may consider 'infinitesimal thickenings' $\mathrm{Spec}(k) \hookrightarrow B$, where B is the spectrum of a local (pro-)Artinian algebra with residue field k, along with an extension

$$\begin{tikzcd} \mathrm{Spec}(k) \arrow[r] \arrow[d, hook] & \mathcal{M}_{FG}. \\ B \arrow[ur, dashed] & \end{tikzcd}$$

This is called a deformation of the formal group. The Lubin–Tate theorem says that a height n formal group admits a universal deformation (a deformation with a unique map from any other deformation), carried by the ring $\mathbb{W}(k)[[v_1, \ldots v_{n-1}]]$. Here, $\mathbb{W}(k)$ denotes the ring of Witt vectors of k. Moreover, the map from $B := \mathrm{Spf}(\mathbb{W}(k)[[v_1, \ldots v_{n-1}]])$ to $\mathcal{M}_{FG}$ is then flat.

The formal groups of interest in elliptic cohomology come from elliptic curves. The Serre–Tate theorem further connects the geometry of $\mathcal{M}_{ell}$ with that of $\mathcal{M}_{FG}$, in the neighborhood of supersingular elliptic curves. According to this theorem, the deformations of a supersingular elliptic curve are equivalent to the deformations of its associated formal group. The formal neighborhood of a point $\mathrm{Spec}(k) \to \mathcal{M}_{ell}$ classifying a supersingular elliptic curve is therefore isomorphic to $\mathrm{Spf}(\mathbb{W}(k)[[v_1]])$, the formal spectrum of the universal deformation ring.

<u>Chapter 8</u>: Goerss–Hopkins obstruction theory. Goerss–Hopkins obstruction theory is a technical apparatus for approaching questions such as the following: for a ring spectrum E and a commutative E_*-algebra A in E_*E-comodules, is there an E_∞-ring spectrum X such that E_*X is equivalent to A? What is the homotopy type of the space of all such E_∞-ring spectra X?

That space is called the realization space of A and denoted $B\mathcal{R}(A)$. The main result here is that there is an obstruction theory for specifying points of $B\mathcal{R}(A)$, and that the obstructions live in certain André–Quillen cohomology groups of A. More precisely, there is a Postnikov-type tower

$$\ldots \to B\mathcal{R}_n(A) \to B\mathcal{R}_{n-1}(A) \to \ldots \to B\mathcal{R}_0(A)$$

with inverse limit $B\mathcal{R}(A)$ whose layers are controlled by the André–Quillen cohomology groups of A, as follows. If we let $\mathcal{H}^{n+2}(A;\Omega^n A)$ be the André–Quillen cohomology space (the Eilenberg–MacLane space for the André–Quillen cohomology group) of the algebra A with coefficients in the nth desuspension of A, then $\mathcal{H}^{n+2}(A;\Omega^n A)$ is acted on by the automorphism group of the pair $(A,\Omega^n A)$ and we can form, by the Borel construction, a space $\widehat{\mathcal{H}}^{n+2}(A;\Omega^n A)$ over the classifying space of $Aut(A,\Omega^n A)$. This is a bundle of pointed spaces and the base points provide a section $BAut(A,\Omega^n A) \to \widehat{\mathcal{H}}^{n+2}(A;\Omega^n A)$. The spaces $B\mathcal{R}_n(A)$ then fit into homotopy pullback squares

$$\begin{tikzcd}
B\mathcal{R}_n(A) \arrow[r] \arrow[d] & BAut(A,\Omega^n A) \arrow[d] \\
B\mathcal{R}_{n-1}(A) \arrow[r] & \widehat{\mathcal{H}}^{n+2}(A;\Omega^n A).
\end{tikzcd}$$

<u>Chapter 9</u>: From spectra to stacks. We have focussed on constructing spectra using stacks, but one can also go the other way, associating stacks to spectra. Given a commutative ring spectrum X, let $\mathcal{M}_X$ be the stack associated to the Hopf algebroid

$$(MU_*X, MU_*MU \otimes_{MU_*} MU_*X).$$

If X is complex orientable, then $\mathcal{M}_X$ is the scheme $\mathrm{Spec}(\pi_*X)$—the stackiness of $\mathcal{M}_X$ therefore measures the failure of complex orientability of X. The canonical Hopf algebroid map $(MU_*, MU_*MU) \to (MU_*X, MU_*MU \otimes_{MU_*} MU_*X)$ induces a map of stacks from $\mathcal{M}_X$ to $\mathcal{M}_{FG}^{(1)}$, the moduli stack of formal groups with first order coordinate. Moreover, under good circumstances, the stack associated to a smash product of two ring spectra is the fiber product over $\mathcal{M}_{FG}^{(1)}$:

$$\mathcal{M}_{X\wedge Y} \cong \mathcal{M}_X \times_{\mathcal{M}_{FG}^{(1)}} \mathcal{M}_Y.$$

It will be instructive to apply the above isomorphism to the case when Y is tmf, and X is one of the spectra in a filtration

$$\mathbb{S}^0 = X(1) \to X(2) \to \cdots X(n) \to \cdots \to MU$$

of the complex cobordism spectrum. By definition, $X(n)$ is the Thom spectrum associated to the subspace $\Omega SU(n)$ of $\Omega SU \simeq BU$; the spectrum $X(n)$ is an E_2-ring spectrum because $\Omega SU(n)$ is a double loop space. Recall that for a complex orientable theory R, multiplicative maps $MU \to R$ correspond to coordinates on the formal group of R. There is a similar story with $X(n)$ in place of MU, where the formal groups are now only defined modulo terms of degree $n+1$, and multiplicative maps $X(n) \to R$ correspond to coordinates up to degree n. Using this description, one can show that $\mathcal{M}_{X(n)}$ is the stack $\mathcal{M}_{FG}^{(n)}$, the classifying stack of formal groups with a coordinate up to degree n. The map from $\mathcal{M}_{FG}^{(n)}$ to $\mathcal{M}_{FG}^{(1)}$ is the obvious forgetful map.

The stack $\mathcal{M}_{tmf}$ associated to *tmf* is the moduli stack of generalized elliptic curves (both multiplicative and additive degenerations allowed) with first order coordinate. The stack $\mathcal{M}_{X(4)\wedge tmf}$ can therefore be identified with the moduli stack of elliptic curves together with a coordinate up to degree 4. The pair of an elliptic curve and such a coordinate identifies a Weierstrass equation for the curve, and so this stack is in fact a scheme:

$$\mathcal{M}_{X(4)\wedge tmf} \cong \operatorname{Spec} \mathbb{Z}[a_1, a_2, a_3, a_4, a_6].$$

Here, a_1, a_2, a_3, a_4, a_6 are the coefficients of the universal Weierstras equation. By considering the products $X(4) \wedge \ldots \wedge X(4) \wedge tmf$, one can furthermore identify the whole $X(4)$-based Adams resolution of *tmf* with the cobar resolution of the Weierstrass Hopf algebroid.

Chapter 10: The string orientation. The string orientation, or σ-orientation of *tmf* is a map of E_∞-ring spectra

$$MO\langle 8\rangle \to tmf.$$

Here, $MO\langle 8\rangle = MString$ is the Thom spectrum of the 7-connected cover of BO, and its homotopy groups are the cobordism groups of string manifolds (manifolds with a chosen lift to $BO\langle 8\rangle$ of their tangent bundle's classifying map). At the level of homotopy groups, the map $MO\langle 8\rangle \to tmf$ is the Witten genus, a homomoprhism $[M] \mapsto \phi_W(M)$ from the string cobordism ring to the ring of integral modular forms. Note that $\phi_W(M)$ being an element of $\pi_*(tmf)$ instead of a mere modular form provides interesting congruences, not visible from the original definition of the Witten genus.

Even before having a proof, there are hints that the σ-orientation should exist. The Steenrod algebra module $H^*(tmf, \mathbb{F}_2) = A/\!/A(2)$ occurs as a summand in $H^*(MString, \mathbb{F}_2)$. This is reminiscent of the situation with the Atiyah–Bott–Shapiro orientation $MSpin \to ko$, where $H^*(ko, \mathbb{F}_2) = A/\!/A(1)$ occurs as a summand of $H^*(MSpin, \mathbb{F}_2)$.

Another hint is that, for any complex oriented cohomology theory E with associated formal group G, multiplicative (not E_∞) maps $MO\langle 8\rangle \to E$ correspond to sections of a line bundle over G^3 subject to a certain cocycle condition. If G is the completion of an elliptic curve C, then that line bundle is naturally the restriction of a bundle over C^3. That bundle is trivial, and because C^3 is proper, its space of sections is one dimensional (and there is even a preferred section). Thus, there is a preferred map $MO\langle 8\rangle \to E$ for every elliptic spectrum E.

A not-necessarily E_∞ orientation $MO\langle 8\rangle \to tmf$ is the same thing as a nullhomotopy of the composite

$$BO\langle 8\rangle \to BO \xrightarrow{J} BGL_1(\mathbb{S}) \to BGL_1(tmf),$$

where $BGL_1(R)$ is the classifying space for rank one R-modules. An E_∞ orientation $MO\langle 8\rangle \to tmf$ is a nullhomotopy of the corresponding map of spectra

$$bo\langle 8\rangle \to bo \xrightarrow{J} \Sigma gl_1(\mathbb{S}) \to \Sigma gl_1(tmf).$$

In order to construct that nullhomotopy, one needs to understand the homotopy type of $gl_1(tmf)$—this is done one prime at a time. The crucial observation is that there is a map of spectra $gl_1(tmf)_p^\wedge \to tmf_p^\wedge$, the 'topological logarithm', and a homotopy pullback square

$$\begin{tikzcd} gl_1(tmf) \arrow[r, "\log"] \arrow[d] & tmf \arrow[d] \\ L_{K(1)}(tmf) \arrow[r, "1-U_p"] & L_{K(1)}(tmf) \end{tikzcd}$$

where U_p is a topological refinement of Atkin's operator on p-adic modular forms. The fiber of the topological logarithm is particularly intriguing: Hopkins speculates that it is related to exotic smooth structures on free loop spaces of spheres.

Chapters 11 and 12: The sheaf of E_∞ ring spectra and **The construction of *tmf*.** We outline a roadmap for the construction of *tmf*, the connective spectrum of topological modular forms. The major steps in the construction are given in reverse order.

- The spectrum *tmf* is the connective cover of the nonconnective spectrum *Tmf*,

$$tmf := \tau_{\geq 0} Tmf,$$

and *Tmf* is the global sections of a sheaf of spectra,

$$Tmf := \mathcal{O}^{top}(\overline{\mathcal{M}}_{ell}),$$

where $\overline{\mathcal{M}}_{ell}$ is the moduli stack of elliptic curves with possibly nodal singularities. This stack is the Deligne–Mumford compactification of the moduli stack of smooth elliptic curves. Here, $\mathcal{O}^{top}$ is a sheaf on $\overline{\mathcal{M}}_{ell}$ in the étale topology. Also, *TMF* is the global sections of $\mathcal{O}^{top}$ over the substack $\mathcal{M}_{ell}$ of smooth elliptic curves,

$$TMF := \mathcal{O}^{top}(\mathcal{M}_{ell}).$$

The uppercase 'T' in *Tmf* signifies that the spectrum is no longer connective (but it is also not periodic). The '*top*' stands for topological, and $\mathcal{O}^{top}$ can be viewed as a kind of structure sheaf for a spectral version of $\overline{\mathcal{M}}_{ell}$.

We are left now to construct the sheaf of spectra $\mathcal{O}^{top}$. The first step is to isolate the problem at every prime p and at $\mathbb{Q}$. That is, one constructs $\mathcal{O}_p^{top}$, a sheaf of spectra on the p-completion $(\overline{\mathcal{M}}_{ell})_p$ and then pushes this sheaf forward along the inclusion map $\iota_p : (\overline{\mathcal{M}}_{ell})_p \to \overline{\mathcal{M}}_{ell}$. One then assembles these pushforwards to obtain $\mathcal{O}^{top}$, as follows.

- The sheaf $\mathcal{O}^{top}$ is the limit in a diagram

$$\begin{tikzcd} \mathcal{O}^{top} \arrow[r] \arrow[d] & \prod_p \iota_{p,*}\mathcal{O}_p^{top} \arrow[d] \\ \iota_{\mathbb{Q},*}\mathcal{O}_{\mathbb{Q}}^{top} \arrow[r] & \left(\prod_p \iota_{p,*}\mathcal{O}_p^{top}\right)_{\mathbb{Q}} \end{tikzcd}$$

for a given choice of map $\alpha_{arith} : \iota_{\mathbb{Q},*}\mathcal{O}_{\mathbb{Q}}^{top} \to \left(\prod_p \iota_{p,*}\mathcal{O}_p^{top}\right)_{\mathbb{Q}}$.

Once $\mathcal{O}^{top}$ has been constructed, it will turn out that $\mathcal{O}^{top}_p$ is the p-completion of $\mathcal{O}^{top}$, and $\mathcal{O}^{top}_{\mathbb{Q}}$ is its rationalization, so that the above diagram is the arithmetic square for $\mathcal{O}^{top}$. This thus leaves one to construct each $\mathcal{O}^{top}_p$ and $\mathcal{O}^{top}_{\mathbb{Q}}$ and the gluing map α_{arith}. The sheaf $\mathcal{O}^{top}_{\mathbb{Q}}$ is not difficult to construct. Its value on an étale map $\mathrm{Spec}(R) \to (\mathcal{M}_{ell})_{\mathbb{Q}}$ is given by $\mathcal{O}^{top}_{\mathbb{Q}}(\mathrm{Spec}\, R) = H(R_*)$, the rational Eilenberg–MacLane spectrum associated to a certain evenly graded ring R_*. This ring is specified by $R_{2t} := \Gamma(\omega^{\otimes t}|_{\mathrm{Spec}\, R})$, where ω is the sheaf of invariant differentials.

The construction of $\mathcal{O}^{top}_p$ is more subtle. The first step in its construction is to employ a natural stratification of $(\overline{\mathcal{M}}_{ell})_p$. Each elliptic curve has an associated formal group which either has height equal to 1 if the curve is *ordinary*, or equal to 2 if the curve is *supersingular*. This gives a stratification of the moduli space with exactly two strata:

$$\mathcal{M}^{ord}_{ell} \xrightarrow{\iota_{ord}} (\overline{\mathcal{M}}_{ell})_p \xleftarrow{\iota_{ss}} \mathcal{M}^{ss}_{ell}.$$

The sheaf $\mathcal{O}^{top}_p$ is presented by a Hasse square, gluing together a sheaf $\mathcal{O}^{top}_{K(1)}$ on $\mathcal{M}^{ord}_{ell}$ and a sheaf $\mathcal{O}^{top}_{K(2)}$ on $\mathcal{M}^{ss}_{ell}$. (This notation is used because the sheaves $\mathcal{O}^{top}_{K(i)}$ are also the $K(i)$-localizations of $\mathcal{O}^{top}$, where $K(i)$ is the ith Morava K-theory at the prime p.)

- $\mathcal{O}^{top}_p$ is the limit

$$\begin{tikzcd}
\mathcal{O}^{top}_p \arrow[r] \arrow[d] & \iota_{ss,*}\mathcal{O}^{top}_{K(2)} \arrow[d] \\
\iota_{ord,*}\mathcal{O}^{top}_{K(1)} \arrow[r] & \left(\iota_{ss,*}\mathcal{O}^{top}_{K(2)}\right)_{K(1)}
\end{tikzcd}$$

  for a certain 'chromatic' attaching map

$$\alpha_{chrom} : \iota_{ord,*}\mathcal{O}^{top}_{K(1)} \longrightarrow \left(\iota_{ss,*}\mathcal{O}^{top}_{K(2)}\right)_{K(1)}.$$

The sheaf $\mathcal{O}^{top}_p$ is thus equivalent to the following triple of data: a sheaf $\mathcal{O}^{top}_{K(1)}$ on $\mathcal{M}^{ord}_{ell}$, a sheaf $\mathcal{O}^{top}_{K(2)}$ on $\mathcal{M}^{ss}_{ell}$, and a gluing map α_{chrom} as above. We have now arrived at the core of the construction of *tmf*: the construction of these three objects. This construction proceeds via Goerss–Hopkins obstruction theory.

That obstruction theory is an approach to solving the following problem: one wants to determine the space of all E_∞-ring spectra subject to some conditions, such as having prescribed homology. More specifically, for any generalized homology theory E_*, and any choice of E_*-algebra A in E_*E-comodules, one can calculate the homotopy type of the moduli space of E_∞-ring spectra with E_*-homology isomorphic to A. Goerss and Hopkins describe that moduli space as the homotopy limit of a sequence of spaces, where the homotopy fibers are certain André–Quillen cohomology spaces of A. As a consequence, there is a sequence of obstructions to specifying a point of the moduli space, i.e., an E_∞-ring spectrum whose E_*-homology is A. The obstructions lie in André–Quillen cohomology groups of E_*-algebras in E_*E-comodules. That obstruction theory is used to build the sheaf $\mathcal{O}^{top}_{K(2)}$.

There is also a 'global' version of this obstruction theory, where one tries to lift a whole diagram I of E_*-algebras in E_*E-comodules to the category of E_∞-ring spectra. Here, in general, the obstructions live in the Hochschild–Mitchell cohomology group of the diagram I with coefficients in André–Quillen cohomology. This diagrammatic enhancement of the obstruction theory is used to build the sheaf $\mathcal{O}^{top}_{K(1)}$.

Obstruction theory for $\mathcal{O}^{top}_{K(2)}$: The stack $\mathcal{M}^{ss}_{ell}$ is a 0-dimensional substack of $\overline{\mathcal{M}}_{ell}$. More precisely, it is the disjoint union of classifying stacks BG where G ranges over the automorphism groups of the various supersingular elliptic curves. The Serre–Tate theorem identifies the formal completion of these groups G with the automorphism groups of the associated formal group. Consequently, to construct the sheaf $\mathcal{O}^{top}_{K(2)}$ on the category of étale affines mapping to $\mathcal{M}^{ss}_{ell}$, it suffices to construct the stalks of the sheaf at each point of $\mathcal{M}^{ss}_{ell}$, together with the action of these automorphism groups. The spectrum associated to a stalk is a Morava E-theory, the uniqueness of which is the Goerss–Hopkins–Miller theorem: that theorem says that there is an essentially unique (unique up to a contractible space of choices) way to construct an E_∞-ring spectrum $E(k,\mathbb{G})$, from a pair $(k,\mathbb{G})$ of a formal group $\mathbb{G}$ of finite height over a perfect field k, whose underlying homology theory is the Landweber exact homology theory associated to $(k,\mathbb{G})$. Altogether then, given a formal affine scheme $\mathrm{Spf}(R)$, with maximal ideal $I \subset R$, and an étale map $\mathrm{Spf}(R) \to \mathcal{M}^{ss}_{ell}$ classifying an elliptic curve C over R, the value of the sheaf $\mathcal{O}^{top}_{K(2)}$ is

$$\mathcal{O}^{top}_{K(2)}(\mathrm{Spf}(R)) := \prod_i E(k_i, \widehat{C}^{(i)}_0).$$

In this formula, the product is indexed by the set i in the expression of the quotient $R/I = \prod_i k_i$ as a product of perfect fields, and $\widehat{C}^i_0$ is the formal group associated to the base change to k_i of the elliptic curve C_0 over R/I.

Obstruction theory for $\mathcal{O}^{top}_{K(1)}$: We first explain the approach described in Chapter 11. Over the stack $\mathcal{M}^{ord}_{ell}$, there is a presheaf of homology theories given by the Landweber exact functor theorem. This presheaf assigns to an elliptic curve classified by an étale map $\mathrm{Spec}(R) \to \mathcal{M}^{ord}_{ell}$ the homology theory $X \mapsto BP_*(X) \otimes_{BP_*} R$. Ordinary elliptic curves have height 1, and so the representing spectrum is $K(1)$-local. In the setup of the Goerss–Hopkins obstruction theory for this situation, we take E_* to be p-adic K-theory, which has the structure of a *θ-algebra*. The moduli problem that we are trying to solve is that of determining the space of all lifts:

$$\begin{tikzcd} & E_\infty\text{-rings}_{K(1)} \arrow[d, "K^\wedge_p"] \\ I := \left(\mathit{Aff}/\mathcal{M}^{ord}_{ell}\right)_{\acute{e}t} \arrow[r] \arrow[ur, dashed] & \mathit{Alg}_\theta. \end{tikzcd}$$

In the general obstruction theory, the obstructions live in certain Hochschild–Mitchell cohomology groups of the diagram I. For this particular diagram, the obstruction groups simplify, and are equivalent to just diagram cohomology of I with coefficients in André–Quillen cohomology. The diagram cohomology of I is in turn isomorphic to the étale cohomology of the stack $\mathcal{M}_{ell}^{ord}$. In the end, the essential calculation is

$$H^s(\mathcal{M}_{ell}^{ord}, \omega^{\otimes k}) = 0 \qquad \text{for} \quad s > 0,$$

where $k \in \mathbb{Z}$ and ω is the line bundle of invariant differentials on $\mathcal{M}_{ell}$. At odd primes, the obstruction groups vanish in the relevant degrees, thus proving the existence and uniqueness of $\mathcal{O}^{top}_{K(1)}$. Unfortunately, the higher homotopy groups of the space of lifts are not all zero, and so one doesn't get a contractible space of choices for the sheaf $\mathcal{O}^{top}_{K(1)}$. At the prime $p = 2$, one needs to use real instead of complex K-theory to get obstruction groups that vanish.

The same obstruction theory for E_∞-ring spectra also applies to E_∞-ring maps, such as the gluing maps α_{chrom} and α_{arith} for the Hasse square and the arithmetic square. For α_{chrom}, one considers the moduli space of all E_∞-ring maps $\iota_{ord,*}\mathcal{O}^{top}_{K(1)} \to \iota_{ss,*}(\mathcal{O}^{top}_{K(2)})_{K(1)}$ whose induced map of theta-algebras is prescribed, and one tries to compute the homotopy groups of this moduli space. The obstruction groups here vanish, and there is an essentially unique map. For α_{arith}, the map is of rational spectra and the analysis is much easier; the obstruction groups vanish, and again there is an essentially unique map.

The approach presented in Chapter 12 is a somewhat different way of constructing $\mathcal{O}^{top}_{K(1)}$. In that other approach, one directly applies the $K(1)$-local obstruction theory to construct $L_{K(1)}tmf$, and then works backwards to construct $\mathcal{O}^{top}$. That approach allows one to avoid the obstruction theory for diagrams, but is more difficult in other steps—for instance, it requires use of level structures on the moduli stack $\overline{\mathcal{M}}_{ell}$ to resolve the obstructions.

<u>Chapter 13</u>: The homotopy groups of *tmf* and of its localizations. The homotopy groups of *tmf* are an elaborate amalgam of the classical ring of modular forms MF_* and certain pieces of the 2- and 3-primary part of the stable homotopy groups of spheres $\pi_*(\mathbb{S})$.

There are two homomorphisms

$$\pi_*(\mathbb{S}) \to \pi_*(tmf) \to MF_*.$$

The first map is the Hurewicz homomorphism, and it is an isomorphism on π_0 through π_6. Conjecturally, this map hits almost all of the interesting torsion classes in $\pi_*(tmf)$ and its image (except for the classes η, η^2, and ν) is periodic with period 576 (arising from a 192-fold periodicity at the prime 2 and a 72-fold periodicity at the prime 3). Among others, the map is nontrivial on the 3-primary stable homotopy classes $\alpha \in \pi_3(\mathbb{S})$ and $\beta \in \pi_{10}(\mathbb{S})$ and the 2-primary stable homotopy classes $\eta, \nu, \epsilon, \kappa, \overline{\kappa}, q \in \pi_*(\mathbb{S})$. The second map in the above display is the composite of the inclusion $\pi_* tmf \to \pi_* Tmf$ with the boundary homomorphism in the elliptic spectral sequence

$$H^s(\overline{\mathcal{M}}_{ell}; \pi_t\mathcal{O}^{\text{top}}) \;\Rightarrow\; \pi_{t-s}(Tmf).$$

This map $\pi_*(tmf) \to MF_* = \mathbb{Z}[c_4, c_6, \Delta]/(c_4^3 - c_6^2 - (12)^3\Delta)$ is an isomorphism after inverting the primes 2 and 3. The kernel of this map is exactly the torsion in $\pi_*(tmf)$ and the cokernel is a cyclic group of order dividing 24 in degrees divisible by 24, along with some number of cyclic groups of order 2 in degrees congruent to 4 mod 8. In particular, the map from $\pi_*(tmf)$ hits the modular forms c_4, $2c_6$, and 24Δ, but c_6 and Δ themselves are not in the image. The localization $\pi_*(tmf)_{(p)}$ at any prime larger than 3 is isomorphic to $(MF_*)_{(p)} \cong \mathbb{Z}_{(p)}[c_4, c_6]$.

The homotopy of *tmf* can be computed directly using the Adams spectral sequence. Alternatively, one can use the elliptic spectral sequence to compute the homotopy of *Tmf*. The Adams spectral sequence has the form

$$E_2 = \mathrm{Ext}_{A_p^{tmf}}(\mathbb{F}_p, \mathbb{F}_p) \Rightarrow \pi_*(tmf)\hat{}_p,$$

where $A_p^{tmf} := \hom_{tmf-\text{modules}}(H\mathbb{F}_p, H\mathbb{F}_p)$ is a *tmf*-analog of the Steenrod algebra. At the prime 2, the map $A_2^{tmf} \to A \equiv A_2$ to the classical Steenrod algebra is injective, and the *tmf*-module Adams spectral sequence can be identified with the classical Adams spectral sequence

$$E_2 = \mathrm{Ext}_A(H^*(tmf), \mathbb{F}_2) = \mathrm{Ext}_A(A/\!/A(2), \mathbb{F}_2) = \mathrm{Ext}_{A(2)}(\mathbb{F}_2, \mathbb{F}_2) \Rightarrow \pi_*(tmf)\hat{}_2.$$

The elliptic spectral sequence has the form $H^s(\overline{\mathcal{M}}_{ell}, \pi_t\mathcal{O}^{\text{top}}) \Rightarrow \pi_{t-s}(Tmf)$. The homotopy $\pi_t\mathcal{O}^{\text{top}}$ is concentrated in even degrees and is the $t/2$-th power of a line bundle ω; the spectral sequence thus has the form $H^q(\overline{\mathcal{M}}_{ell}; \omega^{\otimes p}) \Rightarrow \pi_{2p-q}(Tmf)$.

Part II

The manuscripts. The book concludes with three of the original, previously unpublished, manuscripts on *tmf*: "Elliptic curves and stable homotopy I" (1996) by Hopkins and Miller, "From elliptic curves to homotopy theory" (1998) by Hopkins and Mahowald, and "$K(1)$-local E_∞ ring spectra" (1998) by Hopkins. The first focuses primarily on the construction of the sheaf of (associative) ring spectra on the moduli stack of elliptic curves, the second on the computation of the homotopy of the resulting spectrum of sections around the supersingular elliptic curve at the prime 2, and the third on a direct cellular construction of the $K(1)$-localization of *tmf*. These documents have been left, for the most part, in their original draft form; they retain the attendant roughness and sometimes substantive loose ends, but also the dense, heady insight of their original composition. The preceding chapters of this book can be viewed as a communal exposition, more than fifteen years on, of aspects of these and other primary sources about *tmf*.

4. Reader's guide

This is not a textbook. Though the contents spans all the way from classical aspects of elliptic cohomology to the construction of *tmf*, there are substantive gaps of both exposition and content, and an attempt to use this book for a lecture, seminar, or reading course will require thoughtful supplementation.

Reading straight through the book would require, among much else, some familiarity and comfort with *commutative ring spectra*, *stacks*, and *spectral sequences*. Many of the chapters, though, presume knowledge of none of these topics; instead of thinking of them as prerequisites, we suggest one simply starts reading, and as appropriate or necessary selects from among the following as companion sources:

Commutative ring spectra:

- May, J. Peter. *E_∞ ring spaces and E_∞ ring spectra.* With contributions by Frank Quinn, Nigel Ray, and Jrgen Tornehave. Lecture Notes in Mathematics, Vol. 577. Springer-Verlag, Berlin-New York, 1977.

- Rezk, Charles. *Notes on the Hopkins-Miller theorem.* Homotopy theory via algebraic geometry and group representations (Evanston, IL, 1997), 313–366, Contemp. Math., 220, Amer. Math. Soc., Providence, RI, 1998.
- Schwede, Stefan. *Book project about symmetric spectra.* Book preprint. Available at http://www.math.uni-bonn.de/people/schwede/SymSpec.pdf

Stacks:

- *Complex oriented cohomology theories and the language of stacks* course notes for 18.917, taught by Mike Hopkins (1999), available at http://www.math.rochester.edu/people/faculty/doug/otherpapers/coctalos.pdf
- Naumann, Niko. *The stack of formal groups in stable homotopy theory.* Adv. Math. 215 (2007), no. 2, 569–600.
- *The Stacks project.* Open source textbook, available at http://stacks.math.columbia.edu
- Vistoli, Angelo. *Grothendieck topologies, fibered categories and descent theory.* Fundamental algebraic geometry, 1104, Math. Surveys Monogr., 123, Amer. Math. Soc., Providence, RI, 2005.

Spectral sequences:

- Hatcher, Allen. *Spectral Sequences in Algebraic Topology.* Book preprint. Available at http://www.math.cornell.edu/~hatcher/SSAT/SSATpage.html
- McCleary, John. *A user's guide to spectral sequences.* Second edition. Cambridge Studies in Advanced Mathematics, 58. Cambridge University Press, Cambridge, 2001. xvi+561 pp. ISBN: 0-521-56759-9
- Weibel, Charles A. *An introduction to homological algebra.* Cambridge Studies in Advanced Mathematics, 38. Cambridge University Press, Cambridge, 1994. xiv+450 pp. ISBN: 0-521-43500-5; 0-521-55987-1

The contents of this book span four levels: the first five chapters (elliptic cohomology, elliptic curves, the moduli stack, the exact functor theorem, sheaves in homotopy theory) are more elementary, classical, and expository and we hope will be tractable for all readers and instructive or at least entertaining for all but the experts; the next three chapters (the Hasse square, the local structure of the moduli stack, obstruction theory) are somewhat more sophisticated in both content and tone, and especially for novice and intermediate readers will require more determination, patience, and willingness to repeatedly pause and read other references before proceeding; the last five chapters (from spectra to stacks, string orientation, the sheaf of ring spectra, the construction, the homotopy groups) are distinctly yet more advanced, with Mike Hopkins' reflective account of and perspective on the subject, followed by an extensive technical treatment of the construction and homotopy of *tmf*; finally the three classic manuscripts (Hopkins–Miller, Hopkins–Mahowald, Hopkins) illuminate the original viewpoint on *tmf*—a careful reading of them will require serious dedication even from experts.

In addition to the references listed above, we encourage the reader to consult the following sources about *tmf* more broadly:

- Goerss, Paul. *Topological modular forms [after Hopkins, Miller, and Lurie].* Séminaire Bourbaki (2008/2009). Astérisque No. 332 (2010), 221-255.
- Hopkins, Michael. *Topological modular forms, the Witten genus, and the theorem of the cube.* Proceedings of the International Congress of Mathematicians (Zurich 1994), 554-565, Birkhäuser, Basel, 1995.
- Hopkins, Michael. *Algebraic topology and modular forms.* Proceedings of the International Congress of Mathematicians (Beijing 2002), 291–317, Higher Ed. Press, Beijing, 2002.
- Rezk, Charles. *Supplementary notes for Math 512.* Available at http://www.math.uiuc.edu/~rezk/512-spr2001-notes.pdf

Elliptic genera and elliptic cohomology

Corbett Redden

The goal of this overview is to introduce concepts which underlie elliptic cohomology and reappear in the construction of *tmf*. We begin by defining complex-oriented cohomology theories and looking at the two special cases of complex cobordism and K-theory. We then see that a complex orientation of a cohomology theory naturally leads to a formal group law. Furthermore, Quillen's theorem states that the universal complex-oriented theory (complex cobordism) encodes the universal formal group law. This implies that complex genera, or homomorphisms from the complex cobordism ring to a ring R, are equivalent to formal group laws over R. The group structure on an elliptic curve naturally leads to the notion of an elliptic genus. Finally, we use the Landweber exact functor theorem to produce an elliptic cohomology theory whose formal group law is given by the universal elliptic genus.

Elliptic cohomology was introduced by Landweber, Ravenel, and Stong in the mid-1980's as a cohomological refinement of elliptic genera. The notion of elliptic genera had previously been invented by Ochanine to address conjectured rigidity and vanishing theorems for certain genera on manifolds admitting non-trivial group actions. Witten played an important role in this process by using intuition from string theory to form many of these conjectures. He subsequently interpreted the elliptic genus as the signature of the free loop space of a spin manifold, beginning a long and interesting interaction between theoretical physics and algebraic topology that is still active today. While we don't have the space to adequately tell this story, there are already several excellent references: the introductory article in [**Lan**] gives the history of elliptic genera and elliptic cohomology, [**Seg**] explains how they should be related to more geometric objects, and [**Hop**] summarizes important properties of *tmf*. Finally, both [**Lur**] and [**Goe**] give a detailed survey of elliptic cohomology and *tmf* from the more modern perspective of derived algebraic geometry.

1. Complex-oriented cohomology theories

A generalized cohomology theory E is a functor from (some subcategory of) topological spaces to the category of abelian groups. This functor must satisfy all the Eilenberg–Steenrod axioms except for the dimension axiom, which states the cohomology of a point is only non-trivial in degree 0. Any cohomology theory is represented by a spectrum which we also call E, and from a spectrum the reduced homology and cohomology groups of a finite CW complex X are given by

$$\widetilde{E}_n(X) = \lim_{k\to\infty} \pi_{n+k}(X \wedge E_k),$$
$$\widetilde{E}^n(X) = \lim_{k\to\infty} [\Sigma^k X, E_{n+k}].$$

The coefficient groups are abbreviated by $E^* = E^*(\mathrm{pt})$ and $E_* = E_*(\mathrm{pt})$, and they are naturally related by $\pi_* E = E_* \cong E^{-*}$. We restrict to theories with a graded commutative ring structure $E^i(X) \times E^j(X) \to E^{i+j}(X)$ analogous to the cup product in ordinary cohomology. They are known as multiplicative cohomology theories and are represented by ring spectra.

EXAMPLE 1.1 (Cobordism). A smooth closed (compact with no boundary) manifold M is said to be null-bordant if there exists a compact manifold W whose boundary is M. A singular manifold (M, f) in X, where $f : M \to X$ is a continuous map, is null-bordant if there exists a singular manifold (W, F) with boundary (M, f). The n-th unoriented bordism group of X, denoted by $\Omega_n^O(X)$, is the set of smooth closed singular n-manifolds in X modulo null-bordism; the group structure is given by the disjoint union of manifolds.

Let $\{G_k\}$ be a sequence of topological groups with representations $\{G_k \xrightarrow{\rho_k} O(k)\}$ which are compatible with the inclusion maps. We define a G-structure on M as a stable lift of the structure groups to G_k for the stable normal bundle ν_M. Suppose a manifold W with $\partial W = M$ has a G-structure on ν_W that extends to the G-structure on ν_M. This is considered a null-bordism of M as a G-manifold. The abelian group $\Omega_n^G(X)$ is then defined as before; it is the set of smooth closed singular n-manifolds on X with G-structure on ν, modulo null-bordism. Up to homotopy, G-structures on the stable tangent bundle and stable normal bundle are equivalent; we later use this fact in geometric constructions.

The functors Ω_*^G are examples of generalized homology theories, and the Pontryagin–Thom construction shows they are represented by the Thom spectra $MG = \{MG_k\} = \{Th(\rho_k^* \xi_k)\}$. Here, $\xi_k \to BO(k)$ is the universal k-dimensional vector bundle ($\xi_k = EO(k) \times_{O(k)} \mathbb{R}^k$), and for any vector bundle $V \to X$ the Thom space $Th(V)$ is defined as the unit disc bundle modulo the unit sphere bundle $D(V)/S(V)$. Particularly common examples of G-bordism include oriented bordism, spin bordism, and complex bordism, corresponding to the groups $SO(k)$, $Spin(k)$, and $U(k)$, respectively. Bordism classes in these examples have an orientation, spin structure, or complex structure on the manifold's stable normal bundle (or stable tangent bundle).

The spectrum MG defines a generalized cohomology theory known as G-cobordism. It is also a multiplicative cohomology theory (assuming there are maps $G_{k_1} \times G_{k_2} \to G_{k_1+k_2}$ compatible with the orthogonal representations). The coefficient ring of MG is simply the bordism ring of manifolds with stable G-structure,

$$MG^{-*}(\mathrm{pt}) \cong MG_*(\mathrm{pt}) = \Omega_*^G,$$

and the product structure is induced by the product of manifolds. Of particular interest to us will be oriented cobordism and complex cobordism. The first coefficient calculation is due to Thom, and the second is from Thom, Milnor, and Novikov:

$$MSO^* \otimes \mathbb{Q} \cong \mathbb{Q}[[\mathbb{CP}^2], [\mathbb{CP}^4], \ldots] \tag{1}$$
$$MU^* \cong \mathbb{Z}[a_1, a_2, \ldots]; \quad |a_i| = -2i.$$

Rationally, $MU^* \otimes \mathbb{Q}$ is generated by the complex projective spaces $\mathbb{CP}^i$ for $i \geq 1$. The book [**Sto**] is an excellent source of further information on cobordism.

EXAMPLE 1.2 (Complex K-theory). Isomorphism classes of complex vector bundles over a space X form an abelian monoid via the direct sum $\oplus$ operation. Formally adjoining inverses gives the associated Grothendieck group known as $K(X)$ or $K^0(X)$; elements in $K(X)$ are formal differences of vector bundles up to isomorphism. The reduced group $\widetilde{K}^0(X)$ is naturally isomorphic to $[X, \mathbb{Z} \times BU]$, and Bott periodicity gives a homotopy equivalence $\Omega^2(\mathbb{Z} \times BU) \simeq \mathbb{Z} \times BU$. Therefore, we can extend $\mathbb{Z} \times BU$ to an Ω-spectrum known as K, where

$$\begin{aligned} K_{2n} &= \mathbb{Z} \times BU, \\ K_{2n+1} &= \Omega(\mathbb{Z} \times BU) \simeq U. \end{aligned}$$

This defines the multiplicative cohomology theory known as (complex) K-theory, with ring structure induced by the tensor product of vector bundles. A straightforward evaluation shows that the coefficients $\pi_* K$ are

$$\begin{aligned} K^{2n}(\mathrm{pt}) &\cong \pi_0(\mathbb{Z} \times BU) = \mathbb{Z}, \\ K^{2n+1}(\mathrm{pt}) &\cong \pi_0(U) = 0. \end{aligned}$$

Furthermore, Bott periodicity is manifested in K-theory by the Bott class $\beta = [\xi] - 1 \in \widetilde{K}(S^2) \cong K^{-2}(\mathrm{pt})$, where $\xi \to S^2$ is the Hopf bundle and 1 is the isomorphism class of the trivial line bundle. The class β is invertible in K^*, and multiplication by β and β^{-1} induces the periodicity in general rings $K^*(X)$.

The periodicity in K-theory turns out to be a very convenient property, and it motivates the following definition.

DEFINITION 1.3. A multiplicative cohomology theory E is *even periodic* if $E^i(\mathrm{pt}) = 0$ whenever i is odd and there exists $\beta \in E^{-2}(\mathrm{pt})$ such that β is invertible in $E^*(\mathrm{pt})$.

The existence of $\beta^{-1} \in E^2(\mathrm{pt})$ implies that for general X there are natural isomorphisms

$$\begin{tikzcd} E^{*+2}(X) \arrow[r, shift left, "\cdot\beta"] \arrow[r, phantom, "\cong"] & E^*(X) \arrow[l, shift left, "\cdot\beta^{-1}"] \end{tikzcd} .$$

given by multiplication with β and β^{-1}, so E is periodic with period 2.

A number of cohomology theories, such as ordinary cohomology, are even (i.e. $E^{\mathrm{odd}}(\mathrm{pt}) = 0$) but not periodic. Given an arbitrary even cohomology theory, we can create an even periodic theory A by defining

$$A^n(X) := \prod_{k \in \mathbb{Z}} E^{n+2k}(X).$$

For example, if we perform this construction on ordinary cohomology with coefficients in a ring R, we obtain a theory known as periodic ordinary cohomology. The coefficients of MU in (1) show that MU also is even but not periodic. We define *periodic complex cobordism* MP by

$$MP^n(X) := \prod_{k \in \mathbb{Z}} MU^{n+2k}(X),$$

Letting $|\beta| = -2$, we could equivalently define $MP^n(X) \subset MU^*(X)[\![\beta, \beta^{-1}]\!]$ as formal series which are homogeneous of degree n.

Definition 1.4. In E-cohomology, a Thom class for the vector bundle $V \to X$ (with $\dim_{\mathbb{R}} V = n$) is a class $\mathcal{U}_V \in \widetilde{E}^n(Th(V))$ such that for each $x \in X$ there exists $\varphi_x : \mathbb{R}^n \to V_x$ so that $\mathcal{U}_V \mapsto 1$ under the following composition:

$$\begin{array}{ccccccc} \widetilde{E}^n(Th(V)) & \longrightarrow & \widetilde{E}^n(Th(V_x)) & \xrightarrow[\varphi_x^*]{\cong} & \widetilde{E}^n(S^n) & \xrightarrow{\cong} & E^0(\mathrm{pt}) \\ \mathcal{U}_V & \longmapsto & & & & & 1 \end{array}$$

Thom classes give rise to Thom isomorphisms

$$E^*(X) \overset{\cdot\mathcal{U}_V}{\to} \widetilde{E}^{*+n}(Th(V)).$$

The existence of Thom isomorphisms allows one to construct pushforward maps in cohomology theories, which in turn gives important invariants generalizing the Euler class. Ordinary cohomology with $\mathbb{Z}/2$ coefficients admits Thom classes for all vector bundles, but only oriented bundles have Thom classes in $H^*(-;\mathbb{Z})$. In general, we would like to functorially define Thom classes compatible with the $\oplus$ operation. Such a choice for vector bundles with lifts of the structure group to G_k is called a G-orientation of the cohomology theory E, and E is said to be G-orientable if there exists such an orientation. A specific orientation will be given by universal Thom classes in $\widetilde{E}^n(MG_n)$ and is equivalent (at least up to homotopy) to a map of ring spectra $MG \to E$. We will mostly be concerned with complex orientable theories, and summarizing the above discussion gives the following definition.

Definition 1.5. A complex orientation of E is a natural, multiplicative, collection of Thom classes $\mathcal{U}_V \in \widetilde{E}^{2n}(Th(V))$ for all complex vector bundles $V \to X$, where $\dim_{\mathbb{C}} V = n$. More explicitly, these classes must satisfy

- $f^*(\mathcal{U}_V) = \mathcal{U}_{f^*V}$ for $f : Y \to X$,
- $\mathcal{U}_{V_1 \oplus V_2} = \mathcal{U}_{V_1} \cdot \mathcal{U}_{V_2}$,
- For any $x \in X$, the class $\mathcal{U}_V$ maps to 1 under the composition[1]

$$\widetilde{E}^{2n}(Th(V)) \to \widetilde{E}^{2n}(Th(V_x)) \overset{\cong}{\to} \widetilde{E}^{2n}(S^{2n}) \overset{\cong}{\to} E^0(\mathrm{pt}).$$

Given a complex orientation, we can define Chern classes in the cohomology theory E. Because the zero-section $\mathbb{CP}^\infty \to \xi$ induces a homotopy equivalence $\mathbb{CP}^\infty \overset{\sim}{\to} Th(\xi)$, the universal Thom class for line bundles is naturally a class $c_1 \in \widetilde{E}^2(\mathbb{CP}^\infty)$, and it plays the role of the universal first Chern class. If one computes $E^*(\mathbb{CP}^\infty)$, the existence of c_1 implies the Atiyah–Hirzebruch spectral sequence must collapse at the E_2 page. This implies the first part of the following theorem.

Theorem 1.6. *A complex orientation of E determines an isomorphism*

$$E^*(\mathbb{CP}^\infty) \cong E^*(\mathrm{pt})[\![c_1]\!],$$

and such an isomorphism is equivalent to a complex orientation. Furthermore, any even periodic theory is complex orientable.

[1] The complex structure induces an orientation on V_x, hence there is a canonical homotopy class of map $\varphi_x : \mathbb{R}^{2n} \to V_x$.

In addition to the above proposition, the splitting principle carries over, and the class c_1 uniquely determines isomorphisms

$$\begin{aligned}E^*(BU(n)) &\cong (E^*(\mathbb{CP}^\infty \times \cdots \times \mathbb{CP}^\infty))^{\Sigma_n} \cong \left(E^*(\mathrm{pt})[\![x_1, \cdots, x_n]\!]\right)^{\Sigma_n} \\ &\cong E^*(\mathrm{pt})[\![c_1, \cdots, c_n]\!],\end{aligned}$$

where $c_k \in E^{2k}(BU(n))$ is the k-th elementary symmetric polynomial in the variables x_i. This gives us a theory of Chern classes analogous to the one in ordinary cohomology.

2. Formal group laws and genera

A complex orientation of E determines Chern classes for complex vector bundles. As in ordinary cohomology, the Chern classes satisfy the properties of naturality and additivity. In ordinary cohomology, the first Chern class of a product of line bundles is given by

$$c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2).$$

For a general complex-oriented cohomology theory, this relation no longer holds and leads to an interesting structure.

The universal tensor product is classified by

$$\begin{tikzcd} \xi \otimes \xi \arrow[r] \arrow[d] & \xi \arrow[d] \\ \mathbb{CP}^\infty \times \mathbb{CP}^\infty \arrow[r] & \mathbb{CP}^\infty. \end{tikzcd}$$

The induced map in cohomology,

$$\begin{aligned} E^*(\mathbb{CP}^\infty) \otimes E^*(\mathbb{CP}^\infty) &\longleftarrow E^*(\mathbb{CP}^\infty) \\ F(x_1, x_2) &\longleftarrow\!\shortmid c_1 \end{aligned}$$

where $F(x_1, x_2)$ is a formal power series in two variables over the ring E^*, gives the universal formula

$$c_1(L_1 \otimes L_2) = F(c_1(L_1), c_1(L_2)).$$

This formal power series F is an example of a formal group law over the graded ring E^*.

Definition 2.1. A *formal group law* over a ring R is a formal power series $F \in R[\![x_1, x_2]\!]$ satisfying the following conditions:

- $F(x, 0) = F(0, x) = x$ (Identity)
- $F(x_1, x_2) = F(x_2, x_1)$ (Commutativity)
- $F(F(x_1, x_2), x_3) = F(x_1, F(x_2, x_3))$ (Associativity)

If R is a graded ring, we require F to be homogeneous of degree 2 where $|x_1| = |x_2| = 2$.

One easily verifies that the power series giving $c_1(L_1 \otimes L_2)$ is a formal group law. The three properties in the definition follow immediately from the natural transformations which give the identity, commutativity, and associativity properties of the tensor product.

Example 2.2. As noted above, the formal group law obtained from ordinary cohomology is $F_+ = F(x_1, x_2) = x_1 + x_2$, and is known as the additive formal group law.

Example 2.3. The multiplicative formal group law is defined by

$$F_\times(x_1, x_2) = x_1 + x_2 - x_1 x_2.$$

One can explicitly verify it satisfies the definition of a formal group law. One can also see that it is obtained from the standard complex orientation of K-theory. Since K-theory is even periodic, we place the classes c_1 in degree 0. The resulting formal group law is over the ring $K^0(\mathrm{pt}) = \mathbb{Z}$ and involves no grading. (Though, we could use the Bott element and its inverse to maintain the grading $|c_1| = 2$ if we wish.)

To the universal line bundle $\xi \to \mathbb{CP}^\infty$, we define the universal first Chern class to be $1 - [\xi] \in K^0(\mathbb{CP}^\infty)$. The term 1 is included so that trivial bundles have trivial first Chern class. Hence, for any line bundle $L \to X$,

$$c_1(L) = 1 - [L] \in K^0(X).$$

A simple calculation demonstrates

$$\begin{aligned} c_1(L_1 \otimes L_2) &= 1 - L_1 \otimes L_2 \\ &= (1 - L_1) + (1 - L_2) - (1 - L_1)(1 - L_2) \\ &= c_1(L_1) + c_1(L_2) - c_1(L_1)c_1(L_2), \end{aligned}$$

demonstrating that the multiplicative formal group law is obtained from K-theory.

Any ring homomorphism $R \to S$ induces a map of formal group laws $FGL(R) \to FGL(S)$. In fact, there is a universal formal group law $F_{univ} \in R_{univ}[\![x_1, x_2]\!]$ such that any $F \in FGL(R)$ is induced by a ring homomorphism $R_{univ} \to R$. The existence of R_{univ} is easy, since one can construct it formally by

$$R_{univ} = \mathbb{Z}[a_{ij}]/\sim$$

where a_{ij} is the coefficient of $x_1^i x_2^j$, and $\sim$ represents all equivalence relations induced by the three axioms of a formal group law. Though this description is quite unwieldy, a theorem by Lazard shows that this ring is isomorphic to a polynomial algebra; i.e.

$$R_{univ} \cong L := \mathbb{Z}[a_1, a_2, \ldots]$$

where $|a_i| = -2i$ if we include the grading.

A complex orientation of E therefore induces a map $L \to E^*$ defining the formal group law. Earlier we noted that complex orientations are basically equivalent to maps of ring spectra $MU \to E$, so MU has a canonical complex orientation given by the identity map $MU \to MU$. The following important theorem of Quillen shows that in addition to MU being the universal complex oriented cohomology theory, it is also the home of the universal formal group law. It also explains the grading of the Lazard ring.

Theorem 2.4. *(Quillen) The map $L \to MU^*$ induced from the identity map $MU \to MU$ is an isomorphism.*

To summarize, we have maps

$$\begin{tikzcd}[row sep=small] & \{ MU^* \to E^* \} \arrow[dd, leftrightarrow, "\text{Quillen}"] \\ \{ MU \to E \} \arrow[ur] \arrow[dr] & \\ & FGL(E^*) \end{tikzcd}$$

where E^* can be any graded ring. Given a formal group law, can we construct a complex oriented cohomology theory with that formal group law? We will return to this question in Section 4 and see that in certain cases we can construct such a cohomology theory.

First, we discuss formal group laws from the slightly different viewpoint of complex genera. A *genus* is some multiplicative bordism invariant associated to manifolds. There are two main types of genera, and this is due to the description of the cobordism groups from (1).

DEFINITION 2.5. A *complex genus* is a ring homomorphism

$$\varphi : MU^* \to R.$$

An *oriented genus* (or usually just *genus*) is a ring homomorphism

$$\varphi : MSO^* \otimes \mathbb{Q} \to R,$$

where R is a $\mathbb{Q}$-algebra. More explicitly, $\varphi(M)$ only depends on the cobordism class of M and satisfies

$$\varphi(M_1 \sqcup M_2) = \varphi(M_1) + \varphi(M_2), \quad \varphi(M_1 \times M_2) = \varphi(M_1)\varphi(M_2).$$

Quillen's theorem implies there is a 1-1 correspondence between formal group laws over R and complex genera over R. We introduce some common terminology which will make this correspondence more concrete.

First, a homomorphism between formal group laws $F \xrightarrow{f} G$ (over R) is a power series $f(x) \in R[\![x]\!]$ such that

$$f(F(x_1, x_2)) = G(f(x_1), f(x_2)).$$

If f is invertible then it is considered an isomorphism, and f is a strict isomorphism if $f(x) = x +$ higher order terms.

EXAMPLE 2.6. We could have chosen complex orientation of K-theory so that $c_1(L) = [L] - 1$ as opposed to $1 - [L]$. The resulting formal group law would have been $F(x_1, x_2) = x_1 + x_2 + x_1 x_2$, which is also sometimes defined as the multiplicative formal group law. These two formal group laws are (non-strictly) isomorphic, with isomorphism given by $f(x) = -x$. Our original choice, though, coincides with the Todd genus and with conventions in index theory.

REMARK 2.7. In general, our formal group law depends on the particular complex orientation. Two different orientations will lead to an isomorphism between the formal group laws. More abstractly, to any complex orientable theory is canonically associated a *formal group*. The choice of orientation gives a coordinate for the formal group, and the formal group expanded in this coordinate is the formal group law.

Over a $\mathbb{Q}$-algebra, any formal group law is uniquely (strictly) isomorphic to the additive formal group law F_+. We denote this isomorphism by $\log_F$ and its inverse by $\exp_F$:

$$\begin{tikzcd} F \arrow[r, bend left, "\log_F"] & F_+ \arrow[l, bend left, "\exp_F"] \end{tikzcd}$$

The isomorphism $\log_F$ can be solved by the following:

$$\begin{aligned} f(F(x_1,x_2)) &= f(x_1)+f(x_2) \\ \frac{\partial}{\partial x_2}\bigg|_{x,0}(f(F(x_1,x_2))) &= \frac{\partial}{\partial x_2}\bigg|_{x,0}(f(x_1)+f(x_2)) \\ f'(x)\frac{\partial F}{\partial x_2}(x,0) &= 1 \\ \log_F(x) = f(x) &= \int_0^x \frac{dt}{\frac{\partial F}{\partial x_2}(t,0)} \qquad (2) \end{aligned}$$

Going from the third to fourth line involves inverting a power series, so one must work over a $\mathbb{Q}$-algebra. If R is torsion-free, then $R \hookrightarrow R \otimes \mathbb{Q}$ is an injection, and we lose no information in considering $\log_F$ instead of F itself.

Over the ring $MU^* \otimes \mathbb{Q}$, the universal formal group law F_{MU} coming from complex cobordism has the particularly nice logarithm

$$\log_{F_{MU}}(x) = \sum_{n\geq 0} \frac{[\mathbb{CP}^n]}{n+1}x^n.$$

Therefore, a formal group law F (or a complex genus) induced by $\varphi : MU^* \to R$ has a logarithm

$$\log_F(x) = \sum_{n\geq 0} \frac{\varphi([\mathbb{CP}^n])}{n+1}x^n.$$

While (modulo torsion in R) the logarithm encodes the value of a genus on any complex manifold, in practice it is difficult to decompose the bordism class of a manifold into projective spaces. However, there is an easier approach to calculating genera due to work of Hirzebruch.

Proposition 2.8. *(Hirzebruch) For R a $\mathbb{Q}$-algebra, there are bijections*

$$\{Q(x) = 1 + a_1x + a_2x^2 + \cdots \in R[\![x]\!]\} \longleftrightarrow \{\varphi : MU^* \otimes \mathbb{Q} \to R\}$$
$$\{Q(x) = 1 + a_2x^2 + a_4x^4 + \cdots \in R[\![x]\!] \mid a_{odd} = 0\} \longleftrightarrow \{\varphi : MSO^* \otimes \mathbb{Q} \to R\}$$

The first bijection is given by the following construction. Given $Q(x)$, to a complex line bundle $L \to X$ assign the cohomology class

$$\varphi_Q(L) := Q(c_1(L)) \in H^*(X; R).$$

Using the splitting principle, φ_Q extends to a stable exponential characteristic class on all complex vector bundles. The complex genus φ generated by $Q(x)$ is then defined by

$$\varphi(M) := \langle \varphi_Q(TM), [M] \rangle \in R,$$

where M is a stably almost complex manifold, $\langle , \rangle$ is the natural pairing between cohomology and homology, and $[M]$ is the fundamental class (an almost complex structure induces an orientation). Going the other direction, the series $Q(x)$ is related to the formal group law by

$$Q(x) = \frac{x}{\exp_\varphi(x)},$$

where $\exp_\varphi(x)$ is the inverse to $\log_\varphi(x)$. The second bijection follows in the same manner, but one needs an even power series to define the stable exponential characteristic class for real vector bundles.

EXAMPLE 2.9 (K-theory and $F_\times$). From (2), the logarithm for the multiplicative formal group law $F_\times(x_1, x_2) = x_1 + x_2 - x_1x_2$ is given by

$$\log_\times(x) = \int_0^x \frac{dt}{1-t} = -\log(1-x).$$

Therefore,

$$\exp_\times(x) = 1 - e^{-x},$$

and the associated power series

$$Q_\times(x) = \frac{x}{\exp_\times(x)} = \frac{x}{1-e^{-x}} = 1 + \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720} + \cdots \in \mathbb{Q}[\![x]\!]$$

generates the Todd genus Td. When we evaluate the Todd genus on a Riemann surface M^2 with genus g,

$$\begin{aligned} Td(M^2) &= \langle Q(c_1(TM)), [M]\rangle = \langle 1 + \frac{1}{2}c_1(TM) + \cdots, [M]\rangle \\ &= \frac{1}{2}\langle c_1(TM), [M]\rangle = 1 - g. \end{aligned}$$

In this situation, the Todd genus recovers the standard notion of genus.

Note that even though we started with a $\mathbb{Z}$-valued complex genus, the power series $Q(x)$ has fractional coefficients. If one is only given $Q(x)$, it is quite surprising that Todd genus gives integers when evaluated on manifolds with an almost complex structure on the stable tangent bundle. Another explanation for the integrality is given by the following important index theorem. In fact, most of the common genera are equal to the index of some elliptic operator on a manifold (possibly with G-structure).

THEOREM 2.10. *(Hirzebruch-Riemann-Roch) Let M be a compact complex manifold, and let V be a holomorphic vector bundle. Then, the index of the Dolbeault operator $\bar\partial + \bar\partial^*$ on the Dolbeault complex $\{\Lambda^{0,i} \otimes V\}$, which equals the Euler characteristic in sheaf cohomology $H^*(M, V)$, is given by*

$$\operatorname{index}(\bar\partial + \bar\partial^*) = \chi(M, V) = \langle Td(M)ch(V), [M]\rangle \in \mathbb{Z}.$$

3. Elliptic genera

Another example of a formal group law comes from the group structure of the Jacobi quartic elliptic curve. We first start by working over $\mathbb{C}$. Assume $\delta, \epsilon \in \mathbb{C}$ and the discriminant $\Delta = \epsilon(\delta^2 - \epsilon)^2 \neq 0$. Letting the subscript J stand for Jacobi, we define

$$\log_J(x) := \int_0^x \frac{dt}{\sqrt{1 - 2\delta t^2 + \epsilon t^4}} = \int_0^x \frac{dt}{\sqrt{R(t)}}. \tag{3}$$

Here, $\log_J(x)$ is an example of an elliptic integral, and it naturally arises in physical problems such as modeling the motion of a pendulum. Expanding $\log_J$ as a power series in x produces a formal group law with a nice geometric description. Inverting the function $\log_J(x)$ gives

$$f(z) := \exp_J(z) = (\log_J)^{-1}(z),$$

which is an elliptic function (i.e. periodic with respect to a lattice $\Lambda \subset \mathbb{C}$) satisfying the differential equation $(f')^2(z) = R(z)$. Hence, it parameterizes the elliptic curve C defined by the Jacobi quartic equation

$$y^2 = R(x) = 1 - 2\delta x^2 + \epsilon x^4 \subset \mathbb{CP}^2$$

via the map

$$\begin{aligned} \mathbb{C}/\Lambda &\longrightarrow \mathbb{CP}^2 \\ z &\longmapsto [x(z), y(z), 1] = [f(z), f'(z), 1]. \end{aligned}$$

The additive group structure on the torus $\mathbb{C}/\Lambda$ induces a natural group structure on the elliptic curve C. This group structure coincides with the one given in Chapter 2, defined by $P + Q + R = 0$ for points P, Q, R on a straight line. Near the point $[0, 1, 1]$, the group structure is given in the parameter x by

$$F_J(x_1, x_2) := f(f^{-1}(x_1) + f^{-1}(x_2)) = \exp_J(\log_J(x_1) + \log_J(x_2)).$$

The formal group law F_J defined by the logarithm $\log_J$ can therefore be expressed by

$$\int_0^{x_1} \frac{dt}{\sqrt{R(t)}} + \int_0^{x_2} \frac{dt}{\sqrt{R(t)}} = \int_0^{F_J(x_1,x_2)} \frac{dt}{\sqrt{R(t)}}.$$

Despite the integral $\log_J$ having no closed form solution, the formal group law was solved for explicitly by Euler.

THEOREM 3.1. *(Euler)*

$$F_J(x_1, x_2) = \frac{x_1\sqrt{R(x_2)} + x_2\sqrt{R(x_1)}}{1 - \epsilon x_1^2 x_2^2}.$$

While we previously worked over the field $\mathbb{C}$, the Jacobi quartic is defined over an arbitrary ring, and the universal curve is defined by the same equation over the ring $\mathbb{Z}[\delta, \epsilon]$. The formal group law F_J can be expanded as a power series in the ring $\mathbb{Z}[\frac{1}{2}, \delta, \epsilon]$. Any genus whose logarithm is of the form (3) is called an *elliptic genus*, and the universal elliptic genus φ_J corresponds to Euler's formal group law F_J over $\mathbb{Z}[\frac{1}{2}, \delta, \epsilon]$. When considering the grading, $|\delta| = -4$ and $|\epsilon| = -8$, so φ_J also defines an oriented genus. In fact, one can calculate that

$$\varphi_J(\mathbb{CP}^2) = \delta, \quad \varphi_J(\mathbb{HP}^2) = \epsilon.$$

EXAMPLE 3.2. The geometric description of F_J assumed $\Delta = \epsilon(\delta^2 - \epsilon)^2 \neq 0$ so that the curve C has no singularities. However, the degenerate case $\delta = \epsilon = 1$ gives the L-genus, which equals the signature of an oriented manifold:

$$\begin{aligned} \log(x) &= \int_0^x \frac{dt}{1 - t^2} = \tanh^{-1}(x), \\ Q(x) &= \frac{x}{\tanh x}. \end{aligned}$$

Similarly, letting $\delta = -\frac{1}{8}, \epsilon = 0$, we recover the $\widehat{A}$-genus, which for a spin manifold is the index of the Dirac operator:

$$\log(x) = \int_0^x \frac{dt}{\sqrt{1+(t/2)^2}} = 2\sinh^{-1}(x/2);$$
$$Q(x) = \frac{x/2}{\sinh(x/2)}.$$

The genera L and $\widehat{A}$ are elliptic genera corresponding to singular elliptic curves. This is explicitly seen in the fact that their logarithms invert to singly-periodic functions as opposed to doubly-periodic functions.

The signature was long known to satisfy a stronger form of multiplicativity, known as strict multiplicativity. If M is a fiber bundle over B with fiber F and connected structure group, then $L(M) = L(B)L(F)$. The same statement holds for the $\widehat{A}$-genus when F is a spin manifold. As more examples were discovered, Ochanine introduced the notion of elliptic genera to explain the phenomenon and classify strictly multiplicative genera.

THEOREM 3.3 (Ochanine, Bott–Taubes). *A genus φ satisfies the strict multiplicativity condition $\varphi(M) = \varphi(B)\varphi(F)$ for all bundles of spin manifolds with connected structure group if and only if φ is an elliptic genus.*

There is extra algebraic structure encoded within the values of the universal elliptic genus. Using the Weierstrass $\wp$ function, to a lattice $\Lambda_\tau = \mathbb{Z}\tau + \mathbb{Z}$ we can canonically associate coefficients $\epsilon(\tau)$ and $\delta(\tau)$ satisfying the Jacobi quartic equation. The functions $\epsilon(\tau)$ and $\delta(\tau)$ are modular forms, of weight 2 and 4 respectively, on the subgroup $\Gamma_0(2) := \{A \in SL_2(\mathbb{Z}) \mid A = \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \mod 2\} \subset SL_2(\mathbb{Z})$. Therefore, the elliptic genus φ_J associates to any compact oriented $4k$-manifold a modular form of weight $2k$ on the subgroup $\Gamma_0(2)$. Because modular forms are holomorphic and invariant under the translation $\tau \mapsto \tau + 1$, we can expand them in the variable $q = e^{2\pi i\tau}$ and consider $\varphi_J(M) \in \mathbb{Q}[\![q]\!]$.

Using insights from quantum field theory, Witten gave an alternate interpretation of the elliptic genus which illuminated several of its properties. His definition of φ_J is as follows. Let M be a spin manifold of dimension n with complex spinor bundle $S(TM^{\mathbb{C}})$. To a complex vector bundle $V \to X$, use the symmetric and exterior powers to define the bundle operations

$$S_tV = 1 + tV + t^2V^{\otimes 2} + \cdots \in K(X)[\![t]\!], \quad \Lambda_tV = 1 + tV + t^2\Lambda^2V + \cdots \in K(X)[\![t]\!].$$

Then, the power series in q defined by

$$\left\langle \widehat{A}(TM)ch\Big(S(TM^{\mathbb{C}})\bigotimes_{l=1}^{\infty} S_{q^l}(TM^{\mathbb{C}} - \mathbb{C}^n) \otimes \bigotimes_{l=1}^{\infty} \Lambda_{q^l}(TM^{\mathbb{C}} - \mathbb{C}^n)\Big), [M^n]\right\rangle \in \mathbb{Q}[\![q]\!]$$

is equal to the q-expansion of the elliptic genus φ_J. When M is a spin manifold, Witten formally defined the signature operator on the free loop space LM, and he showed its S^1-equivariant index equals φ_J (up to a normalization factor involving the Dedekind η function). The S^1-action on $LM = Map(S^1, M)$ is induced by the natural action of S^1 on itself.

Witten also defined the following genus, now known as the *Witten genus*:

$$\varphi_W(M) := \left\langle \widehat{A}(TM)ch\Big(\bigotimes_{l=1}^{\infty} S_{q^l}(TM^{\mathbb{C}} - \mathbb{C}^n))\Big), [M^n]\right\rangle \in \mathbb{Q}[\![q]\!].$$

When M admits a string structure (i.e. M is a spin manifold with spin characteristic class $\frac{p_1}{2}(M) = 0 \in H^4(M;\mathbb{Z})$), Witten formally defined the Dirac operator on LM and showed its S^1-equivariant index equals $\varphi_W(M)$, up to a normalization involving η. If M is spin, then the q-series $\varphi_J(M)$ and $\varphi_W(M)$ both have integer coefficients. If M is string, $\varphi_W(M)$ is the q-expansion of a modular form over all of $SL_2(\mathbb{Z})$. Note that the integrality properties can be proven by considering φ_J and φ_W as a power series where each coefficient is the index of a twisted Dirac operator on M. For more detailed information on elliptic genera, an excellent reference is the text [**HBJ**].

4. Elliptic Cohomology

There is an important description of K-theory via Conner-Floyd. As described in Section 2, the formal group law for K-theory is given by a map $MP^0(\mathrm{pt}) \to K^0(\mathrm{pt}) \cong \mathbb{Z}$ or $MU^* \to K^* \cong \mathbb{Z}[\![\beta, \beta^{-1}]\!]$, depending on our desired grading convention. From this map of coefficients encoding the formal group law, one can in fact recover all of K-theory.

THEOREM 4.1. *(Conner-Floyd) For any finite cell complex X,*

$$K^*(X) \cong MP^*(X) \otimes_{MP^0} \mathbb{Z} \cong MU^*(X) \otimes_{MU^*} \mathbb{Z}[\![\beta, \beta^{-1}]\!].$$

In general, Quillen's theorem shows that a formal group law F over a graded ring R is induced by a map $MU^* \to R$. Can we construct a complex-oriented cohomology theory E with formal group law F over $E^* \cong R$? Imitating the Conner-Floyd description of K-theory, we can define

$$E^*(X) := MU^*(X) \otimes_{MU^*} R.$$

While E is a functor satisfying the homotopy, excision, and additivity axioms of a cohomology theory, the "long exact sequence of a pair" will not necessarily be exact. This is due to the fact that exact sequences are not in general exact after tensoring with an arbitrary ring. If R is flat over MU^*, then E will satisfy the long exact sequence of a pair and will be a cohomology theory.

The condition that R is flat over MU^* is very strong and not usually satisfied. However, the Landweber exact functor theorem states that R only needs to satisfy a much weaker set of conditions. This criterion, described in more detail in Chapter 5, states one only needs to check that multiplication by certain elements v_i is injective on certain quotients R/I_i. In the case of the elliptic formal group law, the elements v_1 and v_2 can be given explicitly in terms of ϵ and δ, and the quotients R/I_n are trivial for $n > 2$. Therefore, one can explicitly check Landweber's criterion and conclude the following.

THEOREM 4.2. *(Landweber, Ravenel, Stong) There is a homology theory Ell*

$$Ell_*(X) = MU_*(X) \otimes_{MU_*} \mathbb{Z}[\frac{1}{2}, \delta, \epsilon, \Delta^{-1}]$$

whose associated cohomology theory is complex oriented with formal group law given by the Euler formal group law. For finite CW complexes X,

$$Ell^*(X) = MU^*(X) \otimes_{MU^*} \mathbb{Z}[\frac{1}{2}, \delta, \epsilon, \Delta^{-1}].$$

In Ell^*, $|\delta| = -4, |\epsilon| = -8$.

The theory *Ell* was originally referred to as *elliptic cohomology*, but it is now thought of as a particular elliptic cohomology theory. If we ignore the grading of δ and ϵ, we can form an even periodic theory by $MP(-) \otimes_{MP} \mathbb{Z}[\frac{1}{2}, \delta, \epsilon, \Delta^{-1}]$. This motivates the following definiton.

DEFINITION 4.3. An *elliptic cohomology theory* E consists of:

- A multiplicative cohomology theory E which is even periodic,
- An elliptic curve C over a commutative ring R,
- Isomorphisms $E^0(\mathrm{pt}) \cong R$ and an isomorphism of the formal group from E with the formal group associated to C.

The even periodic theory associated to *Ell* is an elliptic cohomology theory related to the Jacobi quartic curve over $\mathbb{Z}[\frac{1}{2}, \delta, \epsilon, \Delta^{-1}]$. An obvious question is whether there is a universal elliptic cohomology theory; this universal theory should be related to a universal elliptic curve. Any elliptic curve C over R is isomorphic to a curve given in affine coordinates by the Weierstrass equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6, \quad a_i \in R.$$

However, there is no canonical way to do this since the Weierstrass equation has non-trivial automorphisms. There is no single universal elliptic curve, but instead a *moduli stack* of elliptic curves, as seen in Chapter 4. Because of this, there is no universal elliptic cohomology theory in the naive sense.

What one does end up with as the "universal elliptic cohomology theory" is *topological modular forms* or *tmf*. Its mere existence is a difficult and subtle theorem, and it will take the rest of these proceedings to construct *tmf*. Roughly speaking, one uses the Landweber exact functor theorem to form a pre-sheaf of elliptic cohomology theories on the moduli stack of elliptic curves. One then lifts this to a sheaf of E_∞ ring-spectra and takes the global sections to obtain the spectrum *tmf*. While constructed out of elliptic cohomology theories, *tmf* is not an elliptic cohomology theory, as evidenced by the following properties.

There is a homomorphism from the coefficients tmf^{-*} to the ring of modular forms MF. While this map is rationally an isomorphism, it is neither injective nor surjective integrally. In particular, tmf^{-*} contains a large number of torsion groups, many of which are in odd degrees. Topological modular forms is therefore not even, and the periodic version TMF has period $24^2 = 576$ as opposed to 2 (or as opposed to 24, the period of *Ell*). Furthermore, the theory *tmf* is not complex orientable, but instead has an $MO\langle 8\rangle$ or string orientation denoted σ. At the level of coefficients, the induced map $MString^{-*} \to tmf^{-*}$ gives a refinement of the Witten genus φ_W.

$$\begin{tikzcd} & tmf^{-*} \arrow[d] \\ MString^{-*} \arrow[ur, "\sigma"] \arrow[r, "\varphi_W"] & MF \end{tikzcd}$$

While a great deal of information about *tmf* has already been discovered, there are still many things not yet understood. As an example, the index of family of (complex) elliptic operators parameterized by a space X naturally lives in $K(X)$, and topologically this is encoded by the complex orientation of K-theroy. Because of analytic difficulties, there is no good theory of elliptic operators on loop spaces. However, it is believed that families indexes for elliptic operators on loop spaces should naturally live in *tmf* and refine the Witten genus. Making mathematical sense of this would almost certainly require a geometric definition of *tmf*, which still does not yet exist despite efforts including [**Seg, ST**].

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Elliptic Curves and Modular Forms

Carl Mautner

We introduce elliptic curves as certain smooth curves in $\mathbb{P}^2$. We then consider the group law on their points and formulate an equivalent definition of elliptic curves as projective one dimensional group varieties, sketching the equivalence. We conclude by considering the more general notion of a pointed curve of genus 1 over an arbitrary base scheme and define the notion of a modular form. The first two sections follow parts of Silverman's books [**Si1, Si2**] and the third section is based on the short note [**De**] by Deligne.

1. Elliptic curves as cubics in $\mathbb{P}^2$...

Definition 1.1. An *elliptic curve* defined over a field K is a nonsingular curve C in $\mathbb{P}^2$ defined by a cubic equation such that $C \cap \mathbb{P}^1_\infty = [0 : * : 0]$ (where $\mathbb{P}^1_\infty$ is the curve $[* : * : 0]$).

Any such curve can be expressed in projective coordinates $[X, Y, Z]$, after rescaling X and Y, as

$$Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3$$

for some $a_1, a_2, \dots, a_6 \in K$. Conversely, any such equation will cut out a (possibly singular) cubic curve such that $C \cap \mathbb{P}^1_\infty = [0 : * : 0]$.

In affine coordinates, this becomes the so-called Weierstrass form:

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

If $\mathrm{char}(K) \neq 2$, then we can complete the square on the left hand side and use the change of coordinates

$$y = \frac{1}{2}(y_1 - a_1x - a_3)$$

to obtain the equation:

$$y_1^2 = 4x^3 + b_2x^2 + 2b_4x + b_6,$$

$$\text{where} \quad \begin{aligned} b_2 &= a_1^2 + 4a_2, \\ b_4 &= 2a_4 + a_1a_3, \\ b_6 &= a_3^2 + 4a_6. \end{aligned}$$

Of course, these elements of K can be defined without any restrictions the characteristic. Similarly, one can define the following objects associated to a fixed Weierstrass equation.

$$
\begin{aligned}
&b_8 = a_1^2 a_6 + 4a_2 a_6 - a_1 a_3 a_4 + a_2 a_3^2 - a_4^2,\\
&c_4 = b_2^2 - 24 b_4,\\
&c_6 = b_2^3 + 36 b_2 b_4 - 216 b_6,\\
&\Delta = -b_2^2 b_8 - 8 b_4^3 - 27 b_6^2 + 9 b_2 b_4 b_6,\\
&j = c_4^3/\Delta,\\
&\omega = dx/(2y + a_1 x + a_3) = dy/(3x^2 + 2a_2 x + a_4 - a_1 y).
\end{aligned}
$$

The last three have names. Δ is called the discriminant, j is the j-invariant, and ω is the invariant differential.

If the characteristic is neither 2 nor 3, then by the following change of coordinates:

$$
\begin{aligned}
x_2 &= (x_1 - 3b_2)/36\\
y_2 &= y_1/108,
\end{aligned}
$$

we obtain

$$y_2^2 = x_2^3 - 27c_4 x_2 - 54c_6.$$

One is really interested in the curve itself, as opposed to the various equations which cut it out. Thus we must understand how these quantities change under changes of variables preserving Weierstrass form and fixing the point $[0:1:0]$.

Any such change of variables is of the form:

$$
\begin{aligned}
x &= u^2 x' + r\\
y &= u^3 y' + u^2 s x' + t,
\end{aligned}
\tag{1}
$$

where $r, s, t \in K$ and $u \in K^*$. This changes the quantities above as shown in Table 1.

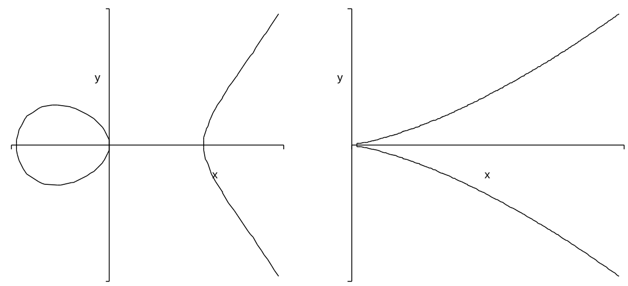
$$
\boxed{
\begin{aligned}
u a_1' &= a_1 + 2s\\
u^2 a_2' &= a_2 - s a_1 + 3r - s^2\\
u^3 a_3' &= a_3 + r a_1 + 2t\\
u^4 a_4' &= a_4 - s a_3 + 2r a_2 - (t + rs) a_1 + 3r^2 - 2st\\
u^6 a_6' &= a_6 + r a_4 + r^2 a_2 + r^3 - t a_3 - t^2 - r t a_1
\end{aligned}}
$$

$$
\boxed{
\begin{aligned}
u^2 b_2' &= b_2 + 12r\\
u^4 b_4' &= b_4 + r b_2 + 6r^2\\
u^6 b_6' &= b_6 + 2r b_4 + r^2 b_2 + 4r^3\\
u^8 b_8' &= b_8 + 3r b_6 + 3r^2 b_4 + r^3 b_2 + 3r^4
\end{aligned}}
\qquad
\boxed{
\begin{aligned}
u^4 c_4' &= c_4\\
u^6 c_6' &= c_6\\
u^{12} \Delta' &= \Delta\\
j' &= j\\
u^{-1} \omega' &= \omega
\end{aligned}}
$$

TABLE 1. a,b,c's under the change of coordinates given in equation (1).

As an excuse to draw some pictures, we include the plots of the real points of some Weierstrass equations:

(2)



Real points of the elliptic curve $y^2 = x^3 - x$ and the singular cubic $y^2 = x^3$.

We conclude the section with a few easy facts, proofs of which can be either provided by the reader or found in [**Si1**]:

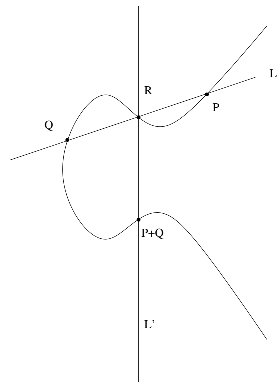
1. A Weierstrass curve is singular if and only if $\Delta = 0$.
2. Two elliptic curves over a field K, C and C', are isomorphic over $\bar{K}$ if and only if the have the same j-invariants, i.e., $j(C) = j(C')$.
3. The invariant differential ω has no poles and no zeros.

2. ...and as one dimensional group varieties

As an elliptic curve C is defined by a cubic, intersecting with any line in $\mathbb{P}^2$ will provide three points (when counted with multiplicity) of C. It turns out that one can define a group law on the points of C by declaring that any three points of C obtained from intersection with a line should sum to the identity element. More directly, given two points P and Q on C, we define their sum as follows:

(1) Find the line L containing P and Q (or tangent to C at P if $P = Q$) and let R be the third point on $L \cap C$. (2) Take the line L' passing through R and $e = [0, 1, 0]$ and define $P + Q$ to be the third point of intersection in $L' \cap C$.

A picture illustrating this group law is shown in Figure 3.

(3) 

Group law on an elliptic curve.

THEOREM 2.1. *The law defined above provides an abelian group structure on the points of C with identity element $e = [0, 1, 0]$. In fact, the maps*

$$+ : E \times E \to E, - : E \to E$$

are morphisms.

Proof: See [**Si1**] III.2-3.

COROLLARY 2.2. *An elliptic curve C over K is a one dimensional group variety over K.*

In fact, there is a rather strong converse to this statement.

THEOREM 2.3. *Let G be a one dimensional (connected) group variety over an algebraically closed field K. Then either:*
(i) $G \equiv \mathbb{G}_a$ the additive group,
(ii) $G \equiv \mathbb{G}_m$ the multiplicative group,
or (iii) G is an elliptic curve.

Remark: Over a non-algebraically closed field, a very similar statement is true, the only modification being that there can be 'more multiplicative groups.'

We will only sketch a proof, for more details see [**Si2**]. We begin by recalling a lemma which we will not prove.

LEMMA 2.4. *Let B be a non-singular projective curve of genus g and $S \subset B$ a finite set of points such that:*
(i) $\#S \geq 3$ if $g = 0$, (ii) $\#S \geq 1$ if $g = 1$, and (iii) S is arbitrary if $g \geq 2$.
Then, $Aut(B; S) := \{\phi \in Aut(B) | \phi(S) \subset S\}$ is a finite set.

Assume first that we knew the lemma held. Note that as G is a group, it is non-singular and irreducible. This together with the fact that G is one dimensional implies that it embeds as a Zariski open subset in a non-singular projective curve $G \subset B$ (cf. [**Ha**] I.6).

Let $S = B - G$. Now each point $P \in G$ provides a different translation automorphism $\tau_P : G \to G$ of G as a variety. This extends to a rational map from B to B and as B is non-singular, an element of $\mathrm{Aut}(B; S)$. This gives an injection of G into $\mathrm{Aut}(B;S)$ so this set can not be finite. Applying the lemma above shows that either G if $\mathbb{P}^1$ with 0,1, or 2 points removed or a genus one surface.

A simple argument shows that $\mathbb{P}^1$ does not admit a group structure (even topologically!), and the only group structure on $\mathbb{A}^1$ (resp. $\mathbb{A}^1 - 0$) is $\mathbb{G}_a$ (resp. $\mathbb{G}_m$). It then remains to show that a genus one curve G with a base point e is isomorphic to an elliptic curve and that the only possible group structures on the genus one surface arise from such an isomorphism. We will ignore the second issue and focus on the first as we will want to generalize the argument later.

THEOREM 2.5. *Let (B,e) be a genus one curve with marked point $e \in B$. Then (B,e) is an elliptic curve, i.e. there exist functions $x, y \in K(B)$ such that the map*

$$\phi : B \to \mathbb{P}^2$$

defined by $\phi = [x, y, 1]$ is an isomorphism of B/K onto the elliptic curve

$$C : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

for some $a_1, \ldots, a_6 \in K$.

To prove this theorem, we use the following special case of Riemann-Roch.

THEOREM 2.6. *Let B be a curve of genus 1. If $D \in \mathrm{Div}(B)$ has positive degree and $\mathscr{L}(D) = \{f \in \bar{K}(C)^* : div(f) \geq -D\} \cup \{0\}$, then $dim\mathscr{L}(D) = degD$.*

Now apply this version of Riemann-Roch to the sequence of divisors: $e, 2e, 3e, \ldots$.

Applied to e we find that $dim\mathscr{L}(e) = 1$ so $\mathscr{L}$ consists of the constant functions $\bar{K}$. $\mathscr{L}(2e)$ is 2 dimensional and thus will contain a function x with pole of order 2 at e. Similarly, $\mathscr{L}(3e)$ is three dimensional and thus have another function y with a pole of order 3 at e. Continuing on, we see that $(1, x, y, x^2)$ is a basis for $\mathscr{L}(4e)$ and $(1, x, y, x^2, xy)$ a basis for $\mathscr{L}(5e)$. However, the dimension of $\mathscr{L}(6e)$ is 6 and it must contain the functions $(1, x, y, x^2, xy, y^2, x^3)$. Thus there must exist a linear combination of these such that the coefficients of y^2 and x^3 are non-zero. Rescalling x and y appropriately, we obtain a Weierstrass equation.

It remains to show that the map ϕ defined in the statement of the theorem is (1) a surjective morphism, (2) a degree 1 map onto its image C, and (3) C is smooth.

To see (1) we note that ϕ is a rational map from a smooth curve, and therefore a morphism. Moreover, any morphism between connected curves is surjective.

For (2) it is equivalent to show that $K(B) = K(x, y)$. First consider the map $[x, 1] : B \to \mathbb{P}^1$. It has a pole of order 2 at e and no others, so is of degree 2, i.e., $[K(B) : K(x)] = 2$. On the other hand, mapping E to $\mathbb{P}^1$ by $[y, 1]$ we see that $[K(B) : K(y)] = 3$. As 2 and 3 are relatively prime, $[K(B) : K(x, y)] = 1$ and thus $K(B) = K(x, y)$.

Lastly, suppose that C were singular. Then there would be a rational map to $\mathbb{P}^1$ of degree one. But composing this with ϕ would give a degree one map between smooth curves from B to $\mathbb{P}^1$. Any degree one map between smooth curves is an isomorphism which would imply that B was of genus zero, a contradiction.

3. Elliptic curves over arbitrary schemes

Now that we have a sense of what an elliptic curve looks like, we would like to understand how they behave in families. Once we do so, it makes sense to allow some singularities. For example, if one wants to understand the rational points of an elliptic curve, say $y^2 = x^3 + 2x^2 + 6$, one might wish to consider the curve reduced modulo a prime. However, in characteristic 2 (respectively 3), the curve becomes singular as $y^2 = x^3 + 2x^2 + 6 \equiv x^3$ (resp. $y^2 = x^3 + 2x^2$). If we think of our curve as being a scheme over $\operatorname{Spec} \mathbb{Z}$, over most primes the geometric fiber will be an elliptic curve, but over 2 and 3 for example, it will become singular.

This motivates the following definition.

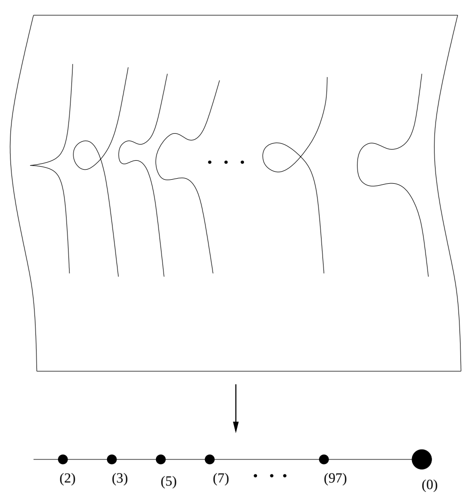
Definition 3.1. A pointed curve of genus 1 over a scheme S is a proper, flat, finitely presented morphism $p : C \to S$ together with a section $e : S \to C$ such that the section is contained in the smooth locus of the fibers and every geometric fiber of p is either

(i) an elliptic curve,
(ii) a singular cubic in $\mathbb{P}^2$ with a node, or
(iii) a singular cubic in $\mathbb{P}^2$ with a cusp.

A fiber of the form (ii) (resp. (iii)) is said to be multiplicative (resp. additive).

Example 3.2. Consider the scheme over $\operatorname{Spec} \mathbb{Z}$ given in affine coordinates as $C_1 : y^2 = x^3 + 2x^2 + 6$. The discriminant of this curve is $-18624 = -2^6 \cdot 3 \cdot 97$. It follows that C_1 is singular only over the primes $(2), (3), (97)$. As we saw above, C has a cusp over (2) and a node over (3), similarly one can check that it also has a node over (97). See Figure (4).

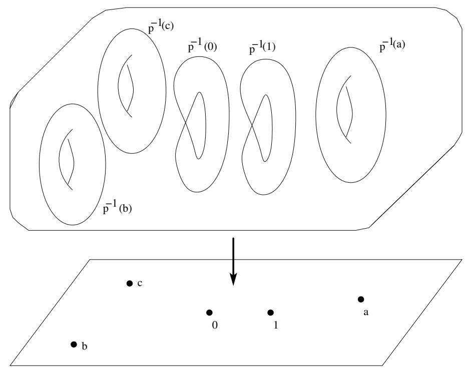
(4)



The curve $C_1 : y^2 = x^3 + 2x^2 + 6$ over $\operatorname{Spec} \mathbb{Z}$.

Example 3.3. Over the base $\operatorname{Spec}(\mathbb{C}(\lambda)) = \mathbb{A}^1$ we can consider the scheme $C_2 : y^2 = x(x-1)(x-\lambda)$. Over 0 and 1, the fibers are multiplicative and all other fibers are smooth. See Figure (5).

(5)



The curve $C_2 : y^2 = x(x-1)(x-\lambda)$ over $\mathrm{Spec}(\mathbb{C}[\lambda]) = \mathbb{A}^1$.

DEFINITION 3.4. Let ω be the invertible sheaf $\omega = e^*\Omega^1_{C/S}$.

In other words, ω is the sheaf over S whose stalk at a closed point $s \in S$ is the restriction of the cotangent space of the geometric fiber over s to $e(s)$.

The section e is a relative Cartier divisor of C over S (i.e., a closed subscheme, flat over S whose ideal sheaf is an invertable $\mathcal{O}_C$-module [**KM**, 1.1]). One has a short exact sequence for all $n > 0$:

$$0 \to \mathcal{O}(ne) \to \mathcal{O}((n+1)e) \to \mathcal{O}_e((n+1)e) \to 0.$$

Using the residue pairing between $\Omega^1_{C/S}$ and $\mathcal{O}_e(e)$, we can identify $p_*\mathcal{O}_e(e)$ and ω^{-1}. Also, one can check fiber-by-fiber, that $R^1p_*\mathcal{O}(ne) = 0$ for all $n > 0$. Thus the long exact sequence obtained by pushing forward will begin with the short exact sequence:

$$0 \to p_*\mathcal{O}(ne) \to p_*\mathcal{O}((n+1)e) \to \omega^{\otimes -(n+1)} \to 0.$$

Riemann-Roch tells us that $p_*\mathcal{O}(ne)$ is locally free of rank n. Further one can check fiber by fiber that $\mathcal{O}_S$ is isomorphic to $p_*\mathcal{O}(e)$. Putting this together we see that we have a filtration of $p_*\mathcal{O}(ne)$ by $p_*\mathcal{O}(me)$ for $1 \le m \le n$ with associated graded

$$\mathrm{Gr}\ p_*\mathcal{O}(ne) = \mathcal{O}_S \oplus \bigoplus_{i=2}^{n} \omega^{\otimes -i}.$$

We will use this to show that we can (locally) embed any pointed curve of genus 1 into $\mathbb{P}^2_S$ by a Weierstrass equation, just as we did for genus 1 curves over algebraically closed fields.

Let π be an invertible section of ω. In analogy to the simpler case, we choose a basis $\{1, x, y\}$ of $p_*\mathcal{O}(3e)$ such that $x \in p_*\mathcal{O}(2e) \subset p_*\mathcal{O}(3e)$ and under the projections to the pieces of the associated graded:

$$\begin{gathered} p_*(\mathcal{O}(3e)) \to \omega^{\otimes -3} \\ y \mapsto \pi^{\otimes -3} \\ p_*(\mathcal{O}(2e)) \to \omega^{\otimes -2} \\ x \mapsto \pi^{\otimes -2} \end{gathered}$$

If instead of π, we had chosen a different section $\pi' = u\pi$ where u is an invertible function, then all such bases with respect to u' can be written in terms of the old as

$$x = u^2 x' + r$$
$$y = u^3 y' + s u^2 x' + t.$$

Note that this is a global version of formula (1) from the first section and that all of the formulas in Table 1 still hold. Moreover, the section of $\omega^{\otimes 4}$ given by $c_4 \pi^{\otimes 4}$ changes to $c_4'(\pi')^{\otimes 4}$, so it does not depend on the choice of π, x, and y. The same goes for $c_6 \pi^{\otimes 6}$ and $\Delta \pi^{\otimes 12}$ which are sections respectively of ω^6 and ω^{12}. This suggests the following definition.

Definition 3.5. An integral modular form of weight n is a law associating to every pointed curve of genus 1 a section of $\omega^{\otimes n}$ in a way compatible with base change.

From the discussion above, examples of integral modular forms of weight 4, 6, and 12 are $c_4 \pi^{\otimes 4}$, $c_6 \pi^{\otimes 6}$, and $\Delta \pi^{\otimes 12}$.

Remark 3.6. The product of two integral modular forms f and g of weights n and m produces an integral modular form of weight $n + m$. If we let modular forms live in the direct sum of the tensor powers of ω, we can then consider integral modular forms as making up a ring.

Remark 3.7. As any pointed curve of genus 1 embeds locally in $\mathbb{P}^2_S$ by a Weierstrass equation, the curve $C : y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6$ defined over $\operatorname{Spec} \mathbb{Z}[a_1, a_2, a_3, a_4, a_6]$ is universal and any modular form will be a polynomial in the a_i.

As we saw in section 1, if we only consider bases in which 2 and 3 are invertible (i.e., if we work over $\mathbb{Z}[1/6]$), then for a fixed choice of π, there exists a unique choice of x, y such that $a_1 = a_2 = a_3 = 0$. In analogy to the previous remark, this says that the curve given by $y^2 = x^3 - 27c_4 x - 54c_6$ defined over $\mathbb{Z}[1/6][c_4, c_6]$ is universal over $\mathbb{Z}[1/6]$. This then implies that every $\mathbb{Z}[1/6]$-modular form is a polynomial in $\mathbb{Z}[1/6][c_4, c_6]$. But as both c_4 and c_6 are integral modular forms as shown above, we are left with the following theorem.

Theorem 3.8. *The ring of $\mathbb{Z}[1/6]$-modular forms is the polynomial ring $\mathbb{Z}[1/6][c_4, c_6]$.*

With a little more work the ring of integral modular forms can be calculated.

Theorem 3.9. *The ring of integral modular forms is generated over $\mathbb{Z}$ by c_4, c_6, and Δ and has only one relation:*

$$c_4^3 - c_6^2 = 1728\Delta.$$

The proof can be found in [**De**].

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The Moduli stack of elliptic curves

André Henriques

1. The geometry of $\mathcal{M}_{ell}$

1.1. $\mathcal{M}_{ell}$ over the complex numbers. Over the complex numbers, any elliptic curve C is isomorphic to $C_\tau := \mathbb{C}/\mathbb{Z}\{1,\tau\}$ for some complex number τ in the upper half plane $\mathbb{H}$. Two elliptic curves C_τ and $C_{\tau'}$ are then isomorphic if and only if $\tau' = g \cdot \tau$ for some element $g \in SL_2(\mathbb{Z})$, where the (non-faithful) action of $SL_2(\mathbb{Z})$ on $\mathbb{H}$ is given by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot \tau = \frac{a\tau+b}{c\tau+d}$. More precisely, there is a bijective correspondence between isomorphisms $C_\tau \xrightarrow{\sim} C_{\tau'}$, and group elements $g \in SL_2(\mathbb{Z})$ satisfying $\tau' = g \cdot \tau$. The map corresponding to $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is then given by $C_\tau \to C_{\tau'} : z \mapsto \frac{z}{c\tau+d}$. It follows that $\mathcal{M}_{ell}$ is the quotient stack

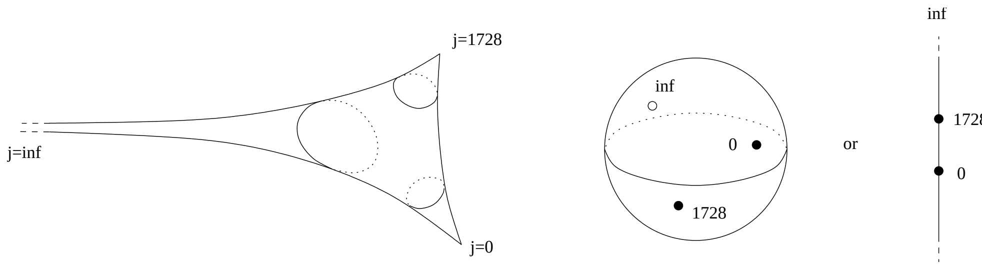
$$\mathcal{M}_{ell} = [\mathbb{H}/SL_2(\mathbb{Z})].$$

The automorphism group of the elliptic curve C_τ then corresponds to the stabilizer group of τ by the action of $SL_2(\mathbb{Z})$.

The j-invariant provides an algebraic map $\mathcal{M}_{ell} \to \mathbb{C}$, expressing $\mathbb{C}$ as the coarse moduli space of $\mathcal{M}_{ell}$. By definition, this means that the above map is initial among all maps from $\mathcal{M}_{ell}$ to a variety. By a generalized elliptic curve we mean a curve that is either an elliptic curve or isomorphic to the multiplicative curve $\mathbb{G}_m = \mathbb{C}^\times$. The moduli space $\overline{\mathcal{M}}_{ell}$ of generalized elliptic curves can then be thought as the one point compactification of $\mathcal{M}_{ell}$, and the j-invariant extends to a map $\overline{\mathcal{M}}_{ell} \to \mathbb{CP}^1$.

There are two distinguished j-invariants, corresponding to curves with extra symmetries. These are $j = 0$, which corresponds to the elliptic curve obtained by modding out $\mathbb{C}$ by an equilateral lattice, and $j = 12^3 = 1728$, which corresponds to the elliptic curve $\mathbb{C}_i = \mathbb{C}/\mathbb{Z}\{1,i\}$. Those elliptic curves have automorphism group $\mathbb{Z}/6$ and $\mathbb{Z}/4$, respectively, while all the other generalized elliptic curves have automorphism group $\mathbb{Z}/2$. Depending on one's taste, one might then draw $\mathcal{M}_{ell}$

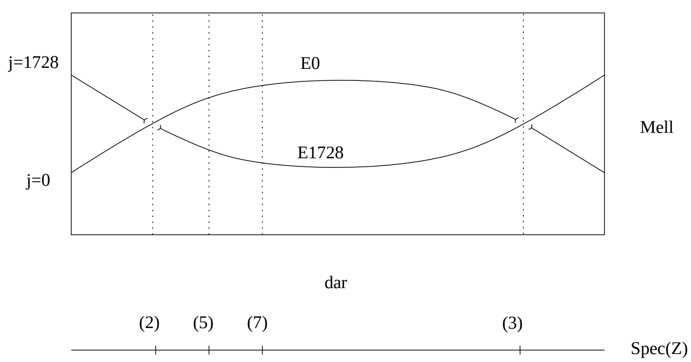
as



The first one of the above pictures is the most geometric, but it is only the third one that we will be able to generalize to the case when our base is $\mathrm{Spec}\,(\mathbb{Z})$.

1.2. $\mathcal{M}_{ell}$ **over the integers.** Over $\mathrm{Spec}\,(\mathbb{Z})$, the coarse moduli spaces of $\mathcal{M}_{ell}$ and $\overline{\mathcal{M}}_{ell}$ are still isomorphic $\mathbb{A}^1$ and $\mathbb{P}^1$, respectively. However, $\mathrm{Spec}\,(\mathbb{Z})$ being one-dimensional, it is now preferable to draw $\mathbb{A}^1_{\mathrm{Spec}\,(\mathbb{Z})}$ as a plane, as opposed to a line. The elements $0, 12^3 \in \mathbb{Z}$ then correspond to sections of the projection map $\mathbb{A}^1_{\mathrm{Spec}\,(\mathbb{Z})} \to \mathrm{Spec}\,(\mathbb{Z})$. Note that, since the difference $12^3 - 0$ is divisible by 2 and 3, the values of those two sections agree over the points $(2), (3) \in \mathrm{Spec}\,(\mathbb{Z})$.

The elliptic curves $\{y^2 + y = x^3\}$ and $\{y^2 = x^3 - x\}$ have j-invariants 0 and 12^3, respectively. Their discriminants are -27 and -64. Therefore, as elliptic curves, they are actually only defined over $\mathbb{Z}_{(3)}$ and $\mathbb{Z}_{(2)}$, respectively. By definition of $\mathcal{M}_{ell}$, an elliptic curve defined over a ring R gives rise to a map $\mathrm{Spec}\,(R) \to \mathcal{M}_{ell}$. So the above elliptic curves correspond to maps $\mathrm{Spec}\,(\mathbb{Z}_{(3)}) \to \mathcal{M}_{ell}$ and $\mathrm{Spec}\,(\mathbb{Z}_{(2)}) \to \mathcal{M}_{ell}$.



Given a variety X defined over $\mathbb{F}_p$, one typically needs to base change to $\bar{\mathbb{F}}_p$ in order to "see" all the automorphisms of X. But for elliptic curves it turns out that $\mathbb{F}_{p^2}$ is always enough. The automorphism groups of an elliptic curve C defined over a finite field $\mathbb{F}$ containing $\mathbb{F}_{p^2}$ are given by:

$$\mathrm{Aut}(C) = \begin{cases} \mathbb{Z}/2 & \text{if} \quad j \neq 0, 12^3 & \\ \mathbb{Z}/6 & \text{if} \quad j = 0\,, & p \neq 2,3 \\ \mathbb{Z}/4 & \text{if} \quad j = 12^3, & p \neq 2,3 \\ \mathbb{Z}/4 \ltimes \mathbb{Z}/3 & \text{if} \quad j = 0 = 12^3, & p = 3 \\ \mathbb{Z}/3 \ltimes Q_8 & \text{if} \quad j = 0 = 12^3, & p = 2 \end{cases}$$

The last group is the semi-direct product of $\mathbb{Z}/3$ with the quaternion group Q_8, where the action permutes the generators i, j and k.

If C is defined over $\mathbb{F}_p$, then it is better to view $\mathrm{Aut}(C)$ as a finite group scheme, as opposed to a mere finite group. In general, the data of a finite group scheme over some field k is equivalent to that of a finite group and an action of $\mathrm{Gal}(K/k)$ by group automorphisms, where K is some finite extension of k. In our case, $k = \mathbb{F}_p$, $K = \mathbb{F}_{p^2}$, and $\mathrm{Aut}(C)$ is given by $\mathrm{Aut}\left(C \times_{\mathrm{Spec}\,(\mathbb{F}_p)} \mathrm{Spec}\,(\mathbb{F}_{p^2})\right)$, along with its action of $\mathbb{Z}/2 = \mathrm{Gal}(\mathbb{F}_{p^2}/\mathbb{F}_p)$. The actions of $\mathbb{Z}/2$ on the above groups is given by:

$\mathbb{Z}/2$	:	the trivial action;
$\mathbb{Z}/6$	:	trivial iff 3 divides $p-1$;
$\mathbb{Z}/4$	:	trivial iff 4 divides $p-1$;
$\mathbb{Z}/4 \ltimes \mathbb{Z}/3$	:	non-trivial on $\mathbb{Z}/4$, trivial on $\mathbb{Z}/3$;
$\mathbb{Z}/3 \ltimes Q_8$	:	non-trivial action on $\mathbb{Z}/3$; the action on Q_8 exchanges $i \leftrightarrow -i$, $j \leftrightarrow -k$, and $k \leftrightarrow -j$.

Note that the group schemes corresponding to the first three rows of the above list are isomorphic to μ_2, μ_6, and μ_4, respectively.

2. Multiplication by p

Let C be an elliptic curve defined over a field $\mathbb{F}$. Like any abelian group, it has a natural endomorphism $[p] : C \to C$ given by $x \mapsto x + \cdots + x$ (p times). Clearly, the derivative of $[p]$ at the identity element is multiplication by p. If p is invertible in $\mathbb{F}$, then the derivative of $[p]$ is non-zero at the identity element, and therefore everywhere non-zero since it is a group homomorphism. On the other hand, if $\mathbb{F}$ has characteristic p, then the derivative of $[p]$ is identically zero.

Let $C[p]$ denote the scheme-theoretical kernel of $[p]$. From the above discussion, we see that $C[p]$ is a reduced scheme if and only if $p \neq 0$. The number of geometric points of $C[p]$ can vary. But if we count points with multiplicities, then that number is always p^2.

THEOREM 2.1. *Let C be an elliptic curve defined over a field $\mathbb{F}$. Then $[p]\colon C \to C$ has degree p^2. Equivalently, the vector space $\Gamma(C[p], \mathcal{O})$ has dimension p^2.*

PROOF. We begin by the observation that, if C_1 and C_2 are smooth curves over a field $\mathbb{F}$, then any non-constant map $f\colon C_1 \to C_2$ is flat. Indeed, flatness can be checked on formal neighborhoods of points. So without loss of generality, we may replace C_1 and C_2 by their completions around some given points. The map f can then be written locally as

$$
\begin{array}{rcll}
f : \mathrm{Spf}\left(\mathbb{F}[[x]]\right) & \to & \mathrm{Spf}\left(\mathbb{F}[[y]]\right) & \\
\mathbb{F}[[x]] & \leftarrow & \mathbb{F}[[y]] & : \ f^* \\
f^*(y) & \leftarrow\!\shortmid & y &
\end{array} \tag{1}
$$

The power series $f^*(y) \in \mathbb{F}[[x]]$ is non-zero by assumption, so can be written as $f^*(y) = ax^d +$ (higher terms) for some $a \neq 0$. One then checks that $\mathbb{F}[[x]]$ is a free $\mathbb{F}[[y]]$ module with basis $\{1, x, \ldots, x^{d-1}\}$. In particular, it is flat.

Now let C be an elliptic curve defined over a field. Since $[p]\colon C \to C$ is not the constant map [**Si**, Prop. III.4.2], it is flat.

It will be useful to allow more general base schemes. Recall [**Gro**, section 11.3.11] that given a diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

where X and Y are flat and of finite type over S, the map f is flat iff for every field $\mathbb{F}$ and every map $\operatorname{Spec}(\mathbb{F}) \to S$, the pullback map $f : X \times_S \operatorname{Spec}(\mathbb{F}) \to Y \times_S \operatorname{Spec}(\mathbb{F})$ is flat. Let C_{Weier} denote the universal Weierstrass elliptic curve, defined over the ring $\mathbb{Z}[a_1, a_2, a_3, a_4, a_6][\Delta^{-1}]$. By the above criterion, and using our knowledge about elliptic curves over fields, we see that

$$\begin{tikzcd} C_{\text{Weier}} \arrow[rr, "{[p]}"] \arrow[dr] & & C_{\text{Weier}} \arrow[dl] \\ & \operatorname{Spec}(\mathbb{Z}[a_i][\Delta^{-1}]) & \end{tikzcd}$$

is flat.

Fix a map $\varphi : \operatorname{Spec}(\mathbb{C}) \to \operatorname{Spec}(\mathbb{Z}[a_i][\Delta^{-1}])$, and let C_φ be the elliptic curve it classifies. Since C_φ is topologically isomorphic to $S^1 \times S^1$, one sees immediately that $C_\varphi[p] \simeq (\mathbb{Z}/p)^2$. Now consider the following commutative diagram:

$$\begin{tikzcd} C_\varphi[p] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & C_{\text{Weier}}[p] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \operatorname{Spec}(\mathbb{Z}[a_i][\Delta^{-1}]) \arrow[d, "\text{zero section}"] \\ C_\varphi \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & C_{\text{Weier}} \arrow[r, "{[p]}"] \arrow[d] & C_{\text{Weier}} \arrow[dl] \\ \operatorname{Spec}(\mathbb{C}) \arrow[r, "\varphi"] & \operatorname{Spec}(\mathbb{Z}[a_i][\Delta^{-1}]) & \end{tikzcd}$$

Being an elliptic curve, C_{Weier} is proper over $\operatorname{Spec}(\mathbb{Z}[a_i][\Delta^{-1}])$. Any map between proper schemes is proper, and so $[p]$ is proper. The projection $C_{\text{Weier}}[p] \to \operatorname{Spec}(\mathbb{Z}[a_i][\Delta^{-1}])$ is pulled back from $[p]$. It is therefore flat and proper, and in particular, it has constant relative dimension. To compute the latter, we note that the relative dimension of a map is left unchanged by pullbacks, and that $C_\varphi[p] \to \operatorname{Spec}(\mathbb{C})$ is of relative dimension zero.

We have shown that $C_{\text{Weier}}[p] \to \operatorname{Spec}(\mathbb{Z}[a_i][\Delta^{-1}])$ is proper, flat, and of relative dimension zero. It is therefore a finite map. Being flat and finitely generated, the $\mathbb{Z}[a_i][\Delta^{-1}]$ module $\Gamma(C_{\text{Weier}}[p]; \mathcal{O})$ is therefore projective of finite rank. The rank of a projective module is stable under pullbacks. It is therefore equal to the complex dimension of $\Gamma(C_\varphi[p]; \mathcal{O})$, namely p^2.

Now let C be an arbitrary elliptic curve, defined over a field $\mathbb{F}$. By the following pullback square

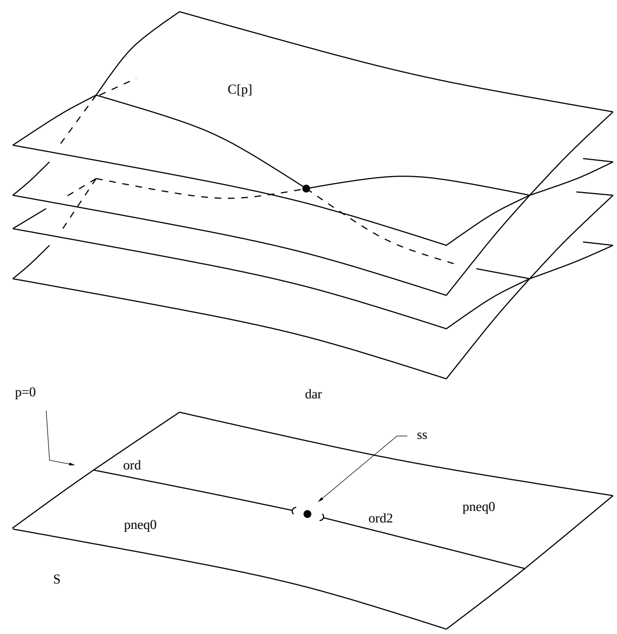
$$\begin{tikzcd} C[p] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & C_{\text{Weier}}[p] \arrow[d] \\ \operatorname{Spec}(\mathbb{F}) \arrow[r] & \operatorname{Spec}(\mathbb{Z}[a_i][\Delta^{-1}]) \end{tikzcd}$$

we see that the dimension of $\Gamma(C[p];\mathcal{O})$ is equal to the rank of $\Gamma(C_{\text{Weier}}[p];\mathcal{O})$, which is p^2. $\square$

The scheme theoretic cardinality of $C[p]$ is always p^2. Since $C[p]$ is a group, the number of connected components of $C[p]$ is therefore either p^2, p, or 1. All those cases can occur. The first one happens iff $C[p]$ is reduced, namely iff the base field is of characteristic different from p. The two other cases happen in characteristic p.

Definition 2.2. Let $\mathbb{F}$ be a field of characteristic p, and C an elliptic curve defined over $\mathbb{F}$. Then C is called *ordinary* if $C[p]$ has p connected components, and *supersingular* if $C[p]$ is connected.

If C is an elliptic curve defined over an arbitrary base scheme S, then there is a natural stratification of S: the stratum over which the fibers of the map $C[p]\to S$ have cardinality p^2, the one over which they have cardinality p, and the one over which they consist of a single thick point. The first of the above three strata is the Zariski open set $\{p\neq 0\}$. The other two strata are called the *ordinary locus* and the *supersingular locus*.



3. The relative Frobenius

In characteristic zero, the derivative of a non-constant map $f: C_1 \to C_2$ can only vanish at a finite number of points. But in characteristic p, the derivative of f can vanish identically without f being constant. For example, this is the case for the map $[p]: C\to C$, where C is an elliptic curve. The prototypical example of a map whose derivative vanishes identically is the relative Frobenius.

Definition 3.1. Let C be a curve defined over a perfect field $\mathbb{F}$ of characteristic p. Define $C^{(p)}$ to be the scheme with same underlying space as C, but with structure sheaf

$$\mathcal{O}_{C^{(p)}}(U) := \{x^p \mid x \in \mathcal{O}_C(U)\}.$$

The relative Frobenius of C is the map

$$\phi : C \twoheadrightarrow C^{(p)}$$

given by the identity on the underlying spaces, and by the inclusion $\mathcal{O}_{C^{(p)}} \hookrightarrow \mathcal{O}_C$ at the level of structure sheaves.

In local coordinates, the relative Frobenius is given by

$$\begin{array}{ccccc} \phi : \mathrm{Spf}(\mathbb{F}[[x]]) & \to & \mathrm{Spf}(\mathbb{F}[[x]])^{(p)} = \mathrm{Spf}(\mathbb{F}[[y]]) & & \text{(where } y = x^p\text{)} \\ x^p & \leftarrow\!\shortmid & y & & \end{array}$$

We see in particular that its derivative ϕ' vanishes identically.

Remark 3.2. If C is defined by equations in $\mathbb{P}^n$, then $C^{(p)}$ can be identified with the curve defined by those same equations, but where all the coefficients have been replaced by their pth powers. In those coordinates, the relative Frobenius is given by $\phi([x_0, \dots, x_n]) = [x_0^p, \dots, x_n^p]$. Using that as our working definition, we could have removed the condition that $\mathbb{F}$ be perfect in Definition 3.1.

In the rest of this chapter, we assume for convenience that the base field $\mathbb{F}$ is perfect. All the statements remain true without that assumption.

Proposition 3.3. *Let C_1, C_2 be curves defined over a perfect field of characteristic p, and let $f : C_1 \to C_2$ be a map whose derivative vanishes identically. Then the map f can then be factored through the relative Frobenius of C_1:*

$$\begin{tikzcd} C_1 \arrow[rr, "f"] \arrow[dr, "\phi"'] & & C_2 . \\ & C_1^{(p)} \arrow[ur, "\bar{f}"'] & \end{tikzcd} \tag{2}$$

Proof. In local coordinates, the map f is given by some power series expansion, as in (1). Since $f^*(y) = \sum a_i x^i \in \mathbb{F}[[x]]$ has zero derivative, it follows that $a_i = 0$ for all i not divisible by p.

Now, let $g \in \Gamma(U, \mathcal{O}_{C_1})$ be a function in the image of $f : f^*\mathcal{O}_{C_2} \to \mathcal{O}_{C_1}$. Its derivative g' is identically zero, so g admits a pth root (a priori, this is only true locally, but the local pth roots are unique, so they assemble to a pth root defined on the whole U). It follows that $g \in \mathcal{O}_{C_1}^{(p)}$. We have shown that $f^*\mathcal{O}_{C_2} \to \mathcal{O}_{C_1}$ factors through $\mathcal{O}_{C_1}^{(p)}$, which is equivalent to the statement that f factors through $C_1^{(p)}$. $\square$

Corollary 3.4. *Let C be an elliptic curve defined over a perfect field of characteristic p, Then the map $[p] : C \to C$ factors through ϕ.*

Define inductively $C^{(p^{n+1})} := (C^{(p^n)})^{(p)}$. Given a non-constant map $f : C_1 \to C_2$, there is a maximal number n such that we get factorizations

$$\begin{tikzcd}
C_1 \arrow[rrrr, "f"] \arrow[dr, "\phi"'] & & & & C_2 \\
& C_1^{(p)} \arrow[urrr] \arrow[dr, "\phi"'] & & \cdots & \\
& & C_1^{(p^2)} \cdots \arrow[uurr] \arrow[rr, "\phi"'] & & C_1^{(p^n)} \arrow[uu, "\bar{f}"']
\end{tikzcd}$$

Note that since $\deg(\phi) = p$, it follows that $\deg(f) = p^n \deg(\bar{f})$.

If C is an elliptic curve and f is the map $[p]: C \to C$, then by Theorem 2.1 two cases can occur. Either C is ordinary, in which case $n = 1$ and the map $\bar{f}: C^{(p)} \to C$ is an étale cover of order p, or C is supersingular, in which case $n = 2$ and $\bar{f}: C^{(p^2)} \to C$ is an isomorphism.

$$(3) \qquad \begin{tikzcd}
C \arrow[rr, "{[p]}"] \arrow[dr, "\phi"'] & & C \\
& C^{(p)} \arrow[ur, "\text{étale cover}" ' , "\text{of degree } p"] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
C \arrow[rr, "{[p]}"] \arrow[d, "\phi"'] & & C \\
C^{(p)} \arrow[urr] \arrow[rr, "\phi"'] & & C^{(p^2)} \arrow[u, "\simeq"']
\end{tikzcd}$$

C is ordinary $\qquad\qquad$ C is supersingular

LEMMA 3.5. *If C is a supersingular elliptic curve defined over a perfect field $\mathbb{F}$ of characteristic p, then its j-invariant is an element of $\mathbb{F} \cap \mathbb{F}_{p^2}$. In particular, there are at most p^2 isomorphism classes of supersingular elliptic curves.*

PROOF. By (3), we see that $C^{(p^2)}$ is isomorphic to C. It follows that $j(C^{(p^2)}) = j(C)^{p^2} = j(C)$. ☐

In fact, there are roughly $p/12$ isomorphism classes of supersingular elliptic curves [**Si**, Theorem V.4.1]. More precisely, if we count each isomorphism class with a multiplicity of $1/|\operatorname{Aut}(C)|$, then one has the following formula:

$$\sum_{\substack{[C]:\, C \text{ is} \\ \text{supersingular}}} \frac{1}{|\operatorname{Aut}(C)|} = \frac{p-1}{24}.$$

For small primes, the number of supersingular curves is recorded in the following table:

$p =$	2	3	5	7	11	13	17	19	23	29	$\cdots$
# of ss curves	1	1	1	1	2	1	2	2	3	3	$\cdots$
$\sum \frac{1}{\lvert\operatorname{Aut}(C)\rvert}$	$\frac{1}{24}$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{4}+\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{2}+\frac{1}{6}$	$\frac{1}{2}+\frac{1}{4}$	$\frac{1}{2}+\frac{1}{4}+\frac{1}{6}$	$\frac{1}{2}+\frac{1}{2}+\frac{1}{6}$	$\cdots$

4. Formal groups

Given an elliptic curve C, let $\widehat{C}$ denote its formal completion at the identity. Then $\widehat{C}$ has the structure of a formal group.

Definition 4.1. A (one dimensional, commutative) formal group over S is a formal scheme $G \to S$ which is isomorphic to the formal completion of a line bundle along its zero section, and which comes equipped with an addition law

$$+ : G \times_S G \to G\,,$$

making it into an abelian group object, with neutral element given by the zero section $S \to G$.

Given a formal group G, one can consider the multiplication-by-p map $[p] : G \to G$.

Lemma 4.2. *Let G be formal group over a perfect field $\mathbb{F}$ of characteristic p, and let us assume that the map $[p] : G \to G$ is not constant. Then, after picking an identification of G with* $\mathrm{Spf}(\mathbb{F}[[x]])$*, the power series expansion of $[p]$ is of the form*

$$[p](x) = a_1 x^{p^n} + a_2 x^{2p^n} + a_3 x^{3p^n} + \dots \tag{4}$$

for some integer $n \geq 1$, and elements $a_i \in \mathbb{F}$, $a_1 \neq 0$.

Proof. The first derivative of $[p]$ vanishes identically, so we can factor $[p]$ as in (2). Letting n be the biggest number for which we get a factorization $[p] = \bar{f} \circ \phi^n$, the power series expansion for $[p]$ then looks as in (4). We need to show that $a_1 \neq 0$.

Since ϕ is a surjective group homomorphism, $\bar{f}$ is also a homomorphism. The first derivative of $\bar{f}$ is therefore either everywhere non-zero or identically zero. If $\bar{f}' = 0$, then by Proposition 3.3, we get a further factorization by ϕ, contradicting the maximality of n. So $\bar{f}'(0) = a_1 \neq 0$. ☐

The power series (4) is called the *p-series* of G. By the above lemma, we see that the first non-zero coefficient of the p-series is always that of some x^{p^n}.

Definition 4.3. Let G be formal group over a field of characteristic p. The *height* of G is the smallest number n such that the coefficient of x^{p^n} in the p-series of G is non-zero. If the p-series is identically zero, then we declare G to have height ∞.

Writing the p-series as $[p](x) = \sum a_i x^i$, we see by Lemma 4.2 that the condition of being of height greater than n is given by exactly n equations:

$$a_p = 0, \qquad a_{p^2} = 0, \qquad a_{p^3} = 0, \qquad \cdots, \qquad a_{p^n} = 0.$$

It is customary to write v_n for the coefficient a_{p^n}.

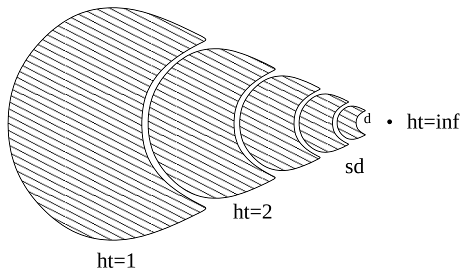
If C is an elliptic curve over a field of characteristic p, then by (3), the only possible heights for $\widehat{C}$ are 1 and 2. The height is 1 if C is ordinary, and 2 if is C is supersingular.

Theorem 4.4 (Lazard [**Laz**]). *Let $\mathbb{F}$ be an algebraically closed field of characteristic p. Then the height provides a bijection between the isomorphism classes of formal groups over $\mathbb{F}$, and the set $\{1, 2, 3, \dots\} \cup \{\infty\}$.* ☐

Given a formal group G defined over a scheme S of characteristic p, one can consider the heights of the fibers $G|_x$ at the various closed points $x \in S$. This yields a partition of S into strata $S_n := \{x \in S : \mathrm{ht}(G|_x) = n\}$. The closed subsets $S_{>n} := \bigcup_{m>n} S_m$ then form a decreasing sequence

$$S = S_{>0} \supset S_{>1} \supset S_{>2} \supset \cdots,$$

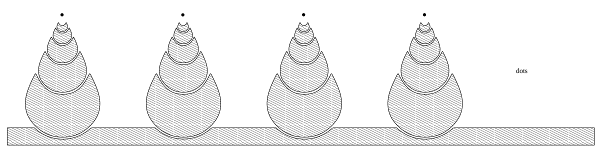
where each one is of codimension at most one in the previous one. Let $\mathcal{M}_{FG}$ denote the the moduli space of formal groups. From the above discussion, we see that $\mathcal{M}_{FG}|_{\mathrm{Spec}\,(\mathbb{F}_p)}$ looks roughly as follows:



Namely, it consists of a countable sequence of stacky points, each point containing all the next ones in its closure, and each point being of codimension one in the previous one.

When considering formal groups over schemes S that are not necessarily of characteristic p, then one has many heights: one for each prime p. By convention, we let the p-height be 0 whenever $p \neq 0$. Here's a picture of $\mathcal{M}_{FG}$ over $\mathrm{Spec}\,(\mathbb{Z})$:

(5)



The assignment $C \mapsto \widehat{C}$ defines a map $\mathcal{M}_{ell} \to \mathcal{M}_{FG}$. As we have seen, that map only hits the first three layers of (5), namely the ones where the p-heights are 0, 1, and 2.

To complete the picture, we give some examples of formal groups of height n. The following result of Lazard will be useful to us.

Proposition 4.5 (Lazard [**Laz**]). *Let R be a ring, and let*

$$\widetilde{+}_F : \mathrm{Spf}(R[[x]])^2 = \mathrm{Spf}(R[[x,y]]) \to \mathrm{Spf}(R[[x]])$$

be a binary operation satisfying the axioms for abelian groups, modulo error terms of total degree $\geq d$. Then there exists another operation $+_F$, which does satisfy the abelian group axioms, and which agrees with $\widetilde{+}_F$ modulo terms of degree $\geq d$. $\square$

Let R be any ring, and consider the example:

$$x \,\widetilde{+}_F\, y \;:=\; x + y + \frac{(x+y)^q - x^q - y^q}{p}\,,$$

where $q = p^n$, and p is a prime. It satisfies commutativity, unitarity, and associativity modulo terms of degree $\geq 2q-1$:

$$\begin{aligned}
\big(x \mathbin{\widetilde{+}_F} y\big) \mathbin{\widetilde{+}_F} z &= x + y + \frac{(x+y)^q - x^q - y^q}{p} + z \\
&\quad + \frac{\big(x + y + \frac{(x+y)^q - x^q - y^q}{p} + z\big)^q - \big(x + y + \frac{(x+y)^q - x^q - y^q}{p}\big)^q - z^q}{p} \\
&= x + y + \frac{(x+y)^q - x^q - y^q}{p} + z + \frac{\big(x+y+z\big)^q - \big(x+y\big)^q - z^q}{p} \qquad \big(\text{mod deg} \geq 2q-1\big) \\
&= x + y + z + \frac{(x+y+z)^q - x^q - y^q - z^q}{p} \\
&= x \mathbin{\widetilde{+}_F} \big(y \mathbin{\widetilde{+}_F} z\big) \qquad \big(\text{mod degree} \geq 2q-1\big)
\end{aligned} \tag{6}$$

So, by the above proposition, there exists an abelian group law on $\mathrm{Spf}(R[[x]])$ of the form:

$$x \ +_F \ y \ := \ x + y + \frac{(x+y)^q - x^q - y^q}{p} \ + \ \big(\text{terms of degree} \geq 2q-1\big). \tag{7}$$

Generalizing the computation (6), it is easy to get the following formula:

$$x_1 \ +_F \ x_2 \ +_F \cdots +_F \ x_r \ = \ \sum x_i + \frac{\big(\sum x_i\big)^q - \sum x_i^q}{p} \qquad \big(\text{mod degree} \geq 2q-1\big).$$

In particular, if $R = \mathbb{F}$ is a field of characteristic p, then the p-series of (7) looks as follows:

$$[p](x) \ = \ px + \frac{(px)^q - px^q}{p} \ = \ -x^q \qquad \big(\text{mod degree} \geq 2q-1\big).$$

The first non-zero coefficient is that of $x^q = x^{p^n}$, therefore (7) defines an abelian group law of height n on $\mathrm{Spf}(\mathbb{F}[[x]])$.

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The Landweber exact functor theorem

Henning Hohnhold

1. Introduction

The goal of this chapter is the construction of the presheaf of elliptic homology theories on the moduli stack of elliptic curves $\mathcal{M}_{ell}$. This sets the stage for many of the later chapters where the objective will be to turn this presheaf into to a sheaf of E_∞-ring spectra (using obstruction theory).

Even though we use the language of stacks, much of this chapter is closely related to the classical story of elliptic cohomology. Constructing a presheaf of homology theories on $\mathcal{M}_{ell}$ means to associate to every elliptic curve $\mathcal{C}$ (satisfying a certain flatness condition) a homology theory $Ell^{\mathcal{C}}$. The construction of $Ell^{\mathcal{C}}$ is based on Landweber's exact functor theorem, and we verify the assumptions of Landweber's theorem using the argument given by Landweber, Stong, Ravenel, and Franke, see [**LRS**], [**Fr**]. However, in the approach we describe (due to Hopkins and Miller) the use of Landweber's theorem is elegantly hidden in the statement that the morphism

$$\mathfrak{F} : \mathcal{M}_{ell} \to \mathcal{M}_{FG}$$

given by associating to an elliptic curve its formal group is flat. This result is the main ingredient in the construction of the presheaf of elliptic homology theories on $\mathcal{M}_{ell}$.

Let us briefly outline the content of this chapter. In Section 2 we first review some generalities concerning even periodic cohomology theories and their formal groups. After that we turn to Quillen's theorem, the fundamental link between complex cobordism and formal group laws. We state a version of the theorem that indicates how the moduli stack of formal groups $\mathcal{M}_{FG}$ comes up in connection with complex cobordism. We conclude Section 2 by describing Landweber's exactness condition and his exact functor theorem which gives a nice criterion for when it is satisfied.

We begin Section 3 by introducing some basics about stacks that are needed for the construction of the presheaf of elliptic homology theories. We then express Landweber's exactness condition in the language of stacks: a formal group law over a ring R is Landweber exact if and only if the corresponding map $\operatorname{Spec} R \to \mathcal{M}_{FG}$ is flat. Using this and the flatness of the map $\mathfrak{F} : \mathcal{M}_{ell} \to \mathcal{M}_{FG}$ we see that every flat morphism $\mathcal{C} : \operatorname{Spec} R \to \mathcal{M}_{ell}$ gives rise to a homology theory $Ell^{\mathcal{C}}$. We also explain how these fit together to give a presheaf of homology theories on $\mathcal{M}_{ell}$.

Finally, in Section 4 we prove, following Hopkins and Miller, that the map $\mathfrak{F}$ is flat. The proof is based on some facts about elliptic curves and their formal group laws and the Landweber exact functor theorem.

Before getting started, let us fix some terminology. Whenever we talk about stacks, we mean stacks defined on the Grothendieck site *Aff* of affine schemes with the flat topology. In this Grothendieck topology, a covering is a faithfully flat map $\operatorname{Spec} S \to \operatorname{Spec} R$. Since we will not be able to supply all the details about stacks that are needed, we will occasionally refer the reader to the papers [**Goe**] and [**Nau**] by Goerss and Naumann. Throughout this chapter, we will refer to 1-morphisms between stacks simply as morphisms.

2. Periodic cohomology theories, formal groups, and the Landweber exact functor theorem

We begin by outlining the relationship between even periodic cohomology theories and formal groups.

Definition 2.1. A multiplicative cohomology theory h^* is *even periodic* if $h^k(pt) = 0$ for odd integers k and if there exists a unit $u \in h^*(pt)$ of degree $|u| = 2$.

Remark 2.2. Let $h^0 := h^0(pt)$.

(1) Using the Atiyah-Hirzebruch spectral sequence we obtain (non-canonical) isomorphisms

$$\iota_k : h^0((\mathbb{CP}^\infty)^k) \stackrel{\cong}{\to} h^0[[x_1, ..., x_k]].$$

The choice of $\iota := \iota_1$ determines in a canonical way a choice of ι_k for $k > 1$.

(2) The map $\mu : \mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$ classifying the (exterior) tensor product of two copies of the universal line bundle induces a map

$$h^0[[x]] \underset{\iota^{-1}}{\cong} h^0(\mathbb{CP}^\infty) \underset{\mu_*}{\to} h^0(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) \underset{\iota_2}{\cong} h^0[[x_1, x_2]].$$

The image $F_h(x_1, x_2)$ of x under this map defines a formal group law over the ring h^0.

(3) The formal group underlying the formal group law $F_h(x_1, x_2)$ is independent of the choice of ι and is called the formal group associated with h^*.

More information about this can be found in [**Re**]. The formal group associated with h^* can also be defined without choosing the coordinate $\iota : h^0(\mathbb{CP}^\infty)) \cong h^0[[x]]$, see [**Lu**].

Examples 2.3.

(1) The formal group associated with the periodic Eilenberg-MacLane spectrum

$$HP := \bigvee_{m \in \mathbb{Z}} \Sigma^{2m} H\mathbb{Z}$$

is the additive formal group.

(2) The formal group associated with complex K-theory is the multiplicative formal group.

(3) The periodic complex cobordism spectrum

$$MP := \bigvee_{m \in \mathbb{Z}} \Sigma^{2m} MU.$$

has a canonical complex orientation and the associated formal group law $F_{MP}(x_1, x_2)$ over $MP_0 = MP^0$ is, according to Quillen's theorem, the universal formal group law.

REMARK 2.4. As in the case of MU_*, one can show that

$$MP_0 \rightrightarrows MP_0 MP$$

is a Hopf algebroid and that $MP_*(X)$ is a $\mathbb{Z}$-graded comodule over it for all spaces X.

Recall that every Hopf algebroid $L \rightrightarrows W$ defines a functor from rings to groupoids by associating to a ring R the groupoid $\mathcal{P}_{(L,W)}(R)$ whose set of objects is $\operatorname{Hom}(L, R)$ and whose set of morphisms is $\operatorname{Hom}(W, R)$. The structure maps of $\mathcal{P}_{(L,W)}(R)$ (for example, the source and target maps $\operatorname{Hom}(W, R) \rightrightarrows \operatorname{Hom}(L, R)$), are induced by the structure maps of $L \rightrightarrows W$. We chose the notation $\mathcal{P}_{(L,W)}$ here, since the association

$$\operatorname{Spec}(R) \mapsto \mathcal{P}_{(L,W)}(R)$$

defines a prestack on *Aff*.

Now, consider the functor FGL from rings to groupoids that associates to a ring R the category of formal group laws over R and their isomorphisms; on morphisms, FGL is defined by pushforward. The following theorem is the fundamental link between formal groups laws and complex cobordism.

In the remainder of this section, we will use the short-hand $(L, W) := (MP_0, MP_0 MP)$ for the Hopf algebroid associated with periodic complex cobordism.

THEOREM 2.5 (Quillen, Landweber, Novikov). *The Hopf algebroid (L, W) corepresents the functor FGL, i.e. we have a natural isomorphism of functors $FGL \cong \mathcal{P}_{(L,W)}$.*

For a proof in the non-periodic case we refer to [**Ad**] and [**Ko**]. More information about the periodic case can be found in [**St**].

Translating the theorem into the language of stacks, we obtain the following result.

COROLLARY 2.6. *The stack of formal groups $\mathcal{M}_{FG}$ is equivalent to the the stack associated to the Hopf algebroid (L, W).*

PROOF. The theorem tells us that the prestack of formal group laws is isomorphic to $\mathcal{P}_{(L,W)}$. Note that there is a morphism from the stackification of the prestack of formal group laws to the stack of formal groups given by forgetting the choice of a (local) coordinate. Furthermore, since every formal group locally admits a coordinate, this morphism is an equivalence. This implies the claim. $\square$

We have seen that every even periodic cohomology theory gives rise to a formal group. Conversely, let us now try to associate to a formal group law F a cohomology theory. Recall that, according to Quillen's theorem, a formal group law over a ring R is the same as a ring homomorphism $F : L \to R$. One way to get a cohomology theory from F is to start with the cohomology theory MP^* corresponding to the universal formal group law and to tensor it with R, using the L-algebra structure on

R defined by F. The functor $R \otimes_L MP^*(\_)$ is again a cohomology theory, provided the exactness of the Mayer-Vietoris sequence is preserved.

We will describe Landweber's theorem in the homological setting, i.e. we address the question when the functor $R \otimes_L MP_*(\_)$ is a homology theory. This is certainly the case if R is flat over L. However, a much weaker condition will do. The point is that the L-modules $MP_*(X)$ that we tensor with R are of a very special type: as we remarked above, they are comodules over (L, W). This motivates the following definition.

Definition 2.7. A formal group law $F : L \to R$ is *Landweber exact* if the functor

$$M \mapsto R \otimes_L M$$

is exact on the category of (L, W)-comodules.

Using this language, we have: if F is a Landweber exact formal group law, then the functor

$$X \mapsto R \otimes_L MP_*(X)$$

defines a homology theory. We now formulate Landweber's theorem, which gives a useful characterization of Landweber exact formal group laws. For any prime p, write the p-series of the universal formal group law F_u over L, $[p]_{F_u}(x) := x +_{F_u} ... +_{F_u} x$ (summing p copies of x), as

$$[p]_{F_u}(x) =: \sum_{n \geq 1} a_n x^n$$

and let $v_i := a_{p^i}$ for all $i \geq 0$. Note that the definition of a formal group law implies $v_0 = p$. Define ideals $I_n \subset R$ by $I_n := v_0 R + ... + v_{n-1} R$ for all $n \geq 0$.

Theorem 2.8 (Landweber, 1973). *If for all primes p and for all integers $n \geq 0$ the map*

$$v_n : R/I_n \to R/I_n$$

is injective, then F is Landweber exact.

For the proof we refer to [**La**]. Note that Landweber only considers the functor $M \mapsto R \otimes_L M$ on the category of (L, W)-comodules that are finitely presented as L-modules. However, since every (L, W)-comodule is a direct limit of comodules of this type, the result holds in general, cf. Remark 9.6 in [**Mi**].

Examples 2.9.

(1) The additive formal group over $\mathbb{Z}$ is not Landweber exact. However, it is Landweber exact over $\mathbb{Q}$. The corresponding cohomology theory is ordinary cohomology with rational coefficients.
(2) The multiplicative formal group over $\mathbb{Z}$ is Landweber exact. The corresponding cohomology theory is complex K-theory.
(3) We will see that many formal group laws arising from elliptic curves are Landweber exact. They give rise to so-called elliptic cohomology theories. In the next section we will see that all these theories can be put together to form a presheaf of (co)homology theories on the moduli stack of elliptic curves.

3. Landweber exactness, stacks, and elliptic (co)homology

The goal of this section is to express Landweber's exactness condition in the language of stacks and to construct the presheaf of elliptic homology theories on the moduli stack of elliptic curves $\mathcal{M}_{ell}$. We will begin with some preliminary material about stacks with a focus on stacks arising from Hopf algebroids. In this section, (L, W) denotes a general Hopf algebroid, but the case we are mainly interested in is $(L, W) = (MP_0, MP_0MP)$, just as in Section 2.

REMARK 3.1. Fibered products[1] exist in the 2-category of stacks. Given morphisms of stacks $\mathcal{F} : \mathcal{M} \to \mathcal{N}$ and $\mathcal{G} : \mathcal{M}' \to \mathcal{N}$ we have a 2-categorical pullback diagram

$$\begin{array}{ccc} \mathcal{M} \times_{\mathcal{N}} \mathcal{M}' & \longrightarrow & \mathcal{M} \\ \Big\downarrow & & \Big\downarrow \mathcal{F} \\ \mathcal{M}' & \xrightarrow{\mathcal{G}} & \mathcal{N}, \end{array}$$

where the fibered product $\mathcal{M} \times_{\mathcal{N}} \mathcal{M}'$ is defined as follows. For an affine scheme U, the objects of the groupoid $\mathcal{M} \times_{\mathcal{N}} \mathcal{M}'(U)$ are given by triples (m, m', ϕ), where $m \in \mathcal{M}(U)$, $m' \in \mathcal{M}'(U)$, and ϕ is an isomorphism $\phi : \mathcal{F}(m) \cong \mathcal{G}(m')$. Morphisms $(m_1, m_1', \phi_1) \to (m_2, m_2', \phi_2)$ are pairs (ψ, ψ'), where $\psi : m_1 \to m_2$ and $\psi' : m_1' \to m_2'$ such that $\mathcal{G}(\psi')\phi_1 = \phi_2\mathcal{F}(\psi)$.

EXAMPLE 3.2. If the stacks involved come from Hopf algebroids, we can compute the fibered product in terms of a pushout of Hopf algebroids, i.e. we have a pullback square

$$\begin{array}{ccc} \mathcal{M}_{(S,T)} & \longrightarrow & \mathcal{M}_{(L_1,W_1)} \\ \Big\downarrow & & \Big\downarrow \\ \mathcal{M}_{(L_2,W_2)} & \longrightarrow & \mathcal{M}_{(L,W)}, \end{array}$$

where $S = L_1 \otimes_L W \otimes_L L_2$ and $T = W_1 \otimes_L W \otimes_L W_2$.

REMARK 3.3. We will frequently use the following fact. If $\mathcal{M}_{(L,W)}$ is a stack associated with a Hopf algebroid and $a : \operatorname{Spec} A \to \mathcal{M}_{(L,W)}$ is any morphism of stacks, then a factors *locally* through the canonical map $c : \operatorname{Spec} L \to \mathcal{M}_{(L,W)}$. This means that there exists a covering, i.e. a faithfully flat map $k : \operatorname{Spec} S \to \operatorname{Spec} A$, such that $ak : \operatorname{Spec} S \to \mathcal{M}_{(L,W)}$ factors through c. This follows from the definition of the stackification functor.

DEFINITION 3.4.

(1) A morphism $\mathcal{F} : \mathcal{M} \to \mathcal{N}$ is *representable* if for all morphisms $a : \operatorname{Spec} A \to \mathcal{N}$ from an affine scheme to $\mathcal{N}$ the fibered product $\mathcal{M} \times_{\mathcal{N}} \operatorname{Spec} A$ is equivalent to an affine scheme $\operatorname{Spec} P$.

(2) In this case, we call $\mathcal{F}$ *flat* (resp. a *covering*) if for all a the morphism of affine schemes $\mathcal{F}_a$ in the pullback diagram

$$\begin{array}{ccc} \operatorname{Spec} P & \longrightarrow & \mathcal{M} \\ \Big\downarrow \mathcal{F}_a & & \Big\downarrow \mathcal{F} \\ \operatorname{Spec} A & \xrightarrow{a} & \mathcal{N} \end{array}$$

is flat (resp. a covering).

[1] Also known as 'pullbacks', 'homotopy pullbacks', or '2-category pullbacks'.

Remark 3.5.

(1) In the case $\mathcal{N} = \mathcal{M}_{(L,W)}$ it suffices to check the representability of $\mathcal{F} : \mathcal{M} \to \mathcal{N}$ on the morphism $c : \operatorname{Spec} L \to \mathcal{M}_{(L,W)} = \mathcal{N}$. To see this, assume that $\mathcal{M} \otimes_{\mathcal{N}} \operatorname{Spec} L \cong \operatorname{Spec} P$ is affine. If the morphism $a : \operatorname{Spec} A \to \mathcal{N}$ factors through $\operatorname{Spec} L$, then $\mathcal{M} \otimes_{\mathcal{N}} \operatorname{Spec} A \cong \operatorname{Spec} P \otimes_{\operatorname{Spec} L} \operatorname{Spec} A$ is affine. In the general case, we can find a covering $k : \operatorname{Spec} S \to \operatorname{Spec} A$ such ak factors through c. Hence the pullback of $\mathcal{M}$ to $\operatorname{Spec} S$ is affine and it follows from descent theory for affine schemes that the same is true for the pullback of $\mathcal{M}$ to $\operatorname{Spec} A$.
(2) Similarly, we can check whether a representable morphism $\mathcal{F} : \mathcal{M} \to \mathcal{M}_{(L,W)}$ is flat (or a covering) by checking on c. Since the properties 'flat' and 'faithfully flat' are local for the flat topology, it suffices to prove that $\mathcal{F}_c$ (faithfully) flat implies that $\mathcal{F}_a$ is flat for any $a : \operatorname{Spec} A \to \mathcal{M}_{(L,W)}$ in the case in which a factors through c. In this case, however, this follows since 'flat' and 'faithfully flat' are stable under base change.

Examples 3.6.

(1) The description of the fibered product of stacks coming from Hopf algebroids given in Example 3.2 implies that every map $\operatorname{Spec} R \to \mathcal{M}_{(L,W)}$ is representable. In particular, it always makes sense to ask whether or not a map from an affine scheme to the stack $\mathcal{M}_{FG} \cong \mathcal{M}_{(MP_0, MP_0MP)}$ is flat.
(2) Let (L, W) be a Hopf algebroid and consider the canonical map $c : \operatorname{Spec} L \to \mathcal{M}_{(L,W)}$. From Example 3.2 we see that

$$\begin{tikzcd} \operatorname{Spec} W \arrow[r, "\eta_L"] \arrow[d, "\eta_R"] & \operatorname{Spec} L \arrow[d, "c"] \\ \operatorname{Spec} L \arrow[r, "c"] & \mathcal{M}_{(L,W)} \end{tikzcd}$$

is a pullback diagram. Hence, if η_L (or, equivalently, η_R) is a faithfully flat map, then $c : \operatorname{Spec} L \to \mathcal{M}_{(L,W)}$ is a covering. In particular, we see that $\operatorname{Spec} MP_0 \to \mathcal{M}_{FG}$ is a cover.

We will now define (pre)sheaves on stacks, more details can be found in [**Goe**]. Define the category $\mathit{Aff}/\mathcal{M}$ of affine schemes over the stack $\mathcal{M}$ as follows. The objects are simply morphisms $\mathcal{F} : \operatorname{Spec} R \to \mathcal{M}$. The morphisms, say from $\mathcal{F}_1$ to $\mathcal{F}_2$, are 2-commutative triangles, i.e. pairs (h, ϕ) consisting of a morphism $h : \operatorname{Spec} R_1 \to \operatorname{Spec} R_2$ and a 2-isomorphism ϕ between $\mathcal{F}_1$ and $\mathcal{F}_2 h$. We make $\mathit{Aff}/\mathcal{M}$ into a Grothendieck site by declaring a morphism (h, ϕ) to be a covering if h is a covering in Aff.

Definition 3.7. A *presheaf* on a stack $\mathcal{M}$ is a contravariant functor M on the category $\mathit{Aff}/\mathcal{M}$. A presheaf S on $\mathcal{M}$ is a *sheaf* if for every covering $V \to U$ in $\mathit{Aff}/\mathcal{M}$ we have an equalizer diagram

$$S(U) \to S(V) \rightrightarrows S(V \times_U V).$$

For example, the structure sheaf $\mathcal{O}_{\mathcal{M}}$ of the stack $\mathcal{M}$ is the sheaf of rings defined by

$$\mathcal{O}_{\mathcal{M}}(r : \operatorname{Spec} R \to \mathcal{M}) := R.$$

Sheaves of modules over $\mathcal{O}_{\mathcal{M}}$ are defined in the obvious way. The notion of quasi-coherent sheaves of modules over $\mathcal{O}_{\mathcal{M}}$ and the correspondence in the following proposition is explained in [**Goe**], Section 1.3; see also [**Nau**], Section 3.4. In the following we assume that $\eta_L : L \to W$ is faithfully flat, i.e. that $c : \operatorname{Spec} L \to \mathcal{M}_{(L,W)}$ is a covering.

PROPOSITION 3.8. *The category of comodules over* (L, W) *is equivalent to the category of quasi-coherent sheaves on* $\mathcal{M}_{(L,W)}$.

This implies that for every space X the comodule $MP_*(X)$ defines a quasi-coherent sheaf over the stack $\mathcal{M}_{FG}$.

For simplicity, we will call a sequence of quasi-coherent sheaves on $\mathcal{M}_{(L,W)}$ exact if the corresponding sequence of (L, W)-comodules is exact. By definition of the correspondence in Proposition 3.8 this is equivalent to asking that the pullback of the sequence to $\operatorname{Spec} L$ under $\operatorname{Spec} L \to \mathcal{M}_{(L,W)}$ is exact. Here and in the following the pullback functor is defined by composition: for $\mathcal{F} : \operatorname{Spec} R \to \mathcal{M}_{(L,W)}$ and a quasi-coherent sheaf S on $\mathcal{M}_{(L,W)}$ define the pullback by $\mathcal{F}^*S(r) := S(\mathcal{F}r)$.

PROPOSITION 3.9. *A morphism* $\mathcal{F} : \operatorname{Spec} R \to \mathcal{M}_{(L,W)}$ *is flat if and only if the pullback functor* $\mathcal{F}^*$ *from the category of quasi-coherent sheaves on* $\mathcal{M}_{(L,W)}$ *to the category of quasi-coherent sheaves on* $\operatorname{Spec} R$ *is exact.*

PROOF. Since both conditions are local with respect to the flat topology, we may assume that $\mathcal{F}$ factors through $\operatorname{Spec} L$ by Remark 3.3. Now consider the pullback diagram

$$\begin{tikzcd}
\operatorname{Spec} R \otimes_L W \arrow[r, "c_{\mathcal{F}}"] \arrow[d, "\mathcal{F}_c"] & \operatorname{Spec} R \arrow[d, "\mathcal{F}"] \\
\operatorname{Spec} L \arrow[r, "c"] & \mathcal{M}_{(L,W)}.
\end{tikzcd}$$

Since c is a covering the map $c_{\mathcal{F}}$ is faithfully flat. Hence $\mathcal{F}^*$ is exact if and only if $(\mathcal{F}c_{\mathcal{F}})^* = (c\mathcal{F}_c)^*$ is exact. By our definition of exact sequences on $\mathcal{M}_{(L,W)}$ this means precisely that $\mathcal{F}_c^*$ is exact on (L, W)-comodules. This is certainly true if $\mathcal{F}$ and hence $\mathcal{F}_c$ are flat. Conversely, if $\mathcal{F}^*$ is exact then the functor $N \mapsto \mathcal{F}^*N = R \otimes_L N$ is exact on (L, W)-comodules. Note that one can use the coalgebra structure of W to make every L-module of the form $W \otimes_L M$ (for some L-module M) into a (L, W)-comodule. Using this and the flatness of $\eta_R : L \to W$ it follows that the functor $M \mapsto R \otimes_L W \otimes_L M$ is exact on L-modules. Hence $\mathcal{F}_c$ is flat, i.e. $\mathcal{F}$ is flat. □

Recall that $\mathcal{M}_{FG} \cong \mathcal{M}_{(MP_0, MP_0MP)}$. Hence, given a formal group law $F : MP_0 \to R$, we have a corresponding morphism of stacks

$$\mathcal{F} : \operatorname{Spec} R \to \operatorname{Spec} MP_0 \xrightarrow{c} \mathcal{M}_{(MP_0, MP_0MP)} \cong \mathcal{M}_{FG}.$$

COROLLARY 3.10. *A formal group law* $F : MP_0 \to R$ *over* R *is Landweber exact if and only if the corresponding morphism* $\mathcal{F} : \operatorname{Spec} R \to \mathcal{M}_{FG}$ *is flat.*

COROLLARY 3.11. *Let* $\mathcal{F} : \operatorname{Spec} R \to \mathcal{M}_{FG}$ *be any flat morphism. Then the functor*

$$X \mapsto \mathcal{F}^* MP_*(X)$$

defines a homology theory.

Now we turn to the definition of the presheaf of elliptic homology theories on $\mathcal{M}_{ell}$. We use the following theorem that will be proved in Section 4.

THEOREM 3.12 (Hopkins, Miller). *The morphism $\mathfrak{F} : \mathcal{M}_{ell} \to \mathcal{M}_{FG}$ that maps a familiy $\mathcal{C}$ of elliptic curves to the associated formal group $\mathcal{F}_{\mathcal{C}}$ is flat.*

Theorem 3.12 implies that for any flat morphism $\mathcal{C} : \operatorname{Spec} R \to \mathcal{M}_{ell}$ the composition

$$\mathcal{F}_{\mathcal{C}} : \operatorname{Spec} R \to \mathcal{M}_{ell} \to \mathcal{M}_{FG}$$

is again a flat. By Corollary 3.11 we hence obtain a homology theory for every such $\mathcal{C}$.

DEFINITION 3.13. Given a flat map $\mathcal{C} : \operatorname{Spec} R \to \mathcal{M}_{ell}$, define by

$$Ell_*^{\mathcal{C}}(X) := \mathcal{F}_{\mathcal{C}}^* MP_*(X)$$

the *elliptic homology theory associated with the elliptic curve $\mathcal{C}$ over R.*

REMARK 3.14. Note that if $\mathcal{F}$ factors through the cover $\operatorname{Spec} L \to \mathcal{M}_{FG}$ then, by construction, $Ell^{\mathcal{C}}$ has coefficients $Ell_*^{\mathcal{C}}(pt) = R \otimes_L MP_*(pt) \cong R[u^{\pm 1}]$ and the associated cohomology theory is even periodic. For a general morphism $\mathcal{F}$, the theory $Ell_*^{\mathcal{C}}(\_)$ might merely be *weakly even periodic*, cf. [**Lu**], Remark 1.6.

The *presheaf of elliptic homology theories* is defined on the Grothendieck site of flat affine schemes over $\mathcal{M}_{ell}$. This is the sub-site of $Aff/\mathcal{M}_{ell}$ whose objects are flat morphisms $\mathcal{C} : \operatorname{Spec} R \to \mathcal{M}_{ell}$. The value of this presheaf on an object $\mathcal{C}$ is defined to be $Ell_*^{\mathcal{C}}$. In other words: for every space X we simply evaluate the quasi-coherent sheaf $\mathfrak{F}^* MP_*(X)$ on $\mathcal{C}$. On morphisms, we define the presheaf in the same way: for every space X and 2-commutative triangle (h, ϕ), say with $h : \mathcal{C}_1 \to \mathcal{C}_2$, we evaluate the sheaf $\mathfrak{F}^* MP_*(X)$ on (h, ϕ) to get an induced map

$$Ell_*^{\mathcal{C}_2}(X) \to Ell_*^{\mathcal{C}_1}(X).$$

This is nice and functorial since $\mathfrak{F}^* MP_*(X)$ is a sheaf on $\mathcal{M}_{ell}$. The work of defining the induced maps is hidden in Proposition 3.8, which allowed us to consider the comodule $MP_*(X)$ as a sheaf on $\mathcal{M}_{FG}$ (the explicit construction of the induced maps can be found in [**Goe**], Section 1.3).

4. The map $\mathcal{M}_{ell} \to \mathcal{M}_{FG}$ is flat

Recall that the moduli stack of elliptic curves $\mathcal{M}_{ell}$ is an open substack of the moduli stack of generalized elliptic curves $\mathcal{M}_{Weier}$. The latter is isomorphic to the stack associated with the Hopf algebroid (A, Γ), where $A := \mathbb{Z}[a_1, a_2, a_3, a_4, a_6]$ is the ring over which the universal Weierstrass curve lives and $\Gamma := A[u^{\pm 1}, r, s, t]$ parametrizes isomorphisms of Weierstrass curves. The open substack of elliptic curves $\mathcal{M}_{ell} \hookrightarrow \mathcal{M}_{Weier}$ is given by the locus where the discriminant Δ is invertible. Consequently, $\mathcal{M}_{ell} \cong \mathcal{M}_{(\tilde{A}, \tilde{\Gamma})}$, where $\tilde{A} := A[\Delta^{-1}]$ and $\tilde{\Gamma} := \Gamma[\Delta^{-1}]$. For more information about (generalized) elliptic curves, see Appendix B in [**AHS**] and the references therein.

In order to make sense of the statement that the map $\mathcal{M}_{ell} \to \mathcal{M}_{FG}$ is flat, we have to show that it is representable. It is easy to see that $\mathcal{M}_{ell} \hookrightarrow \mathcal{M}_{Weier}$ is representable and hence the representability of $\mathcal{M}_{ell} \to \mathcal{M}_{FG}$ is implied by the following theorem.

THEOREM 4.1. *The map $\mathcal{M}_{Weier} \to \mathcal{M}_{FG}$, $\mathcal{C} \mapsto \mathcal{F}_{\mathcal{C}}$, is representable.*

PROOF. We will construct a pullback diagram of prestacks

$$\begin{array}{ccccc}
\operatorname{Spec} A[b_4, b_5, ...] & \longrightarrow & \operatorname{Spec} A & \longrightarrow & \mathcal{P}_{(A,\Gamma)} \\
\downarrow & & \downarrow & & \downarrow \\
\operatorname{Spec} L & \longrightarrow & \mathcal{P}_{FG^{(5)}} & \longrightarrow & \mathcal{P}_{(L,W)}.
\end{array}$$

Stackifying this diagram yields the corresponding pullback diagram of the associated stacks.[2] It will be clear from the definition of the horizontal arrows in the bottom row that their composition is the usual cover $c : \operatorname{Spec} L \to \mathcal{M}_{FG}$. This implies the claim since the representability of $\mathcal{M}_{Weier} \to \mathcal{M}_{FG}$ can be checked on the cover c, see Remark 3.5.

The *prestack* $\mathcal{P}_{FG^{(5)}}$ *of formal group laws with a parameter modulo degree* 5 in the bottom row is corepresented by the Hopf algebroid

$$FG^{(5)} := (L^{(5)}, L^{(5)}[t_0^{\pm 1}, t_1, t_2, ...]),$$

where $L^{(5)} := L[u_1^{\pm 1}, u_2, u_3, u_4]$.

In order to complete the proof it suffices to show that the square on the right hand side is a pullback diagram. It then follows from the formula in Example 3.2 that the prestack in the top left of the diagram is isomorphic to an affine scheme (which turns out to be $\operatorname{Spec} A[b_4, b_5, ...]$).

We need the following lemma.

LEMMA 4.2. *Given a Weierstrass curve $\mathcal{C}$ over R with a local parameter u modulo degree* 5 *near the identity, there exists a unique Weierstrass curve $\tilde{\mathcal{C}}$ over R and a unique isomorphism*

$$\psi : \mathcal{C} \to \tilde{\mathcal{C}}$$

that maps u to the canonical local parameter x/y near the identity (modulo degree 5*).*

PROOF. We will show that there is a unique transformation ψ that maps u to $v := x/y$. $\tilde{\mathcal{C}}$ is then defined of be the image of $\mathcal{C}$ under ψ. Recall that isomorphisms of Weierstrass curves over R are given by transformations of 2-dimensional projective space over R that are of the form

$$M(u, s, r, t) = \begin{pmatrix} u^2 & 0 & r \\ su & u^3 & t \\ 0 & 0 & 1 \end{pmatrix},$$

see [**AHS**]. It is not difficult to compute that the following expansions hold near the identity:

$$\begin{array}{lcrcl}
U_u = M(u, 0, 0, 0) & \text{satisfies} & U_u(v) & = & u^{-1}v + O(v^2) \\
S_s = M(1, 0, s, 0) & \text{satisfies} & S_s(v) & = & v - sv^2 + O(v^3) \\
R_r = M(1, r, 0, 0) & \text{satisfies} & R_r(v) & = & v + rv^3 + O(v^4) \\
T_t = M(1, 0, 0, t) & \text{satisfies} & T_t(v) & = & v - tv^4 + O(v^5).
\end{array}$$

In order to prove the last two, one uses that $y^{-1} = v^3 + O(v^4)$ near the identity, see [**Si**], IV §1. Now, given any local parameter $u = u_1 v + u_2 v^2 + ...$ modulo degree 5, we can apply the transformation U_{u_1} to make $u_1 = 1$. Similarly, we can apply transformations S_s, R_r, and T_t to achieve $u_2 = u_3 = u_4 = 0$. The composition of

[2]Readers comfortable with stacks might prefer to argue directly on the level of stacks, using essentially the same argument we use for the prestacks, but working 'locally' whenever appropriate.

these transformations is the isomorphism ψ we wanted to construct. Furthermore, from the expansions for U_u, S_s, R_r, and T_t given above it is clear that the choices for u, s, r, and t are unique. Since any transformation $M(u, r, s, t)$ can be written as

$$M(u, r, s, t) = T_t R_r S_s U_u$$

for unique elements $t, r, s, u \in R$ this implies that ψ is unique. $\square$

Now we can explain the pullback square on the right hand side of the diagram. Recall that for an affine scheme $U = \operatorname{Spec} R$ the objects of the groupoid $\mathcal{P}_{(A,\Gamma)} \times_{\mathcal{P}_{(L,W)}} \mathcal{P}_{FG^{(5)}}(U)$ are quadruples $(\mathcal{C}, F, u, \phi)$, where $\mathcal{C} : A \to R$ is a Weierstrass curve, $F : L \to R$ is a formal group law, $u : \mathbb{Z}[u_1^{\pm 1}, u_2, u_3, u_4] \to R$ is a local parameter modulo degree 5 for F, and $\phi : F_{\mathcal{C}} \to F$ is an isomorphism of formal group laws. We claim that the functor

$$i_U : \operatorname{Spec} A\,(U) \to \mathcal{P}_{(A,\Gamma)} \times_{\mathcal{P}_{(L,W)}} \mathcal{P}_{FG^{(5)}}(U),\ \mathcal{C} \mapsto (\mathcal{C}, \mathcal{F}_{\mathcal{C}}, x/y, id),$$

is an equivalence of categories. To see this, note that the inclusion of the full subcategory of $\mathcal{P}_{(A,\Gamma)} \times_{\mathcal{P}_{(L,W)}} \mathcal{P}_{FG^{(5)}}(U)$ whose objects are of the form $(\mathcal{C}, F_{\mathcal{C}}, u, id)$ is obviously essentially surjective. This together with the existence of ψ in Lemma 4.2 shows that i_U is essentially surjective. Furthermore, the uniqueness of ψ means that i_U is fully faithful, so i_U is an equivalence of categories. The functors i_U fit together to give an equivalence of prestacks $i : \operatorname{Spec} A \to \mathcal{P}_{(A,\Gamma)} \times_{\mathcal{P}_{(L,W)}} \mathcal{P}_{FG^{(5)}}$. Thus we have constructed the pullback square on the right hand side. $\square$

Now that we have shown that $\mathcal{M}_{ell} \to \mathcal{M}_{FG}$ is representable, we are ready to prove that it is flat. The crucial ingredient is the following lemma.

Lemma 4.3 (Franke, Landweber, Ravenel, Stong). *The formal group law*

$$\tilde{F} : L \to \tilde{A}$$

of the universal smooth Weierstrass curve $\mathcal{C}_{univ}$ over $\tilde{A} := A[\Delta^{-1}]$ is Landweber exact.

Proof. We apply Theorem 2.8, the Landweber exact functor theorem. We have to check that for all primes p and integers $n \geq 0$, the map $v_n : \tilde{A}/I_n \to \tilde{A}/I_n$ is injective. This can be done as follows.

(1) Since $\tilde{A}$ is torsion-free, multiplication by $v_0 = p$ defines an injective endomorphism of $\tilde{A}$.

(2) Consider the endomorphism of $\tilde{A}/p\tilde{A} \cong \mathbb{F}_p[a_1, a_1, ..., a_6, \Delta^{-1}]$ given by v_1. Since $\tilde{A}/p\tilde{A}$ is an integral domain, v_1 is injective if and only if the image of v_1 in $\tilde{A}/p\tilde{A}$ is not zero. This, in turn, is equivalent to the condition that the mod p reduction of $\mathcal{C}_{\text{univ}}$ has a fiber that is not supersingular. This is certainly true; in fact, most elliptic curves are not supersingular.

(3) We finish the proof by showing that the image of v_2 in $\tilde{A}/(p\tilde{A} + v_1\tilde{A})$ is a unit. This shows that multiplication by v_2 is injective as required and that the conditions for all v_n, $n \geq 3$, are trivial, since the ring $\tilde{A}/(p\tilde{A} + v_1\tilde{A} + v_2\tilde{A})$ is zero. Assume the image of v_2 is not a unit in $\tilde{A}/(p\tilde{A} + v_1\tilde{A})$. Then there exists a maximal ideal $\mathfrak{m} \subset A$ containing (the images of) p, v_1, and v_2. Hence the quotient $\tilde{A}/\mathfrak{m}$ is a field (of characteristic p) and the reduction of $\mathcal{C}_{\text{univ}}$ defines an elliptic curve over $\tilde{A}/\mathfrak{m}$ whose associated formal group law has height > 2. This contradicts the fact that the height

of the formal group law of an elliptic curve over a field of characteristic $p > 0$ is either 1 or 2, see [**Si**], Chapter IV, Corollary 7.5. $\square$

Now we are ready to complete the *proof of Theorem 3.12*. Since the structure maps $\tilde{A} \rightrightarrows \tilde{\Gamma}$ of the Hopf algebroid $(\tilde{A}, \tilde{\Gamma})$ are faithfully flat, the canonical map $c : \operatorname{Spec} \tilde{A} \to \mathcal{M}_{ell}$ is a covering. Hence $\mathfrak{F} : \mathcal{M}_{ell} \to \mathcal{M}_{FG}$ is flat if and only the composition

$$\tilde{\mathcal{F}} : \operatorname{Spec} \tilde{A} \to \mathcal{M}_{ell} \to \mathcal{M}_{FG}$$

is flat. By Corollary 3.10, $\tilde{\mathcal{F}}$ is flat if and only if the formal group law $\tilde{F} : L \to \tilde{A}$ is Landweber exact, which was shown in Lemma 4.3. $\square$

REMARK 4.4. In the following chapters we will see that the presheaf *Ell* on $\mathcal{M}_{ell}$ can be turned into a sheaf of E_∞-ring spectra (if we replace the flat topology by the étale topology). The spectrum *TMF* is defined as the global sections of this sheaf. It can be shown that the presheaf *Ell* and the corresponding sheaf of E_∞-ring spectra extend to the Deligne-Mumford compactification $\bar{\mathcal{M}}_{ell}$ of $\mathcal{M}_{ell}$. The (-1)-connected cover of the global sections of this sheaf is *tmf*.

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Sheaves in Homotopy Theory

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Contents

Background and Motivation

In the chapter on the Landweber Exact Functor Theorem, we constructed a presheaf $\mathcal{O}^{\text{hom}}$ of homology theories on the moduli stack of elliptic curves, as follows. A map $f : \text{Spec}(R) \to \mathcal{M}_{ell}$ from an affine scheme to the moduli stack of elliptic curves provided an elliptic curve C over the ring R, and this elliptic curve had an associated formal group $\hat{C} : MP_0 = MU_* \to R$. Provided the map $f : \text{Spec}(R) \to \mathcal{M}_{ell}$ was flat, the functor $Ell_{C/R}(X) = MP_*(X) \otimes_{MP_0} R$ was a homology theory.

The value of the presheaf $\mathcal{O}^{\mathrm{hom}}$ on an elliptic curve was defined to be the homology theory associated to the formal group of that elliptic curve: $\mathcal{O}^{\mathrm{hom}}(f) = Ell_{C/R}$. Recall that such a presheaf is by definition simply a contravariant functor:

$$\begin{array}{ccc} (\text{Affine Schemes}/(\mathcal{M}_{ell}))^{\mathrm{op}} & \xrightarrow{\mathcal{O}^{\mathrm{hom}}} & \text{Homology Theories} \\ C/R & \longmapsto & Ell_{C/R} \end{array}$$

The presheaf $\mathcal{O}^{\mathrm{hom}}$ nicely encodes all the homology theories built from the formal groups of elliptic curves. The only problem is that there are many such theories, and they are related to one another in complicated ways. We would like instead a "global" or "universal" elliptic homology theory. The standard way of building a global object from a presheaf is of course to take global sections. Unfortunately, the site of affine schemes over the moduli stack of elliptic curves has no initial object, and therefore no notion of global sections. We would like to find a homology theory $\mathcal{O}^{\mathrm{hom}}(\mathcal{M}_{ell})$ associated to the whole moduli stack. One might guess that this homology theory should be the limit $\lim_{U\in\mathfrak{U}} \mathcal{O}^{\mathrm{hom}}(U)$ of the theories $\mathcal{O}^{\mathrm{hom}}(U)$, where $\mathfrak{U}$ is an affine cover of the moduli stack. The category of homology theories is not complete, though, and this limit does not exist.

Thanks to Brown representability, we know that homology theories can be represented by spectra. The category of spectra is rather better behaved than the category of homology theories—for instance, it has limits and their homotopically meaningful cousins, homotopy limits. If we can show that the presheaf $\mathcal{O}^{\mathrm{hom}}$ is the presheaf of homology theories associated to a presheaf $\mathcal{O}^{\mathrm{top}}$ of spectra, then we can build a global spectrum, thus have a global homology theory, using a homotopy limit construction. The main theorem is that there is indeed an appropriate presheaf of spectra:

Theorem 0.1 (Goerss-Hopkins-Miller). *There exists a sheaf $\mathcal{O}^{\mathrm{top}}$ of E_∞ ring spectra on $(\mathcal{M}_{ell})_{\acute{e}t}$, the moduli stack of elliptic curves in the étale topology, whose associated presheaf of homology theories is the presheaf $\mathcal{O}^{\mathrm{hom}}$ built using the Landweber Exact Functor Theorem.*

That $\mathcal{O}^{\mathrm{top}}$ is a sheaf and not merely a presheaf entails, for example, that its value $\mathcal{O}^{\mathrm{top}}(\mathcal{M}_{ell})$ on the whole moduli stack is determined as a homotopy limit $\mathrm{holim}_{U\in\mathfrak{U}}\, \mathcal{O}^{\mathrm{top}}(U)$ of its value on the open sets in a cover $\mathfrak{U}$ of the moduli stack. The spectrum $\mathcal{O}^{\mathrm{top}}(\mathcal{M}_{ell})$ represents the homology theory we were hunting for, and warrants a special name:

Definition 0.2. $TMF := \mathcal{O}^{\mathrm{top}}(\mathcal{M}_{ell})$.

The first goal of this chapter is to explain what it means to have a sheaf of E_∞ ring spectra on the moduli stack of elliptic curves. Note that we would have been happy with a sheaf of (not necessarily E_∞ ring) spectra. That the theorem produces a sheaf of E_∞ ring spectra is an artifact of the ingenious proof: it turns out to be easier to handle the obstruction theory for sheaves of E_∞ ring spectra than the obstruction theory for sheaves of ordinary spectra.

Once we have the sheaf $\mathcal{O}^{\mathrm{top}}$, we would like to understand the global homology theory TMF. In particular, we would like to compute the coefficient ring $TMF_* = \pi_*(\mathcal{O}^{\mathrm{top}}(\mathcal{M}_{ell}))$. The spectrum $\mathcal{O}^{\mathrm{top}}(\mathcal{M}_{ell})$ is, as described above, built as a homotopy limit out of smaller pieces $\{\mathcal{O}^{\mathrm{top}}(U)\}_{U\in\mathfrak{U}}$. There is a spectral sequence that computes the homotopy groups of the homotopy limit $\mathcal{O}^{\mathrm{top}}(\mathcal{M}_{ell})$ in terms of the homotopy groups of the pieces $\mathcal{O}^{\mathrm{top}}(U)$. The E_2 term of this spectral sequence is conveniently expressed in terms of the sheaf cohomology of the sheafification of the presheaf on $\mathcal{M}_{ell}$ given by $U \mapsto \pi_*(\mathcal{O}^{\mathrm{top}}(U))$.

PROPOSITION 0.3. *There is a strongly convergent spectral sequence*

$$E_2 = H^q(\mathcal{M}_{ell}, \pi_p^\dagger \mathcal{O}^{\mathrm{top}}) \Longrightarrow \pi_{p-q} TMF.$$

Here $\pi_p^\dagger \mathcal{O}^{\mathrm{top}}$ *is the sheafification of the presheaf* $\pi_p \mathcal{O}^{\mathrm{top}}$.

The second goal of this chapter is to construct this "descent spectral sequence". In the chapter on the homotopy of TMF we will evaluate the E_2 term of this and related spectral sequences, and illustrate how one computes the numerous differentials.

In the following section 1, we review the classical notion of sheaves, discuss a homotopy-theoretic version of sheaves, and describe stacks as sheaves, in this homotopy sense, of groupoids. Then in section 2 we describe what it means to have a sheaf on a stack and recall the notion of homotopy limit needed to make sense, in particular, of sheaves of spectra. In the final section 3, we discuss sheaf cohomology and Cech cohomology on a stack and construct the descent spectral sequence for a sheaf of spectra on a stack.

In writing this chapter, we have benefited enormously from discussions with Andrew Blumberg and Andre Henriques, and from reading work of Dan Dugger, Phil Hirschhorn, Paul Goerss and Rick Jardine, and, of course, Mike Hopkins.

1. Sheaves and Stacks

A sheaf S of sets on a space X is a way of functorially associating a set $S(U)$ to each open subset U of X. It is natural to generalize this notion in two directions, by considering sheaves of objects besides sets and by considering sheaves on objects besides spaces. Stacks are, at root, representing objects for moduli problems in algebraic geometry, and as such might seem to have little to do with sheaves. However, stacks can naturally be viewed as sheaves of groupoids on the category of schemes, and this perspective is useful when discussing, as we will in section 2, sheaves on a stack.

1.1. Sheaves. We begin with the classical notions of presheaves and sheaves of sets on a space.

DEFINITION 1.1. Given a space X, let $\mathcal{X}$ denote the category whose objects are open subsets U of X and whose morphisms are inclusions $U \hookrightarrow V$ of open subsets. A presheaf of sets S on the space X is a contravariant functor from the category $\mathcal{X}$ to the category Set of sets.

Explicitly, the presheaf provides a set $S(U)$ for each open U and a restriction map $S(V) \xrightarrow{r_{VU}} S(U)$ for each inclusion $U \hookrightarrow V$ such that the composite $S(W) \xrightarrow{r_{WV}} S(V) \xrightarrow{r_{VU}} S(U)$ is the restriction map r_{WU}. The prototypical example is the presheaf of real valued functions on the space: $S(U) = \mathrm{Map}(U, \mathbb{R})$; here the restriction maps are restriction of functions. This presheaf has the special property that an element of $S(U)$, that is a function, is uniquely determined by its restriction to any open cover of U by smaller open sets $\{U_i \hookrightarrow U\}_{i \in I}$—such a presheaf is called a sheaf.

DEFINITION 1.2. A sheaf of sets on a space is a presheaf of sets S on a space X such that for all open sets $U \subset X$ and all open covers $\{U_i \hookrightarrow U\}_{i \in I}$ of U, the set $S(U)$ is given by the following limit:

$$S(U) = \lim \left(\prod_i S(U_i) \rightrightarrows \prod_{i,j} S(U_{ij}) \overset{\rightarrow}{\rightrightarrows} \prod_{i,j,k} S(U_{ijk}) \overset{\rightrightarrows}{\rightrightarrows} \cdots \right)$$

Here the intersection $U_i \cap U_j$ is denoted U_{ij}, the triple intersection $U_i \cap U_j \cap U_k$ is denoted U_{ijk}, and so forth. To be clear, the products above occur over unordered tuples of not-necessarily-distinct elements of the indexing set I, and the diagram indexing the limit is the full standard cosimplicial diagram.

REMARK 1.3. We emphasize that the limit diagram in this definition does contain codegeneracy maps, despite their frequent omission from the notation. For example, if the index set has order two, then the limit, written out, is $\lim \Bigg(S(U_1) \times S(U_2) \overset{\rightarrow}{\underset{\rightarrow}{\leftarrow}} S(U_{12}) \times S(U_{11}) \times S(U$

not $\lim \Big(S(U_1) \times S(U_2) \rightrightarrows S(U_{12}) \overset{\rightarrow}{\rightrightarrows} * \overset{\rightrightarrows}{\rightrightarrows} \cdots \Big)$.

REMARK 1.4. The classical definition of a sheaf demands that $S(U)$ be the limit $\lim \Big(\prod_i S(U_i) \rightrightarrows \prod_{i,j} S(U_{ij}) \Big)$. This truncated limit is equal to the limit in definition 1.2. However, only the full limit generalizes well when we consider sheaves of objects other than sets.

We can define a presheaf of sets on a category $\mathcal{C}$, not necessarily the category of open subsets of a space, simply as a contravariant functor from $\mathcal{C}$ to Set. Moreover we can give a definition of sheaves of sets on $\mathcal{C}$ provided we have a notion of covers in the category. A Grothendieck topology on a category $\mathcal{C}$ provides such a notion:

DEFINITION 1.5. A Grothendieck topology on a category $\mathcal{C}$ is a collection of sets of morphisms $\{\{U_i \to U\}_{i \in I}\}$; these sets of morphisms are called covering families. The collection of covering families is required to satisfy the following axioms: 1) $\{f : V \to U\}$ is a covering family if f is an isomorphism; 2) if $\{U_i \to U\}_{i \in I}$ is a covering family, and $g : V \to U$ is a morphism, then $\{g^* U_i \to V\}$ is a covering family; 3) if $\{U_i \to U\}$ is a covering family and $\{V_{ij} \to U_i\}$ is a covering family for each i, then $\{V_{ij} \to U\}$ is a covering family. A pair of a category $\mathcal{C}$ and a Grothendieck topology on $\mathcal{C}$ is called a Grothendieck site.

The basic example of a Grothendieck site is of course the category of open subsets of a space, with morphisms inclusions, together with covering families the sets $\{U_i \to U\}$ where $\{U_i\}$ is an open cover of U. More interesting are the various Grothendieck topologies on the category Sch of schemes. For example, in the étale (respectively flat) topology, the covering families are the sets $\{U_i \to U\}$ such that $\coprod_i U_i \to U$ is an étale (respectively flat) covering map. A sheaf on a Grothendieck site $\mathcal{C}$ is of course a presheaf $S : \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$ such that for all covering families $\{U_i \to U\}$ of the site, the set $S(U)$ is the limit $\lim \Big(\prod_i S(U_i) \rightrightarrows \prod_{i,j} S(U_{ij}) \overset{\rightarrow}{\rightrightarrows} \cdots \Big)$.

Next we consider sheaves on a Grothendieck site $\mathcal{C}$ taking values in a category $\mathcal{D}$ other than sets. We are interested in categories $\mathcal{D}$ that have some notion of homotopy theory—these include the categories of groupoids, spaces, spectra, and E_∞ ring spectra. More specifically, we need the category $\mathcal{D}$ to come equipped with a notion of homotopy limits and a notion of weak equivalences. We will discuss homotopy limits in detail in section 2.3. For now we content ourselves with a brief example illustrating the idea that homotopy limits in, for example, spaces behave like limits with a bit of homotopical wiggle room:

EXAMPLE 1.6. Suppose we are interested in the diagram of spaces $X \rightrightarrows Y$, where the two maps are f and g. The limit of this diagram is the space of points of X whose image in Y is the same under the two maps: $\lim(X \rightrightarrows Y) = \{x \in X \text{ s.t. } f(x) = g(x)\}$. The homotopy limit, by contrast, only expects the two images to be the same up to chosen homotopy:

$$\operatorname{holim}(X \rightrightarrows Y) = \{(x \in X, h_x : [0,1] \to Y) \text{ s.t. } h_x(0) = f(x), h_x(1) = g(x)\}.$$

A presheaf on the site $\mathcal{C}$ with values in the category $\mathcal{D}$ is a contravariant functor $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$. These presheaves are also referred to as presheaves of objects of $\mathcal{D}$ on the site $\mathcal{C}$: for example, "presheaves of sets", "presheaves of spaces", "presheaves of spectra".

DEFINITION 1.7. A sheaf on the site $\mathcal{C}$ with values in the category $\mathcal{D}$ is a presheaf F such that for all objects U of $\mathcal{C}$ and all covers $\{U_i \to U\}_{i \in I}$, the map

$$F(U) \xrightarrow{\simeq} \operatorname{holim}\left(\prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_{ij}) \substack{\rightarrow\\\rightarrow\\\rightarrow} \prod_{i,j,k} F(U_{ijk}) \substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow} \cdots\right)$$

is a weak equivalence. The products here occur over unordered tuples of not-necessarily-distinct elements of the indexing set I; in particular, the indexing diagram does contain codegeneracy maps.

We will be particularly interested in the case where $\mathcal{C}$ is the étale site $(\mathcal{M}_{ell})_{\text{ét}}$ on the moduli stack of elliptic curves and $\mathcal{D}$ is the category of E_∞ ring spectra. We describe this particular site in section 2.2 and discuss homotopy limits of (E_∞ ring) spectra in section 2.3.

1.2. Stacks. A scheme X represents a functor $\mathsf{Sch}^{\mathrm{op}} \to \mathsf{Set}$ by $Y \mapsto \operatorname{Hom}(Y, X)$. A moduli problem, such as "What are the elliptic curves over a scheme?", also associates a set to each scheme, for example by $Y \mapsto \{\text{ell curves}/Y\}/\text{iso}$. Unfortunately, many such moduli problems are not representable by schemes. To manage this situation, we keep track of not just the moduli set but the moduli groupoid. We therefore consider, for example, the association taking $Y \in \mathsf{Sch}$ to $\{\text{ell curves}/Y, \text{with isoms}\} \in \mathsf{Gpd}$. This association is very nearly a presheaf of groupoids on the category of schemes; (it is not a presheaf because pullback is only functorial up to isomorphism.) It moreover has the sheaf-like property that elliptic curves over a scheme Y can be built by gluing together elliptic curves on a cover of Y. Altogether, this suggests that sheaves of groupoids are a reasonable model for studying moduli problems.

DEFINITION 1.8. A stack on the site $\mathcal{C}$ is a sheaf of groupoids on $\mathcal{C}$.

Recall that this definition means that for a presheaf $F : \mathcal{C}^{\mathrm{op}} \to \mathsf{Gpd}$ to be a stack, the map $F(U) \to \operatorname{holim}\left(\prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_{ij}) \substack{\rightarrow\\\rightarrow\\\rightarrow} \cdots\right)$ must be a weak equivalence for all covers $\{U_i \to U\}$. In order to unpack this condition, we need to know what the weak equivalences and the homotopy limits are in the category of groupoids. A weak equivalence of groupoids is simply an equivalence of categories. The following proposition identifies the needed homotopy limit.

Proposition 1.9 ([**Hol**]). *The homotopy limit of groupoids* $\operatorname{holim}\left(\prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_{ij}) \overset{\to}{\underset{\to}{\to}} \cdots\right)$ *associated to a cover* $\mathfrak{U} = \{U_i \to U\}$ *is the groupoid* $\operatorname{Desc}(F, \mathfrak{U})$ *defined as follows. The objects of* $\operatorname{Desc}(F, \mathfrak{U})$ *are collections of objects* a_i *in* $\operatorname{ob}(F(U_i))$ *and morphisms* $\alpha_{ij} : a_i|_{U_{ij}} \to a_j|_{U_{ij}}$ *in* $\operatorname{mor}(F(U_{ij}))$ *such that* $\alpha_{jk}\alpha_{ij} = \alpha_{ik}$*. The morphisms in* $\operatorname{Desc}(F, \mathfrak{U})$ *from* $\{a_i, \alpha_{ij}\}$ *to* $\{b_i, \beta_{ij}\}$ *are collections of morphisms* $m_i : a_i \to b_i$ *in* $\operatorname{mor}(F(U_i))$ *such that* $\beta_{ij} m_i = m_j \alpha_{ij}$*.*

Stacks are often defined to be presheaves of groupoids such that the natural map $F(U) \to \operatorname{Desc}(F, \mathfrak{U})$ is an equivalence of categories for all covers $\mathfrak{U}$ of U. The above proposition establishes that the more conceptual homotopy limit definition agrees with the descent definition.

Example 1.10. We will be concerned primarily with the moduli stack of elliptic curves $\mathcal{M}_{ell}$. This is a stack on the category of schemes in any of the flat, étale, or Zariski topologies. Roughly speaking, the stack associates to a scheme Y the groupoid of elliptic curves over Y. (Precisely, this association must be slightly rigidified, a la [**Hop**, p.26].) Alternately, $\mathcal{M}_{ell}$ is the stack associated to the Hopf algebroid $(A, \Gamma) := (\mathbb{Z}[a_1, a_2, a_3, a_4, a_6][\Delta^{-1}], A[u^{\pm 1}, r, s, t][\Delta^{-1}])$.

2. Sheaves on Stacks

We would like to understand sheaves of spectra on the moduli stack of elliptic curves in the étale topology. First we consider the general notion of a sheaf on a stack, then describe the étale site of the moduli stack of elliptic curves, and finally discuss sheaves of spectra in particular.

2.1. The Site of a Stack.

Suppose $\mathcal{C}$ is a Grothendieck site and X is an object of $\mathcal{C}$. We have a notion of sheaves on X, which are by definition sheaves on the site $\mathcal{C}/X$ whose objects are maps $Y \to X$ in $\mathcal{C}$ and whose covers are inherited from $\mathcal{C}$. We would like a notion of sheaves on a stack $\mathcal{M}$ on the site $\mathcal{C}$. In order to consider objects of $\mathcal{C}$ over $\mathcal{M}$, we need objects of $\mathcal{C}$ to live in the same place as stacks on $\mathcal{C}$. This is accomplished by the following functors:

$$\begin{aligned} \mathcal{C} &\hookrightarrow \operatorname{Pre}\mathcal{C} \hookrightarrow \operatorname{Pre}_{\mathsf{Gpd}}\mathcal{C} \\ U &\mapsto \operatorname{Hom}(-, U) \mapsto \operatorname{Hom}(-, U) \text{ with id} \end{aligned}$$

Thinking of object of $\mathcal{C}$ as the presheaves of groupoids they represent, we can consider the site of objects over $\mathcal{M}$.

Definition 2.1. Let $\mathcal{M}$ be a stack on the site $\mathcal{C}$. The site $\mathcal{C}/\mathcal{M}$ has objects the morphisms $U \to \mathcal{M}$ in presheaves of groupoids on $\mathcal{C}$. The morphisms in $\mathcal{C}/\mathcal{M}$ from $U \xrightarrow{a} \mathcal{M}$ to $V \xrightarrow{b} \mathcal{M}$ are the pairs (c, ϕ) where $c : U \to V$ is a morphism of $\mathcal{C}$ and ϕ is a natural isomorphisms between the functors a and bc. The covering families of $U \to \mathcal{M}$ in $\mathcal{C}/\mathcal{M}$ are the sets of morphisms $\{c_i : U_i \to U, \phi_i\}$ such that

$\{c_i\}$ is a covering family in $\mathcal{C}$. Schematically, the definition is

$$\mathcal{C}/\mathcal{M} = \begin{cases} \text{obj} \;=\; \{U \to \mathcal{M} \text{ in } \mathrm{Pre}_{\mathsf{Gpd}}\,\mathcal{C}\} \\ \text{mor} = \left\{ \begin{tikzcd} U \arrow[dd, "c"'] \arrow[dr, "a"] & \\ \arrow[r, phantom, "\overset{\phi}{\Swarrow}"] & \mathcal{M} \\ V \arrow[ur, "b"'] & \end{tikzcd} \right\} \\ \text{cov} \;=\; \left\{ \left\{ \begin{tikzcd} U_i \arrow[dd] \arrow[dr] & \\ \arrow[r, phantom, "\Swarrow"] & \mathcal{M} \\ U \arrow[ur] & \end{tikzcd} \right\} \text{ s.t. } \left\{ \begin{tikzcd} U_i \arrow[d] \\ U \end{tikzcd} \right\} \text{ cov in } \mathcal{C} \right\} \end{cases}$$

Definition 2.2. For a stack $\mathcal{M}$ on the site $\mathcal{C}$, a sheaf on $\mathcal{M}$ with values in $\mathcal{D}$ is a sheaf on $\mathcal{C}/\mathcal{M}$ with values in $\mathcal{D}$.

For example, we might consider $\mathcal{M}_{ell}$ as a stack on schemes in the étale topology $\mathsf{Sch}_{\text{ét}}$ and then consider sheaves on $\mathcal{M}_{ell}$ with values in the category of spectra. However, in the end this is not the notion of sheaves on the moduli stack that we want, so we need to modify the site $\mathsf{Sch}_{\text{ét}}/\mathcal{M}_{ell}$.

2.2. The site $(\mathcal{M}_{ell})_{\text{ét}}$. We adjust the site $\mathsf{Sch}_{\text{ét}}/\mathcal{M}_{ell}$ in two ways. First, to enable later obstruction theory arguments we need to restrict the objects of our site to be étale, not arbitrary, maps to the moduli stack. Second, it will be convenient if our sheaves take values not only on schemes over the moduli stack but also on stacks over the moduli stack; in particular we will then be able to evaluate a sheaf on the moduli stack itself, producing a spectrum of global sections.

We will be interested in étale maps between stacks and étale covers of stacks. These notions are derived from the corresponding notions for schemes. Recall that a map $X \to Y$ between schemes is étale if it is flat and unramified or equivalently smooth of relative dimension zero. The topologist can think of étale maps as being the algebro-geometric analog of local homeomorphisms. A collection of étale maps $\{U_i \to U\}$ is an étale cover if for all algebraically closed fields k and all maps $f : \operatorname{Spec} k \to U$ there exists an i with a lift of f to a map $\tilde{f} : \operatorname{Spec} k \to U_i$:

$$\left\{ \begin{tikzcd} U_i \arrow[d] \\ U \end{tikzcd} \right\} \text{ étale cover if } \begin{tikzcd} & U_i \arrow[d] \\ \operatorname{Spec} k \arrow[ur, dashed, "\exists \tilde{f}"] \arrow[r, "\forall f"'] & U \end{tikzcd}$$

Note that these collections are the covers in the site $\mathsf{Sch}_{\text{ét}}$.

Definition 2.3. A map of stacks $f : \mathcal{N} \to \mathcal{M}$ is étale if for all maps $V \to \mathcal{M}$ from a scheme to $\mathcal{M}$, the pullback f^*V is a scheme and the induced map $f^*V \to V$ is étale:

$$\mathcal{N} \xrightarrow{f} \mathcal{M} \text{ étale if } \begin{tikzcd} f^*V \arrow[r, "\boxed{\text{ét}}"] \arrow[d] & V \arrow[d, "\forall"] \\ \mathcal{N} \arrow[r, "f"'] & \mathcal{M} \end{tikzcd}$$

Étale maps $\mathcal{N} \to \mathcal{M}_{ell}$ to the moduli stack of elliptic curves will be the objects of the étale site of the moduli stack. Roughly, the morphisms are maps of stacks $\mathcal{N}' \to \mathcal{N}$ over the moduli stack, and covers are collections of maps $\{\mathcal{N}_i \to \mathcal{N}\}$ over the moduli stack that are étale covers in their own right. (A collection of maps of stacks is an étale cover if it satisfies a lifting property precisely analogous to the one for étale covers of schemes.) In more detail, the étale site is defined as follows. Note that we are considering all stacks, the moduli stack $\mathcal{M}_{ell}$ included, as stacks on the étale site of schemes $\mathsf{Sch}_{\text{ét}}$.

Definition 2.4. The objects of the étale site of the moduli stack of elliptic curves $(\mathcal{M}_{ell})_{\text{ét}}$ are the étale morphisms $\mathcal{N} \to \mathcal{M}_{ell}$ from a stack $\mathcal{N}$ to the moduli stack. The morphisms from $\mathcal{N} \xrightarrow{a} \mathcal{M}_{ell}$ to $\mathcal{N}' \xrightarrow{b} \mathcal{M}_{ell}$ are equivalence classes of pairs (c, ϕ), where $c : \mathcal{N} \to \mathcal{N}'$ is a map of stacks and ϕ is a natural isomorphism between a and bc. A natural isomorphism ψ from $c : \mathcal{N} \to \mathcal{N}'$ to $d : \mathcal{N} \to \mathcal{N}'$ can be viewed as an isomorphism from the pair (c, ϕ) to the pair $(d, \psi\phi)$—these two pairs are therefore considered equivalent as morphisms between $\mathcal{N} \to \mathcal{M}_{ell}$ and $\mathcal{N}' \to \mathcal{M}_{ell}$. A collection of morphisms $\{[(c_i : \mathcal{N}_i \to \mathcal{N}, \phi_i)]\}$ of stacks over $\mathcal{M}_{ell}$ is a cover of $(\mathcal{M}_{ell})_{\text{ét}}$ if for all algebraically closed fields k and all maps $f : \operatorname{Spec} k \to \mathcal{N}$, there exists an i, a representative (c, ϕ) of the equivalence class $[(c_i, \phi_i)]$, and a lift of f to a map $\tilde{f} : \operatorname{Spec} k \to \mathcal{N}_i$ such that $f = c\tilde{f}$. Schematically, we have

$$
(\mathcal{M}_{ell})_{\text{ét}} = \begin{cases}
\text{obj} \; = \; \{\mathcal{N} \xrightarrow{\text{ét}} \mathcal{M}_{ell}\} \\
\text{mor} = \left\{ \begin{tikzcd}[row sep=small] \mathcal{N} \arrow[dr, "a"] \arrow[dd, "c"'] & \\ \arrow[r, phantom, "\overset{\phi}{\Swarrow}"] & \mathcal{M}_{ell} \\ \mathcal{N}' \arrow[ur, "b"'] & \end{tikzcd} \right\} \Big/ \left\{ \begin{tikzcd} \mathcal{N} \arrow[d, bend right, "d"'] \arrow[d, bend left, "c"] \arrow[d, phantom, "\overset{\psi}{\Leftarrow}"] \\ \mathcal{N}' \end{tikzcd} \right\} \\
\text{cov} \; = \left\{ \left\{ \begin{tikzcd}[row sep=small] \mathcal{N}_i \arrow[dr] \arrow[dd, "c_i"'] & \\ \arrow[r, phantom, "\overset{\phi_i}{\Swarrow}"] & \mathcal{M}_{ell} \\ \mathcal{N} \arrow[ur] & \end{tikzcd} \right\} \text{ s.t. } \begin{tikzcd} & \mathcal{N}_i \arrow[d] \\ \operatorname{Spec} k \arrow[ur, dashed, "\exists \tilde{f}"] \arrow[r, "\forall f"'] & \mathcal{N} \end{tikzcd} \right\}
\end{cases}
$$

Though the definition of this étale site $(\mathcal{M}_{ell})_{\text{ét}}$ is complicated by the introduction of stacks over the moduli stack, the site $(\mathcal{M}_{ell})_{\text{ét}}$ still contains the fundamental objects of study, namely elliptic curves over schemes; sheaves on $(\mathcal{M}_{ell})_{\text{ét}}$ should be thought of primarily as assignments of sets (or, in a moment, spectra) to these elliptic curves.

We are hunting for a sheaf of spectra $\mathcal{O}^{\text{top}}$ on the étale site $(\mathcal{M}_{ell})_{\text{ét}}$ of the moduli stack of elliptic curves. Such a sheaf was defined in section 1.1 as a presheaf $F : (\mathcal{M}_{ell})_{\text{ét}}^{\text{op}} \to \mathsf{Spec}$ such that, for all objects U of $(\mathcal{M}_{ell})_{\text{ét}}$, the natural map

$$F(U) \to \operatorname{holim} \left(\prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_{ij}) \overset{\rightarrow}{\rightrightarrows} \prod_{i,j,k} F(U_{ijk}) \overset{\rightrightarrows}{\rightrightarrows} \cdots \right)$$

is a weak equivalence. We need therefore to understand in detail the notion of homotopy limits in the category of spectra.

2.3. Homotopy Limits and Sheaves of Spectra. A sheaf of sets on a space X is a functor $F : \{U \subset X\}^{\text{op}} \to \mathsf{Set}$ whose value on large open sets is determined as a limit of the value on smaller open sets. We could take a similar definition for sheaves of spectra $F : \{U \subset X\}^{\text{op}} \to \mathsf{Spec}$, but the limit condition ignores the topological structure of spectra, and the value of the sheaf on large open sets would not capture any information about the topological behavior of the sheaf on small open sets. Instead of "gluing" the values of F together with a limit, we glue them together with a homotopy limit. The homotopy limit takes the various values of F

and thickens them up with a bit of padding, so that they aren't too badly damaged, homotopically speaking, by the gluing process.

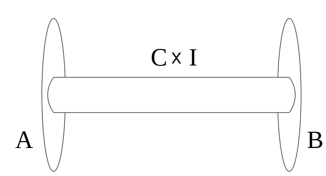
In section 2.3.1, we describe colimits and limits in terms of tensors and cotensors, and use this framework to give a concise description of homotopy colimits and limits. In section 2.3.2, we fess up to the fact that even the homotopy limit is not always appropriately homotopy invariant, and this leads us into a discussion of derived limits and corrected homotopy limits. In section 2.3.3, we specialize to the case of spectra, describing the categories of orthogonal spectra, symmetric spectra, and S-modules and specifying the tensors and cotensors needed for homotopy colimits and limits in these categories.

2.3.1. *Limits and Homotopy Limits.* Limit and colimit are brutal operations in the category of spaces: they tend to destroy homotopical information, and they are not invariant under homotopies of maps. For example, the colimit of the diagram $* \leftarrow S^2 \rightarrow *$ is a point and has no recollection of the homotopy type of the middle space S^2; the limit of the diagram $* \rightrightarrows [0,1]$, where both maps send the point to 0, is also a point, but becomes empty if we deform one of the two maps away from 0.

We would like homotopy versions of limit and colimit that have more respect for the homotopical structure of spaces. We take our cue from two fundamental examples.

Example 2.5. The colimit of the diagram of spaces $A \xleftarrow{f} C \xrightarrow{g} B$ is $(A \sqcup B)/(f(c) \sim g(c), c \in C)$. We can homotopify this construction by, instead of directly identifying $f(c)$ and $g(c)$, putting a path between them. This is the double mapping cylinder construction and is an example of a homotopy colimit:

$$\begin{aligned}\operatorname{hocolim}(A \xleftarrow{f} C \xrightarrow{g} B) &= \operatorname{colim}(A \leftarrow C \rightarrow C \times [0,1] \leftarrow C \rightarrow B) \\ &= (A \sqcup C \times [0,1] \sqcup B)/\{(c,0) \sim f(c), (c,1) \sim g(c)\}.\end{aligned}$$

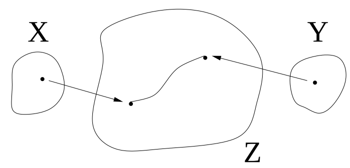


The suspension functor is a special case of this homotopy colimit, when $A = B = *$.

Example 2.6. The limit of the diagram of spaces $X \xrightarrow{f} Z \xleftarrow{g} Y$ is $\{(x,y) \in X \times Y | f(x) = g(y)\}$. Instead of expecting $f(x)$ and $g(y)$ to be equal in this limit, we can merely demand that they be connected by a chosen path. This is the double path space construction and is a homotopy limit:

$$\begin{aligned}\operatorname{holim}(X \xrightarrow{f} Z \xleftarrow{g} Y) &= \lim\left(X \rightarrow Z \leftarrow Z^{[0,1]} \rightarrow Z \leftarrow Y\right) \\ &= \{(x, m : [0,1] \rightarrow Z, y) | f(x) = m(0), g(y) = m(1)\}.\end{aligned}$$

The loop functor is the special case of $X = Y = *$ and $f = g$.



In order to generalize these homotopical constructions to other colimits and limits, it is convenient to have a concise description of the colimit and limit functors. Let $\mathcal{C}$ be the category we are working in, typically spaces or spectra or more generally a simplicial model category, let I be a small category, and let $X : I \to \mathcal{C}$ be a diagram in $\mathcal{C}$ indexed by I. The colimit and limit can be explicitly constructed as follows:

$$\operatorname{colim}_I X \cong X \otimes_I *_I$$
$$\lim_I X \cong \hom^I(*_I, X)$$

Here $*_I$ is the I-diagram of simplicial sets with $*_I(i) = *$ for all $i \in I$. (Full disclosure: here we are using $*_I$ to refer both to this trivial I-diagram and to the trivial I^{op}-diagram.) The constructions $\otimes_I$ and $\hom^I$ are the tensor and cotensor on the diagram category $\mathcal{C}^I$; these are special cases of, respectively, coends and ends, and are discussed in the following remark and example.

Remark 2.7. We recall the tensor and cotensor on the diagram category $\mathcal{C}^I$, following Hirschhorn [**Hi**, §18.3.1]. For X an I-diagram in $\mathcal{C}$ and A an I^{op}-diagram in simplicial sets, the tensor of X and A is as follows:

$$X \otimes_I A := \operatorname{colim} \left(\coprod_{i \xrightarrow{\alpha} j} X(i) \otimes A(j) \rightrightarrows \coprod_i X(i) \otimes A(i) \right)$$

This is the coend $\int^i X(i) \otimes A(i)$. Here $\otimes$ is the tensor action of simplicial sets on the category $\mathcal{C}$.

For X again an I-diagram in $\mathcal{C}$ and A an I-diagram in simplicial sets, the cotensor of A and X is as follows:

$$\hom^I(A, X) := \lim \left(\prod_i X(i)^{A(i)} \rightrightarrows \prod_{j \xrightarrow{\alpha} i} X(i)^{A(j)} \right)$$

This is the end $\int_i X(i)^{A(i)}$. Here the superscript refers to the cotensor coaction of simplicial sets on the category $\mathcal{C}$.

Example 2.8. When $\mathcal{C}$ is the category of spaces, the colimit tensor expression $X \otimes_I *_I$ boils down to the space $\left(\coprod_{i \in I} X(i)\right) / \{x \sim \alpha(x) \ \forall \ (i \xrightarrow{\alpha} j) \in I, x \in X(i)\}$—this is the disjoint union of all the objects in the diagram, mod equivalences introduced by the arrows of the diagram. Similarly, the limit cotensor $\hom^I(*_I, X)$ is simply $\{(x_i) \in \prod_{i \in I} X(i) | \alpha(x_j) = x_i \ \forall \ (j \xrightarrow{\alpha} i) \in I\}$. This last space can conveniently be thought of as the space of maps of diagrams from the trivial I-diagram to X, and this justifies the "hom" notation for the cotensor.

The point of all this abstract hoopla is that we can replace $*_I$ in the constructions $X \otimes_I *_I$ and $\hom^I(*_I, X)$ by a diagram of larger contractible spaces—this replacement gives us the homotopical wiggle room we were looking for and produces the homotopy colimit and homotopy limit. The minimal natural choices for these contractible spaces come from the nerves of over and under categories in the diagram. Specifically we have the following definitions:

$$\begin{aligned}\operatorname{hocolim}_I X &:= X \otimes_I N(-/I)^{\mathrm{op}} \\ \operatorname{holim}_I X &:= \hom^I(N(I/-), X)\end{aligned}$$

Here $N(I/-)$ and $N(-/I)^{\mathrm{op}}$ are respectively the functors taking i to the nerve of the over respectively opposite under categories I/i and $(i/I)^{\mathrm{op}}$. See section 2.3.2 for a discussion of why these are sensible replacements for the trivial diagram $*_I$. It is worth writing out this tensor and cotensor:

Definition 2.9. For X an I-diagram in the simplicial model category $\mathcal{C}$, the homotopy colimit and limit are defined as follows:

$$\operatorname{hocolim}_I X = \operatorname{colim}\left(\coprod_{i \xrightarrow{\alpha} j} X(i) \otimes N(j/I)^{\mathrm{op}} \rightrightarrows \coprod_i X(i) \otimes N(i/I)^{\mathrm{op}}\right)$$

$$\operatorname{holim}_I X = \lim\left(\prod_i X(i)^{N(I/i)} \rightrightarrows \prod_{j \xrightarrow{\alpha} i} X(i)^{N(I/j)}\right).$$

The casual reader can safely ignore the "op" here, referring if desired to Hirschhorn [**Hi**, Remark 18.1.11] for a description of how and why it arises and also for a comparison of these definitions to the original treatment of homotopy colimits and limits by Bousfield and Kan.

The reader is invited to check that this definition specializes to the description of the homotopy colimit and limit in example 2.5 and example 2.6. Such a specialization requires, of course, knowing the tensor and cotensor on spaces, namely, for a space Y and simplicial set B, that $Y \otimes B = Y \times |B|$ and $Y^B = \mathrm{Map}(|B|, Y)$.

To pin down the homotopy limit and colimit of spectra, it remains only to specify the tensor and cotensor on some particular category of spectra—see section 2.3.3 for these constructions in orthogonal spectra, symmetric spectra, and S-modules. Note that once we have a complete picture of homotopy limits of spectra, we have, combining definitions 1.7 and 2.4, our desired notion of sheaves of spectra on the moduli stack of elliptic curves.

2.3.2. *Derived Limits and Corrected Homotopy Limits.* Unfortunately, the above definitions of homotopy limit and colimit do not always behave as well as we would like, particularly when we are working in categories other than spaces or simplicial sets. In particular, they are not always homotopy invariant and so do not induce functors on the level of homotopy categories. Problems tend to arise when the objects of our diagram are not fibrant or not cofibrant. In this section we discuss these technicalities and describe and differentiate the four relevant notions: limits, derived limits, homotopy limits, and corrected homotopy limits (and, of course, their co- analogs).

As before, let $\mathcal{C}$ be a simplicial model category, I a small category, and $X : I \to \mathcal{C}$ an I-diagram in $\mathcal{C}$. As discussed above, the limit is the functor

$$\begin{aligned} \lim : \mathcal{C}^I &\to \mathcal{C} \\ X &\mapsto \hom^I(*_I, X) \end{aligned}$$

This functor is not homotopy invariant, in that an objectwise weak equivalence $X \xrightarrow{\sim} Y$ of I-diagrams need not induce a weak equivalence of their limits. The most straightforward way to attempt to fix this problem is to derive the limit functor.

The diagram category $\mathcal{C}^I$ can itself have a model structure, and it may even have several depending on particular properties of $\mathcal{C}$ and I. If $\mathcal{C}$ is a combinatorial model category, then $\mathcal{C}^I$ has an injective model structure, where the weak equivalences and cofibrations are detected objectwise [**Be, Lur, Shu**]. If $\mathcal{C}$ is merely a cofibrantly generated model category, then $\mathcal{C}^I$ has a projective model structure, where the weak equivalences and fibrations are detected objectwise [**Hi**, p.224]. If the diagram I is Reedy, then for any model category $\mathcal{C}$, the category of diagrams $\mathcal{C}^I$ has a Reedy model structure, where only the weak equivalences are detected objectwise. In the following, we assume without comment that $\mathcal{C}$ and I have appropriate structure to ensure the existence of injective, projective, or Reedy model structures, as needed.

Limit is a right Quillen functor from the injective model structure on $\mathcal{C}^I$ to $\mathcal{C}$ [**Shu**, p.16]. It therefore makes sense to take the total right derived functor of the limit:

$$\begin{aligned} R\lim : \mathrm{Ho}(\mathcal{C}^I) &\to \mathrm{Ho}(\mathcal{C}) \\ [X] &\mapsto [\hom^I(*_I, FX)] \end{aligned}$$

Here F is fibrant replacement in the injective model structure, and brackets refer to the object in the homotopy category. By construction this derived limit is homotopy invariant and so is in a sense the "right" replacement for the limit. The derived limit and the homotopy limit are occasionally conflated in the literature, but they are distinct functors and are easy to tell apart because the derived limit is a functor from $\mathrm{Ho}(\mathcal{C}^I)$ to $\mathrm{Ho}(\mathcal{C})$, while the homotopy limit is a functor from $\mathcal{C}^I$ to $\mathcal{C}$. In retrospect, it might have made more sense to call the derived limit the "homotopy limit" and to have a different name for the particular not-always-homotopy-invariant functor now called the homotopy limit—but it is much too late for a terminological switcheroo.

The biggest disadvantage of the derived limit is that it can be quite difficult to calculate the fibrant replacement FX. In general, such a calculation is hopeless, but if the diagram is particularly simple, we can proceed as follows. Suppose the diagram I is Reedy and has cofibrant constants. (A diagram I is said to have cofibrant constants if the constant I-diagram at any cofibrant object of any model category is Reedy cofibrant.) In this case, the limit is right Quillen not only with respect to the injective model structure, but also with respect to the Reedy model structure [**Hi**, Thm 15.10.8]. The derived limit (which up to homotopy does not depend on the model structure we use) can therefore be described as $R\lim X = [\hom^I(*_I, F_R X)]$, where F_R is fibrant replacement in the Reedy model structure. We can then go about explicitly calculating the Reedy fibrant replacement of our diagram.

EXAMPLE 2.10. If the diagram X is $Y \xrightarrow{f} Z \xleftarrow{g} W$, then a Reedy fibrant replacement $F_R X$ is a diagram $Y' \xrightarrow{f'} Z' \xleftarrow{g'} W'$ with an objectwise weak equivalence $X \to F_R X$ such that Y', Z', and W' are fibrant, and f' and g' are fibrations. The derived limit $R \lim X$ of the original diagram is the limit $\lim F_R X$ of the new diagram. Notice that this method, replacing maps by fibrations and then taking the ordinary pullback, is the usual means for calculating homotopy pullbacks. It happens to be the case that it is often enough to convert only one of the two maps to a fibration.

All this said, we would be better off if we could avoid replacing X in either the injective or Reedy model structures.

The homotopy limit is a compromise solution: it avoids the fibrant replacement that plagues the derived limit and is therefore more explicit and calculable, at the expense of some weakening of homotopy invariance. It has the further advantage that it is a "point-set level" functor, not a functor on homotopy categories. Here the key motivation for the homotopy limit comes from shifting attention from the injective to the projective model structure. The derived limit was $[\hom^I(*_I, FX)]$. Though this cotensor $\hom^I$ is not in fact a mapping space, it behaves rather like one. In particular note that $*_I$ is cofibrant in the injective model structure on sSet^I, and FX is by definition fibrant in the injective model structure on $\mathcal{C}^I$, and so we expect $\hom^I(*_I, FX)$ to have, as it does, a well behaved homotopy type. Suppose that instead of fibrantly replacing X in the injective model structure on $\mathcal{C}^I$, we cofibrantly replace $*_I$ in the projective model structure on sSet^I. That is, consider the cotensor $\hom^I(C(*_I), X)$, where C is cofibrant replacement in the projective model structure. Provided X is objectwise fibrant (therefore fibrant in the projective model structure), we might expect this cotensor to have a reasonable homotopy type. Indeed this is the case:

LEMMA 2.11. *If the I-diagram X in $\mathcal{C}$ is objectwise fibrant, then the cotensors $\hom^I(*_I, FX)$ and $\hom^I(C(*_I), X)$ are weakly equivalent, where F is fibrant replacement in the injective model structure on $\mathcal{C}^I$ and C is cofibrant replacement in the projective model structure on sSet^I.*

The lemma is also true if we substitute the Reedy model structure fibrant replacement $F_R X$ (if it makes sense) in place of the injective model structure fibrant replacement FX.

Note that the construction $\hom^I(C(*_I), X)$ has the huge advantage that the replacement $C(*_I)$ only depends on the category I and not on the category $\mathcal{C}$ or the particular diagram X. We can therefore make such a choice of replacement once and for all. The nerve $N(I/-)$ of the overcategory is a cofibrant object in the projective model structure on sSet^I and so provides such a choice [**Hi**, Prop 14.8.9]. The definition of homotopy limit follows:

$$\begin{aligned} \operatorname{holim} : \mathcal{C}^I &\to \mathcal{C} \\ X &\mapsto \hom^I(N(I/-), X) \end{aligned}$$

We reiterate that this is a point-set level functor, and is functorial both with respect to the diagram X and with respect to the category I; this would have been difficult to arrange using the cotensor $\hom^I(*_I, FX)$ because we would need to have made a choice, compatible for all diagram categories $\mathcal{C}^I$, of a functorial fibrant replacement functor F. Instead, we make use of the simple functorial cofibrant replacement $C(*_I) = N(I/-)$.

The main disadvantage of the homotopy limit is, as one might guess from Lemma 2.11, that it is not homotopy invariant when the diagram X is not objectwise fibrant. In some categories, such as spaces, this does not present a problem (and indeed the proper behavior of holim on spaces probably accounts for its widespread use and the general lack of clarity concerning its deficiencies). As we intend to work in categories of spectra and ring spectra, though, we must correct this lack of invariance by precomposing with a functorial fibrant replacement. The resulting functor is called the corrected homotopy limit:

$$\begin{aligned} \mathrm{corholim} : \mathcal{C}^I &\to \mathcal{C} \\ X &\mapsto \hom^I(N(I/-), F_{\mathrm{obj}} X) \end{aligned}$$

Here F_{obj} is objectwise functorial fibrant replacement. According to the extent that one views objectwise fibrant replacement as a minor adjustment, one might welcome or disdain the occasional conflation of holim and corholim.

The corrected homotopy limit brings us full circle in so far as it represents the derived functor of the limit: by Lemma 2.11, the cotensor $\hom^I(*_I, FX)$ (or $\hom^I(*, F_R X)$ in the case of a Reedy diagram, either of which represent the derived limit) is weakly equivalent to the corrected homotopy limit $\hom^I(N(I/-), F_{\mathrm{obj}} X)$.

Remark 2.12. The reader may be wondering why we did not define sheaves to be presheaves satisfying a corholim, rather than a holim, condition. Indeed, in all respects that probably would have been wiser, but for reasons of convention we stick to the holim definition. We can get away with this because in the end we will restrict our attention to presheaves of fibrant objects, in which case a holim and a corholim condition amount to the same thing.

We briefly describe the colimit analog of the above discussion. The ordinary colimit is, as before, the functor

$$\begin{aligned} \mathrm{colim} : \mathcal{C}^I &\to \mathcal{C} \\ X &\mapsto X \otimes_I *_I \end{aligned}$$

This functor is not homotopy invariant. It is, though, a left Quillen functor from the projective model structure on $\mathcal{C}^I$ to $\mathcal{C}$, and therefore has a total left derived functor:

$$\begin{aligned} L\,\mathrm{colim} : \mathrm{Ho}(\mathcal{C}^I) &\to \mathrm{Ho}(\mathcal{C}) \\ [X] &\mapsto [CX \otimes_I *_I] \end{aligned}$$

Here C is cofibrant replacement in the projective model structure.

This cofibrant replacement can be painful to calculate, so instead of replacing X we replace $*_I$. If X is objectwise cofibrant, then $CX \otimes_I *_I$ and $X \otimes_I C(*_I)$ are weakly equivalent—here CX is, as before, the cofibrant replacement in the projective model structure on $\mathcal{C}^I$, while $C(*_I)$ is the cofibrant replacement in the projective model structure on $\mathsf{sSet}^{(I^{\mathrm{op}})}$. The nerve $N(-/I)^{\mathrm{op}}$ is cofibrant in the projective model structure on $\mathsf{sSet}^{(I^{\mathrm{op}})}$ and so provides a particular choice of the latter cofibrant replacement, and thereby the definition of homotopy colimit:

$$\begin{aligned}\mathrm{hocolim} : \mathcal{C}^I &\to \mathcal{C} \\ X &\mapsto X \otimes_I N(-/I)^{\mathrm{op}}\end{aligned}$$

We can reestablish homotopy invariance by precomposing this functor with an objectwise cofibrant replacement; the result is the corrected homotopy colimit:

$$\begin{aligned}\mathrm{corhocolim} : \mathcal{C}^I &\to \mathcal{C} \\ X &\mapsto C_{\mathrm{obj}} X \otimes_I N(-/I)^{\mathrm{op}}\end{aligned}$$

The corrected homotopy colimit represents the derived functor, as desired: the tensor $CX \otimes_I *_I$ (which represents the derived colimit) is weakly equivalent to the corrected homotopy colimit $C_{\mathrm{obj}} X \otimes_I N(-/I)^{\mathrm{op}}$. Here CX is the cofibrant replacement in the projective model structure on $\mathcal{C}^I$, and C_{obj} is objectwise functorial cofibrant replacement.

Example 2.13. If the diagram I is Reedy and has fibrant constants (that is every constant I-diagram at a fibrant object is Reedy fibrant), then the colimit is a left Quillen functor from the Reedy model structure on $\mathcal{C}^I$ [**Hi**, Thm 15.10.8]. We can therefore calculate the derived colimit using a Reedy cofibrant replacement: $L\,\mathrm{colim}\, X = [C_R X \otimes_I *_I]$. If the diagram X has the form $B \xleftarrow{j} A \xrightarrow{k} C$, a Reedy cofibrant replacement $C_R X$ is a diagram $B' \xleftarrow{j'} A' \xrightarrow{k'} C'$ with an objectwise weak equivalence $C_R X \to X$ such that B', A', and C' are cofibrant and j' and k' are cofibrations. The derived limit is the pushout $\mathrm{colim}\, C_R X$ of this modified diagram. Indeed, replacing maps by cofibrations in such a diagram is the usual way to calculate homotopy pushouts. Note that it is often sufficient to convert one of the two maps to a cofibration.

Remark 2.14. A last important distinction between the homotopy limit and colimit and their corrected versions is that the latter depend on a choice of model structure on the underlying category $\mathcal{C}$, while the former do not.

2.3.3. *Sheaves of Orthogonal Spectra, Symmetric Spectra, and S-Modules.* It is time to bite the bullet and specify the particular categories of spectra in which we intend to work. The relevant options are symmetric spectra, orthogonal spectra, S-modules, and the categories of commutative symmetric ring spectra, commutative orthogonal ring spectra, and commutative S-algebras. We briefly review the definitions of these various categories. Along the way we describe the tensor and cotensor over simplicial sets that we needed in the definition of hocolim and holim. We refer, however, to the chapter on model categories of spectra for the notions of fibrancy and cofibrancy needed for the corrected homotopy colimit and limit.

Remark 2.15. At the end of the day, we are trying to make sense of the notion of a "sheaf of E_∞ ring spectra". By definition an E_∞ ring spectrum in a particular category $\mathcal{S}$ of spectra is an algebra in $\mathcal{S}$ over an E_∞ operad. However, provided $\mathcal{S}$ is for example symmetric or orthogonal spectra or S-modules, the category of E_∞ ring spectra in $\mathcal{S}$ is Quillen equivalent to the category of commutative monoids in $\mathcal{S}$. Therefore, we stick to these various categories of commutative monoids.

The reader may wonder, then, why the notion of E_∞ comes in at all, if the technicalities of sheaves of commutative ring spectra are best handled directly with commutative monoids in spectra. The answer is that the obstruction theory we need to actually construct such spectra uses in a fundamental way the operadic formulation of the commutativity conditions on ring spectra.

Orthogonal spectra and symmetric spectra are both examples of diagram spectra, and as such their formulations are nearly identical. We describe orthogonal spectra and then mention the modifications for symmetric spectra. An excellent reference for diagram spectra is [**MMSS**] and our discussion follows the treatment there.

The basic "diagram" $\mathcal{J}$ for orthogonal spectra is the category of finite-dimensional real inner product spaces, together with orthogonal isomorphisms. From this diagram, we define the category $\mathcal{J}\mathcal{T}$ of $\mathcal{J}$-spaces to be the category of continuous functors from $\mathcal{J}$ to based spaces, together with natural transformations between these functors.

The key observation is that $\mathcal{J}\mathcal{T}$ is a symmetric monoidal category with product as follows:

$$\begin{aligned} &\mathcal{J}\mathcal{T} \times \mathcal{J}\mathcal{T} \xrightarrow{\wedge} \mathcal{J}\mathcal{T} \\ &(X, Y) \mapsto \Big(V \xmapsto{X\wedge Y} \bigvee_{W\subset V} X(W) \wedge Y(V-W)\Big) \end{aligned}$$

Note that the wedge product $\bigvee_{W\subset V} X(W) \wedge Y(V-W)$ is topologized using the ordinary topology on subspaces of V. There is a natural commutative monoid S in $\mathcal{J}\mathcal{T}$, namely $S(V) = S^V$; here S^V denotes the one point compactification of V. The product on S is induced by direct sum of vector spaces: $\bigvee_{W\subset V} S^W \wedge S^{V-W} \to S^V$; here $V-W$ is the orthogonal complement of W in V and the map $W \times (V-W) \to V$ is $(a,b) \mapsto i(a) + j(b)$ for $i : W \to V$ and $j : V-W \to V$ the inclusions. The reader is invited to check that this monoid really is strictly commutative, simply because direct sum of disjoint orthogonal vector subspaces is strictly commutative.

Definition 2.16. An orthogonal spectrum is a $\mathcal{J}$-space with an action by the monoid S. In other words, it is an S-module in $\mathcal{J}\mathcal{T}$. Denote the category of orthogonal spectra by $\mathcal{J}\mathcal{S}$.

Because S is a *commutative* monoid, the category $\mathcal{J}\mathcal{S}$ itself has a symmetric monoidal structure with product denoted $\wedge_S$:

$$X \wedge_S Y := \operatorname{colim}_{\mathcal{J}\mathcal{T}}(X \wedge S \wedge Y \rightrightarrows X \wedge Y)$$

This coequalizer is, of course, the usual way to define tensor products of modules in algebra. Finally we have our desired notions of ring spectra:

Definition 2.17. An orthogonal ring spectrum is a monoid in $\mathcal{J}\mathcal{S}$. A commutative orthogonal ring spectrum is a commutative monoid in $\mathcal{J}\mathcal{S}$.

The tensor and cotensor on the category of orthogonal spectra are particularly simple: they are both levelwise, which is to say that for X an S-module in $\mathcal{J}\mathcal{T}$ and A a (based) space, the tensor $X \otimes A$ is given by $(X \otimes A)(V) = X(V) \wedge A$ and the cotensor X^A is given by $(X^A)(V) = X(V)^A$. Note that these tensors over topological spaces can be extended to simplicial sets via the realization functor. The cotensors on both orthogonal ring spectra and commutative orthogonal ring spectra are also levelwise. However, the tensors on these categories are rather less explicit and we do not discuss them; luckily, we only need the cotensors for our discussion of sheaves.

The definition of symmetric spectra is entirely analogous. The diagram Σ in question is the category of finite sets with isomorphisms. The category $\Sigma\mathcal{T}$ of Σ spaces is symmetric monoidal:

$$\begin{aligned} &\Sigma\mathcal{T} \times \Sigma\mathcal{T} \xrightarrow{\wedge} \Sigma\mathcal{T} \\ &\quad (X, Y) \mapsto (X \wedge Y)(N) = \bigvee_{M \subset N} X(M) \wedge Y(N \backslash M) \end{aligned}$$

The distinguished commutative monoid S in $\Sigma\mathcal{T}$ has $S(N) = S^N$. A symmetric spectrum is a Σ-space with an action of S. The category of symmetric spectra $\Sigma\mathcal{S}$ has a smash product, given by an appropriate coequalizer, and (commutative) symmetric ring spectra are (commutative) monoids in $\Sigma\mathcal{S}$.

Remark 2.18. Symmetric spectra are sometimes defined using the skeleton diagram Σ^{skel} whose objects are the rigid finite sets $\mathbf{n} = \{1, \ldots, n\}$, for $n \geq 0$. This variant may look more elementary but requires a rather less intuitive formula for the smash product of Σ^{skel}-spaces: $(X \wedge Y)(\mathbf{n}) = \bigvee_{m \leq n} \Sigma_{n+} \wedge_{\Sigma_m \times \Sigma_{n-m}} X(\mathbf{m}) \wedge Y(\mathbf{n} - \mathbf{m})$.

The tensor and cotensor on the category of symmetric spectra and the cotensors on symmetric ring spectra and commutative symmetric ring spectra are all levelwise, as in the case of orthogonal spectra. The tensors for (commutative) symmetric ring spectra are not levelwise, and we leave them as a mystery.

The last category of spectra we consider is the category of S-modules. S-modules are somewhat more technical than diagram spectra, and we give only the most cursory treatment, closely following EKMM [**EKMM**]. Fix a universe, that is a real inner product space U isomorphic to $\mathbb{R}^\infty$. A prespectrum is an assignment to each finite dimensional subspace $V \subset U$ a based space $E(V)$ together with compatible (adjoint) structure maps $E(V) \to \Omega^{W-V} E(W)$. Denote the category of prespectra by $\mathcal{P}U$ or simply $\mathcal{P}$. A (Lewis-May-Steinberger) spectrum is a prespectrum in which all the structure maps are homeomorphisms, and the category of such is denoted $\mathcal{S}U$ or $\mathcal{S}$. The forgetful functor $\mathcal{S} \to \mathcal{P}$ has a left adjoint $L : \mathcal{P} \to \mathcal{S}$ called spectrification.

There is an external smash product of spectra $\mathcal{S}U \times \mathcal{S}U' \xrightarrow{\wedge} \mathcal{S}(U \oplus U')$. Given a pair of spectra (E, E'), the assignment $F : V \oplus V' \mapsto E(V) \wedge E'(V')$ defines a prespectrum on the decomposable subspaces of $U \oplus U'$. There is a spectrification functor here as well that produces from F a spectrum LF on the decomposable subspaces of $U \oplus U'$; there is moreover a left adjoint ψ to the restriction to such subspaces, which in turn produces our desired smash $E \wedge E' := \psi LF \in \mathcal{S}(U \oplus U')$ indexed on all finite dimensional subspaces of $U \oplus U'$.

We would like to internalize this smash product, using the space of linear isometries from $U \oplus U$ to U. If we have a linear isometry $f : U \to U'$ we can transport a spectrum $E \in \mathcal{S}U$ to a spectrum $f_* E \in \mathcal{S}U'$: define $f_* E$ to be the spectrification of the prespectrum taking $V' \subset U'$ to $E(V) \wedge S^{V' - f(V)}$, where $V = f^{-1}(V' \cap \operatorname{im} f)$. Given an A-parameter family of linear isometries, that is a map $\alpha : A \to \mathcal{I}(U, U')$, there is a spectrum $A \ltimes E \in \mathcal{S}U'$ called the twisted half smash product, which is in an appropriate sense a union of all the spectra $\alpha(a)_* E$ for $a \in A$.

Let $\mathcal{L}(j) = \mathcal{I}(U^j, U)$ be the space of all internalizing linear isometries. Given $E, F \in \mathcal{S}U$, the twisted half smash $\mathcal{L}(2) \ltimes (E \wedge F)$ is a canonical internal smash product, but it is not associative. We fix this by restricting to $\mathbb{L}$-spectra: an $\mathbb{L}$-spectrum is a spectrum E with an action $\mathcal{L}(1) \ltimes E \to E$ by the isometries $\mathcal{L}(1)$. The smash product of $\mathbb{L}$-spectra is $M \wedge_{\mathcal{L}} N := \mathcal{L}(2) \ltimes_{\mathcal{L}(1)\times\mathcal{L}(1)} (M \wedge N)$. (Here the twisted half smash over $\mathcal{L}(1) \times \mathcal{L}(1)$ is given by the expected coequalizer.) We would be done, except that the category of $\mathbb{L}$-spectra doesn't have a point-set-level unit.

We can conjure up a unit as follows. There is a natural map $\lambda : S \wedge_{\mathcal{L}} M \to M$, where S is the spectrification of the prespectrum $V \mapsto S^V$. An S-module is by definition an $\mathbb{L}$-spectrum such that λ is an isomorphism. The smash product of two S-modules is simply their smash product as $\mathbb{L}$-spectra. The category of S-modules is our desired symmetric monoidal, unital category of spectra. S-algebras and commutative S-algebras are simply monoids and commutative monoids respectively in S-modules.

Given an S-module M and a based space X, the tensor S-module $M \otimes X$ is defined to be the spectrification of the prespectrum $V \mapsto M(V) \wedge X$. The cotensor M^X is defined to be $S \wedge_{\mathcal{L}} \phi(M)^X$, where ϕ forgets from S-modules to spectra, and the cotensor on spectra is $E^X(V) = E(V)^X$.

The cotensors on S-algebras and on commutative S-algebras are simply given by taking the cotensor in S-modules. The tensors on S-algebras and commutative S-algebras by contrast are not created in S-modules. However, the tensor on commutative S-algebras has a convenient description, as follows. Given a finite set $[n]$ and a commutative S-algebra R, define the tensor $R \otimes [n] := R^{\wedge [n]}$. Now for X a finite simplicial set, the tensor $R \otimes X$ in commutative S-algebras is the realization as a simplicial S-module of the levelwise tensor $R \otimes X_*$.

Picking one of the above three models for spectra, and feeding the cotensors back into the construction of homotopy limits, we now have a precise definition of sheaf of spectra, sheaf of ring spectra, and sheaf of commutative ring spectra.

REMARK 2.19. The reader may worry that there could be a confusing difference between sheaves of ring spectra and presheaves of ring spectra that are sheaves of spectra. Luckily, this is not the case: the two notions agree because the sheaf condition is a homotopy limit condition, and this homotopy limit is built using limits and cotensors; these limits and cotensors are, in any of the above categories of ring and commutative ring spectra, simply computed in the underlying category of spectra.

We invite the reader to ruminate on the fact that cosheaves of ring spectra are very different objects from precosheaves of ring spectra that are cosheaves of spectra.

3. The Descent Spectral Sequence

Recall that the main theorem (of Goerss, Hopkins and Miller) is that there exists a sheaf $\mathcal{O}^{\mathrm{top}}$ of spectra on the moduli stack $(\mathcal{M}_{ell})_{\text{ét}}$ of elliptic curves in the étale topology. Sections 1 and 2 described what it means to have such a sheaf. In particular, section 2.2 described the Grothendieck site of the moduli stack in the étale topology, while section 1.1 defined sheaves on such a site with values in a category (as presheaves satisfying a homotopy limit condition), and section 2.3 discussed the homotopy limits of spectra needed for this definition of sheaves.

Given the sheaf $\mathcal{O}^{\mathrm{top}}$, we are primarily interested in understanding its spectrum of global sections $\mathcal{O}^{\mathrm{top}}(\mathcal{M}_{ell})$—recall that this spectrum is called *TMF*. By the definition of a sheaf, information about the global sections $\mathcal{O}^{\mathrm{top}}(\mathcal{M}_{ell})$ is contained in the spectra $\mathcal{O}^{\mathrm{top}}(U)$ associated to small open subsets U of the moduli stack $\mathcal{M}_{ell}$; the goal of this section is describe precisely how this local information is assembled into the desired global information. In particular, we construct a spectral sequence beginning with the sheaf cohomology of the moduli stack with coefficients in the sheafification of the presheaf of homotopy groups of $\mathcal{O}^{\mathrm{top}}$, strongly converging to the homotopy groups of the global spectrum $\mathcal{O}^{\mathrm{top}}(\mathcal{M}_{ell})$:

$$E^2 = H^q(\mathcal{M}_{ell}, \pi_p^{\dagger}\mathcal{O}^{\mathrm{top}}) \Longrightarrow \pi_{p-q}\mathcal{O}^{\mathrm{top}}(\mathcal{M}_{ell})$$

In the chapter on the homotopy groups of *TMF*, we will compute the E^2 term and describe the elaborate pattern of differentials.

We begin in section 3.1 by reviewing the notions of sheaf cohomology and Cech cohomology and discussing how they are related. Then in section 3.2 we construct a spectral sequence beginning with Cech cohomology and converging to the homotopy of the global sections of a sheaf of spectra. Finally, in section 3.3, we specialize to the sheaf $\mathcal{O}^{\mathrm{top}}$, using properties of this particular sheaf to simplify the spectral sequence from the preceding section.

3.1. Sheaf Cohomology and Cech Cohomology. We are studying a sheaf $\mathcal{O}^{\mathrm{top}}$ of spectra on the moduli stack of elliptic curves. We can consider the homotopy groups $\pi_*(\mathcal{O}^{\mathrm{top}}(U))$ of the spectra $\mathcal{O}^{\mathrm{top}}(U)$ associated to particular objects U of the étale site of the moduli stack. These homotopy groups fit together into a good-old down-to-earth presheaf of graded abelian groups. The spectral sequence computing the homotopy groups of *TMF* will begin with the sheaf cohomology of the sheafification of this presheaf. We review sheaf cohomology, and the related Cech cohomology, in some generality.

Let $\mathcal{A}$ be an abelian category. A morphism $f : X \to Y$ in $\mathcal{A}$ is a monomorphism if $fg_1 = fg_2$ implies $g_1 = g_2$. Recall that an object $I \in \mathcal{A}$ is injective if for all maps $m : X \to I$ and all monomorphisms $X \hookrightarrow Y$, there exists an extension of m to Y. The category $\mathcal{A}$ is said to have enough injectives if for all objects $A \in \mathcal{A}$ there is a monomorphism $A \hookrightarrow I$ into an injective object I. We are interested, of course, in the category of sheaves on a site:

Note 3.1. For any site $\mathcal{C}$, the category $\mathsf{Shv}_{\mathsf{Ab}}(\mathcal{C})$ of sheaves of abelian groups on the site is an abelian category with enough injectives.

We can therefore use the usual definition of sheaf cohomology:

Definition 3.2. For $\pi \in \mathsf{Shv}_{\mathsf{Ab}}(\mathcal{C})$ a sheaf of abelian groups on a site $\mathcal{C}$, the sheaf cohomology of an object $X \in \mathcal{C}$ of the site with coefficients in π is

$$H^q(X, \pi) := H_q\big(0 \to I^0(X) \to I^1(X) \to I^2(X) \to \cdots\big)$$

where $0 \to \pi|_X \to I^0 \to I^1 \to I^2 \to \cdots$ is an injective resolution of $\pi|_X$ in $\mathsf{Shv}_{\mathsf{Ab}}(\mathcal{C}/X)$.

Sheaf cohomology has a more concrete cousin, Cech cohomology, which does not involve an abstract resolution:

Definition 3.3. Let $\pi \in \mathsf{Shv}_{\mathsf{Ab}}(\mathcal{C})$ be a sheaf of abelian groups, and let $\mathfrak{U} = \{U_i \to U\}$ be a cover in the site $\mathcal{C}$. The Cech cohomology of U with respect to the cover $\mathfrak{U}$ and with coefficients in π is

$$\check{H}^q_{\mathfrak{U}}(U,\pi) := H_q\Big(0 \to \prod \pi(U_i) \to \prod \pi(U_{ij}) \to \prod \pi(U_{ijk}) \to \cdots\Big)$$

Here U_I refers to the intersection (ie fibre product over U) of the $U_i, i \in I$, and the maps are the alternating sums of the various natural restriction maps.

Cech cohomology is computable by hand, while sheaf cohomology is evidently independent of a particular choice of cover. If the object U of $\mathcal{C}$ has an acyclic cover, then the two theories agree:

Proposition 3.4. *For $\pi \in \mathsf{Shv}_{\mathsf{Ab}}(\mathcal{C})$ a sheaf of abelian groups, and $\mathfrak{U} = \{U_i \to U\}_{i\in I}$ a cover in $\mathcal{C}$ such that $H^q(U_J,\pi) = 0$ for all $J \subset I$ and all $q \geq 1$, sheaf and Cech cohomology agree: $H^q(U,\pi) = \check{H}^q_{\mathfrak{U}}(U,\pi)$.*

The proof is the usual double complex argument: build the double complex $[I^q(\coprod_{|J|=p} U_J)]$ from an injective resolution I^* of π, and show that the two resulting spectral sequences collapse respectively to sheaf and to Cech cohomology.

Remark 3.5. We will be interested in cases where the site $\mathcal{C}$ has a global terminal object, usually denoted (confusingly) $\mathcal{C}$, and from now on we assume we are in that situation.

3.2. The Spectral Sequence for a Sheaf of Spectra. We begin with a sheaf of spectra $\mathcal{O}$ on the étale site $(\mathcal{M}_{ell})_{\text{ét}}$ of the moduli stack of elliptic curves, and we would like to construct a spectral sequence converging to the homotopy groups $\pi_*(\mathcal{O}(\mathcal{M}_{ell}))$. The spectral sequence is meant to start with the local data of $\mathcal{O}$; we therefore chose a cover $\{\mathcal{N}_i \to \mathcal{M}_{ell}\}$ of the moduli stack. In outline, we use this cover to build a simplicial object of the site, then apply the sheaf $\mathcal{O}$ to get a cosimplicial spectrum, from which we get a tower of spectra, which we wrap up into an exact couple, and thereby arrive at our desired spectral sequence:

$$\begin{array}{c}
\text{cover } \mathfrak{N} = \{\mathcal{N}_i \to \mathcal{M}_{ell}\} \text{ in } (\mathcal{M}_{ell})_{\text{ét}} \\
\big\downarrow{\scriptstyle e} \\
\text{simplicial object } \coprod \mathcal{N}_i \leftleftarrows \coprod \mathcal{N}_{ij} \cdots \text{ in } (\mathcal{M}_{ell})_{\text{ét}} \\
\big\downarrow{\scriptstyle d} \\
\text{cosimplicial spectrum } \mathcal{O}^* = \{\mathcal{O}(\coprod \mathcal{N}_i) \rightrightarrows \mathcal{O}(\coprod \mathcal{N}_{ij}) \cdots\} \\
\big\downarrow{\scriptstyle c} \\
\text{tower of spectra } \cdots \to (\operatorname{Tot}\mathcal{O})^2 \xrightarrow{\phi_2} (\operatorname{Tot}\mathcal{O})^1 \xrightarrow{\phi_1} (\operatorname{Tot}\mathcal{O})^0 \\
\big\downarrow{\scriptstyle b} \\
\text{exact couple } \bigoplus \pi_{p+q}(\operatorname{Tot}\mathcal{O})^q \to \bigoplus \pi_{p+q}(\operatorname{Tot}\mathcal{O})^q \to \bigoplus \pi_{p+q}(\operatorname{corhofib}\phi_q) \\
\big\downarrow{\scriptstyle a} \\
\text{spectral sequence } E^2_{pq} = \check{H}^q_{\mathfrak{N}}(\mathcal{M}_{ell},\pi_p\mathcal{O}) \Longrightarrow \pi_{p-q}\mathcal{O}(\mathcal{M}_{ell})
\end{array}$$

In this section we describe this chain of constructions in detail. In section 3.3 we use particular properties of the sheaf $\mathcal{O}^{\mathrm{top}}$ to compare the E^2 term of this spectral sequence for $\pi_*(\mathcal{O}^{\mathrm{top}}(\mathcal{M}_{ell}))$, which is the Cech cohomology of the presheaf $\pi_p\mathcal{O}^{\mathrm{top}}$, to the sheaf cohomology of the sheafification $\pi_p^\dagger\mathcal{O}^{\mathrm{top}}$ of the presheaf $\pi_p\mathcal{O}^{\mathrm{top}}$.

We begin at the end (step a) and work our way back to the beginning (step e). We assume the reader is familiar with the construction of a spectral sequence from an exact couple—see McCleary [**McCl**] for a detailed presentation of the construction and Boardman [**Bo**] for a careful treatment of convergence issues. We therefore proceed to step b, building an exact couple from a tower of spectra.

3.2.1. *The Spectral Sequence of a Filtration of Spectra and of a Tower of Spectra.* We are interested in towers of spectra and their associated inverse limits. Along the way we address the slightly more intuitive situation of a filtration of spectra and its associated direct limit.

Suppose we have a filtration of spectra:

$$\begin{tikzcd} * \arrow[r, "\phi_0"] & F_0 \arrow[dl] \arrow[r, "\phi_1"] & F_1 \arrow[dl] \arrow[r, "\phi_2"] & F_2 \arrow[dl] \arrow[r] & \cdots \\ \operatorname{corhocofib}\phi_0 & \operatorname{corhocofib}\phi_1 & \operatorname{corhocofib}\phi_2 & & \end{tikzcd}$$

Denote the corrected homotopy colimit $\operatorname{corhocolim}_i F_i$ by F. We think of the sequence F_i as a filtration of F, and expect any spectral sequence constructed from the filtration to give information about F. Take homotopy groups of the spectra F_i and of the corrected homotopy cofibres $\operatorname{corhocofib}\phi_i$, and wrap up the resulting triangles into an exact couple. This produces a spectral sequence with $E^1_{pq} = \pi_{p+q}(\operatorname{corhocofib}\phi_q)$. This spectral sequence is a half plane spectral sequence with exiting differentials (in the sense of Boardman [**Bo**]), and converges strongly to $\pi_{p+q}F$:

$$E^1_{pq} = \pi_{p+q}(\operatorname{corhocofib}\phi_q) \underset{\text{strong}}{\Longrightarrow} \pi_{p+q}\operatorname{corhocolim}_i F_i.$$

Note 3.6. Lest there be any confusion, note that for X a spectrum, by $\pi_i X$ we mean the set of maps $\mathrm{Ho}_{\mathsf{Spec}}([S^i],[X])$ in the homotopy category of spectra between the sphere $[S^i]$ and $[X]$, not for instance homotopy classes of maps in the category of spectra from a sphere S^i to X. The above spectral sequence converges, a priori, to $\operatorname{colim}_i \pi_{p+q}F_i$; we have used implicitly the equality

$$\operatorname{colim}_i \pi_{p+q}F_i = \pi_{p+q}\operatorname{corhocolim}_i F_i.$$

In the dual picture, we begin with a tower of spectra:

$$\begin{tikzcd} \cdots \arrow[r] & F^2 \arrow[r, "\phi_2"] & F^1 \arrow[r, "\phi_1"] & F^0 \arrow[r, "\phi_0"] & * \\ & \operatorname{corhofib}\phi_2 \arrow[u] & \operatorname{corhofib}\phi_1 \arrow[u] & \operatorname{corhofib}\phi_0 \arrow[u] & \end{tikzcd}$$

Let F denote the corrected homotopy limit $\operatorname{corholim}_i F^i$. Again we take homotopy groups of the whole diagram of the F^i and $\operatorname{corhofib}\phi_i$, and wrap up the resulting triangles into an exact couple. The spectral sequence associated to this exact couple has $E^1_{pq} = \pi_{p-q}(\operatorname{corhofib}\phi_q)$. It is a half plane spectral sequence with entering differentials and converges conditionally to $\lim_i \pi_{p-q}F^i$:

$$E^1_{pq} = \pi_{p-q}(\operatorname{corhofib}\phi_q) \underset{\text{cond}}{\Longrightarrow} \lim_i \pi_{p-q}F^i.$$

In general, one must address two issues: whether or not this conditionally convergent spectral sequence in fact converges strongly, and whether or not $\lim_i \pi_{p-q} F^i = \pi_{p-q} F$, the latter of which is of course a $\lim^1$ problem.

NOTE 3.7. The spectral sequence of a tower has target $\lim_i \pi_{p-q} F^i$, but we are usually more interested in the homotopy $\pi_{p-q} \operatorname{corholim}_i F^i$ of the corrected homotopy limit of the tower. These are related by the Milnor exact sequence:

$$0 \to \lim_i^1 \pi_{p-q-1} F^i \to \pi_{p-q} \operatorname{corholim}_i F^i \to \lim_i \pi_{p-q} F^i \to 0.$$

3.2.2. *The Realization of a Simplicial Spectrum and Tot of a Cosimplicial Spectrum.* Here we note that a simplicial spectrum leads to a filtration of spectra, and dually that a cosimplicial spectrum leads to a tower of spectra. We also identify the colimit (resp. limit) of the resulting filtration (tower) in terms of the homotopy colimit (limit) of the original (co)simplicial spectrum.

NOTATION 3.8. We begin by fixing some notation. The cosimplicial category Δ has objects $[0], [1], [2], \ldots$, (where we think of $[n]$ as $\{0, 1, \ldots, n\}$) and morphisms weakly order preserving maps. The bold $\boldsymbol{\Delta}$ will denote the standard cosimplicial simplicial set, whose simplicial set of n-cosimplicies $\boldsymbol{\Delta}(n)$ is the simplicial n-simplex $\Delta[n]$; this n-simplex $\Delta[n]$ has k-simplicies $\Delta[n]_k = \Delta([k], [n])$. We use the symbol Δ_0^n to denote the full subcategory of Δ whose objects are $[0], \ldots, [n]$.

Let A be a simplicial spectrum, that is a simplicial object $A : \Delta^{\mathrm{op}} \to \mathsf{Spec}$ in the category of spectra. The realization of A is defined as follows:

$$|A| := A \otimes_{\Delta^{\mathrm{op}}} \boldsymbol{\Delta} = \operatorname{colim}\left(\coprod_{\phi : n \to m} A_m \otimes \Delta[n] \rightrightarrows \coprod_{n \geq 0} A_n \otimes \Delta[n] \right).$$

We construct a filtration whose colimit is the realization $|A|$ as follows. Recall that the n-skeleton $\operatorname{sk}_n A$ of A is the left Kan extension to Δ^{op} of the restriction of A from Δ^{op} to $(\Delta_0^n)^{\mathrm{op}}$. Intuitively, $\operatorname{sk}_n A$ consists of the simplicies of A of dimension less than or equal to n, together with the possible degeneracies of those simplicies; (here "possible" means possible in a simplicial spectrum agreeing with A through dimension n). Let $|A|_n$ denote the realization $|\operatorname{sk}_n A|$. We have the sequence

$$* \to |A|_0 \to |A|_1 \to |A|_2 \to \cdots$$

The colimit of this sequence is, by construction, the realization $|A|$.

NOTE 3.9. We can build the skeleta of A inductively as follows. Define the n-th latching object of A as

$$L_n A = (\operatorname{sk}_{n-1} A)_n.$$

This latching object can, roughly speaking, be thought of as the spectrum of degenerate n-simplicies of A—in general, though, the map $L_n A \to A_n$ need not be a cofibration, which means in particular that the latching object $L_n A$ may record more degenerate simplicies than are present in A itself. We have a pushout diagram [**GJ**, p.367]:

$$\begin{tikzcd}
A_n \otimes \partial\Delta^n \cup_{L_n A \otimes \partial\Delta^n} L_n A \otimes \Delta^n \arrow[r] \arrow[d] & \operatorname{sk}_{n-1} A \arrow[d] \\
A_n \otimes \Delta^n \arrow[r] & \operatorname{sk}_n A
\end{tikzcd}$$

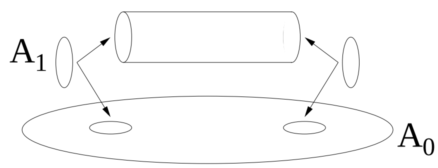
Here only, by "$\otimes$" we mean the external tensor, taking a spectrum and a simplicial set and giving a simplicial spectrum.

Ignoring the terms involving L_nA, the pushout says that the n-skeleton $\mathrm{sk}_n A$ is built by gluing an n-simplex along its boundary onto the $(n-1)$-skeleton $\mathrm{sk}_{n-1} A$, for each "element" of the spectrum A_n of n-simplicies. But in fact, the $(n-1)$-skeleton already includes degenerate n-simplicies, and the pushout accounts for this by quotienting out the latching object $L_nA \otimes \Delta^n$.

Example 3.10. Consider the first stage $|A|_1 = |\mathrm{sk}_1 A|$ of the filtration of $|A|$. Ignoring degeneracies, the realization of the 1-skeleton is roughly speaking the colimit

$$\mathrm{colim}\left(A_1 \otimes \Delta^0 \sqcup A_1 \otimes \Delta^0 \rightrightarrows A_0 \otimes \Delta^0 \sqcup A_1 \otimes \Delta^1\right).$$

We can schematically picture this 1-skeleton glued together as follows:



Notice that the filtration $* \to |A|_0 \to |A|_1 \to \cdots$ above has *colimit* the realization $|A|$. However, recall that the spectral sequence from section 3.2.1 converges to the homotopy groups of the *corrected homotopy colimit* of the filtration. We therefore need to address the issue of when these two agree. Conveniently, the condition ensuring this agreement can be expressed in terms of Reedy cofibrancy, as follows.

Definition 3.11. A simplicial spectrum A is Reedy cofibrant if the maps $L_nA \to A_n$ are cofibrations for all n. Roughly speaking, this is true when the degenerate simplicies of A are freely generated, and for all n the degenerate n-simplicies of A map by a cofibration into all the n-simplicies of A. A sequence of spectra $* \to X_0 \to X_1 \to \cdots$ is Reedy cofibrant if all the maps in the sequence are cofibrations.

If a simplicial spectrum A is Reedy cofibrant, then the morphisms $|A|_{n-1} \to |A|_n$ are cofibrations (see [**GJ**, p.385]), which is to say the realization sequence $* \to |A|_0 \to |A|_1 \to \cdots$ is Reedy cofibrant. In section 2.3.2, we saw that the corrected homotopy colimit represents the derived colimit and observed that in the case of a Reedy cofibrant diagram, this derived colimit is represented by the honest colimit. Altogether, when the simplicial spectrum A is Reedy cofibrant, we have a weak equivalence

$$\mathrm{corhocolim}_i\, |A|_i \underset{\mathrm{RC}}{\simeq} \mathrm{colim}_i\, |A|_i = |A|.$$

The subscript "RC" will serve as a reminder that an equivalence depends on Reedy cofibrancy.

Remark 3.12. When the sequence $* \to |A|_0 \to |A|_1 \to \cdots$ is Reedy cofibrant, all the terms in the sequence are necessarily cofibrant (though this is not, per se, part of the Reedy condition). This objectwise cofibrancy implies that the corrected homotopy colimit $\mathrm{corhocolim}\, |A|_i$ agrees with the usual homotopy colimit $\mathrm{hocolim}\, |A|_i$. We will not, however, need this fact.

Confusingly, there is another homotopy colimit floating around, namely the homotopy colimit of the Δ^{op}-diagram A itself. We also need to compare the realization $|A|$ to this homotopy colimit $\mathrm{hocolim}_{\Delta^{\mathrm{op}}} A$: when we are studying a cosheaf of spectra, we will care about the homotopy colimit of the associated simplicial spectrum, but the spectral sequence constructed in section 3.2.1 converges, if A is Reedy cofibrant, to the homotopy groups of the realization of this simplicial spectrum.

Recall that the homotopy colimit of the Δ^{op}-diagram A was defined as $A \otimes_{\Delta^{\mathrm{op}}} N(-/\Delta^{\mathrm{op}})^{\mathrm{op}}$. We might therefore expect a close relationship between the realization and the homotopy colimit:

PROPOSITION 3.13 ([**Hi**, Thm 18.7.4]). *If the simplicial spectrum A is Reedy cofibrant, then there is a natural weak equivalence (the Bousfield-Kan map)*

$$|A| \xleftarrow{\simeq} \mathrm{hocolim}_{\Delta^{\mathrm{op}}} A.$$

This equivalence is most plausible: the reader can note that, ignoring degeneracy maps, the nerves $N(-/\Delta^{\mathrm{op}})^{\mathrm{op}}$ of the overcategories in Δ^{op} are the barycentric subdivisions of the standard simplicies of $\mathbf{\Delta}$. The Reedy cofibrancy condition will be automatically satisfied (see Lemma 3.20) in the situation we care about, and so need not concern us.

In summary, we have the following chain of equalities, relating the target of our spectral sequence to the homotopy of the homotopy colimit of our simplicial spectrum:

$$\mathrm{colim}_i\, \pi_*|A|_i = \pi_*\, \mathrm{corhocolim}_i\, |A|_i \underset{\mathrm{RC}}{=} \pi_*\, \mathrm{colim}_i\, |A|_i = \pi_*|A| \underset{\mathrm{RC}}{=} \pi_*\, \mathrm{hocolim}_{\Delta^{\mathrm{op}}} A.$$

The indicated equalities depend on A being Reedy cofibrant.

The dual, cosimplicial picture is the one that concerns us more directly. Let $B : \Delta \to \mathit{Spec}$ be a cosimplicial spectrum. The "corealization" is traditionally called the "totalization" of the cosimplicial spectrum and is defined as follows:

$$\mathrm{Tot}\, B := \hom^{\Delta}(\mathbf{\Delta}, B) = \lim \left(\prod_{n\geq 0} (B^n)^{\mathbf{\Delta}(n)} \rightrightarrows \prod_{\phi: n\to m} (B^m)^{\mathbf{\Delta}(n)} \right).$$

Here again $\mathbf{\Delta}$ is the cosimplicial standard simplex with $\mathbf{\Delta}(n)$ the standard simplicial n-simplex $\Delta[n]$.

The key feature of Tot of a cosimplicial spectrum is that it is the inverse limit of a tower of spectra built from the coskeleta of the cosimplicial spectrum, and this tower leads to our desired spectral sequence. We define the n-coskeleton $\mathrm{cosk}^n B$ of a cosimplicial spectrum $B : \Delta \to \mathsf{Spec}$ to be the right Kan extension to Δ of the restriction of B from Δ to Δ_0^n. Intuitively, $\mathrm{cosk}^n B$ consists of the cosimplicies of B of dimension between 0 and n; for $k > n$ the coskeleton has a k-cosimplex for every possible combination of n-cosimplicies of B that could be the image under the codegeneracy maps of a k-cosimplex of a cosimplicial spectrum agreeing with B through dimension n. Let $\mathrm{Tot}^n B$ denote the totalization $\mathrm{Tot}\,\mathrm{cosk}_n B$ of the coskeleton of B. We have a tower

$$\cdots \to \mathrm{Tot}^2 B \to \mathrm{Tot}^1 B \to \mathrm{Tot}^0 B \to *$$

The limit of this sequence is the full totalization $\mathrm{Tot}\, B$.

REMARK 3.14. The above definition of Tot^n is not formally the same as the usual one in the literature, for instance as the definition in Bousfield-Kan or Goerss-Jardine, and we would like to spell out and emphasize the difference.

For a simplicial spectrum A, recall that the n-skeleton $\mathrm{sk}_n A$ and n-coskeleton $\mathrm{cosk}_n A$ of A are defined respectively as the left and right Kan extensions of $A|_{(\Delta_0^n)^{\mathrm{op}}}$ to Δ^{op}. These are both simplicial spectra in their own right. Now, somewhat unconventionally, for a cosimplicial spectrum B, we define the n-skeleton $\mathrm{sk}^n B$ and n-coskeleton $\mathrm{cosk}^n B$ to be respectively the left and right Kan extensions of $B|_{\Delta_0^n}$ to Δ. These are, of course, both cosimplicial spectra.

It is standard to define

$$|A|_n := |\operatorname{sk}_n A| = \operatorname{sk}_n A \otimes_{\Delta^{\mathrm{op}}} \mathbf{\Delta}$$
$$\mathrm{Tot}^n B := \hom^{\Delta}(\operatorname{sk}_n \mathbf{\Delta}, B)$$

This is unsettling for two reasons. First, here $\mathrm{sk}_n \mathbf{\Delta}$ does not refer to a left Kan extension of the cosimplicial object $\mathbf{\Delta}$, but to a levelwise left Kan extension of the simplicial levels of $\mathbf{\Delta}$. Second, it does not express the layers Tot^n as totalizations in their own right, and moreover entirely obscures the precise duality between the skeletal filtration and the totalization tower.

Instead we prefer

$$|A|_n := |\operatorname{sk}_n A| = \operatorname{sk}_n A \otimes_{\Delta^{\mathrm{op}}} \mathbf{\Delta}$$
$$\mathrm{Tot}^n B := \mathrm{Tot}\operatorname{cosk}^n B = \hom^{\Delta}(\mathbf{\Delta}, \operatorname{cosk}^n B)$$

We leave it to the reader to verify that this results in the same spectrum $\mathrm{Tot}^n B$ as the usual formulation.

NOTE 3.15. As we could for the skeleta of a simplicial spectrum, we can build the coskeleta of our cosimplicial spectrum B inductively. In a classic example of mathematical nomenclature, the duals of latching objects are called matching objects:

$$M^n B = (\operatorname{cosk}^{n-1} B)^n.$$

Readers should be warned that this indexing is not the same as that in Goerss-Jardine or Bousfield-Kan; instead we specialize the abstractly consistent scheme of Hirschhorn. The inductive pullback diagram is

$$\begin{tikzcd}
\operatorname{cosk}^n B \arrow[r] \arrow[d] & (B^n)^{\Delta^n} \arrow[d] \\
\operatorname{cosk}^{n-1} B \arrow[r] & (B^n)^{\partial\Delta^n} \times_{(M^n B)^{\partial\Delta^n}} (M^n B)^{\Delta^n}
\end{tikzcd}$$

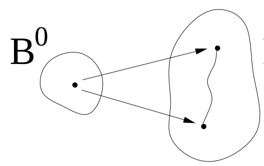
The totalization of this diagram gives a corresponding pullback for $\mathrm{Tot}^n B$ in terms of $\mathrm{Tot}^{n-1} B$.

Ignoring the terms involving matching objects, this pullback would indicate that $\mathrm{Tot}^n B$ can be seen as the pairs of maps $\phi : \Delta^n \to B^n$ and “points” $\psi \in \mathrm{Tot}^{n-1} B$ that agree as maps $\theta : \partial\Delta^n \to B^n$; here, ψ determines θ by the coface maps of B. This idea is illustrated in the following example. More precisely, though, the matching terms in the pullback account for the fact that the $(n-1)$-coskeleton already contains a collection of potential n-cosimplicies.

Example 3.16. We consider the first stage of the Tot tower. Ignoring codegeneracy issues, the totalization of the 1-coskeleton is, roughly, the limit

$$\lim \left((B^0)^{\Delta^0} \times (B^1)^{\Delta^1} \rightrightarrows (B^1)^{\Delta^0} \times (B^1)^{\Delta^0} \right),$$

which is to say a 0-cosimplex, together with a path of 1-cosimplicies agreeing at the ends with the cofaces of the 0-cosimplex:



By construction the limit of the Tot tower is $\operatorname{Tot} B$. There is the pesky issue of whether this limit is the same as the corrected homotopy limit of the tower—recall that, in the absence of $\lim^1$ problems, the spectral sequence of the tower has target the homotopy groups of the corrected homotopy limit. The condition on the cosimplicial spectrum B that tethers the limit and the corrected homotopy limit is, as expected, Reedy fibrancy.

Definition 3.17. A cosimplicial spectrum B is Reedy fibrant if the maps $B^n \to M^n B$ are fibrations. A tower of spectra $\cdots \to Y^1 \to Y^0 \to *$ is Reedy fibrant if the maps in the tower are all fibrations.

When a cosimplicial spectrum B is Reedy fibrant, the maps $\operatorname{Tot}^n B \to \operatorname{Tot}^{n-1} B$ are fibrations, so the Tot tower is Reedy fibrant. We saw in section 2.3.2 that the corrected homotopy limit represents the derived limit; in the case of a Reedy fibrant diagram, the honest limit also represents this derived limit, and the limit and corrected homotopy limit agree. When the cosimplicial spectrum B is Reedy fibrant, we therefore have a weak equivalence

$$\operatorname{corholim}_i \operatorname{Tot}^i B \underset{\mathrm{RF}}{\simeq} \lim_i \operatorname{Tot}^i B = \operatorname{Tot} B.$$

Remark 3.18. When the tower $\cdots \to \operatorname{Tot}^1 B \to \operatorname{Tot}^0 B \to *$ is Reedy fibrant, all the terms in the tower are fibrant, even though this is not explicitly part of the Reedy condition. This objectwise fibrancy implies that the corrected homotopy limit $\operatorname{corholim} \operatorname{Tot}^i B$ is equal to the homotopy limit $\operatorname{holim} \operatorname{Tot}^i B$, though we do not need to consider the latter homotopy limit.

We have a tower of spectra whose limit is the totalization of our cosimplicial spectrum, and a spectral sequence with target the homotopy of this totalization (provide we have Reedy fibrancy, and no $\lim^1$ problem). However, in the end we will be interested in the homotopy limit of the cosimplicial spectrum (not its totalization), because that homotopy limit will carry the global homotopical information in a sheaf of spectra. Therefore, we need to compare the totalization of the tower to the homotopy limit of the cosimplicial diagram itself:

Proposition 3.19 ([**Hi**, Thm 18.7.4]). *If the cosimplicial spectrum B is Reedy fibrant, then there is a natural weak equivalence (again called the Bousfield-Kan map)*

$$\operatorname{Tot} B \xrightarrow{\simeq} \operatorname{holim}_\Delta B.$$

The totalization is $\hom^\Delta(\mathbf{\Delta}, B)$, while the homotopy limit is $\hom^\Delta(N(\Delta/-), B)$. We already remarked that, glossing over degeneracy issues, the nerves of the under-categories $N(\Delta/-)$ are the barycentric subdivisions of the standard simplicies of $\mathbf{\Delta}$, and so this weak equivalence is unsurprising. The cosimplicial spectra coming from our sheaves of spectra will always be Reedy fibrant—see Lemma 3.20.

In summary, we have the following chain relating the target of the spectral sequence of the Tot tower to the homotopy of the homotopy limit of our cosimplicial spectrum:

$$\lim_i \pi_* \operatorname{Tot}^i B \underset{\lim^1}{\rightsquigarrow} \pi_* \operatorname{corholim}_i \operatorname{Tot}^i B \underset{\mathrm{RF}}{=} \pi_* \lim_i \operatorname{Tot}^i B = \pi_* \operatorname{Tot} B \underset{\mathrm{RF}}{=} \pi_* \operatorname{holim}_\Delta B.$$

Here the first arrow refers to the Milnor short exact sequence, and the indicated equalities depend on B being Reedy fibrant.

3.2.3. *Cosheaves and Simplicial Spectra, and Sheaves and Cosimplicial Spectra.* In the last section, we constructed a filtration of spectra out of a simplicial spectrum, and a tower of spectra out of a cosimplicial spectrum. In either case we have an associated exact couple and therefore spectral sequence. In this section, given a cosheaf (resp. sheaf) of spectra, we build a simplicial (resp. cosimplicial) spectrum, and we describe in detail the E^1 and E^2 terms of the resulting spectral sequence.

Let $\mathcal{C}$ be a site, for instance the étale site of the moduli stack of elliptic curves, and let $\mathfrak{U} = \{U_i \to U\}_{i\in I}$ be a cover in $\mathcal{C}$. Assuming we have coproducts in $\mathcal{C}$, this cover yields a simplicial object in $\mathcal{C}$:

$$\mathbf{U}. := \left(\coprod_i U_i \leftleftarrows \coprod_{i,j} U_{ij} \; \overset{\leftarrow}{\underset{\leftarrow}{\leftarrow}} \; \coprod_{i,j,k} U_{ijk} \; \overset{\leftarrow}{\underset{\leftarrow}{\leftleftarrows}} \; \cdots \right).$$

A precosheaf of spectra on $\mathcal{C}$ is a covariant functor $\mathcal{G} : \mathcal{C} \to \mathsf{Spec}$; similarly of course we may have precosheaves with values in other categories. If we apply such a precosheaf $\mathcal{G}$ to the simplicial object $\mathbf{U}.$, we get a simplicial spectrum:

$$\mathcal{G}(\mathbf{U}.) = \left(\mathcal{G}(\coprod U_i) \leftleftarrows \mathcal{G}(\coprod U_{ij}) \; \overset{\leftarrow}{\underset{\leftarrow}{\leftarrow}} \; \cdots \right).$$

A cosheaf of spectra is a precosheaf $\mathcal{G}$ such that for all objects U of the site $\mathcal{C}$ and for all covers $\mathfrak{U}$ of U, the map

$$\mathcal{G}(U) \xleftarrow{\simeq} \operatorname{hocolim}_{\Delta^{\mathrm{op}}} \mathcal{G}(\mathbf{U}.)$$

is a weak equivalence.

The spectral sequence associated to the the simplicial spectrum $\mathcal{G}(\mathbf{U}.)$ has the form

$$E^1_{pq} = \pi_{p+q}(\operatorname{corhocofib}(|\mathcal{G}(\mathbf{U}.)|_{q-1} \to |\mathcal{G}(\mathbf{U}.)|_q)) \Longrightarrow \pi_{p+q} \operatorname{corhocolim}_i |\mathcal{G}(\mathbf{U}.)|_i.$$

In order to better identify both the E^1 term and the target of this spectral sequence, we need to know that simplicial spectra built from precosheaves are well behaved:

Lemma 3.20. *Let $\mathcal{G}$ be a precosheaf on a site $\mathcal{C}$ with values in cofibrant objects of a (model) category $\mathcal{D}$; suppose moreover that $\mathcal{G}$ preserves coproducts. Let $\mathfrak{U} = \{U_i \to U\}$ be a cover in $\mathcal{C}$, and $\mathbf{U}. = (\coprod U_i \leftleftarrows \coprod U_{ij} \overset{\leftarrow}{\underset{\leftarrow}{\leftarrow}} \cdots)$ the associated simplicial object of $\mathcal{C}$. The simplicial object $\mathcal{G}(\mathbf{U}.)$ of $\mathcal{D}$ is Reedy cofibrant. Similarly, if $\mathcal{F}$ is a presheaf on $\mathcal{C}$ with values in the fibrant objects of $\mathcal{D}$, and if $\mathcal{F}$ takes coproducts to products, the associated cosimplicial object $\mathcal{F}(\mathbf{U}.)$ is Reedy fibrant.*

We assume our precosheaf $\mathcal{G}$ takes values in cofibrant spectra and preserves coproducts. The associated simplicial spectrum $\mathcal{G}(\mathbf{U}.)$ is therefore Reedy cofibrant, and the maps $|\mathcal{G}(\mathbf{U}.)|_{q-1} \to |\mathcal{G}(\mathbf{U}.)|_q$ are cofibrations between cofibrant spectra. The corrected homotopy cofibres of these maps, which appear in the E^1 term of the spectral sequence, are then simply ordinary cofibres. We can identify the cofibre of $|\mathcal{G}(\mathbf{U}.)|_{q-1} \to |\mathcal{G}(\mathbf{U}.)|_q$ explicitly as the q-fold suspension of the q-simplicies $\coprod_{|Q|=q} \mathcal{G}(U_Q)$ of the simplicial spectrum $\mathcal{G}(\mathbf{U}.)$:

$$\operatorname{cofib}\left(|\mathcal{G}(\mathbf{U}.)|_{q-1} \to |\mathcal{G}(\mathbf{U}.)|_q\right) = \Sigma^q\left(\coprod_{|Q|=q} \mathcal{G}(U_Q)\right).$$

For example, in picture 3.10 the cofibre of the inclusion of the 0-skeleton is evidently the 1-fold suspension of the 1-simplicies; (note that the 0-skeleton includes degeneracies of the 0-simplicies). The E^1 term of the spectral sequence is therefore $E^1_{pq} = \pi_p(\coprod_{|Q|=q} \mathcal{G}(U_Q))$. Tracing the d^1 differential from the exact couple through to this description of the E^1 term, we find the E^2 term of the spectral sequence is $E^2_{pq} = \check{H}^{\mathfrak{U}}_q(U, \pi_q\mathcal{G})$—the Cech homology of U with respect to the cover $\mathfrak{U}$ with coefficients in the precosheaf of abelian groups $(\pi_p\mathcal{G})(V) := \pi_p(\mathcal{G}(V))$.

Now suppose $\mathcal{G}$ is a cosheaf of cofibrant spectra. The simplicial spectrum $\mathcal{G}(\mathbf{U}.)$ is Reedy cofibrant, and the results of section 3.2.2 identify the target of the spectral sequence:

$$\operatorname{corhocolim}_i |\mathcal{G}(\mathbf{U}.)|_i \simeq \operatorname{colim}_i |\mathcal{G}(\mathbf{U}.)|_i = |\mathcal{G}(\mathbf{U}.)| \simeq \operatorname{hocolim}_{\Delta^{\mathrm{op}}} \mathcal{G}(\mathbf{U}.) \simeq \mathcal{G}(U).$$

The last equivalence here comes from the cosheaf condition. Altogether then, for such a cosheaf of spectra we have a strongly convergent spectral sequence

$$E^2_{pq} = \check{H}^{\mathfrak{U}}_q(U, \pi_p\mathcal{G}) \underset{\text{strong}}{\Longrightarrow} \pi_{p+q}\mathcal{G}(U).$$

The reader can imagine how the dual story progresses. Given a presheaf $\mathcal{F}$ of spectra, we have a cosimplicial spectrum

$$\mathcal{F}(\mathbf{U}.) = \left(\mathcal{F}(\coprod U_i) \rightrightarrows \mathcal{F}(\coprod U_{ij}) \overset{\to}{\rightrightarrows} \cdots\right).$$

The spectral sequence associated to this cosimplicial spectrum has the form:

$$E^1_{pq} = \pi_{p-q}(\operatorname{corhofib} \operatorname{Tot}^q \mathcal{F}(\mathbf{U}.) \to \operatorname{Tot}^{q-1} \mathcal{F}(\mathbf{U}.)) \Longrightarrow \lim_i \pi_{p-q} \operatorname{Tot}^i \mathcal{F}(\mathbf{U}.).$$

Suppose the presheaf $\mathcal{F}$ takes values in fibrant spectra, and takes coproducts to products. The cosimplicial spectrum $\mathcal{F}(\mathbf{U}.)$ is then Reedy fibrant, and the maps $\operatorname{Tot}^q \mathcal{F}(\mathbf{U}.) \to \operatorname{Tot}^{q-1} \mathcal{F}(\mathbf{U}.)$ in the Tot tower are fibrations between fibrant spectra. The corrected homotopy fibres appearing in the E^1 term of the spectral sequence are therefore ordinary fibres, which are explicitly identifiable. The fibre of the map $\operatorname{Tot}^q \mathcal{F}(\mathbf{U}.) \to \operatorname{Tot}^{q-1} \mathcal{F}(\mathbf{U}.)$ is, up to homotopy, the q-fold loop space of the q-cosimplicies $\prod_{|Q|=q} \mathcal{F}(U_Q)$ of our cosimplicial spectrum:

$$\operatorname{fib}(\operatorname{Tot}^q \mathcal{F}(\mathbf{U}.) \to \operatorname{Tot}^{q-1} \mathcal{F}(\mathbf{U}.)) \simeq \Omega^q(\prod_{|Q|=q} \mathcal{F}(U_Q)).$$

For example, for a cosimplicial spectrum B, the map $\operatorname{Tot}^1 B \to \operatorname{Tot}^0 B$ is, roughly speaking, projecting to the 0-cosimplicies B^0; the fibre over a given 0-cosimplex is an end-fixed path space in the 1-cosimplicies B^1, which (provided the

1-cosimplicies B^1 are connected) has the homotopy type of the loop space ΩB^1—consider picture 3.16. The E^1 term of the spectral sequence associated to the cosimplicial spectrum $\mathcal{F}(\mathbf{U}.)$ is now $E^1_{pq} = \pi_p(\prod_{|Q|=q} \mathcal{F}(U_Q))$. The corresponding E^2 term is $E^2_{pq} = \check{H}^q_{\mathfrak{U}}(U, \pi_p\mathcal{F})$, the Cech cohomology with coefficients in the presheaf $(\pi_p\mathcal{F})(V) := \pi_p(\mathcal{F}(V))$.

Provided there is no $\lim^1$ problem, that is if $\lim^1_i \pi_r \operatorname{Tot}^i \mathcal{F}(\mathbf{U}.) = 0$, the target of the spectral sequence is $\lim_i \pi_{p-q} \operatorname{Tot}^i \mathcal{F}(\mathbf{U}.) = \pi_{p-q} \operatorname{corholim}_i \operatorname{Tot}^i \mathcal{F}(\mathbf{U}.)$. Finally suppose that our presheaf $\mathcal{F}$ is actually a sheaf taking values in fibrant spectra. The associated cosimplicial spectrum is Reedy fibrant, and using the results of section 3.2.2 this allows us to further identify the target spectrum:

$$\operatorname{corholim}_i \operatorname{Tot}^i \mathcal{F}(\mathbf{U}.) \simeq \lim_i \operatorname{Tot}^i \mathcal{F}(\mathbf{U}.) = \operatorname{Tot} \mathcal{F}(\mathbf{U}.) \simeq \operatorname{holim}_\Delta \mathcal{F}(\mathbf{U}.) \simeq \mathcal{F}(U).$$

The last equivalence is the sheaf condition on $\mathcal{F}$. The spectral sequence, at long last, is

$$E^2_{pq} = \check{H}^q_{\mathfrak{U}}(U, \pi_p\mathcal{F}) \underset{\text{cond,no}\lim^1}{\Longrightarrow} \pi_{p-q}\mathcal{F}(U).$$

3.3. The spectral sequence for $\pi_* TMF$. We specialize the spectral sequence of Section 3.2 to the particular sheaf $\mathcal{O}^{\text{top}}$ of (fibrant) spectra on the moduli stack $(\mathcal{M}_{ell})_{\text{ét}}$ of elliptic curves in the étale topology. We use particular properties of this sheaf and the results of Section 3.1 to identify the E^2 term as a sheaf cohomology of the moduli stack. Moreover, we address the $\lim^1$ problem for the relevant tower of spectra, thereby pinning down the target of the spectral sequence as the spectrum of global sections $\mathcal{O}^{\text{top}}(\mathcal{M}_{ell})$.

For a cover $\mathfrak{U} = \{U_i \to \mathcal{M}_{ell}\}$ of $\mathcal{M}_{ell}$ in the site $(\mathcal{M}_{ell})_{\text{ét}}$, and $\mathbf{U}.$ the associated simplicial object of $(\mathcal{M}_{ell})_{\text{ét}}$, the spectral sequence associated to $\mathcal{O}^{\text{top}}$ and $\mathfrak{U}$ has the form

$$E^2_{pq} = \check{H}^q_{\mathfrak{U}}(\mathcal{M}_{ell}, \pi_p\mathcal{O}^{\text{top}}) \underset{\text{cond}}{\Longrightarrow} \lim_i \pi_{p-q} \operatorname{Tot}^i \mathcal{O}^{\text{top}}(\mathbf{U}.).$$

We can remove the dependence of the E^2 term on the particular cover $\mathfrak{U}$ by restricting attention to covers of the moduli stack by affine schemes.

Proposition 3.21. *Suppose the cover $\mathfrak{U} = \{U_i \to \mathcal{M}_{ell}\}$ is by affine schemes U_i. In this case, for any collection of indices $J = \{i_1, \ldots, i_j\}$, the value of the presheaf $\pi_p\mathcal{O}^{\text{top}}$ on U_J is the same as the corresponding value of the sheafification of $\pi_p\mathcal{O}^{\text{top}}$:*

$$\pi_p\mathcal{O}^{\text{top}}(U_J) = (\pi^\dagger_p\mathcal{O}^{\text{top}})(U_J).$$

It follows from this proposition that there is an equality of Cech cohomology groups:

$$\check{H}^q_{\mathfrak{U}}(\mathcal{M}_{ell}, \pi_p\mathcal{O}^{\text{top}}) = \check{H}^q_{\mathfrak{U}}(\mathcal{M}_{ell}, \pi^\dagger_p\mathcal{O}^{\text{top}}).$$

The same affine condition on the cover gets us the rest of the way to sheaf cohomology:

Proposition 3.22. *Suppose again the cover $\mathfrak{U}$ is by affine schemes. Then for all J, the intersection U_J is acyclic for the sheaf $\pi^\dagger_p\mathcal{O}^{\text{top}}$, that is*

$$H^i(U_J, \pi^\dagger_p\mathcal{O}^{\text{top}}) = 0, \quad i > 0.$$

By the general Proposition 3.4, the acyclicity of U_J for the sheaf $\pi^\dagger_p\mathcal{O}^{\text{top}}$ implies that in this case Cech and sheaf cohomology agree:

$$\check{H}^q_{\mathfrak{U}}(\mathcal{M}_{ell}, \pi^\dagger_p\mathcal{O}^{\text{top}}) = H^q(\mathcal{M}_{ell}, \pi^\dagger_p\mathcal{O}^{\text{top}}).$$

Having identified the E^2 term, we reconsider the target of the spectral sequence. Both the convergence properties of the spectral sequence and the potential $\lim^1$ term are controlled by the same finiteness condition on the differentials—see Boardman [**Bo**]:

PROPOSITION 3.23. *Let $E^1_{pq} = \pi_{p-q}(\operatorname{corhofib}\phi_q) \Longrightarrow \lim_i \pi_{p-q} F^i$ be the spectral sequence associated to the tower of spectra $\cdots \to F^2 \xrightarrow{\phi_2} F^1 \xrightarrow{\phi_1} F^0 \xrightarrow{\phi_0} *$. If for all p and q there are only finitely many r such that the differential originating at E^r_{pq} is nonzero, then the spectral sequence convergences strongly and $\lim^1_i \pi_{p-q-1} F^i = 0$.*

The spectral sequence associated to the sheaf $\mathcal{O}^{\mathrm{top}}$ evaluated on an affine cover of the moduli stack $\mathcal{M}_{ell}$ has the feature that there are only finitely many nonzero differentials originating at any term.

COROLLARY 3.24. *For $\mathbf{U}.$ the simplicial object of $(\mathcal{M}_{ell})_{\acute{e}t}$ associated to an affine cover $\mathfrak{U}$ of the moduli stack $\mathcal{M}_{ell}$, the $\lim^1$ term in the Milnor sequence for $\operatorname{Tot}^* \mathcal{O}^{\mathrm{top}}(\mathbf{U}.)$ vanishes:*

$$\lim^1_i \pi_{p-q-1} \operatorname{Tot}^i \mathcal{O}^{\mathrm{top}}(\mathbf{U}.) = 0.$$

The target $\lim_i \pi_{p-q} \operatorname{Tot}^i \mathcal{O}^{\mathrm{top}}(\mathbf{U}.)$ of our spectral sequence is therefore equal to $\pi_{p-q} \operatorname{corholim}_i \operatorname{Tot}^i \mathcal{O}^{\mathrm{top}}(\mathbf{U}.)$, which is, in turn, equal to $\pi_{p-q}\mathcal{O}^{\mathrm{top}}(\mathcal{M}_{ell})$. The spectrum *TMF* is by definition this spectrum $\mathcal{O}^{\mathrm{top}}(\mathcal{M}_{ell})$ of global sections of the sheaf $\mathcal{O}^{\mathrm{top}}$.

Though we have restricted our attention to the moduli stack $\mathcal{M}_{ell}$ of smooth elliptic curves, the sheaf $\mathcal{O}^{\mathrm{top}}$ extends to the Deligne-Mumford compactification $\overline{\mathcal{M}}_{ell}$. The process we have described also provides a spectral sequence for the homotopy of the spectrum $\mathcal{O}^{\mathrm{top}}(\overline{\mathcal{M}}_{ell})$ of global sections over the compactification—this spectrum is denoted *Tmf*. Altogether then, we have reached the end of our road:

PROPOSITION 3.25. *There are strongly convergent spectral sequences*

$$E^2_{pq} = H^q(\mathcal{M}_{ell}, \pi^\dagger_p \mathcal{O}^{\mathrm{top}}) \Longrightarrow \pi_{p-q}\, TMF,$$
$$E^2_{pq} = H^q(\overline{\mathcal{M}}_{ell}, \pi^\dagger_p \mathcal{O}^{\mathrm{top}}) \Longrightarrow \pi_{p-q}\, Tmf.$$

As the spectrum *tmf* is by definition the connective cover of *Tmf*, the second spectral sequence gives in particular a means of computing the homotopy groups of *tmf*.

Appendix. Degenerate simplicies and codegenerase cosimplicies

We hope we are not alone in thinking that cosimplicial objects, totalization, matching objects, and Reedy fibrancy appear at first more abstruse than the dual notions of simplicial objects, realization, latching objects, and Reedy cofibrancy. We think this is in part due to some missing terminology, which we advertise here. We restrict attention to pointed model categories $\mathcal{C}$ that behave like the category of based spaces in the sense that fibrations are categorical epimorphisms, and the image of a cofibration $P \hookrightarrow Q$ is isomorphic to its source P.

Let A be a simplicial object in such a model category $\mathcal{C}$. The object A_n is called, of course, the *n-simplicies* of A. This object receives degeneracy maps from the k-simplicies A_k for $k < n$. We would like to build an object $\operatorname{Dgnt}_n A$, the *degenerate n-simplicies*, that contains precisely the *targets* of these degeneracy maps in A. In order to do this, we first use a left Kan extension to glue together the possible n-simplicies that could be targets of degeneracy maps from the k-simplicies A_k for $k < n$—this forms the latching object L_nA, which we think of as the *possible degenerate n-simplicies* of A. This latching object was described in section 3.2.2, where we noted that "possible" refers to n-simplicies that may exist in simplicial objects agreeing with A in levels less than n.

There is a natural map $\lambda_n : L_nA \to A_n$ from the n-th latching object to the n-simplicies. Define the object $\operatorname{Dgnt}_n A$ of degenerate n-simplicies of A to be the fibre of the cofibre (that is the image) of the map λ_n. The degenerate n-simplicies $\operatorname{Dgnt}_n A$ are the possible degenerate simplicies that actually occur in A. (Note that in an arbitrary model category, it would be wiser to define the degenerate n-simplicies to be the cofibre of the fibre (that is the coimage) of λ_n; however, in topological contexts coimage rarely coincides with our intuition about the target of a map, and we scuttle the coimage formulation.)

Recall that A is Reedy cofibrant if $\lambda_n : L_nA \to A_n$ is a cofibration. This occurs precisely when the map $L_nA \to \operatorname{Dgnt}_n A$ is an isomorphism and the map $\operatorname{Dgnt}_n A \to A_n$ is a cofibration—the first condition says that all possible degeneracies actually occur, which is to say the degenerate simplicies are freely generated, while the second condition says that the degenerate simplicies include into all simplicies by a cofibration. This pair of conditions is a convenient mnemonic for Reedy cofibrancy of simplicial diagrams.

Now let B be a cosimplicial object in the category $\mathcal{C}$. The object B^n is called the *n-cosimplicies* of B. The n-cosimplicies map by codegeneracy maps to the k-cosimplicies, for $k < n$. We would like to build an object that encodes information about the *sources* of these codegeneracy maps—we will call the resulting object $\operatorname{Codgns}^n B$, the *codegenerase n-cosimplicies*. Note well that these are not the "codegenerate n-cosimplicies", a term which would refer to n-cosimplicies in the target of codegeneracy maps, and this object also has nothing to do with coface maps. In order to build this codegenerase object, we first use a right Kan extension to assemble the possible n-cosimplicies that could be sources of codegeneracy maps to the k-cosimplicies B^k for $k < n$—this forms the matching object M^nB, which we think of as the *possible codegenerase n-cosimplicies* of B. Matching objects were defined in section 3.2.2; the "possible" here refers to cosimplicies that could appear in cosimplicial objects agreeing with B below level n. (Pedantically speaking, we could emphasize the left versus right Kan extension by thinking of L_nA as the "copossible degenerate" n-simplicies and M^nB as the "possible codegenerase" n-cosimplicies, but we draw a line before the term "copossible".)

There is a natural map $\mu^n : B^n \to M^nB$ from the n-cosimplicies to the n-th matching object. Define the object $\operatorname{Codgns}^n B$ of codegenerase n-cosimplicies of B to be the fibre of the cofibre (that is the image) of the map μ^n. (Note that we do use the image here, not as one might expect the coimage.) The codegenerase n-cosimplicies $\operatorname{Codgns}^n B$ is the object of possible codegenerase cosimplicies that actually occur in B.

The cosimplicial object B is Reedy fibrant if $\mu^n : B^n \to M^nB$ is a fibration. This happens exactly when the map $B^n \to \operatorname{Codgns}^n B$ is a fibration and the map $\operatorname{Codgns}^n B \to M^n B$ is an isomorphism—the first condition is that the n-cosimplicies map by a fibration onto the codegenerase cosimplicies, and the second condition is that all possible codegenerase cosimplicies occur, which is to say that the codegenerase cosimplicies are cofreely generated. This provides a convenient perspective on the meaning of Reedy fibrancy for cosimplicial diagrams.

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Bousfield localization and the Hasse square

Tilman Bauer

1. Bousfield localization

The general idea of localization at a spectrum E is to associate to any spectrum X the "part of X that E can see", denoted by $L_E X$. In particular, it is desirable that L_E is a functor with the following equivalent properties:

$$E \wedge X \simeq * \Longrightarrow L_E X \simeq *$$

$$\text{If } X \to Y \text{ induces an equivalence } E \wedge X \to E \wedge Y \text{ then } L_E X \xrightarrow{\simeq} L_E Y.$$

Definition 1.1. A spectrum X is called *E-acyclic* if $E \wedge X \simeq *$. It is called *E-local* if for each E-acyclic T, $[T, X] = 0$, where $[T, X]$ denotes the group of stable homotopy classes. A map of spectra $f\colon X \to Y$ is called an *E-equivalence* if $E \wedge f\colon E \wedge X \to E \wedge Y$ is a homotopy equivalence. It is immediate that a spectrum X is E-local iff for each E-equivalence $S \to T$, the induced map $[T, X] \to [S, X]$ is an isomorphism.

A spectrum Y with a map $X \to Y$ is called an *E-localization* of X if Y is E-local and $X \to Y$ is an E-equivalence.

If a localization of X exists, then it is unique up to homotopy and will be denoted by $X \xrightarrow{\eta_E} L_E X$.

Localizations of this kind were first studied by Adams [**Ad**], but set-theoretic difficulties prevented him from actually constructing them. Bousfield found a way of overcoming these problems in the unstable category [**Bou1**]; for spectra, he showed in [**Bou2**] that localization functors exist for arbitrary E.

We start by collecting a couple of easy facts about localizations.

Lemma 1.2. *Module spectra over a ring spectrum E are E-local.*

Proof. Since any map from a spectrum W into an E-module spectrum M can be factored through $E \wedge W$, it follows that all maps from an E-acyclic W into M are nullhomotopic. □

Lemma 1.3. *If $v \in \pi_*(E)$ is an element of a ring spectrum E (of an arbitrary but homogeneous degree), then $L_E \simeq L_{v^{-1}E \vee E/v}$, where E/v denotes the cofiber of multiplication with v and*

$$v^{-1}E = \operatorname{colim}\left(E \xrightarrow{v} E \xrightarrow{v} \cdots\right)$$

the mapping telescope.

Proof. It suffices to show that the class of E-acyclics agrees with the class of $(v^{-1}E \vee E/v)$-acyclics. Since $L_{v^{-1}E\vee E/v}$ is a module spectrum over E, E-acyclics are clearly $(v^{-1}E\vee E/v)$-acyclic; conversely, if $E/v\wedge W \simeq *$ then $v\colon E\wedge W \to E\wedge W$ is a homotopy equivalence, hence $E \wedge W \simeq v^{-1}E \wedge W$. Thus if also $v^{-1}E \wedge W \simeq *$, W is E-acyclic. $\square$

Lemma 1.4. *Homotopy limits and retracts of E-local spectra are E-local.*

Proof. The statement about retracts is obvious. For the statement about limits, first observe that a spectrum X is E-local if and only if the mapping spectrum $\mathrm{Map}(T, X)$ is contractible for all E-acyclic T. This is obvious because $\pi_k \mathrm{Map}(T, X) = [\Sigma^k T, X]$, and if T is E-acyclic then so are all its suspensions.

Now if $F\colon I \to \{\text{spectra}\}$ is a diagram of E-local spectra, the claim follows from the equivalence

$$\mathrm{Map}(T, \mathrm{holim}\, F) \simeq \mathrm{holim}\, \mathrm{Map}(T, F)$$

$\square$

The following lemma characterizes E-localizations.

Lemma 1.5. *The following are equivalent for a map of spectra $X \to Y$:*

- *$X \to Y$ is an E-localization;*
- ① *$X \to Y$ is the initial map into an E-local target;*
- ② *$X \to Y$ is the terminal map which is an E-equivalence.*

Proof. Obvious from the axioms. $\square$

This characterization suggests two ways of constructing $X \to L_E X$:

① $L_E X = \underset{\substack{X\to Y \\ Y\, E\text{-local}}}{\mathrm{holim}}\, Y$ or

② $L_E X = \underset{\substack{X\to Y \\ E\text{-equivalence}}}{\mathrm{hocolim}}\, Y$.

In both cases, these limits are not guaranteed to exist because the indexing categories are not small. This is more than a set-theoretic nuisance and requires a deeper study of the structure of the background categories.

I will first briefly discuss what can be done with approach ①. The main construction will be closer to method ②.

① **Localizations as limits.** For a ring spectrum E, instead of indexing the homotopy limit over all $X \to Y$ with Y E-local, we could use the spaces in the Adams tower for E:

$$X \to \mathrm{Tot}^n\left(E^{\wedge(\bullet+1)} \wedge X\right),$$

which is a subdiagram because $E \wedge X$ is E-local for any X by Lemma 1.2, and E-locality satisfies the 2-out-of-3 property for cofibration sequences of spectra. For this cosimplicial construction to make sense, the ring spectrum E has to be associative in a strict sense (e.g. in the framework of [**EKMM**]) or at least A_∞ [**?**]), or one can restrict to *cofacial* spectra: A cofacial spectrum is a functor from Δ_f to spectra, where Δ_f is the subcategory of Δ with the same objects but only injective maps. In that case, Tot is just defined as the homotopy limit, and one can show that this agrees with the cosimplicial Tot if the cofacial spectrum is the underlying cofacial spectrum of a cosimplicial spectrum. Note that in this approach, no multiplication on E is needed whatsoever – this works with any coaugmented spectrum $\mathbb{S} \to E$.

If we are lucky, $X \to X_{\hat{E}} =_{\mathrm{def}} \mathrm{Tot}(E^{\wedge(\bullet+1)} \wedge X)$ is an E-localization. This is not always the case – $X \to X_{\hat{E}}$ sometimes fails to be an E-equivalence. Whether or not $L_E X \simeq X_{\hat{E}}$, the latter is what the E-based Adams-Novikov spectral sequence converges to and thus is of independent interest. If $L_E X$ can be built from E-module spectra by a finite sequence of cofiber extensions and retracts, then $L_E X \simeq X_{\hat{E}}$ [**Bou2**, Thm 6.10] (such spectra are called E-prenilpotent). For some spectra E, every X is E-prenilpotent; these spectra have the characterizing property that their Adams spectral sequence has a common horizontal vanishing line at E_∞ and a horizontal stabilization line at every E_r for every finite CW-spectrum [**Bou2**, Thm 6.12]. A necessary condition for this is that E is *smashing*, i.e., that $L_E X = X \wedge L_E S^0$ for every spectrum X.

② **Localizations as colimits.** Bousfield's approach to constructing localizations uses colimits. The basic idea for cutting down the size of the diagram the colimit is formed over is the following observation:

To check if X is E-local, it is enough to show that for any E-equivalence $S \to T$ with $\#S$, $\#T < \kappa$ for some cardinal κ depending only on E, $[T, X] \xrightarrow{\cong} [S, X]$.

At this point, it is not crucial what exactly we mean by $\#S$. For a construction of $L_E X$ that is functorial up to homotopy, it is enough to define $\#S$ to be the number of stable cells.

Given this observation, $L_E X$ can be constructed in a small-object-argument-like fashion by forming homotopy pushouts

$$\begin{tikzcd} \coprod\limits_{\substack{S \to T \\ E\text{-eq.}}} S \arrow[r] \arrow[d] & X \arrow[d] \\ \coprod\limits_{\substack{S \to T \\ E\text{-eq.}}} T \arrow[r] & X_{(1)} \end{tikzcd}$$

and iterating this transfinitely (using colimits at limit ordinals). When the cardinal κ is reached, $X_{(\kappa)}$ is E-local because it satisfies the lifting condition for "small" $S \to T$.

THEOREM 1.6. *The category of spectra has a model structure with*

- *cofibrations the usual cofibrations of spectra, i.e. levelwise cofibrations $A_n \to B_n$ such that*
$$\mathbb{S}^1 \wedge B_n \cup_{\mathbb{S}^1 \wedge A_n} A_{n+1} \to B_{n+1}$$
are also cofibrations;
- *weak equivalences the (stable) E-equivalences;*
- *fibrations given by the lifting property*

The fibrant objects in this model structure are the E-local Ω-spectra.

Here are some explicit examples of localization functors.

EXAMPLE 1.7. (1) $E = \mathbb{S}^0$. In this case, L_E is the functor that replaces a spectrum by an equivalent Ω-spectrum.
(2) $E = M(\mathbb{Z}_{(p)})$ = Moore spectrum. In this case $L_E X \simeq X_{(p)}$ is the Bousfield p-localization. This is an example of a *smashing localization*, i.e. $L_E X \simeq X \wedge L_E \mathbb{S}^0$, which in this case is also the same as $X \wedge E$.

(3) $E = M(\mathbb{Z}/p)$. For connective X, $L_E X \simeq X_p^\wedge$ is the p-completion functor

$$X_p^\wedge = \operatorname{holim}\{\cdots \to X \wedge M(\mathbb{Z}/p^2) \to X \wedge M(\mathbb{Z}/p)\}.$$

We write $X \xrightarrow{\eta_p} L_p X$ for this localization $X \xrightarrow{\eta_E} L_E X$.

(4) $E = M(\mathbb{Q}) = H\mathbb{Q}$. As in (2), $L_E X = X \wedge L_{\mathbb{Q}}\mathbb{S}^0 = X \wedge H\mathbb{Q}$ is smashing; it is the rationalization of X. As in the previous case, we write $X \xrightarrow{\eta_{\mathbb{Q}}} L_{\mathbb{Q}} X$ for this localization $X \xrightarrow{\eta_E} L_E X$.

2. The Sullivan arithmetic square

The arithmetic square is a homotopy cartesian square that allows one to reconstruct a space if, roughly, all of its mod-p-localizations and its rationalization are known. For the case of nilpotent spaces, which is similar to spectra, this was first observed by Sullivan [**Sul**].

LEMMA 2.1. *For any spectrum X, the following diagram is a homotopy pullback square:*

$$\begin{tikzcd} X \arrow[r, "\prod \eta_p"] \arrow[d, "\eta_{\mathbb{Q}}"'] & \prod_p L_p X \arrow[d, "\eta_{\mathbb{Q}}"] \\ L_{\mathbb{Q}} X \arrow[r, "L_{\mathbb{Q}}(\prod \eta_p)"] & L_{\mathbb{Q}}\left(\prod_p L_p X\right) \end{tikzcd}$$

This is a special case of

PROPOSITION 2.2. *Let E, F, X be spectra with $E_*(L_F X) = 0$. Then there is a homotopy pullback square*

$$\begin{tikzcd} L_{E\vee F} X \arrow[r, "\eta_E"] \arrow[d, "\eta_F"'] & L_E X \arrow[d, "\eta_F"] \\ L_F X \arrow[r, "L_F(\eta_E)"] & L_F L_E X \end{tikzcd}$$

In the case of Prop. 2.1, $E = \bigvee_p M(\mathbb{Z}/p)$, $F = H\mathbb{Q}$. To see that $L_E = \prod_p L_p$, we have to show that there are no nontrivial homotopy classes from an E-acyclic space to a spectrum of the form $\prod_p L_p X$, which is immediate, and that

$$M(\mathbb{Z}/p)_*(X) \xrightarrow{\cong} M(\mathbb{Z}/p)_*\left(\prod_l L_l X\right)$$

is an isomorphism for all p. The latter holds because smashing with $M(\mathbb{Z}/p)$ commutes with products since $M(\mathbb{Z}/p)$ is a finite (two-cell) spectrum (use Spanier-Whitehead duality).

Furthermore, the condition $E_*(L_F X) = E_*(H\mathbb{Q} \wedge X) = 0$ is satisfied because $H_*(M(\mathbb{Z}/p); \mathbb{Q}) = 0$.

PROOF OF THE PROPOSITION. Note that the map denoted η_E in the diagram is the unique factorization of $\eta_E \colon X \to L_E X$ through $L_{E\vee F} X$, which exists because $X \to L_{E\vee F} X$ is an E-equivalence. The same holds for η_F, and furthermore, these maps are E- and F-equivalences, respectively. Now let P be the pullback. We need to see that (1) P is $(E \vee F)$-local and (2) the induced map $X \to P$ is an E- and an F-equivalence. For (1), take a spectrum T with $E_*T = F_*T = 0$. Then in the Mayer-Vietoris sequence for the pullback,

$$\cdots \to [T, P] \to [T, L_E X] \oplus [T, L_F X] \to [T, L_F L_E X] \to \ldots,$$

the two terms on the right are zero, hence so is $[T, P]$.

For (2), observe that $P \to L_F X$ is an F-equivalence because it is the pullback of η_F on $L_E X$, and since $X \to L_F X$ is also an F-equivalence, so is $X \to P$. The same argument works for $P \to L_E X$ except that here, the bottom map is an E-equivalence for the trivial reason that both terms are E-acyclic by the assumption. □

3. Morava K-theories and related ring spectra

Given a complex oriented even ring spectrum E and an element $v \in \pi_* E$, we would like to construct a new complex oriented ring spectrum E/v such that $\pi_*(E/v) = (\pi_* E)/(v)$. This is clearly not always possible. The machinery of commutative $\mathbb{S}$-algebras of [**EKMM**] (or any other construction of a symmetric monoidal category of spectra, such as symmetric spectra) allows us to make this work in many cases where more classical homotopy theory has to rely on ad-hoc constructions (such as the Baas-Sullivan theory of bordism of manifolds with singularities).

In this section, let E be a complex oriented even commutative $\mathbb{S}$-algebra and A an E-module spectrum with a commutative ring structure in the homotopy category of E-modules, and which is also a complex oriented even ring spectrum. Let us call this an E-even ring spectrum. A commutative E-algebra would of course be fine, but we need the greater generality.

THEOREM 3.1 ([**EKMM**, Chapter V]). *For any $v \in \pi_* E$, $v^{-1}A$ carries the structure of an E-even ring spectrum. Furthermore, if v is a non-zero divisor then A/v is also an E-even ring spectrum.*

Even if A is a commutative E-algebra (for example, $A = E$), the resulting spectrum is usually not a commutative $\mathbb{S}$-algebra.

Of course, this construction can be iterated to give

COROLLARY 3.2. *Given a graded ideal $I \lhd \pi_* E$ generated by a regular sequence and a graded multiplicative set $S \subset \pi_* E$, one can construct an E-even ring spectrum $S^{-1}A/I$ with $\pi_* S^{-1}A/I = S^{-1}(\pi_* A)/I$.*

In particular, this can be done for $E = MU$. For example, BP can be constructed in this way by taking $I = \ker(MU_* \to BP_*)$, which is generated by a regular sequence. It is currently not known whether BP is a commutative $\mathbb{S}$-algebra. However, the methods above allow us to construct all the customary BP-ring spectra by pulling regular sequence back to $E = MU_*$ and letting $A = BP$. For example,

$$\begin{aligned} E(n) &= v_n^{-1} BP/(v_{n+1}, v_{n+2}, \ldots) \\ K(n) &= v_n^{-1} BP/(p, v_1, \ldots, v_{n-1}, v_{n+1}, \ldots) \\ P(n) &= BP/(p, v_1, \ldots, v_{n-1}) \\ B(n) &= v_n^{-1} BP/(p, v_1, \ldots, v_{n-1}). \end{aligned}$$

Any MU-even ring spectrum A gives rise to a Hopf algebroid (A_*, A_*A) and an Adams-Novikov spectral sequence

$$E^2_{**} = \mathrm{Cotor}_{A_*A}(A_*, A_*X) \Longrightarrow \pi_* X\hat{_A}.$$

If $\mathcal{M}_A$ denotes the stack associated to the Hopf algebroid (A_*, A_*A) and F_X the graded sheaf associated with the comodule A_*X, this E^2-term can be expressed as

$$E^2_{**} = H^{**}(\mathcal{M}_A, F_X),$$

which is the cohomology of the stack $\mathcal{M}_A$ with coefficients in the sheaf F_X.

In particular, if $f\colon A \to B$ is a morphism of MU-even ring spectra, we get a morphism of spectral sequences, and if f induces an equivalence of the associated stacks, then f induces an isomorphism of spectral sequences from the E_2-term on. In particular, in this case, $X\hat{_A} \simeq X\hat{_B}$ if we can assure that the spectral sequences converge strongly. Note that we do not need an inverse map $B \to A$.

Theorem 3.3. *If $f\colon A \to B$ is a morphism of MU-even ring spectra inducing an equivalence of associated stacks, then $L_A \simeq L_B$.*

Proof. The argument outline above gives an almost-proof of this fact, but it puts us at the mercy of the convergence of the Adams-Novikov spectral sequences to the localizations $L_A X$ and $L_B X$. We give an argument that doesn't require such additional assumptions. Note that it is sufficient to show that $A_*X = 0$ if and only if $B_*X = 0$. Assume $A_*X = 0$. Then the A-based Adams-Novikov spectral sequence is 0 from E^1 on, thus the B-based Adams-Novikov spectral sequence is also trivial from E^2 on. This time, the spectral sequence converges strongly because it is conditionally convergent in the sense of Boardman [**Boa**], which implies strong convergence if the derived E_∞-term is 0 – but this is automatic since the E_r-terms are all trivial for $r \geq 0$. Thus $X\hat{_B}$ is contractible.

Now the Hurewicz map $X \to B \wedge X$ factors as $X \to L_B X \to X\hat{_B} \to B \wedge X$ by the universal property ① of the localization, since $X\hat{_B}$ is B-local. Thus $X \to B \wedge X$ is trivial. Using the ring spectrum structure on B, we see that $B \wedge X \to B \wedge B \wedge X \xrightarrow{\mu} B \wedge X$, which is the identity, is also trivial, so $B \wedge X \simeq *$. □

In particular, this applies to the following cases:

Theorem 3.4. *We have*

$$L_{B(n)} \simeq L_{K(n)}$$

Let $I_n = (p, v_1, \dots, v_{n-1}) \lhd BP_*$ and $E(k,n) = E(n)/I_k$ for $0 \leq k \leq n \leq \infty$. Then

$$L_{v_k^{-1}E(k,n)} \simeq L_{K(k)}.$$

Proof. The first part is due to Ravenel [**Rav**] and Johnson-Wilson [**JW**], but they give a different proof without the Adams-Novikov spectral sequence.

To apply Theorem 3.3, it is useful to extend the ground ring of the homology theories in question from $\mathbb{F}_p$ to $\mathbb{F}_{p^n}$, which does not change their localization functors. The Hopf algebroids for $B(n) \otimes \mathbb{F}_{p^n}$ and $K(n) \otimes \mathbb{F}_{p^n}$ both classify formal groups of height n. By Lazard's theorem, there is only one such group over $\mathbb{F}_{p^n}$ up to isomorphism, which shows that the quotient map $B(n) \otimes \mathbb{F}_{p^n} \to K(n) \otimes \mathbb{F}_{p^n}$ induces an isomorphism of Hopf algebroids.

The second part works similarly by considering the maps of Hopf algebroids induced from

$$v_k^{-1}E(k,n) \leftarrow B(k)/(v_{n+1}, v_{n+2}, \dots) \to K(k)$$

which again all represent the stack of formal groups of height k.

$\square$

THEOREM 3.5. *We have that*

$$L_{E(n)} \simeq L_{K(0)\vee K(1)\vee\cdots\vee K(n)} \simeq L_{v_n^{-1}BP}.$$

PROOF. With the notation of Theorem 3.4, since $E(n,n) = K(n)$ and $E(0,n) = E(n)$, it suffices to show that

$$L_{E(k,n)} \simeq L_{K(k)\vee E(k+1,n)}.$$

By Lemma 1.3, $L_{E(k,n)} \simeq L_{v_k^{-1}E(k,n)\vee E(k+1,n)}$. By Theorem 3.4, $L_{v_k^{-1}E(k,n)} \simeq L_{K(k)}$, and the result follows by induction.

The second equivalence can be proved by a similar argument, not needed here, and left to the reader. $\square$

THEOREM 3.6. *There is a homotopy pullback square*

$$\begin{tikzcd} L_{K(1)\vee K(2)}X \arrow[r, "\eta_{K(2)}"] \arrow[d, "\eta_{K(1)}"'] & L_{K(2)}X \arrow[d, "\eta_{K(1)}"] \\ L_{K(1)}X \arrow[r, "L_{K(1)}(\eta_{K(2)})"] & L_{K(1)}L_{K(2)}X \end{tikzcd}$$

PROOF. This is an application of Prop. 2.2. We need to see that $K(2)_*(L_{K(1)}X) = 0$ for any X. To see this, let $\alpha\colon \Sigma^k M(\mathbb{Z}/p) \to M(\mathbb{Z}/p)$ be the Adams map, which induces multiplication with a power of v_1 in $K(1)$ and is trivial in $K(2)$. Here $k = 2p-2$ for odd p and $k = 8$ for $p = 2$).

Let X be $K(1)$-local. Then so is $X \wedge M(\mathbb{Z}/p)$, and since $\Sigma^k X \wedge M(\mathbb{Z}/p) \xrightarrow{\alpha} X \wedge M(\mathbb{Z}/p)$ is a $K(1)$-isomorphism, it is a homotopy equivalence. On the other hand, $\alpha_*\colon K(2)_*(\Sigma^k X\wedge M(\mathbb{Z}/p)) \to K(2)_*(X\wedge M(\mathbb{Z}/p))$ is trivial, thus $K(2)_*(X\wedge M(\mathbb{Z}/p)) = 0$. By the Künneth isomorphism, $K(2)_*(X) = 0$. $\square$

The same result holds true for any $K(m)$ and $K(n)$ with $m < n$; $M(\mathbb{Z}/p)$ and α then have to be replaced by a type-m complex and its v_m-self map in the argument. We briefly recall some basic facts around the periodicity theorem.

DEFINITION 3.7. A finite p-local CW-spectrum X has *type n* if $K(n)_*(X) \neq 0$ but $K(k)_*(X) = 0$ for $k < n$. For example, the sphere has type 0, the Moore spectrum $M(\mathbb{Z}/p)$ has type 1, and the the cofiber of the Adams map has type 2.

THEOREM 3.8 ([**DHS, HS**]). *Every type-n spectrum X admits a v_n-*self map*, i. e. a map $f\colon \Sigma^? X \to X$ which induces multiplication by a power of v_n in $K(n)_*(X)$.*

The periodicity theorem implies that there exist type-n complexes for every $n \in \mathbb{N}$. They can be constructed iteratively, starting with the sphere, by taking cofibers of v_k-self maps. Thus, there exist multi-indices $I = (i_0, \ldots, i_{n-1})$ and spectra $\mathbb{S}^0/(v^I)$ such that $BP_*(\mathbb{S}^0/(v^I)) = BP_*/(v^I)$, where $(v^I) = (p^{i_0}, v_1^{i_1}, \ldots, v_{n-1}^{i_{n-1}})$. These are sometimes called *generalized Moore spectra*. It is an open question what the minimal values of I are (they certainly depend on the prime.)

4. The Hasse square

In this section, we will study algebraic interpretations of $K(n)$-localization in terms of formal groups and elliptic curves.

PROPOSITION 4.1. *Let E be a complex oriented ring spectrum over $\mathbb{S}_{(p)}$ and define*

$$E' = \operatorname*{holim}_{(i_0,\ldots,i_{n-1})\in\mathbb{N}^n} v_n^{-1}E/(p^{i_0}, v_1^{i_1}, \ldots, v_{n-1}^{i_{n-1}}).$$

Here v_i are the images of the classes in BP of the same name under the orientation $BP \to E$. Then $L_{K(n)}E \simeq E'$.

PROOF. We also denote by $I_n \triangleleft E_*$ the image of the ideal of the same name in BP. As $v_n^{-1}E/I_n$ is a $B(n)$-module spectrum, it is $B(n)$-local by Lemma 1.2, thus by Theorem 3.4 also $K(n)$-local. Each spectrum $v_n^{-1}E/(v^I)$ (using multi-index notation) is constructed from $v_n^{-1}E/I$ by a finite number of cofibration sequences, thus it is also $K(n)$-local. Since homotopy limits of local spectra are again local (Lemma 1.4), E' is $K(n)$-local, and it remains to show that $K(n)_*(E) \cong K(n)_*(E')$. The coefficient rings of the Morava K-theories $K(n)$ are graded fields, hence they have Künneth isomorphisms. Thus it suffices to show that $E \wedge X \to E' \wedge X$ is a $K(n)$-equivalence for some X with nontrivial $K(n)_*(X)$. Choose $X = \mathbb{S}^0/(v^J)$ to be a generalized Moore spectrum of type n, for some multi-index J. Then

$$E' \wedge X \simeq \operatorname*{holim}_{I\in\mathbb{N}^n} \left(v_n^{-1}E/(v^I) \wedge \mathbb{S}^0/(v^J)\right) \simeq v_n^{-1}E/(v^J).$$

Thus $K(n)_*(E \wedge X) = K(n)_*(E/v^J) = K(n)_*(v_n^{-1}E/v^J) = K(n)_*(E' \wedge X)$. □

Now we will specialize to an elliptic spectrum E defined over the ring E_0 with associated elliptic curve C_E over $\operatorname{Spec} E_0$. Proposition 4.1 in particular tells us that

$$\pi_0 L_{K(1)}E \cong \lim_i v_1^{-1}E_0/(p^i),$$

which is the ring of functions on $\operatorname{Spf}((E_0)\hat{_p})^{\mathrm{ord}}$, the ordinary locus of the formal competion of $\operatorname{Spec} E_0$ at p, i.e. the sub-formal scheme over which C_E is ordinary. In particular, if E_0 is an $\mathbb{F}_p$-algebra, $\pi_0 L_{K(1)}E \cong v_1^{-1}E_0$ is just the (non-formal) ordinary locus of E_0. Similarly,

$$\pi_0 L_{K(2)}E \cong \lim_{i_0,i_1} v_2^{-1}E_0/(p^{i_0}, v_1^{i_1}) = \lim_{i_0,i_1} E_0/(p^{i_0}, v_1^{i_1})$$

is the ring of functions on the formal completion of $\operatorname{Spec} E_0$ at the supersingular locus at p. The last equality holds because any elliptic curve has height either 1 or 2 over $\mathbb{F}_p$, thus v_2 is a unit in $E_0/(p, v_1)$ and hence in $E_0/(p^{i_0}, v_1^{i_1})$.

LEMMA 4.2. *Any p-local elliptic spectrum E is $E(2)$-local.*

PROOF. We need to show that for any W with $E(2)_*W = 0$, we have that $E_*W = 0$. By Theorems 3.4 and 3.5, this is equivalent to $B(i) = 0$ for $0 \le i \le 2$. That is,

$$\begin{aligned} p^{-1}BP \wedge W &\simeq * \\ v_1^{-1}BP/p \wedge W &\simeq * \\ v_2^{-1}BP/(p, v_1) \wedge W &\simeq *. \end{aligned}$$

Now since E is a BP-ring spectrum, the same equalities hold with BP replaced by E. It follows from Lemma 1.3 that

$$\begin{aligned} E/(p,v_1)\wedge W \simeq v_2^{-1}E/(p,v_1)\wedge W \simeq * \text{ and } v_1^{-1}E/p\wedge W \simeq * \quad &\Rightarrow \quad E/p\wedge W\simeq * \\ E/p\wedge W\simeq * \text{ and } p^{-1}E\wedge W\simeq * \quad &\Rightarrow \quad E\wedge W\simeq *. \end{aligned}$$

□

Corollary 4.3 (the "Hasse square"). *For any elliptic spectrum E, there is a pullback square*

$$\begin{tikzcd} E^{\hat{}}_p \arrow[r] \arrow[d] & L_{K(2)}E \arrow[d] \\ L_{K(1)}E \arrow[r] & L_{K(1)}L_{K(2)}E. \end{tikzcd}$$

Proof. It follows from Lemma 3.6 that the pullback is $L_{K(1)\vee K(2)}E$. Now consider the arithmetic square

$$\begin{tikzcd} L_{K(0)\vee K(1)\vee K(2)}E \arrow[r] \arrow[d] & L_{K(1)\vee K(2)}E \arrow[d] \\ L_{K(0)}E \arrow[r] & L_{K(0)}L_{K(1)\vee K(2)}E. \end{tikzcd}$$

Since $L_pL_{K(0)}X = L_pL_{\mathbb{Q}}X = *$, applying the p-completion functor L_p, we see that top horizontal map

$$L_pE\simeq L_pL_{K(0)\vee K(1)\vee K(2)}E \to L_pL_{K(1)\vee K(2)}E\simeq L_{K(1)\vee K(2)}E$$

is an equivalence, hence the result. □

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The Local Structure of the Moduli Stack of Formal Groups

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Let $\mathcal{M}_{FGL}$ denote the moduli stack of formal groups (always assumed to be commutative and one-dimensional): this can be identified with the functor which assigns, to every commutative ring R, the groupoid $\mathcal{M}_{FGL}(R)$ of formal groups over R (and isomorphisms between them). We wish to understand the structure of $\mathcal{M}_{FGL}$. A natural starting point is to identify the *points* of $\mathcal{M}_{FGL}$; that is, maps $\operatorname{Spec} k \to \mathcal{M}_{FGL}$, where k is a field. For these we have the following classification (at least in the case where k is separably closed):

- If k is a separably closed field of characteristic $p > 0$, then formal groups $\mathbb{G}$ over k are classified by an invariant $h = h(\mathbb{G})$, called the *height* of $\mathbb{G}$. The height of the formal additive group $\widehat{\mathbb{G}}_a$ is ∞; otherwise h is a positive integer, uniquely determined by the requirement that the (finite) group scheme of p-torsion points of $\mathbb{G}$ has rank p^h.
- If k is a field of characteristic zero, then every formal group $\mathbb{G}$ over k is isomorphic to the formal additive group $\widehat{\mathbb{G}}_a$. In this case, we agree by convention that the height of $\mathbb{G}$ is equal to zero.

Our interest in the moduli stack $\mathcal{M}_{FGL}$ is primarily that it can be used to construct cohomology theories. More precisely, given a commutative ring R and a *flat* map $\operatorname{Spec} R \to \mathcal{M}_{FGL}$ (classified by a formal group $\mathbb{G}$ over R), we can construct a (weakly) even periodic homology theory A. If $\mathbb{G}$ admits a coordinate (in other words, if the Lie algebra of $\mathbb{G}$ is trivial), then $\mathbb{G}$ together with this coordinate are described by a map $L \to R$, where $L \simeq Z[u_1, u_2, \ldots,]$ is the Lazard ring; in this case A can be described by the formula $A_0(X) = MU_*(X) \otimes_L R$, where we implicitly use the isomorphism $L \simeq MU^*(*)$ provided by Quillen's theorem.

Unfortunately, a geometric point $\eta : \operatorname{Spec} k \to \mathcal{M}_{FGL}$ is almost never flat (this holds only when k is of characteristic zero). This is consistent with classical geometric intuition: if X is a finite dimensional algebraic variety containing a closed point x, then the inclusion $i : \{x\} \subseteq X$ is usually not a flat map. However, there is a canonical factorization of i as a composition

$$\{x\} \subseteq \operatorname{Spec} \widehat{\mathcal{O}}_{X,x} \to X$$

where the second map is flat. Here $\widehat{\mathcal{O}}_{X,x}$ denotes the completion of the structure sheaf of X at the point x, which encodes the structure of a formal neighborhood of x in X.

There exists an analogous description of a formal neighborhood of η in $\mathcal{M}_{FGL}$. Let $\mathbb{G}_0$ be the formal group over k classified by the map $\eta : \operatorname{Spec} k \to \mathcal{M}_{FGL}$. Let R be a local Artin ring with maximal ideal $\mathfrak{m}$, and let $\alpha : R/\mathfrak{m} \simeq k$ be an isomorphism.

We let $\mathcal{C}(R)$ denote the category of pairs $(\mathbb{G}, \overline{\alpha})$, where $\mathbb{G}$ is a formal group over R and $\overline{\alpha}$ is an isomorphism of $\mathbb{G} \times_{\operatorname{Spec} R} \operatorname{Spec} k$ with $\mathbb{G}_0$. (Here we regard $\operatorname{Spec} k$ as an R-scheme via the map α, but suppress mention of this in our notation.)

THEOREM 0.1 (Lubin, Tate). *Let k be a perfect field of characteristic p, and let $\mathbb{G}_0$ be a formal group of height $n < \infty$ over k. Then:*

(1) *For every local Artin ring R with residue field k, the groupoid $\mathcal{C}(R)$ is discrete (that is, the automorphism group of every object of $\mathcal{C}(R)$ is trivial). Let $\pi_0\mathcal{C}(R)$ denote the set of isomorphism classes of objects of $\mathcal{C}(R)$.*

(2) *There exists a complete local Noetherian ring A with residue field (isomorphic to) k and a formal group $\mathbb{G}$ over A such that, for every local Artin ring having residue field k, the base change of $\mathbb{G}$ induces a bijection*

$$\operatorname{Hom}_0(A, R) \to \pi_0\mathcal{C}(R).$$

Here $\operatorname{Hom}_0(A, R)$ denotes the subset of $\operatorname{Hom}(A, R)$ consisting of local homomorphisms which induce the identity map from k to itself.

(3) *The ring A is (noncanonically) isomorphic to $W(k)[[v_1, \ldots, v_{n-1}]]$ where $W(k)$ denotes the ring of (p-typical) Witt vectors of k.*

REMARK 0.2. We can rephrase assertion (1) of Theorem 0.1 as follows. Let R be a local Artin ring, and $\mathbb{G}$ a formal group over R. Suppose that the residue field k of R is a perfect field of positive characteristic, and that $\mathbb{G}_0 = \mathbb{G} \times_{\operatorname{Spec} R} \operatorname{Spec} k$ has finite height. Then an automorphism of $\mathbb{G}$ is trivial if and only if its restriction to $\mathbb{G}_0$ is trivial. Note that this assertion fails drastically if k is of characteristic zero, or if the height of $\mathbb{G}_0$ is infinite: the additive group $\widehat{\mathbb{G}}_a$ admits a nontrivial action of the multiplicative group $R^\times$, and the reduction map $R^\times \to k^\times$ is usually not injective.

REMARK 0.3. The moduli stack $\mathcal{M}_{FGL}$ is far from being a scheme, since formal groups over a ring R generally admit many automorphisms. However, assertion (1) of Theorem 0.1 can be interpreted as saying that the "stackiness" of $\mathcal{M}_{FGL}$ is not visible in a formal neighborhood of most points of $\mathcal{M}_{FGL}$.

REMARK 0.4. Once assertion (2) of Theorem 0.1 has been established, assertion (3) is equivalent to the statement that the ring A is formally smooth over $\mathbb{Z}$. To prove this, one can argue as follows: the moduli stack $\mathcal{M}_{FGL}$ admits a (pro)smooth cover by the moduli space $\mathcal{M}_{FGL}$ of formal group laws. A theorem of Lazard asserts that $\mathcal{M}_{FGL}$ is isomorphic to an infinite dimensional affine space $\operatorname{Spec} \mathbb{Z}[u_1, u_2, \ldots]$, which is manifestly (pro)smooth over $\operatorname{Spec} \mathbb{Z}$.

REMARK 0.5. Using assertion (3) of Theorem 0.1 and Landweber's criterion, it is easy to see that the formal group $\mathbb{G}$ over A classifies a flat map $\operatorname{Spec} A \to \mathcal{M}_{FGL}$.

For applications to elliptic cohomology, it is important to understand the relationship between the moduli stack $\mathcal{M}_{FGL}$ of formal group laws and the moduli stack $\mathcal{M}_{ell}$ of elliptic curves. There is an evident map $\psi : \mathcal{M}_{ell} \to \mathcal{M}_{FGL}$, which associates to each elliptic curve $E \to \operatorname{Spec} R$ the formal completion of E along its identity section. We now analyze the fiber of ψ over a geometric point $\eta : \operatorname{Spec} k \to \mathcal{M}_{FGL}$. There are several cases to consider:

(a) Let k be of characteristic zero. Without loss of generality, we may assume that η classifies the formal additive group $\widehat{\mathbb{G}}_a$. In this case, the fiber product $\mathcal{M}_{ell} \times_{\mathcal{M}_{FGL}} \operatorname{Spec} k$ is actually an affine algebraic variety of dimension

2 over k, which classifies elliptic curves E over k equipped with a global 1-form.

(b) Let k be a field of characteristic p, and let η classify a formal group of height 1 over k. The fiber product $\mathcal{M}_{ell} \times_{\mathcal{M}_{FGL}} \operatorname{Spec} k$ is again a scheme, which is a pro-etale cover of the moduli stack of ordinary elliptic curves defined over k (which is of Krull dimension 1).

(c) Let k be a field of characteristic p, and let η classify a formal group of height 2 over k. The fiber product $\mathcal{M}_{ell} \times_{\mathcal{M}_{FGL}} \operatorname{Spec} k$ is pro-etale over k: in other words, it is isomorphic to an affine scheme $\operatorname{Spec} k'$, where k' is an inductive limit of products of separable field extensions of k. The induced map $\operatorname{Spec} k' \to \mathcal{M}_{ell}$ surjects onto the locus of supersingular elliptic curves in characteristic p.

(d) Let k be a field of characteristic p, and let η classify a formal group of height > 2 over k. Then the fiber product $\mathcal{M}_{ell} \times_{\mathcal{M}_{FGL}} \operatorname{Spec} k$ is empty.

We conclude from this that $\mathcal{M}_{ell}$ is a rather good approximation to $\mathcal{M}_{FGL}$ in height 2 (that is, near a supersingular elliptic curve), but not elsewhere. In fact, one can prove a stronger assertion of this kind, which describes not only the fiber of the map ψ over a formal group of height 2, but a formal neighborhood of that fiber:

THEOREM 0.6 (Serre, Tate). *Let R be a Noetherian ring, p a prime number which is nilpotent in R, and I an nilpotent ideal in R. Say that an elliptic curve E defined over R (respectively R/I) is everywhere supersingular if, for every residue field k of R (respectively R/I), the fiber product $E \times_{\operatorname{Spec} R} \operatorname{Spec} k$ is a supersingular elliptic curve. Then the diagram of categories*

$$\begin{tikzcd} \{\textit{everywhere supersingular elliptic curves } E \to \operatorname{Spec} R\} \arrow[r] \arrow[d] & \{\textit{formal groups } \widehat{E} \to \operatorname{Spec} R\} \arrow[d] \\ \{\textit{ everywhere supersingular elliptic curves } E_0 \to \operatorname{Spec} R/I\} \arrow[r] & \{\textit{formal groups } \widehat{E}_0 \to \operatorname{Spec} R/I\} \end{tikzcd}$$

is a homotopy pullback. Here the vertical arrows are given by reduction modulo I, and the horizontal arrows are given by formal completion.

In other words, given an elliptic curve E_0 over R/I and a formal group $\widehat{E}$ over R which give rise to the same formal group over R/I, we can cobble together an elliptic curve E over R lifting E_0, having formal group $\widehat{E}$; moreover, E is uniquely determined up to unique isomorphism.

COROLLARY 0.7. *Let E be a supersingular elliptic curve over a perfect field k of characteristic p, classifying a point $\eta : \operatorname{Spec} k \to \mathcal{M}_{ell}$. Then the map ψ induces an isomorphism of a neighborhood of η with the formal scheme $\operatorname{Spf} A$, where $A \simeq W(k)[[v_1]]$ is the universal deformation ring for the formal group $\widehat{E}$ (see Theorem 0.1).*

Goerss-Hopkins obstruction theory

Vigleik Angeltveit

We develop an obstruction theory for answering the following question. Given a spectrum E with certain nice properties and a commutative E_*-algebra A in E_*E-comodules, is there a commutative S-algebra X with $E_*X \cong A$?

1. k-invariants

Let X be a 1-connected space. We usually think of the n'th k-invariant as a map $P_{n-1}X \to K(\pi_n X, n+1)$ with P_nX as the homotopy fiber. If we know all the k-invariants of X we can recontruct X as the inverse limit of its Postnikov tower. We wish to adapt this theory to build commutative S-algebras.

There are two main points we need to address when setting up the theory. First, Postnikov towers for ring spectra work best when the spectra have no negative homotopy groups. But many of the spectra we are interested in have lots of negative homotopy groups. We finesse this problem by introducting an additional simplicial direction and building our Postnikover tower in that direction.

Second, we cannot obtain the n'th stage simply as the homotopy fiber of a map from the $(n-1)$'st stage. This complication happens for spaces as well. If X is not 1-connected, we are forced to instead think of the k-invariant as a map $P_{n-1}X \to K(\pi_nX, n+1) \times_{\pi_1 X} E\pi_1 X$ making the following diagram into a homotopy pullback square:

$$\begin{array}{ccc} P_nX & \longrightarrow & B\pi_1 X \\ \downarrow & & \downarrow \\ P_{n-1}X & \longrightarrow & K(\pi_nX, n+1) \times_{\pi_1 X} E\pi_1 X \end{array}$$

The idea of the obstruction theory is to set up a situation where our commutative S-algebra X (if it exists) is the inverse limit of its Postnikov sections P_nX and such that the existence or non-existence of the correct k-invariant for building P_nX from $P_{n-1}X$ can be calculated algebraically.

Let E be a homotopy commutative ring spectrum satisfying certain technical conditions. Given a commutative E_*-algebra A in E_*E-comodules, we want to build a commutative S-algebra X with $E_*X \cong A$ as comodule algebras. We put the following technical conditions on E:

(1) E_*E should be flat over E_*. We need this for the category of E_*E-comodules to be well behaved.

(2) E should satisfy the Adams condition: $E = \operatorname{holim} E_\alpha$, where each E_α is a finite cellular spectrum such that E_*DE_α is projective over E_* and such that the natural map $[DE_\alpha, M] \to Hom_{E_*-mod}(E_*DE_\alpha, M_*)$ is an isomorphism for every E-module M. Here $DE_\alpha = F_S(E_\alpha, S)$ is the S-module (or Spanier-Whitehead) dual. This condition is necessary to guarantee that E has a Künneth spectral sequence.

Instead of simply asking for the existence of such an X, we can try to understand the realization category of all such X. An object in $\mathcal{R}(A)$ is a commutative S-algebra X with $E_*X \cong A$ as comodule algebras, and a morphism in $\mathcal{R}(A)$ is a map $X \to X'$ of commutative S-algebras inducing an isomorphism $E_*X \cong E_*X'$ as comodule algebras.

Note that the isomorphism $E_*X \to A$ is not part of the data. This builds the automorphisms of X into the category. We can then consider the moduli space $\mathcal{BR}(A)$, which is the geometric realization of the nerve of the category $\mathcal{R}(A)$. By a theorem of Dwyer and Kan [**DK**],

$$\mathcal{BR}(A) \cong \coprod_{[X]} BAut(X),$$

where X runs over the E_*-isomorphism classes of objects in $\mathcal{BR}(A)$ and $Aut(X)$ is the monoid of self equivalences of a cofibrant-fibrant model for X.

The question becomes "is $\mathcal{BR}(A)$ nonempty?"

As a typical example, indeed the main example, we can let $E = E_n$ be the n'th Morava E-theory spectrum. This is the spectrum associated to the universal deformation ring of a height n formal group law over a perfect field k as discussed in the previous chapter. To be specific, one often considers the height n Honda formal group law over $\mathbb{F}_{p^n}$, though the obstruction theory works just as well in the more general situation. By the Lubin-Tate theorem, the ring of universal deformations of such a formal group law is isomorphic to

$$\mathbb{W}(k)[[u_1, \ldots, u_{n-1}]],$$

where $\mathbb{W}(k)$ denotes the Witt vectors over k. Then the Landweber Exact Functor Theorem tells us that we can construct a 2-periodic spectrum E_n with

$$\pi_* E_n \cong \mathbb{W}(k)[[u_1, \ldots, u_{n-1}]][u, u^{-1}]$$

with $|u| = 2$. It is not too hard to show that E_n is a homotopy commutative ring spectrum. Thus this is an appropriate starting point.

2. Simplicial operads

We will attempt to construct X as an algebra over some E_∞ operad, using that E_∞-ring spectra are equivalent to commutative S-algebras. If $\mathcal{O}$ is an E_∞ operad, it is hard to understand the set of maps

$$\mathcal{O}(k)_+ \wedge_{\Sigma_k} X^{(k)} \to X.$$

To get around this problem, we introduce a simplicial direction. Let T be a simplicial operad, i.e., a sequence $\{T_n\}_{n\geq 0}$ of operads connected by the usual structure maps of a simplicial object. We will assume that

(1) Each $T_n(k)$ is Σ_k-free. This means that $T_n(k)_+ \wedge_{\Sigma_k} X^{(k)} \to X$ is easy to understand.

(2) For each k the geometric realization $|n \mapsto T_n(k)|$ is equivalent to $E\Sigma_k$, i.e., a contractible space with a free Σ_k-action.

For example, we can let $T_n(k) = \Sigma_k^{n+1}$. Then $|T(k)| = B(\Sigma_k, \Sigma_k, *)$ is the usual model for $E\Sigma_k$. (The Barratt-Eccles operad.)

Now we can conider the category of T-algebras in spectra. An object in T-alg is a simplicial spectrum $X_\bullet$ such that each X_n is a T_n-algebra. It follows that $|X_\bullet|$ is a $|T|$-algebra spectrum, or in other words an E_∞ ring spectrum.

Suppose we have a simplicial spectrum $X_\bullet$. Then we get a simplicial graded abelian group $E_*X_\bullet = \{n \mapsto E_*X_n\}_{n\geq 0}$, and we can use the simplicial structure maps to define the homotopy groups $\pi_*E_*X_\bullet$ of this object. Note that even though each E_*X_n might be nonzero in negative degrees, $\pi_iE_*X_\bullet$ is only defined for $i \geq 0$. This simplicial direction will be our Postnikov tower direction.

Recall that A is a commutative E_*-algebra in E_*E-comodules. Let $\mathcal{R}_\infty(A)$ be the realization category corresponding to the above setup. An object is a simplicial spectrum $X_\bullet \in T$-alg such that

$$\pi_iE_*X_\bullet \cong \begin{cases} A & \text{if } i = 0 \\ 0 & \text{if } i > 0 \end{cases}$$

The morphisms are maps which induce an isomorphism after applying $\pi_*E_*(-)$. Let $\mathcal{BR}_\infty(A)$ be the geometric realization (in the nerve direction, not the other simplicial direction) of the nerve of this category.

Theorem 2.1. *(Goerss-Hopkins* [**GH**]*) Geometric realization and the constant simplicial spectrum functors gives a weak equivalence*

$$\mathcal{BR}_\infty(A) \leftrightarrows \mathcal{BR}(A).$$

3. The spiral exact sequence

Given a simplicial spectrum like $E \wedge X_\bullet$ we can define two types of bigraded homotopy groups. Type I, which is the one we discussed above, defined as

$$\pi_pE_qX_\bullet = \pi_p(\pi_q(E \wedge X_\bullet))$$

and type II, which uses the simplicial sphere $\Delta^p/\partial\Delta^p$, defined as

$$\pi_{p,q}(E \wedge X_\bullet) = [\Delta^p/\partial\Delta^p \wedge S^q, E \wedge X_\bullet].$$

As usual, we have a spectral sequence

$$E^2_{p,q} = \pi_pE_qX_\bullet \Rightarrow E_{p+q}|X_\bullet|.$$

This spectral sequence comes from the skeletal filtration of $X_\bullet$. If we package the spectral sequence as an exact couple, the D^2 and E^2 terms of the exact couple form a long exact sequence called the spiral exact sequence:

$$\ldots \to \pi_{p-1,q+1}E \wedge X \to \pi_{p,q}E \wedge X \to \pi_pE_qX \to \pi_{p-2,q+1}E \wedge X \to \ldots$$

See [**DKS**] for more on how to construct the spiral exact sequence. This is where we use that E satisfies Adams' condition. If $X \in \mathcal{BR}_\infty(A)$ then every third term in the above long exact sequence is zero, and so $\pi_{p,*}E \wedge X \cong \Omega^pA$ for all $p \geq 0$.

4. Building $\mathcal{BR}_\infty(A)$ inductively

Now we make the following definition. Let $\mathcal{BR}_n(A)$ be the nerve of the category with objects $X \in T$-alg with $\pi_{p,*}E \wedge X \cong \Omega^p A$ if $p \leq n$ and 0 otherwise. It is with respect to this type of homotopy groups that the category of T-algebras has Postnikov sections. Given a T-algebra X it is possible to construct a new T-algebra P_nX with $\pi_{i,*}P_nX \cong \pi_{i,*}X$ if $i \leq n$ and 0 otherwise by gluing on algebra cells, and similarly after smashing with E.

We say that X is a potential n-stage for A if

$$\pi_i E_* X \cong \begin{cases} A & \text{if } i = 0 \\ 0 & \text{if } 1 \leq i \leq n+1 \end{cases}$$

and $\pi_{i,*}E \wedge X = 0$ for $i > n$. The spiral exact sequence then implies that

$$\pi_i E_* X \cong \begin{cases} A & \text{if } i = 0 \\ \Omega^{n+1} A & \text{if } i = n+2 \\ 0 & \text{if } i \neq 0, n+2 \end{cases}$$

and

$$\pi_{i,*}E \wedge X \cong \begin{cases} \Omega^i A & \text{if } 0 \leq i \leq n \\ 0 & \text{if } i > n \end{cases}$$

Now we can try to build $X \in \mathcal{BR}_\infty(A)$ inductively. To get started, consider the functor

$$sAlg_T^{op} \to \text{Sets}$$

given by $Y \mapsto Hom_{Alg_{E_*}/E_*E}(\pi_0 E_* Y, A)$. A version of Brown representability implies that this is represented by some $B_A \in \mathcal{BR}_0(A)$.

If we have an A-module M in E_*E-comodules, we get an object $B_A(M,n)$ with type II homotopy A in degree 0, and M in degree n. On the algebra side, we can construct $K_A(M,n) = K(M,n) \rtimes A$.

Proposition 4.1. *There is a natural map $E_*B_A(M,n) \to K_A(M,n)$ which induces an isomorphism*

$$Hom_{sAlg_T/B_A}(X, B_A(M,n)) \to Hom_{sAlg_{E_*T/E_*E}/A}(E_*X, K_A(M,n)).$$

This is where we need T to be a simplicial operad with each $T_m(k)$ Σ_k-free, to control the homotopy type of the mapping space

$$Map_{Alg_{T_m}}(X_m, B_A(M,n)_m).$$

The right hand side of Proposition 4.1 is some kind of derived functors of derivations, or André-Quillen cohomology. Now we define

$$\mathcal{H}^n(A;M) = Hom_{sAlg_{E_*T/E_*E}/A}(A, K_A(M,n))$$

and

$$\widehat{\mathcal{H}}^n(A;M) = \mathcal{H}^n(A;M) \times_{Aut(A,M)} EAut(A,M).$$

Theorem 4.2. *(Goerss-Hopkins [**GH**]) The following is a homotopy pullback square:*

$$\begin{tikzcd} \mathcal{BR}_n(A) \arrow[r] \arrow[d] & BAut(A, \Omega^n A) \arrow[d] \\ \mathcal{BR}_{n-1}(A) \arrow[r] & \widehat{\mathcal{H}}^{n+2}(A; \Omega^n A) \end{tikzcd}$$

Proof. (Sketch) Given $Y \in \mathcal{BR}_{n-1}(A)$, it is enough to produce a map $Y \to B_A(\Omega^n A, n+1)$ of T-algebras over B_A, inducing an isomorphism on $\pi_{n+1,*}$. Then the fiber will have the right $\pi_{*,*}$ and be a T-algebra. By Proposition 4.1, it is enough to produce a map

$$E_*Y \xrightarrow{\simeq} K_A(\Omega^n A, n+1)$$

of E_*T-algebras in E_*E-comodules.

E_*Y is a 2-stage Postnikov tower, so it is determined by a map

$$A \simeq P_0 E_* Y \to K_A(\Omega^n A, n+2).$$

The class of this map in

$$\pi_0 \widehat{\mathcal{H}}^{n+2}(A; \Omega^n A)$$

is the obstruction. $\square$

Corollary 4.3. *There are obstructions*

$$\theta_n \in Hom_{sAlg_{E_*T/E_*E}/A}(A, K_A(A, n+2))$$

to existence and

$$\theta'_n \in Hom_{sAlg_{E_*T/E_*E}/A}(A, K_A(A, n+1))$$

to uniqueness of a commutative S-algebra X with $E_*X \cong A$.

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From spectra to stacks

Lecture by Michael Hopkins,
Typed by Michael A. Hill

This talk aims to explain why the language of stacks, language which was relatively unused even in algebraic geometry around the time *tmf* was constructed, plays such a prominent role in the theorem expressing the existence of TMF.

Let me try to explain where it all came from... I come from an era in homotopy theory when all that anybody did was to compute things. Doug Ravenel and others tried to make sense of all these computations, but it was only the people in my generation who started to think about how to understand these computations in a more conceptual way. Nevertheless, I still tend to see everything through a computational lens.

For me, TMF came out of what I call "designer spectra" or "designer homotopy types". The first examples of such designer objects are the Eilenberg-MacLane spaces: they are spaces designed to have certain homotopy theoretic properties, but with otherwise not much geometry. The appearance of those homotopy types marked the point when people started to make actual progress, e.g., Serre's first computations of homotopy groups of spheres. Before that, just using geometric methods, all that one could prove (using simplicial approximation) was that these homotopy groups are countable. From then on, every advance in homotpy theory was spearheaded by the appearance of some special designer objects: spectra, various forms of K-theory, the Adams spectral sequence, Brown-Gitler spectra, etc.

One of the things about TMF that we were really proud of is that, suddenly, this was something that we could explain in a language that did not involve any corky, weird homotopy types, so that someone who did not know too much about homotopy theory could come in and do stuff with TMF.

Even tough the TMF story might make more sense from the point of view of elliptic curves, I am going to tell you today about some of those older designer objects, and I hope that they will help fit the story together.

Let us start by recalling the main theorem:

THEOREM 0.1. *There exists a unique sheaf $\mathcal{O}^{top}$ of E_∞-ring spectra on $(\mathcal{M}_{ell})_{\text{ét}}$ such that for an étale map $f : \operatorname{Spec} R \to \mathcal{M}_{ell}$, one has that $\mathcal{O}^{top}(\operatorname{Spec} R)$ is the Landweber exact theory corresponding to the composite $\operatorname{Spec} R \to \mathcal{M}_{ell} \to \mathcal{M}_{FG}$.*

To be more precise, the spectrum $\mathcal{O}^{top}(\operatorname{Spec} R)$ is even and periodic, its π_0 is identified with R, and the formal group corresponding to it is identified with the formal completion of the elliptic curve classified by f.

The argument for the existence of such a sheaf is via obstruction theory: instead of showing that the sheaf exists, one shows that it cannot *not* exist. Though the resulting sheaf is unique up to isomorphism, it has automorphisms, and so it fails to be unique up to unique isomorphism. It is therefore only unique in a weak sense. It is however possible to uniquely specify such a sheaf, and this was accomplished by Jacob Lurie's construction using derived elliptic curves. We should also note that the initial version of the theorem only produced a sheaf of A_∞-ring spectra, and it was Jacob Lurie who pointed out that $\mathcal{O}^{top}$ is actually a sheaf of E_∞-ring spectra. Later on, the obstruction theory could be adapted to prove that stronger result, but this was not part of the original approach.

Since Quillen's landmark result that MU carries the universal formal group law, the correspondence between complex-oriented cohomology theories and formal group laws has informed most of the results in stable homotopy theory. The triumph of Miller-Ravenel-Wilson, Landweber, and many mathematicians of that generation was finding good ways to coordinatize the formal groups and their associated cohomology theories. They used these coordinates to successfully carry out incredible computations, revealing many subtleties of the stable homotopy category. For the later generations, it was impossible to "out-compute" the earlier mathematicians, so people sought more conceptual formulations of what was known. This lead naturally to looking in a coordinate-free context, focusing on formal groups rather than formal group laws, and the culmination of that point of view was to realize that one should really be thinking about the moduli stack $\mathcal{M}_{FG}$.

1. Ravenel's Filtration of MU

Given a ring spectrum R, we can identify multiplicative maps $MU \to R$, that is, maps of ring spectra (not to be confused with A_∞ or E_∞ maps), with coordinates on the formal group associated to R. If there is a multiplicative map at all then there is a formal group law on R, and any two multiplicative maps define isomorphic formal groups. The formal group is therefore intrinsic to R, and it exists if and only if there exits a multiplicative map from MU into it, that is, if and only if R is complex orientable. For example, if a multiplicative cohomology theory is even and periodic, then there exists such a map, which means that there exists a formal group associated to R. There is however no preferred map $MU \to R$, and so the formal group does not come with any preferred choice of coordinate.

If R is complex orientable, then, by the Thom isomorphism, there is an isomorphism of R_*-algebras

$$R_*MU \;\cong\; R_*BU.$$

The latter is in turn the symmetric algebra over R_* of $R_*\mathbb{C}P^\infty$ (the generator of $R_0\mathbb{C}P^\infty$ being identified with the multiplicative unit). Since R is complex orientable, we also know that

$$Hom^{\otimes}(MU, R) \;\cong\; \mathrm{Alg}(R_*MU, R_*),$$

where $Hom^{\otimes}$ denotes maps of ring spectra and Alg denotes R_*-algebra maps. A multiplicative map $MU \to R$ is therefore equivalent to an R_*-module homomorphism from $R_*\mathbb{C}P^\infty$ (reduced homology) to R_*. There is also a geometric way of stating the above facts. Recall that there is an important map $\mathbb{C}P^\infty \to BU$ that classifies the virtual vector bundle $L-1$, where L is the canonical line bundle, and

1 the trivial complex line bundle. Passing to Thom spectra, we get a map

$$\Sigma^{-2}\mathbb{C}P^\infty \cong (\mathbb{C}P^\infty)^{(L-1)} \to MU$$

that realizes the copy of $R_*\mathbb{C}P^\infty$ that we see inside R_*MU (for $n \geq 0$, the generator of $R_{2n}\mathbb{C}P^\infty$ corresponds to the $2n$-cell of $\Sigma^{-2}\mathbb{C}P^\infty$). The cells of $\Sigma^{-2}\mathbb{C}P^\infty$ therefore correspond to the terms in the formal power series expansion of the coordinate on the formal group.

Ravenel introduced an important filtration $\{X(n)\}_{n\geq 1}$ of MU, which played a key role in the nilpotence and periodicity theorems, and also in his approach to computing the stable homotopy groups of spheres. It is defined as follows. First of all, by Bott periodicity, we know that $BU = \Omega SU$. If we filter SU by the Lie groups $SU(n)$, then we get a filtration

$$\{*\} = \Omega SU(1) \subset \cdots \subset \Omega SU(n) \subset \cdots \subset \Omega SU = BU.$$

Recall that MU is the Thom spectrum of the universal virtual bundle over BU. Applying the Thom spectrum construction to the above filtration of BU produces a filtration

$$\mathbb{S}^0 = X(1) \to X(2) \to \cdots \to X(n) \to \cdots \to MU$$

of MU. The spectra $X(n)$ are homotopy commutative and their homotopy groups are roughly as complicated as those of the sphere. They are actually E_2-ring spectra because $\Omega SU(n) \to \Omega SU = BU$ is a 2-fold loop map. More generally, the Thom spectrum of a k-fold loop map is always an E_k-ring spectrum. For example:

Proposition 1.1. *If S is an H-space and $\zeta\colon S \to BU$ is an H-map, then S^ζ is a ring spectrum.*

Proof. We have a commutative square

$$\begin{tikzcd} S \times S \arrow[r, "\zeta\times\zeta"] \arrow[d, "\mu"'] & BU \times BU \arrow[d, "\oplus"] \\ S \arrow[r, "\zeta"'] & BU \end{tikzcd}$$

which gives us two ways to describe Thom spectrum over $S \times S$. Naturality of the Thom spectrum produces a map

$$(S \times S)^{\zeta\circ\mu} \to S^\zeta.$$

However, since $\zeta \circ \mu = \zeta \oplus \zeta$, we learn that

$$(S \times S)^{\zeta\circ\mu} = (S \times S)^{\zeta\oplus\zeta} = S^\zeta \wedge S^\zeta.$$

Thus we have that S^ζ is a ring spectrum. □

Although the spectra $X(n)$ are not complex orientable (only MU is), there is a similar story that involves them, and that has something to do with complex orientations and formal groups. We first isolate the copy of $\mathbb{C}P^{n-1}$ inside $\mathbb{C}P^\infty$, and build a map $S^1 \times \mathbb{C}P^{n-1} \to SU(n)$, as follows. Let $r\colon S^1 \times \mathbb{C}P^{n-1} \to U(n)$ be the map which associates to a pair (λ, ℓ) the rotation of $\mathbb{C}^n$ in the line ℓ with angle of rotation λ. If we let ℓ_0 correspond to the base point of $\mathbb{C}P^{n-1}$, then we can make a map to $SU(n)$ by dividing by the value at ℓ_0: our map is therefore given by $(\lambda, \ell) \mapsto r(\lambda, \ell) r(\lambda, \ell_0)^{-1}$. The points $S^1 \times *$ and $* \times \mathbb{C}P^{n-1}$ all go to the base-point in $SU(n)$, and so this descends to a map

$$S^1 \wedge \mathbb{C}P^{n-1} \to SU(n).$$

Taking the adjoint then gives us a map $\mathbb{C}P^{n-1} \to \Omega SU(n)$ with the property that, when composed with the inclusion $\Omega SU(n) \hookrightarrow \Omega SU \cong BU$, it yields the classifying map for the virtual bundle $L - 1$ over $\mathbb{C}P^{n-1}$ (this is part of an even dimensional cell structure on $\Omega SU(n)$). Finally, applying the Thom spectrum construction to the inclusion $\mathbb{C}P^{n-1} \to \Omega SU(n)$ gives us a map

$$\Sigma^{-2}\mathbb{C}P^n \cong (\mathbb{C}P^{n-1})^{L-1} \to X(n). \tag{1}$$

If R is complex orientable, then, by the Thom isomorphism, there is an isomorphism of R_*-algebras

$$R_*X(n) = R_*\Omega SU(n) = \mathrm{Sym}_{R_*}(R_*\mathbb{C}P^{n-1}) = R_*[b_1, \ldots, b_{n-1}].$$

Moreover, multiplicative maps $X(n) \to R$ corresponds bijectively to coordinates up to degree n (that is, modulo degree $n+1$) on the formal group of R.

Now, even if R is not complex orientable, we can still make sense of multiplicative maps from $X(n)$ to R. They yield "formal group law chunks", that is, formal group laws up to degree n. Assuming the existence of a multiplicative map $X(n) \to R$, we can therefore associate to R a formal group up to degree n.

Recall that a ring spectrum X is called *flat* if X_*X is flat as an X_*-module. Even though the homotopy groups of $X(n)$ are not known for any finite value of n, these spectra are known to be flat, and so there is a sort of Thom isomorphism

$$X(n)_*X(n) = X(n)_*\Omega SU(n) = X(n)_*[b_1, \ldots, b_{n-1}], \qquad |b_i| = 2i.$$

In particular, the left hand side is a free $X(n)_*$-module, and it follows that

$$X(n) \wedge X(n) = X(n)[b_1, \ldots, b_{n-1}],$$

where the last term is a wedge of suspensions of $X(n)$ indexed by the monomials in $\mathbb{Z}[b_1, \ldots, b_{n-1}]$. Like with MU, we can recast this as saying that

$$X(n)_*X(n) = \mathrm{Sym}_{X(n)_*}(X(n)_*\mathbb{C}P^{n-1}).$$

More precisely, $X(n)_*X(n)$ is the symmetric algebra in $X(n)_*$-modules on the $X(n)$-homology of $\Sigma^{-2}\mathbb{C}P^n$, where the latter is understood modulo its (-2)-cell. Also, the 0-cell corresponds to $b_0 = 1$, and so doesn't appear among the generators. Note that the map including $X(n)_*\Sigma^{-2}\mathbb{C}P^n$ into $X(n)_*X(n)$ is induced by the map (1) above. As was the case for MU, the pair $\big(X(n)_*, X(n)_*X(n)\big)$ forms a Hopf algebroid. In particular, though we cannot compute the homotopy groups of $X(n)$, we can hope to study algebraically the $X(n)$-based Adams Novikov spectral sequence.

2. Stacks from Spectra

In the old days, in order to talk about formal groups in homotopy theory, one necessarily had to have a complex oriented cohomology theory. However, in retrospect, one can get something having to do with formal groups associated to any spectrum. In this section, given a ring spectrum X, typically not complex orientable, we will associate to it a stack $\mathcal{M}_X$ over $\mathcal{M}_{FG}^{(1)}$.

Here, $\mathcal{M}_{FG}^{(1)}$ denotes the stack classifying formal groups equipped with a first order coordinate or, in other words, a non-zero tangent vector. Recall that the stack associated to the Hopf algebroid (MU_*, MU_*MU), equivalently, to the groupoid $\mathrm{Spec}(MU_*MU) \rightrightarrows \mathrm{Spec}(MU_*)$, is almost but not quite the stack of formal groups. It is $\mathcal{M}_{FG}^{(1)}$. The rings MU_* and MU_*MU being $\mathbb{Z}$-graded, there is an action of $\mathbb{G}_m$ on that stack. The action rescales the tangent vector, and modding out by it yields $\mathcal{M}_{FG}$.

We first consider the case when X is just a spectrum (not a ring spectrum), and use the MU-based Adams resolution

(2)

$$MU_*X \rightrightarrows MU_*MU \otimes_{MU_*} MU_*X \;\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}\; MU_*MU \otimes_{MU_*} MU_*MU \otimes_{MU_*} MU_*X \;\cdots$$

to construct a quasicoherent sheaf $\mathcal{F}_X$ over $\mathcal{M}_{FG}^{(1)}$. We recall how this resolution arises. Start with the cosimplicial spectrum whose k^{th} stage is given by $MU^{\wedge(k+1)} \wedge X$ (inserting the sphere spectrum in various places and then applying the unit map gives the coface maps, and the codegeneracies come from the multiplication on MU), we then apply $\pi_*(-)$ and use repeatedly the isomorphism

$$\pi_*(MU \wedge MU \wedge X) \cong MU_*MU \otimes_{MU_*} MU_*X.$$

The first term of the above resolution is MU_*X, an MU_*-module, and to it corresponds a quasicoherent sheaf over Spec of the Lazard ring $L = MU_*$. Similarly, the next term $MU_*MU \otimes_{MU_*} MU_*X$ gives a sheaf over $\mathrm{Spec}(MU_*MU) = \mathrm{Spec}(L) \times_{\mathcal{M}_{FG}^{(1)}} \mathrm{Spec}(L)$, the moduli scheme that parametrizes a pair of formal group laws, and an isomorphism between the corresponding formal groups that respects the chosen tangent vectors. The two maps

$$MU_*X \rightrightarrows MU_*MU \otimes_{MU_*} MU_*X$$

(the counit and coaction maps) are exactly what one needs to give descent data, and so we get our sheaf $\mathcal{F}_X$ over $\mathcal{M}_{FG}^{(1)}$. Note that everything is $\mathbb{Z}$-graded, which is to say, everything is acted on by $\mathbb{G}_m$, and so one can also get sheaf over $\mathcal{M}_{FG}$ by modding out that action.

If now X is a homotopy commutative ring spectrum, then the terms in the Adams resolution (2) are commutative rings, and $(MU_*X, MU_*MU \otimes_{MU_*} MU_*X)$ is a Hopf algebroid. We define $\mathcal{M}_X$ to be the associated stack. By construction, it comes equipped with a map

$$\mathcal{M}_X \to \mathcal{M}_{FG}^{(1)}.$$

If we want, we could also mod out the action of $\mathbb{G}_m$ to get a stack over $\mathcal{M}_{FG}$. Here are two examples:

- $\mathcal{M}_{S^0} = \mathcal{M}_{FG}^{(1)}$, essentially by definition.
- $\mathcal{M}_{X(n)} = \mathcal{M}_{FG}^{(n)}$, the stack of formal groups together with an n-jet, that is, a coordinate modulo degree $n+1$.

In some sense, the above construction brings all ring spectra into the world of complex orientable cohomology theories, even if they are not themselves complex orientable. If X is complex orientable, then $\mathcal{M}_X$ is actually a scheme: the complex orientation provides a contracting homotopy (in the form of a $(-1)^{\text{st}}$ codegeneracy map), and the Adams-Novikov resolution collapses to π_*X. This means that $\mathcal{M}_X = \mathrm{Spec}(\pi_*X)$, with no stackiness. We can therefore view the stackiness of $\mathcal{M}_X$ as a measure of the failure of complex orientability of X.

If X and Y are ring spectra, then we would like $\mathcal{M}_{X\wedge Y}$ to be the stacky pullback (also called the 2-categorical pullback: a point of the pullback consists of a point of $\mathcal{M}_X$, a point of $\mathcal{M}_Y$, and an isomorphism between their images in $\mathcal{M}_{FG}^{(1)}$)

$$
\begin{tikzcd}
\mathcal{M}_{X\wedge Y} \arrow[r] \arrow[d] & \mathcal{M}_X \arrow[d] \\
\mathcal{M}_Y \arrow[r] & \mathcal{M}_{FG}^{(1)}
\end{tikzcd}
\tag{3}
$$

as this would allow us to do some key computations. For that, we need to know that $MU_*(X\wedge Y) \cong MU_*(X)\otimes_{MU_*} MU_*(Y)$. If either $MU_*(X)$ or $MU_*(Y)$ is flat as an MU_*-module, then the Künneth spectral sequence for MU collapses, giving us the desired isomorphism. In this case, when we form the resolution for $X\wedge Y$, we see that it is just the tensor product of the resolutions for X and for Y, and this is exactly the desired stacky pullback statement. In short, if MU_*X or MU_*Y is flat as MU_*-module, then (3) is a stacky pullback.

Recall that $\mathcal{M}_X$ is the stack associated to the Hopf algebroid encoded in (2):

$$\mathcal{M}_X = \mathit{Stack}\big(MU_*X, (MU\wedge MU)_*X\big). \tag{4}$$

Here is something that is quite natural from the point of view of stacks: the above presentation of $\mathcal{M}_X$ might be a really inefficient one, and it might be better to replace it with a smaller, more efficient one before attempting any calculations. At the end of the day we care only about the underlying stack (really, just its cohomology), not the particular Hopf algebroid presentation. This gives us lots of flexibility. For example, one doesn't necessarily need to use MU above: if R is a any commutative ring spectrum satisfying the conditions (*i, ii, iii*) below, then we can use R instead of MU in (4), and get an equivalent stack:

$$\mathcal{M}_X = \mathit{Stack}\big(R_*X, (R\wedge R)_*X\big). \tag{5}$$

In the next section, we will apply this trick with $R = X(n)$. This fact, that for any such R, the above resolution gives the same underlying stack, is a reformulation of many of the classical change-of-rings theorems used for computations with the Adams spectral sequence.

Finally, we list the technical conditions that R needs to satisfy for the above story to work:

(*i*) $R\wedge X$ is complex orientable.

(*ii*) R_*R is a flat R_*-module: this guarantees that $(R_*X, (R\wedge R)_*X)$ is a Hopf algebroid.

(*iii*) X is R-local, that is, $R\wedge X \rightrightarrows R\wedge R\wedge X \,\substack{\rightarrow\\[-1em]\rightarrow\\[-1em]\rightarrow} \ldots$ is a resolution of X. If one prefers a condition that does not depend on X, then one can ask for the sheaf $\mathcal{F}_R$ to have full support (for example, if $\pi_0(R) = \mathbb{Z}$ and R is connective, then this condition is automatically satisfied).

3. Two Important Examples

3.1. The $X(4)$-homology of *tmf*. This computation will tie the Weierstrass Hopf algebroid firmly to *tmf* for topological reasons, and it will allow us to see the usefulness of the stacky pullback square (3). We first recall two things:

- $\mathcal{M}_{X(4)}$ is the moduli stack $\mathcal{M}_{FG}^{(4)}$ of formal groups equipped with a 4-jet.

- $\mathcal{M}_{tmf}$ is the moduli stack $\overline{\mathcal{M}}_{ell}^{+(1)}$ of elliptic curves with a 1-jet, where both multiplicative and additive degenerations allowed.[1]

Since $MU_*X(n)$ is flat as MU_*-module, the discussion in Section 2 applies, and so we have a stacky pullback square

$$\begin{tikzcd} \mathcal{M}_{X(4)\wedge tmf} \arrow[r] \arrow[d] & \mathcal{M}_{tmf} \arrow[d] \\ \mathcal{M}_{X(4)} \arrow[r] & \mathcal{M}_{FG}^{(1)}. \end{tikzcd}$$

The data classified by $\mathcal{M}_{X(4)\wedge tmf}$ is then an elliptic curve C together with a 4-jet (a coordinate modulo degree 5) on its formal group or, equivalently, a 4-jet on C. This data is exactly what is needed to identify a Weierstrass equation for C. We therefore learn that $\mathcal{M}_{X(4)\wedge tmf}$ is affine:

$$\mathcal{M}_{X(4)\wedge tmf} = \operatorname{Spec} \mathbb{Z}[a_1, a_2, a_3, a_4, a_6],$$

and that

$$\pi_*(X(4) \wedge tmf) = \mathbb{Z}[a_1, a_2, a_3, a_4, a_6].$$

Running a similar analysis for $X(4) \wedge \ldots \wedge X(4) \wedge tmf$, we also learn that the $X(4)$-based Adams resolution of tmf is exactly the cobar resolution of the Weierstrass Hopf algebroid.

3.2. The Stack $\mathcal{M}_{bo}$. We now investigate the spectrum bo, the (-1)-connected cover of KO. This spectrum has been extremely useful in stable homotopy, as computations with it tend to be quite tractable. Its zeroth space is $\mathbb{Z} \times BO$, its 8^{th} space is the 7-connected cover of BO, etc. Its relationship to bu, the spectrum of connective complex K-theory, is given by the following result:

Proposition 3.1. *We have an equivalence* $\Sigma^{-2}\mathbb{C}P^2 \wedge bo \simeq bu$.

Proof. Recall that $\Sigma^{-2}\mathbb{C}P^2$ is a cell complex with exactly two cells (a 0-cell and a 2-cell), and attaching map $\eta \in \pi_1(\mathbb{S}^0)$. By the work of Bott, we know that $\Omega(U/O) = \mathbb{Z} \times BO$. The fibration of infinite loop spaces $U/O \to \mathbb{Z} \times BO \to \mathbb{Z} \times BU$ therefore corresponds to a fibration $\Sigma bo \to bo \to bu$ of connective spectra. The first map is multiplication by $\eta \in \pi_1(\mathbb{S}^0) = \pi_1(bo)$, and so

$$bu = \operatorname{cofib}(\eta : \Sigma bo \to bo) = \operatorname{cofib}(\eta : \mathbb{S}^1 \to \mathbb{S}^0) \wedge bo = \Sigma^{-2}\mathbb{C}P^2 \wedge bo.$$

$\square$

Thus, although bo is not complex orientable, the spectrum $bo \wedge \Sigma^{-2}\mathbb{C}P^2$ is. It is therefore tempting to try to use $\Sigma^{-2}\mathbb{C}P^2$ for R in equation (5) in order to compute $\mathcal{M}_{bo}$. Moreover, since $\Sigma^{-2}\mathbb{C}P^2$ has exactly 2 cells, this indicates that $\mathcal{M}_{bo}$ should admit a double cover by the scheme $\mathcal{M}_{bu} = \operatorname{Spec}(\pi_* bu)$. Unfortunately, that idea doesn't quite work, because $\Sigma^{-2}\mathbb{C}P^2$ is not a ring spectrum. It is however possible to embed $\Sigma^{-2}\mathbb{C}P^2$ in a nice ring spectrum subject to the conditions (*i, ii, iii*) above, namely $R = X(2)$.

We first explain why $X(2) \wedge bo$ is complex orientable. Recall that by the Thom isomorphism $H_*X(2) = H_*\Omega SU(2) = \mathbb{Z}[b_1]$. It follows that $X(2)$ is a cell complex

[1] In our notation, the bar indicates that multiplicative degenerations are allowed, and the plus indicates that additive degenerations are also allowed.

with one cell in each even degree. In fact, there exists a cell complex Y_4 with a single cell in each degree multiple of 4,

$$H_*Y_4 = \mathbb{Z}[c], \quad |c| = 4,$$

such that $X(2) = Y_4 \wedge \Sigma^{-2}\mathbb{C}P^2$, and therefore

$$X(2) \wedge bo = Y_4 \wedge \Sigma^{-2}\mathbb{C}P^2 \wedge bo = Y_4 \wedge bu$$

is complex orientable.

COROLLARY 3.2. *As rings*

$$X(2)_*bo = \mathbb{Z}[b, c], \qquad |b| = 2, \ |c| = 4,$$

and

$$X(2)_*X(2) \otimes_{X(2)_*} X(2)_*bo = \mathbb{Z}[b, c, r], \quad |r| = 2.$$

PROOF. Since $X(2) \wedge bo = Y_4 \wedge bu$, it follows from the Atiyah Hirzebruch spectral sequence that $X(2)_*bo = bu_*Y_4 = \mathbb{Z}[b, c]$, where $|b| = 2$ and $|c| = 4$. Furthermore, since $X(2)_*X(2) = X(2)_*[b_1]$, base-changing and renaming b_1 to r gives the second part. $\square$

This shows that, while we cannot hope to understand $\pi_*X(2)$, we can understand every piece of the resolution

$$X(2)_*bo \rightrightarrows X(2)_*X(2) \otimes_{X(2)_*} X(2)_*bo \Rrightarrow \dots .$$

Our next goal is to use this to understand the stack

$$\mathcal{M}_{bo} = \mathit{Stack}(X(2)_*bo, (X(2) \wedge X(2))_*bo) = \mathit{Stack}(\mathbb{Z}[b, c], \mathbb{Z}[b, c, r]).$$

Since everything is torsion free, computing the right unit of the Hopf algebroid $\big(X(2)_*bo, X(2)_*X(2) \otimes_{X(2)_*} X(2)_*bo\big)$ turns out not to be too hard: it is given by $b \mapsto b + 2r$ and $c \mapsto c + br + r^2$ (and the left unit is course just $b \mapsto b$, $c \mapsto c$). Those formulas are exactly the transformation rules of the coefficients of $x^2 + bx + c$ under the change of variable $x \mapsto x + r$, and so this Hopf algebroid corepresents the following algebro-geometric objects: curves

$$y = x^2 + bx + c,$$

together with the changes of coordinate $x \mapsto x + r$. Generically, such an equation can be thought of as describing a form of $\mathbb{G}_m$: first projectvize the curve by adding a point at infinity (the neutral element), and then remove the locus where $y = 0$. Moreover, this curve is naturally equipped with an invariant differential $\omega = dx/y$, or equivalently, a 1-jet at infinity. Note the similarity with elliptic curves: if we take the above equation $y^{d-1} = x^d + bx + c$ and set $d = 3$ instead of 2, then we get (generically) an elliptic curve.

We recall what the stack associated to $(\mathbb{Z}[b, c], \mathbb{Z}[b, c, r])$ is. By definition, a stack is given in terms of what it means to map into it. In our case, a map $\operatorname{Spec} A \to \mathcal{M}_{bo}$ consists of 3 pieces of data:

(1) a faithfully flat extension $A \to B$,
(2) a ring homomoprhism $\mathbb{Z}[b, c] \to B$, equivalently, two elements $b, c \in B$, which we'll interpret as giving a curve $y = x^2 + bx + c$ over B,
(3) a ring homomoprhism $\mathbb{Z}[b, c, r] \to B \otimes_A B$ compatible with the left/right actions of $\mathbb{Z}[b, c]$, equivalently, an element $r \in B \otimes_A B$ such that the change of variable $x \mapsto x + r$ yields an isomorphism between the curves over $B \otimes_A B$ gotten by base change along $B \rightrightarrows B \otimes_A B$,

subject to a cocycle condition in $B \otimes_A B \otimes_A B$. In other words, a map $\operatorname{Spec} A \to \mathcal{M}_{bo}$ is descent data for a (form of the) multiplicative group over $\operatorname{Spec} A$, along with a 1-jet. We denote this stack by $\mathcal{M}_{\mathbb{G}_m}^{+(1)}$. It is the moduli stack of multiplicative groups with a 1-jet, where additive degenerations are allowed.

Replacing MU by $X(2)$ in the definition (4) of $\mathcal{M}_{bo}$ was already a huge simplification, and allowed us to identify it with $\mathcal{M}_{\mathbb{G}_m}^{+(1)}$, but there is an even simpler Hopf algebroid that represents this stack. That smaller presentation will also make it evident that the map $\operatorname{Spec} \mathbb{Z}[b] = \mathcal{M}_{bu} \to \mathcal{M}_{bo}$ is finite of degree two, reflecting the fact that $\Sigma^{-2}\mathbb{C}P^2$ has only 2 cells. Consider the map $\operatorname{Spec} \mathbb{Z}[b] \to \mathcal{M}_{bo}$ representing the curve $y = x^2 + bx$. We want to restrict $(\mathbb{Z}[b,c], \mathbb{Z}[b,c,r])$ to curves of only that form, to get a smaller but equivalent Hopf algebroid. To do so, we look at those coordinate transformations $x \mapsto x + r$ that preserve the property $c = 0$, namely those for which r satisfies $r^2 + br = 0$. We therefore get a new Hopf algebroid

$$(\mathbb{Z}[b], \mathbb{Z}[b,r]/r^2 + 2r), \quad b \mapsto b + 2r$$

that represents $\mathcal{M}_{bo}$.

To know that this represents the same stack $\mathcal{M}_{bo}$, we still need to check that $\operatorname{Spec} \mathbb{Z}[b] \to \mathcal{M}_{bo}$ is a flat cover. This can be checked after pulling back along the cover $\operatorname{Spec} \mathbb{Z}[b,c] \to \mathcal{M}_{bo}$ (representing the universal curve $y = x^2 + bx + c$):

$$\begin{tikzcd}
\operatorname{Spec} \mathbb{Z}[b,c,r]/r^2 + br + c \arrow[r] \arrow[d, "\binom{b}{c} \mapsto \binom{b+2r}{0}"'] & \operatorname{Spec} \mathbb{Z}[b] \arrow[d, "y=x^2+bx"] \\
\operatorname{Spec} \mathbb{Z}[b,c] \arrow[r, "y=x^2+bx+c"'] & \mathcal{M}_{bo}
\end{tikzcd}$$

The stacky pullback represents pairs of curves $(y = x^2 + b_1 x, y = x'^2 + bx' + c)$, together with a coordinate transformation $x' = x + r$ that identifies them. This works if and only if the constant coefficient of $(x+r)^2 + b(x+r) + c = x^2 + b_1 x$ is zero, that is, if r is a root of $x^2 + bx + c$. The pullback is therefore given by $\operatorname{Spec} \mathbb{Z}[b, b_1, c, r]/(b + 2r - b_1, r^2 + br + c) = \operatorname{Spec} \mathbb{Z}[b,c,r]/r^2 + br + c$. It is a free module of rank two over $\operatorname{Spec} \mathbb{Z}[b,c]$, and in particular, it is faithfully flat. More generally, adjoining a root of a monic polynomial always produces a faithfully flat extension.

This example with *bo* is an important toy model for understanding how *tmf* works. At $p = 2$, there is an 8-fold cover of the moduli stack $\mathcal{M}_{tmf} = \overline{\mathcal{M}}_{ell}^{+(1)}$ which corresponds to the existence of an 8-cell complex X such that $tmf \wedge X$ is complex orientable (there is also an interesting 24-fold cover). Similarly, at $p = 3$, there is a 3-fold cover of $\mathcal{M}_{tmf}$, corresponding to a 3-cell complex Y such that $tmf \wedge Y$ is complex orientable. These covers and the associated presentations of the moduli stack of elliptic curves are very important for carrying out computations with *tmf*.

The string orientation

Lecture by Michael Hopkins,
Typed by André Henriques

Our goal is to construct an E_∞ map

$$MO\langle 8\rangle \to tmf. \tag{1}$$

Here, $MO\langle 8\rangle$ is a synonym of $MString$, and stands for the Thom spectrum of the 7-connected cover of BO. Recall that the connected covers of $\mathbb{Z}\times BO$ fit in a tower

$$\mathbb{Z}\times BO \leftarrow BO \leftarrow BSO \leftarrow BSpin \leftarrow BString.$$

The space BO is obtained from $\mathbb{Z}\times BO$ by killing its π_0. BSO is obtained from BO by killing $\pi_1(BO)=\mathbb{Z}/2$. $BSpin$ is obtained from BSO by killing $\pi_2(BSO)=\mathbb{Z}/2$. Finally, $BString$ is obtained from $BSpin$ by killing its first non-vanishing homotopy group, namely

$$\pi_4(BSpin)=\mathbb{Z}.$$

The first Pontryagin class of the universal vector bundle over $BSpin$ being twice the generator of $H^4(BSpin)=\mathbb{Z}$, that generator is usually called $\frac{p_1}{2}$. The space $BString$ is the fiber of the map $\frac{p_1}{2}: BSpin \to K(\mathbb{Z},4)$. It was previously called $BO\langle 8\rangle$ and its current optimistic name is due to connections with string theory.

The group $\pi_{2n}(MO\langle 8\rangle)$ is the bordism group of string manifolds of dimension $2n$. The group $\pi_{2n}(tmf)$ has a natural maps to MF_n, the group of modular forms of weight n. At the level of homotopy groups, the map (1) should then send a string manifold M to its Witten genus $\phi_W(M)$

$$\begin{array}{ccccc}\pi_{2n}(MO\langle 8\rangle) & \to & \pi_{2n}(tmf) & \to & MF_n\\ [M] & & \mapsto & & \phi_W(M).\end{array}$$

The cohomology theory tmf was known to us (actually its 2-completion, and under the name eo_2) before we knew about the connection with modular forms. We were then looking for the map (1) because of the following cohomology calculation. The cohomology of tmf at the prime 2 is the cyclic module $A/\langle Sq^1, Sq^2, Sq^4\rangle$ over the Steenrod algebra A. That same cyclic module occurs as a sumand in the cohomology of $MO\langle 8\rangle$, which lead us to believe that there should be such a map. The connection with elliptic curves was made while trying to construct the map $MO\langle 8\rangle \to tmf$. Indeed, producing a map from $MO\langle 8\rangle$ into a complex oriented cohomology theory E is something that one can do easily if the formal group associated to E comes from an elliptic curve. So the whole story of tmf had to do with that orientation. It is only in retrospect that we noted that the map (1) reproduces the Witten genus.

One interesting fact is that the map $\pi_{2n}(tmf) \to MF_n$ is not quite an isomorphism. It is an isomorphism after tensoring with $\mathbb{Z}[\frac{1}{2}, \frac{1}{3}]$, but it contains some torsion in its kernel, and its image is only a subgroup of finite index. So that way, one learns some things about the Witten genus that one might not have known before. For example, looking at π_{24}, we see a map

$$\pi_{24}(tmf) \simeq \mathbb{Z} \oplus \mathbb{Z} \longrightarrow MF_{12} \simeq \mathbb{Z} \oplus \mathbb{Z}. \tag{2}$$

The group MF_{12} has two generators c_4^3 and Δ, and it is interesting to note that the image of (2) is the subgroup generated by c_4^3 and 24Δ. So this gives a restriction on the possible values of the Witten genus. Translating it back into geometry, it says for example that on a 24 dimensional string manifold, the index of the Dirac operator with coefficients in the complexified tangent bundle is always divisible by 24

$$\widehat{A}\big(M; T_{\mathbb{C}}\big) \equiv 0 \pmod{24}. \tag{3}$$

The torsion in $\pi_*(tmf)$ also give interesting "mod 24 Witten genera", which are analogs of the mod 2 indices of the Dirac operator in KO-theory (those exist for manifolds of dimension 1 and 2 mod 8). These are all facts for which there is no explanation in terms of index theory, or even string theory.

In short, one reason for wanting (1) is that one gets more refined geometric information about the Witten genus of string manifolds. According to string theory, the Witten genus is supposed to be the index of the Dirac operator on the loop space LM of M. There ought to be some kind of structure on LM that accounts for the factor of 24 in (3), but up to now, there is no explanation using the geometric approach.

$\sim \quad \sim \quad \sim$

We now explain why E_∞ maps out of $MO\langle 8\rangle$, have anything to do with elliptic curves. In some sense, there is a very natural reason to expect a map like (1). To simplify the analysis, we work with the complex analog of $MO\langle 8\rangle$, namely $MU\langle 6\rangle$.

Considering the various connected covers of BU, one gets a tower of spaces

$$\mathbb{Z} \times BU \leftarrow BU \leftarrow BSU \leftarrow BU\langle 6\rangle$$

whose last term is the fiber of $c_2 : BSU \to K(\mathbb{Z}, 4)$. These have companion bordism theories, and $MU\langle 6\rangle$ is the one corresponding to $BU\langle 6\rangle$. In other words, $MU\langle 6\rangle$ is the Thom spectrum of the universal bundle over $BU\langle 6\rangle$.

We recall Quillen's theorem in its formulation via formal groups. Roughly speaking, it says that multiplicative maps from MU into an (even periodic) complex orientable cohomology theory E correspond to coordinates on the formal group $G := \mathrm{spf}(E^0(\mathbb{CP}^\infty))$ associated to E. In fact, the above statement is not quite accurate. It is true that a coordinate gives you a map $MU \to E$, but the latter encode slightly more data.

To understand the subtlety, we begin with an analogy. Multiplicative maps from the suspension spectrum of BU into E also correspond to some structures on G. The important thing about BU is that there is an inclusion $\mathbb{CP}^\infty \hookrightarrow BU$ exhibiting $E_*(BU)$ as the free commutative E_*-algebra on the E_*-module $E_*(\mathbb{CP}^\infty)$. The sloppy analysis goes as follows. A multiplicative map

$$\Sigma^\infty BU_+ \to E \tag{4}$$

corresponds to an E_*-algebra homomorphism

$$E_*(BU) \to E_*,$$

and since $E_*(BU)$ is the symmetric algebra on $E_*(\mathbb{CP}^\infty)$, those correspond to E_*-module maps

$$E_*(\mathbb{CP}^\infty) \to E_*.$$

The latter are then elements of $E^0(\mathbb{CP}^\infty)$, in other words functions on G. The problem with the above reasoning is that if we want (4) to be a multiplicative map, the base point has to go to 1. So the base point of $\mathbb{CP}^\infty$ has to go to $1 \in E$, and therefore we don't get all functions f on G, but only those satisfying $f(e) = 1$, where $e \in G$ is the unit. Now if we run all this through the Thom isomorphism, we find that multiplicative maps

$$MU \to E$$

are expressions of the form $f(z)/dz$ on G, such that the residue of $dz/f(z)$ is 1.

Let $\mathcal{O}$ denote the structure sheaf of G, let $e : S \to G$ be the identity section, and let $p : G \to S$ be the projection of G onto the base scheme $S := \operatorname{spec}(\pi_0 E)$. Let also $\mathcal{O}(-e)$ be the line bundle over G, whose sections are functions vanishing at zero. Expressions of the form $f(z)/dz$ can then be understood as sections of the line bundle

$$\mathcal{O}(-e) \otimes p^* e^* \mathcal{O}(-e)^{-1}. \tag{5}$$

The fiber of (5) over the origin is canonically trivialized, and the condition that $\operatorname{res}(dz/f(z)) = 1$ means that the value at e of our section is equal to $1 \in e^*\big(\mathcal{O}(-e) \otimes p^* e^* \mathcal{O}(-e)^{-1}\big)$.

Now, we would like to do something analogous for BSU instead of BU. Let L denote the tautological line bundle over $\mathbb{CP}^\infty$. The thing that allowed us to do the previous computation was the map

$$\mathbb{CP}^\infty \to BU$$

classifying $1 - L$. That map doesn't lift to BSU because $c_1(1 - L) \neq 0$, but the problem goes away as soon as one takes the product of two such bundles.

Let L_1 and L_2 denote the line bundles over $\mathbb{CP}^\infty \times \mathbb{CP}^\infty$ given by $L_1 := L \times 1$, and $L_2 := 1 \times L$ respectively. Since $c_1((1 - L_1) \otimes (1 - L_2)) = 0$, we get a map

$$\mathbb{CP}^\infty \times \mathbb{CP}^\infty \xrightarrow{(1-L_1)\otimes(1-L_2)} BSU.$$

If E is a complex orientable cohomology theory, then we find that multiplicative maps

$$\Sigma^\infty BSU_+ \to E,$$

equivalently ring homomorphisms

$$E_*(BSU) \to E_*,$$

give rise to functions $f(x, y)$ on $G \times G$ satisfying

$$\begin{aligned} & f(e, e) = 1, \\ & f(x, y) = f(y, x), \\ \text{and} \qquad & f(y, z) f(x, y +_G z) = f(x, y) f(x +_G y, z). \end{aligned} \tag{6}$$

In other words, they are functions on G with values in the multiplicative group, that are rigid, and that are symmetric 2-cocycles. The last condition in (6) is obtained by expanding the virtual bundle $(1-L_1)\otimes(1-L_2)\otimes(1-L_3)$ over $(\mathbb{CP}^\infty)^3$ in the following two ways:

$$\begin{aligned}
&(1-L_1)(1-L_2)(1-L_3)\\
=&(1-L_1)(1-L_3)+(1-L_1)(1-L_2)-(1-L_1)(1-L_2L_3)\\
=&(1-L_1)(1-L_3)+(1-L_2)(1-L_3)-(1-L_1L_2)(1-L_3),
\end{aligned}$$

from which we deduce that

$$(1-L_2)(1-L_3)+(1-L_1)(1-L_2L_3)=(1-L_1)(1-L_2)+(1-L_1L_2)(1-L_3).$$

In fact, the conditions (6) characterize homomorphisms from $E_*(BSU)$ into any E_*-algebra.

In the case of $BU\langle 6\rangle$, multiplicative maps

$$\Sigma^\infty BU\langle 6\rangle_+ \to E,$$

equivalently ring homomorphisms

$$E_*(BU\langle 6\rangle)\to E_*,$$

give rise to functions of three variables $f : G^3 \to \mathbb{G}_m$ that satisfy the following conditions: they are rigid, meaning that $f(e,e,e)=1$, they are symmetric, and they are 2-cocycles in any two of the three variables.

Remark 0.1. That kind of analysis stops at $BU\langle 6\rangle$ because it is the last connected cover of BU with only even dimensional cells.

Remark 0.2. There is an interesting analogy with classical group theory. Let Γ be a group, and let $I\subset\mathbb{Z}[\Gamma]$ denote its augmentation ideal. A function $I\to A$ is the same as a function $f:\Gamma\to A$ satisfying $f(e)=0$. A function $I^2\to A$ is the same thing as a function on $\Gamma\times\Gamma$, that is rigid, and a symmetric 2-cocycle. Finally, a function $I^3\to A$ is the same thing as a function $\Gamma^3\to A$ that is rigid, symmetric, and a 2-cocycle in any two of the three variables. So, in some sense, the connected covers of BU correspond to taking powers of the augmentation ideal of the group ring of a formal group.

We now proceed to $MU\langle 6\rangle$. Recall from our discussion about MU that, under the Thom isomorphism, functions get replaced by sections of a line bundle. Since multiplicative maps $BU\langle 6\rangle\to E$ corresponded to functions on G^3, we expect multiplicative maps $MU\langle 6\rangle\to E$ to produce sections of some line bundle Θ over G^3. Letting $\mathcal{L}:=\mathcal{O}(-e)$, we can describe Θ by writing down its fiber $\Theta_{(x,y,z)}$ over a point[1] $(x,y,z)\in G^3$. It is given by

$$\Theta_{(x,y,z)} := \frac{\mathcal{L}_{x+y+z}\otimes\mathcal{L}_x\otimes\mathcal{L}_y\otimes\mathcal{L}_y}{\mathcal{L}_{x+y}\otimes\mathcal{L}_{x+z}\otimes\mathcal{L}_{y+z}\otimes\mathcal{L}_e}, \tag{7}$$

where $+$ denotes to the operation of G. Multiplicative maps

$$MU\langle 6\rangle\to E$$

[1] G^3 is a formal scheme, and so it doesn't have many points. To make sense of (7) and (8), one can use the formalism of "functor of points" and work with maps $T\to G^3$, that one views as T-parametrized families of points.

then correspond to sections s of Θ that are rigid, symmetric, and 2-cocycles in any two of the three variables. These conditions make sense because the two sides of each one of the equations

$$
\begin{gathered}
s(e,e,e) = 1 \\
s(x,y,z) = s(y,x,z) = s(x,z,y) \\
s(y,z,v)s(x,y+z,v) = s(x,y,v)s(x+y,z,v),
\end{gathered}
\tag{8}
$$

are sections of (canonically) isomorphic line bundles. For example, the first of the above equations makes sense because the fiber $\Theta_{(e,e,e)}$ is canonically trivialized.

Now here's the thing that was inspiring to us: if J is an elliptic curve, then the line bundle Θ is trivial over J^3. This is a special case of a general fact for abelian varieties called the theorem of the cube.

To understand why Θ is a trivial line bundle, recall that given a divisor D on J of degree zero, there is a meromorphic function f with that given divisor iff the points of D add up to zero (taking multiplicities into account). In particular, given two points x, y on our elliptic curve J, there exists a meromorphic function f with simple poles at $-x$ and $-y$, and a simple zeroes at $-x-y$ and e. But that function is only well defined up to scale, and there is no canonical choice for it. In other words, the line bundle

$$
\mathcal{O}\big(-[-x-y]+[-x]+[-y]-[e]\big),
\tag{9}
$$

is trivial, but not trivialized. On the other hand, if we divide (9) by its fiber over zero, then it acquires a canonical trivialization. Fix points $x, y \in J$, and consider the restriction of Θ to the subscheme $\{x\} \times \{y\} \times J$. We then have a canonical isomorphism between $\Theta|_{\{x\}\times\{y\}\times J}$ and the quotient of (9) by its value at zero. So we get canonical trivializations of each such restriction. These trivializations then assemble to a trivialization of Θ.

Note that Θ is not just trivial, it is canonically trivialized. Therefore it makes sense to talk about the section "1" of Θ. If we take that section, and pull it back along any map $J^m \to J^3$ then we'll always get the section "1" of (another) canonically trivial line bundle. So any conditions one might decide to impose on a section of Θ, for example the conditions (8), will be automatically satisfied by our distinguished section "1".

The consequence of the above discussion is that if the formal group of E comes from an elliptic curve, then we get a canonical solution to the equations (8). In particular, we get a canonical multiplicative map

$$
MU\langle 6\rangle \to E.
$$

Now, there is an analog of that for $MO\langle 8\rangle$ which involves adding one more condition to the list (8), namely, the condition $s(x,y,-x-y) = 1$. That condition is automatically satisfied for the same reasons as above, and one finds that there is a canonical multiplicative map

$$
MO\langle 8\rangle \to E
\tag{10}
$$

as soon as the formal group of E comes from an elliptic curve.

If J is an elliptic curve defined over a ring R, and $\varphi : R \to R'$ is a ring homomorphism, one gets an induced elliptic curve J' over R'. Let E, E' be complex orientable cohomology theories whose associated formal groups are the formal completions of J and J'. If $f : E \to E'$ is a map of spectra with $\pi_0(\varphi) = f$, then the maps (10) induce a homotopy commutative diagram

$$\begin{tikzcd} & MO\langle 8\rangle \arrow[dl] \arrow[dr] & \\ E \arrow[rr, "f"] & & E'. \end{tikzcd} \tag{11}$$

This is what led to the idea of assembling all cohomology theories coming from elliptic curves[2] into a single cohomology theory

$$tmf = \underset{\text{elliptic curves } J}{\operatorname{holim}} \Big(\text{cohomology theory } E \text{ associated to } J\Big). \tag{12}$$

The maps (11) then assemble into a map $MO\langle 8\rangle \to tmf$. That map reproduces the Witten genus at the level of homotopy groups, and is then an explanation of why the Witten genus of a string manifold is a modular form.

∼ ∼ ∼

So far, we have addressed the questions of why one should be interested in a map like (1), and why one could expect there to be one. We now describe a homotopy theoretic way of producing that map.

To make the previous arguments actually work, one would need to do things in a much more rigid way. Indeed, the maps (10) are only multiplicative up to homotopy, and the triangles (11) only commute up to homotopy. To get a map of spectra $MO\langle 8\rangle \to tmf$, one would need to rigidify the triangles (11). And then, to get it to be a map of E_∞ ring spectra, one would need to upgrade the maps (10) to E_∞ maps. That shall not be our strategy. Instead, we will produce the map (1) directly, by using the decomposition of tmf into its various p-completions and $K(n)$-localizations.

Another example of something that one can construct using the same methods, is the map from spin bordism to KO theory

$$MSpin \to KO \tag{13}$$

that sends a spin manifold to its $\widehat{A}$-genus, namely the index of the Dirac operator. Equivalently, (13) is the KO-theory orientation of spin bundles. Note that before our techniques, there was no way known of producing the map (13) using the methods of homotopy theory: one had to use geometry. The construction of (13) will be our warm-up, before trying to get the more interesting one

$$MO\langle 8\rangle \to tmf.$$

Given a vector bundle ζ over a space X, we let X^ζ denote the associated Thom space. One should think of X^ζ as a twisted form of X. Actually, if we view X^ζ as a spectrum, then we should rather say that X^ζ is a twisted form of the suspension spectrum $\Sigma^\infty X_+$. From now on, we shall abuse notation, and write $X \wedge -$ instead of $\Sigma^\infty X_+ \wedge -$.

Similarly, given a multiplicative cohomology theory E, the spectrum $X^\zeta \wedge E$ is a twisted form of $X \wedge E$. An E-orientation is then a trivialization of the bundle of module spectra obtained by fiberwise smashing E with the (compactified) fibers of ζ. The Thom isomorphism is the induced equivalence between $X^\zeta \wedge E$ and $X \wedge E$.

[2]In the statement (12), one should include the multiplicative group $\mathbb{G}_m$, as well as the additive group $\mathbb{G}_a$, in our definition of "elliptic curves".

In fact, we didn't need to start with a vector bundle: a spherical fibration is enough to produce a Thom spectrum.

If ζ is just a single bundle, then there are no further conditions that one could impose. But if we're trying to compatibly orient a whole class (such as spin bundles, or string bundles) with a notion of direct sum of bundles, and if E is an E_∞ ring spectrum, then

$$(X, \zeta) \mapsto X^\zeta \wedge E \qquad \text{and} \qquad (X, \zeta) \mapsto X \wedge E$$

are symmetric monoidal functors, and one can ask for the equivalence between them to be symmetric monoidal. That's the concept of an E_∞ orientation. One reason for looking for E_∞ orientation instead of just orientations, is that it simplifies the computations.

If R is a spectrum equipped with a homotopy associative product $R \wedge R \to R$, we define $GL_1(R) \subset \Omega R$ to be the subset of the zeroth space of R where we only take the connected components corresponding to the units of $\pi_0(R)$. It satisfies

$$[X, GL_1(R)] = R^0(X)^\times$$

for all unbased spaces X. If X has a base point, then the above equation is not quite correct. Since the base point of X has to go to the base point $1 \in GL_1(R)$, the better way to write this is

$$[X, GL_1(R)] = \left(1 + \widetilde{R}^0(X)\right)^\times.$$

In words, it is the group of invertible elements of $R^0(X)$ that restrict to 1 at the base point. If R is an A_∞-ring spectrum, then $GL_1(R)$ is an A_∞-space, and thus a loop space. We let $BGL_1(R)$ denote its classifying space.

Example 0.3. $BGL_1(\mathbb{S})$ is the classifying space for spherical fibrations.

If we have a map

$$\zeta : X \to BGL_1(\mathbb{S}),$$

then we get a spherical fibration over X, and thus a corresponding Thom spectrum X^ζ. Now, if we start instead with a map

$$\zeta : X \to BGL_1(R),$$

then we can form an analogous construction in the world of R-modules. Let P be the $GL_1(R)$-principal bundle associated to ζ

$$\begin{tikzcd} GL_1(R) \arrow[r] & P \arrow[d] & \\ & X \arrow[r, "\zeta"] & BGL_1(R). \end{tikzcd}$$

We then define

$$X^\zeta := \Sigma^\infty P_+ \underset{\Sigma^\infty GL_1(R)_+}{\wedge} R.$$

That construction can be understood as follows. Each one of the fibers of P is a copy of $GL_1(R)$, more precisely a torsor for $GL_1(R)$. The operation $- \wedge_{\Sigma^\infty GL_1(R)_+} R$ then converts that torsor into a copy of R. So for each point of X, one gets a copy of R. If ζ is the trivial map, one has $X^\zeta = X \wedge R$. So for general ζ, the R-module spectrum X^ζ should be thought of as a twisted form of $X \wedge R$. The above construction is functorial in the following sense. Given a map $R \to S$ of A_∞-ring spectra, one gets a corresponding map

$$f : BGL_1(R) \to BGL_1(S).$$

If $\zeta : X \to BGL_1(R)$ is as above, then one finds that

$$X^{f\circ\zeta} = X^\zeta \wedge_R S.$$

Now consider the J-homomorphism $O \to GL_1(\mathbb{S})$ that sends a linear automorphism of a vector space to the corresponding self-homotopy equivalence of its one-point compactification. Its delooping

$$J : BO \to BGL_1(S)$$

associates to a vector bundle V a spherical fibration $Sph(V)$. Let $\iota : BGL_1(\mathbb{S}) \to BGL_1(R)$ be induced by the unit map $\mathbb{S} \to R$. If V is a vector bundle and $\zeta := Sph(V)$ the corresponding spherical fibration, then nullhomotopies of $\iota \circ \zeta$ correspond to R-orientations of V:

$$\begin{tikzcd} BO \arrow[r, "J"] & BGL_1(\mathbb{S}) \arrow[r, "\iota"] & BGL_1(E) \\ X \arrow[u, "V"] \arrow[ur, "\zeta"'] & & \end{tikzcd}$$

Indeed, X^ζ is the usual Thom spectrum of V, and $X^{\iota\circ\zeta} = X^\zeta \wedge R$ is the spectrum that we want to trivialize. A homotopy $\iota \circ \zeta \simeq *$ induces an isomorphism $X^\zeta \wedge R \simeq X \wedge R$.

Now suppose that we want to functorially R-orient every vector bundle. Then we would need to chose a nullhomotopy for the composite

$$BO \xrightarrow{J} BGL_1(\mathbb{S}) \xrightarrow{\iota} BGL_1(R).$$

If we only wanted to functorially R-orient spin bundles, then we would need a nullhomotopy of

$$BSpin \to BO \xrightarrow{J} BGL_1(\mathbb{S}) \xrightarrow{\iota} BGL_1(R). \tag{14}$$

Similarly, if we wanted to R-orient string bundles, then we would need a nullhomotopy of

$$BO\langle 8\rangle \to BO \xrightarrow{J} BGL_1(\mathbb{S}) \xrightarrow{\iota} BGL_1(R).$$

At this point, finding those nullhomotopies is still a rather hard problem. For example, if R was KO-theory, and if we were trying to construct the Atiyah-Bott-Shapiro orientation (13), then we wouldn't be able to handle that. The reason is that the space

$$\mathrm{Map}\big(BSpin, BGL_1(KO)\big)$$

is very big. It is hard to tell which map $BSpin \to BGL_1(KO)$ one is looking at. And in particular, it is hard to tell if a map is null.

To solve that problem, we impose more conditions: we ask that (14) be nullhomotopic through E_∞ maps! Of course, that condition doesn't make any sense yet, because we had only assumed that R was A_∞. But if R is an E_∞-ring spectrum, then $GL_1(R)$ is an E_∞-space and the condition does make sense.

So let's assume that R is E_∞, and let $gl_1(R)$ be the spectrum associated to $GL_1(R)$. Since $gl_1(R)$ is obtained by delooping an E_∞-space, it is necessarily (-1)-connected (there are also ways of adding negative homotopy groups to $gl_1(R)$, but that's not relevant for the present discussion).

Let Y be a (-1)-connected spectrum and let $X := \Omega^\infty Y$ be its zeroth space. We can then consider infinite loop maps

$$\zeta : X \to BGL_1(R),$$

or equivalently, maps of spectra

$$\zeta^\infty : Y \to \Sigma\, gl_1(R).$$

Since ζ is an infinite loop map, the Thom spectrum X^ζ is an E_∞-ring spectrum. E_∞-orientations then correspond to nullhomotopies of ζ through infinite loop maps. Equivalently, they correspond to nullhomotopies of ζ^∞. In our case of interest, we see that E_∞ maps $MSpin \to KO$ correspond to nullhomotopies of the composite

$$bspin \to bo \xrightarrow{J} \Sigma\, gl_1(\mathbb{S}) \to \Sigma\, gl_1(KO). \tag{15}$$

Similarly, E_∞ maps $MString \to tmf$ correspond to nullhomotopies of

$$bo\langle 8\rangle \to bo \xrightarrow{J} \Sigma\, gl_1(\mathbb{S}) \to \Sigma\, gl_1(tmf).$$

A nullhomotopy of a composite $f \circ g$ is the same thing as an extension of f over the cone of g. So a nullhomotopy of (15) is equivalent to a dotted arrow

$$\begin{tikzcd} \Sigma^{-1}bspin \arrow[r, "J"] & gl_1(\mathbb{S}) \arrow[r] \arrow[d] & C \arrow[dl, dotted] \\ & gl_1(KO) & \end{tikzcd} \tag{16}$$

making the diagram commute. Similarly, dotted arrows

$$\begin{tikzcd} \Sigma^{-1}bo\langle 8\rangle \arrow[r, "J"] & gl_1(\mathbb{S}) \arrow[r] \arrow[d] & D \arrow[dl, dotted] \\ & gl_1(tmf) & \end{tikzcd} \tag{17}$$

correspond to E_∞ orientations $MString \to tmf$. The horizontal lines in (16) and (17) are cofiber sequences, and we have desuspended all our spectra to simplify the notation. The set of extensions (17) is either empty, or a torsor over the group $\big[bo\langle 8\rangle, gl_1(tmf)\big]$.

From now on, we pick a prime p, and assume implicitly that all spectra are p-completed. Fix $n \geq 1$. Bousfield and Kuhn constructed a functor Φ from spaces to spectra, that factors $K(n)$-localization as

$$\begin{tikzcd} \text{Spectra} \arrow[rr, "L_{K(n)}"] \arrow[dr, "\Omega^\infty"'] & & \text{Spectra.} \\ & \text{Spaces} \arrow[ur, "\Phi"'] & \end{tikzcd}$$

Apart from the difference of π_0, the zeroth space $\Omega^\infty(gl_1(R)) = GL_1(R)$ of the spectrum $gl_1(R)$ is the same as the zeroth space of R. Since $L_{K(n)}$ kills Eilenberg-McLane spectra, the difference of π_0 doesn't matter, and so we get

$$L_{K(n)}\big(gl_1(R)\big) \simeq L_{K(n)}(R).$$

More generally, if X and Y are spectra such that for some $m \geq 0$, the m-th connected cover of $\Omega^\infty X$ agrees with the the m-th connected cover of $\Omega^\infty Y$, then $L_{K(n)}(X) \simeq L_{K(n)}(Y)$.

Since KO is $K(1)$-local, the localization map $gl_1(KO) \to L_{K(1)}\big(gl_1(KO)\big)$ induces a map

$$L : gl_1(KO) \to KO.$$

The spectrum $gl_1(KO)$ being connected, there is no hope for L to be an isomorphism. But Bousfield proved that it is a π_*-isomorphism for $* > 2$. Going back to (16), we note that the first non-zero homotopy group of $\Sigma^{-1}bspin$ is in dimension 3. That is exactly the range above which $gl_1(KO)$ looks like KO. So the obstruction to constructing our orientation (13) may be taken to be the composite

$$\Sigma^{-1}bspin \to gl_1(KO) \xrightarrow{L} KO.$$

It lives in $[\Sigma^{-1}bspin, KO] = KO^1(bspin)$, which is zero. The calculation

$$KO^1\big(bspin\big) = 0$$

is actually not too hard: it follows from the knowledge of the cohomology operations in KO-theory, which is something that one computes using Landweber exactness. So one gets the existence of a dotted arrow in (16).

The above discussion was done at the cost of completing at a prime. To do things correctly, one should complete at each prime individually, do something rationally, and then assemble the results using Sullivan's arithmetic square. Eventually, one sees that the map $\Sigma^{-1}bspin \to gl_1(KO)$ is null (no completion any more). With a little bit more work, one can parametrize the set of nullhomotopies of (15), which amounts to parametrizing the E_∞-orientations (13).

We now proceed to the case of tmf. That's a more complicated story, but it also leads to something more interesting. First of all, one can generalize Bousfield's result and show that the fiber of the localization map

$$L : gl_1(tmf) \to L_{K(1)\vee K(2)}\big(gl_1(tmf)\big)$$

has no homotopy groups in dimensions 4 and above. So, as far as mapping $\Sigma^{-1}bo\langle 8\rangle$ is concerned, we may replace $gl_1(tmf)$ by its $K(1) \vee K(2)$-localization. As a consequence, if we want to produce a nullhomotopy for our map

$$\Sigma^{-1}bo\langle 8\rangle \to gl_1(tmf),$$

we may as well produce one for the composite

$$\Sigma^{-1}bo\langle 8\rangle \to gl_1(tmf) \xrightarrow{L} L_{K(1)\vee K(2)}\big(gl_1(tmf)\big).$$

Since we understand better the localizations at individual Morava K-theories, we consider the Hasse pullback square

$$
\begin{tikzcd}
gl_1(tmf) \arrow[r, dash, "\substack{\pi_*\text{-iso.}\\ \text{for } *\geq 4\\ \downarrow}"] & L_{K(1)\vee K(2)}\big(gl_1(tmf)\big) \arrow[r] \arrow[d] & L_{K(2)}\big(gl_1(tmf)\big) \arrow[d] \\
& L_{K(1)}\big(gl_1(tmf)\big) \arrow[r] & L_{K(1)}L_{K(2)}\big(gl_1(tmf)\big).
\end{tikzcd}
\tag{18}
$$

The spectrum $\Sigma^{-1}bo\langle 8\rangle$ doesn't have any $K(2)$ cohomology, and therefore

$$\Big[\Sigma^{-1}bo\langle 8\rangle, L_{K(2)}\big(gl_1(tmf)\big)\Big]_* = 0.$$

So as far as mapping $\Sigma^{-1}bo\langle 8\rangle$ into it, the square (18) behaves like a fiber sequence, and we get a long exact sequence

$$\ldots \to \Big[bo\langle 8\rangle, L_{K(1)}L_{K(2)}\big(gl_1(tmf)\big)\Big] \to$$
$$\to \Big[\Sigma^{-1}bo\langle 8\rangle, gl_1(tmf)\Big] \to \Big[\Sigma^{-1}bo\langle 8\rangle, L_{K(1)}\big(gl_1(tmf)\big)\Big] \to \ldots$$

Now, we wish to apply Bousfield and Kuhn's result to identify the above spectra. At first glance, things look pretty good because both $L_{K(1)}(gl_1(tmf))$ and $L_{K(1)}L_{K(2)}(gl_1(tmf))$ are $K(1)$-local, and hence "look a lot like K-theory". If we apply the argument with the Bousfield Kuhn functor to the diagram (18), we get a square

$$
\begin{tikzcd}
gl_1(tmf) \arrow[r] \arrow[d] & L_{K(2)}(tmf) \arrow[d] \\
L_{K(1)}(tmf) \arrow[r] & L_{K(1)}L_{K(2)}(tmf)
\end{tikzcd}
\tag{19}
$$

that is a pullback square above dimension 4. The above long exact sequence then becomes

$$\ldots \Big[bo\langle 8\rangle, L_{K(1)}L_{K(2)}(tmf)\Big] \to \Big[\Sigma^{-1}bo\langle 8\rangle, gl_1(tmf)\Big] \to \Big[\Sigma^{-1}bo\langle 8\rangle, L_{K(1)}(tmf)\Big] \ldots$$

We note that, $K(1)$-locally, the orientation obstruction group $[\Sigma^{-1}bo\langle 8\rangle, L_{K(1)}(tmf)]$ vanishes, and so $K(1)$-local orientations exist. The question is whether there exists one which lifts to tmf.

The diagram (19) is very similar to the Hasse square for tmf. The only difference is that it has $gl_1(tmf)$ insted of tmf in its upper left corner. The right vertical map

$$L_{K(2)}(tmf) \to L_{K(1)}L_{K(2)}(tmf)$$

is simply the localization map $X \to L_{K(1)}(X)$ applied to $X = L_{K(2)}(tmf)$, but the lower map

$$L_{K(1)}(tmf) \to L_{K(1)}L_{K(2)}(tmf) \tag{20}$$

is much more subtle. In some sense, the whole game of producing the string orientation, is to understand that map.

At the beginning, we had a few of ad-hoc ways of understanding that map. Later, Charles Rezk found extremely beautiful formula for it. It says that, at the level of homotopy groups, the map (20) is given by $1 - U_p$, where U_p is the Atkin operator. Knowing that fact, we may replace (19) by

$$\begin{tikzcd} gl_1(tmf) \arrow[r] \arrow[d] & tmf \arrow[d] \\ L_{K(1)}(tmf) \arrow[r, "1-U_p"] & L_{K(1)}(tmf). \end{tikzcd} \tag{21}$$

The above square ends up giving us enough understanding of the homotopy type of $gl_1(tmf)$. We refer the reader to [**AHR**] for the actual computations.

$\sim \quad \sim \quad \sim$

Let me emphasize the number of really amazing things that come out of the square (21). First of all, it gives you the string orientation. The question of orientation boils down to the question of understanding the homotopy type of $gl_1(tmf)$; using the square (21) and the description of its bottom arrow, one can then make the required calculations. Secondly, the homotopy fiber of $1 - U_p : L_{K(1)}(tmf) \to L_{K(1)}(tmf)$ is an extremely interesting spectrum. One can describe its homotopy groups in terms of modular forms, but if one tries to compute them explicitly, one encounters some unsolved problems in number theory. Finally, we conjecture that there is a relationship between that homotopy fiber and smooth structures on free loop spaces of spheres.

$\sim \quad \sim \quad \sim$

Let us go back to the construction of the string orientation. Recall that we have the following diagram

$$\begin{tikzcd} \Sigma^{-1}bstring \arrow[r] & gl_1(\mathbb{S}) \arrow[r] \arrow[d] & C \arrow[r] \arrow[dl, dotted] & bstring, \\ & gl_1(tmf) & & \end{tikzcd}$$

and that the string orientation is equivalent to having a map $C \to gl_1(tmf)$. Here, as before, $gl_1(\mathbb{S})$ is the infinite delooping of GL_1 of the sphere spectrum, and C is the mapping cone of the map from $\Sigma^{-1}bstring$. The set of such maps, if non-empty, is a torsor over $[bstring, gl_1(tmf)]$.

Because of the particular homotopy type of the spectra C and $gl_1(tmf)$, the dotted map in the above diagram is completely determined by what it does on rational homotopy groups, which allows us to relate a map like this to the classical theory of characteristic series. This is quite remarkable, because such a map actually corresponds to an E_∞ map $MString \to tmf$, wheras, to specify the charcteristic series, you just need to write down a sequence of numbers. Roughly speaking, both C and $gl_1(tmf)$ "look like K-theory" (or can be localized to look like K-theory), and maps from the K-theory spectrum to itself are determined by their effect on homotopy groups, so you can identity the $[bstring, gl_1(tmf)]$-torsor of above maps by looking at what this does on rational homotopy group.

For simplicity, we treat the complex version of the above diagram. Let us also put an arbitrary E_∞-spectrum R in the place of *tmf*:

$$\begin{tikzcd} \Sigma^{-1}bu \arrow[r] & gl_1(\mathbb{S}) \arrow[r] \arrow[d] & C \arrow[r, "\simeq_{\mathbb{Q}}"] \arrow[dl, dotted, "\mathfrak{U}"] & bu, \arrow[dl, dotted] \\ & gl_1(R) \arrow[r] & gl_1(R)_{\mathbb{Q}} & \end{tikzcd} \tag{22}$$

Note that, since $\pi_0(gl_1(\mathbb{S})) = \mathbb{Z}^\times$ and $\pi_{\geq 1}(gl_1(\mathbb{S})) = \pi_{\geq 1}(\mathbb{S})$, all the homotopy groups of $gl_1(\mathbb{S})$ are finite, and it is thus rationally trivial. It follows that C is rationally homotopy equivalent to bu, which is why one gets a map $bu \to gl_1(R)_{\mathbb{Q}}$ from the "Thom class" $\mathfrak{U}$. For $n \geq 1$, let $b_n \in \pi_{2n}(gl_1(R))_{\mathbb{Q}} = \pi_{2n}R \otimes \mathbb{Q}$ be the images of the Bott generators in $\pi_{2n}(bu)$. In all the cases of interest (i.e., $R = KO$ or *tmf* (in which case, one would have to replace bu by *bspin* or *bstring*, respectively) or, more generally, any spectrum that represents a cohomology theory whose job in life is to really be a cohomolgoy theory, i.e., anything that is build in a simple way out of Lanweber exact theories; not something like the image of J-spectrum), the map $\mathfrak{U}$ is detected rationally and thus completely determined by the sequence of b_n's.

We now relate this to the classical theory of characteristic series. As explained earlier, the map $\mathfrak{U}$ corresponds to an E_∞-map $U : MU \to R$. After rationalizing R, we get two E_∞-maps from MU: the first one is the rationalization of U, and the second is the map that factors through rational homology. These two maps are typically not equal to each other, and the following diagram

$$\begin{tikzcd} MU \arrow[r, "U"] \arrow[d] & R \arrow[d] \\ H\mathbb{Q} \arrow[r] & R_{\mathbb{Q}} \end{tikzcd}$$

is therefore not commutative. The ratio of these two Thom classes, i.e., the failure of the above diagram to commute is a unit in the rational R-cohomology of $(\mathbb{Z}\times BU)^\times$. In other words, it is a stable exponential characteristic class.

By the splitting principle, the above class is determined by its image $K_\Phi(z) \in R^*_{\mathbb{Q}}[[z]]$ under the map

$$R^0_{\mathbb{Q}}\big((\mathbb{Z} \times BU)^\times\big) \to R^0_{\mathbb{Q}}(\mathbb{CP}^\infty) \subset (R_* \otimes \mathbb{Q}[[z]])^\times$$

coming from the universal line bundle $\mathbb{CP}^\infty \to (\mathbb{Z} \times BU)^\times$. Thinking of U as a genus, the Hirzebruch characteristic series $K_\Phi(z)$ is the Chern character of that class, i.e., its value on the universal line bundle over $\mathbb{CP}^\infty$.

Applying the zeroth space functor to $bu \to gl_1(R)_{\mathbb{Q}}$, we get an infinite loop map

$$\mathbb{Z} \times BU \to GL_1R_{\mathbb{Q}} \tag{23}$$

which we then interpret as an invariant of stable bundles that carries Whitney sums to products, namely, a stable exponential characteristic class. Now we need to do the following simple calculation: we have the map (23); we know the composite $\mathbb{CP}^\infty \to \mathbb{Z} \times BU \to GL_1R_{\mathbb{Q}}$; we know that the map is multiplicative, and would like to calculate the effect of that map on π_{2n}. The answer turns out to be

$$K_\Phi(z) = \exp\left(\sum_{n=1}^{\infty} b_n \frac{z^n}{n!}\right), \tag{24}$$

where the constants b_n are the same as the ones coming from the rightmost dotted arrow in (22).

If you're trying to construct an orientation that realizes an old friend, such as the Witten genus, or the genus coming from the theorem of the cube, the formula (24) tells you which map you have to try to pick. In our case, the b_n's were the coefficients in the log of the Weierstrass sigma function, which are Eisenstein series.

Let us take the example of K-theory and of the Todd genus $MU \to K$. The characteristic series is then given by

$$K_\Phi(x) = \frac{x}{1-e^{-x}}.$$

If we want to write it in the form (24), then we get

$$d\log\big(K_\Phi(x)\big) = \frac{1}{x} - \frac{e^{-x}}{1-e^{-x}} = \frac{1}{x} - \frac{1}{e^x - 1} = -\sum_{n\geq 0} B_n \frac{x^{n-1}}{n!},$$

where B_n are the Bernoulli numbers. By integrating and then exponentiating, we finally get $b_n = -\frac{B_n}{n}$. If you do all that for the $\widehat{A}$-genus (after replacing bu by bo in (22)), and once you decide what the canonical generators of $\pi_{4n}(bo)_{\mathbb{Q}}$ are, you get $\frac{B_{2n}}{4n}$. These are the famous numbers from differential topology, whose denominators tell you the order of the image of J (at odd primes), and whose numerators[3] tell you the order of group of smooth structure on S^{4n-1} that bound parallelizable manifolds. All this is already a hint that there is going to be some connection to differential topology.

In geometric topology, the following groups play a central role: O – the stable rotation group, PL – the group of piecewise linear equivalences of the sphere, and $G = GL_1(\mathbb{S})$ – the self homotopy equivalences of the sphere. All of them are infinite loop spaces, and there is a corresponding sequence of classifying spaces

$$BO \to BPL \to BG = BGL_1(\mathbb{S}).$$

We also have associated spectra: $\Sigma^{-1}bo \to \Sigma^{-1}bpl \to gl_1(\mathbb{S})$. The corresponding homogeneous spaces form a fibration sequence

$$PL/O \to G/O \to G/PL, \tag{25}$$

and they all have nice interpretations: the homotopy groups of G/O are called the structure sets of smooth structures on the sphere. The homotopy groups of G/PL are $\mathbb{Z}$ in every fourth degree and zero everywhere else (at least at odd primes), and the map $G/O \to G/PL$ sends an element of the structure set to its surgery obstruction, which is the signature. Finally, the trivializations of those obstructions are controlled by the space PL/O, whose homotopy groups are the Kervaire-Milnor groups of smooth structures on the sphere.

Sullivan showed that the signature also induces a PL orientation $MPL \to KO[\frac{1}{2}]$. That map is E_∞, and one of Sullivan's results is that the corresponding

[3] The order of that group of exotic spheres is $2^{2n-2}(2^{2n-1}-1)$ times the numerator of $\frac{|B_{2n}|}{4n}$.

map

$$\begin{tikzcd} \Sigma^{-1}bpl \arrow[r] & gl_1(\mathbb{S}) \arrow[r] \arrow[d] & C = g/pl \arrow[dl, dotted, "signature"] \\ & gl_1 KO[\frac{1}{2}] & \end{tikzcd}$$

is a homotopy equivalence away from 2: $g/pl \simeq gl_1 KO[\frac{1}{2}]$. If we now run the same story for oriented manifolds, we get

$$\begin{tikzcd} \Sigma^{-1}bso \arrow[r] & gl_1(\mathbb{S}) \arrow[r] \arrow[d] & C = g/so \arrow[dl, dotted, "signature"] \\ & gl_1 KO[\frac{1}{2}] & \end{tikzcd} \tag{26}$$

and the dotted map $C \to gl_1 KO[\frac{1}{2}]$ can be identified with the map $g/so \to g/pl$ form the sequence (25). The upshot of that discussion is that the above map has something to do with geometric topology, and that its fiber is related to groups of exotic spheres.

So far, this was all about K-theory. But there is a conjectural analog for tmf that goes as follows. Let us now invert the prime 2 and consider $tmf_0(2)$, which is the value of the sheaf $\mathcal{O}^{\mathrm{top}}$ on the moduli space of elliptic curves with a point of order two (which is étale over the moduli space of elliptic curves away from $p = 2$). The same argument that refined the Witten genus will also refine the Ochanine genus, and will produce a map

$$\begin{tikzcd} \Sigma^{-1}bso \arrow[r] & gl_1(\mathbb{S}) \arrow[r] \arrow[d] & g/so \arrow[dl, dotted, "Ochanine\ genus"] \\ & gl_1 tmf_0(2)[\frac{1}{2}] & \end{tikzcd} \tag{27}$$

For $p > 2$, there is a pullback square

$$\begin{tikzcd} F \arrow[r] \arrow[d, equal] & gl_1 tmf_0(2)\hat{}_p \arrow[r, "\ell"] \arrow[d] & tmf_0(2)\hat{}_p \arrow[d] \\ F \arrow[r] & L_{K(1)} tmf_0(2) \arrow[r] & L_{K(1)} tmf_0(2) \end{tikzcd}$$

similar to (21). Here, $F = \mathrm{fib}(\ell)$, and the map ℓ is called the topolgogical logarithm as it maps the units of $tmf_0(2)$ back to $tmf_0(2)$. By construction, the Ochanine genus $g/so \to gl_1 tmf_0(2)[\frac{1}{2}]$ maps to zero in $tmf_0(2)\hat{}_p$, and thus lifts to a map $g/so \to F$. The natural map from the diagram (26) to the diagram (27) therefore factors thought the fiber of the topological logarithm.

In the K-theory story, the map $g/so \to gl_1(KO)[\frac{1}{2}]$ was intimately related to the group of exotic spheres. That story was built out of the signature. The Ochanine genus is supposed to be the signature of the loop space, so $\pi_*(F)$ should have something to do with exotic structures on the free loop space of spheres. Moreover, given an exotic structure on S^n we clearly get one on LS^n, and that's what the map from (26) to (27) should be. Of course, this all completely conjectural... But this would be really neat prospect for tmf.

We now switch back to tmf instead of $tmf_0(2)$, and denote by F the fiber of the topological logarithm $gl_1(tmf)^{\wedge}_p \to tmf^{\wedge}_p$. The homotopy groups of F are related to the spectrum of the Atkin operator, which has been intensely studied in number theory.

Let us look at $\pi_{23}(F)$. That group is equal to $\mathbb{Z}_p$ for $p = 691$, and is equal to $\mathbb{Z}_p \oplus \mathbb{Z}_p/(\tau(p)-1)$ otherwise, where τ is the Ramanujan τ-function given by

$$q \prod_{n=1}^{\infty}(1-q^n)^{24} = \sum \tau(n)q^n.$$

In particular, we see that $\pi_{23}(F)$ has a torsion group if and only if $\tau(p)$ is congruent to 1 mod p. I have done a computer search up to $p \approx 10^9$, and the only solutions that I found are $p = 11$, 23, and 691. The number theorists believe that there should be infinitely many such primes, and that they should be log(log) distributed, but nobody really known how many solutions this equation has. Somehow, this is probably related to smooth structure on the loop space of the 23-sphere...

There are other strange things about the homotopy type of F. Aside from a K-theory summand that corresponds to classical geometric topology, the spectrum F contains some p-adic suspensions of the image of J spectrum. The first place one of them occurs (that you don't understand) is at the prime 47. I tried to work out what p-adic suspension that was, computed nine 47-adic digits, but I couldn't figure out what that number is. The number theorists to whom I showed it also couldn't recognize it, which is strange, as one doesn't expect homotopy theory to make up "random numbers"...

At present, we have to make the space F at every prime separately. But if it really had to do with something *awesome*, like smooth structures on LS^n, then it would probably come from a space that is just over $\mathbb{Z}$. If that was the case, then it would also be of finite type, and its homotopy groups would thus be finitely generated. As a consequence this would mean that there are only finitely many primes for which $\tau(p)$ is congruent to 1 mod p. Number theorists don't really believe that this is possible... but who knows? I have no real reason to think that the spectra F aren't just reinvented at every prime.

Anyways, it would be really interesting to get to the bottom of that story, and blossom the relationship between number theory and smooth structures on spheres into new territory.

References

[AHR] M. Ando, M. J. Hopkins, and C. Rezk. *Multiplicative Orientations of KO-theory and of the spectrum of topological modular forms.* available at http://www.math.uiuc.edu/ mando/node1.html.

The sheaf of E_∞ ring spectra

Lecture by Michael Hopkins,
Typed by André Henriques

What are we trying to do in constructing *tmf*? We have the following category

$$I := \mathit{Aff}/\mathcal{M}_{ell}^{\acute{e}t}$$

which consists of certain kinds of affine open subsets over the moduli stack of elliptic curves in the étale topology. We can think of this more colloquiually as follows. An object of I is a pair (R, C), where R is a ring, and C is an elliptic curve over R. And a morphism $(R, C) \to (R', C')$ is a ring map $\phi : R \to R'$ together with an isomorphism $\phi^* C \cong C'$. But we don't want to consider all such pairs, namely, the map $\mathrm{Spec}(R) \to \mathcal{M}_{ell}$ induced by C should be étale. One puts the étaleness condition because it implies that R is Landweber exact (i.e., that the corresponding map $\mathrm{Spec}(R) \to \mathcal{M}_{FG}$ is flat).

By applying the Landweber exact functor theorem, we get a functor from I to the category of multiplicative homology theories. That functor sends (R, C) to the Landweber exact theory associated to the formal completion of C. The lifting problem is expressed as

$$\begin{tikzcd} & E_\infty\text{-ring spectra} \arrow[d] \\ I \arrow[r, "\mathcal{D}"] \arrow[ur, dashed] & \{\text{multiplicative homology theories}\} \end{tikzcd} \tag{1}$$

We didn't necessarily want to lift this to E_∞-ring spectra. We would have been happy to make it to spectra. All we wanted was to rigidify this diagram so as to be able to take its inverse limit.

You can say this in the language of stacks but you can state our task in a more down-to-earth way and in fact we chose that language when I wrote about this in my ICM talk in Zurich because I thought that the language of stacks was too out there. It was even exotic in the algebraic geometry literature. At this level, stacks are just a convenient expository device.

Lifting to E_∞-ring spectra might make the problem seem harder, but really it cuts the space of lifts down to a manageable size and actually makes the problem easier. Here, stacks first arose as a mainly expository device, but there is a real thing that made them essential.

We want to understand if there is a lift of the diagram (1), and whether any two of them are homotopy equivalent. In other words, the goal of our obstruction theory is to understand the realization space $r\mathcal{D}$ of this diagram. This is the nerve of the category of lifts, i.e., whose objects are lifts as above and whose morphisms are natural transformations which are homotopy equivalences on objects. We want to show that this space is nonempty and connected. It would be even better if that space turned out to be contractible but, unfortunatly, that turns out to be not quite right. That's where Jacob Lurie's point of of view improves things.

What you need is for any object $i \in I$ an E_∞-ring spectrum E, and for any morphism $i \to j$ a map $E \to F$ of E_∞-ring spectra. Thus, you are bound to encounter an invariant of the shape of the category I, as well as the homotopy groups of the mapping space $\mathrm{Map}_{E_\infty}(E, F)$. Assuming for a moment that we already have an E_∞-ring spectrum associated to each object i of I, then we would get a diagram

$$\begin{aligned} I^{op} \times I &\to \{\text{abelian groups}\} \\ (i, j) &\mapsto \pi_* \mathrm{Map}_{E_\infty}(E, F) \end{aligned}$$

where E and F are the respective images of i and j. At its crudest level, our obstruction theory will use the cohomology theory of these abelian group valued diagrams.

We've cheated a little bit by assuming that we already had lifts to E_∞-ring spectra of the multiplicative homology theories associated to objects $i \in I$, but the obstruction theory for lifting multiplicative homology theories to E_∞-ring spectra is similar to the obstruction theory we are dealing with.

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There is a general pattern that any obstruction theory follows:

(1) Make an algebraic approximation to your topological setup.
(2) Build resolutions of whatever you are trying to study by pieces for which the algebraic approximation is exactly right.

Some key words that fall into category (1) are "derivations" and "ψ-θ-algebras". On the other hand, "resolution model categories" are the way of making sense of (2). They lead to obstruction groups involving derived functors of derivations, and Hochschild homology. The art of the game is to pose the right problem and find the right kind of algebraic approximation so that the obstruction groups can be handled (the best situation being of course when they all vanish).

In our case, we need an algebraic approximation to a map of E_∞-ring spectra $E \xrightarrow{f} F$. Smashing both sides with F gives $f' : F \wedge E \to F \wedge F \to F$, where the last map is the multiplication of F. Applying π_* gives a map of F_*-algebras, $F_*E \to F_*$, which could be our algebraic approximation to an element of $f \in \pi_0 \mathrm{Map}_{E_\infty}(E, F)$. Here is a clever trick that I learned from Bill Dwyer: to extend this and get approximations for elements of $\pi_t \mathrm{Map}_{E_\infty}(E, F)$ based at f, note that a pointed map $S^t \to \mathrm{Map}_{E_\infty}(E, F)$ is equivalent to a E_∞-ring map $E \to F^{S^t}$ over

F, i.e., a diagram

$$\begin{tikzcd} E \arrow[r] \arrow[dr, "f"'] & F^{S^t} \arrow[d] \\ & F \end{tikzcd}$$

Here, F^{S^t} is the cotensor of F with the sphere, using the topological structure on the category of E_∞-ring spectra. The underlying spectrum of F^{S^t} is just the mapping spectrum from S^t into F, and it is homotopy equivalent to $F \vee \Sigma^{-t}F$. As an F_*-algebra, we have $\pi_* F^{S^t} \cong F_*[\epsilon]/\epsilon^2$, where ϵ is in degree $-t$. Our algebraic approximation to an element in $\pi_t \operatorname{Map}_{E_\infty}(E, F)$ thus becomes a commuting diagram of F_*-algebras

$$\begin{tikzcd} F_*E \arrow[r] \arrow[dr, "f'"'] & F_*[\epsilon]/\epsilon^2 \arrow[d] \\ & F_* \end{tikzcd}$$

The ring homomorphism in the top line is something of the form $f' + D(-)\epsilon$, where D is an F_*-linear derivation $D : F_*E \to F_*$. The group of these derivations is $\operatorname{Der}_{F_*}(F_*E, F_*)$, which is equivalent to $\operatorname{Der}_{F_0}(F_0E, F_0)$. In general, if $f' : B \to A$ is an augmented A-algebra, there is an equivalence $\operatorname{Der}_A(B, A) \simeq \operatorname{Hom}_B(\Omega^1_{B/A}, A) \simeq \operatorname{Hom}_A(f'^*\Omega^1_{B/A}, A)$, where $\Omega^1_{B/A}$ is the B-module of relative Kähler differentials (and the last equivalence is just by adjunction). In this instance, we may set $B = F_0E$ and $A = F_0$, to obtain that this module of derivations is equivalent to $\operatorname{Hom}_{F_0}(f'^*\Omega^1_{F_0E/F_0}, F_0)$.

This is the essential idea. If we were just working with A_∞-ring spectra, this would be a sufficiently good algebraic approximation. The commutative case is a little more involved than the associative one, so a bit more structure needs to be added, such as Dyer-Lashof operations, but this extra stuff winds up "coming along for free". The real point is to come to grip with these derivations.

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When I was first thinking about these groups, it was one of these experiences where every day, I had to reinvent the wheel. I couldn't hold the obstruction groups in my head in an organized manner. And this is where the language of stacks suddenly came in and clarified everything.

In our case, E and F are Landweber exact, which means that the associated maps from $U := \operatorname{Spec} E_0$ and $V := \operatorname{Spec} F_0$ to $\mathcal{M}_{FG}$ are flat. Then we can form the pullback $V \times_{\mathcal{M}_{FG}} U$ which encodes the smash product of E and F:

$$V \times_{\mathcal{M}_{FG}} U \cong \operatorname{Spec} F_0E = \operatorname{Spec} \pi_0(F \wedge E).$$

The language of stacks allows us to reorganize this picture so that U and V eventually just drop out. The map $E \to F$ induces a map $V \to U$ again denoted f, and so we get a diagram

$$\begin{tikzcd}
\mathrm{Spec}(F_0E) \simeq V \times_{\mathcal{M}_{FG}} U \arrow[r] \arrow[d] & U \arrow[d] \\
V \arrow[u, bend left, "i"] \arrow[ur, "f"] \arrow[r] & \mathcal{M}_{FG}\ ,
\end{tikzcd}$$

where i is the map induced by $f' : \pi_0(F \wedge E) \to \pi_0(F)$. Our module of derivations can be written more geometrically as $\mathrm{Hom}(i^*\Omega^1_{V\times_{\mathcal{M}_{FG}}U/V}, \mathcal{O}_V)$. Since this is a pullback square, these relative differentials are just pulled back from the differentials on U relative $\mathcal{M}_{FG}$. In other words, our group of derivations is equivalent to $\mathrm{Hom}(f^*\Omega^1_{U/\mathcal{M}_{FG}}, \mathcal{O}_V)$.

These maps to the moduli stack of formal groups factor through the moduli stack of elliptic curves, and we can write this picture as

$$\begin{tikzcd}
V \times_{\mathcal{M}_{FG}} U \arrow[r] \arrow[d] & U \arrow[d, "C_E"'] \arrow[ddr] & \\
V \arrow[ur, "f"] \arrow[r, "C_F"] \arrow[drr] & \mathcal{M}_{ell} \arrow[dr] & \\
& & \mathcal{M}_{FG}
\end{tikzcd}$$

Since C_E is an étale elliptic curve, the map C_E is étale. Consequently, we have an equivalence $\Omega^1_{U/\mathcal{M}_{FG}} \simeq C_E^*\Omega^1_{\mathcal{M}_{ell}/\mathcal{M}_{FG}}$ (since the relative differentials of an étale map are trivial). The composite $C_E \circ f$ is equal to C_F, the map classifying the elliptic curve associated to our elliptic cohomology theory F. Thus, the sequence of pullbacks $f^*C_E^*\Omega^1_{U/\mathcal{M}_{ell}}$ is equivalent to $C_F^*\Omega^1_{U/\mathcal{M}_{ell}}$. This presents our group of derivations as $\mathrm{Hom}(\Omega^1_{\mathcal{M}_{ell}/\mathcal{M}_{FG}}, \mathcal{O}_{\mathcal{M}_{ell}})$ restricted to the subset $C_F : V \to \mathcal{M}_{ell}$, which is great because U and V have now really fallen out of the picture.

Thus, in the business end of the obstruction theory, the $I^{op} \times I$ diagram cohomology is just

$$H^*(\mathcal{M}_{ell}, (\Omega^1_{\mathcal{M}_{ell}/\mathcal{M}_{FG}})^\vee \otimes \omega^t), \tag{2}$$

where I've put back a power of the canonical sheaf ω because the derivation we started with actually had degree $-t$. Note: In the obstruction theory of Dwyer, Kan, Smith, this is called a centric diagram, which is why the "twisted-arrow" or Hochschild-Mitchell cohomology reduces to this.

This whole story we talked about is similar to something that you encounter when studying basic abstract differential geometry. Namely, for M a smooth manifold and U an open subset, then you can define vector fields on U as derivations from functions on U to functions on U:

$$\mathrm{Vect}(U) = \mathrm{Der}(\mathcal{O}_U, \mathcal{O}_U).$$

If now $V \subset U$ is an open subset of U, it is intuitively obvious that you can restrict a vector field. But how do you restrict a derivation? From the point of view of derivations, it is not obvious that one can restrict them:

$$\begin{tikzcd} \mathcal{O}_U \arrow[r, "D"] \arrow[d] & \mathcal{O}_U \arrow[d] \\ \mathcal{O}_V \arrow[r, dashed, "?"] & \mathcal{O}_V. \end{tikzcd}$$

They seem to rather be bivariant. The way to solve this problem is to note that $\mathrm{Der}(\mathcal{O}_U, \mathcal{O}_U) \cong \mathrm{Hom}_U(\Omega^1_U, \mathcal{O}_U)$ and that for an étale map $i : V \to U$, we have an equivalence $\Omega^1_V \cong i^*\Omega^1_U$. This allows us to define restriction as the composite

$$\mathrm{Hom}_U(\Omega^1_U, \mathcal{O}_U) \to \mathrm{Hom}_V(i^*\Omega^1_U, i^*\mathcal{O}_U) \cong \mathrm{Hom}_V(\Omega^1_V, \mathcal{O}_V).$$

Summarizing, if you define vector fields to be derivations, the thing that makes them into a sheaf is a very special property: it is the isomorphism $\Omega^1_V \cong i^*\Omega^1_U$, which comes from the fact the the map is étale. The fact that our obstruction groups could be rephrased as some variant of (2) is a consequence of that very same property.

Now let us think about how to analyze the obstruction groups (2), which can also be rewritten as

$$\mathrm{RHom}(\Omega^1_{\mathcal{M}_{ell}/\mathcal{M}_{FG}}, \omega^t). \tag{3}$$

Reducing mod p, we can express the relative cotangent complex $\Omega^1_{\mathcal{M}_{ell}/\mathcal{M}_{FG}} \otimes \mathbb{F}_p$ as the pushforward $i_*\Omega^1_{\mathcal{M}_{ell}^{ord}}$ of the cotangent space of the ordinary locus, where i denotes the inclusion $\mathcal{M}_{ell}^{ord} \hookrightarrow \mathcal{M}_{ell}$. Since locally, $\mathcal{M}_{ell}^{ord}$ is obtained from $\mathcal{M}_{ell}$ by inverting one element, the above group is similar to $\mathrm{Ext}_{\mathbb{Z}[x]}(\mathbb{Z}[x^{\pm 1}], \mathbb{Z}[x])$, which has a somewhat big and ugly Ext^1. So it is a little bit cumbersome to deal with (3). In order to handle the structure of $\Omega^1_{\mathcal{M}_{ell}/\mathcal{M}_{FG}}$, it is cleaner to break the problem into two calculations, one over $\mathcal{M}_{ell}^{ord}$, and another one over $\mathcal{M}_{ell}$.

Recall that our problem is to find a lift in the following diagram:

$$\begin{tikzcd} & & E_\infty\text{–rings} \arrow[d] \\ (\mathit{Aff}/\mathcal{M}_{ell})_{\acute{e}t} \arrow[r, equal] & I \arrow[r] \arrow[ur, dashed, "\mathcal{O}^{top}"] & \{\text{homology theories}\} \end{tikzcd}$$

Here, we can break up the realization problems into pieces by functorially putting every spectrum we encouter into a Hasse pullback square. Geometrically, this corresponds to using the stratification of the moduli stack $\overline{\mathcal{M}}_{ell}$ in terms of ordinary and supersingluar loci:

$$\begin{tikzcd} \mathcal{O}^{top} \arrow[r] \arrow[d] & L_{K(2)}\mathcal{O}^{top} \arrow[d] \\ L_{K(1)}\mathcal{O}^{top} \arrow[r] & L_{K(1)}L_{K(2)}\mathcal{O}^{top}. \end{tikzcd}$$

Note that this is still a diagram of sheaves on *the whole* stack $\overline{\mathcal{M}}_{ell}$. We can fracture the problem of understanding the space of lifts into understanding the space of three compatible things:

a. the moduli space of possible $L_{K(2)}\mathcal{O}^{top}$, *b.* the moduli space of possible $L_{K(1)}\mathcal{O}^{top}$, and

c. the moduli space of maps $L_{K(1)}\mathcal{O}^{top} \to L_{K(1)}L_{K(2)}\mathcal{O}^{top}$.

The moduli space of possible $\mathcal{O}^{top}$ is then a pullback of the first two moduli spaces over the third moduli space. We will use different obstruction theories to analyze each one of the above problems. For *a.*, we use that the appropriate Der groups are zero, which gives us that for every supersingular elliplic curve we can realize in a functorial way its universal deformation, along with the action of the automorphism group of the curve. This, along with the fact that the Hasse invariant has distinct zeros allows us to construct a sheaf of E_∞-spectra over $\overline{\mathcal{M}}_{ell}$, supported on $\mathcal{M}_{ell}^{ss}$. For *b.* and *c.*, we use the $K(1)$-local obstruction theory. The obstruction groups become $H^*(\mathcal{M}_{ell}^{ord}, Der_{\theta,\psi})$ and these are zero too, at least at odd primes. At the prime 2, the ordinary locus does have some cohomology, but we can change our algebraic approximation by using KO instead of K, rewrite the algebra differently and, that way, you also don't encounter any obstruction groups. Note that we are not changing the moduli stack: all the sheaves are on $\overline{\mathcal{M}}_{ell}$. It turns out however, that the obstruction groups can be calculated on the stacks $\mathcal{M}_{ell}^{ord}$ and $\mathcal{M}_{ell}^{ss}$.

The construction of *tmf*

Mark Behrens

Contents

1. Introduction

In these notes I will sketch the construction of *tmf* using Goerss-Hopkins obstruction theory. Let $\overline{\mathcal{M}}_{ell}$ denote the moduli stack of generalized elliptic curves over $\mathrm{Spec}(\mathbb{Z})$. For us, unless we specifically specify otherwise, a generalized elliptic curve is implicitly assumed to have irreducible geometric fibers (i.e. no Néron n-gons for $n > 1$). That is to say, $\overline{\mathcal{M}}_{ell}$ is the moduli stack of pointed curves whose fibers are either elliptic curves, or possess a nodal singularity. Our aim is to prove the following theorem.

Theorem 1.1. *There is a presheaf $\mathcal{O}^{top}$ of E_∞-ring spectra on the site $(\overline{\mathcal{M}}_{ell})_{et}$, which is fibrant as a presheaf of spectra in the Jardine model structure. Given an affine étale open*

$$\mathrm{Spec}(R) \xrightarrow{C} \overline{\mathcal{M}}_{ell}$$

classifying a generalized elliptic curve C/R, the spectrum of sections $E = \mathcal{O}^{top}(\mathrm{Spec}(R))$ is a weakly even periodic ring spectrum satisfying:

(1) $\pi_0(E) \cong R$,
(2) $\mathbb{G}_E \cong \widehat{C}$.

Here, $\widehat{C}$ is the formal group of C.

Remark 1.2. A ring spectrum E is *weakly even periodic* if $\pi_* E$ is concentrated in even degrees, $\pi_2 E$ is an invertible $\pi_0 E$-module, and the natural map

$$\pi_2 E \otimes \pi_{2t} E \cong \pi_{2t+2} E$$

is an isomorphism. The spectrum E is automatically complex orientable, and we let $\mathbb{G}_E$ denote the formal group over $\pi_0 E$ associated to E. It then follows that there is a canonical isomorphism

$$\pi_{2t} E \cong \Gamma \omega_{\mathbb{G}_E}^{\otimes t}$$

where $\omega_{\mathbb{G}_E}$ is the line bundle (over $\mathrm{Spec}(R)$) of invariant 1-forms on $\mathbb{G}_E$.

Remark 1.3. The properties of the spectrum of sections of $E = \mathcal{O}^{top}(\mathrm{Spec}(R))$ enumerated in Theorem 1.1 make E an *elliptic spectrum* associated to the generalized elliptic curve C/R in the sense of Hopkins and Miller [**Hop2**]. Thus Theorem 1.1 gives a functorial collection of E_∞-elliptic spectra associated to the collection of generalized elliptic curves whose classifying maps are étale.

Remark 1.4. This theorem practically determines $\mathcal{O}^{top}$, at least as a diagram in the stable homotopy category. Given an affine étale open $\mathrm{Spec}(R) \xrightarrow{C} \overline{\mathcal{M}}_{ell}$, the composite

$$\mathrm{Spec}(R) \xrightarrow{C} \overline{\mathcal{M}}_{ell} \to \mathcal{M}_{FG}$$

is flat, since the map $\overline{\mathcal{M}}_{ell} \to \mathcal{M}_{FG}$ classifying the formal group of the universal generalized elliptic curve is flat (this can be verified using Serre-Tate theory, see [**BL**, Lemma 9.1.6]). Thus the spectrum of sections $E = \mathcal{O}^{top}(R)$ is Landweber exact [**Nau**]. Fibrant presheaves of spectra satisfy homotopy descent, and so the values of the presheaf are determined by values on the affine opens using étale descent.

Remark 1.5. The spectrum *tmf* is defined to be the connective cover of the global sections of this sheaf:

$$tmf = \tau_{\geq 0} \mathcal{O}^{top}(\overline{\mathcal{M}}_{ell}).$$

We give an outline of the argument we shall give. Consider the substacks

$$(\overline{\mathcal{M}}_{ell})_p \xrightarrow{\iota_p} \overline{\mathcal{M}}_{ell},$$
$$(\overline{\mathcal{M}}_{ell})_{\mathbb{Q}} \xrightarrow{\iota_{\mathbb{Q}}} \overline{\mathcal{M}}_{ell},$$

where:

$$(\overline{\mathcal{M}}_{ell})_p = p\text{-completion of } \overline{\mathcal{M}}_{ell},$$
$$(\overline{\mathcal{M}}_{ell})_{\mathbb{Q}} = \overline{\mathcal{M}}_{ell} \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Remark 1.6. We pause to make two important comments on our use of formal geometry in this paper.

(1) The object $(\overline{\mathcal{M}}_{ell})_p$ is a *formal Deligne-Mumford stack*. We shall use these throughout this paper — we refer the reader to the appendix of [**Har**] for some of the basic definitions. Given a formal Deligne-Mumford stack $\mathcal{X}$ and a ring R complete with respect to an ideal I, we define the R-points of $\mathcal{X}$ by $\mathcal{X}(R) = \varprojlim_i \mathcal{X}(R/I^i)$.

(2) If R is complete with respect to an ideal I, a generalized elliptic curve $C/\mathrm{Spf}(R)$ is a compatible ind-system $C_m/\operatorname{Spec}(R/I^m)$. There is, however, a canonical "algebraization" $C^{alg}/\operatorname{Spec}(R)$ where C^{alg} is a generalized elliptic curve which restricts to C_m over $\operatorname{Spec}(R/I^m)$ [**Con**, Cor. 2.2.4]. With this in mind, we shall in these notes always regard $C/\mathrm{Spf}(R)$ as being represented by an honest generalized elliptic curve over the ring R.

We shall construct $\mathcal{O}^{top}$ as the homotopy pullback of an arithmetic square of presheaves of E_∞-ring spectra

$$\begin{tikzcd}
\mathcal{O}^{top} \arrow[r] \arrow[d] & \prod_{p \text{ prime}} (\iota_p)_* \mathcal{O}^{top}_p \arrow[d] \\
(\iota_{\mathbb{Q}})_* \mathcal{O}^{top}_{\mathbb{Q}} \arrow[r, "\alpha_{\text{arith}}"'] & \left(\prod_{p \text{ prime}} (\iota_p)_* \mathcal{O}^{top}_p\right)_{\mathbb{Q}}
\end{tikzcd}$$

Here, $\mathcal{O}^{top}_p$ is a presheaf on $(\overline{\mathcal{M}}_{ell})_p$, and $\mathcal{O}^{top}_{\mathbb{Q}}$ is a presheaf on $(\overline{\mathcal{M}}_{ell})_{\mathbb{Q}}$. The presheaf

$$\left(\prod_{p \text{ prime}} (\iota_p)_* \mathcal{O}^{top}_p\right)_{\mathbb{Q}}$$

is the (sectionwise) rationalization of the presheaf $\prod_{p \text{ prime}} (\iota_p)_* \mathcal{O}^{top}_p$. The presheaf $\mathcal{O}^{top}_{\mathbb{Q}}$ will be constructed using rational homotopy theory, as will the map α_{arith}.

It remains to construct the presheaves $\mathcal{O}^{top}_p$ for each prime p. Define

$$(\overline{\mathcal{M}}_{ell})_{\mathbb{F}_p} = \overline{\mathcal{M}}_{ell} \otimes_{\mathbb{Z}} \mathbb{F}_p.$$

Let

$$(\mathcal{M}^{ord}_{ell})_{\mathbb{F}_p} \subset (\overline{\mathcal{M}}_{ell})_{\mathbb{F}_p}$$

denote the locus of ordinary generalized elliptic curves in characteristic p, and let

$$(\mathcal{M}^{ss}_{ell})_{\mathbb{F}_p} = (\overline{\mathcal{M}}_{ell})_{\mathbb{F}_p} - (\mathcal{M}^{ord}_{ell})_{\mathbb{F}_p}$$

denote the locus of supersingular elliptic curves in characteristic p. Consider the substacks

$$\mathcal{M}^{ord}_{ell} \xrightarrow{\iota_{ord}} (\overline{\mathcal{M}}_{ell})_p, \tag{1.1}$$

$$\mathcal{M}^{ss}_{ell} \xrightarrow{\iota_{ss}} (\overline{\mathcal{M}}_{ell})_p, \tag{1.2}$$

where

$\mathcal{M}^{ord}_{ell}$ = moduli stack of generalized elliptic curves over p-complete rings with ordinary reduction,

$\mathcal{M}^{ss}_{ell}$ = completion of $\overline{\mathcal{M}}_{ell}$ at $(\mathcal{M}^{ss}_{ell})_{\mathbb{F}_p}$.

The presheaves $\mathcal{O}^{top}_p$ will be constructed as homotopy pullbacks:

$$\begin{tikzcd}
\mathcal{O}^{top}_p \arrow[r] \arrow[d] & (\iota_{ss})_* \mathcal{O}^{top}_{K(2)} \arrow[d] \\
(\iota_{ord})_* \mathcal{O}^{top}_{K(1)} \arrow[r, "\alpha_{\text{chrom}}"'] & \left((\iota_{ss})_* \mathcal{O}^{top}_{K(2)}\right)_{K(1)}
\end{tikzcd}$$

Here,

$$\left((\iota_{ss})_*\mathcal{O}^{top}_{K(2)}\right)_{K(1)}$$

denotes the (sectionwise) $K(1)$-localization of the presheaf $(\iota_{ss})_*\mathcal{O}^{top}_{K(2)}$. (The reader wondering at this point why these localizations are related to the ordinary and supersingular loci is invited to glance at Lemma 8.1.)

The presheaf $\mathcal{O}^{top}_{K(2)}$ will be constructed using the Goerss-Hopkins-Miller Theorem — its spectra of sections are given by homotopy fixed points of Morava E-theories with respect to finite group actions.

The presheaf $\mathcal{O}^{top}_{K(1)}$ will be constructed using explicit Goerss-Hopkins obstruction theory. The map α_{chrom} will be be produced from an analysis of the $K(1)$-local mapping spaces, and the θ-algebra structure inherent in certain rings of p-adic modular forms.

Figure 1 shows a diagram which summarizes the above discussion. Many thanks to Aaron Mazel-Gee for creating this diagram, and making it available for inclusion here.

2. Descent lemmas for presheaves of spectra

For a small Grothendieck site $\mathcal{C}$ with enough points, let $\mathrm{PreSp}_{\mathcal{C}}$ denote the category of presheaves of symmetric spectra of simplicial sets. The category $\mathrm{PreSp}_{\mathcal{C}}$ has a Jardine model category structure [**Jar**], where

(1) The cofibrations are the sectionwise cofibrations of symmetric spectra,
(2) The weak equivalences are the stalkwise stable equivalences of symmetric spectra,
(3) The fibrant objects are those objects which are fibrant in the injective model structure of the underlying diagram model category structure, and which satisfy descent with respect to hypercovers [**DHI**].

The following lemma will be useful.

Lemma 2.1.

(1) *If $\mathcal{F} \in \mathrm{PreSp}_{\mathcal{C}}$ satisfies homotopy descent with respect to hypercovers, then the fibrant replacement in the Jardine model structure*

$$\mathcal{F} \to \mathcal{F}'$$

is a sectionwise weak equivalence.

(2) *If $f : \mathcal{F} \to \mathcal{G}$ is a stalkwise weak equivalence in $\mathrm{PreSp}_{\mathcal{C}}$, and $\mathcal{F}$ and $\mathcal{G}$ satisfy homotopy descent with respect to hypercovers, then f is a sectionwise weak equivalence.*

Proof. (1) The Jardine model category structure is a localization of the injective model category structure on $\mathrm{PreSp}_{\mathcal{C}}$. In the injective model structure, weak equivalences are sectionwise. Let

$$\mathcal{F} \to \mathcal{F}'$$

be the fibrant replacement in the injective model category structure. This map is necessarily a sectionwise weak equivalence. By the Dugger-Hollander-Isaksen criterion, to see that $\mathcal{F}'$ is fibrant in the Jardine model structure, it suffices to show

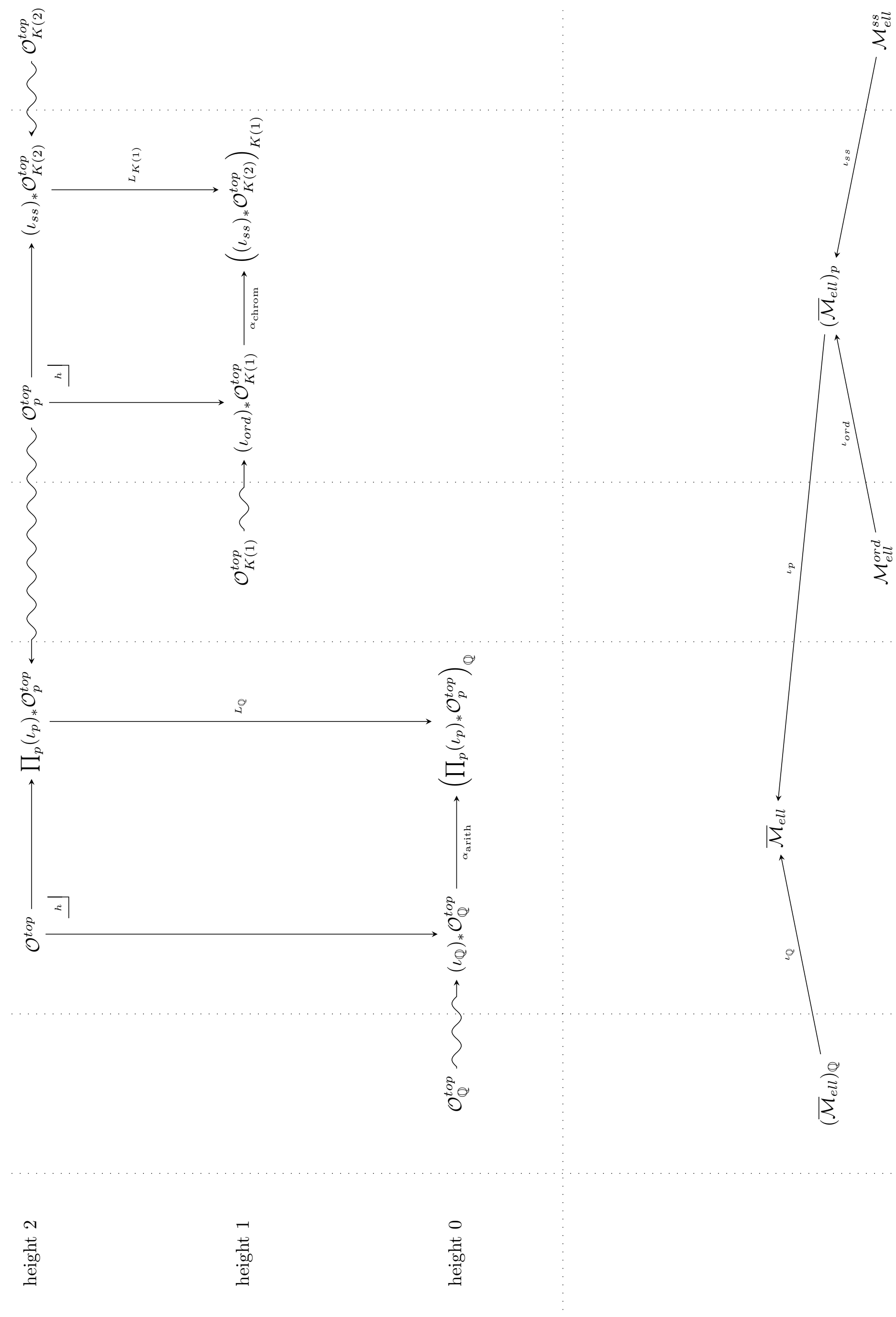

FIGURE 1. Summary of the construction of tmf [courtesy of Aaron Mazel-Gee]

that $\mathcal{F}'$ satisfies homotopy descent with respect to hypercovers. Let $U \in \mathcal{C}$ and let $U_\bullet$ be a hypercover of U. Consider the diagram

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\simeq"] \arrow[d, "\simeq"'] & \operatorname{holim}_\Delta \mathcal{F}(U_\bullet) \arrow[d, "\simeq"] \\ \mathcal{F}'(U) \arrow[r] & \operatorname{holim}_\Delta \mathcal{F}'(U_\bullet) \end{tikzcd}$$

We deduce that the bottom arrow is an equivalence. Thus $\mathcal{F}'$ satisfies descent with respect to hypercovers, and is fibrant in the Jardine model category structure.

(2) Consider the diagram of Jardine fibrant replacements:

$$\begin{tikzcd} \mathcal{F} \arrow[r, "f"] \arrow[d, "u"'] & \mathcal{G} \arrow[d, "v"] \\ \mathcal{F}' \arrow[r, "f'"'] & \mathcal{G}' \end{tikzcd}$$

By (1), the maps u and v are sectionwise equivalences. The map f' is a stalkwise weak equivalence between Jardine fibrant objects. Because the Jardine model structure is a localization of the injective model structure, we deduce that f' is a sectionwise weak equivalence. We therefore conclude that f is a sectionwise weak equivalence. $\square$

Let $\mathcal{X}$ be a Deligne-Mumford stack, and consider the site $\mathcal{X}_{et}$. Being a Deligne-Mumford stack, $\mathcal{X}$ possesses an affine étale cover. The full subcategory

$$\mathcal{X}_{et,aff} \xrightarrow{i} \mathcal{X}_{et}$$

consisting of only the affine étale opens is also a Grothendieck site. The map i induces an adjoint pair of functors

$$i^* : \mathrm{PreSp}_{\mathcal{X}_{et}} \leftrightarrows \mathrm{PreSp}_{\mathcal{X}_{et,aff}} : i_*$$

where i^* is the functor given by precomposition with i, and i_* is the right Kan extension.

Lemma 2.2.

(1) *The adjoint pair (i^*, i_*) is a Quillen equivalence.*
(2) *To construct a fibrant presheaf of spectra on $\mathcal{X}_{et}$, it suffices to construct a fibrant presheaf on $\mathcal{X}_{et,aff}$ and apply the functor i_*.*

Proof. By [**Hov**, Cor. 1.3.16], to check (1) it suffices to check that (i^*, i_*) is a Quillen pair, that i^* reflects weak equivalences, and that the map

$$i_* Li^* X \to X$$

is a weak equivalence. The functor i^* is easily seen to preserve cofibrations, and it preserves and reflects all weak equivalences, since the sites $\mathcal{X}_{et}$ and $\mathcal{X}_{et,aff}$ have the same points. Since the functor i_* preserves stalks, the map above is a stalkwise weak equivalence, hence is an equivalence. Therefore (i^*, i_*) is a Quillen equivalence. (2) In particular, the functor i_* preserves fibrant objects. $\square$

The following construction formalizes the idea that a Jardine fibrant presheaf on $\mathcal{X}_{et}$ is determined by its sections on étale affine opens.

Construction 2.3.
Input: A presheaf $\mathcal{F}$ on $\mathcal{X}_{et,\mathit{aff}}$ that satisfies hyperdescent.

Output: A Jardine fibrant presheaf $\mathcal{G}$ on $\mathcal{X}_{et}$, and a zig-zag of sectionwise weak equivalences between $\mathcal{F}$ and $i^*\mathcal{G}$.

We explain this construction. Let

$$u : \mathcal{F} \to \mathcal{F}'$$

be the Jardine fibrant replacement of $\mathcal{F}$. By Lemma 2.1, u is a sectionwise weak equivalence. Let $\mathcal{G}$ be the presheaf $i_*\mathcal{F}'$. By Lemma 2.2, $\mathcal{G}$ is Jardine fibrant. The counit of the adjunction

$$\epsilon : i^*\mathcal{G} = i^*i_*\mathcal{F}' \to \mathcal{F}'$$

is a stalkwise weak equivalence since, by Lemma 2.2, the adjoint pair (i^*, i_*) is a Quillen equivalence. The sheaf $i^*\mathcal{G}$ is easily seen to satisfy hyperdescent — it is the restriction of $\mathcal{G}$ to a subcategory. Therefore, by Lemma 2.1, the map ϵ is a sectionwise weak equivalence. Thus we have a zig-zag of sectionwise equivalences

$$i^*\mathcal{G} \to \mathcal{F}' \leftarrow \mathcal{F}.$$

Construction 2.3 requires a presheaf $\mathcal{F}$ on $\mathcal{X}_{et,\mathit{aff}}$ which satisfies homotopy descent with respect to hypercovers. The following lemma gives a useful criterion for verifying that $\mathcal{F}$ has this property.

Lemma 2.4. *Suppose that $\mathcal{F}$ is an object of $\mathrm{PreSp}_{\mathcal{X}_{et,\mathit{aff}}}$, and suppose that there is a graded quasi-coherent sheaf $\mathcal{A}_*$ on $\mathcal{X}$ and natural isomorphisms*

$$f_U : \mathcal{A}_*(U) \xrightarrow{\cong} \pi_*\mathcal{F}(U)$$

for all affine étale opens $U \to \mathcal{X}$. Then $\mathcal{F}$ satisfies homotopy descent with respect to hypercovers.

Proof. Suppose that $U \to \mathcal{X}$ is an affine étale open, and that $U_\bullet$ is a hypercover of U. Consider the Bousfield-Kan spectral sequence

$$E_2^{s,t} = \pi^s \mathcal{A}_t(U_\bullet) \Rightarrow \pi_{t-s} \operatorname{holim}_\Delta \mathcal{F}(U_\bullet).$$

Since $\mathcal{A}_*$ quasi-coherent, it satisfies étale hyperdescent, and we deduce that the E_2-term computes the quasi-coherent cohomology

$$E_2^{s,t} \cong H^s(U, \mathcal{A}_t)$$

and since U is affine, there is no higher cohomology. The E_2-term of this spectral sequence is therefore concentrated in $s = 0$. The spectral sequence collapses to give a diagram of isomorphisms

$$\begin{tikzcd} \mathcal{A}_*(U) \arrow[d, "f_U"', "\cong"] \arrow[dr, "\cong"] & \\ \pi_*\mathcal{F}(U) \arrow[r] & \pi_* \operatorname{holim}_\Delta \mathcal{F}(U_\bullet) \end{tikzcd}$$

We deduce that the map

$$\mathcal{F}(U) \to \operatorname{holim}_\Delta \mathcal{F}(U_\bullet)$$

is an equivalence. ∎

Remark 2.5. Construction 2.3 shows that to construct the presheaf $\mathcal{O}^{top}$, it suffices to construct $\mathcal{O}^{top}(U)$ functorially for affine étale opens $U \to \overline{\mathcal{M}}_{ell}$, as long as the resulting values $\mathcal{O}^{top}(U)$ satisfy homotopy descent with respect to affine hypercovers. This is automatic: there is an isomorphism

$$\pi_{2t}\mathcal{O}^{top}(U) \cong \omega^{\otimes t}(U)$$

for an invertible sheaf ω on $\overline{\mathcal{M}}_{ell}$. Lemma 2.4 implies that $\mathcal{O}^{top}$ satisfies the required hyperdescent conditions.

3. p-divisible groups of elliptic curves

Let C be an elliptic curve over R, a p-complete ring. The p-divisible group $C(p)$ is the ind-finite group-scheme over R given by

$$C(p) = \varinjlim_k C[p^k].$$

Here, the finite group scheme $C[p^k]/R$ is the kernel of the p^k-power map on C.

Let $\widehat{C}$ be the formal group of C. If the height of the mod p-reduction of $\widehat{C}$ is constant, then over $\mathrm{Spf}(R)$ there is short exact sequence

$$0 \to \widehat{C} \to C(p) \to C(p)_{et} \to 0$$

where $C(p)_{et}$ is an ind-étale divisible group-scheme over R.

If $R = k$, a field of characteristic p, then we have

$$2 = \mathrm{height}(C(p)) = \mathrm{height}(\widehat{C}) + \mathrm{height}(C(p)_{et}).$$

The height of $\widehat{C}$ is the height of the formal group. The height of $C(p)_{et}$ is the corank of the corresponding divisible group. There are two possibilities:

(1) C is *ordinary*: $\widehat{C}$ has height 1, and the divisible group $C(p)_{et}$ has corank 1.
(2) C is *supersingular*: $\widehat{C}$ has height 2, and the divisible group $C(p)_{et}$ is trivial.

Theorem 3.1 (Serre-Tate). *Suppose that R is a complete local ring with residue field k of characteristic p. Suppose that C is an elliptic curve over k. Then the functor*

$$\begin{gathered}\{\textit{deformations of } C \textit{ to } R\}\\ \downarrow \\ \{\textit{deformations of } C(p) \textit{ to } R\}\end{gathered}$$

is an equivalence of categories.

4. Construction of $\mathcal{O}^{top}_{K(2)}$

Lubin and Tate identified the formal neighborhood of a finite height formal group in $\mathcal{M}_{FG}$:

Theorem 4.1 (Lubin-Tate). *Suppose that $\mathbb{G}$ is a formal group of finite height n over k, a perfect field of characteristic p. Then the formal moduli of deformations of $\mathbb{G}$ is given by*

$$\mathrm{Def}_{\mathbb{G}} \cong \mathrm{Spf}(B(k, \mathbb{G}))$$

where there is an isomorphism

$$B(k,\mathbb{G}) \cong \mathbb{W}(k)[[u_1,\dots,u_{n-1}]].$$

(Here, $\mathbb{W}(k)$ is the Witt ring of k.)

Let $\widetilde{\mathbb{G}}/B(k,\mathbb{G})$ denote the universal deformation of $\mathbb{G}$. The following theorem was proven by Goerss, Hopkins, and Miller [**GH**].

THEOREM 4.2 (Goerss-Hopkins-Miller). *Let $\mathcal{C}$ be the category of pairs $(k,\mathbb{G})$ where k is a perfect field of characteristic p and $\mathbb{G}$ is a formal group of finite height over k. There is a functor*

$$\begin{aligned} \mathcal{C} &\to E_\infty \textit{ ring spectra} \\ (k,\mathbb{G}) &\mapsto E(k,\mathbb{G}) \end{aligned}$$

where $E(k,\mathbb{G})$ is Landweber exact and even periodic, and

(1) $\pi_0 E(k,\mathbb{G}) = B(k,\mathbb{G})$,
(2) $\mathbb{G}_{E(k,\mathbb{G})} \cong \widetilde{\mathbb{G}}$.

Theorem 3.1 and Theorem 4.1 give the following.

COROLLARY 4.3.

(1) *Suppose that C is a supersingular elliptic curve over a field k of characteristic p. There is an isomorphism*

$$\mathrm{Def}_C \cong \mathrm{Spf}(B(k,\widehat{C})).$$

(2) *The substack $(\mathcal{M}_{ell}^{ss})_{\mathbb{F}_p} \subset (\overline{\mathcal{M}}_{ell})_p$ is zero dimensional.*

PROOF. If C is a supersingular curve, then the inclusion of p-divisible groups $\widehat{C} \to C(p)$ is an isomorphism. Therefore, Theorem 3.1 implies that there is an isomorphism

$$\mathrm{Def}_C \cong \mathrm{Def}_{\widehat{C}}$$

and Theorem 4.1 identifies $\mathrm{Def}_{\widehat{C}}$.

To compute the dimension of $(\mathcal{M}_{ell}^{ss})_{\mathbb{F}_p}$ it suffices to do so étale locally. Let k be a finite field, and suppose that C is a supersingular elliptic curve over k. The completion of $\overline{\mathcal{M}}_{ell}$ along the map classifying C is the deformation space $\mathrm{Def}_C \cong \mathrm{Spf}(B(k,\widehat{C}))$, and there is an isomorphism

$$B(k,\widehat{C}) \cong \mathbb{W}(k)[[u_1]].$$

Since we have

$$u_1 \equiv v_1 \mod p,$$

the locus where $\widehat{C}$ has height 2 is given by the ideal (p,u_1). The quotient $B(k,\widehat{C})/(p,u_1)$ is k, and is therefore zero dimensional. □

We now construct the values of the presheaf $\mathcal{O}_{K(2)}^{top}$ on formal affine étale opens

$$f : \mathrm{Spf}(R) \to \mathcal{M}_{ell}^{ss}.$$

Here R is complete with respect to an ideal I. This suffices to construct the presheaf $\mathcal{O}_{K(2)}^{top}$ on $\mathcal{M}_{ell}^{ss}$ by Construction 2.3.

The induced map of special fibers

$$f_0 : \mathrm{Spec}(R/I) \to (\mathcal{M}_{ell}^{ss})_{\mathbb{F}_p}$$

is étale. Since $(\mathcal{M}_{ell}^{ss})_{\mathbb{F}_p}$ is smooth and zero-dimensional, $\mathrm{Spec}(R/I)$ must be étale over $\mathbb{F}_p$. We deduce that there is an isomorphism

$$R/I \cong \prod_i k_i,$$

a finite product of finite fields of characteristic p. Let C be the elliptic curve classified by f, and let C_0 be the elliptic curve classified by f_0. The decomposition of R/I induces a decomposition

$$C_0 \cong \coprod_i C_0^{(i)}.$$

Since f is étale, the elliptic curve C is a universal deformation of the elliptic curve C_0, and hence by Corollary 4.3 there is an isomorphism

$$R \cong \prod_i B(k_i, \widehat{C}_0^{(i)}).$$

We define

$$\mathcal{O}_{K(2)}^{top}(\mathrm{Spf}(R)) := \prod_i E(k_i, \widehat{C}_0^{(i)}).$$

Let $\mathbb{G}$ be the formal group of this even periodic ring spectrum. By Theorem 3.1, since $\mathbb{G}$ is a universal deformation of $\widehat{C}_0$ and C is a universal deformation of C_0, there is an isomorphism

$$\mathbb{G} \cong \widehat{C}.$$

We have therefore verified

Proposition 4.4. *The spectrum of sections $\mathcal{O}_{K(2)}^{top}(\mathrm{Spf}(R))$ is an elliptic spectrum associated to the elliptic curve $C/\mathrm{Spf}(R)$.*

5. The Igusa tower

If C is a generalized elliptic curve over a p-complete ring R, let C_{ns} denote the non-singular locus of $C \to \mathrm{Spf}(R)$. Then C_{ns} is a group scheme over R. Given a closed point $x \in \mathrm{Spf}(R)$, the fiber $(C_{ns})_x$ is given by

$$(C_{ns})_x = \begin{cases} C_x & C_x \text{ nonsingular,} \\ \mathbb{G}_m & C_x \text{ singular.} \end{cases}$$

The formal group $\widehat{C}$ is the formal group $\widehat{C}_{ns}$. We still may consider the ind-quasi-finite group-scheme

$$C(p) = \varinjlim_k C_{ns}[p^k].$$

$C(p)$ is technically *not* a p-divisible group, because its height is not uniform. Rather, we have the following table:

C_x	$ht(C(p)_x)$	$ht((C(p)_x)_{et})$	$ht(\widehat{C}_x)$
supersingular	2	0	2
ordinary	2	1	1
singular	1	0	1

If the classifying map

$$C : \mathrm{Spf}(R) \to (\overline{\mathcal{M}}_{ell})_p$$

factors through $\mathcal{M}_{ell}^{ord}$, then C has no supersingular fibers, but may have singular fibers. We shall call such a generalized elliptic curve C *ordinary*.

Let $\mathcal{M}_{ell}^{ord}(p^k)$ be the moduli stack whose R-points (for a p-complete ring R) is the groupoid of pairs (C, η) where

$$C/R = \text{ordinary generalized elliptic curve,}$$
$$(\eta : \mu_{p^k} \xrightarrow{\cong} \widehat{C}[p^k]) = \text{isomorphism of finite group schemes.}$$

The isomorphism η is a p^k-level structure. The stacks $\mathcal{M}_{ell}^{ord}(p^k)$ are representable by Deligne-Mumford stacks.

A p^{k+1}-level structure induces a canonical p^k-level structure, inducing a map

$$\mathcal{M}_{ell}^{ord}(p^{k+1}) \to \mathcal{M}_{ell}^{ord}(p^k). \tag{5.1}$$

LEMMA 5.1. *The map $\mathcal{M}_{ell}^{ord}(p^{k+1}) \to \mathcal{M}_{ell}^{ord}(p^k)$ is an étale $\mathbb{Z}/p$-torsor (an étale $(\mathbb{Z}/p)^\times$-torsor if $k = 0$).*

PROOF. (This proof is stolen from Paul Goerss.) By Lubin-Tate theory, the p-completed moduli stack $\mathcal{M}_{FG}^{mult}$ of multiplicative formal groups admits a presentation

$$\mathrm{Spf}(\mathbb{Z}_p) \to \mathcal{M}_{FG}^{mult}$$

which is a pro-étale torsor for the group:

$$\mathrm{Aut}(\widehat{\mathbb{G}}_m/\mathbb{Z}_p) = \mathbb{Z}_p^\times.$$

Associated to the closed subgroup $1 + p^k\mathbb{Z}_p \subset \mathbb{Z}_p^\times$ is the étale torsor

$$\mathcal{M}_{FG}^{mult}(p^k) \to \mathcal{M}_{FG}^{mult}$$

for the group $(\mathbb{Z}/p^k)^\times$. The intermediate cover

$$\mathcal{M}_{FG}^{mult}(p^{k+1}) \to \mathcal{M}_{FG}^{mult}(p^k)$$

is an étale $\mathbb{Z}/p$-torsor (it is an étale $(\mathbb{Z}/p)^\times$-torsor if $k = 0$). The R-points of $\mathcal{M}_{FG}^{mult}(p^k)$ is the groupoid whose objects are pairs $(\mathbb{G}, \eta)$ where $\mathbb{G}$ is a formal group over $\mathrm{Spf}(R)$ locally isomorphic to $\widehat{\mathbb{G}}_m$, and η is a level p^k-structure:

$$\eta : \mu_{p^k} \xrightarrow{\cong} \mathbb{G}[p^k].$$

The stacks $\mathcal{M}_{ell}^{ord}(p^k)$ are therefore given by the pullbacks

$$\begin{tikzcd}
\mathcal{M}_{ell}^{ord}(p^{k+1}) \arrow[r] \arrow[d] & \mathcal{M}_{ell}^{ord}(p^k) \arrow[r] \arrow[d] & \mathcal{M}_{ell}^{ord} \arrow[d] \\
\mathcal{M}_{FG}^{mult}(p^{k+1}) \arrow[r] & \mathcal{M}_{FG}^{mult}(p^k) \arrow[r] & \mathcal{M}_{FG}^{mult}
\end{tikzcd}$$

where the map $\mathcal{M}_{ell}^{ord} \to \mathcal{M}_{FG}^{mult}$ classifies the formal group of the universal ordinary generalized elliptic curve. The result follows. $\square$

Thus we have a tower of étale covers:

$$\begin{array}{c} \vdots \\ \downarrow \\ \mathcal{M}_{ell}^{ord}(p^{k+1}) \\ \downarrow \\ \mathcal{M}_{ell}^{ord}(p^k) \\ \downarrow \\ \vdots \\ \downarrow \\ \mathcal{M}_{ell}^{ord} \end{array}$$

This is the *Igusa tower.*

LEMMA 5.2. *For $k \geq 1$ ($k \geq 2$ if $p = 2$) the stack $\mathcal{M}_{ell}^{ord}(p^k)$ is formally affine: there is a p-complete ring V_k such that*

$$\mathcal{M}_{ell}^{ord}(p^k) = \mathrm{Spf}(V_k).$$

PROOF. This is actually well known — see, for instance, Theorem 2.9.4, and the discussion at the beginning of Section 3.2.2 of [**Hid2**]. However, the proof of Theorem 2.9.4 in the above cited book only addresses the case where $p > 3$. The idea there is that the ordinary locus of the moduli of elliptic curves, with sufficient level structure, is affine. The result then follows from geometric invariant theory *provided one can show that the moduli problem $\mathcal{M}_{ell}^{ord}(p^k)$ is rigid (i.e. it takes values in sets, not groupoids).* Since p generates the ideal of definition of the formal stack $\mathcal{M}_{ell}^{ord}(p^k)$, it follows from Proposition 3.5.1 of [**BL**] that it suffices to show that the mod p reduction $\mathcal{M}_{ell}^{ord}(p^k)_{\mathbb{F}_p}$ is rigid.

Let $(\mathcal{M}_{ell}^{ord})^n_{\mathbb{F}_p}$ denote the moduli stack (over $\mathbb{F}_p$) of ordinary elliptic curves with the structure of an n-jet at the basepoint. (Note that jets on an elliptic curve are the same thing as jets on the formal group of the elliptic curve.) By fixing a coordinate T_0 of $\widehat{\mathbb{G}}_m$, we observe that there is a closed inclusion

$$\mathcal{M}_{ell}^{ord}(p^k)_{\mathbb{F}_p} \hookrightarrow (\mathcal{M}_{ell}^{ord})^{p^k-1}_{\mathbb{F}_p}$$

as a level p^k-structure η gives an elliptic curve the structure of a (p^k-1)-jet $\eta_* T_0$, and this jet uniquely determines the level structure. Thus it suffices to show that $(\mathcal{M}_{ell}^{ord})^{p^k-1}_{\mathbb{F}_p}$ is rigid. In fact, we will establish that it is affine.

Case 1: $p > 3$. Let R be an $\mathbb{F}_p$-algebra. Suppose that (C, T) is an object of $(\mathcal{M}_{ell}^{ord})^{p^k-1}_{\mathbb{F}_p}(R)$ for $k \geq 1$. Zariski-locally over $\mathrm{Spec}(R)$, there is a Weierstrass parameterization

$$C = C_{\mathbf{a}} : y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6.$$

The Weierstrass curve $C_{\mathbf{a}}$ has a canonical coordinate at infinity given by $T_{\mathbf{a}} = -x/y$. Suppose that T is a (p^k-1)-jet on $C_{\mathbf{a}}$, given by

$$T = m_0 T_{\mathbf{a}} + m_1 T_{\mathbf{a}}^2 + \cdots + m_{p^k-2} T_{\mathbf{a}}^{p^k-1} + O(T_{\mathbf{a}}^{p^k}).$$

According to [**Rez**, Rmk. 20.3], there are unique values

$$\begin{aligned} \lambda &= \lambda(m_0) \\ s &= s(m_0, m_1) \\ r &= r(m_0, m_1, m_2) \\ t &= t(m_0, m_1, m_2, m_3) \end{aligned}$$

such that under the transformation

$$\begin{aligned} f_{\lambda,s,r,t} : C_{\mathbf{a}} &\to C_{\mathbf{a}'} \\ x &\mapsto \lambda^2 x + r \\ y &\mapsto \lambda^3 y + sx + t \end{aligned}$$

the induced level (p^k-1)-jet $T' = (f_{\lambda,s,r,t})_* T$ is of the form

$$T' = T_{\mathbf{a}'} + m_4' T_{\mathbf{a}'}^5 + \cdots + m_{p^k-2}' T_{\mathbf{a}'}^{p^k-1} + O(T_{\mathbf{a}'}^{p^k}).$$

We have shown that the pair (C, η) is (Zariski locally) uniquely representable by a pair $(C_{\mathbf{a}}, T)$ where

$$T = T_{\mathbf{a}} + m_4 T_{\mathbf{a}}^5 + \cdots m_{p^k-2} T_{\mathbf{a}}^{p^k-1} + O(T_{\mathbf{a}}^{p^k}).$$

The only morphism $f_{\lambda,s,r,t} : C_{\mathbf{a}} \to C_{\mathbf{a}'}$ which satisfies

$$f^*_{\lambda,s,r,t} T_{\mathbf{a}'} = T_{\mathbf{a}} + O(T_{\mathbf{a}}^5)$$

has $\lambda = 1$ and $s = r = t = 0$. Thus (C, T) is determined, Zariski locally, up to unique isomorphism, by the functions

$$a_1, a_2, a_3, a_4, a_6, m_4, \ldots, m_{p^k-2}.$$

The uniqueness of these functions implies that they are compatible on the intersections of a Zariski open cover, and hence patch to give global invariants of (C, T). Expressing the Eisenstein series (Hasse invariant) E_{p-1} of $C_{\mathbf{a}}$ as

$$E_{p-1} = E_{p-1}(a_1, a_2, a_3, a_4, a_6),$$

it follows that we have

$$(\mathcal{M}_{ell}^{ord})_{\mathbb{F}_p}^{p^k-1} \cong \operatorname{Spec}(\mathbb{F}_p[a_1, a_2, a_3, a_4, a_6, m_4, \ldots, m_{p^k-2}])[E_{p-1}^{-1}]).$$

With minor modification, the method for $p > 3$ extends to the cases $p = 2, 3$. The canonical forms for (C, T) just change slightly.

Case 2: $p = 3$. Suppose that (C, T) is an object of $(\mathcal{M}_{ell}^{ord})_{\mathbb{F}_3}^{3^k-1}$ for $k \geq 1$. If $k > 1$, then $3^k - 1 \geq 4$, and thus (C, T) admits a canonical Weierstrass presentation.

If $k = 1$, then this no longer holds. Instead, choosing

$$\begin{aligned} \lambda &= \lambda(m_0) \\ s &= s(m_0, m_1) \end{aligned}$$

there exists (Zariski locally) a Weierstrass curve $(C_{\mathbf{a}}, T) \cong (C, T)$ such that

$$T = T_{\mathbf{a}} + O(T_{\mathbf{a}}^3).$$

Choosing t_0 accordingly, there is a transformation

$$\begin{aligned} f_{t_0} : C_{\mathbf{a}} &\to C_{\mathbf{a}'} \\ (x, y) &\mapsto (x, y + t_0) \end{aligned}$$

such that $a_3' = 0$. The induced 2-jet $T' = (f_{t_0})_* T$ still satisfies $T' \equiv T_{\mathbf{a}'} \mod T_{\mathbf{a}'}^3$. The automorphisms $f_{\lambda,s,r,t}$ of $(C_{a'}, T')$ preserving the property that $a_3 = 0$, and the trivialization of the 2-jet, satisfy

$$\begin{aligned} \lambda &= 1 \\ s &= 0 \\ t &= -a_1 r/2. \end{aligned}$$

Under such a transformation, we find that

$$a_4 \mapsto a_4 + 2b_2 r + 3r^2$$

where $b_2 = a_2 + a_1^2/4$. Because C is assumed to be ordinary, the element b_2 is a unit. Because R is an $\mathbb{F}_3$-algebra, there is a unique r such that $a_4 \mapsto 0$. Thus we have shown that there is a canonical Weierstrass presentation which trivializes the 2-jet, and for which $a_3 = a_4 = 0$.

Case 3: $p = 2$. Assume that $k = 2$ (for $k > 2$, we have $2^k - 1 \geq 4$, and therefore an elliptic curve with a $2^k - 1$-jet admits a canonical Weierstrass presentation). Let (C, T) be an object of $(\mathcal{M}_{ell}^{ord})^3_{\mathbb{F}_2}$. Choose (Zariski locally) a Weierstrass presentation $(C_{\mathbf{a}}, T) \cong (C, T)$. Choosing

$$\begin{aligned} \lambda &= \lambda(m_0) \\ s &= s(m_0, m_1) \\ r &= r(m_0, m_1, m_2) \end{aligned}$$

we may assume that T satisfies

$$T = T_{\mathbf{a}} + O(T_{\mathbf{a}}^4).$$

The automorphisms $f_{\lambda,s,r,t}$ of (C_a, T) preserving the trivialization of the 3-jet satisfy

$$\begin{aligned} \lambda &= 1, \\ s &= 0, \\ r &= 0. \end{aligned}$$

Under such a transformation, we find that

$$a_4 \mapsto a_4 - a_1 t.$$

Because C is assumed to be ordinary, the element a_1 is a unit. Letting $t = a_4/a_1$, we have $a_4 \mapsto 0$. Thus we have shown that there is a canonical Weierstrass presentation which trivializes the 3-jet, and for which $a_4 = 0$. $\square$

Define

$$V_\infty^\wedge := \varprojlim_m \varinjlim_k V_k / p^m V_k.$$

The ring $V_\infty^\wedge$ is the ring of *generalized p-adic modular functions (of level 1)*.

Let $\mathcal{M}_{ell}^{ord}(p^\infty)$ be the formal scheme $\mathrm{Spf}(V_\infty^\wedge)$. There is an isomorphism between the R-points $\mathcal{M}_{ell}^{ord}(p^\infty)(R)$ and isomorphism classes of pairs (C, η) where

$$\begin{aligned} C &= \text{a generalized elliptic curve over } R, \\ (\eta : \widehat{\mathbb{G}}_m \cong \mu_{p^\infty} \xrightarrow{\cong} \widehat{C}) &= \text{an isomorphism of formal groups.} \end{aligned}$$

(Note that the existence of η implies that C has ordinary reduction modulo p.)

The ring $V_\infty^\wedge$ possesses a special structure: it is a θ-algebra (see [**GH**]). That is, it has actions of operators

$$\begin{aligned}
\psi^k, &\quad k \in \mathbb{Z}_p^\times, \\
\psi^p, &\quad \text{lift of the Frobenius}, \\
\theta, &\quad \text{satisfying } \psi^p(x) = x^p + p\theta(x).
\end{aligned}$$

The operations ψ^k and ψ^p are ring homomorphisms. The operation θ is determined from ψ^p, since $V_\infty^\wedge$ is torsion-free and we have

$$\psi^p(x) \equiv x^p \mod p.$$

We determine ψ^k and ψ^p on the functors of points of $\mathcal{M}_{ell}^{ord}(p^\infty)$. Suppose that R is a p-complete ring. Note that

$$\mathrm{Aut}_{\mathbb{Z}_p}(\widehat{\mathbb{G}}_m) \cong \mathbb{Z}_p^\times.$$

We may therefore regard $\mathbb{Z}_p^\times$ as acting on $\widehat{\mathbb{G}}_m/R$. Let $[k]$ be the automorphism corresponding to $k \in \mathbb{Z}_p^\times$. Define

$$\begin{aligned}
(\psi^k)^* : \mathcal{M}_{ell}^{ord}(p^\infty)(R) &\to \mathcal{M}_{ell}^{ord}(p^\infty)(R) \\
(C, \eta) &\mapsto (C, \eta \circ [k]).
\end{aligned}$$

The map $(\psi^k)^*$ is represented by a map

$$\psi^k : V_\infty^\wedge \to V_\infty^\wedge.$$

Suppose that (C, η) is an R-point of $\mathcal{M}_{ell}^{ord}(p^\infty)$. Since C has ordinary reduction mod p, the pth power endomorphism of C_{ns} factors as

$$\begin{tikzcd}
C_{ns} \arrow[rr, "{[p]}"] \arrow[dr, "\Phi_{insep}"'] & & C_{ns} \\
& C_{ns}^{(p)} \arrow[ur, "\Phi_{sep}"'] &
\end{tikzcd} \tag{5.2}$$

where Φ_{insep} is purely inseparable. The morphism Φ_{sep} is not, in general, étale, but $\ker \Phi_{sep}$ is an étale group scheme over R. On the non-singular fibers of C, Φ_{sep} has degree p, whereas on the singular fibers it has degree 1.

These morphisms, and their kernels, fit into a 3×3 diagram of short exact sequences of group schemes:

$$\begin{tikzcd}
\widehat{C}[p] \arrow[r] \arrow[d, equal] & C_{ns}[p] \arrow[r] \arrow[d] & C[p]_{et} \arrow[d] \\
\widehat{C}[p] \arrow[r] \arrow[d] & C_{ns} \arrow[r, "\Phi_{insep}"'] \arrow[d, "{[p]}"'] & C_{ns}^{(p)} \arrow[d, "\Phi_{sep}"] \\
0 \arrow[r] & C_{ns} \arrow[r, equal] & C_{ns}
\end{tikzcd}$$

where $\widehat{C}[p]$ is the p-torsion subgroup of the height 1 formal group $\widehat{C}$ and $C[p]_{et}$ is the p-torsion of the ind-finite group scheme $C(p)_{et}$.

Given a uniformization

$$\eta : \widehat{\mathbb{G}}_m \xrightarrow{\cong} \widehat{C}$$

we get an induced uniformization $\eta^{(p)}$:

$$\begin{tikzcd}
\mu_p \arrow[r] \arrow[d, "\cong"'] & \widehat{\mathbb{G}}_m \arrow[r, "{[p]}"] \arrow[d, "\cong"', "\eta"] & \widehat{\mathbb{G}}_m \arrow[d, dotted, "\eta^{(p)}"] \\
\widehat{C}[p] \arrow[r] & \widehat{C} \arrow[r, "(\Phi_{insep})_*"'] & \widehat{C^{(p)}}
\end{tikzcd} \tag{5.3}$$

REMARK 5.3. The uniformization $\eta^{(p)}$ admits a different characterization: it is the unique isomorphism of formal groups making the following diagram commute:

$$\begin{tikzcd}
 & \widehat{\mathbb{G}}_m \arrow[dl, dotted, "\eta^{(p)}"'] \arrow[d, "\eta"] \\
\widehat{C^{(p)}} \arrow[r, "(\Phi_{sep})_*"'] & \widehat{C}
\end{tikzcd}$$

(The isogeny Φ_{sep} induces an isomorphism on formal groups.) The equivalence of this definition of $\eta^{(p)}$ with the previous is proved by the following diagram.

$$\begin{tikzcd}
\widehat{\mathbb{G}}_m \arrow[r, "{[p]}"] \arrow[d, "\eta"'] & \widehat{\mathbb{G}}_m \arrow[r, equal] \arrow[d, "\eta^{(p)}"'] & \widehat{\mathbb{G}}_m \arrow[d, "\eta"] \\
\widehat{C} \arrow[r, "(\Phi_{insep})_*"] \arrow[rr, bend right, "{[p]}"'] & \widehat{C^{(p)}} \arrow[r, "(\Phi_{sep})_*", "\cong"'] & \widehat{C}
\end{tikzcd}$$

We get a map on R-points

$$\begin{aligned}
(\psi^p)^* : \mathcal{M}_{ell}^{ord}(p^\infty)(R) &\to \mathcal{M}_{ell}^{ord}(p^\infty)(R) \\
(C, \eta) &\mapsto (C^{(p)}, \eta^{(p)})
\end{aligned}$$

which is represented by a ring map

$$\psi^p : V_\infty^\wedge \to V_\infty^\wedge.$$

It is easy to see that ψ^p commutes with ψ^k. To show that ψ^p induces a θ-algebra structure on $V_\infty^\wedge$, it suffices to prove the following:

LEMMA 5.4. *We have $\psi^p(x) \equiv x^p$ mod p.*

PROOF. It suffices to show that $(\psi^p)^*$ is represented by the Frobenius when restricted to characteristic p. That is, we must show that if R is an $\mathbb{F}_p$-algebra, and (C, η) is an R point of $\mathcal{M}_{ell}^{ord}(p^\infty)$, then the Frobenius

$$\begin{aligned}
\sigma : R &\to R \\
x &\mapsto x^p
\end{aligned}$$

gives rise to an isomorphism

$$(C^{(p)}, \eta^{(p)}) \cong (\sigma^* C, \sigma^* \eta).$$

We briefly introduce some notation: if X is a scheme over R, then we have the following diagram of morphisms.

$$\begin{tikzcd}
X \arrow[dr, "Fr^{rel}"] \arrow[drr, bend left, "Fr^{tot}"] \arrow[ddr, bend right] & & \\
& \sigma^* X \arrow[r, "Fr"] \arrow[d] & X \arrow[d] \\
& \mathrm{Spec}(R) \arrow[r, "\sigma^*"'] & \mathrm{Spec}(R)
\end{tikzcd}$$

The square is a pullback square, and Fr is the pullback of σ. The map Fr^{tot} is the total Frobenius, and the universal property of the pullback induces the relative Frobenius Fr^{rel}.

Because the isogeny Fr^{rel} has degree p, we have a factorization

$$\begin{tikzcd}
C_{ns} \arrow[rr, "{[p]}"] \arrow[dr, "Fr^{rel}"'] & & C_{ns} \\
& \sigma^* C_{ns} \arrow[ur, "\widehat{Fr^{rel}}"'] &
\end{tikzcd}$$

Because C has no supersingular fibers, the dual isogeny $\widehat{Fr^{rel}}$ has separable kernel (see, for instance, [**Sil**, Thm. V.3.1]).

Therefore, we have

$$\begin{aligned}
\sigma^* C &\cong C^{(p)}, \\
\Phi_{insep} &\cong Fr^{rel}, \\
\Phi_{sep} &\cong \widehat{Fr^{rel}}.
\end{aligned}$$

We just have to show that under these isomorphisms, we have $\sigma^*\eta \cong \eta^{(p)}$. We have the following diagram of formal groups.

$$\begin{tikzcd}
\widehat{\mathbb{G}}_m \arrow[r, "Fr^{rel}"] \arrow[d, "\eta"'] & \widehat{\mathbb{G}}_m \arrow[r, "Fr"] \arrow[d, "\sigma^*\eta"'] & \widehat{\mathbb{G}}_m \arrow[d, "\eta"] \\
\widehat{C} \arrow[r, "Fr^{rel}"'] & \widehat{\sigma^* C} \arrow[r, "Fr"'] & \widehat{C}
\end{tikzcd}$$

On $\mathbb{G}_m$, the relative Frobenius is the pth power map. Therefore, by the definition of $\eta^{(p)}$, we have $\sigma^*\eta \cong \eta^{(p)}$ under the isomorphism $\sigma^* C \cong C^{(p)}$. $\square$

More generally, letting ω_∞ denote the canonical line bundle over $\mathrm{Spf}(V_\infty^\wedge)$, then the graded algebra

$$(V_\infty^\wedge)_{2*} := \Gamma \omega_\infty^{\otimes *}$$

inherits the structure of an even periodic graded θ-algebra. The θ-algebra structure may be described by the isomorphism

$$(V_\infty^\wedge)_* \cong (K_p)_* \otimes_{\mathbb{Z}_p} V_\infty^\wedge.$$

Here $(K_p)_*$ is the coefficients of p-adic K-theory, and the θ-algebra structure is induced from the diagonal action of the Adams operations.

REMARK 5.5. By defining ψ^p on $V_\infty^\wedge$ using its modular interpretation, I have glossed over several technical issues related to extending the quotient by the canonical subgroup of ordinary elliptic curves to the singular fibers of a generalized elliptic curve. The careful reader could instead choose a different path to defining the operation ψ^p: define it just as I have on the non-singular fibers, and then explicitly define its effect on q-expansions to extend this definition over the cusp. (The effect of ψ^p on q expansions is to raise q to its pth power.)

6. $K(1)$ local elliptic spectra

In this section we will investigate the abstract properties satisfied by a $K(1)$-local elliptic spectrum. Throughout this section, suppose that (E, α, C) be an elliptic spectrum. That is to say, E is a $K(1)$-local weakly even periodic ring spectrum, C is a generalized elliptic curve over $R = \pi_0 E$, and α is an isomorphism of formal groups

$$\alpha : \mathbb{G}_E \to \widehat{C}.$$

We shall furthermore assume that R is p-complete, and that the classifying map

$$f : \mathrm{Spf}(R) \to (\mathcal{M}_{ell})_p$$

is flat. This implies:

(1) E is Landweber exact (Remark 1.4),
(2) C has ordinary reduction modulo p (Lemma 8.1).

There are three distinct subjects we shall address in this section.

(1) The p-adic K-theory of $K(1)$-local elliptic spectra.
(2) θ-compatible $K(1)$-local elliptic E_∞-ring spectra.
(3) The θ-algebra structure of the p-adic K-theory of a supersingular elliptic E_∞-ring spectrum.

The p-adic K-theory of $K(1)$-local elliptic spectra.

Let

$$(K_p^\wedge)_* E := \pi_*((K \wedge E)_p)$$

denote the p-adic K-homology of E. It is geometrically determined by the following standard proposition.

PROPOSITION 6.1. *Let $\mathrm{Spf}(W)$ be the pullback of $\mathrm{Spf}(R)$ over $\mathcal{M}_{ell}^{ord}(p^\infty)$. Then there is an isomorphism*

$$(K_p^\wedge)_0 E \cong W.$$

This isomorphism is $\mathbb{Z}_p^\times$-equivariant, where the $\mathbb{Z}_p^\times$-acts on the left hand side through stable Adams operations, and it acts on the right hand side due to the fact that $\mathrm{Spf}(W)$ is an ind-étale $\mathbb{Z}_p^\times$-torsor over $\mathrm{Spf}(R)$.

PROOF. By Landweber exactness, choosing complex orientations for K_p and E, we have

$$(K_p^\wedge)_0 E = ((K_p)_0 \otimes_{MUP_0} MUP_0 MUP \otimes_{MUP_0} R)_p^\wedge.$$

Using the fact that $\mathrm{Spec}(MUP_0 MUP) = \mathrm{Spec}(MUP_0) \times_{\mathcal{M}_{FG}} \mathrm{Spec}(MUP_0)$, it is not hard to deduce from this that we have

$$\mathrm{Spf}((K_p^\wedge)_0 E) \cong \mathrm{Spf}(R) \times_{\mathcal{M}_{FG}} \mathrm{Spf}((K_p)_0).$$

Consider the induced diagram

$$\begin{tikzcd}
\mathrm{Spf}((K_p^\wedge)_0 E) \arrow[r] \arrow[d] & \mathcal{M}_{ell}^{ord}(p^\infty) \arrow[r] \arrow[d] & \mathrm{Spf}((K_p^\wedge)_0) \arrow[d] \\
\mathrm{Spf}(R) \arrow[r, "f"'] & \mathcal{M}_{ell}^{ord} \arrow[r] & \mathcal{M}_{FG}
\end{tikzcd}$$

The right-hand square is a pullback by the proof of Lemma 5.1, and we have established that the composite is a pullback. We deduce that the left-hand side is a pullback, which completes the proof. $\square$

θ-compatible $K(1)$-local elliptic E_∞-ring spectra.

If E is an E_∞-ring spectrum, then the completed K_p-homology

$$(K_p^\wedge)_* E := \pi_*((K \wedge E)_p)$$

naturally carries the structure of a θ-algebra: for $k \in \mathbb{Z}_p^\times$, the operations ψ^k are the stable Adams operations in K_p-theory, and the operation θ arises from the action of the $K(1)$-local Dyer-Lashof algebra [**GH**].

If the classifying map

$$f : \mathrm{Spf}(R) \to \mathcal{M}_{ell}^{ord}$$

is étale, then the pullback W of Proposition 6.1 carries naturally the structure of a θ-algebra which we now explain. Since $\mathrm{Spf}(R)$ is étale over $\mathcal{M}_{ell}^{ord}$, the pullback $\mathrm{Spf}(W)$ is étale over $\mathcal{M}_{ell}^{ord}(p^\infty) = \mathrm{Spf}(V_\infty^\wedge)$. It is in particular formally étale, and therefore there exists a unique lift

$$\begin{tikzcd}
\mathrm{Spec}(W/p) \arrow[r, "\sigma^*"] \arrow[d] & \mathrm{Spec}(W/p) \arrow[r] & \mathrm{Spf}(W) \arrow[d] \\
\mathrm{Spf}(W) \arrow[r] \arrow[urr, dotted, "(\psi^p)^*"] & \mathcal{M}_{ell}^{ord}(p^\infty) \arrow[r, "(\psi^p)^*"'] & \mathcal{M}_{ell}^{ord}(p^\infty)
\end{tikzcd}$$

where

$$(\psi^p)^* : \mathcal{M}_{ell}^{ord}(p^\infty) \to \mathcal{M}_{ell}^{ord}(p^\infty)$$

is the lift of the Frobenius coming from θ-algebra structure of $V_\infty^\wedge$ and $\sigma : W/p \to W/p$ is the Frobenius. Note that because $\mathrm{Spf}(R)$ is étale over $\mathcal{M}_{ell}^{ord}$, it is in particular flat, and so W must be torsion-free. Therefore, the induced homomorphism

$$\psi^p : W \to W$$

determines a unique θ-algebra structure on W.

If E is E_∞, and the classifying map f is étale, it is *not* necessarily the case that the isomorphism

$$(K_p^\wedge)_0 E \cong W$$

of Proposition 6.1 preserves the operation ψ^p. This is rather a reflection of the choice of E_∞-structure on E. We therefore make the following definition.

Definition 6.2. Suppose that E is a $K(1)$-local E_∞ elliptic spectrum associated to an elliptic curve C/R, and suppose that the classifying map

$$\mathrm{Spf}(R) \to (\overline{\mathcal{M}}_{ell})_p$$

is étale. If the isomorphism $(K_p^\wedge)_0 E \cong W$ is a map of θ-algebras, then we shall say that (E, C) is a *θ-compatible.*

REMARK 6.3. As a *side-effect* of our construction of $\mathcal{O}^{top}_{K(1)}$ it will be the case that the E_∞-structure on the spectrum of sections $E = \mathcal{O}^{top}_{K(1)}(\mathrm{Spf}(R))$ is θ-compatible.

REMARK 6.4. In [**AHS**], the authors define the notion of an H_∞-elliptic ring spectrum, which is a stronger notion than that of an elliptic H_∞-ring spectrum in that they require a compatibility between the H_∞-structure and the elliptic structure. It is easily seen that every $K(1)$-local H_∞-elliptic spectrum whose classifying map is étale over the p-completion of the moduli stack of elliptic curves is θ-compatible.

The θ-algebra structure of the p-adic K-theory of supersingular elliptic E_∞-ring spectra.

In [**AHS**, Sec. 3], previous work of Ando and Strickland is condensed into an elegant perspective on Dyer-Lashof operations on an even periodic complex orientable H_∞-ring spectrum T. Namely, suppose that

(1) T is *homogeneous* — it is a homotopy commutative algebra spectrum over an even periodic E_∞-ring spectrum (such as MU).
(2) $\pi_0 T$ is a complete local ring with residue field of characteristic p.
(3) The reduction $\bar{\mathbb{G}}_T$ of $\mathbb{G}_T$ modulo the maximal ideal has finite height.
(4) $\mathbb{G}_T$ is *Noetherian* — it is obtained by pullback from a formal group over $\mathrm{Spf}(S)$ where S is Noetherian.

Then, for every morphism

$$i : \mathrm{Spf}(R) \to \mathrm{Spf}(\pi_0 T)$$

and every finite subgroup $H < i^*\mathbb{G}_T$ (i.e. H is an effective relative Cartier divisor of $i^*\mathbb{G}_T$ represented by a subgroup-scheme) there is a new morphism

$$\psi_H : \mathrm{Spf}(R) \to \mathrm{Spf}(\pi_0 T)$$

and an isogeny of formal groups

$$f_H : i^*\mathbb{G}_T \to \psi_H^* \mathbb{G}_T$$

with kernel H. This structure is called *descent data for subgroups.*

REMARK 6.5. The authors of [**AHS**] actually describe the structure of *descent data for level structures.* However, their treatment carries over to subgroups (see [**AHS**, Rmk. 3.12]).

EXAMPLE 6.6. Suppose that T is a $K(1)$-local E_∞-ring spectrum. Then the formal group $\mathbb{G}_T$ must have height 1 (see the proof of Lemma 8.1), and it follows that $\mathbb{G}_T$ has a unique subgroup of order p, given by the p-torsion subgroup $\mathbb{G}_T[p]$. Taking i to be the identity map, we get an operation

$$\psi_{\mathbb{G}_T[p]} : \pi_0 T \to \pi_0 T.$$

This operation coincides with ψ^p. We shall let f_p denote the associated degree p isogeny

$$f_p = f_{\mathbb{G}_T[p]} : \mathbb{G}_T \to (\psi^p)^* \mathbb{G}_T.$$

Example 6.7. Suppose that $T = E(k, \mathbb{G})$ is the Morava E-theory associated to a height n formal group $\mathbb{G}/k$, with universal deformation $\tilde{\mathbb{G}}/B$, and the E_∞-structure of Goerss and Hopkins [**GH**]. Then in [**AHS**] it is proven that the associated descent data for subgroups is given as follows. Let

$$i : \mathrm{Spf}(R) \to \mathrm{Spf}(B)$$

be a morphism classifying a deformation $i^*\tilde{\mathbb{G}}/R$ of $i^*\mathbb{G}/k'$ (where $k' := R/\mathfrak{m}_R$). Suppose that $\tilde{H} < i^*\tilde{\mathbb{G}}$ is a finite subgroup, and let H denote the restriction of $\tilde{H}$ to $i^*\mathbb{G}$. Because $\mathbb{G}$ is a formal group of finite height over a field of characteristic p, the only subgroups of $\mathbb{G}$ are of the form

$$H_r = \ker((Fr^{rel})^r : i^*\mathbb{G} \to i^*\mathbb{G})$$

where Fr^{rel} is the relative Frobenius. Therefore, we have $H = H_r$ for some r. The quotient $(i^*\mathbb{G})/H_r$ is the pullback of $\mathbb{G}$ under the composite $i^{(p^r)}$:

$$i^{(p^r)} : \mathrm{Spf}(k') \xrightarrow{(\sigma^r)^*} \mathrm{Spf}(k') \xrightarrow{i} \mathrm{Spf}(k)$$

where σ is the Frobenius. The quotient $i^*\tilde{\mathbb{G}}/\tilde{H}$ is then a deformation of $(i^*\mathbb{G})/H \cong (i^{(p^r)})^*\mathbb{G}$, hence is classified by a morphism

$$\psi_{\tilde{H}} : \mathrm{Spf}(R) \to \mathrm{Spf}(B).$$

This determines the operation $\psi_{\tilde{H}}$. The morphism $f_{\tilde{H}}$ is given by

$$f_{\tilde{H}} : i^*\tilde{\mathbb{G}} \to (i^*\tilde{\mathbb{G}})/\tilde{H} \cong (\psi_{\tilde{H}})^*\tilde{\mathbb{G}}.$$

Suppose now that k is a finite field, and that C/k is a supersingular elliptic curve. Then, by Serre-Tate theory, there is a unique elliptic curve $\tilde{C}$ over the universal deformation ring

$$B := B(k, \widehat{C}) \cong \mathbb{W}(k)[[u_1]]$$

such that the formal group $\tilde{C}^\wedge$ is the universal deformation of the formal group $\widehat{C}$. Furthermore, we have seen that the Goerss-Hopkins-Miller theorem associates to $\widehat{C}/k$ an elliptic E_∞-ring spectrum

$$E := E(k, \widehat{C}) = \mathcal{O}^{top}_{K(2)}(\mathrm{Spf}(B))$$

with associated elliptic curve $\tilde{C}$.

The curve $\tilde{C}$ is to be regarded as an elliptic curve over $\mathrm{Spf}(B)$, but by Remark 1.6, there is a unique elliptic curve $\tilde{C}^{alg}$ over $\mathrm{Spec}(B)$ which restricts to $\tilde{C}/\mathrm{Spf}(B)$. Let B^{ord} be the ring

$$B^{ord} = B[u_1^{-1}]^\wedge_p.$$

We regard B^{ord} as being complete with respect to the ideal (p). Let $\tilde{C}^{ord}$ be the restriction of $\tilde{C}^{alg}$ to $\mathrm{Spf}(B^{ord})$. The following lemma follows from Lemma 8.1.

Lemma 6.8. *The spectrum $E_{K(1)}$ is an elliptic spectrum for the elliptic curve $\tilde{C}^{ord}/B^{ord}$.*

The Goerss-Hopkins E_∞-structure on E induces an E_∞ structure on the $K(1)$-localized spectrum $E_{K(1)}$, and there is an induced operation

$$\psi^p : B^{ord} \to B^{ord}$$

on $B^{ord} = \pi_0 E_{K(1)}$ which lifts the Frobenius in characteristic p. We have the following proposition.

Proposition 6.9. *There is an isomorphism*

$$(\psi^p)^* \tilde{C}^{ord} \cong (\tilde{C}^{ord})^{(p)}$$

(where $(\tilde{C}^{ord})^{(p)}$ is the quotient of $\tilde{C}^{ord}$ of Diagram (5.2)) making the following diagram of isogenies of formal groups commute.

$$\begin{tikzcd} (\tilde{C}^{ord})^\wedge \arrow[r, "f_p"] \arrow[dr, "(\Phi_{insep})_*"'] & (\psi^p)^*(\tilde{C}^{ord})^\wedge \arrow[d, "\cong"] \\ & ((\tilde{C}^{ord})^{(p)})^\wedge \end{tikzcd} \tag{6.1}$$

Proof. (In some sense, this proposition is one of the most important ingredients to the construction of *tmf*, and I would have gotten it wrong except for the help of Niko Naumann and Charles Rezk.) Let

$$i : \mathrm{Sub}_p(\tilde{C}) \to \mathrm{Spf}(B)$$

be the formal scheme of "subgroups of $\tilde{C}$ of order p" (see, for instance, [**Str**]). The formal scheme $\mathrm{Sub}_p(\tilde{C})$ is of the form

$$\mathrm{Spf}(\Gamma_0(p)(\tilde{C})).$$

Observe that because the p-divisible group of $\tilde{C}$ is entirely formal, we have

$$\Gamma_0(p)(\tilde{C}) = \Gamma_0(p)(\tilde{C}^\wedge).$$

Let $\tilde{H}_{can}$ be the universal degree p subgroup of $i^*\tilde{C}$. There is a corresponding operation

$$\psi_{\tilde{H}_{can}} : B = \pi_0 E \to \Gamma_0(p)(\tilde{C})$$

and an isomorphism

$$\psi^*_{\tilde{H}_{can}} \tilde{C}^\wedge \cong (i^*\tilde{C}^\wedge)/\tilde{H}_{can}. \tag{6.2}$$

This operation arises topologically from the total power operation

$$\psi_{\tilde{H}_{can}} : B = \pi_0 E \xrightarrow{\mathcal{P}_E} \pi_0 E^{B\Sigma_{p+}} \twoheadrightarrow \Gamma_0(p)(\tilde{C}^\wedge)$$

where the surjection is the quotient by the image of the transfer morphisms [**AHS**, Rmk. 3.12]).

Forgetting the topology on the ring B, we can regard $\tilde{C}^\wedge$ simply as a formal group over the ring B, and we get an induced formal group $(\tilde{C}^{ord})^\wedge/B^{ord}$ and degree p subgroup

$$\tilde{H}^{ord}_{can} < i^*(\tilde{C}^{ord})^\wedge$$

over

$$\Gamma_0(p)(\tilde{C})^{ord} := \Gamma_0(p)(\tilde{C})[u_1^{-1}]^\wedge_p \cong \Gamma_0(p)(\tilde{C}^{ord}).$$

The last isomorphism follows from the fact that *forgetting* about formal schemes, there is an isomorphism

$$\mathrm{Sub}_p(\tilde{C}^{alg}) \times_{\mathrm{Spec}(B)} \mathrm{Spec}(B^{ord}) \cong \mathrm{Sub}_p(\tilde{C}^{alg} \times_{\mathrm{Spec}(B)} \mathrm{Spec}(B^{ord})).$$

Let

$$c : \Gamma_0(p)(\tilde{C}^{ord}) \to \Gamma_0(p)((\tilde{C}^{ord})^\wedge) \cong B^{ord}$$

be the map classifying the subgroup $\tilde{H}^{ord}_{can}$ of order p of $\Gamma_0(p)((\tilde{C}^{ord})^\wedge)$, regarded as a subgroup of $\tilde{C}^{ord}$. The isomorphism $\Gamma_0(p)((\tilde{C}^{ord})^\wedge) \cong B^{ord}$ reflects the fact that there is one and only one degree p-subgroup of a deformation of a height 1 formal group. Thus

$$\tilde{H}^{ord}_{can} = i^*(\tilde{C}^{ord})^\wedge[p]. \tag{6.3}$$

By Example 6.6, the corresponding operation

$$\psi_{c^*\tilde{H}^{ord}_{can}} : B^{ord} \to \Gamma_0(p)((\tilde{C}^{ord})^\wedge) \cong B^{ord}$$

is nothing more than the operation ψ^p on the $K(1)$-local E_∞ ring spectrum $E_{K(1)}$. Since localization $E \to E_{K(1)}$ is a map of E_∞-ring spectra, E and $E_{K(1)}$ have compatible descent data for subgroups, and we deduce that there is a commutative diagram:

(6.4)

$$\begin{tikzcd}
B \arrow[r] \arrow[d, equal] \arrow[ddd, bend right, "\psi_{\tilde{H}_{can}}"'] & B^{ord} \arrow[d, equal] \arrow[dddrr, bend left, "\psi^p"] & & \\
\pi_0 E \arrow[r] \arrow[d, "\mathcal{P}_E"] & \pi_0 E_{K(1)} \arrow[d, "(\mathcal{P}_E)_{K(1)}"'] \arrow[dr, "\mathcal{P}_{E_{K(1)}}"] & & \\
\pi_0 E^{B\Sigma_{p+}} \arrow[r] \arrow[d] & \pi_0 (E^{B\Sigma_{p+}})_{K(1)} \arrow[r] \arrow[d] & \pi_0 (E_{K(1)})^{B\Sigma_{p+}} \arrow[d] & \\
\Gamma_0(p)(\tilde{C}) \arrow[r, "j"'] & \Gamma_0(p)(\tilde{C}^{ord}) \arrow[r, "c"'] & \Gamma_0(p)((\tilde{C}^{ord})^\wedge) \arrow[r, equal] & B^{ord}
\end{tikzcd}$$

Using Serre-Tate theory, the isomorphism (6.2) lifts to an isomorphism

$$(i^*\tilde{C})/\tilde{H}_{can} \cong (\psi_{\tilde{H}_{can}})^*\tilde{C}.$$

Using the isomorphism above, equality (6.3), and the commutativity of Diagram 6.4, we deduce that there are isomorphisms

$$\begin{aligned}
(\tilde{C}^{ord})^{(p)} &= \tilde{C}^{ord}/c^*\tilde{H}^{ord}_{can} \\
&= c^*j^*((i^*\tilde{C})/\tilde{H}_{can}) \\
&\cong c^*j^*(\psi_{\tilde{H}_{can}})^*\tilde{C} \\
&= (\psi^p)^*\tilde{C}^{ord}.
\end{aligned}$$

Diagram (6.1) commutes because both f_p and $(\Phi_{insep})_*$ are lifts of the relative Frobenius with the same kernel. $\square$

Consider the pullback diagram

$$\begin{tikzcd}
\mathrm{Spf}((K_p^\wedge)_0 E_{K(1)}) \arrow[r, "g"] \arrow[d, "q"'] & \mathrm{Spf}(V_\infty^\wedge) \arrow[d, "q'"] \\
\mathrm{Spf}(\pi_0 E_{K(1)}) \arrow[r, "g'"'] & \mathcal{M}^{ord}_{ell}
\end{tikzcd} \tag{6.5}$$

of Proposition 6.1. The following theorem will be essential in our construction of the map α_{chrom}.

THEOREM 6.10. *The map*

$$g : V_\infty^\wedge \to (K_p^\wedge)_0 E_{K(1)}$$

of Diagram (6.5) is a map of θ-algebras.

PROOF. We just need to check that g commutes with ψ^p (we already know from Proposition 6.1 that g commutes with the action of $\mathbb{Z}_p^\times$). The map g classifies a level structure

$$\eta : \widehat{\mathbb{G}}_m \xrightarrow{\cong} q^*(\tilde{C}^{ord})^\wedge.$$

We need to verify that there is an isomorphism

$$((\psi^p)^* q^* \tilde{C}^{ord}, (\psi^p)^*\eta) \cong ((q^*\tilde{C}^{ord})^{(p)}, \eta^{(p)}).$$

The descent data for level structures arising from E_∞-structures is natural with respect to maps of E_∞-ring spectra (see [**AHS**]). It follows that the maps of E_∞-ring spectra:

$$K_p \xrightarrow{r} (K_p \wedge E_{K(1)})_p \xleftarrow{q} E_{K(1)}.$$

induce a diagram

$$\begin{tikzcd} r^*\widehat{\mathbb{G}}_m \arrow[r, "r^*f_p"] \arrow[d, "\eta"'] & r^*(\psi^p)^*\widehat{\mathbb{G}}_m \arrow[d, "(\psi^p)^*\eta"] \\ (q^*\tilde{C}^{ord})^\wedge \arrow[r, "q^*f_p"'] & q^*(\psi^p)^*(\tilde{C}^{ord})^\wedge \end{tikzcd}$$

The E_∞-structure on p-adic K-theory associates to the formal subgroup $\mu_p < \widehat{\mathbb{G}}_m$ over $\mathbb{Z}_p$ the pth power isogeny

$$f_p = [p] : \widehat{\mathbb{G}}_m \to \widehat{\mathbb{G}}_m$$

(this is a special case of Example 6.7). Combined with Diagram (6.1), we have a diagram

$$\begin{tikzcd} \widehat{\mathbb{G}}_m \arrow[r, "{[p]}"] \arrow[d, "\eta"'] & \widehat{\mathbb{G}}_m \arrow[d, "(\psi^p)^*\eta"] \\ q^*(\tilde{C}^{ord})^\wedge \arrow[r, "(\Phi_{insep})_*"'] & q^*((\tilde{C}^{ord})^{(p)})^\wedge \end{tikzcd}$$

We deduce from Diagram (5.3) that with respect to the isomorphism $(\psi^p)^*\tilde{C}^{ord} \cong (\tilde{C}^{ord})^{(p)}$ we have $(\psi^p)^*\eta \cong \eta^{(p)}$. $\square$

7. Construction of $\mathcal{O}^{top}_{K(1)}$

For a θ-algebra A/k and a θ-A-module M, let

$$H^*_{Alg_\theta}(A/k, M)$$

denote the θ-algebra Andre-Quillen cohomology of A with coefficients in M. In [**GH**] (see also [**GH**]), an obstruction theory for $K(1)$-local E_∞ ring spectra is developed. We summarize their main results:

THEOREM 7.1 (Goerss-Hopkins).

(1) *Given a graded θ-algebra A_*, the obstructions to the existence of a $K(1)$-local E_∞-ring spectrum E, for which there is an isomorphism*

$$(K_p^\wedge)_*E \cong A_*$$

of θ-algebras, lie in

$$H^s_{Alg_\theta}(A_*/(K_p)_*, A_*[-s+2]), \qquad s \geq 3.$$

The obstructions to uniqueness lie in

$$H^s_{Alg_\theta}(A_*/(K_p)_*, A_*[-s+1]), \qquad s \geq 2.$$

(2) *Given $K(1)$-local E_∞-ring spectra E_1, E_2 such that $K_*^\wedge E_i$ is p-complete, and a map of graded θ-algebras*

$$f_* : (K_p^\wedge)_*E_1 \to (K_p^\wedge)_*E_2,$$

the obstructions to the existence of a map $f : E_1 \to E_2$ of E_∞-ring spectra which induces f_* on p-adic K-homology lie in

$$H^s_{Alg_\theta}((K_p^\wedge)_*E_1/(K_p)_*, (K_p^\wedge)_*E_2[-s+1]), \qquad s \geq 2.$$

(Here, the θ-$(K_p^\wedge)_*E_1$-module structure on $(K_p^\wedge)_*E_2$ arises from the map f_*.) The obstructions to uniqueness lie in

$$H^s_{Alg_\theta}((K_p^\wedge)_*E_1/(K_p)_*, (K_p^\wedge)_*E_2[-s]), \qquad s \geq 1.$$

(3) *Given such a map f above, there is a spectral sequence which computes the higher homotopy groups of the space $E_\infty(E_1, E_2)$ of E_∞ maps:*

$$H^s_{Alg_\theta}((K_p^\wedge)_*E_1/(K_p)_*, (K_p^\wedge)_*E_2[t]) \Rightarrow \pi_{-t-s}(E_\infty(E_1, E_2), f).$$

Remark 7.2. The notation $A_*[u]$ corresponds to the notation $\Omega^{-u}A_*$ in [**GH**], [**GH**].

Remark 7.3. To simplify notation in the remainder of this paper, we will write

$$H^*_{Alg_\theta}(A_*, M_*) := H^*_{Alg_\theta}(A_*/(K_p)_*, M_*).$$

(That is, we will always be taking our Andre-Quillen cohomology groups in the category of graded θ-$(K_p)_*$-algebras unless we specify a different base explicitly.)

Remark 7.4. The homotopy groups of a $K(1)$-local E_∞-ring spectrum E are recovered from its p-adic K-homology by an Adams-Novikov spectral sequence. Assuming that the pro-system

$$\{K_t(E \wedge M(p^i))\}_i$$

is Mittag-Leffler (see [**Dav06**, Thm. 10.2]) this spectral sequence takes the form

$$H^s_c(\mathbb{Z}_p^\times, (K_p^\wedge)_tE) \Rightarrow \pi_{t-s}E. \tag{7.1}$$

Let A_* be a graded even periodic θ-algebra, and M_* be a graded θ-A_*-module. In [**GH**, Sec. 2.4.3], it is explained how the cohomology of the cotangent complex $\mathbb{L}(A_0/\mathbb{Z}_p)$ inherits a canonical θ-A_0-module structure from that of A_0, and that there is a spectral sequence

$$\mathrm{Ext}^s_{Mod^\theta_{A_0}}(H_t(\mathbb{L}(A_0/\mathbb{Z}_p)), M_*) \Rightarrow H^{s+t}_{Alg_\theta}(A_*, M_*). \tag{7.2}$$

The following lemma simplifies the computation of these Andre-Quillen cohomology groups.

Lemma 7.5. *Suppose that $A_*/(K_p)_*$ is a torsion-free graded θ-algebra, and that M_* is a torsion-free graded θ-A_*-module. Let A^k_* denote the fixed points $A_*^{1+p^k\mathbb{Z}_p}$ ($A^0_* = A_*^{\mathbb{Z}_p^\times}$). Note that we have*

$$A_* = \varprojlim_m \varinjlim_k A^k_*/p^m A^k_*.$$

Let $\bar{A}_*$ (respectively $\bar{A}^0_*$ and $\bar{M}_*$) denote A_*/pA_* (respectively A^0_*/pA^0_* and M_*/pM_*). Assume that:

1. *A_* and M_* are even periodic,*
2. *$\bar{A}^0_0$ is formally smooth over $\mathbb{F}_p$,*
3. *$H^s_c(\mathbb{Z}_p^\times, \bar{M}_0) = 0$ for $s > 0$,*
4. *$\bar{A}_0$ is ind-étale over $\bar{A}^0_0$.*

Then we have:

$$H^s_{Alg_\theta}(A_*, M_*[t]) = 0$$

if $s > 1$ or t is odd.

Proof. By [**GH**, Prop. 6.8], there is a spectral sequence

$$H^s_{Alg^\theta_{\mathbb{F}_p}}(\bar{A}_*, p^m M_*/p^{m+1}M_*[t]) \Rightarrow H^s_{Alg_\theta}(A_*, M_*[t]).$$

Thus it suffices to prove the mod p result. Note that because M is torsion-free, there is an isomorphism

$$\bar{M}_* \cong p^m M_*/p^{m+1}M_*.$$

Since $\bar{A}^0_0$ is formally smooth over $\mathbb{F}_p$, and since $\bar{A}_0$ is ind-étale over $\bar{A}^0_0$, we deduce that $\bar{A}_0$ is formally smooth over $\mathbb{F}_p$. Therefore, the spectral sequence

$$\mathrm{Ext}^s_{Mod^\theta_{\bar{A}_0}}(H_t(\mathbb{L}(\bar{A}_0/\mathbb{F}_p)), \bar{M}_*[t]) \Rightarrow H^{s+t}_{Alg^\theta_{\mathbb{F}_p}}(\bar{A}_*, \bar{M}_*)$$

collapses to give an isomorphism

$$\mathrm{Ext}^s_{Mod^\theta_{\bar{A}_0}}(\Omega_{\bar{A}_0/\mathbb{F}_p}, \bar{M}_{-t}) \cong H^s_{Alg^\theta_{\mathbb{F}_p}}(\bar{A}_*, \bar{M}_*[t]).$$

Since $\bar{A}_0$ is ind-étale over $\bar{A}^0_0$, there is an isomorphism

$$\Omega_{\bar{A}_0/\mathbb{F}_p} \cong \bar{A}_0 \otimes_{\bar{A}^0_0} \Omega_{\bar{A}^0_0/\mathbb{F}_p}$$

of θ-$\bar{A}_0$-modules. Because $\bar{A}_0$ is flat over $\bar{A}^0_0$, this induces a change of rings isomorphism

$$\mathrm{Ext}^s_{Mod^\theta_{\bar{A}_0}}(\Omega_{\bar{A}_0/\mathbb{F}_p}, \bar{M}_{-t}) \cong \mathrm{Ext}^s_{Mod^\theta_{\bar{A}^0_0}}(\Omega_{\bar{A}^0_0/\mathbb{F}_p}, \bar{M}_{-t}).$$

There is a composite functors spectral sequence

$$\mathrm{Ext}^s_{\bar{A}^0_0[\theta]}(\Omega_{\bar{A}^0_0/\mathbb{F}_p}, H^t_c(\mathbb{Z}_p^\times, \bar{M}_u)) \Rightarrow \mathrm{Ext}^{s+t}_{Mod^\theta_{\bar{A}^0_0}}(\Omega_{\bar{A}^0_0/\mathbb{F}_p}, \bar{M}_u)$$

which, by our hypotheses, collapses to an isomorphism

$$\mathrm{Ext}^s_{\bar{A}^0_0[\theta]}(\Omega_{\bar{A}^0_0/\mathbb{F}_p}, \bar{M}_u^{\mathbb{Z}_p^\times}) \cong \mathrm{Ext}^s_{Mod^\theta_{\bar{A}^0_0}}(\Omega_{\bar{A}^0_0/\mathbb{F}_p}, \bar{M}_u). \tag{7.3}$$

Because $\bar{A}^0_0$ is formally smooth over $\mathbb{F}_p$, the module of Kähler differentials $\Omega_{\bar{A}^0_0/\mathbb{F}_p}$ is projective as an $\bar{A}^0_0$-module. The Ext groups in the left hand side of (7.3) therefore vanish for $s > 1$, and, since M_* is concentrated in even degrees, for u odd. □

There is a relative form of Theorem 7.1. Fix a $K(1)$-local E_∞-ring spectrum E. The entire statement of Theorem 7.1 is valid if you work in the category of $K(1)$-local commutative E-algebras instead of $K(1)$-local E_∞-ring spectra. The obstructions live in the Andre-Quillen cohomology groups for graded θ-W_*-algebras:

$$H^s_{Alg^\theta_{W_*}}(A_*, M_*)$$

where $W_* = (K_p^\wedge)_* E$.

LEMMA 7.6. *Suppose that W_* and A_* are even periodic, and that A_0 is étale over W_0. Then for all s,*

$$H^s_{Alg^\theta_{W_*}}(A_*, M_*) = 0.$$

PROOF. Consider the spectral sequence

$$\mathrm{Ext}^s_{Mod^\theta_{A_*}}(H_t(\mathbb{L}(A_*/W_*)), M_*) \Rightarrow H^{s+t}_{Alg^\theta_{W_*}}(A_*, M_*).$$

Because A_* is étale over W_*, the cotangent complex is contractible, and the spectral sequence collapses to zero. □

We outline our construction of $\mathcal{O}^{top}_{K(1)}$:

Step 1: We will construct a $K(1)$-local E_∞-ring spectrum $tmf(p)^{ord}$. This will be our candidate for the spectrum of sections of $\mathcal{O}^{top}_{K(1)}$ over the étale cover

$$\begin{array}{c} \mathcal{M}^{ord}_{ell}(p) \\ \downarrow \gamma \\ \mathcal{M}^{ord}_{ell} \end{array}$$

This cover is Galois, with Galois group $(\mathbb{Z}/p)^\times$. We will show that there is a corresponding action of $(\mathbb{Z}/p)^\times$ on the spectrum $tmf(p)^{ord}$ by E_∞-ring maps. We will define $tmf_{K(1)}$ to be homotopy fixed points

$$tmf_{K(1)} := (tmf(p)^{ord})^{h(\mathbb{Z}/p)^\times}.$$

Step 2: We will construct the sheaf $\mathcal{O}^{top}_{K(1)}$ in the category of commutative $tmf_{K(1)}$-algebras.

We now give the details of our constructions.

Step 1: construction of $tmf_{K(1)}$.

Case 1: assume that p is odd.

Let $\mathcal{X}$ be the formal pullback

(7.4)
$$\begin{tikzcd} \mathcal{X} \arrow[r] \arrow[d] & \mathcal{M}^{ord}_{ell}(p^\infty) \arrow[d] \\ \mathcal{M}^{ord}_{ell}(p) \arrow[r] & \mathcal{M}^{ord}_{ell} \end{tikzcd}$$

For a p-complete ring R, the R-points of $\mathcal{X}$ are given by

$$\mathcal{X} = \{(C, \eta, \eta')\}$$

where the data is given by:

$$\begin{aligned} C & \quad \text{a generalized elliptic curve over } R, \\ \eta : \widehat{\mathbb{G}}_m \xrightarrow{\cong} \widehat{C} & \quad \text{an isomorphism of formal groups,} \\ \eta' : \mu_p \xrightarrow{\cong} \widehat{C}[p] & \quad \text{an isomorphism of finite group schemes.} \end{aligned}$$

Since $\mathcal{M}_{ell}^{ord}(p) = \mathrm{Spf}(V_1)$ is formally affine, we deduce that $\mathcal{X} = \mathrm{Spf}(W)$ for some ring W. Since $\mathcal{M}_{ell}^{ord}(p)$ is étale over $\mathcal{M}_{ell}^{ord}$, the ring W possesses a canonical θ-algebra structure extending that of $V_\infty^\wedge$. For $k \in \mathbb{Z}_p^\times$, the operations ψ^k are induced by the natural transformation on R-points:

$$\begin{aligned} (\psi^k)^* \mathcal{X}(R) &\to \mathcal{X}(R) \\ (C, \eta, \eta') &\mapsto (C, \eta \circ [k], \eta') \end{aligned}$$

The operation ψ^p is induced by the natural transformation

$$\begin{aligned} (\psi^p)^* \mathcal{X}(R) &\to \mathcal{X}(R) \\ (C, \eta, \eta') &\mapsto (C^{(p)}, \eta^{(p)}, (\eta')^{(p)}) \end{aligned}$$

Here, given η', the level structure $(\eta')^{(p)}$ is the one making the following diagram commute (see Remark 5.3).

$$\begin{tikzcd} & \mu_p \arrow[dl, dotted, "(\eta')^{(p)}"'] \arrow[d, "\eta'"] \\ \widehat{C^{(p)}}[p] \arrow[r, "(\Phi_{sep})_*"'] & \widehat{C}[p] \end{tikzcd}$$

Taking $\omega_{\infty,1}$ to be the canonical line bundle over $\mathcal{X}$, we can construct an evenly graded θ-algebra W_* as

$$W_{2*} := \Gamma \omega_{\infty,1}^{\otimes *}.$$

THEOREM 7.7. *There is a $(\mathbb{Z}/p)^\times$-equivariant, even periodic, $K(1)$-local E_∞-ring spectrum $tmf(p)^{ord}$ such that*

1. $\pi_0 tmf(p)^{ord} \cong V_1$,
2. *Letting $(\mathbf{C}_1, \boldsymbol{\eta}_1)$ be the universal tuple over $\mathcal{M}_{ell}^{ord}(p)$, there is an isomorphism of formal groups $\mathbb{G}_{tmf(p)^{ord}} \cong \widehat{\mathbf{C}}_1$.*
3. *There is an isomorphism of θ-algebras*
$$(K_p^\wedge)_* tmf(p)^{ord} \cong W_*.$$

PROOF. Observe the following.

1. W_* is concentrated in even degrees.
2. W is ind-etale over $W^{\mathbb{Z}_p^\times} = V_1$, and V_1 is smooth over $\mathbb{Z}_p$. This is because in the following pullback
$$\begin{tikzcd} \mathcal{X} \arrow[r] \arrow[d] & \mathcal{M}_{ell}^{ord}(p^\infty) \arrow[d] \\ \mathcal{M}_{ell}^{ord}(p) \arrow[r] & \mathcal{M}_{ell}^{ord} \end{tikzcd}$$
we have $\mathcal{M}_{ell}^{ord}(p^\infty)$ ind-etale over $\mathcal{M}_{ell}^{ord}$, thus $\mathcal{X} = \mathrm{Spf}(W)$ is ind-etale over $\mathcal{M}_{ell}^{ord}(p) = \mathrm{Spf}(V_1)$, and $\mathcal{M}_{ell}^{ord}(p)$ is smooth over $\mathrm{Spf}(\mathbb{Z}_p)$.

(3) $H_c^s(\mathbb{Z}_p^\times, W) = 0$ for $s > 0$. This is because W is is an ind-étale $\mathbb{Z}_p^\times$-torsor over V_1.

We deduce, from Lemma 7.5, that there exists a $K(1)$-local E_∞-ring spectrum $tmf(p)^{ord}$ such that we have an isomorphism

$$(K_p^\wedge)_* tmf(p)^{ord} \cong W_*$$

of graded θ-algebras. As a consequence of (3) above, we deduce that the spectral sequence (7.1) collapses to give an isomorphism

$$\pi_* tmf(p)^{ord} \cong (V_1)_*$$

where, if ω_1 is the canonical line bundle over $\mathcal{M}_{ell}^{ord}(p)$, then

$$(V_1)_{2*} = \Gamma\omega_1^{\otimes *}.$$

Let $(\mathbf{C}_1, \boldsymbol{\eta}_1)$ be the universal tuple over $\mathcal{M}_{ell}^{ord}(p)$. The existence of the isomorphism

$$\boldsymbol{\eta}_1 : \mu_p \xrightarrow{\cong} \widehat{\mathbf{C}}_1[p]$$

implies that ω_1 admits a trivialization. In particular, $tmf(p)^{ord}$ is even periodic.

We now show that the formal group of $\mathbb{G}_{tmf(p)^{ord}}$ is isomorphic to the formal group $\widehat{\mathbf{C}}_1$. Choose complex orientations Φ_K, $\Phi_{tmf(p)^{ord}}$ of K and $tmf(p)^{ord}$. Consider the following diagram.

$$\begin{tikzcd}
MUP_0 \arrow[r, "\Phi_{tmf(p)^{ord}}"] \arrow[d, "\eta_R"'] & \pi_0 tmf(p)^{ord} \arrow[r, equal] \arrow[d, "\eta_R"] & V_1 \arrow[d, hook] \\
MUP_0 MUP \arrow[r, "\Phi_K \wedge \Phi_{tmf(p)^{ord}}"'] & (K_p^\wedge)_0 tmf(p)^{ord} \arrow[r, equal] & W
\end{tikzcd}$$

The map $\Phi_K \wedge \Phi_{tmf(p)^{ord}}$ classifies an isomorphism of formal groups

$$\alpha : \eta_L^* \widehat{\mathbb{G}}_m \xrightarrow{\cong} \eta_R^* \mathbb{G}_{tmf(p)^{ord}}.$$

over W. At the same time, the universal tuple $(\mathbf{C}, \boldsymbol{\eta}, \boldsymbol{\eta}')$ over W has as part of its data an isomorphism of formal groups

$$\boldsymbol{\eta} : \widehat{\mathbb{G}}_m \xrightarrow{\cong} \widehat{\mathbf{C}}.$$

The generalized elliptic curve $\mathbf{C}$ over W is a pullback of the elliptic curve $\mathbf{C}_1$ over V_1 — thus it is invariant under the action of $\mathbb{Z}_p^\times$. The same holds for the formal group $\eta_R^* \mathbb{G}_{tmf(p)^{ord}}$ — it is tautologically the pullback of $\mathbb{G}_{tmf(p)^{ord}}$. Under the action of an element $k \in \mathbb{Z}_p^\times$, the isomorphisms α and $\boldsymbol{\eta}$ transform as

$$[k]^*\alpha = \alpha \circ [k],$$
$$[k]^*\boldsymbol{\eta} = \boldsymbol{\eta} \circ [k].$$

The isomorphism

$$\boldsymbol{\eta} \circ \alpha^{-1} : \eta_R^* \mathbb{G}_{tmf(p)^{ord}} \xrightarrow{\cong} \widehat{\mathbf{C}}$$

is therefore invariant under the action of $\mathbb{Z}_p^\times$. Thus it descends to an isomorphism

$$\alpha_1 : \mathbb{G}_{tmf(p)^{ord}} \xrightarrow{\cong} \widehat{\mathbf{C}}_1.$$

The Galois group $(\mathbb{Z}/p)^\times$ of $\mathcal{M}_{ell}^{ord}(p)$ over $\mathcal{M}_{ell}^{ord}$ acts on V_1. The last thing we need to show is that this action lifts to a point-set level action of $(\mathbb{Z}/p)^\times$ by E_∞-ring maps. Because W_* satisfies the hypotheses of Lemma 7.5, we may deduce from Theorem 7.1 that the K_p-Hurewicz map

$$[tmf(p)^{ord}, tmf(p)^{ord}]_{E_\infty} \to \mathrm{Hom}_{Alg_\theta}(W_*, W_*)$$

is an isomorphism. The action of $(\mathbb{Z}/p)^\times$ on V_1 lifts to W in an obvious way: on the R-points of $\mathrm{Spf}(W) = \mathcal{X}$, an element $[k] \in (\mathbb{Z}/p)^\times$ acts by

$$\begin{aligned} [k]^* : \mathcal{X}(R) &\to \mathcal{X}(R) \\ (C, \eta, \eta') &\mapsto (C, \eta, \eta' \circ [k]) \end{aligned}$$

This action is easily seen to commute with the action of ψ^l for $l \in \mathbb{Z}_p^\times$, and ψ^p. Thus $(\mathbb{Z}/p)^\times$ acts on W through maps of θ-algebras. We deduce that there is a map of groups

$$(\mathbb{Z}/p)^\times \to [tmf(p)^{ord}, tmf(p)^{ord}]_{E_\infty}^\times.$$

The obstructions to lifting this homotopy action to a point-set action may be identified using the obstruction theory of Cooke [**Coo**] (adapted to the topological category of E_∞-ring spectra). Namely, the obstructions lie in the group cohomology

$$H^s((\mathbb{Z}/p)^\times, \pi_{s-2}(E_\infty(tmf(p)^{ord}, tmf(p)^{ord}), \mathrm{Id})), \qquad s \geq 3.$$

Since the space $E_\infty(tmf(p)^{ord}, tmf(p)^{ord})$ is p-complete, and the order of the group $(\mathbb{Z}/p)^\times$ is prime to p, these obstructions must vanish. □

Define

$$tmf_{K(1)} := (tmf(p)^{ord})^{h(\mathbb{Z}/p)^\times}.$$

The following lemma is a useful corollary of a theorem of N. Kuhn.

LEMMA 7.8. *Suppose that G is a finite group which acts on a $K(n)$-local E_∞-ring spectrum E through E_∞-ring maps. Then the Tate spectrum E^{tG} is $K(n)$-acyclic, and the norm map*

$$N : E_{hG} \to E^{hG}$$

is a $K(n)$-local equivalence.

PROOF. Kuhn proves that the localized Tate spectrum $((S_{T(n)})^{tG})_{T(n)}$ is acyclic [**Kuh**, Thm. 1.5], where $T(n)$ is the telescope of a v_n-periodic map on a type n complex. The Tate spectrum $(E^{tG})_{K(n)}$ is an algebra spectrum over $((S_{T(n)})^{tG})_{T(n)}$. In particular, it is a module spectrum over an acyclic ring spectrum, and hence must be acyclic. □

LEMMA 7.9. *There is an isomorphism of θ-algebras $(K_p^\wedge)_* tmf_{K(1)} \cong (V_\infty^\wedge)_*$.*

PROOF. By Lemma 7.8, the natural map

$$(K_p \wedge (tmf(p)^{ord})^{h(\mathbb{Z}/p)^\times})_{K(1)} \to (K_p \wedge tmf(p)^{ord})_{K(1)}^{h(\mathbb{Z}/p)^\times}$$

is an equivalence (the homotopy fixed points are commuted past the smash product by changing them to homotopy orbits). The homotopy fixed point spectral sequence computing the homotopy groups of the latter collapses to give an isomorphism:

$$(V_\infty^\wedge)_* \cong (W_*)^{(\mathbb{Z}/p)^\times} \cong (K_p^\wedge)_* tmf_{K(1)}.$$

(The first isomorphism above comes from the fact that $\mathcal{X}$ is an étale $(\mathbb{Z}/p)^\times$-torsor over $\mathcal{M}_{ell}^{ord}(p^\infty)$.) $\square$

Case 2: $p = 2$.

If one were to try to duplicate the odd-primary argument, one would do the following: the first stack in the 2-primary Igusa tower which is formally affine is

$$\mathcal{M}_{ell}^{ord}(4) = \mathrm{Spf}(V_2).$$

The cover $\mathcal{M}_{ell}^{ord}(4) \xrightarrow{\gamma} \mathcal{M}_{ell}^{ord}$ is Galois with Galois group $(\mathbb{Z}/4)^\times$. One must begin by constructing the $K(1)$-local E_∞-ring spectrum $tmf(4)^{ord}$. One would like to use the obstruction theory of Cooke to make this spectrum $(\mathbb{Z}/4)^\times$-equivariant, but the order of the group is 2, so we cannot conclude that the obstructions vanish.

We instead replace K with KO. Define a graded *reduced θ-algebra* to be a graded θ-algebra over KO_* where the action of $\mathbb{Z}_2^\times$ is replaced with an action of $\mathbb{Z}_2^\times/\{\pm 1\}$.

Suppose that V is a θ-algebra, and that the subgroup $\{\pm 1\} \subset \mathbb{Z}_2^\times$ acts trivially on V. Then V may be regarded as a reduced θ-algebra. One may form a corresponding graded reduced θ-algebra by taking

$$W_* = KO_* \otimes V. \tag{7.5}$$

DEFINITION 7.10. We shall say that a graded reduced θ-algebra W_* is *Bott periodic* if it takes the form (7.5). We shall say that a $K(1)$-local E_∞ ring spectrum is *Bott periodic* if

(1) $(K_2^\wedge)_* E$ is torsion-free and concentrated in even degrees.
(2) The map $(KO_2^\wedge)_0 E \to (K_2^\wedge)_0 E$ is an isomorphism.

The relevance of this definition is given by the following lemma.

LEMMA 7.11. *Suppose that E is a Bott periodic $K(1)$-local E_∞-ring spectrum. Then we have*

$$(KO_2^\wedge)_* E \cong KO_* \otimes (K_2^\wedge)_0 E$$

In particular, the graded reduced θ-algebra $(KO_2^\wedge)_* E$ is Bott periodic. Conversely, if E is an E_∞ ring spectrum with

$$(KO_2^\wedge)_* E \cong KO_* \otimes V$$

as KO_*-modules, then E is Bott periodic, and $(K_2^\wedge)_0 E \cong V$.

PROOF. The first part uses the homotopy fixed point spectral sequence

$$H^s(\mathbb{Z}/2, (K_2^\wedge)_t E) \Rightarrow (KO_2^\wedge)_{t-s} E.$$

The second part follows easily from the Künneth spectral sequence. $\square$

REMARK 7.12. Both KO_2 and $tmf_{K(1)}$ (once we construct it) are Bott periodic.

Unfortunately the homology theory $KO_2^\wedge$ does not seem to satisfy all of the hypotheses required for the Goerss-Hopkins obstruction theory to apply. Nevertheless, when restricted to Bott periodic spectra with vanishing positive cohomology as a $\mathbb{Z}_2^\times/\{\pm 1\}$-module, it can be made to work. This is discussed in Appendix A. There it is shown that given a Bott periodic graded reduced θ-algebra W_* satisfying

$$H_c^s(\mathbb{Z}_2^\times/\{\pm 1\}, W_0) = 0 \text{ for } s > 0,$$

the obstructions to the existence of a $K(1)$-local E_∞-ring spectrum E with $(KO_2^\wedge)_*E \cong W_*$ lie in the cohomology groups

$$H^s_{Alg_\theta^{red}}(W_*, W_*[-s+2]), \quad s \geq 3.$$

Given Bott periodic $K(1)$-local E_∞-ring spectra E_1 and E_2, the obstructions to realizing a map of graded reduced θ-algebras

$$(KO_2^\wedge)_*E_1 \to (KO_2^\wedge)_*E_2$$

lie in

$$H^s_{Alg_\theta^{red}}((KO_2^\wedge)_*E_1, (KO_2^\wedge)_*E_2[-s+1]), \quad s \geq 2.$$

We have the following analog of Lemma 7.5.

Lemma 7.13. *Suppose that $A_*/(KO_2)_*$ is a graded 2-complete reduced θ-algebra, and that M_* is a graded 2-complete reduced θ-A_*-module. Let A_*^k denote the fixed points $A_*^{1+2^k\mathbb{Z}_2}$ ($A_*^0 = A_*^{\mathbb{Z}_2^\times}$). Note that we have*

$$A_* = \varprojlim_m \varinjlim_k A_*^k/p^m A_*^k.$$

Let $\bar{A}_*$ (respectively $\bar{A}_*^0$ and $\bar{M}_*$) denote the mod 2 reduction. Assume that:

1. *A_* and M_* are Bott periodic,*
2. *$\bar{A}_0^0$ is formally smooth over $\mathbb{F}_2$,*
3. *$H_c^s(\mathbb{Z}_2^\times/\{\pm 1\}, \bar{M}_0) = 0$ for $s > 0$,*
4. *$\bar{A}_0$ is ind-étale over $\bar{A}_0^0$.*

Then we have:

$$H^s_{Alg_\theta^{red}}(A_*, M_*[t]) = 0$$

if either $s > 1$ or $-t \equiv 3, 5, 6, 7 \mod 8$.

The following lemma is of crucial importance.

Lemma 7.14. *Let $V_\infty^\wedge$ be the representing ring for $\mathcal{M}_{ell}^{ord}(2^\infty)$ (a.k.a. the θ-algebra of generalized 2-adic modular functions).*

1. *The element $[-1] \in \mathbb{Z}_2^\times$ acts trivially on $V_\infty^\wedge$.*
2. *The subring $V_2 \subset V_\infty$ is isomorphic to the fixed points under the induced action of the group $\mathbb{Z}_2^\times/\{\pm 1\}$.*
3. *We have $H_c^s(\mathbb{Z}_2^\times/\{\pm 1\}, V_\infty^\wedge/2V_\infty^\wedge) = 0$ for $s > 0$.*

Proof. The stack $\mathcal{M}_{ell}^{ord}(2^\infty)$ represents pairs (η, C) where

$$\eta : \widehat{\mathbb{G}}_m \to \widehat{C}$$

is an isomorphism. However, we have $([-1]^*\eta, C) \cong (\eta, C)$:

$$\begin{tikzcd}
\widehat{\mathbb{G}}_m \arrow[r, "{[-1]}"] \arrow[rr, bend left, "{[-1]^*\eta}"] \arrow[drr, "\eta"'] & \widehat{\mathbb{G}}_m \arrow[r, "\eta"] & \widehat{C} \arrow[d, "{[-1]}"] \\
& & \widehat{C}
\end{tikzcd}$$

This proves (1). Under the isomorphism given by the composite

$$1+4\mathbb{Z}_2 \hookrightarrow \mathbb{Z}_2^\times \to \mathbb{Z}_2^\times/\{\pm 1\}$$

the action of the subgroup $1+4\mathbb{Z}_2$ agrees with the induced action of $\mathbb{Z}_2^\times/\{\pm 1\}$ on $V_\infty^\wedge$. But $V_\infty^\wedge/2V_\infty^\wedge$ is ind-Galois over $V_2/2V_2$ (the representing ring for $\mathcal{M}_{ell}^{ord}(4)\otimes\mathbb{F}_2$) with Galois group $1+4\mathbb{Z}_2$. This proves (2) and (3). $\square$

The algebra $V_\infty^\wedge/2V_\infty^\wedge$ is ind-etale over $V_2/2V_2$, and $\mathcal{M}_{ell}^{ord}(4)\otimes\mathbb{F}_2$ is smooth. Lemma 7.13 implies that the groups

$$H^s_{Alg_\theta^{red}}(KO_*\otimes V_\infty^\wedge, KO_*\otimes V_\infty^\wedge[u])$$

vanish for $s>1$ and $-u \equiv 3,5,6,7 \mod 8$. This is enough to deduce that there exists a $K(1)$-local E_∞-ring spectrum $tmf_{K(1)}$ such that there is an isomorphism of graded reduced θ-algebras

$$(KO_2^\wedge)_* tmf_{K(1)} \cong KO_*\otimes V_\infty^\wedge.$$

REMARK 7.15. There is another construction of $tmf_{K(1)}$ at $p=2$ which is described in [**Lau**] (see also [**Hop1**]). The spectrum is explicitly constructed by attaching two $K(1)$-local E_∞-cells to the $K(1)$-local sphere. Unfortunately, it seems that this approach does not generalize to primes $p\geq 5$, though it does work at $p=3$ as well [**Hop1**].

Step 2: construction of the presheaf $\mathcal{O}^{top}_{K(1)}$. We shall now construct the sections of a presheaf $\mathcal{O}^{top}_{K(1)}$ on $(\mathcal{M}_{ell}^{ord})_{et}$. By Remark 2.5, it suffices to produce the values of $\mathcal{O}^{top}_{K(1)}$ on étale formal affine opens of $\mathcal{M}_{ell}^{ord}$.

Let $\mathrm{Spf}(R) \xrightarrow{f} \mathcal{M}_{ell}^{ord}$ be an étale formal affine open. Consider the pullback:

$$\begin{tikzcd}
\mathrm{Spf}(W) \arrow[r, "f'"] \arrow[d] & \mathcal{M}_{ell}^{ord}(p^\infty) \arrow[d] \\
\mathrm{Spf}(R) \arrow[r, "f"'] & \mathcal{M}_{ell}^{ord}
\end{tikzcd}$$

Since f is étale, W is an étale $V_\infty^\wedge$-algebra, and W carries a canonical θ-algebra structure (Section 6). We have an associated even periodic graded θ-$(V_\infty^\wedge)_*$-algebra W_*.

The relative form of Theorem 7.1 indicates that the obstructions to the existence and uniqueness of a $K(1)$-local commutative $tmf_{K(1)}$-algebra E such that there is an isomorphism

$$(K_p^\wedge)_*E \cong W_*$$

of θ-$(V_\infty^\wedge)_*$-algebras lie in the Andre-Quillen cohomology groups

$$H^s_{Alg^\theta_{(V_\infty^\wedge)_*}}(W_*, W_*[u]).$$

These cohomology groups vanish by Lemma 7.6.

Given a map

$$g : \mathrm{Spf}(R_2) \to \mathrm{Spf}(R_1)$$

in $(\mathcal{M}_{ell}^{ord})_{et}$, we get an induced map

$$g^* : (W_1)_* \to (W_2)_*$$

of the corresponding θ-$(V_\infty^\wedge)_*$-algebras. Let E_1, E_2 be the corresponding $K(1)$-local commutative $tmf_{K(1)}$-algebras. The obstructions for existence and uniqueness of a map of $tmf_{K(1)}$-algebras

$$\tilde{g}^* : E_1 \to E_2$$

realizing the map g^* on K_p-homology lie in the groups

$$H^s_{Alg^\theta_{(V_\infty^\wedge)_*}}((W_1)_*, (W_2)_*[u]).$$

Furthermore, given the existence of $\tilde{g}^*$, there is a spectral sequence

$$H^s_{Alg^\theta_{(V_\infty^\wedge)_*}}((W_1)_*, (W_2)_*[u]) \Rightarrow \pi_{-u-s}(\mathrm{Alg}_{tmf_{K(1)}}(E_1, E_2), \tilde{g}^*).$$

Again, these cohomology groups all vanish by Lemma 7.6. We deduce that:

(1) The K_p-Hurewicz map

$$[E_1, E_2]_{Alg_{tmf_{K(1)}}} \to \mathrm{Hom}_{Alg^\theta_{(V_\infty^\wedge)_*}}((W_1)_*, (W_2)_*)$$

is an isomorphism.

(2) The mapping spaces $\mathrm{Alg}_{tmf_{K(1)}}(E_1, E_2)$ have contractible components.

We have constructed a functor

$$\bar{\mathcal{O}}^{top}_{K(1)} : ((\mathcal{M}^{ord}_{ell})_{et,aff})^{op} \to Ho(\text{Commutative } tmf_{K(1)}\text{-algebras}).$$

Since the mapping spaces are contractible, this functor lifts to give a presheaf (see [**DKS**])

$$\mathcal{O}^{top}_{K(1)} : ((\mathcal{M}^{ord}_{ell})_{et,aff})^{op} \to \text{Commutative } tmf_{K(1)}\text{-algebras}.$$

The same argument used to prove part (2) of Theorem 7.7 proves the following.

PROPOSITION 7.16. *Suppose that* $\mathrm{Spf}(R) \to \mathcal{M}^{ord}_{ell}$ *is an étale open classifying a generalized elliptic curve* C/R*. Then the associated spectrum of sections* $\mathcal{O}^{top}_{K(1)}$ *is an elliptic spectrum for the curve* C/R*.*

8. Construction of $\mathcal{O}^{top}_p$

To construct $\mathcal{O}^{top}_p$ it suffices to construct the map

$$\alpha_{chrom} : (i_{ord})_* \mathcal{O}^{top}_{K(1)} \to ((i_{ss})_* \mathcal{O}^{top}_{K(2)})_{K(1)}.$$

Our strategy will be to do this in two steps:

Step 1: We will construct

$$\alpha_{chrom} : tmf_{K(1)} \to (tmf_{K(2)})_{K(1)}$$

where

$$tmf_{K(2)} := \mathcal{O}^{top}_{K(2)}(\mathcal{M}^{ss}_{ell}).$$

Step 2: We will use the $K(1)$-local obstruction theory in the category of $tmf_{K(1)}$-algebra spectra to show that this map can be extended to a map of presheaves of spectra:

$$(\iota_{ord})_* \mathcal{O}^{top}_{K(1)} \to ((\iota_{ss})_* \mathcal{O}^{top}_{K(2)})_{K(1)}.$$

We will need the following lemma.

LEMMA 8.1. *Suppose that* C *is a generalized elliptic curve over a ring* R*, and that* E *is an elliptic spectrum associated with* C*. Then*

(1) *E is $E(2)$-local.*
(2) *Suppose that R is p-complete, and that the classifying map*

$$\mathrm{Spf}(R) \to (\overline{\mathcal{M}}_{ell})_p$$

for C is flat. Then there is an equivalence

$$E_{K(1)} \simeq E[v_1^{-1}]_p.$$

Proof. Greenlees and May [**GM**] proved that there is an equivalence

$$E_{E(n)} \simeq E[I_{n+1}^{-1}].$$

They also showed there is a spectral sequence

$$H^s(\mathrm{Spec}(R) - X_n, \omega^{\otimes t}) \Rightarrow \pi_{2t-s} E[I_{n+1}^{-1}] \tag{8.1}$$

where $X_n = \mathrm{Spec}(R/I_{n+1})$ is the locus of $\mathrm{Spec}(R/p)$ where the formal group of E has height greater than n. (1) therefore follows from the fact that $\widehat{C}$ never has height greater than 2. For (2), since R is assumed to be p-complete, there is an isomorphism

$$\pi_0(E[v_1^{-1}]_p) \cong R[v_1^{-1}]_p.$$

Over $R[v_1^{-1}]/pR[v_1^{-1}]$, the generalized elliptic curve C is ordinary, hence X_1 is empty and the spectral sequence (8.1) collapses to show that $E[v_1^{-1}]_p$ is $E(1)$-local. It is also p-complete by construction, and since $K(1)$-localization is the p-completion of $E(1)$-localization, we deduce that $E[v_1^{-1}]_p$ is $K(1)$-local. It therefore suffices to show that the map

$$E \to E[v_1^{-1}]_p$$

is a $K(1)$-equivalence. It suffices to show that it yields an equivalence on p-adic K-theory. However, by Proposition 6.1, both $(K_p^\wedge)_0 E$ and $(K_p^\wedge)_0(E[v_1^{-1}]_p)$ are given by W, where we have pullback squares:

$$\begin{tikzcd}
\mathrm{Spf}(W) \arrow[r] \arrow[d] & \mathrm{Spf}(R[v_1^{-1}]_p^\wedge) \arrow[r] \arrow[d] & \mathrm{Spf}(R) \arrow[d] \\
\mathcal{M}_{ell}^{ord}(p^\infty) \arrow[r] & \mathcal{M}_{ell}^{ord} \arrow[r] & (\overline{\mathcal{M}}_{ell})_p
\end{tikzcd}$$

□

Step 1: construction of $\boldsymbol{\alpha_{chrom} : tmf_{K(1)} \to (tmf_{K(2)})_{K(1)}}$.

We shall temporarily assume that p is odd. After we complete Step 1 for odd primes, we shall address the changes necessary for the prime 2.

Fix N to be a positive integer greater than or equal to 3 and coprime to p. Let $\mathcal{M}_{ell}(N)/\mathbb{Z}[1/N]$ denote the moduli stack of pairs (C, ρ) where C is an elliptic curve and ρ is a “full level N structure”:

$$\rho : (\mathbb{Z}/N)^2 \xrightarrow{\cong} C[N].$$

Since N is greater than 3, this stack is a scheme [**DR**, Cor. 2.9]. The cover

$$\mathcal{M}_{ell}(N) \to \mathcal{M}_{ell} \otimes \mathbb{Z}[1/N]$$

given by forgetting the level structure is an étale $GL_2(\mathbb{Z}/N)$-torsor. Let $\mathcal{M}_{ell}(N)_p$ denote the completion of $\mathcal{M}_{ell}(N)$ at p, and let $\mathcal{M}_{ell}^{ss}(N)$ denote the pullback

$$\begin{tikzcd} \mathcal{M}_{ell}^{ss}(N) \arrow[r] \arrow[d] & \mathcal{M}_{ell}(N)_p \arrow[d] \\ \mathcal{M}_{ell}^{ss} \arrow[r] & (\mathcal{M}_{ell})_p \end{tikzcd}$$

Since $\mathcal{M}_{ell}(N)_p$ is a formal scheme, $\mathcal{M}_{ell}^{ss}(N)$ is also a formal scheme. By Serre-Tate theory, the formal scheme $\mathcal{M}_{ell}^{ss}(N)$ is given by

$$\mathcal{M}_{ell}^{ss}(N) = \coprod_i \mathrm{Spf}(\mathbb{W}(k_i)[[u_1]])$$

for a finite set of finite fields $\{k_i\}$ (this set of finite fields depends on N). Let A_N denote the representing ring

$$A_N := \prod_i \mathbb{W}(k_i)[[u_1]]$$

and let B_N be the ring

$$B_N := A_N[u_1^{-1}]_p^\wedge = \prod_i \mathbb{W}(k_i)((u_1))_p^\wedge.$$

(Elements in the ring $\mathbb{W}(k_i)((u_1))_p^\wedge$ are bi-infinite Laurent series

$$\sum_{j\in\mathbb{Z}} a_j u_1^j$$

where we require that $a_j \to 0$ as $j \to -\infty$.) We shall use the notation

$$\mathcal{M}_{ell}^{ss}(N) = \mathrm{Spf}_{(p,u_1)}(A_N)$$

to indicate that Spf is taken with respect to the ideal of definition (p, u_1). Define $\mathcal{M}_{ell}^{ss}(N)^{ord}$ to be the formal scheme given by

$$\mathcal{M}_{ell}^{ss}(N)^{ord} = \mathrm{Spf}_{(p)}(B_N).$$

Let $(C_N^{ss}, \eta_N^{ss})/\mathcal{M}_{ell}^{ss}(N)$ be the elliptic curve with full level structure classified by the map

$$\mathcal{M}_{ell}^{ss}(N) \to \mathcal{M}_{ell}(N).$$

We regard $\mathcal{M}_{ell}^{ss}(N)^{ord}$ as the "ordinary locus" of C_N^{ss}. This does not actually make sense in the context of formal schemes — $\mathcal{M}_{ell}^{ss}(N)^{ord}$ is *not* a formal subscheme of $\mathcal{M}_{ell}^{ss}(N)$. Nevertheless, by Remark 1.6, there is a canonical elliptic curve (with level structure) $((C_N^{ss})^{alg}, \eta_N^{ss})$ which lies over $\mathcal{M}_{ell}^{ss}(N)^{alg} := \mathrm{Spec}(A_N)$, and restricts to $C_N^{ss}/\mathrm{Spf}_{(p,u_1)}(A_N)$. The formal scheme $\mathcal{M}_{ell}^{ss}(N)^{ord}$ is given by the pullback

$$\begin{tikzcd} \mathcal{M}_{ell}^{ss}(N)^{ord} \arrow[r] \arrow[d] & \mathcal{M}_{ell}^{ord} \arrow[d] \\ \mathcal{M}_{ell}^{ss}(N)^{alg} \arrow[r] & \overline{\mathcal{M}}_{ell} \end{tikzcd}$$

We let $((C_N^{ss})^{ord}, \eta_N^{ss})$ denote the restriction of the pair $((C_N^{ss})^{alg}, \eta_N^{ss})$ to $\mathcal{M}_{ell}^{ss}(N)^{ord}$. We define $\mathcal{M}_{ell}^{ss}(N,p)^{ord}$ to be the pullback

$$\begin{tikzcd} \mathcal{M}_{ell}^{ss}(N,p)^{ord} \arrow[r] \arrow[d] & \mathcal{M}_{ell}^{ord}(p) \arrow[d] \\ \mathcal{M}_{ell}^{ss}(N)^{ord} \arrow[r] & \mathcal{M}_{ell}^{ord} \end{tikzcd}$$

and denote the pullback of $(C_N^{ss})^{ord}$ to $\mathcal{M}_{ell}^{ss}(N,p)^{ord}$ by $(C_{N,1}^{ss})^{ord}$. Since $\mathcal{M}_{ell}^{ss}(N)^{ord}$ and $\mathcal{M}_{ell}^{ord}(p)$ are formally affine, we deduce that $\mathcal{M}_{ell}^{ss}(N,p)^{ord}$ is formally affine, and is of the form $\mathrm{Spf}_{(p)}(B_{N,1})$.

Let $\mathcal{M}_{ell}^{ord}(p)^{ns}$ denote the locus of the formal affine scheme $\mathcal{M}_{ell}^{ord}(p)$ where the universal curve is nonsingular; it is covered by an étale $GL_2(\mathbb{Z}/N)$-torsor given by the pullback

$$\begin{tikzcd} \mathcal{M}_{ell}^{ord}(N,p)^{ns} \arrow[r] \arrow[d] & \mathcal{M}_{ell}(N)_p \arrow[d] \\ \mathcal{M}_{ell}^{ord}(p)^{ns} \arrow[r] & (\mathcal{M}_{ell})_p \end{tikzcd}$$

The action of $GL_2(\mathbb{Z}/N)$ on the formal affine scheme $\mathcal{M}_{ell}^{ss}(N,p)^{ord} = \mathrm{Spf}_{(p)}(B_{N,1})$ over $\mathcal{M}_{ell}^{ord}(N,p)^{ns}$, gives descent data which, by faithfully flat descent (see, for instance, [**Hid1**, Sec. 1.11.3]), yields a new formal affine scheme

$$\mathcal{M}_{ell}^{ss}(p)^{ord} = \mathrm{Spf}_{(p)}(B_1)$$

over $\mathcal{M}_{ell}^{ord}(p)^{ns}$ (where $B_1 = B_{N,1}^{GL_2(\mathbb{Z}/N)}$) together with a pullback diagram

$$\begin{tikzcd} \mathcal{M}_{ell}^{ss}(N,p)^{ord} \arrow[r] \arrow[d] & \mathcal{M}_{ell}^{ss}(p)^{ord} \arrow[d] \\ \mathcal{M}_{ell}^{ord}(N,p)^{ns} \arrow[r] & \mathcal{M}_{ell}^{ord}(p)^{ns} \end{tikzcd}$$

Define $(V_\infty^\wedge)^{ss}$ to be the pullback

$$\begin{tikzcd} \mathrm{Spf}_{(p)}((V_\infty^\wedge)^{ss}) \arrow[r] \arrow[d] & \mathcal{M}_{ell}^{ord}(p^\infty) \arrow[d] \\ \mathcal{M}_{ell}^{ss}(p)^{ord} \arrow[r] & \mathcal{M}_{ell}^{ord}(p) \end{tikzcd}$$

and define W^{ss} and $\tilde{W}^{ss}$ to be the pullbacks

$$\begin{tikzcd} \mathrm{Spf}_{(p)}(\tilde{W}^{ss}) \arrow[r] \arrow[d] & \mathrm{Spf}_{(p)}(W^{ss}) \arrow[r] \arrow[d] & \mathrm{Spf}_{(p)}((V_\infty^\wedge)^{ss}) \arrow[d] \\ \mathcal{M}_{ell}^{ord}(N,p)^{ns} \arrow[r] & \mathcal{M}_{ell}^{ord}(p)^{ns} \arrow[r] & (\mathcal{M}_{ell}^{ord})^{ns} \end{tikzcd} \tag{8.2}$$

By faithfully flat descent, we have

$$W^{ss} = (\tilde{W}^{ss})^{GL_2(\mathbb{Z}/N)},$$

$$(V_\infty^\wedge)^{ss} = (W^{ss})^{(\mathbb{Z}/p)^\times}.$$

Remark 8.2. Both $\tilde{W}^{ss}$ and W^{ss} possess alternative descriptions. They are given by pullbacks

$$\begin{tikzcd} \mathrm{Spf}(\tilde{W}^{ss}) \arrow[r] \arrow[d] & \mathrm{Spf}(W^{ss}) \arrow[r] \arrow[d] & \mathcal{M}_{ell}^{ord}(p^\infty) \arrow[d] \\ \mathcal{M}_{ell}^{ss}(N,p)^{ord} \arrow[r] & \mathcal{M}_{ell}^{ss}(p)^{ord} \arrow[r] & \mathcal{M}_{ell}^{ord} \end{tikzcd}$$

Let $tmf(N)_{K(2)}$ be the spectrum of sections

$$tmf(N)_{K(2)} := \mathcal{O}^{top}_{K(2)}(\mathcal{M}^{ss}_{ell}(N)).$$

The action of $GL_2(\mathbb{Z}/N)$ on the torsor $\mathcal{M}^{ss}_{ell}(N)$ induces an action of $GL_2(\mathbb{Z}/N)$ on $tmf(N)_{K(2)}$. Since the sheaf $\mathcal{O}^{top}_{K(2)}$ satisfies homotopy decent, we have

$$(tmf(N)_{K(2)})^{hGL_2(\mathbb{Z}/N)} \simeq tmf_{K(2)}.$$

Lemma 8.3. *There is an equivalence*

$$((tmf(N)_{K(2)})_{K(1)})^{hGL_2(\mathbb{Z}/N)} \simeq (tmf_{K(2)})_{K(1)}.$$

Proof. Using Lemma 7.8, and descent, we may deduce that there are equivalences

$$\begin{aligned} ((tmf(N)_{K(2)})_{K(1)})^{hGL_2(\mathbb{Z}/N)} &\simeq ((tmf_{K(2)})^{hGL_2(\mathbb{Z}/N)})_{K(1)} \\ &\simeq (tmf_{K(2)})_{K(1)}. \end{aligned}$$

$\square$

Consider the finite $(\mathbb{Z}/p)^\times$ Galois extension $E_1^{h(1+p\mathbb{Z}_p)}$ of $S_{K(1)}$ given by the homotopy fixed points of E_1-theory with respect to the open subgroup $1+p\mathbb{Z}_p \subset \mathbb{Z}_p^\times$ (see [**DH**], [**Rog**]). Note that we have

$$(E_1^{h(1+p\mathbb{Z}_p)})^{h(\mathbb{Z}/p)^\times} \simeq S_{K(1)}. \tag{8.3}$$

Define spectra

$$\begin{aligned} (tmf(N,p)_{K(2)})_{K(1)} &:= (tmf(N)_{K(2)})_{K(1)} \wedge_{S_{K(1)}} E_1^{h(1+p\mathbb{Z}_p)} \\ (tmf(p)_{K(2)})_{K(1)} &:= (tmf_{K(2)})_{K(1)} \wedge_{S_{K(1)}} E_1^{h(1+p\mathbb{Z}_p)} \end{aligned}$$

These spectra inherit an action by the group $(\mathbb{Z}/p)^\times = \mathbb{Z}_p^\times/1+p\mathbb{Z}_p$.

Using Lemma 7.8, Lemma 8.3 and Equation (8.3), we have the following.

Lemma 8.4. *There are equivalences of E_∞-ring spectra*

$$\begin{aligned} ((tmf(N,p)_{K(2)})_{K(1)})^{hGL_2(\mathbb{Z}/N)} &\simeq (tmf(p)_{K(2)})_{K(1)} \\ ((tmf(p)_{K(2)})_{K(1)})^{h(\mathbb{Z}/p)^\times} &\simeq (tmf_{K(2)})_{K(1)} \end{aligned}$$

We now link up some homotopy calculations with our previous algebro-geometric constructions.

Lemma 8.5. *There is an $GL_2(\mathbb{Z}/N) \times (\mathbb{Z}/p)^\times$-equivariant isomorphism*

$$\pi_0(tmf(N,p)_{K(2)})_{K(1)} \cong B_{N,1}$$

making $(tmf(N,p)_{K(2)})_{K(1)}$ an elliptic spectrum with associated elliptic curve $(C^{ss}_{N,1})^{ord}$.

Proof. By construction, there is a $GL_2(\mathbb{Z}/N)$-equivariant isomorphism

$$\pi_0 tmf(N)_{K(2)} \cong A_N$$

making $tmf(N)_{K(2)}$ an elliptic spectrum with associated elliptic curve C_N^{ss}. By Lemma 8.1, this gives rise to an isomorphism

$$\pi_0(tmf(N)_{K(2)})_{K(1)} \cong B_N$$

making the pair $((tmf(N)_{K(2)})_{K(1)}, (C_N^{ss})^{ord})$ an elliptic spectrum. For any $K(1)$-local even periodic Landweber exact cohomology theory E, the homotopy groups of

$$E' = E \wedge_{S_{K(1)}} E_1^{h(1+p\mathbb{Z}_p)}$$

are given by the pullback

$$\begin{tikzcd} \mathrm{Spf}(\pi_0 E') \arrow[r] \arrow[d] & \mathcal{M}_{FG}^{mult}(p) \arrow[d] \\ \mathrm{Spf}(\pi_0 E) \arrow[r] & \mathcal{M}_{FG}^{mult} \end{tikzcd}$$

(where the notation here is the same as in the proof of Lemma 5.1). This is easily deduced from the cofiber sequence

$$E' \to (K_p \wedge E)_p \xrightarrow{\psi^k - 1} (K_p \wedge E)_p$$

where k is chosen to be a topological generator of the subgroup $1 + \mathbb{Z}_p \subseteq \mathbb{Z}_p^\times$. In particular, we have the desired isomorphism

$$\pi_0(tmf(N,p)_{K(2)})_{K(1)} \simeq B_{N,1}.$$

The formal group of E' is the pullback of the formal group of E along the map $\pi_0 E \to \pi_0 E'$. The elliptic curve $(C_{N,1}^{ss})^{ord}$ is the pullback of $(C_N^{ss})^{ord}$ under the same homomorphism. The canonical isomorphism between the formal group of E and the formal group of $(C_N^{ss})^{ord}$ thus pulls back to give the required isomorphism between the formal group of E' and the formal group of $(C_{N,1}^{ss})^{ord}$. □

Lemma 8.6. *There are isomorphisms*

$$\begin{aligned} (K_p^\wedge)_*(tmf(N,p)_{K(2)})_{K(1)} &\cong (K_p)_* \otimes_{\mathbb{Z}_p} \tilde{W}^{ss} \\ (K_p^\wedge)_*(tmf(p)_{K(2)})_{K(1)} &\cong (K_p)_* \otimes_{\mathbb{Z}_p} W^{ss} \\ (K_p^\wedge)_*(tmf_{K(2)})_{K(1)} &\cong (K_p)_* \otimes_{\mathbb{Z}_p} (V_\infty^\wedge)^{ss} \end{aligned}$$

(We shall denote these graded objects as $\tilde{W}_*^{ss}$, W_*^{ss}, and $(V_\infty^\wedge)_*^{ss}$, respectively.)

Proof. We deduce the first isomorphism by combining Proposition 6.1 with Remark 8.2. Using Lemma 8.4, and Lemma 7.8, we have equivalences

$$\begin{aligned} ((K_p \wedge (tmf(N,p)_{K(2)})_{K(1)})_p)^{hGL_2(\mathbb{Z}/N)} &\simeq (K_p \wedge (tmf(p)_{K(2)})_{K(1)})_p \\ ((K_p \wedge (tmf(p)_{K(2)})_{K(1)})_p)^{h(\mathbb{Z}/p)^\times} &\simeq (K_p \wedge (tmf(p)_{K(2)})_{K(1)})_p \end{aligned}$$

The pullback diagram (8.2) implies that $\tilde{W}^{ss}$ is an étale $GL_2(\mathbb{Z}/N)$-torsor over W^{ss}, and W^{ss} is an étale $(\mathbb{Z}/p)^\times$-torsor over $(V_\infty^\wedge)^{ss}$. The resulting homotopy fixed point spectral sequence

$$H^*(GL_2(\mathbb{Z}/N), (K_p^\wedge)_*(tmf(N,p)_{K(2)})_{K(1)}) \Rightarrow (K_p^\wedge)_*(tmf(p)_{K(2)})_{K(1)}$$

therefore collapses to give the required isomorphism

$$(K_p^\wedge)_*(tmf(p)_{K(2)})_{K(1)} \cong (\tilde{W}^{ss})_*^{GL_2(\mathbb{Z}/N)} = W_*^{ss}.$$

This in turn allows us to conclude that the homotopy fixed point spectral sequence

$$H^*((\mathbb{Z}/p)^\times, (K_p^\wedge)_*(tmf(p)_{K(2)})_{K(1)}) \Rightarrow (K_p^\wedge)_*(tmf_{K(2)})_{K(1)}$$

collapses to give the isomorphism

$$(K_p^\wedge)_*(tmf_{K(2)})_{K(1)} \cong (W_*^{ss})^{(\mathbb{Z}/p)^\times} = (V_\infty^\wedge)_*^{ss}.$$

$\square$

The universal property of the pullback, together with the diagram of Remark 8.2, gives a $(\mathbb{Z}/p)^\times$-equivariant map $\tilde{\alpha}^*$:

$$\begin{tikzcd}
\mathrm{Spf}(W^{ss}) \arrow[d] \arrow[dr, dotted, "\tilde{\alpha}^*"] \arrow[drr, bend left] & & \\
\mathcal{M}_{ell}^{ss}(p)^{ord} \arrow[dr] & \mathrm{Spf}(W) \arrow[r] \arrow[d] & \mathcal{M}_{ell}^{ord}(p^\infty) \arrow[d] \\
& \mathcal{M}_{ell}^{ord}(p) \arrow[r] & \mathcal{M}_{ell}^{ord}
\end{tikzcd}$$

Here, $\mathrm{Spf}(W) = \mathcal{X}$ is the pro-Galois cover of $\mathcal{M}_{ell}^{ord}(p)$ given by Diagram (7.4).

To construct our desired map

$$\alpha_{chrom} : tmf_{K(1)} \to (tmf_{K(2)})_{K(1)}$$

it suffices to construct a $(\mathbb{Z}/p)^\times$-equivariant map

$$\alpha'_{chrom} : tmf(p)_{K(1)} \to (tmf(p)_{K(2)})_{K(1)}.$$

The map α_{chrom} is then recovered by taking homotopy fixed point spectra.

The map $\tilde{\alpha}^*$ induces a map

$$\tilde{\alpha} : W_* \to W_*^{ss}$$

of graded θ-algebras. The obstructions to the existence of a map of $K(1)$-local E_∞-ring spectra

$$\alpha'_{chrom} : tmf(p)_{K(1)} \to (tmf(p)_{K(2)})_{K(1)}$$

inducing the map $\tilde{\alpha}$ on p-adic K-theory lie in:

$$H^s_{Alg_\theta}(W_*, W_*^{ss}[-s+1]) \qquad s \geq 2.$$

These groups are seen to vanish using Lemma 7.5. The obstructions to uniqueness (that is, uniqueness up to homotopy) lie in

$$H^s_{Alg_\theta}(W_*, W_*^{ss}[-s]) \qquad s \geq 1,$$

and these groups are also zero. Because $\tilde{\alpha}$ is $(\mathbb{Z}/p)^\times$-equivariant, we deduce that the map α'_{chrom} commutes with the action of $(\mathbb{Z}/p)^\times$ *in the homotopy category of E_∞-ring spectra*. Because we are working in an injective diagram model category structure, after performing a suitable fibrant replacement of $(tmf(p)_{K(2)})_{K(1)}$, there is an equivalence of (derived) mapping spaces

$$E_\infty(tmf(p)_{K(1)}, (tmf(p)_{K(2)})_{K(1)})_{(\mathbb{Z}/p)^\times\text{-equivariant}} \simeq E_\infty(tmf(p)_{K(1)}, (tmf(p)_{K(2)})_{K(1)})^{h(\mathbb{Z}/p)^\times}.$$

Because the order of $(\mathbb{Z}/p)^\times$ is prime to p, the spectral sequence

$$H^s((\mathbb{Z}/p)^\times, \pi_t E_\infty(tmf(p)_{K(1)}, (tmf(p)_{K(2)})_{K(1)})) \Rightarrow \pi_{t-s} E_\infty(tmf(p)_{K(1)}, (tmf(p)_{K(2)})_{K(1)})^{h(\mathbb{Z}/p)^\times}$$

collapses to show that the natural map

$$[tmf(p)_{K(1)}, (tmf(p)_{K(2)})_{K(1)}]_{\substack{E_\infty \\ (\mathbb{Z}/p)^\times-equivariant}} \to [tmf(p)_{K(1)}, (tmf(p)_{K(2)})_{K(1)}]^{(\mathbb{Z}/p)^\times}_{E_\infty}$$

is an isomorphism. In particular, we may choose α'_{chrom} to be a $(\mathbb{Z}/p)^\times$-equivariant map of E_∞-ring spectra.

Modifications for the prime 2.

At the prime 2, the first stage of the Igusa tower which is a formal affine scheme is $\mathcal{M}^{ord}_{ell}(4)$. All of the algebro-geometric constructions such as $\mathcal{M}^{ss}_{ell}(N,p)^{ord}$, $\mathcal{M}^{ss}_{ell}(p)^{ord}$, etc for p an odd prime go through for the prime 2 with $\mathcal{M}^{ord}_{ell}(p)$ replaced by $\mathcal{M}^{ord}_{ell}(4)$ to produce formal affine schemes $\mathcal{M}^{ss}_{ell}(N,4)^{ord}$ and $\mathcal{M}^{ss}_{ell}(4)^{ord}$. One then defines $(V^\wedge_\infty)^{ss}$ as the pullback

$$\begin{tikzcd} \mathrm{Spf}((V^\wedge_\infty)^{ss}) \arrow[r, "\alpha^*"] \arrow[d] & \mathcal{M}^{ord}_{ell}(2^\infty) \arrow[d] \\ \mathcal{M}^{ss}_{ell}(4)^{ord} \arrow[r] & \mathcal{M}^{ord}_{ell}(4) \end{tikzcd} \tag{8.4}$$

Define

$$\begin{aligned} (tmf(N)_{K(2)})_{K(1)} &:= (\mathcal{O}^{top}_{K(2)}(\mathcal{M}^{ss}_{ell}(N)) \\ (tmf(N,4)_{K(2)})_{K(1)} &:= (tmf(N)_{K(2)})_{K(1)} \wedge_{S_{K(1)}} E_1^{h(1+4\mathbb{Z}_2)} \\ (tmf(4)_{K(2)})_{K(1)} &:= (tmf_{K(2)})_{K(1)} \wedge_{S_{K(1)}} E_1^{h(1+4\mathbb{Z}_2)} \end{aligned}$$

Just as in the odd primary case, argue (in this order) that we have

$$\begin{aligned} (K_2^\wedge)_0(tmf(N,4)_{K(2)})_{K(1)} &\cong \tilde{W}^{ss} \\ (K_2^\wedge)_0(tmf(4)_{K(2)})_{K(1)} &\cong W^{ss} \\ (K_2^\wedge)_0(tmf_{K(2)})_{K(1)} &\cong (V^\wedge_\infty)^{ss} \end{aligned}$$

where $\tilde{W}^{ss}$ and W^{ss} are given as the pullbacks

$$\begin{tikzcd} \mathrm{Spf}(\tilde{W}^{ss}) \arrow[r] \arrow[d] & \mathrm{Spf}(W^{ss}) \arrow[r] \arrow[d] & \mathrm{Spf}((V^\wedge_\infty)^{ss}) \arrow[d] \\ \mathcal{M}^{ord}_{ell}(N,4)^{ns} \arrow[r] & \mathcal{M}^{ord}_{ell}(4)^{ns} \arrow[r] & (\mathcal{M}^{ord}_{ell})^{ns} \end{tikzcd}$$

Note that the homotopy groups of $(tmf_{K(2)})_{K(1)}$ are easily computed by inverting c_4 in the homotopy fixed point spectral sequence for EO_2:

$$\pi_*(tmf_{K(2)})_{K(1)} = KO_*((j^{-1}))^\wedge_2.$$

It follows that the hypotheses of Lemma 7.11 are satisfied, and we have an isomorphism

$$(KO_2^\wedge)_*(tmf_{K(2)})_{K(1)} \cong KO_2 \otimes_{\mathbb{Z}_2} (V^\wedge_\infty)^{ss}.$$

The map α^* of Equation (8.4) induces a map

$$\alpha : KO_* \otimes V_\infty^\wedge \to KO_* \otimes (V_\infty^\wedge)^{ss}$$

of graded reduced Bott periodic θ-algebras. The obstructions to the existence of a map of $K(1)$-local E_∞-ring spectra

$$\alpha_{chrom} : tmf_{K(1)} \to (tmf_{K(2)})_{K(1)}$$

inducing the map α on 2-adic KO-theory lie in:

$$H^s_{Alg_\theta^{red}}(KO_* \otimes V_\infty^\wedge, KO_* \otimes (V_\infty^\wedge)^{ss}[-s+1]) \qquad s \geq 2.$$

These groups are seen to vanish using Lemmas 7.11 and 7.13.

Step 2: construction of α_{chrom} as a map of presheaves over $\overline{\mathcal{M}}_{ell}$.

We will now construct a map of presheaves

$$\alpha_{chrom} : (\iota_{ord})_* \mathcal{O}^{top}_{K(1)} \to ((\iota_{ss})_* \mathcal{O}^{top}_{K(2)})_{K(1)}.$$

By the results of Section 2, it suffices to construct this map on the sections of formal affine étale opens of $\overline{\mathcal{M}}_{ell}$.

Let R be a p-complete ring, and let

$$\mathrm{Spf}_{(p)}(R) \to (\overline{\mathcal{M}}_{ell})_p$$

be a formal affine étale open, classifying a generalized elliptic curve C/R. Let ω_R be the pullback of the line bundle ω over $\overline{\mathcal{M}}_{ell}$. The invertible sheaf corresponds to an invertible R-module I. Let R_* denote the evenly graded ring where

$$R_{2t} = I^{\otimes_R t}.$$

Consider the pullbacks:

$$\text{(8.5)} \qquad \begin{tikzcd} \mathrm{Spf}(R^{ord}) \arrow[r] \arrow[d] & \mathrm{Spf}(R) \arrow[d] \\ \mathcal{M}_{ell}^{ord} \arrow[r] & (\overline{\mathcal{M}}_{ell})_p \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Spf}(R^{ss}) \arrow[r] \arrow[d] & \mathrm{Spf}(R) \arrow[d] \\ \mathcal{M}_{ell}^{ss} \arrow[r] & (\overline{\mathcal{M}}_{ell})_p \end{tikzcd}$$

REMARK 8.7. It is not immediately clear why these pullbacks are formal affine schemes.

(1) The pullback of $\mathrm{Spf}(R)$ over $\mathcal{M}_{ell}^{ord}$ is a formal affine scheme because the Hasse invariant can be regarded as a section of the restriction of the line bundle $\omega_R^{\otimes p-1}$ to $\mathrm{Spec}(R/p)$. Indeed, if $v_1 \in I^{\otimes_R (p-1)}$ is a lift of the Hasse invariant, then R^{ord} is the zeroth graded piece of the graded ring

$$R_*^{ord} := (R_*)[v_1^{-1}]_p^\wedge.$$

(2) The pullback of $\mathrm{Spf}(R)$ over $\mathcal{M}_{ell}^{ss}$ is formally affine because, by Serre-Tate theory, and the fact that the classifying map is étale, we know that

$$R^{ss} \cong \prod_i W(k_i)[[u_1]],$$

where $\{k_i\}$ is a finite set of finite fields. In Diagram (8.5), $\mathrm{Spf}(R^{ss})$ is taken with respect to the ideal $(p, u_1) \subset R^{ss}$, while $\mathrm{Spf}(R)$ is taken with respect to the ideal $(p) \subset R$. The ring R^{ss} has an alternative characterization: it is the zeroth graded piece of the completion

$$R^{ss}_* := (R_*)^\wedge_{(v_1)}.$$

Define

$$(R^{ss})^{ord}_* := (R^{ss}_*[v_1^{-1}])^\wedge_p$$

and let $(R^{ss})^{ord} \cong R^{ss}[u_1^{-1}]^\wedge_p$ be the zeroth graded piece. Define generalized elliptic curves:

$$\begin{aligned} C^{ord} &= C \otimes_R R^{ord} \\ C^{ss} &= C \otimes_R R^{ss} \\ (C^{ss})^{ord} &= C^{ss} \otimes_{R^{ss}} (R^{ss})^{ord} \end{aligned}$$

Since the image of v_1 is invertible in $(R^{ss})^{ord}_*$, the curve $(C^{ss}_R)^{ord}$ has ordinary reduction modulo p, and there exists a factorization

(8.6)
$$\begin{tikzcd} R_* \arrow[r] \arrow[d] & R^{ss}_* \arrow[r] & (R^{ss})^{ord}_* \\ R^{ord}_* \arrow[urr, dotted, "\bar{g}"'] & & \end{tikzcd}$$

We have $K(1)$-local E_∞-ring spectra:

$$\begin{aligned} E^{ord} &:= (\iota_{ord})_* \mathcal{O}^{top}_{K(1)}(\mathrm{Spf}(R)), \\ E^{ss} &:= (\iota_{ss})_* \mathcal{O}^{top}_{K(2)}(\mathrm{Spf}(R)), \\ (E^{ss})^{ord} &:= E^{ss}_{K(1)}. \end{aligned}$$

Combining Propositions 4.4 and 7.16 with Lemma 8.1, we have the following.

Lemma 8.8. *The spectra E^{ord}, E^{ss}, and $(E^{ss})^{ord}$ are elliptic with respect to the generalized elliptic curves C^{ord}/R^{ord}, C^{ss}/R^{ss}, and $(C^{ss})^{ord}/(R^{ss})^{ord}$, respectively.*

Consider the pullbacks

$$\begin{tikzcd} \mathrm{Spf}((W_{ss})^{ord}) \arrow[r, "g"] \arrow[d] & \mathrm{Spf}(W^{ord}) \arrow[r] \arrow[d] & \mathcal{M}^{ord}_{ell}(p^\infty) \arrow[d] \\ \mathrm{Spf}((R_{ss})^{ord}) \arrow[r, "\bar{g}"'] & \mathrm{Spf}(R^{ord}) \arrow[r] & \mathcal{M}^{ord}_{ell} \end{tikzcd}$$

We have, by Proposition 6.1, the following isomorphisms of graded θ-$(V^\wedge_\infty)_*$-algebras:

$$\begin{aligned} (K^\wedge_p)_* E^{ord} &\cong W^{ord}_* \\ (K^\wedge_p)_* (E_{ss})^{ord} &\cong (W_{ss})^{ord}_* \end{aligned}$$

where W^{ord}_* and $(W_{ss})^{ord}_*$ are the even periodic graded θ-algebras associated to the θ-algebras W^{ord} and $(W_{ss})^{ord}$.

We wish to construct a map:

$$\begin{tikzcd}
tmf_{K(1)} \arrow[r, "\alpha_{chrom}"] \arrow[d] & (tmf_{K(2)})_{K(1)} \arrow[d] \\
E^{ord} \arrow[r, dotted, "\alpha_{chrom}"'] & (E_{ss})^{ord}
\end{tikzcd}$$

The map g induces a map of graded θ-$(V_\infty^\wedge)_*$-algebras

$$g : W_*^{ord} \to (W_{ss})_*^{ord}.$$

The obstructions to realizing this map to the desired map

$$\alpha_{chrom} : E^{ord} \to (E_{ss})^{ord}$$

of $K(1)$-local commutative $tmf_{K(1)}$-algebras lie in

$$H^s_{Alg^\theta_{(V_\infty^\wedge)_*}}(W_*^{ord}, (W_{ss})_*^{ord}[-s+1]), \qquad s > 1.$$

Because W^{ord} is étale over $V_\infty^\wedge$, Lemma 7.6 implies that these obstruction groups all vanish. Thus the realization α_{chrom} exists.

Suppose that we are given a pair of étale formal affine opens

$$\mathrm{Spf}(R_i) \to \overline{\mathcal{M}}_{ell}, \qquad i = 1, 2.$$

Associated to these are $K(1)$-local commutative $tmf_{K(1)}$-algebras

$$E_i^{ord} := (\iota_{ord})_* \mathcal{O}^{top}_{K(1)}(\mathrm{Spf}(R_i)),$$
$$(E_{i,ss})^{ord} := (\iota_{ss})_* \mathcal{O}^{top}_{K(2)}(\mathrm{Spf}(R_i))_{K(1)}.$$

and graded θ-$(V_\infty^\wedge)_*$-algebras

$$(K_p^\wedge)_* E_i^{ord} \cong (W_i)_*^{ord},$$
$$(K_p^\wedge)_*(E_{i,ss})^{ord} \cong (W_{i,ss})_*^{ord}.$$

Again, Lemma 7.6 implies that

$$H^s_{Alg^\theta_{(V_\infty^\wedge)_*}}((W_1)_*^{ord}, (W_{2,ss})_*^{ord}[u]) = 0.$$

We deduce that

(1) the Hurewicz map

$$[E_1^{ord}, (E_{2,ss})^{ord}]_{\mathrm{Alg}_{tmf_{K(1)}}} \to \mathrm{Hom}_{Alg^\theta_{(V_\infty^\wedge)_*}}((W_1)_*^{ord}, (W_{2,ss})_*^{ord})$$

is an isomorphism.

(2) The mapping spaces $\mathrm{Alg}_{tmf_{K(1)}}(E_1^{ord}, (E_{2,ss})^{ord})$ have contractible components.

We conclude that:

(1) The maps α_{chrom} assemble to give a natural transformation

$$\alpha_{chrom} : (\iota_{ord})_* \bar{\mathcal{O}}^{top}_{K(1)} \to ((\iota_{ss})_* \bar{\mathcal{O}}^{top}_{K(2)})_{K(1)}.$$

of the associated homotopy functors

$$(\iota_{ord})_* \bar{\mathcal{O}}^{top}_{K(1)} : ((\overline{\mathcal{M}}_{ell})_{p,et,aff})^{op} \to Ho(\text{Comm } tmf_{K(1)}\text{-algebras}),$$
$$((\iota_{ss})_* \bar{\mathcal{O}}^{top}_{K(2)})_{K(1)} : ((\overline{\mathcal{M}}_{ell})_{p,et,aff})^{op} \to Ho(\text{Comm } tmf_{K(1)}\text{-algebras}).$$

(2) The contractibility of the mapping spaces implies that the maps α_{chrom} may be chosen to induce a strict natural transformation of functors:

$$\alpha_{chrom} : (\iota_{ord})_* \mathcal{O}^{top}_{K(1)} \to ((\iota_{ss})_* \mathcal{O}^{top}_{K(2)})_{K(1)}.$$

Putting the pieces together.

Define $\mathcal{O}^{top}_p$ to be the presheaf of E_∞ ring spectra given by the pullback

$$\begin{tikzcd} \mathcal{O}^{top}_p \arrow[r] \arrow[d] & (\iota_{ss})_* \mathcal{O}^{top}_{K(2)} \arrow[d] \\ (\iota_{ord})_* \mathcal{O}^{top}_{K(1)} \arrow[r, "\alpha_{chrom}"'] & ((\iota_{ss})_* \mathcal{O}^{top}_{K(2)})_{K(1)} \end{tikzcd}$$

Let R be a p-complete ring and suppose that

$$\mathrm{Spf}(R) \to (\overline{\mathcal{M}}_{ell})_p$$

is an étale open classifying a generalized elliptic curve C/R. Using the same notation as we have been using, there are associated elliptic spectra E^{ord}, E^{ss}, and $(E^{ss})^{ord}$. The spectrum of sections $E := \mathcal{O}^{top}_p(\mathrm{Spf}(R))$ is given by the homotopy pullback

$$\begin{tikzcd} E \arrow[r] \arrow[d] & E^{ss} \arrow[d] \\ E^{ord} \arrow[r, "\alpha_{chrom}"'] & (E^{ss})^{ord} \end{tikzcd}$$

We then have the following.

Proposition 8.9. *The spectrum E is elliptic for the curve C/R.*

We first need the following lemma.

Lemma 8.10. *Suppose that A is a ring and that $x \in A$ is not a zero-divisor. Then the following square is a pullback.*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A^\wedge_{(x)} \arrow[d] \\ A[x^{-1}] \arrow[r] & A^\wedge_{(x)}[x^{-1}] \end{tikzcd}$$

Proof. Because of our assumption, the map $A \to A[x^{-1}]$ is an injection. The result then follows from the fact that the induced map of the cokernels of the vertical maps

$$A/x^\infty \to A^\wedge_{(x)}/x^\infty$$

is an isomorphism. $\square$

Remark 8.11. Lemma 8.10 is true in greater generality, at least provided that A is Noetherian, but this is the only case we need.

Proof of Proposition 8.9. The proposition reduces to verifying that the diagram

$$\begin{tikzcd} R_* \arrow[r] \arrow[d] & R_*^{ss} \arrow[d] \\ R_*^{ord} \arrow[r, "g"'] & (R^{ss})_*^{ord} \end{tikzcd} \tag{8.7}$$

is a pullback. Since $\mathrm{Spf}(R) \to (\overline{\mathcal{M}}_{ell})_p$ is étale, and the map $(\overline{\mathcal{M}}_{ell})_p \to (\mathcal{M}_{FG})_p$ is flat (Remark 1.4), the composite

$$\mathrm{Spf}(R) \to (\overline{\mathcal{M}}_{ell})_p \to (\mathcal{M}_{FG})_p$$

is flat. In particular, by Landweber's criterion, the sequence $(p, v_1) \subset R_*$ is regular. Therefore R_* is p-torsion-free, and v_1 is not a zero divisor in R_*/pR_*. Using the facts that R_* is p-complete and p-torsion-free, it may be deduced that v_1 is not a zero divisor in R_*. Therefore, by Lemma 8.10, the following square is a pullback.

$$\begin{tikzcd} R_* \arrow[r] \arrow[d] & (R_*)^\wedge_{(v_1)} \arrow[d] \\ R_*[v_1^{-1}] \arrow[r] & (R_*)^\wedge_{(v_1)}[v_1^{-1}] \end{tikzcd}$$

The square (8.7) is the p-completion of the above square. Since p-completion is exact on p-torsion-free modules, we deduce that (8.7) is a pullback diagram, as desired. $\square$

9. Construction of $\mathcal{O}^{top}_{\mathbb{Q}}$ and $\mathcal{O}^{top}$

In this section we will construct the presheaf $\mathcal{O}^{top}_{\mathbb{Q}}$, and the map

$$\alpha_{arith} : (\iota_{\mathbb{Q}})_* \mathcal{O}^{top}_{\mathbb{Q}} \to \left(\prod_p (\iota_p)_* \mathcal{O}^{top}_p \right)_{\mathbb{Q}}.$$

By the results of Section 2, it suffices to restrict our attention to affine étale opens.

The Eilenberg-MacLane functor associates to a graded $\mathbb{Q}$-algebra A_* a commutative $H\mathbb{Q}$-algebra $H(A_*)$. Suppose that

$$f : \mathrm{Spec}(R) \to (\overline{\mathcal{M}}_{ell})_{\mathbb{Q}}$$

is an affine étale open. Define an evenly graded ring R_* by

$$R_{2t} := \Gamma f^* \omega^{\otimes t}.$$

We define

$$\mathcal{O}^{top}_{\mathbb{Q}}(\mathrm{Spec}(R)) = H(R_*).$$

The functoriality of $H(-)$ makes this a presheaf of commutative $H\mathbb{Q}$-algebras.

PROPOSITION 9.1. *Let C/R be the generalized elliptic curve classified by f. Then the spectrum $H(R_*)$ uniquely admits the structure of an elliptic spectrum for the curve C.*

Proof. We just need to show that there is a unique isomorphism of formal groups

$$\widehat{C} \xrightarrow{\cong} \mathbb{G}_{H(R_*)}.$$

It suffices to show that there is a uniue isomorphism Zariski locally on $\operatorname{Spec} R$. Thus it suffices to consider the case where the line bundle $f^*\omega$ is trivial. In this case, the formal group $\mathbb{G}_{H(R_*)}$ is just the additive formal group. Since we are working over $\mathbb{Q}$, there is a unique isomorphism given by the logarithm. □

Because its sections are rational, the presheaf $(\prod_p (\iota_p)_* \mathcal{O}_p^{top})_{\mathbb{Q}}$ is a presheaf of commutative $H\mathbb{Q}$-algebras.

There is an alternative perspective to the homotopy groups of an elliptic spectrum that we shall employ. Let $(\overline{\mathcal{M}}_{ell})^1$ denote the moduli stack of pairs (C, v) where C is a generalized elliptic curve and v is a tangent vector to the identity. Then the forgetful map

$$f : (\overline{\mathcal{M}}_{ell})^1 \to \overline{\mathcal{M}}_{ell}$$

is a $\mathbb{G}_m$-torsor. There is a canonical isomorphism

$$f_* \mathcal{O}_{(\overline{\mathcal{M}}_{ell})^1} \cong \bigoplus_{t \in \mathbb{Z}} \omega^{\otimes t} \tag{9.1}$$

which gives the weight decomposition of $\mathcal{O}_{(\overline{\mathcal{M}}_{ell})^1}$ induced by the $\mathbb{G}_m$-action. We deduce the following lemma.

Lemma 9.2. *For any étale open*

$$U \to (\overline{\mathcal{M}}_{ell})_{\mathbb{Q}}$$

for which the pullback

$$f^*U \to (\overline{\mathcal{M}}_{ell})^1_{\mathbb{Q}}$$

is an affine scheme, there is a natural isomorphism

$$\pi_* \mathcal{O}^{top}_{\mathbb{Q}}(U) \cong \mathcal{O}_{(\overline{\mathcal{M}}_{ell})^1_{\mathbb{Q}}}(f^*U).$$

Consider the substacks:

$$\overline{\mathcal{M}}_{ell}[c_4^{-1}] \subset \overline{\mathcal{M}}_{ell},$$
$$\overline{\mathcal{M}}_{ell}[\Delta^{-1}] \subset \overline{\mathcal{M}}_{ell}.$$

A Weierstrass curve is non-singular if and only if Δ is invertible, whereas a singular Weierstrass curve ($\Delta = 0$) has no cuspidal singularities if and only if c_4 is invertible. Thus the pair $\overline{\mathcal{M}}_{ell}[c_4^{-1}]$, $\overline{\mathcal{M}}_{ell}[\Delta^{-1}]$ form an open cover of $\overline{\mathcal{M}}_{ell}$. Consider the induced cover

$$\{(\overline{\mathcal{M}}_{ell})^1_{\mathbb{Q}}[c_4^{-1}], (\overline{\mathcal{M}}_{ell})^1_{\mathbb{Q}}[\Delta^{-1}]\}.$$

The following lemma is a corollary of the computation of the ring of modular forms of level 1 over $\mathbb{Q}$.

Lemma 9.3. *The stack $(\overline{\mathcal{M}}_{ell})^1_{\mathbb{Q}}$ is the open subscheme of*

$$\operatorname{Spec}(\mathbb{Q}[c_4, c_6])$$

given by the union of the affine subschemes

$$(\overline{\mathcal{M}}_{ell})^1_{\mathbb{Q}}[c_4^{-1}] = \operatorname{Spec}(\mathbb{Q}[c_4^{\pm 1}, c_6]),$$
$$(\overline{\mathcal{M}}_{ell})^1_{\mathbb{Q}}[\Delta^{-1}] = \operatorname{Spec}(\mathbb{Q}[c_4, c_6, \Delta^{-1}]).$$

where $\Delta = (c_4^3 - c_6^2)/1728$.

Let $\overline{\mathcal{M}}_{ell}[c_4^{-1}, \Delta^{-1}]$ denote the intersection (pullback)

$$\overline{\mathcal{M}}_{ell}[c_4^{-1}] \cap \overline{\mathcal{M}}_{ell}[\Delta^{-1}] \hookrightarrow \overline{\mathcal{M}}_{ell}.$$

For a presheaf $\mathcal{F}$ on $\overline{\mathcal{M}}_{ell}$, let

$$\mathcal{F}[c_4^{-1}], \quad \mathcal{F}[\Delta^{-1}], \quad \mathcal{F}[c_4^{-1}, \Delta^{-1}]$$

denote the presheaves on $\overline{\mathcal{M}}_{ell}$ obtained by taking the pushforwards of the restrictions of $\mathcal{F}$ to the open substacks

$$\overline{\mathcal{M}}_{ell}[c_4^{-1}], \quad \overline{\mathcal{M}}_{ell}[\Delta^{-1}], \quad \overline{\mathcal{M}}_{ell}[c_4^{-1}, \Delta^{-1}],$$

respectively. By descent, to construct α_{arith}, it suffices to construct a diagram of presheaves of $H\mathbb{Q}$-algebras:

$$\begin{tikzcd}
(\iota_{\mathbb{Q}})_* \mathcal{O}^{top}_{\mathbb{Q}}[c_4^{-1}] \arrow[r, "\alpha_{arith}"] \arrow[d] & \left(\prod_p (\iota_p)_* \mathcal{O}^{top}_p\right)_{\mathbb{Q}}[c_4^{-1}] \arrow[d] \\
(\iota_{\mathbb{Q}})_* \mathcal{O}^{top}_{\mathbb{Q}}[c_4^{-1}, \Delta^{-1}] \arrow[r, "\alpha_{arith}"] & \left(\prod_p (\iota_p)_* \mathcal{O}^{top}_p\right)_{\mathbb{Q}}[c_4^{-1}, \Delta^{-1}] \\
(\iota_{\mathbb{Q}})_* \mathcal{O}^{top}_{\mathbb{Q}}[\Delta^{-1}] \arrow[r, "\alpha_{arith}"'] \arrow[u] & \left(\prod_p (\iota_p)_* \mathcal{O}^{top}_p\right)_{\mathbb{Q}}[\Delta^{-1}] \arrow[u]
\end{tikzcd} \tag{9.2}$$

We accomplish this in two steps:

Step 1: Construct compatible maps on the sections over $\overline{\mathcal{M}}_{ell}[\Delta^{-1}]$, $\overline{\mathcal{M}}_{ell}[c_4^{-1}]$, and $\overline{\mathcal{M}}_{ell}[c_4^{-1}, \Delta^{-1}]$.

Step 2: Construct corresponding maps of presheaves.

Step 1: Construction of the α_{arith} on certain sections.

Define commutative $H\mathbb{Q}$-algebras

$$\begin{aligned}
tmf_{\mathbb{Q}}[c_4^{-1}] &:= (\iota_{\mathbb{Q}})_* \mathcal{O}^{top}_{\mathbb{Q}}(\overline{\mathcal{M}}_{ell}[c_4^{-1}]) \\
tmf_{\mathbb{Q}}[c_4^{-1}, \Delta^{-1}] &:= (\iota_{\mathbb{Q}})_* \mathcal{O}^{top}_{\mathbb{Q}}(\overline{\mathcal{M}}_{ell}[c_4^{-1}, \Delta^{-1}]) \\
tmf_{\mathbb{Q}}[\Delta^{-1}] &:= (\iota_{\mathbb{Q}})_* \mathcal{O}^{top}_{\mathbb{Q}}(\overline{\mathcal{M}}_{ell}[\Delta^{-1}]) \\
tmf_{\mathbb{A}_f}[c_4^{-1}] &:= \left(\prod_p (\iota_p)_* \mathcal{O}^{top}_p(\overline{\mathcal{M}}_{ell}[c_4^{-1}])\right)_{\mathbb{Q}} \\
tmf_{\mathbb{A}_f}[c_4^{-1}, \Delta^{-1}] &:= \left(\prod_p (\iota_p)_* \mathcal{O}^{top}_p(\overline{\mathcal{M}}_{ell}[c_4^{-1}, \Delta^{-1}])\right)_{\mathbb{Q}} \\
tmf_{\mathbb{A}_f}[\Delta^{-1}] &:= \left(\prod_p (\iota_p)_* \mathcal{O}^{top}_p(\overline{\mathcal{M}}_{ell}[\Delta^{-1}])\right)_{\mathbb{Q}}
\end{aligned}$$

Observe that we have

$$\pi_* tmf_{\mathbb{A}_f}[-] \cong \pi_* tmf_{\mathbb{Q}}[-] \otimes_{\mathbb{Q}} \mathbb{A}_f$$

where $\mathbb{A}_f = \left(\prod_p \mathbb{Z}_p\right) \otimes \mathbb{Q}$ is the ring of finite adeles. Therefore there are natural maps of commutative $\mathbb{Q}$-algebras

$$\bar{\alpha}_{arith} : \pi_* tmf_{\mathbb{Q}}[-] \to \pi_* tmf_{\mathbb{A}_f}[-].$$

The Goerss-Hopkins obstructions to existence and uniqueness of maps

$$\alpha_{arith} : tmf_{\mathbb{Q}}[-] \to tmf_{\mathbb{A}_f}[-]$$

of commutative $H\mathbb{Q}$-algebras realizing the maps $\bar{\alpha}_{arith}$ lie in the Andre-Quillen cohomology of commutative $\mathbb{Q}$-algebras:

$$H^s_{comm_{\mathbb{Q}}}(\pi_* tmf_{\mathbb{Q}}[-], \pi_* tmf_{\mathbb{A}_f}[-][-s+1]), \qquad s > 1.$$

Because

$$\pi_* tmf_{\mathbb{Q}}[-] = \mathbb{Q}[c_4, c_6][-]$$

is a smooth $\mathbb{Q}$-algebra, we have

$$H^s_{comm_{\mathbb{Q}}}(\pi_* tmf_{\mathbb{Q}}[-], \pi_* tmf_{\mathbb{A}_f}[-][u]) = 0, \qquad s > 0.$$

We deduce that the Hurewicz map

$$[tmf_{\mathbb{Q}}[-], tmf_{\mathbb{A}_f}[-]]_{Alg_{H\mathbb{Q}}} \to \mathrm{Hom}_{comm_{\mathbb{Q}}}(\pi_* tmf_{\mathbb{Q}}[-], \pi_* tmf_{\mathbb{A}_f}[-])$$

is an isomorphism. In particular, the maps α_{arith} exist.

We similarly find that we have

$$H^s_{comm_{\mathbb{Q}}}(\pi_* tmf_{\mathbb{Q}}[c_4^{-1}], \pi_* tmf_{\mathbb{A}_f}[c_4^{-1}, \Delta^{-1}][u]) = 0, \qquad s > 0,$$
$$H^s_{comm_{\mathbb{Q}}}(\pi_* tmf_{\mathbb{Q}}[\Delta^{-1}], \pi_* tmf_{\mathbb{A}_f}[c_4^{-1}, \Delta^{-1}][u]) = 0, \qquad s > 0.$$

This implies that the diagram

$$\begin{tikzcd}
tmf_{\mathbb{Q}}[c_4^{-1}] \arrow[r, "\alpha_{arith}"] \arrow[d] & tmf_{\mathbb{A}_f}[c_4^{-1}] \arrow[d, "r_1"] \\
tmf_{\mathbb{Q}}[c_4^{-1}, \Delta^{-1}] \arrow[r, "\alpha_{arith}"] & tmf_{\mathbb{A}_f}[c_4^{-1}, \Delta^{-1}] \\
tmf_{\mathbb{Q}}[\Delta^{-1}] \arrow[r, "\alpha_{arith}"] \arrow[u] & tmf_{\mathbb{A}_f}[\Delta^{-1}] \arrow[u, "r_2"]
\end{tikzcd} \tag{9.3}$$

commutes *up to homotopy* in the category of commutative $H\mathbb{Q}$-algebras.

Because the presheaves $\mathcal{O}_p^{top}$ are fibrant in the Jardine model structure, the maps r_1 and r_2 in Diagram 9.3 are fibrations of commutative $H\mathbb{Q}$-algebras. The following lemma implies that we can rectify Diagram (9.3) to a point-set level commutative diagram of commutative $H\mathbb{Q}$-algebras.

LEMMA 9.4. *Suppose that $\mathcal{C}$ is a simplicial model category, and that*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "p"'] & X \arrow[d, "q"] \\
B \arrow[r, "g"'] & Y
\end{tikzcd}$$

is a homotopy commutative diagram with A cofibrant and q a fibration. Then there exists a map f', homotopic to f, such that the diagram

$$\begin{tikzcd} A \arrow[r, "f'"] \arrow[d, "p"'] & X \arrow[d, "q"] \\ B \arrow[r, "g"'] & Y \end{tikzcd}$$

strictly commutes.

Proof. Let H be a homotopy that makes the diagram commute, and take a lift

$$\begin{tikzcd} A \otimes 0 \arrow[r, "f"] \arrow[d, hook] & X \arrow[d, "q"] \\ A \otimes \Delta^1 \arrow[r, "H"'] \arrow[ur, dotted, "\tilde{H}"] & Y \end{tikzcd}$$

Take $f' = \tilde{H}_1$. $\square$

Step 2: construction of Diagram 9.2.

It suffices to construct the diagram on affine opens. Suppose that

$$\mathrm{Spec}(R) \to \overline{\mathcal{M}}_{ell}$$

is an affine étale open. Define commutative $H\mathbb{Q}$-algebras

$$\begin{aligned} T[c_4^{-1}] &:= (\iota_{\mathbb{Q}})_* \mathcal{O}^{top}_{\mathbb{Q}}(\mathrm{Spec}(R[c_4^{-1}])) \\ T[c_4^{-1}, \Delta^{-1}] &:= (\iota_{\mathbb{Q}})_* \mathcal{O}^{top}_{\mathbb{Q}}(\mathrm{Spec}(R[c_4^{-1}, \Delta^{-1}])) \\ T[\Delta^{-1}] &:= (\iota_{\mathbb{Q}})_* \mathcal{O}^{top}_{\mathbb{Q}}(\mathrm{Spec}(R[\Delta^{-1}])) \\ T'[c_4^{-1}] &:= \left(\prod_p (\iota_p)_* \mathcal{O}^{top}_p(\mathrm{Spec}(R[c_4^{-1}])) \right)_{\mathbb{Q}} \\ T'[c_4^{-1}, \Delta^{-1}] &:= \left(\prod_p (\iota_p)_* \mathcal{O}^{top}_p(\mathrm{Spec}(R[c_4^{-1}, \Delta^{-1}])) \right)_{\mathbb{Q}} \\ T'[\Delta^{-1}] &:= \left(\prod_p (\iota_p)_* \mathcal{O}^{top}_p(\mathrm{Spec}(R[\Delta^{-1}])) \right)_{\mathbb{Q}} \end{aligned}$$

Let T' be any commutative $tmf_{\mathbb{A}_f}[-]$-algebra, and let

$$\pi_* tmf_{\mathbb{Q}}[-] \to \pi_* T'$$

be a map of $\pi_* tmf_{\mathbb{Q}}[-]$-algebras. We have the following pullback diagram.

$$\begin{tikzcd} \mathrm{Spec}(\pi_* T[-]) \arrow[rr] \arrow[d] & & \mathrm{Spec}(R \otimes \mathbb{Q}) \arrow[d] \\ \mathrm{Spec}(\mathbb{Q}[c_4, c_6][-]) \arrow[r, equal] & (\overline{\mathcal{M}}_{ell})^1_{\mathbb{Q}}[-] \arrow[r] & (\overline{\mathcal{M}}_{ell})_{\mathbb{Q}} \end{tikzcd}$$

In particular, we deduce that $\pi_* T[-]$ is étale over

$$\pi_* tmf_{\mathbb{Q}}[-] = \mathbb{Q}[c_4, c_6][-].$$

Therefore, the spectral sequence

$$\mathrm{Ext}^s_{\pi_* T[-]}(H_t(\mathbb{L}(\pi_* T[-]/\pi_* tmf_{\mathbb{Q}}[-])), \pi_* T'[u]) \Rightarrow H^{s+t}_{comm_{\pi_* tmf_{\mathbb{Q}}[-]}}(\pi_* T[-], \pi_* T'[u])$$

collapses to give

$$H^s_{comm_{\pi_* tmf_{\mathbb{Q}}[-]}}(\pi_* T[-], \pi_* T'[u]) = 0.$$

We deduce that

(1) The Hurewicz maps

$$[T[-], T']_{Alg_{tmf_{\mathbb{Q}}[-]}} \to \mathrm{Hom}_{comm_{\pi_* tmf_{\mathbb{Q}}[-]}}(\pi_* T[-], \pi_* T')$$

are isomorphisms.

(2) The mapping spaces $\mathrm{Alg}_{tmf_{\mathbb{Q}}[-]}(T[-], T')$ have contractible components.

This is enough to conclude that there exist maps α_{arith}, functorial in R, making the following diagrams commute

$$\begin{tikzcd}
T[c_4^{-1}] \arrow[r, "\alpha_{arith}"] \arrow[d] & T'[c_4^{-1}] \arrow[d] \\
T[c_4^{-1}, \Delta^{-1}] \arrow[r, "\alpha_{arith}"] & T'[c_4^{-1}, \Delta^{-1}] \\
T[\Delta^{-1}] \arrow[r, "\alpha_{arith}"] \arrow[u] & T'[\Delta^{-1}] \arrow[u]
\end{tikzcd}$$

Since, by homotopy descent, there are homotopy pullbacks

$$\begin{tikzcd}
(\iota_{\mathbb{Q}})_* \mathcal{O}^{top}_{\mathbb{Q}}(\mathrm{Spec}(R)) \arrow[r] \arrow[d] & T[c_4^{-1}] \arrow[d] \\
T[\Delta^{-1}] \arrow[r] & T[c_4^{-1}, \Delta^{-1}]
\end{tikzcd}
\qquad
\begin{tikzcd}
\left(\prod_p (\iota_p)_* \mathcal{O}^{top}_p(\mathrm{Spec}(R))\right)_{\mathbb{Q}} \arrow[r] \arrow[d] & T'[c_4^{-1}] \arrow[d] \\
T'[\Delta^{-1}] \arrow[r] & T'[c_4^{-1}, \Delta^{-1}]
\end{tikzcd}$$

We get an induced map on pullbacks

$$\alpha_{arith} : (\iota_{\mathbb{Q}})_* \mathcal{O}^{top}_{\mathbb{Q}}(\mathrm{Spec}(R)) \to \left(\prod_p (\iota_p)_* \mathcal{O}^{top}_p(\mathrm{Spec}(R))\right)_{\mathbb{Q}}.$$

which is natural in $\mathrm{Spec}(R)$.

We define $\mathcal{O}^{top}$ to be the presheaf on $\overline{\mathcal{M}}_{ell}$ whose sections over $\mathrm{Spec}(R)$ are given by the pullback

$$\begin{tikzcd}
\mathcal{O}^{top}(\mathrm{Spec}(R)) \arrow[r] \arrow[d] & \prod_p (\iota_p)_* \mathcal{O}^{top}_p(\mathrm{Spec}(R)) \arrow[d] \\
(\iota_{\mathbb{Q}})_* \mathcal{O}^{top}_{\mathbb{Q}}(\mathrm{Spec}(R)) \arrow[r, "\alpha_{arith}"'] & \left(\prod_p (\iota_p)_* \mathcal{O}^{top}_p(\mathrm{Spec}(R))\right)_{\mathbb{Q}}
\end{tikzcd}$$

The following proposition concludes our proof of Theorem 1.1.

Proposition 9.5. *The spectrum $\mathcal{O}^{top}(\mathrm{Spec}(R))$ is elliptic with respect to the elliptic curve C/R.*

Proof. The proposition follows from Propositions 8.9 and 9.1, and the pullback

$$
\begin{tikzcd}
R \arrow[r] \arrow[d] & \prod_p R_p^\wedge \arrow[d] \\
R \otimes \mathbb{Q} \arrow[r] & \left(\prod_p R_p^\wedge\right) \otimes \mathbb{Q}
\end{tikzcd}
$$

□

10. Acknowledgements

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Appendix A. $K(1)$-local Goerss-Hopkins obstruction theory for the prime 2

Theorem 7.1 provides an obstruction theory for producing $K(1)$-local E_∞-ring spectra, and maps between them, at all primes. These obstructions lie in the Andre-Quillen cohomology groups based on p-adic K-homology. Unfortunately, as indicated in Section 7, the K-theoretic obstruction theory is insufficient to produce the sheaf $\mathcal{O}^{top}_{K(1)}$ at the prime 2. At the prime 2 we instead must use a variant of the theory based on 2-adic *real* K-theory. The material in this Appendix is the product of some enlightening discussions with Tyler Lawson.

For a spectrum E, the E-based obstruction theory of [**GH**] requires the homology theory to be "adapted" to the E_∞ operad. Unfortunately, $KO_2^\wedge$ does not seem to be adapted to the E_∞-operad. While the $KO_2^\wedge$-homology of a free E_∞ algebra generated by the 0-sphere is the free graded reduced θ-algebra on one generator, this fails to occur for spheres of every dimension. Nevertheless, we will show that the obstruction theory can be manually implemented when the spaces and spectra involved are Bott periodic (Definition 7.10).

Theorem A.1.

(1) *Given a Bott-periodic graded reduced θ-algebra A_* satisfying*

$$H^s_c(\mathbb{Z}_2^\times/\{\pm 1\}, A_*) = 0, \text{ for } s > 0, \tag{A.1}$$

the obstructions to the existence of a $K(1)$-local E_∞-ring spectrum E, for which there is an isomorphism

$$(KO_2^\wedge)_* E \cong A_*$$

of graded reduced θ-algebras, lie in

$$H^s_{Alg^{red}_\theta}(A_*/(KO_2)_*, A_*[-s+2]), \qquad s \geq 3.$$

(2) *Given Bott periodic $K(1)$-local E_∞-ring spectra E_1, E_2, and a map of graded θ-algebras*

$$f_* : (KO_2^\wedge)_* E_1 \to (KO_2^\wedge)_* E_2,$$

the obstructions to the existence of a map $f : E_1 \to E_2$ of E_∞-ring spectra which induces f_* on 2-adic KO-homology lie in

$$H^s_{Alg^{red}_\theta}((KO_2^\wedge)_* E_1/(KO_2)_*, (KO_2^\wedge)_* E_2[-s+1]), \qquad s \geq 2.$$

(Here, the θ-$(KO_2^\wedge)_* E_1$-module structure on $(KO_2^\wedge)_* E_2$ arises from the map f_*.) The obstructions to uniqueness lie in

$$H^s_{Alg^{red}_\theta}((KO_2^\wedge)_* E_1/(KO_2)_*, (KO_2^\wedge)_* E_2[-s]), \qquad s \geq 1.$$

(3) *Given such a map f above, there is a spectral sequence which computes the higher homotopy groups of the space $E_\infty(E_1, E_2)$ of E_∞ maps:*

$$H^s_{Alg^{red}_\theta}((KO_2^\wedge)_* E_1/(KO_2)_*, (KO_2^\wedge)_* E_2[t]) \Rightarrow \pi_{-t-s}(E_\infty(E_1, E_2), f).$$

Remark A.2. The author believes that Condition (A.1) is unnecessary, but it makes the proof of the theorem much easier to write down, and is satisfied by in the cases needed in this paper.

The remainder of this section will be devoted to proving the theorem above. Most of the work is in proving (1). As in [**GH**], consider the category $s\mathrm{Alg}^{K(1)}_{E_\infty}$ of simplicial objects in the $K(1)$-local category of E_∞-ring spectra. Endow this category with a $\mathcal{P}$-resolution model structure[1] with projectives given by

$$\mathcal{P} = \{\Sigma^i T_j\}_{i \in \mathbb{Z}, j > 1}$$

where the spectra T_j are the finite Galois extensions of $S_{K(1)}$ given by

$$T_j = KO_2^{hG_j}$$

for

$$G_j = 1 + 2^j \mathbb{Z}_2 \subset \mathbb{Z}_2^\times/\{\pm 1\} =: \Gamma.$$

Note that T_j is $K(1)$-locally dualizible (in fact, it is self-dual), and we have

$$KO_2 \simeq_{K(1)} \varinjlim_j T_j.$$

The forgetful functor $\mathrm{Alg}^{red}_\theta \to \mathrm{Mod}_{\mathbb{Z}_2[[\Gamma]]}$ has a left adjoint — call it $\mathbb{P}_\theta$. Let $\mathbb{P}$ denote the free $K(1)$-local E_∞-algebra functor. Then the natural map is an isomorphism:

$$KO_* \otimes \mathbb{P}_\theta (KO_2^\wedge)_0(S^0) \to (KO_2^\wedge)_*(\mathbb{P}S^0).$$

In fact, the same holds when S^0 is replaced by the spectrum T_j.

As in [**GH**], an object $X_\bullet$ of $s\mathrm{Alg}^{K(1)}_{E_\infty}$ has two kinds of homotopy groups associated to an object $P \in \mathcal{P}$: the E_2-homotopy groups

$$\pi_{s,t}(X_\bullet; P) := \pi_s[\Sigma^t P, X_\bullet]_{\mathrm{Sp}_{K(1)}}$$

[1] To be precise, we are endowing the category of simplicial *spectra* with the $\mathcal{P}$-resolution model structure associated to the $K(1)$-local model structure on spectra, and then lifting this to a model structure on simplicial commutative ring spectra.

given as the homotopy groups of the simplicial abelian group, and the natural homotopy groups

$$\pi^\natural_{s,t}(X_\bullet; P) := [\Sigma^t P \otimes \Delta^s/\partial\Delta^s, X_\bullet]_{s\mathrm{Sp}_{K(1)}}$$

given as the homotopy classes of maps computed in the homotopy category $h(s\mathrm{Sp}_{K(1)})$. These homotopy groups are related by the spiral exact sequence

$$\cdots \to \pi^\natural_{s-1,t+1}(X_\bullet; P) \to \pi^\natural_{s,t}(X_\bullet; P) \to \pi_{s,t}(X_\bullet; P) \to \pi^\natural_{s-2,t+1}(X_\bullet; P) \to \cdots.$$

We shall omit P from the notation when $P = S^0$.

We will closely follow the explicit treatment of obstruction theory given by Blanc-Johnson-Turner [**BJT**], adapted to our setting. Namely, we will produce a free simplicial resolution $W_\bullet$ of the reduced theta algebra A_0, and then analyze the obstructions to inductively producing an explicit object $X_\bullet \in s\mathrm{Alg}^{K(1)}_{E_\infty}$ with

$$(KO^\wedge_2)_* X_\bullet \cong KO_* \otimes W_\bullet.$$

The desired E_∞ ring spectrum will then be given by $E := |X_\bullet|$.

Both of the resolutions $W_\bullet$ and $X_\bullet$ will be *CW-objects* in the sense of [**BJT**, Defn. 1.20] — the spaces of n-simplices take the form:

$$W_n = \bar{W}_n \widehat{\otimes} L_n W_\bullet,$$
$$X_n = (\bar{X}_n \wedge L_n X_\bullet)_{K(1)}.$$

(where $L_n(-)$ denotes the nth latching object). The 'cells' $\bar{W}_n$ (resp. $\bar{X}_n$) will be free reduced θ-algebras (respectively free $K(1)$-local E_∞ rings) and are thus augmented.

For $Y_\bullet$ denoting either $W_\bullet$ or $X_\bullet$, we require that for $i > 0$, the map d_i is the augmentation when restricted to $\bar{Y}_n$. The simplicial structure is then completely determined by the 'attaching maps'

$$\bar{d}^{Y_n}_0 : \bar{Y}_n \to Y_{n-1}.$$

and the simplicial identities. Saying that an attaching map $\bar{d}^{Y_n}_0$ satisfies the simplicial identities is equivalent to requiring that the composites $d_i \bar{d}^{Y_n}_0$ factor through the augmentation.

Given such a simplicial free θ-algebra resolution $W_\bullet$ of A_0, and a θ-A_0-module M, the André-Quillen cohomology of A_0 with coefficients in M may be computed as follows. Let QW_n denote the indecomposibles of the augmented free θ-algebra W_n. Then $QW_\bullet$ is a simplicial reduced Morava module, and the Moore chains $(C_* QW_\bullet, d_0)$ form a chain complex of Morava modules. The André-Quillen cohomology is given by the hypercohomology

$$H^n_{\mathrm{Alg}^{red}_\theta}(A_0, M) = \mathbb{H}^n(\mathrm{Hom}^c_{\mathbb{Z}_2[[\Gamma]]}(C_* QW_\bullet), I^*)$$

where I^* is an injective resolution of M in the category of reduced Morava modules. However, if M satisfies

$$H^s_c(\Gamma; M) = 0, \quad s > 0$$

then one can dispense with the injective resolution I^*, and we simply have

$$H^n_{\mathrm{Alg}^{red}_\theta}(A_0, M) = H^n(\mathrm{Hom}^c_{\mathbb{Z}_2[[\Gamma]]}(C_* QW_\bullet, M).$$

We produce $W_\bullet$ and $X_\bullet$ simultaneously and inductively so that $KO_0 X_\bullet = W_\bullet$, so that $W_\bullet$ is a resolution of A_0. Start by taking a set of topological generators $\{\alpha^i_0\}$ of A_0 as a θ-algebra. We may take these generators to have open isotropy subgroups in Γ: then there exist j_i so that the isotropy of α_0^i is contained in the image of $1+2^j\mathbb{Z}_2$ in Γ. Note that since there are isomorphisms of Morava modules

$$(KO_2^\wedge)_0 T_j \cong \mathbb{Z}_2[(\mathbb{Z}/2^j)^\times/\{\pm 1\}],$$

the generators $\{\alpha_0^i\}$ may be viewed as giving a surjection of θ-algebras

$$\{\alpha_0^i\} : \mathbb{P}_\theta (KO_2^\wedge)_0 \bar{Y}_0 \to A$$

for $\bar{Y}_0 = \bigvee_{\alpha_0^i} T_{j_i}$. Define

$$W_0 = \mathbb{P}_\theta (KO_2^\wedge)_0 \bar{Y}_0, \qquad X_0 = \mathbb{P}\bar{Y}_0.$$

Then take a collection of open isotropy topological generators $\{\alpha_1^i\}$ (as a Morava module) of the kernel of the map

$$\{\alpha_0^i\} : W_0 \to A_0.$$

Realize these as maps

$$\bar{\alpha}_1^i : S^0 \to (KO_2 \wedge X_0)_{K(1)}.$$

Suppose that α_1^i factors through $T_{j_i} \wedge X_0$. Then, since T_{j_i} is $K(1)$-locally Spanier-Whitehead self-dual, there will be resulting maps

$$\tilde{\alpha}_1^i : T_{j_i} \to X_0.$$

Take

$$\bar{Y}_1 = \bigvee_{\tilde{\alpha}_1^i} T_{j_i}, \quad \bar{W}_1 = \mathbb{P}_\theta (KO_2^\wedge)_0 \bar{Y}_1, \quad \bar{X}_1 = \mathbb{P}\bar{Y}_1.$$

and let $\bar{d}_0^{X_1}$ be the map induced from $\{\tilde{\alpha}_1^i\}$. Suppose inductively that we have defined the skeleta $W_\bullet^{[n-1]}$ and $X_\bullet^{[n-1]}$. Note that since

$$\pi_{s,*}(KO \wedge X_\bullet^{[n-1]}) = \begin{cases} A, & s = 0, \\ 0, & 0 < s < n-1 \end{cases}$$

we can deduce from the spiral exact sequence that

$$\pi^\natural_{s,*}(KO \wedge X_\bullet^{[n-1]}) \cong A[-s] \quad 0 \le s \le n-3.$$

Consider the portion of the spiral exact sequence

$$\pi^\natural_{n-1,0}(KO \wedge X_\bullet^{[n-1]}) \to \pi_{n-1,0}(KO \wedge X_\bullet^{[n-1]}) \xrightarrow{\beta_n} \pi^\natural_{n-3,1}(KO \wedge X_\bullet^{[n-1]}) \cong A[-n+2]_0.$$

The map of Morava modules β_n will represent our nth obstruction. Indeed, β_n may be regarded as a map of graded Morava modules

$$\beta_n : \pi_{n-1,*}(KO \wedge X_\bullet^{[n-1]}) \to A[-n+2].$$

Since A satisfies Hypothesis (A.1), there is a short exact sequence

(A.2)

$$\begin{aligned}\mathrm{Hom}^c_{\mathbb{Z}_2[[\Gamma]]}(C_{n-1}QW_\bullet^{[n-1]}, A[-n+2]_0) \xrightarrow{u} \mathrm{Hom}^c_{\mathbb{Z}_2[[\Gamma]]}(\pi_{n-1,0}(KO \wedge X), A[-n+2]_0) \\ \to H^n_{\mathrm{Alg}_\theta^{red}}(A; A[-n+2]) \to 0\end{aligned}$$

and this gives a corresponding class $[\beta_n] \in H^n_{\mathrm{Alg}_\theta^{red}}(A; A[-n+2])$.

Suppose that β_n was zero on the nose. Take a collection $\{\alpha_n^i\}$ of open isotropy topological generators of the Morava module $\pi_{n-1,0}(KO \wedge X_\bullet^{[n-1]})$. Since β_n is zero, these lift to elements

$$\bar{\alpha}_n^i \in \pi^\natural_{n-1,0}(KO \wedge X_\bullet^{[n-1]}).$$

Assume the lifts also have open isotropy. Then for j_i sufficiently large, the maps

$$\alpha_n^i : S^0 \otimes \Delta^{n-1}/\partial\Delta^{n-1} \to KO \wedge X_\bullet^{[n-1]}$$

come from maps

$$\tilde{\alpha}_n^i : T_{j_i} \otimes \Delta^{n-1}/\partial\Delta^{n-1} \to X_\bullet^{[n-1]}.$$

Define

$$\bar{Y}_n = \bigvee_{\tilde{\alpha}_n^i} T_{j_i}, \quad \bar{W}_n = \mathbb{P}_\theta(KO_2^\wedge)_0\bar{Y}_n, \quad \bar{X}_n = \mathbb{P}\bar{Y}_n.$$

We define a map of simplicial E_∞-algebras

$$\phi_n : \bar{X}_n \otimes \partial\Delta^n \to X_\bullet^{[n-1]}$$

where the restriction

$$\phi_n|_{\Lambda_0^n} : \bar{X}_n \otimes \Lambda_0^n \to X_\bullet^{[n-1]}$$

is taken to be the map which is given by the augmentation on each of the faces of Λ_0^n. The map ϕ_n is then determined by specifying a candidate for the restriction on the 0-face

$$\bar{d}_0^{X_n} = \phi_n|_{\Delta^{n-1}} : \bar{X}_n \otimes \Delta^{n-1} \to X_\bullet^{[n-1]}$$

which restricts to the augmentation on each of the faces of $\partial\Delta^{n-1}$. Thus we just need to produce an appropriate class

$$[\bar{d}_0^{X_n}] \in \pi_{n-1,0}^\natural(X_\bullet^{[n-1]}; \bar{Y}_n).$$

We take $[\bar{d}_0^{X_n}]$ to be the map given by $\{\tilde{\alpha}_n^i\}$. Then we define $X_\bullet^{[n]}$ to be the pushout

$$\begin{array}{ccc} \bar{X}_n \otimes \partial\Delta^n & \xrightarrow{\phi_n} & X_\bullet^{[n-1]} \\ \downarrow & & \downarrow \\ \bar{X}_n \otimes \Delta^n & \longrightarrow & X_\bullet^{[n]} \end{array}$$

in $s\mathrm{Alg}_{E_\infty}^{K(1)}$, and define $W_\bullet^{[n]} := (KO_2^\wedge)_0 X_\bullet^{[n]}$.

However, we claim that if the *cohomology class* $[\beta_n]$ vanishes, then there exists a different choice of ϕ_{n-1} one level down, which will yield a different $(n-1)$-skeleton $X_\bullet^{[n-1]'}$, whose associated obstruction β_n' vanishes on the nose. Backing up a level, different choices ϕ_{n-1}, ϕ_{n-1}' correspond to different lifts of $\{\alpha_{n-1}^i\}$. By the spiral exact sequence, any two lifts differ by an element δ_{n-1}, as depicted in the following diagram in the category of Morava modules:

$$\begin{tikzcd} & Q\bar{W}_{n-1} \arrow[dl, "\{\alpha_{n-1}^i\}"'] \arrow[d, shift left] \arrow[d, shift right] \arrow[dr, dotted, "\delta_{n-1}"] & \\ \pi_{n-2,0}(KO \wedge X_\bullet^{[n-2]}) & \pi_{n-2,0}^\natural(KO \wedge X_\bullet^{[n-2]}) \arrow[l] & \pi_{n-3,1}^\natural(KO \wedge X_\bullet^{[n-2]}) \arrow[l] \end{tikzcd}$$

The fact that $\beta_{n-1} = 0$, together with the spiral exact sequence

$$\pi_{n-2,*}(KO \wedge X_\bullet^{[n-2]}) \xrightarrow{\beta_{n-1}} \pi_{n-4,*+1}^\natural(KO \wedge X_\bullet^{[n-2]}) \to \pi_{n-3,*}^\natural(KO \wedge X_\bullet^{[n-2]}) \to 0$$

tells us that there is an isomorphism

$$\pi_{n-3,*}^\natural(KO \wedge X_\bullet^{[n-2]}) \xleftarrow{\cong} \pi_{n-4,*+1}^\natural(KO \wedge X_\bullet^{[n-2]}) \cong A[-n+3]$$

and in particular that we can regard δ_{n-1} to lie in (compare [**BJT**, Lem. 2.11]):

$$\mathrm{Hom}^c_{\mathbb{Z}_2[[\Gamma]]}(Q\bar{W}_{n-1}, A[-n+2]) \cong \mathrm{Hom}^c_{\mathbb{Z}_2[[\Gamma]]}(C_{n-1}QW_\bullet^{[n-1]}, A[-n+2]_0).$$

Let $X_\bullet^{[n-1]'}$ denote the $(n-1)$-skeleton obtained by using the attaching map ϕ'_{n-1}, with associated obstruction β'_n. The difference $\beta_n - \beta'_n$ is the image of δ_{n-1} under the map u of (A.2). Therefore, if the cohomology class $[\beta_n]$ vanishes, then there exists δ_{n-1} such that $u(\delta_{n-1}) = \beta_n$, and a corresponding ϕ'_n, whose associated obstruction $\beta'_n = 0$. This completes the inductive step.

The spectral sequence (3) is the Bousfield-Kan spectral sequence associated to the (diagonal) cosimplicial space

$$E_\infty(B(\mathbb{P}, \mathbb{P}, E_1), KO_2^{\bullet+1} \wedge E_2).$$

The identification of the E_2-term relies on the fact that since E_1 is Bott-periodic,

$$(KO_2^\wedge)_*\mathbb{P}^{\bullet+1}E_1 \cong \mathbb{P}_\theta^{\bullet+1}(KO_2^\wedge)_*E_1.$$

The obstruction theory (2) is just the usual Bousfield obstruction theory [**Bou**] specialized to this cosimplicial space.

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The homotopy groups of *tmf* and of its localizations

André Henriques

In this survey, I present a compilation of the homotopy groups of *tmf* and of its various localizations. This work was an exercise in collecting the diffuse knowledge from my mathematical surroundings.

1. The homotopy of *tmf*

The spectrum *tmf* is connective, which means that the ring $\pi_n(tmf)$ is zero for $n < 0$. Vaguely speaking, its homotopy ring $\pi_*(tmf)$ is an amalgam of $MF_* = \mathbb{Z}[c_4, c_6, \Delta]/(c_4^3 - c_6^2 - (12)^3\Delta)$, the ring of classical modular forms, and part of $\pi_*(\mathbb{S})$, the ring of stable homotopy groups of spheres. More concretely, there are two ring homomorphisms

$$\pi_*(\mathbb{S}) \longrightarrow \pi_*(tmf) \longrightarrow MF_* \tag{1}$$

The map from $\pi_*(\mathbb{S})$ captures information about the torsion part of $\pi_*(tmf)$, while the second map is almost an isomorphism between the non-torsion part of $\pi_*(tmf)$ and MF_*.

The first map in (1) is the Hurewicz homomorphism. Since *tmf* is a ring spectrum, it admits a unit map from the sphere spectrum $\mathbb{S}$. This induces a map in homotopy $\pi_*(\mathbb{S}) \to \pi_*(tmf)$, which is an isomorphism on $\pi_0, \pi_1, \ldots, \pi_6$. The only torsion in $\pi_*(tmf)$ is 2-torsion and 3-torsion; it is at those primes that *tmf* resembles the sphere spectrum. The 3-primary part of the Hurewicz image is 72-periodic and is given by

$$\mathrm{im}\Big(\pi_*(\mathbb{S}) \to \pi_*(tmf)\Big)_{(3)} = \mathbb{Z}_{(3)} \oplus \alpha\,\mathbb{Z}/3\mathbb{Z} \oplus \bigoplus_{k\geq 0} \Delta^{3k}\{\beta, \alpha\beta, \beta^2, \beta^3, \beta^4/\alpha, \beta^4\}\,\mathbb{Z}/3\mathbb{Z}, \tag{2}$$

where α has degree 3, and $\beta = \langle \alpha, \alpha, \alpha \rangle$ has degree 10. The Hurewicz image contains most but not all the 3-torsion of $\pi_*(tmf)$: the classes in dimensions $27 + 72k$ and $75 + 72k$ for $k \geq 0$ are not hit by elements of $\pi_*(\mathbb{S})$. The 2-torsion of $\mathrm{im}(\pi_*(\mathbb{S}) \to \pi_*(tmf))$ is substantially more complicated. It exhibits very rich patterns including two distinct periodicity phenomena. The first one is a periodicity by $c_4 \in \pi_8(tmf)$, which corresponds to v_1^4; the second one is a periodicity by $\Delta^8 \in \pi_{192}(tmf)$, which corresponds to v_2^{32}.

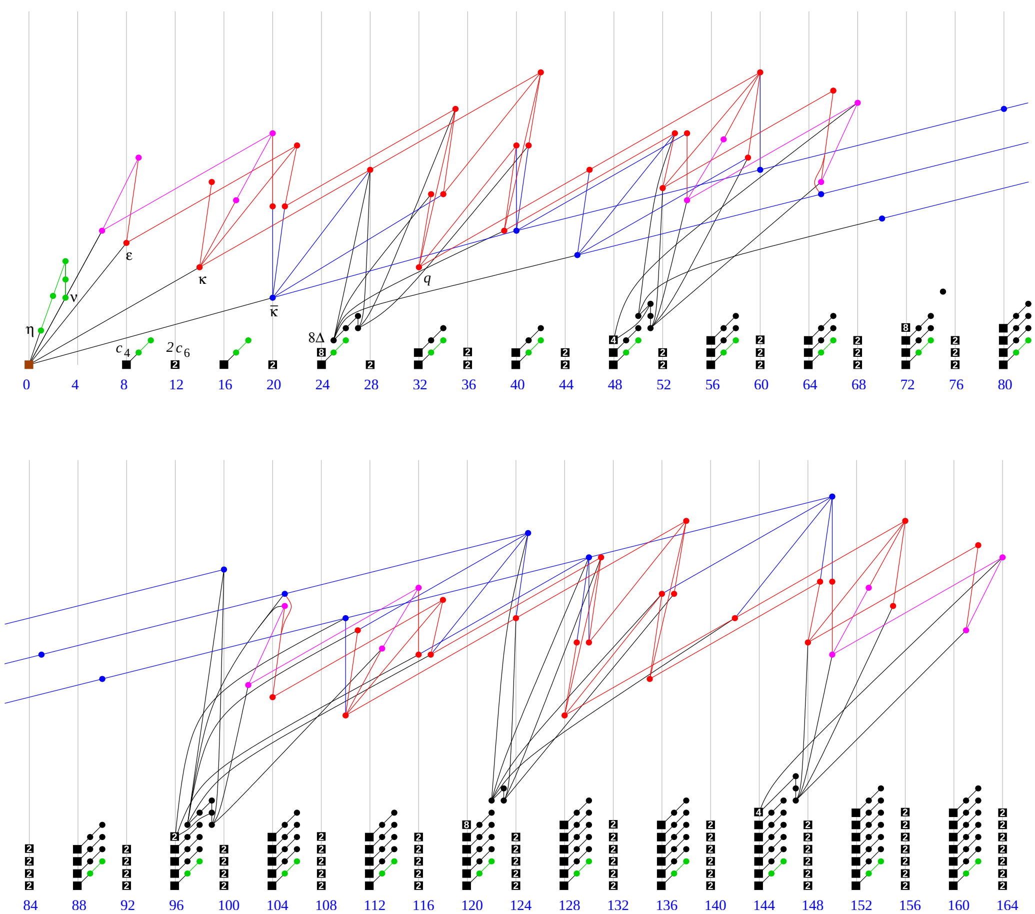
The second map in (1) is the boundary homomorphism of the elliptic spectral sequence. Under that map, a class in $\pi_n(tmf)$ maps to a modular form of weight $n/2$ (and to zero if n is odd). That map is an isomorphism after inverting the primes 2 and 3, which means that both its kernel and its cokernel are 2- and 3-

torsion. Its cokernel can be described explicitly

$$\operatorname{coker}\left(\pi_n(tmf) \to MF_{\frac{n}{2}}\right) = \begin{cases} \mathbb{Z}\Big/\frac{24}{\gcd(k,24)}\,\mathbb{Z}, & \text{if} \quad n = 24k \\ (\mathbb{Z}/2\mathbb{Z})^{\lceil\frac{n-8}{24}\rceil} & \text{if} \quad n \equiv 4 \pmod 8) \\ 0 & \text{otherwise,} \end{cases}$$

where the first cyclic group is generated by Δ^k and the second group is generated by $\Delta^a c_4^b c_6$ for integers a and b satisfying $24a + 8b + 12 = n$. Its kernel agrees with the torsion in $\pi_*(tmf)$ and is much more complicated – it resembles the stable homotopy groups of spheres. The 3-primary component of the kernel is at most $\mathbb{Z}/3\mathbb{Z}$ in any given degree. The 2-primary component is a direct sum of $(\mathbb{Z}/2\mathbb{Z})^\ell$ for some ℓ (corresponding to v_1-periodic elements) with a group isomorphic to $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z}$, $\mathbb{Z}/8\mathbb{Z}$, or $(\mathbb{Z}/2\mathbb{Z})^2$ (corresponding to v_2-periodic elements).

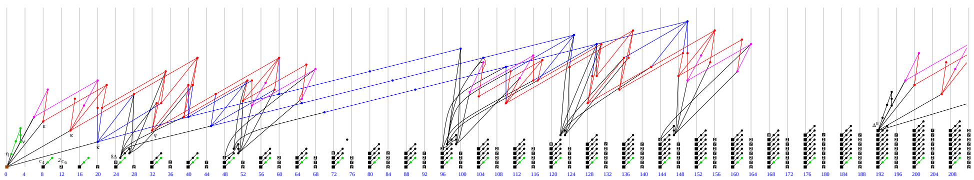
The following picture represents the homotopy ring of *tmf* at the prime 2:



The homotopy groups of *tmf* at the prime 2

The vertical direction has no meaning. Bullets represent $\mathbb{Z}/2\mathbb{Z}$'s while squares represent $\mathbb{Z}_{(2)}$'s. A chain of n bullets connected by vertical lines represent a $\mathbb{Z}/2^n\mathbb{Z}$. The bullets are named by the classes in $\pi_*(\mathbb{S})$ of which they are the image (η, ν, ε, $\bar{\kappa}$, κ, q are standard names), while the squares are named after their images in MF_*. The slanted lines represent multiplication by η, ν, ε, κ, and $\bar{\kappa}$.

The top part of the diagram is 192-periodic with polynomial generator Δ^8, as can be seen in the following zoomed-out picture:



We have colored the conjectural image of the Hurewicz homomorphism (conjectured by Mark Mahowald) as follows: the v_1-periodic classes are in green, and the v_2-periodic classes are in pink, red, and blue, depending on their periodicity. The green classes are v_1^4-periodic in the sphere, and, except for ν, they remain periodic in *tmf* via the identification $c_4 = v_1^4$. The v_2^8-periodic classes are pink, the v_2^{16}-periodic red, and the v_2^{32}-periodic blue. They remain periodic in *tmf* via the identification $\Delta^8 = v_2^{32}$.

The tiny white numbers written in the squares indicate the size of the groups $\operatorname{coker}(\pi_*(tmf) \to MF_*)_{(2)}$. The $\mathbb{Z}_{(2)}$-algebra $\pi_*(tmf)_{(2)}$ is finitely generated, with generators

degree:	1	3	8	8	12	14	20	24	25	27	32	32
name:	η	ν	c_4	ε	$\{2c_6\}$	κ	$\bar{\kappa}$	$\{8\Delta\}$	$\{\eta\Delta\}$	$\{2\nu\Delta\}$	q	$\{c_4\Delta\}$

36	48	51	56	60	72	80	84
$\{2c_6\Delta\}$	$\{4\Delta^2\}$	$\{\nu\Delta^2\}$	$\{c_4\Delta^2\}$	$\{2c_6\Delta^2\}$	$\{8\Delta^3\}$	$\{c_4\Delta^3\}$	$\{2c_6\Delta^3\}$
96	97	99	104	104	108	110	120
$\{2\Delta^4\}$	$\{\eta\Delta^4\}$	$\{\nu\Delta^4\}$	$\{\varepsilon\Delta^4\}$	$\{c_4\Delta^4\}$	$\{2c_6\Delta^4\}$	$\{\kappa\Delta^4\}$	$\{8\Delta^5\}$
123	128	128	132	144	147	152	156
$\{\nu\Delta^5\}$	$\{q\Delta^4\}$	$\{c_4\Delta^5\}$	$\{2c_6\Delta^5\}$	$\{4\Delta^6\}$	$\{\nu\Delta^6\}$	$\{c_4\Delta^6\}$	$\{2c_6\Delta^6\}$
168	176	180	192				
$\{8\Delta^7\}$	$\{c_4\Delta^7\}$	$\{2c_6\Delta^7\}$	$\{\Delta^8\}$				

Hereafter, we list the multiplications that are neither indicated in our chart, nor implied by the ring homomorphism $\pi_*(tmf) \to MF_*$. On the top are the generators, and on the bottom the degrees of the classes that support non-trivial multiplications by those generators (the images never involve $\eta^a c_4^b \Delta^c$ with $b \geq 1$, or $2c_4^a c_6 \Delta^b$):

ε	κ	$\bar{\kappa}$	q
1	1, 26.	1, 2, 8, 14, 15, 21, 22, 26, 98, 22, 32, 33, 34, 39, 46, 104, 110, 111, 116, 117, 118, 128, 129, 130, 135, 136, 142.	1, 3, 8, 14, 20, 21, 25, 27, 28, 34, 97, 99, 104, 110, 117, 118, 123, 124, 130.

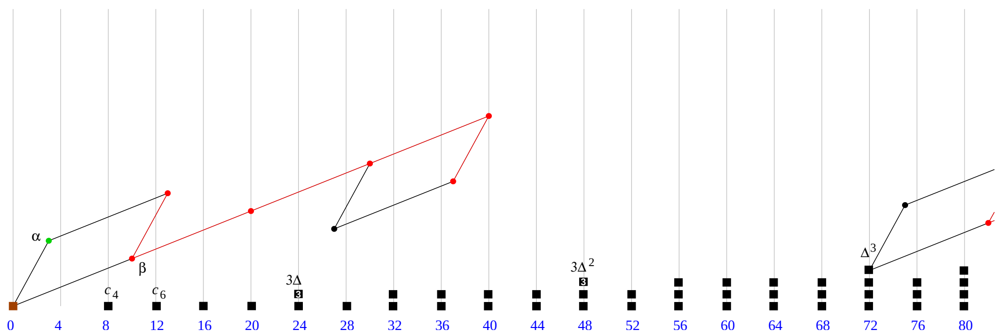
$\{\eta\Delta\}$	$\{2\nu\Delta\}$
1, 2, 3, 8, 14, 15, 20, 21, 25, 26, 27, 28, 32, 34, 35, 40, 41, 45, 50, 60, 65, 75, 80, 85, 97, 98, 99, 100, 104, 105, 110, 111, 113, 117, 122, 123, 124, 125, 128, 130, 131, 137.	1, 8, 14, 15, 25, 26, 27, 32, 33, 39, 96, 97, 98, 104, 110, 111, 122, 123, 128, 129, 135.

$\{\nu\Delta^2\}$	
$1,2,3,6,8,9,14,15,17,51,54^\dagger,65^\dagger,96,97,$ 98, 99, 102, 104, 105, 110, 111, 113, 116.	$\dagger\colon \nu\{\nu\Delta^2\}\mapsto \nu^2\{\nu\Delta^4\}$. $\kappa\{\nu\Delta^2\}\mapsto \nu\kappa\{\nu\Delta^4\}$.

Acting with ε, κ, $\bar\kappa$, q, $\{\eta\Delta\}$, $\{2\nu\Delta\}$, $\{\nu\Delta^2\}$ on $\eta^a c_4^b\Delta^c$ with $b\ge 1$, or $2c_4^a c_6\Delta^b$ always gives zero, except $\{\eta\Delta\}\eta^a c_4^b\Delta^c = \eta^{a+1}c_4^b\Delta^{c+1}$.

To finish, we emphasize the two relations that cannot be deduced from the information contained in our chart: $\{\eta\Delta\}^4 = \bar\kappa^5$, $\{2\nu\Delta\}^2 = \kappa\bar\kappa^2$.

The homotopy groups of *tmf* at the prime 3 exhibit similar phenomena to those at the prime 2. The following is an illustration of $\pi_*(tmf)_{(3)}$:



The homotopy groups of *tmf* at the prime 3

The bullets represent $\mathbb{Z}/3\mathbb{Z}$'s and are named after the corresponding elements of $\pi_*(\mathbb{S})$. The squares represent $\mathbb{Z}_{(3)}$'s and are named after their image in MF_*. The slanted lines indicate multiplication by α and β. The top part of the diagram is 72-periodic, with polynomial generator Δ^3. We have drawn the image of the Hurewicz in color: in green is the unique v_1-periodic class α, and in red are the v_2-periodic classes. The latter remain periodic in *tmf* through the identification $v_2^9 = \Delta^6$ (or maybe $v_2^{9/2} = \Delta^3$?). Once again, the white numbers in the squares indicate the size of $\mathrm{coker}(\pi_*(tmf)\to MF_*)_{(3)}$. The algebra $\pi_*(tmf)_{(3)}$ is finitely generated, with generators

degree:	3	8	10	12	24	27	32	36	48	56	60	72
name:	α	c_4	β	c_6	$\{3\Delta\}$	$\{\alpha\Delta\}$	$\{c_4\Delta\}$	$\{c_6\Delta\}$	$\{3\Delta^2\}$	$\{c_4\Delta^2\}$	$\{c_6\Delta^2\}$	$\{\Delta^3\}$

and many relations. It is also worthwhile noting that the classes in dimensions 3, 13, 20, 30 (mod 72) support non-trivial $\langle\alpha,\alpha,-\rangle$ Massey products.

When localized at a prime $p\ge 5$, the homotopy ring of *tmf* becomes isomorphic to MF_*. Since $\Delta\in MF_{12}$ is a $\mathbb{Z}_{(p)}$-linear combination of c_4^3 and c_6^2, this ring then simplifies to $\pi_*(tmf)_{(p)} = \mathbb{Z}_{(p)}[c_4, c_6]$.

2. Localizations of *tmf*

The periodic version of *tmf* goes by the name *TMF*. Its homotopy groups are given by

$$\pi_*(TMF) = \pi_*(tmf)\big[\{\Delta^{24}\}^{-1}\big].$$

The groups $\pi_n(TMF)$ are finitely generated abelian groups, except for $n \equiv 0, 1, 2, 4 \pmod 8$ in which case they contain summands isomorphic to $\mathbb{Z}[x]$, $(\mathbb{Z}/2\mathbb{Z})[x]$, $(\mathbb{Z}/2\mathbb{Z})[x]$, and $\mathbb{Z}[x]$, respectively.

Fix a prime p, and let $K(n)$ denote the nth Morava K-theory at that prime (p is omitted from the notation). We can then consider the $K(n)$-localization $L_{K(n)}tmf$ of the spectrum tmf. The spectrum $L_{K(0)}tmf$ is simply the rationalisation of tmf (and does not depend on p). Its homotopy ring is therefore given by

$$\pi_*(L_{K(0)}tmf) = \pi_*(tmf) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

The homotopy groups of $L_{K(1)}tmf$ are easiest to describe at the primes 2 and 3. In those cases, they are given by

$$\begin{aligned} \pi_*(L_{K(1)}tmf) &= \Big(\pi_*(KO)[j^{-1}]\Big)^{\hat{}}_p \\ &= \pi_*(KO)^{\hat{}}_p \,\langle j^{-1}\rangle \end{aligned} \qquad p = 2 \text{ or } 3. \tag{3}$$

Here, the notation $R\langle x\rangle$ refers to powers series $\sum_{k=0}^{\infty} a_k x^k$ whose coefficients $a_k \in R$ tend to zero p-adically as $k \to \infty$. The variable is called j^{-1} because its inverse corresponds to the j-invariant of elliptic curves. The reason why (3) is simpler at $p = 2$ and 3 is that, at those primes, there exists only one supersingular elliptic curve and its j-invariant is equal to zero. For general prime $p \geq 3$, let $\alpha_1, \ldots, \alpha_n$ denote the supersingular j-values. Each element α_i is a priori only an element of $\mathbb{F}_{p^n}$ (actually in $\mathbb{F}_{p^2}$), however, their union $S := \{\alpha_1, \ldots, \alpha_n\}$ is always a scheme over $\mathbb{F}_p$. Let $\tilde{S}$ denote any sheme over $\mathbb{Z}$ whose reduction mod p is S. The homotopy groups of $L_{K(1)}tmf$ are then given by

$$\pi_*(L_{K(1)}tmf) = \Big(\text{functions on } \ \mathbb{P}^1_{\mathbb{Z}} \setminus \tilde{S}\Big)^{\hat{}}_p [\, b^{\pm 1}\,], \qquad p \geq 3,$$

where b is a class in degree 4.

The homotopy ring of $L_{K(2)}tmf$ is the completion of $\pi_* TMF$ at the ideal generated by p and by the Hasse invariant E_{p-1}:

$$\pi_*(L_{K(2)}tmf) = \pi_*(TMF)^{\hat{}}_{(p,E_{p-1})}, \qquad p \ \text{arbitrary},$$

where the Hasse invariant is a polynomial in c_4 and c_6 whose zeroes correspond to the supersingular elliptic curves. Once again, given the fact that there is a unique supersingular elliptic curve at $p = 2$ and 3, the above formula simplifies to

$$\pi_*(L_{K(2)}tmf) = \pi_*(TMF)^{\hat{}}_{(p,c_4)} \qquad p = 2, 3.$$

For $n > 2$, the localization $L_{K(n)}tmf$ is trivial, and therefore satisfies

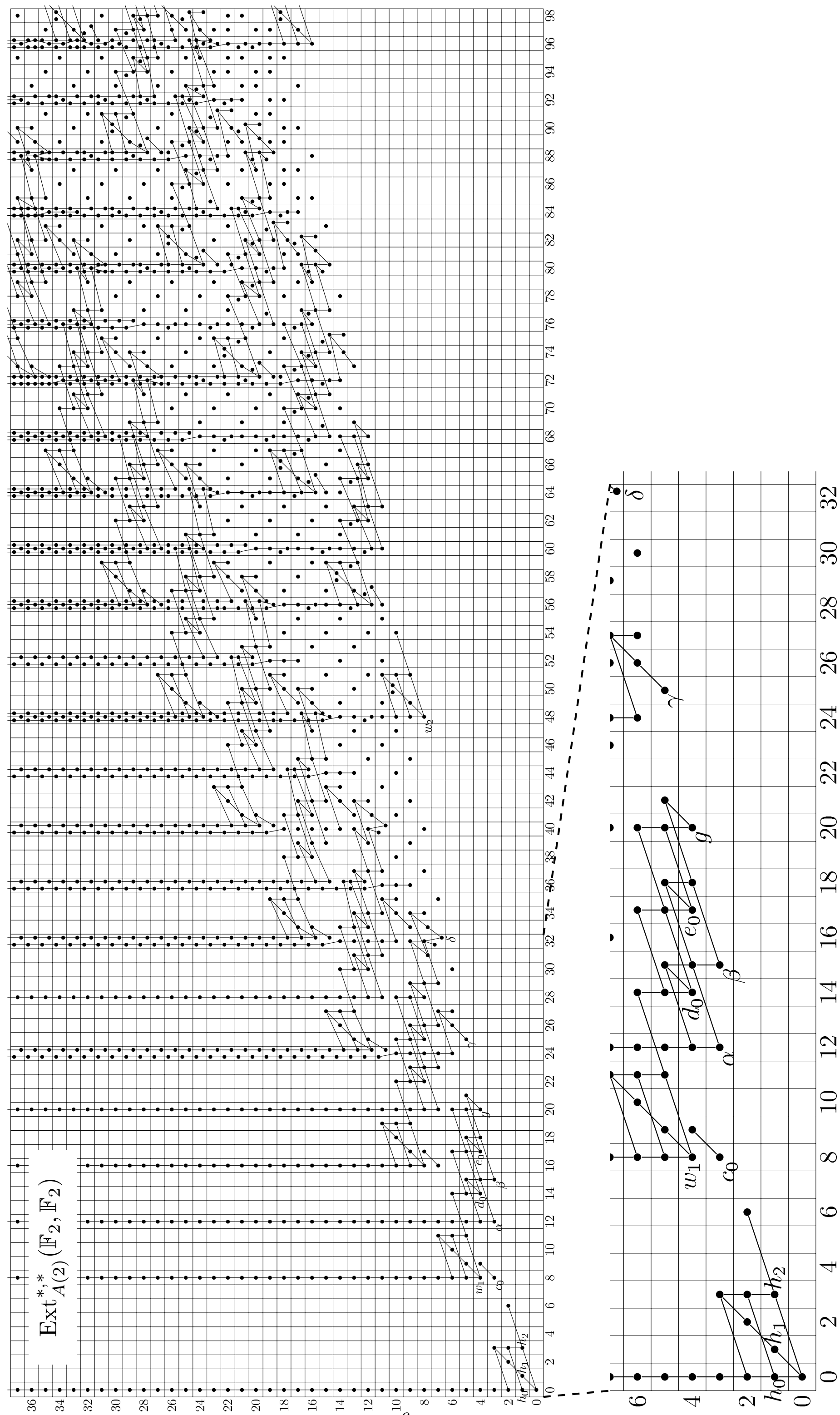
$$\pi_*(L_{K(n)}tmf) = 0.$$

3. The Adams spectral sequence

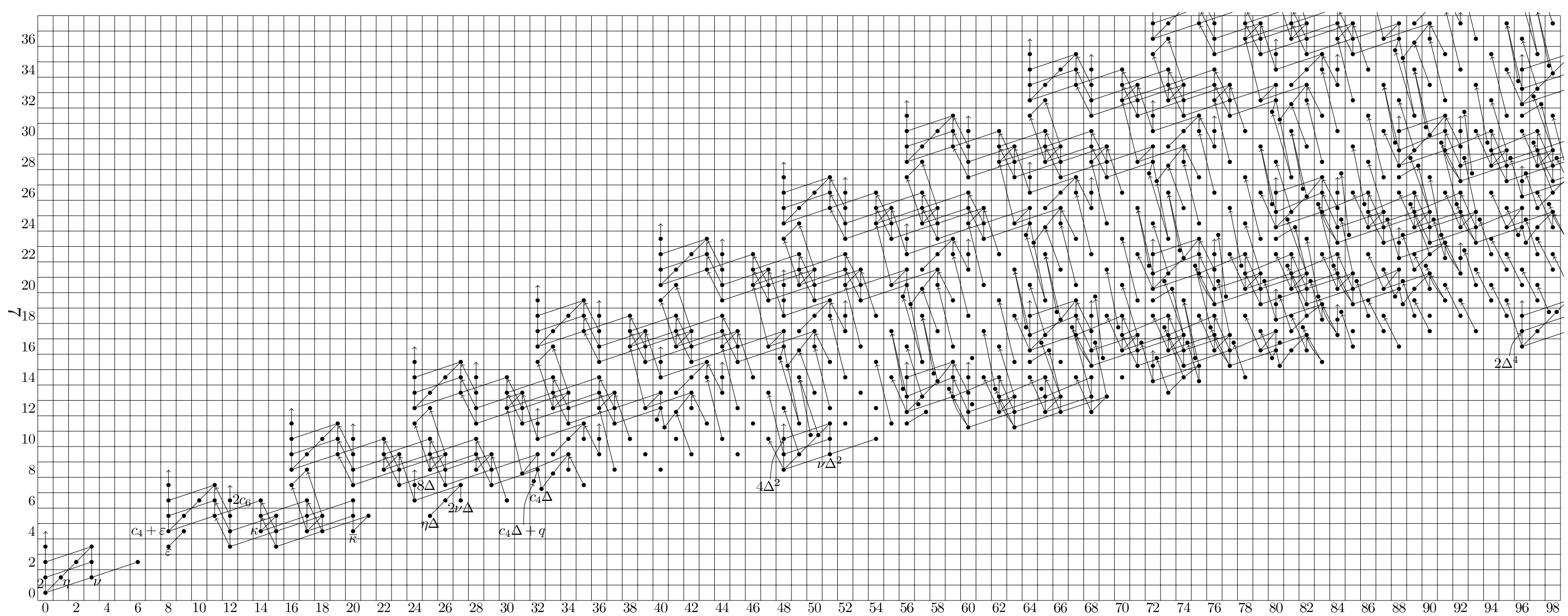
Given a fixed prime p, the Adams spectral sequence for tmf is a spectral sequence that converges to $\pi_*(tmf)^{\hat{}}_p$, see [**BL**]. Its E_2 page is given by $\mathrm{Ext}_{A_p^{tmf}}(\mathbb{F}_p, \mathbb{F}_p)$, where A_p^{tmf} is a finite dimensional $\mathbb{F}_p$-algebra that is a tmf-analog of the Steenrod algebra:

$$A_p^{tmf} := \hom_{tmf-\text{modules}}(H\mathbb{F}_p, H\mathbb{F}_p).$$

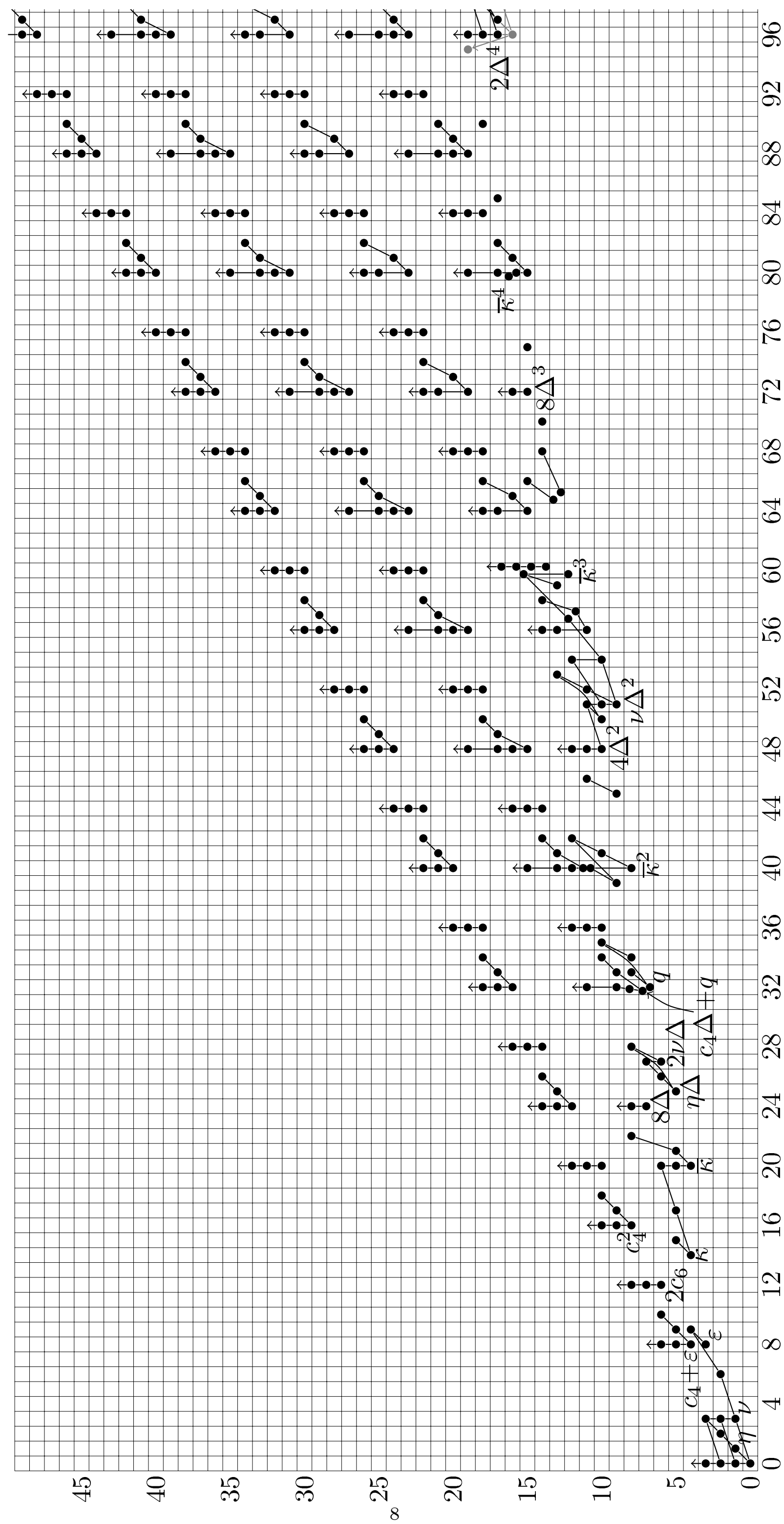
The E_2 page of the Adams spectral sequence for *tmf* at the prime 2, with classes named as in (4).



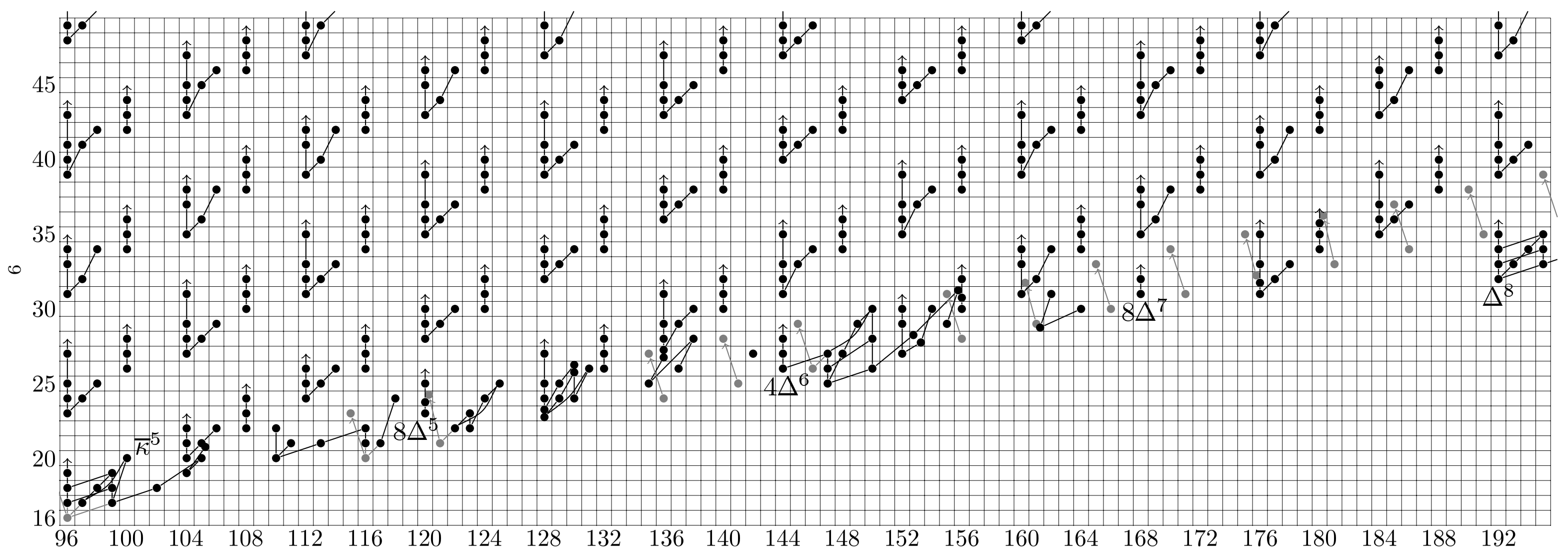
The differentials in the Adams spectral sequence for tmf at the prime 2.



The E_∞ page of the Adams spectral sequence for tmf at the prime 2.



The E_∞ page of the Adams spectral sequence for tmf, along with the most important differentials (continued).



At the prime 2, the natural map $A_2^{tmf} \to A \equiv A_2$ to the Steenrod algebra is injective. Its image is the subalgebra $A(2) \subset A$ generated by Sq^1, Sq^2 and Sq^4. That algebra is of dimension 64 over $\mathbb{F}_2$, and defined by the relations

$$Sq^1Sq^1 = 0, \quad Sq^2Sq^2 = Sq^1Sq^2Sq^1,$$
$$Sq^1Sq^4 + Sq^4Sq^1 + Sq^2Sq^1Sq^2 = 0, \text{ and}$$
$$Sq^4Sq^4 + Sq^2Sq^2Sq^4 + Sq^2Sq^4Sq^2 = 0.$$

By the change of rings theorem, the Adams spectral sequence for *tmf* can then be identified with the classical Adams spectral sequence (see [**Rez**])

$$\begin{aligned} E_2 &= \mathrm{Ext}_A\big(H^*(tmf), \mathbb{F}_2\big) \\ &= \mathrm{Ext}_A\big(A/\!\!/A(2), \mathbb{F}_2\big) \Rightarrow \pi_*(tmf)\hat{_2}. \end{aligned}$$

The bigraded ring $\mathrm{Ext}_{A(2)}(\mathbb{F}_2, \mathbb{F}_2)$ is generated by the classes:

(4)

bidegree:	(0,1)	(1,1)	(3,1)	(8,4)	(8,3)	(12,3)	(14,4)	(15,3)	(17,4)	(20,4)	(25,5)	(32,7)	(48,8)
name:	h_0	h_1	h_2	w_1	c_0	α	d_0	β	e_0	g	γ	δ	w_2

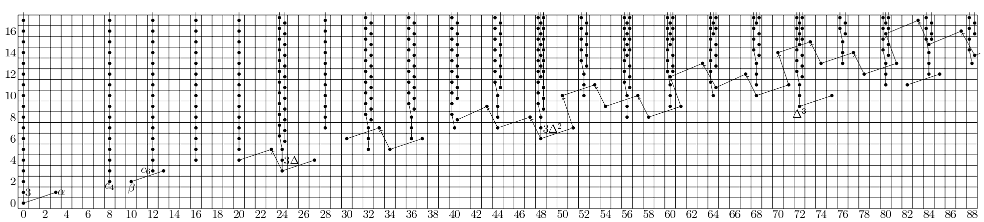
subject to the following complete set of relations:

$h_0h_1 = 0$, $h_1h_2 = 0$, $h_0^2h_2 = h_1^3$, $h_0h_2^2 = 0$, $h_2^3 = 0$, $h_0c_0 = 0$, $h_1^2c_0 = 0$, $h_2c_0 = 0$, $c_0^2 = 0$, $c_0d_0 = 0$, $c_0e_0 = 0$, $c_0g = 0$, $c_0\alpha = h_0^2g$, $c_0\beta = 0$, $h_0^2d_0 = h_2^2w_1$, $h_1d_0 = h_0^2\beta$, $h_2d_0 = h_0e_0$, $d_0^2 = w_1g$, $d_0g = e_0^2$, $h_0\alpha d_0 = h_2\beta w_1$, $\alpha^2d_0 = \beta^2w_1$, $\beta d_0 = \alpha e_0$, $h_1e_0 = h_2^2\alpha$, $h_2e_0 = h_0g$, $\beta e_0 = \alpha g$, $h_1g = h_2^2\beta$, $h_2g = 0$, $e_0g = \alpha\gamma$, $g^2 = \beta\gamma$, $h_1\alpha = 0$, $h_1\beta = 0$, $h_2\alpha = h_0\beta$, $h_0\beta^2 = 0$, $h_2\beta^2 = 0$, $\alpha^4 = h_0^4w_2 + g^2w_1$, $h_0\gamma = 0$, $h_1^2\gamma = h_2\alpha^2$, $h_2\gamma = 0$, $c_0\gamma = h_1\delta$, $d_0\gamma = \alpha^2\beta$, $e_0\gamma = \alpha\beta^2$, $g\gamma = \beta^3$, $\gamma^2 = h_1^2w_2 + \beta^2 g$, $h_0\delta = h_0\alpha g$, $h_1^2\delta = h_0d_0g$, $h_2\delta = 0$, $c_0\delta = 0$, $d_0\delta = 0$, $e_0\delta = 0$, $g\delta = 0$, $\alpha\delta = 0$, $\beta\delta = 0$ $\gamma\delta = h_1c_0w_2$, $\delta^2 = 0$.

At the prime 3, the map from A_3^{tmf} to the Steenrod algebra not injective. Indeed, the algebra A_3^{tmf} is 24 dimensional, while its image in the Steenrod algebra is the 12 dmensional subalgebra generated by β and $\mathcal{P}^1$. Naming the generators by their images in A_3, the following relations define A_3^{tmf} (see [**Hil**]):

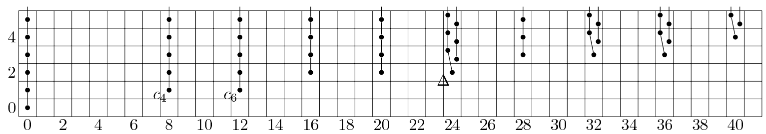
$$\beta^2 = 0, \quad (\mathcal{P}^1)^3 = 0,$$
$$\beta\,\mathcal{P}^1\beta\,\mathcal{P}^1 + \mathcal{P}^1\beta\,\mathcal{P}^1\beta = \beta\,(\mathcal{P}^1)^2\beta.$$

Note that the relation $\beta(\mathcal{P}^1)^2 + \mathcal{P}^1\beta\,\mathcal{P}^1 + (\mathcal{P}^1)^2\beta = 0$ holds in A_3, but not in A_3^{tmf}.



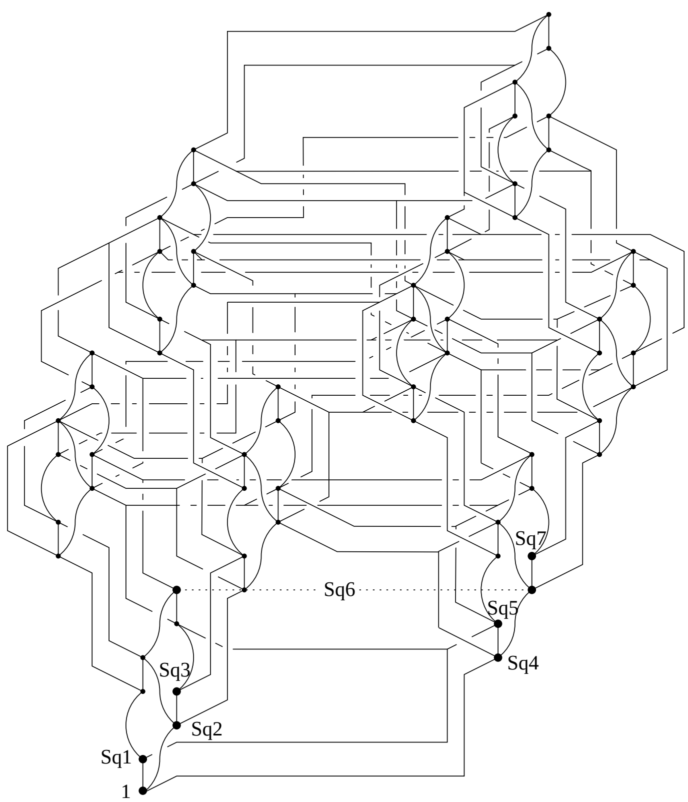
The Adams spectral sequence $\mathrm{Ext}_{A_3^{tmf}}(\mathbb{F}_3, \mathbb{F}_3) \Rightarrow \pi_*(tmf)\hat{_3}$.

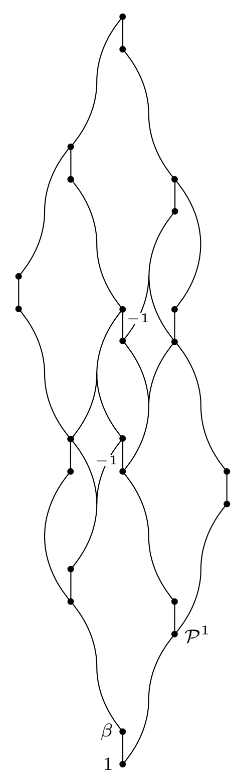
At primes $p > 3$, the algebra A_p^{tmf} is an exterior algebra on generators in degrees 1, 9, and 13. The ring $\mathrm{Ext}_{A_p^{tmf}}(\mathbb{F}_p, \mathbb{F}_p)$ is a polynomial algebra on classes in bidegrees $(0,1)$, $(8,1)$, and $(12,1)$ and the Adams spectal sequence for *tmf* collapses.



The Adams spectral sequence for *tmf* at a prime $p > 3$.

It is interesting to note that regardless of the prime, the algebra A_p^{tmf} has its top dimensional class in degree 23. Below, we picture the algebras A_p^{tmf} for the primes 2 and 3:





4. The elliptic spectral sequence

The spectrum *TMF* is the global sections of a sheaf of E_∞ ring spectra over the moduli space of elliptic curves $\mathcal{M}_{ell}$. As far as we know, there is no moduli space yielding *tmf* that way. To construct *tmf*, one first considers the sheaf $\mathcal{O}^{\mathrm{top}}$ of E_∞ ring spectra over the Deligne-Mumford compactification $\overline{\mathcal{M}}_{ell}$. The spectrum of global sections

$$Tmf := \Gamma(\overline{\mathcal{M}}_{ell}; \mathcal{O}^{\mathrm{top}})$$

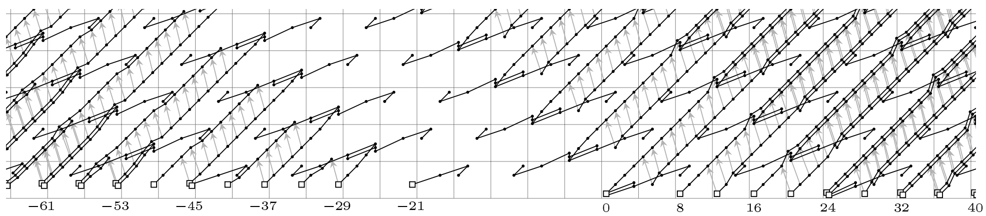
is not connective, and its connective cover is tmf. One can also recover $\pi_*(Tmf)$ from $\pi_*(tmf)$ by the following "Serre duality" short exact sequence:

$$0 \to \operatorname{Ext}^1\big(\pi_{n-22}(tmf), \mathbb{Z}\big) \to \pi_{-n}(Tmf) \to \operatorname{Hom}\big(\pi_{n-21}(tmf), \mathbb{Z}\big) \to 0, \quad n > 0.$$

The "elliptic spectral sequence" is the descent spectral sequence

$$H^s(\overline{\mathcal{M}}_{ell}; \pi_t \mathcal{O}^{\mathrm{top}}) = H^s\left(\overline{\mathcal{M}}_{ell}\,; \begin{Bmatrix} \omega^{t/2} & \text{if } t \text{ is even} \\ 0 & \text{if } t \text{ is odd} \end{Bmatrix}\right) \;\Rightarrow\; \pi_{t-s}(Tmf).$$

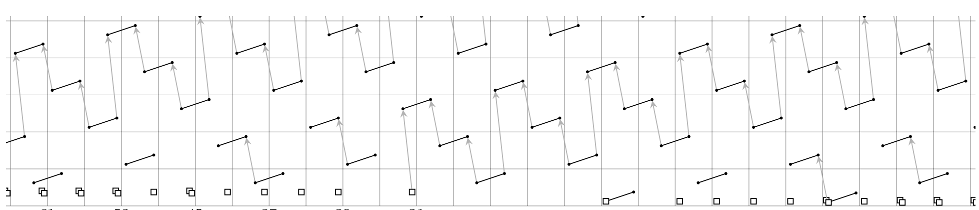
Its E_2 page at $p = 2$ is as follows (see [**Bau, Kon**]):



The E_2 page of the elliptic spectral sequence at $p = 2$.

In the above chart, squares indicate copies of $\mathbb{Z}_{(2)}$ and bullets indicate copies of $\mathbb{Z}/2\mathbb{Z}$. Two (three) bullets stacked vertically onto each other indicate a copy of $\mathbb{Z}/4\mathbb{Z}$ ($\mathbb{Z}/8\mathbb{Z}$). The d_3 differentials are drawn in gray; the remaining differentials $d_5, d_7, d_9, \ldots, d_{25}$ are drawn on the charts on pages 13 − 16. The colors on those charts indicate the periodicity by which the various sub-patterns of differentials repeat.

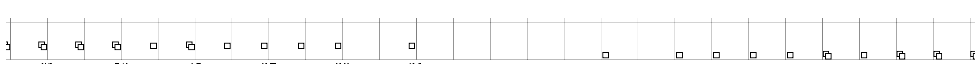
At $p = 3$, the elliptic spectral sequence looks as follows:



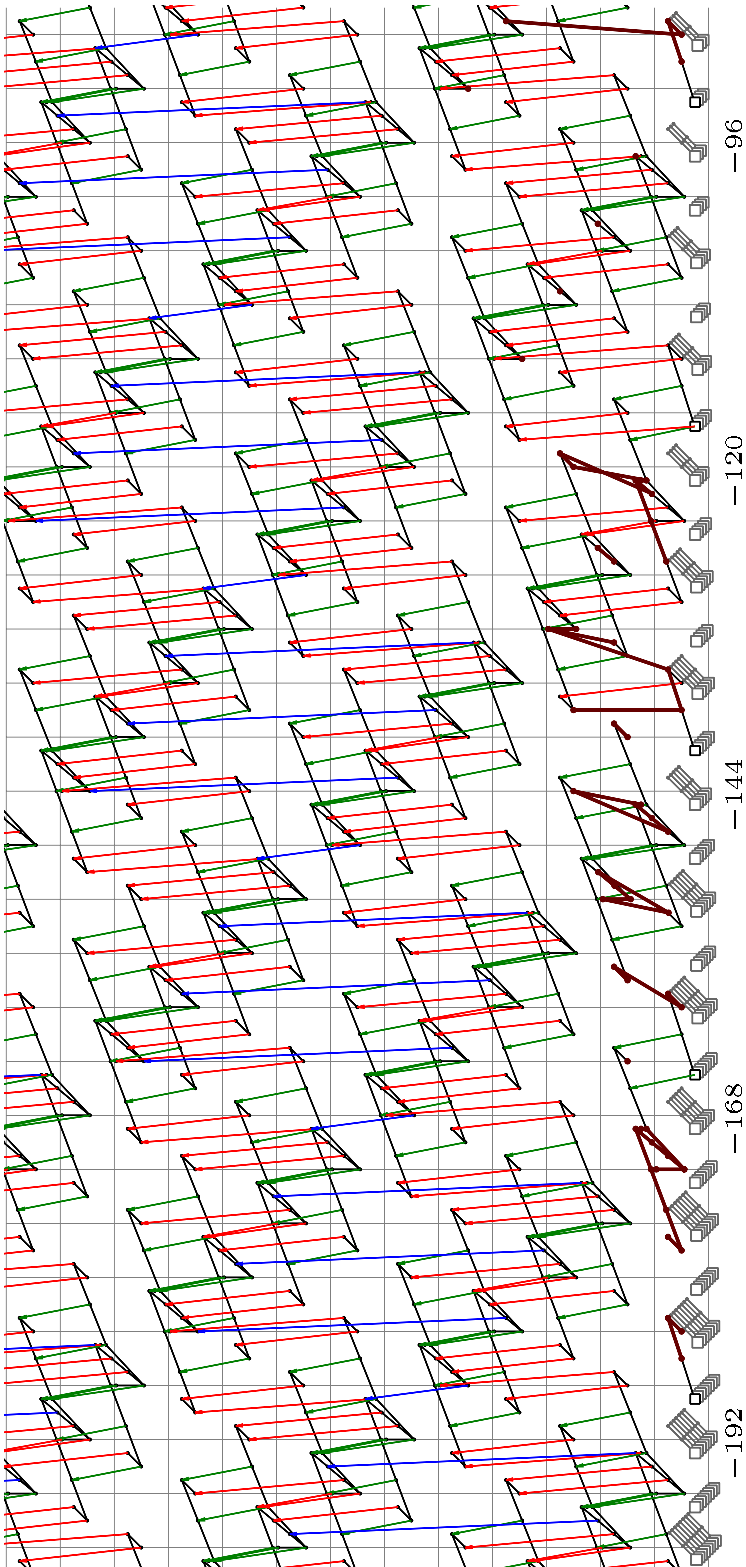
The elliptic spectral sequence at $p = 3$.

The squares indicate copies of $\mathbb{Z}_{(3)}$, and the bullets indicate copies of $\mathbb{Z}/3\mathbb{Z}$.

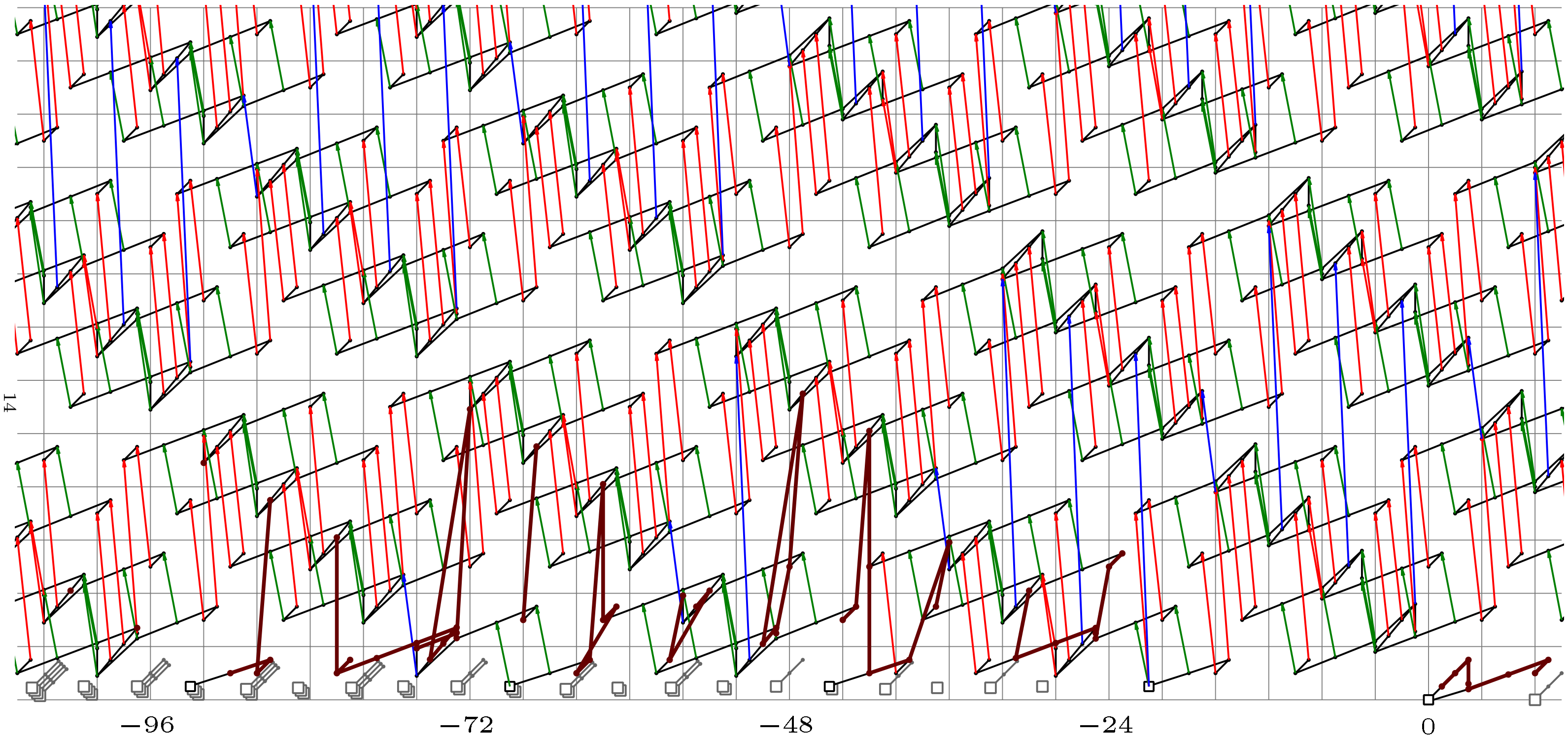
At $p \geq 5$, there is no p-torsion and the elliptic spectral sequence collapses:



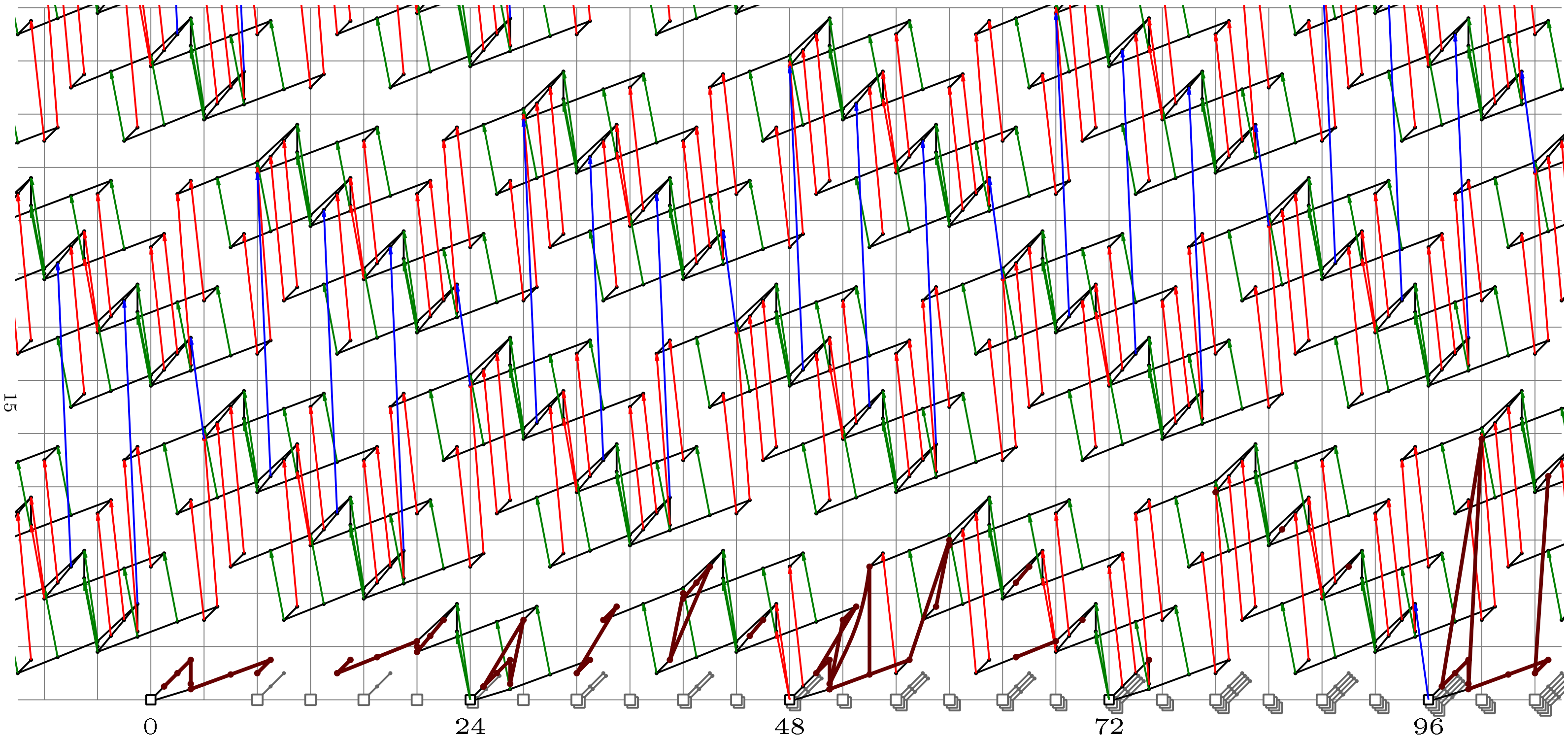
The elliptic spectral sequence at $p \geq 5$.



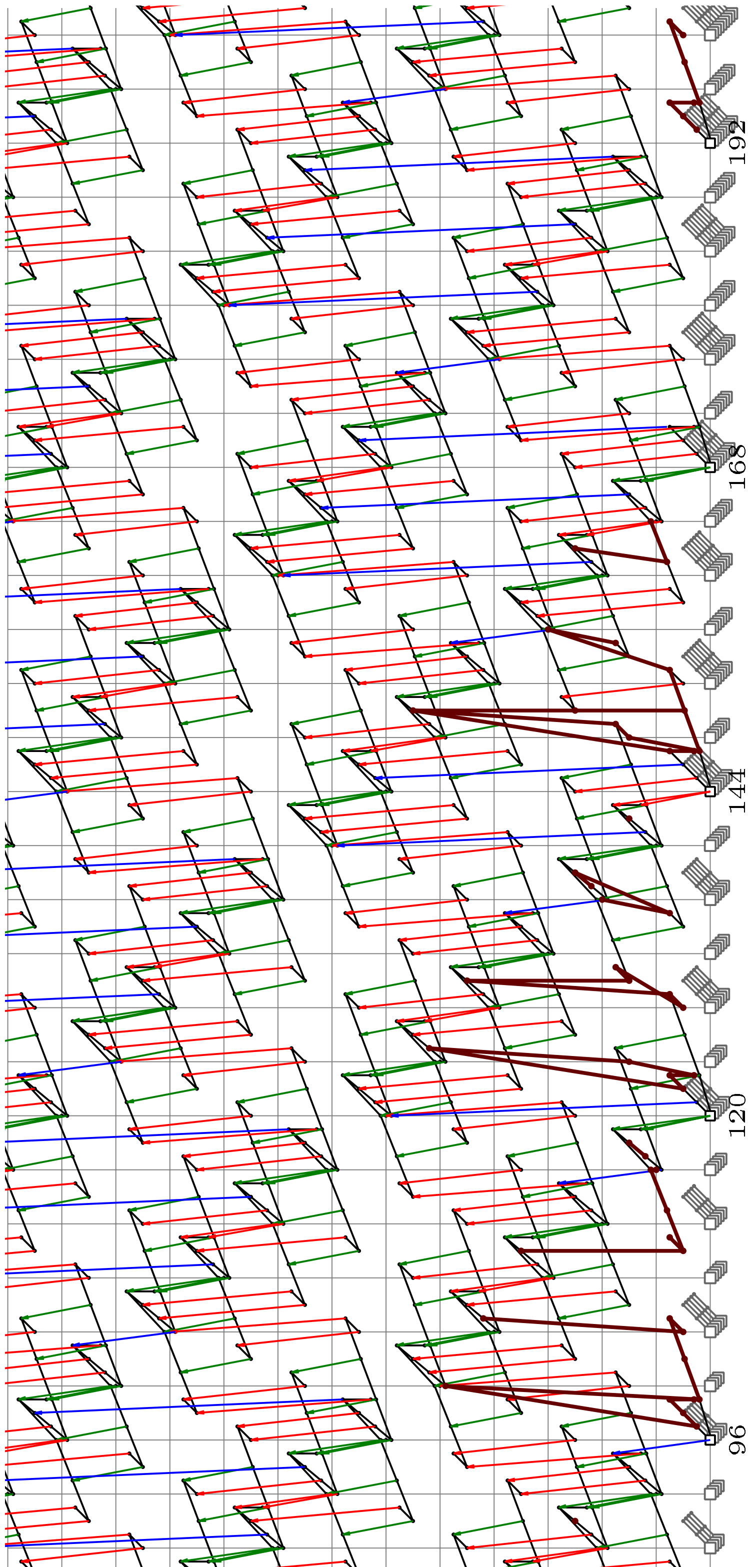
The elliptic spectral sequence $H^q(\overline{\mathcal{M}}_{Ell};\omega^{\otimes p}) \Rightarrow \pi_{2p-q}(Tmf)$ for $2p-q$ between -202 and -86.



The elliptic spectral sequence $H^q(\overline{\mathcal{M}}_{Ell};\omega^{\otimes p}) \Rightarrow \pi_{2p-q}(Tmf)$ for $2p-q$ between -106 and 10.



The elliptic spectral sequence $H^q(\overline{\mathcal{M}}_{Ell}; \omega^{\otimes p}) \Rightarrow \pi_{2p-q}(Tmf)$ for $2p - q$ between -10 and 106.



The elliptic spectral sequence $H^q(\overline{\mathcal{M}}_{Ell}; \omega^{\otimes p}) \Rightarrow \pi_{2p-q}(Tmf)$ for $2p-q$ between 86 and 202.

5. acknowledgements

I am seriously indebted to many people for the completion of this survey on the homotopy groups of *tmf*. First of all, I would like to thank Tilman Bauer for spending an enormous amount of time with me sorting out the multiplicative structure of $\pi_*(tmf)$. Based on a computer program of Christian Nassau, Tilman is also the one who created the first draft of the $\mathrm{Ext}_{A(2)}$ charts. I thank Mark Behrens for sending me his calculations of the Adams spectral sequence, and for answering many questions about the multiplicative structure of $\pi_*(tmf)$. The defining relations of $\mathrm{Ext}_{A(2)}$ were taken from a calculation of Peter May, who was made available to me by Mark Behrens. Next, I would like to thank Mark Mahowald for some very informative email exchanges. I am especially grateful to him for explaining me how to detect v_2^{32}-periodic families of elements in the image of Hurewicz homomorphism, and for agreeing to make the following conjecture:

Conjecture (Mark Mahowald). *The image of the Hurewicz homomorphism $\pi_*(\mathbb{S}) \to \pi_*(tmf)$ is given by the classes drawn in color in the pictures on page 2.*

I thank Vesna Stojanoska for comments on an early draft, and Niko Nauman for useful conversations. I am indebted to Mike Hopkins for putting me in contact with Mark Mahowald. I also thank my master student Johan Konter for working out the structure of the elliptic spectral sequence at $p = 2$, following suggestions by Mike Hill. Finally, I would like to thank Mike Hill (who was the first to figure out the structure of A_3^{tmf}) for the numerous conversations that we had while we were graduate students at MIT.

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Elliptic curves and stable homotopy I

M. J. Hopkins and H. R. Miller

Abstract. In this note a new cohomology theory will be constructed. From the point of view of homotopy theory, it is the most natural candidate for elliptic cohomology. The construction is via obstruction theory.

Contents

1. Introduction

The point of this document is to make use of A_∞ moduli spaces to rigidify the diagram of étale elliptic spectra. The main theorem is

Theorem 1.1. *Let $\mathcal{M}_{\mathrm{Ell}}$ be the moduli stack of generalized elliptic curves, and $(\mathcal{M}_{\mathrm{Ell}})_{\mathrm{et}}$ the category of étale open subsets of $\mathcal{M}_{\mathrm{Ell}}$. Let $(\mathcal{M}_{\mathrm{Ell}})'_{\mathrm{et}}$ denote the*

Date: Jan 5, 1996.

category of affine étale open subsets of $\mathcal{M}_{\mathrm{Ell}}$ *over which the sheaf of invariant differentials is trivial. There is a presheaf*

$$E : (\mathcal{M}_{\mathrm{Ell}})_{\mathrm{et}} \to A_\infty^{\mathrm{Ell}}$$

with values in the category of cofibrant-fibrant A_∞ *elliptic spectra (see below) which has cohomological descent and, when restricted to* $(\mathcal{M}_{\mathrm{Ell}})'_{\mathrm{et}}$ *comes equipped with an isomorphism of functors*

$$\operatorname{spec} \pi_0 E(U) \approx 1.$$

Any two such presheaves are naturally weakly equivalent.

Definition 1.2. An *elliptic spectrum* is a triple (E, A, t) in which

(1) E is a multiplicative spectrum with the property that $\pi_* E$ commutative, concentrated in even degrees, and $\pi_2 E$ contains a unit;

(It follows that E is complex orientable and has a formal group G canonically defined over $\pi_0 E$.)

(2) A is a generalized elliptic curve [**DR**] over $\pi_0 E$

(3) $t : G \to \hat{A}$ is an isomorphism of the formal group G with the formal completion of A.

The elliptic spectra form a category in which a map

$$f : (E, A, t) \to (E', A', t')$$

consists of a multiplicative map $f_1 : E \to E'$, and a map of generalized elliptic curves $f_2 : A' \to \pi_0 f_1 A$ such that the following diagram commutes

$$\begin{tikzcd} G' \arrow[r, "f_1"] \arrow[d, "t'"'] & \pi_0 f_1 G \arrow[d, "t"] \\ \hat{A}' \arrow[r, "\hat{f}_2"'] & \pi_0 f_1 \hat{A} \end{tikzcd}$$

Let $\mathcal{M}_{\mathrm{Ell}}$ be the moduli stack of generalized elliptic curves with integral geometric fibers (§ 6). An elliptic spectrum (E, A, t) comes equipped with a map

$$\operatorname{spec} \pi_0 E \to \mathcal{M}_{\mathrm{Ell}}$$

classifying the elliptic curve A.

Definition 1.3. An elliptic spectrum (E, A, t) is *étale* if the map $\operatorname{spec} \pi_0 E \to \mathcal{M}_{\mathrm{Ell}}$ is an open subset of $\mathcal{M}_{\mathrm{Ell}}$ is the étale topology.

For a stack $\mathcal{M}$ let $(\mathcal{M})_{\mathrm{et}}$ be the category of étale open subsets of $\mathcal{M}$ of finite type. Attaching to each étale elliptic spectrum the map classifying its elliptic curve thus defines a functor from the category of étale elliptic spectra to the category $(\mathcal{M}_{\mathrm{Ell}})_{\mathrm{et}}$. One goal of this paper is to “reverse” this process and construct a kind of sheaf on $(\mathcal{M}_{\mathrm{Ell}})_{\mathrm{et}}$ with values in A_∞ ring spectra.

Let A_∞^{Ell} be the (topological) category consisting of triples (E, A, t) consisting of an A_∞ ring spectrum E together with an étale elliptic structure on its underlying spectrum, and whose morphism space from (E_1, A_1, t_1) to (E_2, A_2, t_2) is the fiber product

$$\begin{tikzcd}
A_\infty^{\mathrm{Ell}}\left((E_1, A_1, t_1), (E_2, A_2, t_2)\right) \arrow[r] \arrow[d] & \mathcal{S}^{\mathrm{Ell}}\left((E_1, A_1, t_1), (E_2, A_2, t_2)\right) \arrow[d] \\
A_\infty\left(E_1, E_2\right) \arrow[r] & \mathcal{S}\left(E_1, E_2\right).
\end{tikzcd}$$

Then the composite $\operatorname{spec}\pi_0$ defines a functor

$$\operatorname{ho} A_\infty^{\mathrm{Ell}} \to \left(\mathcal{M}_{\mathrm{Ell}}\right)_{\mathrm{et}}.$$

Occasionally the symbol $\operatorname{spec}\pi_0$ will also refer to the composite functor $A_\infty^{\mathrm{Ell}} \to \left(\mathcal{M}_{\mathrm{Ell}}\right)_{\mathrm{et}}$.

Let $\left(\mathcal{M}_{\mathrm{Ell}}\right)'_{\mathrm{et}}$ be the subcategory of $\left(\mathcal{M}_{\mathrm{Ell}}\right)_{\mathrm{et}}$ consisting of affine $U \to \mathcal{M}_{\mathrm{Ell}}$ for which the sheaf ω_U admits a nowhere vanishing global section (ie the restriction of the line bundle ω to U is trivializable). Our first main result is that the functor $\operatorname{spec}\pi_0$ admits a more or less canonical section over $\left(\mathcal{M}_{\mathrm{Ell}}\right)'_{\mathrm{et}}$

$$\left(\mathcal{M}_{\mathrm{Ell}}\right)'_{\mathrm{et}} \to \operatorname{ho} A_\infty^{\mathrm{Ell}}.$$

Next we will apply the Dwyer-Kan lifting machinery to the the problem

$$\begin{tikzcd}
 & & A_\infty \arrow[d] \\
\left(\mathcal{M}_{\mathrm{Ell}}\right)'_{\mathrm{et}} \arrow[r] \arrow[urr, dashed] & \operatorname{ho} A_\infty^{\mathrm{Ell}} \arrow[r] & \operatorname{ho} A_\infty.
\end{tikzcd}$$

We will see that the space of lifts is non-empty and connected, thus giving an essentially unique diagram. This diagram of spectra *is* elliptic cohomology.

2. A_∞-structures and maps

2.1. A_∞-ring spectra. We will use the theory of spectra and ring spectra as developed by May et. al. Let A_∞ denote the category of A_∞-ring spectra indexed on a fixed universe $\mathcal{U}$. This is a closed topological model category and is generated by small objects. The functor

$$X \mapsto TX = \bigvee \mathcal{L}(n) \rtimes X^{(n)},$$

is a monad (or triple), and an A_∞-ring spectrum is an algebra over this monad.

There are adjoint functors

$$T : \mathcal{S} \rightleftarrows A_\infty : F,$$

where by a mild abuse of notation, the functor T is regarded as taking its values in A_∞, and F is simply the forgetful functor. As the diagram suggests, the functor T is left adjoint to F. The morphism objects of A_∞, being subsets of the morphism objects of $\mathcal{S}$ are naturally topological spaces.

The following result also holds with A_∞ replaced by M-algebras, where M is any operad over the linear isometries operad:

THEOREM 2.1. i) *The category A_∞ is canonically enriched over topological spaces. In other words, there are bifunctors*

$$(S,X) \mapsto S \otimes X : \mathfrak{T} \times A_\infty \to A_\infty$$
$$(Y,S) \mapsto Y^S : A_\infty \times \mathfrak{T}^{op} \to A_\infty$$

and natural homeomorphisms

$$A_\infty(X, Y^S) \approx A_\infty(S \otimes X, Y) \approx \mathfrak{T}(S, A_\infty(X,Y)).$$

ii) *The category A_∞ is cartesian closed in the sense of enriched categories: it contains all small indexed limits and colimits;*

iii) *The category A_∞ has a unique closed model category structure in which a map $X \to Y$ is a weak equivalence (resp. fibration) if and only if the map $FX \to FY$ is. This model category structure is generated by small objects and is a topological model category in the sense that Quillen's axiom SM7* [**Qu**] *holds, with "simplicial set" replaced by "toplogical space."*

iv) *If $X_\bullet$ is a simplicial object in A_∞, then the natural map*

$$|FX_\bullet|_{\mathcal{S}} \to F|X_\bullet|_{A_\infty}$$

is an isomorphism.

DEFINITION 2.2. Suppose that X is a spectrum. The space $A_\infty\{X\}$ is the nerve of the category whose objects are weak equivalences $F(R) \to X$, and whose morphisms are commutative squares of weak equivalences

$$\begin{tikzcd} F(R) \arrow[r, "F(f)"] \arrow[d] & F(S) \arrow[d] \\ X \arrow[r, equal] & X. \end{tikzcd}$$

In other words, if $(A_\infty)_w$ denotes the topological category of A_∞ ring spectra and weak equivalences, then $A_\infty\{X\}$ is the nerve of the category of $(A_\infty)_w$ objects over X.

In case X comes equipped with a homotopy associative multiplication μ we will let $A_\infty\{X\}_\mu$ denote the nerve of the full subcategory with objects f which are maps of homotopy associative ring spectra. The space $A_\infty\{X\}$ is the disjoint union of the spaces $A_\infty\{X\}_\mu$ as μ ranges over the different homotopy associative multiplications on X (monoid structures on X in $\mathrm{ho}\,\mathcal{S}$), since the same is true for the over-category. In case the multiplication μ is an integral part of the ambient structure of X (like when X is elliptic) the subscript will be dropped from the notation.

Rezk has shown that the space $A_\infty\{X\}$ is weakly equivalent to the space of operad maps from the A_∞ operad to the endomorphism operad of X.

2.2. The moduli spaces in a useful case. We are interested in determining the homotopy types of the spaces $A_\infty\{X\}_\mu$ and $A_\infty(E,F)$. It turns out that a lot can be said about these spaces in the presence of certain special conditions. Suppose that F is a ring spectrum satisfying the following condition of Adams [**Ad**, Condition 13.3 (page 284)]:

CONDITION 2.3. *Each map of spectra*

$$X \to F$$

with X finite, factors as

$$X \to V \to F$$

where V is finite and both F_*V ane F^*V are projective over F_*.

Examples of spectra satisfying this abound. In fact

PROPOSITION 2.4. *Condition 2.3 holds if F is Landweber exact.*

PROPOSITION 2.5. *Suppose that (F, μ) is a ring spectrum satisfying Condition 2.3. There is a spectral sequence*

$$E_2^{s,t} \Longrightarrow \pi_{t-s}(A_\infty \{F\}_\mu, f),$$

with

$$E_2^{s,t} = \begin{cases} \mathrm{Der}^{s+1}_{\pi_* F}\left(F_*F, \tilde{F}^0 S^t\right) & s \geq 0, \quad t \geq 1 \\ * & (s,t) = (0,0) \\ \textit{undefined} & \textit{otherwise} \end{cases}$$

The group $\mathrm{Der}^s(\,-\,,\,-\,)$ is Quillen's sth derived functor of associative algebra derivations, and is described in § 5.

PROPOSITION 2.6. *Suppose that E and F are A_∞ ring spectra, F satisfies Condition 2.3, and $f : E \to F$ is a homotopy class of maps of the underlying ring spectra. There is a spectral sequence*

$$E_2^{s,t} \Longrightarrow \pi_{t-s}(A_\infty(E, F); \tilde{f})$$

with

$$E_2^{s,t} = \begin{cases} \mathrm{Der}^{s}_{F_*}\left(F_*E, \tilde{F}^0 S^t\right) & t \geq 1 \\ * & (s,t) = (0,0) \\ \textit{undefined} & t = 0, \quad s > 0 \end{cases}$$

3. Lifting diagrams

This section consists of a collection of material from [**DK**].

3.1. The realization space. Let $\mathbf{S}$ be a closed simplicial (or topological) model category, and I a small category. The situation relevant to this paper is that I is the opposite category of $(\mathcal{M}_{\mathrm{Ell}})'_{\mathrm{et}}$, and $\mathbf{S}$ is the category A_∞. We suppose given an I-diagram in $\mathrm{ho}\,\mathbf{S}$ (ie, an object of $(\mathrm{ho}\,\mathbf{S})^I$, and we want to consider the problem of lifting it to an object of $\mathbf{S}^I$. This problem can be attacked by studying the realization space $\mathrm{r}(X)$ of X.

DEFINITION 3.1. With the above notation, the *realization category* of X is category whose objects are pairs (Y, f) consisting of an object Y of $\mathbf{S}^I$, together with an isomorphism $f : \pi Y \to X$ in $(\mathrm{ho}\,\mathbf{S})^I$. A morphism in $\mathrm{r}(X)$ from (Y_1, f_1) to

(Y_2, f_2) is a map $g : Y_1 \to Y_2$ (necessarily a weak equivalence) making the following diagram commute

$$\begin{tikzcd} \pi Y_1 \arrow[r, "\pi g"] \arrow[d, "f_1"'] & \pi Y_2 \arrow[d, "f_2"] \\ X \arrow[r, equal] & X. \end{tikzcd}$$

The *realization space*, is the classifying space of the realization category. Both the category and its classifying space are denoted $\mathrm{r}(X)$.

There is another useful space to consider in this regard. Let $\mathrm{lift}(X)$ be the nerve of the category whose objects are diagrams Y with the property that πY is isomorphic to X, and whose morphism are weak equivalences. There is a fibration

$$\mathrm{r}(X) \to \mathrm{lift}(X) \to B\,\mathrm{Aut}(X).$$

By making a simplicial resolution of I by free categories, the spaces $\mathrm{r}(X)$ and $\mathrm{lift}(X)$ can be expressed as the total complexes of cosimplicial spaces which we will call $\mathrm{r}(X)_\bullet$ and $\mathrm{lift}(X)_\bullet$ respectively. While the cosimplicial spaces depend on the choice of resolution, the homotopy spectral sequence is independent of the resolution from the E_2-term on. There is a termwise fibration

$$\mathrm{r}(X)_\bullet \to \mathrm{lift}(X)_\bullet \to B\,\mathrm{Aut}\,X$$

where $B\,\mathrm{Aut}\,X$ is thought of as a constant cosimplicial space. The homotopy spectral sequences for $\mathrm{r}(X)_\bullet$ and $\mathrm{lift}(X)_\bullet$ therefore coincide except possibly in the $E_2^{0,0}$-term where the latter is the quotient of the former by the action of $\pi_0\,\mathrm{haut}(X)$. In fact, the spectral sequences coincide anyway, since before the spectral sequence can be defined, one must choose a component of the 0-space which is equalized by d^0 and d^1, and then restrict to this component.

3.2. Centric diagrams. A diagram X as above is said to be *centric* if for each morphism $f : d \to d'$ in I, the map

$$\mathbf{S}(Xd', Xd')_1 \to \mathbf{S}(Xd, Xd')_f$$

is a weak equivalence ([**DK**]). In the case of a centric diagram there is a simple description of the E_2-term of the spectral sequence.

PROPOSITION 3.2. [**DK**, §3] *Let $X : I \to \mathrm{ho}\,\mathbf{S}$ be a centric diagram. There is a spectral sequence*

$$E_2^{s,t} \implies \pi_{t-s}\,\mathrm{r}(X)\,,$$

with

$$E_2^{s,t} = \begin{cases} \varprojlim^s \pi_t\,\mathrm{Bhaut}_1\,X(-) & s \geq 0 \textit{ and } t \geq 2 \\ \{1\} & (s,t) = (0,1) \\ * & (s,t) = (0,0) \textit{ or } (1,1) \\ \textit{undefined} & \textit{otherwise.} \end{cases}$$

REMARK 3.3. It is only the identity component of $\mathrm{haut}(X)$ that comes in, so that the spaces $\mathrm{Bhaut}_1(X)$ are, in fact, simply connected.

4. Quillen (co-)homology

The basic reference for this and the following section is [**Qu**].

4.1. Generalities. Fix a ground ring k and let $i : A \to B$ be a map of commutative k-algebras. Quillen defines a homology object $L_{B/A}$ as follows. First he shows that the category of simplicial A-algebras is a closed model category when one defines weak equivalence to be quasi-isomorphism, and one takes all objects to be fibrant. To define the homology object, regard B and A as constant simplicial objects, and let $A \hookrightarrow P_\bullet \xrightarrow{\simeq} B$ be a factorization into a cofibration followed by a weak equivalence. The complex $L_{B/A}$ is then defined by

$$L_{B/A} = \Omega^1_{P_\bullet/A} \underset{P_\bullet}{\otimes} B,$$

where Ω^1 is the $P_\bullet$-module of relative Kähler differentials. The quasi-isomorphism class of $L_{B/A}$ is independent of the resolution.

The association $B/A \mapsto L_{B/A}$ is functorial: given a map of pairs

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & B \arrow[d, "j"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

there is a natural map $L_{j/i} : j^* L_{B/A} \to L_{D/C}$. The map associated to a composite of maps of pairs is chain homotopic to the composite of the maps associated to each pair.

Actually, for fixed A, the association $L_{B/A}$ can be made into a functor with values in the category of chain complexes of B-modules. To do this one makes use of the fact the the category of simplicial A-modules is generated by small objects, and so the factorization into a cofibration followed by acyclic fibration can be made functorially. We will make use of this rigidified functoriality. One can also simply use the standard resolution based on the triple "free A-algebra generated by the underlying point set of B."

We will often write the absolute homology $L_{B/k}$ as simply L_B.

Given a sequence $A \to B \xrightarrow{f} C$, Quillen shows that the natural sequence

$$f^* L_{B/A} \to L_{C/A} \to L_{C/B}$$

is a cofibration. This cofibration is known as the *transitivity sequence*. It is useful to write down this sequence in the case $A = k$ (which is no loss of generality, since no use has been made so far of k):

$$f^* L_B \to L_C \to L_{C/B}.$$

It shows that the relative homology object is the cofiber of the map of absolute homology objects, and that the homology of the transitivity sequence can be thought of as the long exact sequence of a pair.

Another important result is the following..

PROPOSITION 4.1 (Flat base change [**Qu**]). *Suppose that*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "j"'] & B \arrow[d, "j"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

is a pushout square with the property that the groups $\mathrm{Tor}_t^A(C,B)$ *are* 0 *for* $t>0$. *Then the natural map* $g^*L_{B/A}\to L_{D/C}$ *is an equivalence.*

Note that the condition on Tor holds if either i or f is flat.

Proof: Factor f as $A\to P_\bullet\to B$ into a cofibration followed by an acyclic fibration. Then $C\to i^*P_\bullet=C\otimes_A P_\bullet$ is a cofibration, and the natural map $i^*P_\bullet\to D$ is a weak equivalence since the s^{th} homology group of $i^*P_\bullet$ is $\mathrm{Tor}_s^A(C,B)$. The result follows easily from this. □

The transitivity sequence and the flat base change result combine to give a Meyer-Vietoris sequence.

Corollary 4.2. *Suppose given a pushout square as in Proposition 4.1. Then the square*

$$\begin{tikzcd}
(jf=gi)^*L_A \arrow[r, "j^*L_f"] \arrow[d, "g^*L_i"'] & j^*B \arrow[d, "L_j"] \\
g^*L_C \arrow[r, "L_g"'] & L_D
\end{tikzcd}$$

is a homotopy pushout. It follows that there is a cofibration

$$L_A\xrightarrow{[j^*L_f,g^*L_i]^T} j^*L_B\oplus g^*L_C\xrightarrow{[-L_j,L_g]} L_D.$$

Proof: The map of transitivity cofibrations for f and g can be written as

$$\begin{tikzcd}
j^*f^*L_A \arrow[r] \arrow[d] & j^*L_B \arrow[r] \arrow[d] & j^*L_{B/A} \arrow[d, "L_{j/i}"] \\
g^*L_C \arrow[r] & L_D \arrow[r] & L_{D/C}.
\end{tikzcd}$$

Proposition 4.1 implies that the map $L_{j/i}$ is an equivalence. The result follows. □

5. Quillen associative algebra (co-)homology

5.1. Associative algebras. For each map of associative algebras $A\to B$, let

$$D_{B/A}$$

be the complex of B bi-modules which is the Quillen homology object of B/A. In case B is flat over A (and projective as a module), this complex is weakly equivalent to the complex

$$\begin{array}{cc}
B\underset{A}{\otimes}B^{\mathrm{op}} & 0 \\
\downarrow & \\
B & -1
\end{array}$$

The object $D_{B/A}$ is functorial in B/A in the sense that a diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & A' \arrow[d] \\
B \arrow[r, "f"'] & B'
\end{tikzcd} \tag{5.1}$$

gives rise to a map $(f \otimes f)^* D_{B/A} \to D_{B'/A'}$. Given,

$$A \to B \xrightarrow{f} C$$

the sequence

$$f^* D_{B/A} \to D_{C/A} \to D_{C/B}$$

is a cofibration. The long exact sequences derived from this cofibration are known as *transitivity sequences.*

Another important tool is the following base change result.

LEMMA 5.2. *Suppose that A is commutative and that B is a flat associative A-algebra containing A in its center. For any map*

$$A \to A',$$

write

$$B' = B \underset{A}{\otimes} A',$$

and let j be the map

$$\begin{aligned} j : B \underset{A}{\otimes} B &\to B' \underset{A'}{\otimes} B' \\ b_1 \otimes b_2 &\mapsto b_1 \otimes 1 \otimes b_2 \otimes 1. \end{aligned}$$

Then the map (5.1)

$$j^* D_{B/A} \to D_{B'/A'}$$

is a weak equivalence.

Proof: To ease the notation, the symbol $\otimes$, unless indicated otherwise, will refer to $\otimes_A$. Since B/A is flat, so is B'/A', and we can make use of the cofiber sequences

$$\begin{aligned} D_{B/A} &\to B \otimes B \to B \\ D_{B'/A'} &\to B' \underset{A'}{\otimes} B' \to B'. \end{aligned}$$

Let $P. \to B$ be a projective resolution of B as a $B \otimes B$-module. Then

$$P \underset{B \otimes B}{\otimes} \left(B' \underset{A'}{\otimes} B' \right) \approx P \underset{B \otimes B}{\otimes} (B \otimes B \otimes A') \approx P \otimes A' \tag{5.3}$$

is a complex of projective $B' \otimes_{A'} B'$-modules, and its homology is

$$\operatorname{Tor}^*_A(B, A') = \begin{cases} B' & \text{if } * = 0 \\ 0 & \text{otherwise.} \end{cases}$$

It follows that (5.3) is a projective resolution of B'. Lifting the map $B \otimes B \to B$ to $P.$ gives a cofibration

$$D_{B/A} \to B \otimes B \to P.,$$

and hence a map of cofibrations

$$\begin{array}{ccccc} j^* D_{B/A} & \longrightarrow & B' \underset{A'}{\otimes} B' & \longrightarrow & j^* P. \\ \downarrow & & \downarrow & & \downarrow \\ D_{B'/A'} & \longrightarrow & B' \underset{A'}{\otimes} B' & \longrightarrow & B'. \end{array}$$

Since the second and third vertical maps are weak equivalences, so is the first. The result follows. □

5.2. Commutative rings. From now on suppose that $A \to B$ is a map of commutative rings. In this case we define a complex of B-modules

$$\Lambda_{B/A} = \mu^* D_{B/A} = B \overset{L}{\underset{B\otimes B}{\otimes}} D_{B/A}.$$

The L indicates that the tensor product is taken in the homotopy invariant sense, so that, even though the homology of the complex $D_{B/A}$ is concentrated in dimension zero, the homology of $\Lambda_{B/A}$ need not be. A sequence $A \to B \to C$ gives rise to a cofibration in the homotopy theory of chain complexes of B-modules.

$$f^*\Lambda_{B/A} \to \Lambda_{C/A} \to \Lambda_{C/B}.$$

REMARK 5.4. This is a slightly misleading expression. It is important to remember that μ is the map from $C \otimes_A C$ to C (as opposed to the map from $C \otimes_B C$ to C.) The consequence of this is that, in the case where both maps are flat, the homology groups of $\mu^*\Lambda_{C/B}$ are *not* the groups $\mathrm{Tor}_{i-1}^{C\otimes_B C}(C, C)$. Of course, as in the comment below, if the complex $D_{C/B}$ is acyclic (as in the application below), then the complex $\Lambda_{C/B}$ is also acyclic.

Note that if $C = S^{-1}B$, for some multiplicatively closed subset $S \subset A$, then $D_{B/C}$ is acyclic, and so by the above exact sequence, the maps

$$\begin{aligned} (S \times S)^{-1} D_{B/A} &\to D_{C/A} \\ S^{-1}\Lambda_{B/A} &\to \Lambda_{C/A} \end{aligned} \tag{5.5}$$

are isomorphisms.

LEMMA 5.6. *Suppose that P is a complex of R-modules and $s \in R$ an element which is not a divisor of zero. Let $\pi : R \to R/(s)$ be the quotient map. If π^*P is contractible, then*

$$P \to s^{-1}P$$

is a weak equivalence.

Proof: The complex $s^{-1}P$ is the colimit of

$$P \xrightarrow{s} P \xrightarrow{s} \cdots .$$

Since s is not a divisor of zero, the cofiber of $P \xrightarrow{s} P$ is π^*P. Since π^*P is contractible, each map in the above sequence is a weak equivalence. It follows that the map $P \to s^{-1}P$ is a weak equivalence since homology commutes with directed colimits. □

REMARK 5.7. This can be considerably generalized, but this version is so simple, and is all that is needed in the present paper.

5.3. Schemes. Suppose now that X/S is a flat scheme, and define complexes

$$\begin{aligned} D_{X/S} &= \left(\mathcal{O}_X \underset{\mathcal{O}_S}{\otimes} \mathcal{O}_X \to \mathcal{O}_X \right) [-1] \\ \Lambda_{X/S} &= R\Delta^* D_{X/S}. \end{aligned}$$

The complex $D_{X/S}$ is a complex of coherent sheaves on $X \times X$, and $\Lambda_{X/S}$ is its (derived) pullback to the diagonal. By the isomorphism (5.5) if $i : U \subset X$ is the inclusion of an open subscheme, the restriction maps

$$\begin{aligned} i^* D_{X/S} &\to D_{U/S} \\ i^* \Lambda_{X/S} &\to \Lambda_{U/S} \end{aligned}$$

are weak equivalences.

In the context of schemes, the fundamental exact sequences take the following form. Given

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[d] \\ & S \end{tikzcd}$$

there are cofibrations

$$\begin{aligned} (f \times f)^* D_{Y/S} \to D_{X/S} \to D_{X/Y} \\ f^* \Lambda_{Y/S} \to \Lambda_{X/S} \to \Lambda_{X/Y}. \end{aligned} \tag{5.8}$$

As far as computation goes, we have the following result.

Lemma 5.9. *If X/S is smooth, there is an isomorphism*

$$H_i \Lambda_{X/S} \approx \Omega^{2i}_{X/S}$$

Proof: The question is local on X/S (provided we make a map) and so we reduced to the case that $X = \mathcal{A}^n_S$. In this case one can proceed by direct calculation. □

Lemma 5.10. *Suppose that X is a scheme, $j : Z \subset X$ is a closed subscheme, and $i : U \subset X$ its complement. If P is s complex of coherent $\mathcal{O}_X$-modules and $j^* P$ is acyclic, then the map*

$$P \to i_* i^* P$$

is a weak equivalence.

Lemma 5.11. *Suppose that X is a scheme $j : Z \subset X$ is a closed subscheme, and P a complex of coherent $\mathcal{O}_X$-modules. If each of the sheaves $H_i P$ is supported on Z and $j^* P$ is contractible, then P is contractible.*

5.4. Vanishing criteria.

Proposition 5.12. *Let $A \to B$ be a map of commutative rings. The complex of B-modules $\Lambda_{B/A}$ is acyclic if and only if the complex $L_{B/A}$ is acyclic.*

Now suppose that A and B are algebras over $\mathbb{Z}/p$. Let ϕ be the absolute Frobenius map

$$\begin{array}{ccc} B & \xrightarrow{\phi} & B \\ \uparrow & & \uparrow \\ A & \xrightarrow{\phi} & A \end{array} \qquad \phi(x) = x^p$$

and

$$\phi_{B/A} : \phi^* B \to B$$

the relative Frobenius. More generally, write $\phi^n_{B/A} : (\phi^n)^* B \to B$ for the iterated relative Frobenius.

Lemma 5.13. *For each $n > 0$, the map*

$$L_{((\phi^n)^* B)/A} \to L_{B/A},$$

induced by the iterated relative Frobenius, is null.

Proof: The result is clear if B is a polynomial algebra over A, since $Dx^{p^n} = 0$. It follows that if $P_\bullet$ is a simplicial A-algebra, which is dimension-wise a polynomial algebra, then the map

$$L_{\phi^* P_\bullet/A} \to L_{P_\bullet/A},$$

is zero. Taking $P_\bullet \to B$ to be a resolution gives the desired result. □

Corollary 5.14. *If for some n, the map $\phi^n_{B/A}$ is an isomorphism, then $L_{B/A}$ and $D_{B/A}$ are acyclic.*

Proof: With these assumptions, the relative Frobenius map

$$L_{\phi^* B/A} \to L_{B/A},$$

is both an isomorphism and null. It follows that $L_{B/A}$ is acyclic. The fact that $D_{B/A}$ is acyclic follows from Proposition 5.12. □

6. Stacks

Let $\mathcal{C}$ be a category with a Grothendieck topology.

Definition 6.1. A *presheaf of groupoids* on $\mathcal{C}$ is a rule associating to each object X of $\mathcal{C}$ a groupoid $\mathfrak{G}_X$, to each map $f : X \to Y$ a functor $\mathfrak{G}f = f^* : \mathfrak{G}_Y \to \mathfrak{G}_X$ and to each composable pair

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

a natural transformation $\epsilon(f, g) : f^* g^* \to (g \circ f)^*$, satisfying the associativity condition

$$\epsilon(\epsilon(f, g), h) = \epsilon(f, \epsilon(g, h)). \tag{6.2}$$

The collection of presheafs of groupoids over $\mathcal{C}$ forms a 2-category.

Now suppose $\mathfrak{G}$ is a presheaf of groupoids on $\mathcal{C}$ and that $U \to X$ is a covering. The category $\mathrm{Desc}(U/X)$ of *descent data* relative to U is the category whose objects are objects E of $\mathfrak{G}(U)$ together with an isomorphism $s : \pi_1^* E \to \pi_2^* E$ satisfying the cocycle condition identical to the one described below.

A presheaf of groupoids $\mathfrak{G}$ is a *sheaf* if for each covering $U \to X$ of $\mathcal{C}$ the map

$$\mathfrak{G}(X) \to \mathrm{Desc}(U/X)$$

is an equivalence of groupoids.

DEFINITION 6.3. A *stack* in $\mathcal{C}$ is a sheaf of groupoids.

Given a stack $\mathfrak{G}$ over $\mathcal{C}$ one can form a category $\mathfrak{E}\mathfrak{G}$ in which an object consists of a pair (E, X) with X an object of $\mathcal{C}$ and E an object of $\mathfrak{G}(X)$. A map from (E_1, X_1) to (E_2, X_2) consists of a map $f : X_1 \to X_2$ together with a morphism $g : E_1 \to f^* E_2$. It is convenient to picture this data as

$$\begin{array}{ccc} E_1 & \xrightarrow{g} & E_2 \\ \downarrow & & \downarrow \\ X_1 & \xrightarrow[f]{} & X_2. \end{array}$$

The fact that $\mathfrak{E}\mathfrak{G}$ is a category makes use of the associativity condition 6.2. The category $\mathfrak{E}\mathfrak{G}$ is fibered in groupoids over $\mathcal{C}$.

The collection of stacks over $\mathcal{C}$ forms a 2-category, $\mathrm{St}\mathcal{C}$.

A 1-morphism $\mathcal{M}_1 \to \mathcal{M}_2$ of stacks is said to be *representable* if for each 1-morphism $X \to \mathcal{M}_2$ with X representable, the fibre product

$$X \underset{\mathcal{M}_2}{\times} \mathcal{M}_1$$

is representable.

If $\mathbf{P}$ is a property of morphisms in $\mathcal{C}$ which is stable under change of base, and which is local in nature, then a 1-morphism $\mathcal{M}_1 \to \mathcal{M}_2$ will be said to have property $\mathbf{P}$ if it is representable, and if for each $X \to \mathcal{M}_2$, the map

$$X \underset{\mathcal{M}_2}{\times} \mathcal{M}_1 \to X \tag{6.4}$$

has property $\mathbf{P}$. One important example is the case in which $\mathbf{P}$ is the property of being a covering.

We can define a kind of Grothendieck topology on the 2-category of stacks by letting the covers be the 1-morphisms which are coverings in the above sense.

By construction, the coverings of $\mathrm{St}\mathcal{C}$ satisfy the following axioms:

(1) If $T_1 \to T_2$ is a *representable equivalence* then it is a covering;

(2) If $T_1 \to T_2$ is a covering, and $U \to T_2$ is any 1-morphism, then the 1-morphism $U \underset{T_2}{\times} T_1 \to U$ is a covering;

(3) The class of coverings is stable under formation of coproducts and composition.

A *presheaf* on $\mathrm{St}\mathcal{C}$ is a contravariant functor from $\mathrm{St}\mathcal{C}$ to $\mathfrak{Sets}$. A presheaf on a stack $\mathcal{M}$ is a presheaf on the 2-categry of stacks in $\mathcal{C}$ over M. Here we must regard $\mathfrak{Sets}$ as a 2-cagegory in which the only 2-morphisms are identity maps. This has the consequence that if P is a presheaf, and f and g are two 1-morphisms which are related by a 2-morphism, then $Pf = Pg$.

A presheaf is a *sheaf* if it satisfies the sheaf axiom: for each covering $T_1 \to T_2$, the sequence

$$P(T_2) \to P(T_1) \rightrightarrows (T_1 \underset{T_2}{\times} T_1)$$

is an equalizer.

6.1. Inverse limits and Čech cohomology. Again, let $\mathcal{C}$ be a category with a Grothendieck topology, and F a presheaf of abelain groups. The point of this section is to identify the groups

$$\varprojlim{}^s_{\mathcal{C}'} F$$

where $\mathcal{C}' \subset \mathcal{C}$ is a full subcategory. We are interested in full subcategories generated by collections $\mathcal{U} \subset \operatorname{ob} \mathcal{C}$ which are *coverings of the terminal object* in the sense that for each $X \in \operatorname{ob} \mathcal{C}$, the map

$$\coprod_{U \in \mathcal{U}} U \times X \to X$$

is a covering. The Čech complex of $\mathcal{U}$ is defined to be the normalized complex associated to the cosimplicial abelian group

$$\prod_{U \in \mathcal{U}} F(U) \rightrightarrows \prod_{(U_0, U_1) \in \mathcal{U}^2} F(U_0 \times U_1) \ldots .$$

The cohomology of the Čech complex is denoted $H^*(\mathcal{U}; F)$. If $\mathcal{U} \subset \mathcal{U}'$, then the natural map

$$H^*(\mathcal{U}) \to H^*(\mathcal{U}')$$

is an isomorphism.

Now suppose that the full subcategory $\mathcal{C}'$ generated by $\mathcal{U}$ happens to be stable under formation of products. Then we can compare the Čech complex of $\mathcal{U}$ with the cosimplicial replacement of the functor F, which is defined to be the complex

$$\prod_{U_0} FU_0 \rightrightarrows \prod_{U_0 \leftarrow U_1} FU_1 \ldots .$$

The natural map goes from the Čech complex to the cosimplicial replacement, and is in degree n, the map whose component with index $U_0 \leftarrow \cdots \leftarrow U_n$ is the projection mapping to the factor with index $(U_0, \ldots, U_n)$.

Proposition 6.5. *With the above assumptions, the natural map from the Čech complex to the cosimplicial replacement is an isomorphism.*

Proof: The natural map is an isomorphism of H^0. Since the Čech cohomology groups take a short exace sequence of presheaves to a long exact sequence in cohomology, it suffices to show that the Čech cohomology groups are effaceable. For this it is enough to show that the groups $H^i(\mathcal{U}; F)$, $i > 0$, are zero when F is the right Kan extension of functor which assigns to a fixed object $V \in \mathcal{U}$ an abelian group A. More explicitly, the presheaf F is given by

$$FU = \prod_{V \to U} A.$$

Now the let S be the set $\{f : V \to U \mid U \in \mathcal{U}\}$. Then the Čech complex of F is just the complex

$$A^S \rightrightarrows A^{S\times S} \dots.$$

For any set S, the simplicial set $ES = S \leftleftarrows S \times \dots$ is contractible. It follows that the groups $H^i(\mathcal{U}; F)$ are the cohomology groups $H^i(ES; A)$ which are zero for $i > 0$. This completes the proof. □

6.2. The quasi-étale topology. Now let $\underline{\mathrm{Aff}}$ be the opposite of the category of commutative rings. There are several topologies we need to consider. A collection $\{U_i \to U\}$ is a covering in the flat topology if the map $\coprod U_i \to U$ is faithfully flat. This implies, in particular, that each $U_i \to U$ is flat. The fppf topology is obtained by requiring that the collection $\{U_i \to U\}$ be finite, and that each $U_i \to U$ be finitely presented. The *quasi-étale* topology is obtained by requiring that the complexes $\Lambda_{U_i/U}$ be acyclic.

Definition 6.6. A map $R \to S$ of rings is *quasi-unramified* if the complex of S-modules $\Lambda_{S/R}$ is acyclic. A map is *quasi-étale* if it is quasi-unramified and and flat.

6.3. The associated stack. Now suppose that $\mathcal{C}$ is a Grothendieck topology and F is a contravariant functor from $\mathcal{C}$ to groupoids. The point of this section is to associate a stack to F.

Let $\mathbf{J} = \{U_i \to U\}$ be a covering in $\mathcal{C}$. A *descent datum* for F relative to $\mathbf{J}$ consists of a collection of objects $x_i \in \operatorname{ob} F(U_i)$, together with isomorphisms $t_{i,j} : \pi_i^* x_i \to \pi_j^* x_j$ satisfying the cocycle condition

$$\pi_i^* t_{j,k} \circ \pi_k^* t_{i,j} = \pi_j^* t_{i,k}.$$

For the covering $\mathbf{J}$, define

$$\tilde{U}_0 = \coprod U_i$$
$$\tilde{U}_1 = \coprod U_i \underset{Y}{\times} U_j.$$

Then $\tilde{J} = (\tilde{U}_0, \tilde{U}_1)$ is naturally a groupoid object of $\mathcal{C}$, which we may regard as a contravariant functor from $\mathcal{C}$ to groupoids. The category of descent data for F relative to J is the category of morphisms of groupoids from $\tilde{J}$ to F.

6.4. The cotangent complex of a stack. In this section we will consider the cotangent complex of a morphism $\mathcal{M} \to \mathcal{N}$ of stacks. This object will be a chain complex of quasicoherent $\mathcal{O}_{\mathcal{M}}$-modules, and is determined by insisting on the transitivity and flat base change properties. Unlike in the case of rings, it will not be the case that $H_i L_{\mathcal{M}/\mathcal{N}} = 0$ if $i < 0$.

By the transitivity sequence, the complex $L_{\mathcal{M}/\mathcal{N}}$ will be the cofiber of the map $L_{\mathcal{M}} \to L_{\mathcal{N}}$, and so we need only define the absolute complex. We will consider the case of a stack of the form $\mathcal{M}(A, \Gamma)$.

We will need a lemma. For a Hopf-algebroid (A, Γ) let $B(A, \Gamma)^\bullet$ be the cosimplicial object which represents the nerve of the groupoid represented by (A, Γ). One has

$$B(A, \Gamma)^n = \Gamma^{\otimes n}$$

where $\otimes$ means "tensor over A," and $\Gamma^{\otimes 0} = A$. Given an (A, Γ) co-module M, let $B(M)^\bullet$ be cosimplicial module over $B(A, \Gamma)$ with $B(M)^n = \Gamma^{\otimes n} \otimes M$.

Lemma 6.7. *The functor* B *from* (A, Γ) *co-modules to cosimplicial* $B(A, \Gamma)^\bullet$ *modules has a right adjoint* Rcom.

Remark 6.8. There is another description which is useful. Suppose we are given a stack $\mathcal{M}$ whose diagonal is representable, and a cover $f : \operatorname{spec} A \to \mathcal{M}$. Write

$$\operatorname{spec} \Gamma \sim \operatorname{spec} A \underset{\mathcal{M}}{\times} \operatorname{spec} A.$$

Then (A, Γ) is a Hopf-algebroid. Set $f_\bullet : \operatorname{spec} B(A, \Gamma) \to \mathcal{M}$ be the augmentation. Strictly speaking, this map does not exist, but if one first converts the map f into a "fibration," then one can find an equivalent simplicial stack, and a map to $\mathcal{M}$. If we identify (A, Γ) co-modules with quasi-coherent $\mathcal{M}$-modules, then the functor denoted B above would naturally be denoted $f_\bullet^*$. Its right adjoint is then $(f_\bullet)_*$.

Making use of the functorial L, we define

$$L_{\mathcal{M}(A,\Gamma)} := \operatorname{Rcom} L_{B(A,\Gamma)}.$$

We have to verify some of the basic properties.

Proposition 6.9. *If the* 1*-morphism* $\mathcal{M} \to \mathcal{N}$ *is representable, then the sheaves* $H_i L_{\mathcal{M}/\mathcal{N}}$ *are zero for* $i < 0$.

Proof: Choose a faithfully flat cover $f : \operatorname{spec} A \to \mathcal{N}$ and consider the 2-category pullback

$$\begin{tikzcd} \operatorname{spec} B \arrow[r, "g"] \arrow[d] & \mathcal{M} \arrow[d] \\ \operatorname{spec} A \arrow[r, "f"'] & \mathcal{N}. \end{tikzcd}$$

Then by flat base change, $g^* L_{\mathcal{M}/\mathcal{N}} \sim L_{B/A}$. Since f is faithfully flat, so is g, and so

$$\begin{aligned} H_i L_{\mathcal{M}/\mathcal{N}} = 0 &\Leftrightarrow g^* H_i L_{\mathcal{M}/\mathcal{N}} = 0 \\ &\Leftrightarrow H_i g^* L_{\mathcal{M}/\mathcal{N}} = 0 \\ &\Leftrightarrow H_i L_{B/A} = 0. \end{aligned}$$

But $H_i L_{B/A} = 0$ for $i < 0$. □

7. The moduli stack of formal groups

The point of this section is to establish some basic notation, and give the Landweber criteria for a map to be flat.

Define $\mathcal{M}_{\mathrm{FG}}^{[n]}$ to be the moduli stack of formal groups of height greater than or equal to n. Over $\mathcal{M}_{\mathrm{FG}}^{[n]}$ there is a global section v_n of ω^{p^n-1}. The zero set of v_n is $\mathcal{M}_{\mathrm{FG}}^{[n+1]}$, and its complement is $\mathcal{M}_{\mathrm{FG}}^{(n)}$. The stack $\mathcal{M}_{\mathrm{FG}}^{(n)}$ is the moduli stack over $\mathbb{Z}/p$ of formal group laws, such that locally, in a local coordinate, "multiplication by p" is represented by a power series of the form $\lambda x^{p^n} + \cdots$.

Remark 7.1. The moduli stack of formal group laws, all of whose geometric fibers have height greater than or equal to n does not come from a Hopf algebroid. It is a colimit of ones that do, though. It is the colimit of $(\operatorname{spec} L/I_n^k, \operatorname{spec} W/I_n^k)$.

7.1. The Frobenius morphism. Now let $G \to S$ be a formal group over a scheme S, which is, itself a scheme over $\operatorname{spec} \mathbb{Z}/p$. The Frobenius map ϕ gives a commutative diagram

$$\begin{CD} G @>{\phi}>> G \\ @VVV @VVV \\ S @>>{\phi}> S, \end{CD}$$

and hence a map $\mathbf{f}_G : G \to \phi^* G$. If x is a local coordinate on G, then $\mathbf{f}_G(x) = x^p$.

Write ω for the invariant differential on G. In a local coordinate x write "multiplication by p" as $[p](x)$. Since $p^*\omega = p\omega = 0$ it follows that $[p]'(x) = 0$, and hence that $[p](x) = g(x^p)$ for some power series g. This means that we can write $p = \mathbf{v}_G \mathbf{f}_G$ for some $\mathbf{v}_G : \phi^* G \to G$. Since p is surjective, the map $\mathbf{v}_G$ is uniquely determined by this equation.

Now suppose that the height of G is $\geq n$, with $n > 1$. Then, in a local coordinate, $[p](x) = cx^{p^n} + \dots$ for some c. It follows that $\mathbf{v}_G(x) = cx^{p^{n-1}} + \dots,$ and that $\mathbf{v}_G'(0) = 0$. But this means that $\mathbf{v}_G^* \omega = 0$ and hence that $\mathbf{v}_G'(x) = 0$, and so we can write $\mathbf{v}_G(x) = \mathbf{v}_G^{(2)}(x^p)$ for some h. Writing this globally we see that there is a unique map $\mathbf{v}_G^{(2)} : (\phi^2)^* G \to G$ with $\mathbf{v}_G^{(2)} \circ \mathbf{f}_G^{\phi} \circ \mathbf{f}_G$. Iterating this discussion leads to the following result.

Proposition 7.2. *If G has height $\geq n$ then there is a unique map $\mathbf{v}_G^{(n)}$: $(\phi^n)^* G \to G$ with the property that*

$$\mathbf{v}_G^{(n)} \mathbf{f}_G^{\phi^{n-1}} \cdots \mathbf{f}_G = p : G \to G.$$

If the height of G is equal to n, then the map $\mathbf{v}_G^{(n)}$ is a isomorphism.

Proposition 7.3. *For $n \geq 1$, the 1-morphism $\mathcal{M}_{\mathrm{FG}}^{(n)} \to \operatorname{spec} \mathbb{Z}/p$ is quasi-étale.*

Proof: The map is flat, so it suffices to show that it is quasi-unramified. We will use the criteria of Proposition 5.14. We need to show that $\phi^n : \mathcal{M}_{\mathrm{FG}}^{(n)} \to \mathcal{M}_{\mathrm{FG}}^{(n)}$ is an equivalence. But this is the 1-morphism sending a formal group G to the formal group $(\phi^n)^* G$. The map $\mathbf{v}_G^{(n)}$ defines a natural isomorphism from this functor to the identity functor. □

8. Elliptic curves

8.1. The moduli stack. Define a contravariant groupoid valued functor $\mathcal{M}_{\mathrm{Weier}}$ on $\underline{\mathrm{Aff}}$ by taking the objects to be the set of projective plane curves given by a Weierstrass equation

$$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6,$$

which we will also write in the form $F(x, y) = 0$, where

$$F(x, y) = y^2 + a_1 xy + a_3 y - (x^3 + a_2 x^2 + a_4 x + a_6).$$

A morphism from the curve defined by $F_1(x, y) = 0$ to the curve defined by $F_2(x, y) = 0$ is a transformation

$$\begin{aligned} x &\mapsto \lambda^2(x + r) \\ y &\mapsto \lambda^3(y + sx + t) \end{aligned} \qquad \lambda \text{ a unit}$$

sending a solution of $F_1(x,y)=0$ to a solution to $F_2(x,y)=0$. If (x,y) is a solution of $F(x,y)=0$, then the point

$$\left(\lambda^2(x+r), \lambda^3(y+sx+t)\right)$$

is a solution to

$$\lambda^6 F\left(\lambda^{-2}x - r, \lambda^{-3}y - s(\lambda^{-2}x-r) - t\right),$$

which can be rewritten as

$$y^2 + \eta_R(a_1)xy + \eta_R(a_3)y = x^3 + \eta_R(a_2)x^2 + a_4x + \eta_R(a_6),$$

where

$$\begin{aligned}(8.1)\quad & y^2 + \lambda(a_1-2s)xy + \lambda^3(a_3 - ra_1 - 2t - 2rs)y = \\ & x^3 + \lambda^2(a_2 + sa_1 - s^2 - 3r)x^2 + \lambda^4(a_4 + sa_3 - 2ra_2 + (t-2rs)a_1 - 2st + 2rs^2 + 3r^2)x + \\ & \quad \lambda^6(a_6 - a_4r + a_2r^2 - r^3 - a_3rs + a_1r^2s - r^2s^2 + a_3t - a_1rt + 2rst - t^2)\end{aligned}$$

8.2. Height and the Frobenius isogeny. Let A be a p-divisible group over a base S, and

$$\mathbf{f} = \mathbf{f}_{A/S} : A \to \phi^* A$$

the relative Frobenius map. Since multiplication by p is is surjective and kills the kernel of $\mathbf{f}$, there is a unique map

$$\mathbf{v} = \mathbf{v}_{A/S} : \phi^* A \to A,$$

satisfying $\mathbf{vf} = p$. Since $\mathbf{fvf} = p\mathbf{f} = \mathbf{f}p$, and since $\mathbf{f}$ is surjective, it also follows that $\mathbf{fv} = p$.

Let E be a generalized elliptic curve over a base S, E^f its formal completion, and $i : E^f \to E$ the canonical map. The map i gives rise to an isomorphism $\ker(\mathbf{f}_{E^f}) \to \ker(\mathbf{f}_E)$. When S is the spectrum of a field, and E is supersingular, the map i restricts to an isomorphism between the kernels of multiplication by p. This implies that the map $\ker(\mathbf{v}_{E^f}) \to \ker(\mathbf{v}_E)$ induced by i, is also an isomorphism. Finally, it follows for general S, that if all of the geometric fibers of E are supersingular, then the map $\ker(\mathbf{v}_{E^f}) \to \ker(\mathbf{v}_E)$ is an isomorphism. This is proved by reducing to the case where S is Noetherian and affine (for example, the universal case) and then using the fact that a map of finitely generated modules over a Noetherian ring is an isomorphism if and only if it is over every geometric fiber.

8.3. The invariant differential. Let E be the elliptic curve given by the Weierstrass equation

$$F(x,y) = y^2 + a_1xy + a_3y - (x^3 + a_2x^2 + a_4x + a_6) = 0. \tag{8.2}$$

Introduce the coordinates $z = -x/y$, $w = -1/y$. Then z is a local parameter near the identity section (the point at infinity). The invariant differential is given by

$$\omega = \frac{dx}{F_y} = -\frac{dy}{F_x}.$$

We need to expand ω as a power series in z.

In terms of w and z, the invariant differential ω is

$$\begin{aligned}\omega &= \frac{w\,dz}{(3z^3-2w)+2a_1wz+2a_2wz^2+a_3w^2+a_4w^2z} \\ &= \frac{dz}{(3z^3/w-2)+2a_1z+2a_2z^2+a_3w+a_4wz},\end{aligned}$$

and the equation (8.2) becomes

$$w = z^3 + a_1wz + a_2wz^2 + a_3w^2 + a_4w^2z + a_6w^3. \tag{8.3}$$

It is also useful to rewrite this equation as

$$\frac{z^3}{w} = 1 - a_1z - a_2z^2 - a_3w - a_4wz - a_6w^2. \tag{8.4}$$

From these two equations one concludes

$$\begin{aligned} w &= z^3 + O(z^4) \\ \frac{z^3}{w} &= 1 - a_1z - a_2z^2 - a_3z^3 + O(z^4),\end{aligned}$$

and finally

$$\omega = \frac{dz}{1 - a_1z - a_2z^3 - 2a_3z^3 + O(z^4)}.$$

Lemma 8.5. *Suppose that E is an elliptic curve over a ring R with identity section e, and*

$$u \in H^0(\mathcal{O}(-e)/\mathcal{O}(-5e))$$

is a local parameter at e modulo degree 5. There are unique sections $x \in H^0(\mathcal{O}(2e))$ and $y \in H^0(\mathcal{O}(3e))$ satisfying

$$x \equiv u^{-2} \mod \mathcal{O}(e) \tag{8.6}$$

$$y \equiv u^{-3} \mod \mathcal{O}(2e) \tag{8.7}$$

$$x/y \equiv u \mod \mathcal{O}(5e). \tag{8.8}$$

Proof: The question is local in R so we may suppose that E is given by a Weierstrass equation

$$y^2 + a_1x + a_3y = x^3 + a_2x^2 + a_4x + a_6,$$

with x and y satisfying (8.6) and (8.7). For convenience write $z = x/y$. Then z is a local parameter near e, and $z \equiv u$ modulo $\mathcal{O}(2e)$. Write

$$z \equiv u + cu^2 + O(u)^3,$$

and consider a change of variables of the form $y \mapsto y + sx$. Then

$$z \mapsto z - sz^2 + O(z)^3,$$

and so there is a choice of x and y with the property that $x/y \equiv u$ modulo $\mathcal{O}(3e)$. This choice of x and y is unique up to a change of variables of the form

$$\begin{aligned} x &\mapsto x + r \\ y &\mapsto y + t.\end{aligned}$$

Now write $z \equiv u + cu^3 + O(u^4)$. Replacing x by $x + r$ replaces z with

$$z + r\frac{1}{y} \equiv z + ru^3 + O(u^4).$$

It follows that x and y can be chosen so that $x/y \equiv u$ modulo $\mathcal{O}(4e)$, and that this choice of x and y is unique up to a change of variables of the form

$$y \mapsto y + t.$$

Finally, write $z = u + cu^4 + O(u^5)$. The transformation $y \mapsto y + t$ sends z to

$$\frac{x}{y+t} = \frac{x}{y(1+\frac{t}{y})} \equiv z - tz^4 + O(z^5).$$

It follows that x and y can be chosen so that $z/y \equiv u$ modulo $\mathcal{O}(-5e)$, and that this choice is unique. □

COROLLARY 8.9. *The ring $A = \mathbb{Z}[a_1, a_2, a_3, a_4, a_6]$ represents the functor "elliptic curves together with a local parameter at e modulo degree 5". The Hopf-algebroid (A, Γ) can be identified with the one representing the functor "elliptic curves with a local parameter at e modulo degree 5, and changes of local parameter."* □

8.4. Representability of $\mathcal{M}_{\mathrm{Ell}} \to \mathcal{M}_{\mathrm{FG}}$. The point of this section is to give a proof of the following result.

PROPOSITION 8.10. *The 1-morphism $\mathcal{M}_{\mathrm{Ell}} \to \mathcal{M}_{\mathrm{FG}}$ is representable.*

The proof makes use of an auxilliary stack.

DEFINITION 8.11. A *formal group law chunk* modulo degree n over a ring R is an element

$$F(x,y) \in R[x,y]/(x,y)^n$$

satisfying

$$\begin{aligned} F(x,y) &= F(y,x) \\ F(x,0) = F(0,x) &= x \\ F(F(x,y),z) &= F(x,F(y,x)). \end{aligned}$$

A *homomorphism* $f : F_1 \to F_2$ of formal group law chunks is an element $f \in R[t]/(t)^n$ with the property that

$$f(F_1(x,y)) = F_2(f(x), f(y)).$$

The functor "formal group law chunks over R modulo degree n and isomorphisms" is represented by a Hopf-algebroid. The theory is much the same as in the case of formal group laws. The ring representing the functor "set of formal group law chunks modulo degree $n+2$" is isomorphic to

$$\mathbb{Z}[x_1, \dots, x_n],$$

and the universal "group chunk" is given, modulo products of the x_i's, by

$$x + y + \sum x_n c_n(x,y),$$

where $c_n(x,y) = d_n((x+y)^n - x^n - y^n)$, and

$$d_n = \begin{cases} \frac{1}{p} & \text{if } n = p^j,\ p \text{ a prime} \\ 1 & \text{otherwise.} \end{cases}$$

The universal isomorphism between group chunks is given by the change of variables

$$x \mapsto t_0 x + \cdots + t_n x^{n+1} + O(x^{n+2}),$$

with t_0 a unit.

The stack associated to the Hopf algebroid of formal group law chunks modulo degree n will be denoted $\mathcal{M}_{\mathrm{FG}(n)}$. A 1-morphism $\operatorname{spec} R \to \mathcal{M}_{\mathrm{FG}(n)}$ will be called a formal group chunk modulo degree n over R. By construction, there are 1-morphisms

$$\cdots \to \mathcal{M}_{\mathrm{FG}(n+1)} \to \mathcal{M}_{\mathrm{FG}(n)} \to \ldots,$$

and the "homotopy inverse limit" of this tower is just $\mathcal{M}_{\mathrm{FG}}$. Since they come from Hopf algebroids, the diagonal map of each $\mathcal{M}_{\mathrm{FG}(n)}$ is representable.

We will be interested in the case $n = 5$. In this case the universal group law chunk can be written as

$$F(x,y) = x + y + x_1 c_2 + x_2 c_3 + (x_3 c_4 + x_2 x_1 c_2^2) + O(x,y)^5.$$

The universal invariant differential is

$$\begin{aligned}\frac{dx}{F_y(x,0)} &= \frac{dx}{1 + x_1 x + x_2 x^2 + 2x_3 x^3 + O(x)^4} \\ &= (1 - x_1 x + (x_1^2 - x_2)x^2 \\ &\qquad + (-x_1^3 + 2x_1 x_2 - 2x_3)x^3 + O(x)^4)\, dx.\end{aligned}$$

LEMMA 8.12. *The 2-category pulback of*

$$\begin{array}{ccc} & & \mathcal{M}_{\mathrm{Ell}} \\ & & \downarrow \\ \operatorname{spec} \mathbb{Z}[x_1, x_2, x_3] & \longrightarrow & \mathcal{M}_{\mathrm{FG}(5)} \end{array}$$

is representable. It is represented by the elliptic curve

$$y^2 + a_2 x + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6$$

over the ring A with local parameter $t = x/y$ modulo t^5.

Proof: The pullback stack is the stack which associates to each ring R an elliptic curve over R, a formal group law chunk over R modulo degree 5, together with an isomorphism between the associated formal group chunk and the formal completion of the elliptic curve modulo degree 5. This data is clearly equivalent to giving an elliptic curve over R and together with a local parameter at e modulo degree 5. The result follows from Corollary 8.9 $\square$

REMARK 8.13. Working over A, we have explicitly allowed "additive" reduction. This fiber must be removed when we consider the homotopy theory, specifically when we verify flatness of the map to the moduli stack of formal groups.

8.5. Some vanishing results. Fix a prime p, and let $\mathcal{M}^{\mathrm{ord}}_{\mathrm{Ell},p}$ (respectively $\mathcal{M}^{\mathrm{ss}}_{\mathrm{Ell},p}$) be the moduli stacks of ordinary (respectively supersingular) elliptic curves in charasteric p. There are therefore 2-cartesian squares

$$\begin{array}{ccc} \mathcal{M}^{\mathrm{ord}}_{\mathrm{Ell},p} & \longrightarrow & \mathcal{M}_{\mathrm{Ell}} \\ \downarrow & & \downarrow \\ \mathcal{M}^{(1)}_{\mathrm{FG}} & \longrightarrow & \mathcal{M}_{\mathrm{FG}} \end{array} \quad \text{and} \quad \begin{array}{ccccc} \mathcal{M}^{\mathrm{ss}}_{\mathrm{Ell},p} & = & \mathcal{M}^{\mathrm{ss}}_{\mathrm{Ell},p} & \longrightarrow & \mathcal{M}_{\mathrm{Ell}} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{M}^{(2)}_{\mathrm{FG}} & \longrightarrow & \mathcal{M}^{[2]}_{\mathrm{FG}} & \longrightarrow & \mathcal{M}_{\mathrm{FG}} \end{array}$$

PROPOSITION 8.14. *The 1-morphism $\mathcal{M}^{\mathrm{ss}}_{\mathrm{Ell},p} \to \mathcal{M}^{(2)}_{\mathrm{FG}}$ is quasi-étale.*

Proof: Being the base change of a flat map it is flat, so it suffices to show that it quasi-unramified. Since $\mathcal{M}_{\mathrm{FG}}^{(2)} \to \operatorname{spec} \mathbb{Z}/p$ is quasi-unramified, it suffices, by the transitivity sequence to show that $\mathcal{M}_{\mathrm{Ell},p}^{\mathrm{ss}} \to \operatorname{spec} \mathbb{Z}/p$ is quasi-unramified. The proof of this is similar to that of Proposition 7.3, and uses the fact that for an supersingular elliptic curve A, the map $\mathbf{v}_A^{(2)} : (\phi^2)^* A \to A$ is an isomorphism. □

8.6. A quasi-étale cover. The point of this section is to exhibit a quasi-étale cover of $\mathcal{M}_{\mathrm{Ell}}$. This is needed in order to know that the cohomology of the Čech complexes we are computing actually give the cohomology of the whole stack. It is also useful in identifying the cotangent complex of ℓ with ω^{10}, since, among other things, it shows that the cotangent complex has all of its homology in dimension 0. The cover will be exhibited in pieces. We will consider separately the cases when 2 is inverted, and when 3 is inverted. Over each of these opens we will consider separately the case when Δ is inverted, and when c_4 is inverted.

We will need a couple of algebraic lemmas.

Let R be a ring. For an R-module M, let $M[x]$ denote the R-module $M \otimes_R R[x]$, and write elements of $M[x]$ as polynomials with coefficients in M.

Lemma 8.15. *Let $f(x) = a_n x^n + \dots a_0$ be an element of $R[x]$. If the ideal generated by the a_i is the unit ideal, then f does not annihilate any non-zero element of $M[x]$.*

Proof: Suppose that the ideal $(a_0, \dots, a_n)$ is the unit ideal and let

$$g(x) = m_k x^k + \dots + m_0 \in M[x]$$

be an element of minimal degree which is annihilated by f. Let $j \geq 0$ be the smallest number with the property that the elements $a_n, \dots, a_j$ annihilate g. If $j > 0$ then

$$f(x)g(x) = a_{j-1} m_k x^{k+j-1} + \text{lower terms.}$$

It follows that $a_{j-1} m_k = 0$, and so $a_{j-1} g(x)$ has lower degree than g and is annihilated by f, and so must be zero. It follows that $j = 0$ and that g is annihilated by all of the a_i. But this implies that $g = 0$ since the a_i generate the unit ideal. □

Lemma 8.16. *Suppose that R is a ring, and that*

$$f(x) = a_n x^n + \dots + a_0 \in R[x]$$

is a polynomial. The module $R[x]/(f(x))$ is flat if and only if the ideal $(a_0, \dots, a_n)$ is the unit ideal.

Proof: We just do the "if" part, so suppose that the a_i generate the unit ideal. Then Lemma 8.15 with $M = R$ shows that the sequence

$$R[x] \xrightarrow{\cdot f} R[x] \to R[x]/(f)$$

is short exact, and hence a free resolution. Tensoring with an arbitrary R-module M and once again using Lemma 8.15 gives the result. □

Corollary 8.17. *Suppose that R is a ring and that $f \in R[x]$ is not a divisor of zero. If the ideal (f, f') is the unit ideal then $B = R[x]/(f)$ is quasi-étale over R.*

Proof: The fact that (f, f') is the unit ideal implies that the coefficients of f generate the unit ideal. It follows from Lemma 8.16 that $R[x]/(f)$ is flat over R and from Lemma 8.15 that f is not a divisor of zero. The Meyer-Vietoris sequence (Corollary 4.2) then shows that the complex $L_{B/R}$ is quasi-isomorphic to

$$B \xrightarrow{\cdot f'} B$$

which is acyclic, since f' is a unit in B. □

The following is a useful special case.

COROLLARY 8.18. *Suppose R is a ring, n is an integer, and that a and $n \cdot 1$ are units in R. Then the map $R \to R[x]/(x^n - a)$ is quasi-étale.* □

8.6.1. *When 2 and Δ are units.* Consider the elliptic curve

$$y^2 = x^3 + \tau x^2 + x.$$

over the ring $R = \mathbb{Z}[\frac{1}{2}][\tau, (\tau^2 - 4)^{-1}]$. With 2 inverted, the stack $\mathcal{M}_{\mathrm{Ell}}$ is presented by the Hopf algebroid (B, Γ_B), with $B = \mathbb{Z}[\frac{1}{2}][b_2, b_4, b_6]$, $\Gamma = B[r, \lambda^{\pm 1}]$. The map $\operatorname{spec} B \to \mathcal{M}_{\mathrm{Ell}}$ is the one classifying the elliptic curve

$$y^2 = x^3 + b_2 x^2 + b_4 x + b_6 = f(x),$$

and the universal isomorphism is given by the transformation

$$x \mapsto \lambda^{-2} x + r.$$

Replacing x with $x + r$ changes b_6 to $f(r)$, and b_4 to $f'(r)$. Replacing x with $\lambda^{-2}x$ sends b_4 to $\lambda^4 b_4$. The 2-category fiber product of $\operatorname{spec} R$ and $\operatorname{spec} B$ over $\mathcal{M}_{\mathrm{Ell}}$ is therefore represented by $B[t, \alpha]/(f(t), \alpha^4 - f'(t))$. The ring $B[t]/f(t)$ is étale over B because the discriminant of f is a unit. Over $B[t]/f(t)$, the element $f'(t)$ is a unit, and so the addition of α defines a further quasi-étale extension.

8.6.2. *When 2 and c_4 are units.* Consider the elliptic curve

$$y^2 = x^3 + x^2 + \tau.$$

over the ring $R = \mathbb{Z}[\frac{1}{2}][\tau, (\tau^2 - 4)^{-1}]$. The value of c_4 is 16 for this curve, and so is a unit since 2 has been inverted. The value of c_4 for the universal curve is $16(b_2^2 - 3b_4)$. Making the change of variables $x \mapsto x + r$ changes b_4 to $f'(r)$, and sends b_2 to $b_2 + 3r$. The 2-category fiber product is therefore represented by

$$B[t, \alpha]/(f'(t), \alpha^2 - (b_2 + 3t)).$$

Adjoining t is flat, since the coefficients of f' generate the unit ideal in R. It is unramified, since the discriminant of f' is $4(b_2^2 - 3b_4) = \frac{c_4}{4}$, which is a unit. The element $b_2 + 3t$ is a unit since it is $\frac{1}{2}f''(t)$, so adjoining α is étale.

8.6.3. *When 3 and Δ are units.* Now consider the curve

$$y^2 + \tau xy + y = x^3.$$

The discriminant is $\tau^3 - 27$, so it is non-singular over the ring

$$R = \mathbb{Z}[\tfrac{1}{3}][\tau, (\tau^3 - 27)^{-1}].$$

8.6.4. *When* 3 *and* c_4 *are units.* Now consider the curve

$$y^2 + xy = x^3 + \tau. \tag{8.19}$$

The value of c_4 is 1 and the value of c_6 is $1 + 864\tau$. It follows that the value of Δ is $-\tau(1 + 432\tau)$. We will consider this curve over the ring

$$R = \mathbb{Z}[\tfrac{1}{3}][\tau].$$

To verify that it is quasi-étale over $\mathcal{M}_{\mathrm{Ell}}$ it suffices, since $\mathcal{M}_{\mathrm{Ell}}$ is flat over $\mathbb{Z}$, to verify that it is quasi-étale after reducing mod 2, and after inverting 2.

Modulo 2, the value of $c_4 = a_1^4$ is a unit. There is a unique transformation $x \mapsto x + r$ (namely $r = -a_3/a_1$) which gets rid of the term $a_3 y$. There is a unique transformation $y \mapsto y + t$ (namely $t = a_4/a_1$) which gets rid of the term $a_4 x^2$. Finally, there is a unique value of λ which brings a_1 to 1. This shows that, modulo 2 and with c_4 inverted, the stack we are interested is associated to the curve

$$y^2 + xy = x^3 + \alpha_2 x^2 + \alpha_6,$$

and the transformation $y \mapsto y + sx$. Making the substitution $y \mapsto y + sx$ leaves α_6 unchanged, and sends α_2 to $s^2 + s + \alpha_2$. It follows that the 2-category pullback

$$\begin{tikzcd}
 & \operatorname{spec} \mathbb{Z}/2[\alpha_2, \alpha_6] \arrow[d] \\
\operatorname{spec} \mathbb{Z}/2[d] \arrow[r] & \mathcal{M}_{\mathrm{Ell}}
\end{tikzcd}$$

is $\operatorname{spec} \mathbb{Z}/2[\alpha_2, \alpha_6][s]/(f(s))$, where $f(s) = s^2 + s + \alpha_2$. Since $f'(s) = 1$ the map is quasi-étale by Corollary 8.17.

Now suppose that 2 has been inverted, so that, in fact, 6 is a unit. The map

$$\operatorname{spec} \mathbb{Z}[\tfrac{1}{6}][a_6] \to \mathcal{M}_{\mathrm{Ell}}[\tfrac{1}{6}]$$

classifying the curve

$$y^2 = x^3 + x + a_6 \tag{8.20}$$

is a faithfully flat cover of the locus where c_4 is a unit. To compute the 2-category pullback of

$$\begin{tikzcd}
 & \operatorname{spec} \mathbb{Z}[\tfrac{1}{6}][a_6] \arrow[d] \\
\operatorname{spec} \mathbb{Z}[\tfrac{1}{6}][\tau] \arrow[r] & c_4^{-1}\mathcal{M}_{\mathrm{Ell}}[\tfrac{1}{6}]
\end{tikzcd}$$

first note that in order that a transformation

$$x \mapsto l^{-2}(x + r)$$
$$y \mapsto l^{-3}(y + sx + t)$$

send an equation of the form (8.20) into one of the form (8.19) one must have

$$\begin{array}{ll} t = 0 & s^4 = -3 \\ 3r = s^3 & l = 1/2s. \end{array}$$

For the record, the value of τ is then $(3s^2 a_6 - 2)/1728$. This means that the 2-category pullback is given by $\operatorname{spec} \mathbb{Z}[\tfrac{1}{6}][a_6][s]/(s^4 + 3)$. This is quasi-étale over $\mathbb{Z}[\tfrac{1}{6}][a_6]$ by 8.18.

9. Determination of certain cotangent complexes

9.1. The elliptic moduli stack. In this section let $\mathcal{M}$ be the stack $\mathcal{M}_{\mathrm{Ell}}$.

PROPOSITION 9.1. *The natural map $L_{\mathcal{M}_{\mathrm{Ell}}} \to \Omega^1_{\mathcal{M}_{\mathrm{Ell}}}$ is a quasi-isomorphism.*

Proof: This is immediate from the fact that the stack $\mathcal{M}_{\mathrm{Ell}}$ admits a quasi-étale cover by smooth algebras. □

PROPOSITION 9.2. *There is a unique 1-form ν on $\mathcal{M}_{\mathrm{Ell}}$ with*

$$1728 \cdot \nu = 2c_4\, dc_6 - 3c_6\, dc_4.$$

The form ν gives an isomorphism

$$\omega^{\pm 10} \approx \Omega^1_{\mathcal{M}_{\mathrm{Ell}}}.$$

Proof: It suffices to check this locally on the cover given in section 8.6. For the curve

$$y^2 = x^3 + \tau x^2 + x$$

over $\mathbb{Z}[\frac{1}{2}][\tau, (\tau^2-4)^{-1}]$, the values of c_4 and c_6 are given by

$$c_4 = 16\tau^2 - 48$$
$$c_6 = 64\tau^3 - 288\tau$$

and so $\nu = 16\, d\tau$ exists, is unique, and is a nowhere vanishing section of the cotangent bundle.

For the curve

$$y^2 = x^3 + x^2 + \tau,$$

over $\mathbb{Z}[\frac{1}{2}][\tau]$, the values of c_4 and c_6 are given by

$$c_4 = 16$$
$$c_6 = 32(27\tau + 2),$$

and so $\nu = 16\, d\tau$ exists, is unique, and is a nowhere vanishing section of the cotangent bundle.

For the curve

$$y^2 + \tau xy + y = x^3$$

over $\mathbb{Z}[\frac{1}{3}][\tau, (\tau^3-27)^{-1}]$, the values of c_4 and c_6 are given by

$$c_4 = \tau^4 - 24\tau$$
$$c_6 = \tau^6 - 36\tau^3 + 216$$

and so $\nu = 9\, d\tau$ exists, is unique, and is a nowhere vanishing section of the cotangent bundle.

Finally, for the curve

$$y^2 + xy = x^3 + \tau$$

over $\mathbb{Z}[\frac{1}{3}][\tau]$, the values of c_4 and c_6 are given by

$$c_4 = 1$$
$$c_6 = 864\tau + 1$$

and so $\nu = d\tau$ exists, is unique, and is a nowhere vanishing section of the cotangent bundle. □

Corollary 9.3. *The complex $L_{\mathcal{M}^{\mathrm{ord}}_{\mathrm{Ell},p}}$ is quasi-isomorphic to $\omega^{\pm 10}$.*

Proof: We know that the assertion is true of $\mathcal{M}_{\mathrm{Ell},p}$, by flat base change. It then follows for $\mathcal{M}^{\mathrm{ord}}_{\mathrm{Ell},p}$ since the inclusion $\mathcal{M}^{\mathrm{ord}}_{\mathrm{Ell},p} \to \mathcal{M}_{\mathrm{Ell},p}$ is quasi-étale. $\square$

9.2. The cotangent complex of $\mathcal{M}_{\mathrm{Ell}}$ over $\mathcal{M}_{\mathrm{FG}}$. To make the notation less cluttered, for this section write $\mathcal{M} = \mathcal{M}_{\mathrm{Ell}}$ and $\mathcal{N} = \mathcal{M}_{\mathrm{FG}}$. The point of this section is to identify $L_{\mathcal{M}/\mathcal{N}} \otimes \mathbb{Z}/p$ and $L_{\mathcal{M}/\mathcal{N}} \otimes \mathbb{Q}$.

Proposition 9.4. *For each prime p, the complex $L_{\mathcal{M}/\mathcal{N}} \otimes \mathbb{Z}/p$ is quasi-isomorphic to $i_*\omega^{\pm 10}$, where $i : \mathcal{M}^{\mathrm{ord}}_{\mathrm{Ell},p} \to \mathcal{M}$ is the inclusion.*

Proof: It is useful to write $\mathcal{M}_p = \mathcal{M} \otimes \mathbb{Z}/p$, $\mathcal{N}_p = \mathcal{N} \otimes \mathbb{Z}/p$ and look at the diagrams

$$\begin{tikzcd}
\mathcal{M}^{\mathrm{ss}}_{\mathrm{Ell},p} \arrow[r, "k"] \arrow[d] & \mathcal{M}_p \arrow[d] & \mathcal{M}^{\mathrm{ord}}_{\mathrm{Ell},p} \arrow[l, "l"'] \arrow[d] \\
\mathcal{M}^{(2)}_{\mathrm{FG}} \arrow[r] & \mathcal{N}_p & \mathcal{M}^{(1)}_{\mathrm{FG}}, \arrow[l]
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathcal{M}_p \arrow[r, "j"] \arrow[d] & \mathcal{M} \arrow[d] \\
\mathcal{N}_p \arrow[r] & \mathcal{N}.
\end{tikzcd}$$

in which the squares are 2-cartesian, and all of the vertical maps are flat. In this notation, the map i is the composite jl. Now, by definition, $L_{\mathcal{M}/\mathcal{N}} \otimes \mathbb{Z}/p \sim j_*j^*L_{\mathcal{M}/\mathcal{N}}$. By flat base change, $j^*L_{\mathcal{M}/\mathcal{N}} \sim L_{\mathcal{M}_p/\mathcal{N}_p}$. Again, by flat base change, we have a weak equivalence

$$k^*L_{\mathcal{M}_p/\mathcal{N}_p} \to L_{\mathcal{M}^{\mathrm{ss}}_{\mathrm{Ell},p}/\mathcal{M}^{(2)}_{\mathrm{FG}}}.$$

But this latter complex is acyclic, by Proposition 7.3. Since k is locally defined by the vanishing of a single function which is not a divisor of zero, the means that the map

$$L_{\mathcal{M}_p/\mathcal{N}_p} \to l_*l^*L_{\mathcal{M}_p/\mathcal{N}_p}$$

is a weak equivalence. By flat base change, the map

$$l^*L_{\mathcal{M}_p/\mathcal{N}_p} \to L_{\mathcal{M}^{\mathrm{ord}}_{\mathrm{Ell},p}/\mathcal{M}^{(1)}_{\mathrm{FG}}}$$

is a weak eqivalence. By Proposition 7.3 the map $\mathcal{M}^{(1)}_{\mathrm{FG}} \to \operatorname{spec} \mathbb{Z}/p$ is quasi-étale. It follows from the transitivity sequence that

$$L_{\mathcal{M}^{\mathrm{ord}}_{\mathrm{Ell},p}} \to L_{\mathcal{M}^{\mathrm{ord}}_{\mathrm{Ell},p}/\mathcal{M}^{(1)}_{\mathrm{FG}}}$$

is an equivalence. But $L_{\mathcal{M}^{\mathrm{ord}}_{\mathrm{Ell},p}}$ is quasi-isomorphic to ω^{10} by Corollary 9.3. Collecting these isomorphisms completes the proof. $\square$

Proposition 9.5. *There is a natural quasi-isomorphism*

$$L_{\mathcal{M}/\mathcal{N}} \otimes \mathbb{Q} \sim (\omega^4 \oplus \omega^6) \otimes \mathbb{Q}.$$

10. Elliptic spectra

DEFINITION 10.1. An elliptic spectrum E is *étale* if the map

$$\operatorname{spec} \pi_0 E \to \mathcal{M}_{\mathrm{Ell}}$$

classifying the elliptic curve associated to E is étale open of finite type.

For a map $U \to \mathcal{M}_{\mathrm{Ell}}$, define a graded ring E_{U*} by

$$\begin{aligned} E_{U\,2k} &= \Gamma_U(\omega^k) \\ E_{U\,2k+1} &= 0 \end{aligned}$$

LEMMA 10.2. *If U is affine, $U \to \mathcal{M}$ is an étale open, and ω_U admits a nowhere vanishing global section, then the pair (E_{U*}, A_U) satisfies the conditions of the Landweber exact functor theorem. It has a unique homotopy associative multiplication.*

Proof: It suffices to check these after fppf base change. Do it on the standard cover. □

Thus the étale elliptic spectra are those of the form E_U.

11. Factoring and lifting

11.1. The section $(\mathcal{M}_{\mathrm{Ell}})'_{\mathrm{et}} \to \operatorname{ho} A_\infty^{\mathrm{Ell}}$.

PROPOSITION 11.1. *The functor* $\operatorname{spec} \pi_0 : \operatorname{ho} A_\infty \to (\mathcal{M}_{\mathrm{Ell}})'_{\mathrm{et}}$ *has a unique section up to vertical natural equivalence.*

LEMMA 11.2. *For an object $U \to \mathcal{M}_{\mathrm{Ell}}$ of $(\mathcal{M}_{\mathrm{Ell}})'_{\mathrm{et}}$, the space*

$$A_\infty \{E_U\}$$

is non-empty and connected.

Proof: The group

$$\operatorname{Der}^2_{\pi_0 E_U} \left(\pi_0(E_U \wedge E_U), E^0_U S^1\right)$$

is zero because $E^0_U S^1 = 0$. According to Theorem 12.12 the groups

$$\operatorname{Der}^{s+1}_{\pi_0 E_U} \left(\pi_0(E_U \wedge E_U), E^0_U S^t\right)$$

are zero for $s > 1$. The result therefore follows from the spectral sequence of Proposition 2.5. □

LEMMA 11.3. *Suppose that E and F are cofibrant-fibrant objects of A_∞^{Ell} with* $\operatorname{spec} \pi_0 E$ *and* $\operatorname{spec} \pi_0 F$ *corresponding to $U \to \mathcal{M}_{\mathrm{Ell}}$ and $V \to \mathcal{M}_{\mathrm{Ell}}$ respectively. Then*

i) *The map $A_\infty^{\mathrm{Ell}}(E, F) \to \mathcal{M}_{\mathrm{Ell}}(V, U)$ is surjective.*

ii) *Given* $f \in \mathcal{M}_{\mathrm{Ell}}(U,V)$ *let*

$$A_\infty^{\mathrm{Ell}}(E,F)_f = (\operatorname{spec}\pi_0)^{-1}\{f\}.$$

Then the set of components of $A_\infty^{\mathrm{Ell}}(E,F)_f$ *is a principal homogeneous space for*

$$\begin{aligned}\pi_0 A_\infty^{\mathrm{Ell}}(F,F)_1 &\approx \mathrm{Der}^2_{\pi_0 F}(\pi_0 F \wedge E, F^0 S^2)\\ &\approx \left(\Omega^1_V\right)^{-1} \otimes \prod_p (\mathcal{O}_V)_p/\mathcal{O}_V.\end{aligned}$$

Proof: This all follows from the spectral sequence of Proposition 2.6 and the computation of Theorem 12.12. Indeed for any choice of basepoint the spectral sequence collapses at E_2 and π_0 is given by the principal homogeneous space above. □

Proof of Proposition 11.1: By Proposition A.6 the obstruction to the existence of a section lies in

$$H^2\left(\mathcal{M}_{\mathrm{Ell}}; \left(\Omega^1_V\right)^{-1} \otimes \prod_p (\mathcal{O}_V)_p/\mathcal{O}_V\right) = 0.$$

This proves the existence of a section. Given that there is a section, Proposition A.6 then asserts that the set of vertical natural equivalence classes of sections is a principal homogeneous space for

$$H^1\left(\mathcal{M}_{\mathrm{Ell}}; \left(\Omega^1_V\right)^{-1} \otimes \prod_p (\mathcal{O}_V)_p/\mathcal{O}_V\right) = 0.$$

This completes the proof. □

Remark 11.4. Two objects of A_∞^{Ell} are isomorphic in $\mathrm{ho}\, A_\infty$ if and only if their images in $(\mathcal{M}_{\mathrm{Ell}})'_{\mathrm{et}}$ are isomorphic.

Remark 11.5. An object or map of A_∞^{Ell} is of type "P," if its image in A_∞ has P, where P is one of cofibrant, cofibration, fibrant, fibration, weak equivalence, etc, and $\mathrm{ho}\, A_\infty$ to be the category with objects the cofibrant-fibrant objects of A_∞ and with $\mathrm{ho}\, A_\infty^{\mathrm{Ell}}(E,F) = \pi_0 A_\infty^{\mathrm{Ell}}(E,F)$.

11.2. The lift to A_∞. Proposition 11.1 gives a unique (up to isomorphism) functor

$$D : (\mathcal{M}_{\mathrm{Ell}})'_{\mathrm{et}} \to \mathrm{ho}\, A_\infty.$$

We now turn to the problem of lifting it to a functor

$$\tilde{D} : (\mathcal{M}_{\mathrm{Ell}})'_{\mathrm{et}} \to A_\infty.$$

The functor $\tilde{D}$ will automatically factor through A_∞^{Ell} since the diagram

$$\begin{tikzcd} A_\infty^{\mathrm{Ell}} \arrow[r] \arrow[d] & A_\infty \arrow[d] \\ \mathrm{ho}\, A_\infty^{\mathrm{Ell}} \arrow[r] & \mathrm{ho}\, A_\infty \end{tikzcd}$$

is a pullback.

PROPOSITION 11.6. *The diagram D is a centric diagram: for each map $f : V \to U$ in $(\mathcal{M}_{\mathrm{Ell}})'_{\mathrm{et}}$, the map*

$$A_\infty(DV, DV)_1 \to A_\infty(DU, DV)_f$$

is a weak equivalence.

Proof: We may (and must) suppose that DU and DV are cofibrant-fibrant obects of A_∞. The homotopy groups of the two spaces in question can be calculated with the spectral sequence of Proposition 2.6. By Corollary 12.14, the map of E_2-terms is an isomorphism. The result follows. □

The homotopy groups of the realization space $\mathrm{r}(D)$ can now be computed using the spectral sequence of Proposition 3.2.

PROPOSITION 11.7. *For each t, the functor*

$$U \mapsto \pi_t \,\mathrm{Bhaut}_1\, DU$$

is a sheaf (of abelian groups) on $(\mathcal{M}_{\mathrm{Ell}})'_{\mathrm{et}}$. There is, for $n > 1$, an isomorphism

$$\pi_{2n-1}\big(\mathrm{Bhaut}_1\, D(\,-\,)\big) \approx \omega^n \otimes (\Omega^1)^{-1} \otimes \prod_p \left(\mathcal{O}_{\mathcal{M}_{\mathrm{Ell}}}\right)_p / \mathcal{O}_{\mathcal{M}_{\mathrm{Ell}}},$$

and an exact sequence

$$0 \to \omega^n \otimes M \to \pi_{2n}\big(\mathrm{Bhaut}_1\, D(\,-\,)\big) \to \omega^n \otimes \prod_p \left(v^{-1}\hat{\mathcal{O}}_V/\mathcal{O}_V\right)_p \to 0,$$

where M lies in the (splittable) exact sequence

$$0 \to \left(\Omega^1_V\right)^{-1} \otimes \prod_p (\mathcal{O}_V)_p/\mathcal{O}_V \to M \to \prod_p (\mathcal{O}_V)_p/\mathcal{O}_V \to 0.$$

PROPOSITION 11.8. *For $s > 0$, the groups*

$$H^s\big(\mathcal{M}_{\mathrm{Ell}}; \pi_{2n-1}\,\mathrm{Bhaut}_1\, D(\,-\,)\big)$$

are 0, and the map

$$H^s\big(\mathcal{M}_{\mathrm{Ell}}; \pi_{2n}\,\mathrm{Bhaut}_1\, D(\,-\,)\big) \to H^s\left(\mathcal{M}_{\mathrm{Ell}}; \omega^n \otimes (\Omega^1)^{-1} \otimes \left(v^{-1}\hat{\mathcal{O}}_V/\mathcal{O}_V\right)_p\right).$$

is an isomorphism. The bigraded group $H^s\big(\mathcal{M}_{\mathrm{Ell}}; \pi_t\,\mathrm{Bhaut}_1\, D(\,-\,)\big)$ is therefore isomorphic to the bigraded abelian group

$$a_6 \cdot \left(a_1^{-2}\mathbb{Z}_2[\![a_1^2]\!][a_6, \eta, a_1\eta]/a_6\mathbb{Z}_2[a_2^{\pm 1}, a_6, \eta]\right)$$

in which

$$|a_i| = (0, 2i) \qquad |\eta| = (1, 2).$$

and in which, to simplify notation, the relation $2\eta = 0$ is implied, but not mentioned.

12. Computations of Quillen cohomology

The main purpose of this section is to calculate

$$\mathrm{Der}_{\pi_0 F}(\pi_0 F \wedge E, F^0 S^t) = \mathrm{Rhom}^i(\Lambda_{(V \underset{\mathcal{M}_{\mathrm{FGL}}}{\times} U)/V}, s_* \mathcal{O}_V),$$

with respect to a map $f : E \to F$ of elliptic spectra, with

$$\begin{array}{rcl} (\operatorname{spec} \pi_0 E, A_E) & \leftrightarrow & U \to \mathcal{M}_{\mathrm{Ell}} \\ (\operatorname{spec} \pi_0 F, A_F) & \leftrightarrow & V \to \mathcal{M}_{\mathrm{Ell}} \\ f & \leftrightarrow & s : V \to V \underset{\mathcal{M}_{\mathrm{FGL}}}{\times} U \approx \mathrm{Isom}(\pi_1^* A_V^f, \pi_2^* A_U^f). \end{array}$$

The main results are Corollaries 12.7, 12.10, and Theorem 12.12 below.

12.1. Computations modulo a prime power. Fix a prime p, and let $\mathcal{M}_{\mathrm{Ell},m}$ and $\mathcal{M}_{\mathrm{FGL},m}$ be the moduli stacks obtained from $\mathcal{M}_{\mathrm{Ell}}$ and $\mathcal{M}_{\mathrm{FGL}}$ by change of base to $\mathbb{Z}/(p^m)$.

Suppose given objects

$$U \to \mathcal{M}_{\mathrm{Ell},m},\ V \to \mathcal{M}_{\mathrm{Ell},m} \in \operatorname{ob}(\mathcal{M}_{\mathrm{Ell},m})'_{\mathrm{et}},$$

and a section $s : V \to V \underset{\mathcal{M}_{\mathrm{Ell},m}}{\times} U$. Let $V^{\mathrm{ord}} \subseteq V$ be the subscheme of V defined by the condition that all of the geometric fibers of A_V are ordinary elliptic curves. Thus

$$V^{\mathrm{ord}} = V \underset{\mathcal{M}_{\mathrm{Ell},m}}{\times} \mathcal{M}^{\mathrm{ord}}_{\mathrm{Ell},m}$$

The complement of V^{ord} is V^{ss}. The closed subscheme V^{ss} is defined by the vanishing of a single function $v \in \mathcal{O}_V$. In fact we can take $v = u^{-p^{m-1}(p-1)} E_{p-1}^{p^{m-1}}$ (or even $v = u^{1-p} E_{p-1}$), where u is a nowhere vanishing section of ω, and $E_{p-1} = v_1$ is the Eisenstein series. The function v is not a divisor of zero, by the assumptions we have made.

PROPOSITION 12.1. *The complex of sheaves $\Lambda_{(V \underset{\mathcal{M}_{\mathrm{FGL},m}}{\times} U)/V}$ is quasi-isomorphic to the sheaf*

$$i_* \pi_2^* \Omega^1_U = (v \times 1)^{-1} \pi_2^* \Omega^1_U,$$

where

$$\begin{aligned} i &: V^{\mathrm{ord}} \underset{\mathcal{M}_{\mathrm{FGL},m}}{\times} U \to V \underset{\mathcal{M}_{\mathrm{FGL},m}}{\times} U \quad \textit{and} \\ \pi_2 &: V^{\mathrm{ord}} \underset{\mathcal{M}_{\mathrm{FGL},m}}{\times} U \to U \end{aligned}$$

are the inclusion and projection respectively.

The proof depends on a couple of other results.

LEMMA 12.2. i) *The map*

$$V^{\mathrm{ord}} \underset{\mathcal{M}_{\mathrm{FGL},m}}{\times} U \to V^{\mathrm{ord}} \times U$$

is formally étale. It factors as a formally étale surjective map

$$\pi : V^{\mathrm{ord}} \underset{\mathcal{M}_{\mathrm{FGL},m}}{\times} U = V^{\mathrm{ord}} \underset{\mathcal{M}_{\mathrm{FGL},m}}{\times} U^{\mathrm{ord}} \to V^{\mathrm{ord}} \times U^{\mathrm{ord}}$$

followed by a closed immersion.

ii) *The map*

$$V^{\mathrm{ss}} \underset{\mathcal{M}_{\mathrm{FGL},m}}{\times} U \to V^{\mathrm{ss}}$$

is formally étale.

proof of Proposition 12.1, given Lemma 12.2: Let

$$j : V^{\mathrm{ss}} \underset{\mathcal{M}_{\mathrm{FGL},m}}{\times} U \to V \underset{\mathcal{M}_{\mathrm{FGL},m}}{\times} U$$

be the inclusion. Since $V \underset{\mathcal{M}_{\mathrm{FGL},m}}{\times} U \to V$ is flat, the map

$$j^* \Lambda_{(V \underset{\mathcal{M}_{\mathrm{FGL},m}}{\times} U)/V} \to \Lambda_{(V^{\mathrm{ss}} \underset{\mathcal{M}_{\mathrm{FGL},m}}{\times} U)/V}$$

is a quasi-isomorphism (Lemma 5.2). By part ii) and an algebraic lemma, the complex $\Lambda_{(V^{\mathrm{ss}} \underset{\mathcal{M}_{\mathrm{FGL},m}}{\times} U)/V}$ is acyclic, so by Lemma 5.6 the map

$$\Lambda_{(V \underset{\mathcal{M}_{\mathrm{FGL},m}}{\times} U)/V} \to i_* \Lambda_{(V^{\mathrm{ord}} \underset{\mathcal{M}_{\mathrm{FGL},m}}{\times} U)/V} = i_* \Lambda_{(V^{\mathrm{ord}} \underset{\mathcal{M}_{\mathrm{FGL},m}}{\times} U)/V^{\mathrm{ord}}}$$

is an equivalence. By the transitivity sequence, and part i) of Lemma 12.2, the map

$$\pi^* \Lambda_{(V^{\mathrm{ord}} \times U)/V^{\mathrm{ord}}} \to \Lambda_{(V^{\mathrm{ord}} \underset{\mathcal{M}_{\mathrm{FGL},m}}{\times} U)/V^{\mathrm{ord}}}$$

is an equivalence. Finally, since U is smooth of dimension 1, the computation of Lemma 5.9 gives a weak equivalence

$$\Lambda_{(V^{\mathrm{ord}} \times U)/V^{\mathrm{ord}}} \approx \mathcal{O}_{V^{\mathrm{ord}}} \otimes \Omega^1_U.$$

The result now follows by assembling this chain of isomorphisms. $\square$

LEMMA 12.3. *Suppose that R is a ring, and $v \in R$ an element which is not a divisor of zero. If P is a projective module and M is a module for which $v : M \to M$ is a monomorphism, then*

$$\mathrm{Ext}^i_R(v^{-1}P, M) = \begin{cases} \hom_R(P, v^{-1}\hat{M}/v^{-1}M) & i = 1 \\ 0 & i > 1, \end{cases}$$

where

$$\hat{M} = \varprojlim_k M/v^k M.$$

Proof: The long exact sequence in Ext_R coming from

$$M \rightarrowtail v^{-1}M \twoheadrightarrow v^{-1}M/M$$

gives (with hom being hom of R-modules)

$$\hom_R(v^{-1}P, M) \to \hom_R(v^{-1}P, v^{-1}M) \to$$
$$\hom_R(v^{-1}P, v^{-1}M/M) \to \mathrm{Ext}_R(v^{-1}P, M) \to 0.$$

Now there are isomorphisms

$$\hom_{R[v^{-1}]}(v^{-1}P, v^{-1}M) \approx \hom_R(P, v^{-1}M) \tag{12.4}$$

$$v^{-1}\hat{M} \xrightarrow{\approx} \varprojlim_n v^{-1}M/v^nM \tag{12.5}$$

$$\hom(P, \varprojlim_n v^{-1}M/v^nM) \xrightarrow{\approx} \hom(v^{-1}P, v^{-1}M/M), \tag{12.6}$$

and with respect to these the map

$$\begin{gathered} v^{-1}M \to \hom(v^{-1}R, v^{-1}M/M) \\ m \mapsto f_m \qquad f_m(r) = rm \end{gathered}$$

corresponds to the natural map $v^{-1}M \to v^{-1}\hat{M}$.

This completes the proof, save verifying the above isomorphisms. The first and third are clear. To see that (12.5) is surjective, suppose that $\mathbf{a} = (a_n) \in \varprojlim_n v^{-1}M/v^nM$ is a sequence of elements of $v^{-1}M$ with

$$a_{n+1} \equiv a_n \mod v^nM.$$

Then for some N, $v^N a_0 \in M$, so for all $n \geq 0$, $v^N a_n \in M$. It follows that $v^N\mathbf{a} \in \hat{M}$, and so $\mathbf{a}$ is in the image of $v^{-1}\hat{M}$. To see that (12.5) is injective, note that the kernel of

$$v^{-1}\hat{M} \to v^{-1}M/v^nM$$

is $v^n\hat{M}$. It follows that the kernel of (12.6) is $\cap v^n\hat{M} = 0$.

Finally, the Milnor sequence for the colimit describing $v^{-1}P$ shows that the higher Ext groups vanish. $\square$

Corollary 12.7. *There is an isomorphism of quasi coherent sheaves*

$$\mathrm{Rhom}^i(\Lambda_{(V \underset{\mathcal{M}_{\mathrm{FGL},m}}{\times} U)/V}, s_*\mathcal{O}_V) \approx \begin{cases} v^{-1}\hat{\mathcal{O}}_V/v^{-1}\mathcal{O}_V \otimes (\Omega^1_V)^{-1} & i = 1 \\ 0 & \textit{otherwise,} \end{cases}$$

where $\hat{\mathcal{O}}_V = \varprojlim \mathcal{O}_V/v^n\mathcal{O}_V$.

Proof: Write

$$\Lambda = \Lambda_{(V \underset{\mathcal{M}_{\mathrm{FGL},m}}{\times} U)/V}.$$

Then

$$\begin{aligned} \mathrm{Rhom}^i(\Lambda, s_*\mathcal{O}_V) &\approx \mathrm{Ext}^i((v \times 1)^{-1}\pi_2^*\Omega^1_U, s_*\mathcal{O}_V) \\ &\approx \mathrm{Ext}^i_V((v)^{-1}s^*\pi_2^*\Omega^1_U, \mathcal{O}_V). \end{aligned}$$

Since $\pi_2 \circ s$ is étale, the natural map $s^*\pi_2^*\Omega^1_U \to \Omega^1_V$ is an isomorphism. Since V is smooth of dimension 1, this latter sheaf is locally free of rank one, and so its module of global sections is projective. The result now follows from Lemma 12.3, with $R = \mathcal{O}_V$, $v = v$, P the module of global sections of Ω^1_V, and $M = \mathcal{O}_V$. $\square$

12.2. Computations over a $\mathbb{Q}$-algebra. Now let $\mathcal{M}_{\mathrm{Ell},0}$ and $\mathcal{M}_{\mathrm{FGL},0}$ be the moduli stacks obtained from $\mathcal{M}_{\mathrm{Ell}}$ and $\mathcal{M}_{\mathrm{FGL}}$ by change of base to $\mathbb{Q}$. Fix

$$U \to \mathcal{M}_{\mathrm{Ell}} \text{ and } V \to \mathcal{M}_{\mathrm{Ell}} \in \mathrm{ob}\,(\mathcal{M}_{\mathrm{Ell}})'_{\mathrm{et}}\,,$$

and a section $s : V \to V \underset{\mathcal{M}_{\mathrm{Ell},0}}{\times} U$ as before. For ease of notation, set

$$\Lambda = \Lambda_{V \underset{\mathcal{M}_{\mathrm{Ell},0}}{\times} U/V}$$

Lemma 12.8. *The scheme $V \underset{\mathcal{M}_{\mathrm{Ell},0}}{\times} U$ is a trivial G_m-torsor over $V \times U$. It is therefore non-canonically isomorphic to*

$$V \times U \times G_m$$

and in particular is smooth over V of dimension 2.

Proof: Since we are over a $\mathbb{Q}$-algebra, the formal groups $\hat{A}_U$ and $\hat{A}_V$ are isomorphic to the formal completions of the additive groups ω_U and ω_V respectively. It follows that

$$\begin{aligned} V \underset{\mathcal{M}_{\mathrm{Ell},0}}{\times} U &\approx \mathrm{Isom}(\pi_1^*\omega_V, \pi_2^*\omega_U) \\ &\approx \mathrm{Isom}(\pi_1^*\hat{\omega}_V, \pi_2^*\hat{\omega}_U), \end{aligned}$$

and so it is a G_m-torsor over $V \times U$. Since we have assumed that the bundles ω are trivializable, the bundles $\pi_1^*\omega_V$ and $\pi_2^*\omega_U$ are isomorphic, and so this torsor does admit a section. □

Corollary 12.9. *The transitivity sequence*

$$\pi^*\Omega^1_{(V\times U)/U} \to H_0\Lambda \to \Omega^1_{(V \underset{\mathcal{M}_{\mathrm{FGL}}}{\times} U)/(V\times U)}$$

is short exact and splittable. It can be identified with

$$0 \to \pi_2^*\Omega^1_U \to H_0\Lambda \to \hom(\pi_1^*\omega_V, \pi_2^*\omega_U) \to 0.$$

There is an isomorphism

$$H_1\Lambda = \pi_2^*\Omega^1_U \otimes \hom(\pi_1^*\omega_V, \pi_2^*\omega_U).$$

The sheaves $H_i\Lambda$ are 0 for $i > 1$.

Proof: This is immediate from Lemmas 12.8 and 5.9. □

Corollary 12.10. *The groups*

$$\mathrm{Der}^i = \mathrm{Der}^i_V(\mathcal{O}_{(V \underset{\mathcal{M}_{\mathrm{FGL},0}}{\times} U)}, s_*\mathcal{O}_V) = \mathrm{Ext}^i(\Lambda, s_*\mathcal{O}_V)$$

vanish for $i > 1$. There is an exact sequence

$$0 \to \mathcal{O}_V \to \mathrm{Der}^0 \to (\Omega^1_V)^{-1} \to 0$$

and an isomorphism

$$\mathrm{Der}^1 \approx (\Omega^1_V)^{-1}.$$

Proof: Since the homology groups of Λ are locally free sheaves, there are isomorphism

$$\begin{aligned}\mathrm{Der}^i &\approx \hom_{(V \underset{\mathcal{M}_{\mathrm{FGL},0}}{\times} U)}(H_i\Lambda, s_*\mathcal{O}_V)\\ &\approx \hom_V(s^* H_i\Lambda, \mathcal{O}_V).\end{aligned}$$

Now there are canonical isomorphism

$$\begin{aligned}s^*\pi_2^*\omega_U &\approx \omega_V\\ s^*\pi_2^*\Omega_U^1 &\approx \Omega_V^1.\end{aligned}$$

The result then follows from Corollary 12.9. $\square$

12.3. Computations over $\mathbb{Z}$. Now return to the situation of maps

$$U \to \mathcal{M}_{\mathrm{Ell}} \text{ and } V \to \mathcal{M}_{\mathrm{Ell}} \in \mathrm{ob}\,(\mathcal{M}_{\mathrm{Ell}})'_{\mathrm{et}}\,,$$

and a section $s : V \to V \underset{\mathcal{M}_{\mathrm{Ell}}}{\times} U$. The results of the previous two sections will now be assembled to give calculations of

$$H_i\Lambda = H_i\Lambda_{(V \underset{\mathcal{M}_{\mathrm{FGL}}}{\times} U)/V}$$

and

$$\begin{aligned}\mathrm{Der}^i &= \mathrm{Der}^i_V(\mathcal{O}_{(V \underset{\mathcal{M}_{\mathrm{FGL}}}{\times} U)}, s_*\mathcal{O}_V)\\ &= \mathrm{Ext}^i(\Lambda, s_*\mathcal{O}_V).\end{aligned}$$

Theorem 12.11. *The sheaves $H_i\Lambda$ are zero for $i \geq 2$. There are exact sequences*

$$0 \to \mathbb{Q} \otimes \hom(\pi_1^*\omega_V, \pi_2^*\omega_U) \to H_0\Lambda \to \mathbb{Q} \otimes \pi_2^*\Omega_U^1 \to 0$$

and

$$\begin{aligned}0 \to \bigoplus_p \mathbb{Q}/\mathbb{Z}_{(p)} \otimes \Omega_V^1 \otimes v^{-1}\hat{\mathcal{O}}_V/v^{-1}\mathcal{O}_V&\\ \to H_1\Lambda \to \mathbb{Q} \otimes \pi_2^*\Omega_V^1 \otimes \hom(\pi_1^*\omega_V, \pi_2^*\omega_U) \to 0.&\end{aligned}$$

Proof: Consider the long exact sequence coming from the cofibration

$$\Lambda \to \Lambda \otimes \mathbb{Q} \to \Lambda \otimes \mathbb{Q}/\mathbb{Z}.$$

$\square$

Theorem 12.12. *The groups Der^i are zero for $i = 0$ and $i > 2$. There is an exact sequence*

$$0 \to M \to \mathrm{Der}^1 \to \prod_p \left(v^{-1}\hat{\mathcal{O}}_V/\mathcal{O}_V\right)_p \to 0$$

where M lies in the (splittable) exact sequence

$$0 \to \left(\Omega_V^1\right)^{-1} \otimes \prod_p (\mathcal{O}_V)_p/\mathcal{O}_V \to M \to \prod_p (\mathcal{O}_V)_p/\mathcal{O}_V \to 0.$$

There is also an isomorphism

$$\mathrm{Der}^2 \approx \left(\Omega^1_V\right)^{-1} \otimes \prod_p (\mathcal{O}_V)_p/\mathcal{O}_V.$$

REMARK 12.13. Note that Der^2 is a vector space over $\mathbb{Q}$.

COROLLARY 12.14. *For each morphism*

$$s : V \to V \underset{\mathcal{M}_{\mathrm{Ell}}}{\times} U$$

of $(\mathcal{M}_{\mathrm{Ell}})'_{\mathrm{et}}$, *and each* $i \geq 0$, *the map*

$$\begin{aligned}\mathrm{Der}^i_V(\mathcal{O}_{V \underset{\mathcal{M}_{\mathrm{FGL}}}{\times} U}, s_*\mathcal{O}_V) &\to \mathrm{Der}^i_V\left((1\times\pi_2)^*\mathcal{O}_{V \underset{\mathcal{M}_{\mathrm{FGL}}}{\times} U}, s_*\mathcal{O}_V\right)\\ &\approx \mathrm{Der}^i_V(\mathcal{O}_{V \underset{\mathcal{M}_{\mathrm{FGL}}}{\times} V}, \Delta_*\mathcal{O}_V)\end{aligned}$$

derived from

$$\begin{tikzcd} V \underset{\mathcal{M}_{\mathrm{FGL}}}{\times} U \arrow[r, "1\times\pi_2 s"] & V \underset{\mathcal{M}_{\mathrm{FGL}}}{\times} V \\ V \arrow[u, "s"] \arrow[r, equal] & V \arrow[u, "\Delta"'] \end{tikzcd}$$

is an isomorphism. □

13. Coherent cohomology of the stack $\mathcal{M}_{\mathrm{Ell}}$

For the moment we consider the stack which is covered by the union of the Hopf-algebroids

$$\begin{aligned}\mathbb{Z}[a_1,\ldots,a_6][c_4^{-1}] &\to \mathbb{Z}[a_1,\ldots,a_6][c_4^{-1}][r,s,t,\lambda^{\pm1}]\\ \mathbb{Z}[a_1,\ldots,a_6][\Delta^{-1}] &\to \mathbb{Z}[a_1,\ldots,a_6][\Delta^{-1}][r,s,t,\lambda^{\pm1}].\end{aligned}$$

We take this to be the definition of the stack $\mathcal{M}_{\mathrm{Ell}}$. This is the part of the moduli stack on which the function j is defined. It is also the part over which an affine étale open U over which ω is trivializable gives rise to a Landweber exact cohomology. These covers allow us to compute the coherent cohomology.

13.1. When 6 is invertible.

Let M be a coherent sheaf on $\mathcal{M}_{\mathrm{Ell}}$ and suppose that $6 \cdot 1_M$ is an isomorphism. Let

$$\begin{aligned}A &= \mathbb{Z}[\tfrac{1}{6}][\gamma_4,\gamma_6]\\ \delta &= \frac{1}{1728}(\gamma_4^3-\gamma_6^2) \in A\end{aligned}$$

and write $U_0 = \operatorname{spec} A[\gamma_4^{-1}]$, $U_\infty = \operatorname{spec} A[\delta^{-1}]$, and $U_{0,\infty} = U_0 \cap U_\infty = \operatorname{spec} A[(\gamma_4\delta)^{-1}]$. The curve

$$y^2 = x^3 + \gamma_4 x + \gamma_6$$

becomes an elliptic curve when restricted to U_0 and U_∞, and $U_{0,\infty}$ and so defines maps

$$U_0 \xrightarrow{f_0} \mathcal{M}_{\mathrm{Ell}}$$
$$U_\infty \xrightarrow{f_\infty} \mathcal{M}_{\mathrm{Ell}}$$
$$U_{0,\infty} \xrightarrow{f_{0,\infty}} \mathcal{M}_{\mathrm{Ell}}.$$

It is not hard to check (details later) that

$$\begin{aligned} U_0 \underset{\mathcal{M}_{\mathrm{Ell}}}{\times} U_0 &= \operatorname{spec} \mathcal{O}_{U_0}[\lambda^{\pm 1}] \\ U_\infty \underset{\mathcal{M}_{\mathrm{Ell}}}{\times} U_\infty &= \operatorname{spec} \mathcal{O}_{U_\infty}[\lambda^{\pm 1}] \\ U_0 \underset{\mathcal{M}_{\mathrm{Ell}}}{\times} U_\infty &= \operatorname{spec} A[(\gamma_4\delta)^{-1}][\lambda^{\pm 1}], \end{aligned}$$

and that in each case, the isomorphism $\pi_1^* A_{(\ -\ )} \to \pi_2^* A_{(\ -\ )}$ is given by

$$\begin{aligned} x &\mapsto \lambda^2 x & \gamma_4 &\mapsto \eta_R(\gamma_4) = \lambda^4\gamma_4 \\ y &\mapsto \lambda^3 y & \gamma_6 &\mapsto \eta_R\gamma_6 = \lambda^6\gamma_6. \end{aligned} \tag{13.1}$$

For $i = 0$, $i = \infty$ or $i = 0, \infty$, let M_i be the module of global sections of $f_i^* M$. Then each M_i comes equipped with an η_R-linear map

$$\psi : M_i \to M_i[\lambda^{\pm 1}].$$

Let $N_i \subset M_i$ be the subset of elements m for which $\psi(m) = m$.

Proposition 13.2. *If "multiplication by* 6*": $M \to M$ is an isomorphism, then*

$$H^s(\mathcal{M}_{\mathrm{Ell}}; M_*) = 0 \qquad \text{if } s > 1.$$

There is an exact sequence

$$0 \to H^0(M) \to N_0 \oplus N_\infty \to N_{0,\infty} \to H^1(M) \to 0.$$

For example, take M_* to be the sheaf of graded rings

$$M_* = \bigoplus_{n\in\mathbb{Z}} \omega^n,$$

graded so that ω^n has degree $2n$. The form dx/y defines over each U_i a nowhere vanishing section u of ω. It follows from (13.1) that $\eta_R(u) = \lambda^{-1}u$. Let $\mathcal{O}_*$ be the graded ring

$$\begin{aligned} \mathcal{O}_* &= \mathbb{Z}[\tfrac{1}{6}][c_4, c_6] \\ c_4 &= u^4\gamma_4 & |c_4| &= 8 \\ c_6 &= u^6\gamma_6 & |c_6| &= 12. \end{aligned}$$

Then one easily calculates

$$\begin{aligned} N_0 &= \mathcal{O}_*[c_4^{-1}] \\ N_\infty &= \mathcal{O}_*[\Delta^{-1}] \\ N_{0,\infty} &= \mathcal{O}_*[(c_4\Delta)^{-1}]. \end{aligned}$$

It follows that $H^0(M) = \mathcal{O}_*$, and that H^1 is concentrated in degrees ≤ 14.

REMARK 13.3. It seems that the target of

$$H^0(\omega^n) \otimes H^1(\omega^{-7-n}) \to H^1(\omega^{-7})$$

is isomorphic to $\mathbb{Z}[\frac{1}{6}]$ and generated by $c_6/(c_4\Delta)$.

13.2. When 3 is nilpotent, and all geometric fibers are ordinary. Now suppose that $3 \cdot 1_M = 0$, and consider the elliptic curve

$$y^2 = x^3 + b_2' x^2 + b_6'$$

over the ring

$$A = \mathbb{F}_3[{b_2'}^{\pm 1}, b_6'].$$

Set $U = \operatorname{spec} A$. Then the curve above defines a map

$$f : U \to \mathcal{M}_{\mathrm{Ell}}.$$

It is not hard to check that

$$U \underset{\mathcal{M}_{\mathrm{Ell}}}{\times} U = \operatorname{spec} A[\lambda^{\pm 1}].$$

As before, if we let M_0 be the module of global sections of f^*M, then M_0 comes equipped with an η_R-linear map

$$\psi : M_0 \to M_0[\lambda^{\pm 1}].$$

Let $N \subset M_0$ be the subset of elements for which $\psi(n) = n$.

PROPOSITION 13.4. *With the above notation, if $3 \cdot 1_M = 0$, and $v_1 : \omega^{p-1} \otimes \mathcal{M}_* \to M_*$ is an isomorphism, then*

$$H^s(\mathcal{M}_{\mathrm{Ell}}; M_*) = \begin{cases} N & s = 0 \\ 0 & s > 0. \end{cases}$$

Again consider the example of the sheaf of graded rings $M = \bigoplus \omega^n$. One easily calculates that $H^0 M$ is the ring

$$\mathbb{F}_3[b_2^{\pm 1}, b_6]$$

where

$$b_2 = u^2 b_2' \qquad b_6 = u^6 b_6'.$$

13.3. When 2 is nilpotent and all geometric fibers are ordinary. Suppose now that $2 \cdot 1_{M_*}$ is nilpotent, and that $c_4 : \omega^4 \otimes M_* \to M_*$ is an isomorphism. Let $V = \operatorname{spec} \mathbb{Z}_2[\alpha_1^{\pm 1}, \alpha_6] = \operatorname{spec} A$ and let $f : V \to \mathcal{M}_{\mathrm{Ell}}$ be the map classifying the elliptic curve A with equation

$$y^2 + \alpha_1 xy = x^3 + \alpha_6.$$

Define a G_m-action $\mu : G_m \times V \to V$ by

$$\alpha_1 \mapsto \lambda \alpha_1$$
$$\alpha_6 \mapsto \lambda^6 \alpha_6$$

Over the $G_m \times V$, the map ϵ

$$\epsilon(x) = \lambda^2 x \qquad (13.5)$$
$$\epsilon(y) = \lambda^3 y$$

sends a point on the elliptic curve $\pi_2^* A$ defined by the equation $y^2 + \alpha_1 xy = x^3 + \alpha_6$ to a point on the elliptic curve $\mu^* A$ defined by the equation $y^2 + (\lambda\alpha_1)xy = x^3 + (\lambda^6 \alpha_6)$.

This gives a 2-morphism (also denoted ϵ) from $f \circ \pi_2$ to $f \circ \mu$, and therefore an isomorphism $\epsilon^* : \pi_2^* f^* M \to \mu^* f_* M$. The abelian group $H^0 f^* M$ therefore becomes a co-module over $\mathbb{Z}[\lambda^{\pm 1}]$, and hence acquires a grading. The homogeneous part $H^0 f^* M[n]$ of degree n is the set of sections a of $f^* M$ satisfying $\mu^* s = \lambda^n \pi_2^* s$. The image of $H^0(\mathcal{M}_{\mathrm{Ell}}; M)$ in $H^0(V; f^* M)$ is contained in $H^0 f^* M[0]$.

The line bundle $f^*\omega$ has the nowhere vanishing section

$$u = \frac{dx}{2y + \alpha_1 x} = \frac{dy}{3x^2 - \alpha_1 y}.$$

Under the map (13.5) it is sent to $\lambda^{-1} u$. It follows that "multiplication by u" gives an isomorphism

$$\left(H^0 f^* M\right)[n] \approx \left(H^0 f^* \omega \otimes M\right)[n-1].$$

The entire graded module $H^0 f^* M$ can therefore be identified with the $\lambda = 1$ eigenspace in $\bigoplus H^0 f^* \omega^k \otimes M$.

Next consider the scheme $Z = \operatorname{spec} A[s]/(s^2 + \alpha_1 s)$, and define maps $\eta_R, \eta_L : Z \to V$ by

$$\begin{aligned} \operatorname{spec} \eta_L(\alpha_1) &= \alpha_1 \\ \operatorname{spec} \eta_L(\alpha_6) &= \alpha_6 \end{aligned} \qquad \text{and} \qquad \begin{aligned} \operatorname{spec} \eta_R(\alpha_1) &= \alpha_1 + 2s \\ \operatorname{spec} \eta_R(\alpha_6) &= \alpha_6. \end{aligned}$$

Define a map $\epsilon : \eta_L^* A \to \eta_R^* A$ by

$$\begin{aligned} x &\mapsto x \\ y &\mapsto y + sx \end{aligned}$$

This gives a 2-morphism (also denoted ϵ) from $f \circ \eta_L \to f \circ \eta_R$, and equips the sheaf $f^* M$ with a map $\epsilon^* : \eta_L^* f^* M \to \eta_R^* f^* M$.

The ring $\mathcal{O}_Z$ is free of rank 2 as a module over $\mathcal{O}_V$, with basis $\{1, s\}$. Let's use the map "multiplication by s" to identify the cokernel of

$$\eta_L^* : H^0(f^* M) \to H^0(\eta_L^* f^* M) = \mathcal{O}_Z \underset{\mathcal{O}_V, \eta_L}{\otimes} H^0(f^* M)$$

with $H^0(f^* M)$. Let $d : H^0(f^* M) \to H^0(f^* M)$ be the composite

$$\begin{aligned} H^0(f^* M) \xrightarrow{\eta_R^*} H^0(\eta_R^* f^* M) \xrightarrow{\epsilon} H^0(\eta_L^* f^* M) \\ \to H^0(\eta_L^* f^* M)/H^0(f^* M) \approx H^0(f^* M). \end{aligned}$$

Lemma 13.6. *The map d^2 is 0.*

Next we need to investigate the effect of the map d on the grading. Define an action of G_m on Z by

$$\begin{aligned} \alpha_i &\mapsto \lambda^i \alpha_i \\ s &\mapsto \lambda s. \end{aligned}$$

With respect to this action, both maps η_L and η_R are equivariant.

Lemma 13.7. *The map d sends an element of $H^0(f^* M)[n]$ to an element of $H^0(f^* M)[n+1]$.*

Proof: The map d is essentially "multiplication by s". The result follows easily. □

PROPOSITION 13.8. *Suppose that M is a quasi-coherent sheaf over $\mathcal{M}_{\mathrm{Ell}}$ with $2^k 1_M = 0$ for some k, and with the property that $c_4 : \omega^4 \otimes M \to M$ is an isomorphism. Write*

$$N_k = H^0(f^* M)[k] \approx H^0(f^* \omega^k \otimes M)[0].$$

The cohomology group $H^t(M)$ is naturally isomorphic to the t^{th} cohomology group of the complex

$$N_0 \xrightarrow{d} N_1 \to \cdots \to N_{k-1} \xrightarrow{d} N_k \to \dots.$$

EXAMPLE 13.9. Consider the example of

$$M_k = c_4^{-1} \bigoplus_{n \in \mathbb{Z}} \omega^n \mathcal{O}_{\mathcal{M}_{\mathrm{Ell}}}/2^k$$

In case $k = 1$ the differential d is zero since $\eta_L = \eta_R \bmod 2$. The ring $H^* M$ can be identified as the graded ring $\mathbb{Z}/2[a_1^{\pm 1}, a_6][\eta]$, with $a_i = u^i \alpha_i$, and $\eta \in H^1(f^* \omega \otimes M)$. The Bockstein operator is given by $\beta(a_1) = \eta$, and one can identify

$$H^*(\varprojlim M_k)$$

with the subring of $\mathbb{Z}_2[a_1^{\pm 1}, a_6, \eta]/(2\eta)$ generated by $\{a_1^2, a_6, \eta, a_1\eta\}$. As an abstract ring it is isomorphic to

$$\mathbb{Z}_2[a_1^{\pm 2}, a_6, \eta, \beta]/(2\eta, \beta^2 - a_1^2 \eta^2).$$

EXAMPLE 13.10. Another important example is the case in which

$$M_k = c_4^{-1} \bigoplus_{n \in \mathbb{Z}} \omega^n (\widehat{\mathcal{O}_{\mathcal{M}_{\mathrm{Ell}}}/2^k})$$

where the "hat" indicates completion along the supersingular locus. The ring of global sections of $f^* M$ is

$$a_1^{-1} \mathbb{Z}/2^k [\![a_1]\!][a_6, u^{\pm 1}],$$

where $u \in H^0(f^* \omega)$ is the invariant section mentioned above. It is not difficult to calculate that

$$H^*(f^* \varprojlim_k M_k)$$

is isomorphic to the subring of $a_1^{-1} \mathbb{Z}_2 [\![a_1]\!][a_6, \eta]/(2\eta)$ which is topologically generated by $\{a_1^2, a_6, \eta, a_1\eta\}$. As an abstract ring it is

$$a_1^{-2} \mathbb{Z}_2 [\![a_1^2]\!][a_6, \eta, \beta]/(2\eta, \beta^2 - a_1^2 \eta^2).$$

Appendix A. Low dimensional cohomology of categories

Consider the following situation. Suppose that $\mathbf{C}$ and $\mathbf{D}$ are categories with the same object set, and that $F : \mathbf{C} \to \mathbf{D}$ is a functor. Given a morphism $f \in \mathbf{D}(Fd_1, Fd_0)$ let

$$\mathbf{C}(d_1, d_0)_f$$

be the subset of $g \in \mathbf{C}(d_1, d_0)$ with $Fg = f$. In addition assume that the functor F creates isomorphisms (ie. a map g is an isomorphism if and only if Fg is). Then the set $\mathbf{C}(d_0, d_0)_1$ is a group, and it acts on the left of $\mathbf{C}(d_1, d_0)$. Suppose that for each pair of objects d_0 and d_1, the set $\mathbf{C}(d_1, d_0)_f$ is a principal homogeneous space for the group $\mathbf{C}(d_0, d_0)_1$. It will also make life easier if we assume that this group is abelian.

We wish to investigate the obstruction to the existence of a section, the enumeration of sections, and the automorphism group of a section. It is useful consider the case when the categories $\mathbf{C}$ and $\mathbf{D}$ have only one object, and the morphism sets form a group. In this case the situation is that of a surjective map of groups with abelian kernel. The obstruction to the existence of a section is H^2, the set of sections up to vertical equivalence is a principal homogeneous space for H^1, and the automorphism group of any section is H^0.

We thank Charles Rezk here for his help.

Let's write $A(d)$ for the (abelian) group $\mathbf{C}(d,d)_1$. The first thing to notice is that $d \mapsto A(d)$ is a covariant functor on $\mathbf{D}$ in a natural way. Indeed, suppose that $f : d_1 \to d_0$ is a map in $\mathbf{D}$ and pick $f_1 \in \mathbf{C}(d_1, d_0)_f$. Then by assumption, given $\alpha \in A(d_1)$ there is a unique $\beta \in A(d_0)$ for which the following diagram commutes:

$$\begin{tikzcd} d_1 \arrow[r, "f_1"] \arrow[d, "\alpha"'] & d_0 \arrow[d, "\beta"] \\ d_1 \arrow[r, "f_1"'] & d_0. \end{tikzcd}$$

Define

$$A(f) : A(d_1) \to A(d_0)$$

by $A(f)(\alpha) = \beta$.

It is easily checked that $A(f)$ is a homomorphism and it remains to verify its independence of the choice of f_1. For this, suppose that f_2 is another element of $\mathbf{C}(d_1, d_0)_f$. Then $f_2 = \gamma f_1$ for some $\gamma \in A(d_0)$. Let's temporarily write A_i, $i = 1, 2$ for the map based on the choice f_i. Then $A_i(\alpha)$ is the unique solution to the equation

$$A_i(\alpha) f_i = f_i \alpha.$$

Now compute

$$\begin{aligned} A_2(\alpha) f_2 &= f_2 \alpha \\ A_2(\alpha)\gamma f_1 &= \gamma f_1 \alpha \\ \gamma^{-1} A_2(\alpha) \gamma f_1 &= f_1 \alpha \\ A_2(\alpha) f_1 &= f_1 \alpha, \end{aligned}$$

where the last step makes use of the commutativity of the $A(d)$. It follows that $A_2(\alpha)$ satisfies the defining equation of $A_1(\alpha)$ and so the two must coincide.

Suppose instead that we are given a functor A from $\mathbf{D}$ to the category of abelian groups. Form the category

$$A \rtimes \mathbf{D}$$

whose objects are the objects of $\mathbf{D}$ and whose morphisms from d_1 to d_0 are the set $A(d_0) \times \mathbf{D}(d_1, d_0)$. The composition law is:

$$(\beta, g) \circ (\alpha, f) = (\beta \cdot A(g)(\alpha), g \circ f).$$

The construction of this category does not make use of the fact that the groups $A(d)$ are commutative.

The category $A \rtimes \mathbf{D}$ is an abelian group object in the category of categories over $\mathbf{D}$ with a fixed object set. The category $\mathbf{C}$ is a principal homogeneous space for $A \rtimes \mathbf{D}$ and so it follows that the obstruction to the existence of a section is in Quillen H^1 of $\mathbf{D}$ with values in the abelian group object $A \rtimes \mathbf{D}$. If this obstruction vanishes, then the set of sections is, by defintion, Quillen H^0.

In this situation, there is, for $i > 0$, an isomorphism of Quillen H^i with $\varprojlim^{i+1} A$. The isomorphism is easily established by imitating the discussion of Schur in the case of groups.

Suppose then that we wish to write down a section of F. Choose then, for each $f \in \mathbf{D}(d_1, d_0)$ a map $s(f) \in \mathbf{C}(d_1, d_0)_f$. Given a composable pair

$$d_0 \xleftarrow{f_0} d_1 \xleftarrow{f_1} d_2,$$

let $\delta(f_0, f_1) \in A(d_0)$ be the unique solution to the equation

$$\begin{tikzcd} d_2 \arrow[r, "s(f_1)"] \arrow[d, "s(f_2 f_1)"'] & d_1 \arrow[d, "s(f_0)"] \\ d_0 \arrow[r, "\delta(f_0,f_1)"] & d_0. \end{tikzcd}$$

$$\delta(f_0, f_1) s(f_0 f_1) = s(f_0) s(f_1).$$

By considering the situation

$$d_0 \xleftarrow{f_0} d_1 \xleftarrow{f_1} d_2 \xleftarrow{f_2} d_3$$

it is easy to check that the function δ satisfies the "cocycle" condition

$$Af_0(\delta(f_1, f_2)) + \delta(f_0, f_1 f_2) = \delta(f_0, f_1) + \delta(f_0 f_1, f_2), \tag{A.1}$$

where the symbol "+" has been used to indicate composition, in order to emphasize that the group $A(d_0)$ is abelian. Conversely, given a "cocycle" δ it is easy to construct a category $\mathbf{C}$ as above.

Now let's compare two choices s_1 and s_2 of s. For each $f \in \mathbf{D}(d_1, d_0)$ there is a unique $\alpha(f) \in A(d_0)$, with $s_2(f) = \alpha(f)s_1(f)$. Let δ_i, $i = 1, 2$ be the associated cocycles. Then

$$\begin{aligned} \delta_2(f_0, f_1) s_2(f_0 f_1) &= s_2(f_0) s_2(f_1) \\ \delta_2(f_0, f_1) \alpha(f_0 f_1) s_1(f_0 f_1) &= \alpha(f_0) s_1(f_0) \alpha(f_1) s_1(f_1) \\ \delta_2(f_0, f_1) \alpha(f_0 f_1) s_1(f_0 f_1) &= \alpha(f_0) A f_0(\alpha(f_1)) \circ s_1(f_0) s_1(f_1) \end{aligned}$$

and so

$$\delta_2(f_0, f_1) \alpha(f_0 f_1) = \alpha(f_0) A f_0(\alpha(f_1)) \circ \delta_1(f_0, f_1).$$

Define an equivalence relation on the set of cocycles δ by declaring δ_1 to be equivalent to δ_2 if there is a function α with

$$\delta_2(f_0, f_1) + \alpha(f_0 f_1) = \alpha(f_0) + A f_0(\alpha(f_1)) + \delta_1(f_0, f_1). \tag{A.2}$$

Then it is not hard to check that the set of equivalence classes of "extensions" of $\mathbf{D}$ by A is naturally in one to one correspondence with the set of cocycles δ modulo the equivalence relation (A.2).

Now suppose that functor $\mathbf{C} \to \mathbf{D}$ admits a section. Given a section s one easily constructs an equivalence of categories

$$\mathbf{C} \to A \rtimes \mathbf{D}$$

by sending $\tilde{f} \in \mathbf{C}(d_1, d_0)_f$ to (α, f), where α is the unique solution to the equation

$$\alpha s(f) = \tilde{f}.$$

Suppose now that s_1 and s_2 are two sections of F. Write

$$s_1(f) = \alpha(f)s_2(f).$$

It follows from the fact that s_1 and s_2 preserve the composition law that α satisfies the "crossed homomorphism" identity

$$\alpha(f_0f_1) = Af_0(\alpha(f_1)) + \alpha(f_0). \tag{A.3}$$

The set of sections is thus a principal homogeneous space for the abelian group of functions α satisfying the identity (A.3).

Next consider the situation where sections s_1 and s_2 are related by a vertical natural equivalence. Then there is a function $d \to \beta(d) \in A(d)$ representing the natural transformation.

Given $f : d_1 \to d_0$, the natural transformation property

$$\begin{tikzcd} s_1(d_1) \arrow[r, "\beta(d_1)"] \arrow[d, "s_1(f)"'] & s_2(d_1) \arrow[d, "s_2(f)"] \\ s_1(d_0) \arrow[r, "\beta(d_0)"'] & s_2(d_0) \end{tikzcd}$$

becomes the identity

$$\begin{aligned} \beta(d_0)s_1(f) &= s_2(f)\beta(d_1) \\ &= Af\left(\beta(d_1)\right)s_2(f). \end{aligned} \tag{A.4}$$

It follows that the function $\alpha(f)$ satisfies the equation

$$Af(\beta(d_1)) + \alpha(f) = \beta(d_0). \tag{A.5}$$

The set of sections of F up to vertical natural equivalence can therefore be identified with the quotient of the abelian group of functions α satisfying (A.3), modulo the subgroup consisting of those satisfying the identity (A.5) for some β.

Finally, consider the above situation with $s_1 = s_2 = s$. Then $\alpha(f) = 0$, and equation (A.5) becomes

$$Af(\beta(d_1)) = \beta(d_0).$$

In other words β is an element of $\varprojlim^0 A$. It follows that the group of "vertical" automorphisms of any section is naturally isomorphic to $\varprojlim^0 A$.

The situation becomes much clearer when expressed in terms of the *cosimplicial replacement* [**BK**, X1,§5] of the functor A. The cosimplicial replacement $C^\bullet(\mathbf{D}; A)$ is a cosimplical abelian group whose s^{th} cohomology group (or, more precisely, cohomotopy group) is the derived functor $\varprojlim^s A$. It is defined by

$$C^n = C^n(\mathbf{D}; A) = \prod_{\underline{f}} A(d_0) \qquad n > 0$$

where

$$\underline{f} = d_0 \xleftarrow{f_0} d_1 \xleftarrow{f_1} \cdots d_{n-1} \xleftarrow{f_{n-1}} d_n$$

is a sequence of composable maps. It is best to think of elements of C^n as "functions" α on the set of composable n-tuples of maps of $\mathbf{D}$ (when $n = 0$ this is interpreted as "functions" on the set of objects). The coface maps are given for $\alpha \in C^n$ with $n > 0$ by

$$d^i\alpha(f_0,\dots,f_n) = \begin{cases} Af_0\left(\alpha(f_1,\dots,f_n)\right) & i=0 \\ \alpha(\dots,f_{i-1}f_i,\dots) & 0<i\le n \\ \alpha(f_0,\dots,f_{n-1}) & i=n+1, \end{cases}$$

and for $\alpha \in C^0$ by

$$\begin{aligned} d^0\alpha(f_0) &= Af_0\left(\alpha(d_1)\right) \\ d^1\alpha(f_0) &= \alpha(d_0). \end{aligned}$$

The codegeneracy maps are give on $\alpha \in C^{n+1}$ by

$$s^i\alpha(f_0,\dots,f_n) = \alpha(\dots,1_{d_i},f_i,\dots).$$

Now it is easy to check that the cocycle condition (A.1) on a function δ is precisely the condition that δ be a 2-cocycle in NC^2. The equivalence relation (A.2) is the condition that δ_1 and δ_2 differ by a coboundary. Thus the set of "extensions" of $\mathbf{D}$ by A can be identified with $\varprojlim^2 A$.

Given a section $s : \mathbf{D} \to \mathbf{C}$ it is not hard to check that any other section differs by a 1-cocycle in NC^1. It follows that the set of sections is precisely the set of 1-cocycles. Two sections are vertically equivalent if and only if the corresponding 1-cocycles differ by a coboundary. It follows that the set of equivalence classes of sections can be identified with $\varprojlim^1 A$.

This discussion is summarized in the following proposition

Proposition A.6. *With the notation of this appendix, the functor*

$$F : \mathbf{C} \to \mathbf{D}$$

is classified by an element of $\varprojlim^2 A$. The vanishing of this class is a necessary and sufficient condition for the existence of a section. The set of sections is a principal homogeneous space for the group of 1-cocycles in the cosimplicial replacement of the functor A. The set of sections modulo (vertical) natural equivalence is a principal homogeneous space for $\varprojlim^1 A$, and the group of vertical automorphisms of any section is canonically isomorphic to $\varprojlim A$. □

Appendix B. A spectral sequence for A_∞-maps

B.1. The spectral sequence. Let's start with a ring spectrum F satisfying the following condition of Adams [**Ad**, Condition 13.3 (page 284)]

(1) Each map of spectra

$$X \to F$$

with X finite, factors as

$$X \to V \to F$$

where V is finite and $F_*DV \approx F^*V$ is projective over F_*.

We are now going to describe a simplicial model category structure on the category sA_∞ of simplicial A_∞-rings, that will make systematic the kind of resolutions we wish to use. This construction is somewhat ad hoc. It is chosen so that the following result holds.

Proposition B.1. *The F^α model category structure on sA_∞ satisfies*

i) *If $X_\bullet \to Y_\bullet$ is a weak equivalence (resp. fibration), then for each t the maps*

$$F_t X_\bullet \to F_t Y_\bullet$$
$$\pi_t X_\bullet \to \pi_t Y_\bullet$$

are weak equivalences (resp. fibrations) of simplicial groups;

ii) *If $W_\bullet$ is cofibrant, then each W_s is a free A_∞ ring on a spectrum V_s which has the property that $F_* V_s$ is a projective F_*-module.*

REMARK B.2. It follows from the second property that $F_* W_s$ is the free associative F_*-algebra on $F_* V_s$ and that the maps

$$F^* W_s \to \hom_{F_*}(F_* W_s, F_*)$$
$$\pi_t A_\infty(W_s, F) \to \hom_{F_*}(F_* V_s, \tilde{F}^0 S^t)$$

are isomorphisms.

This result will be proved in B.4. Given this result, we are in position to investigate the function space $A_\infty(E, F)$ when the ring spectrum underlying F satisfies the condition above. To do this, regard E and S^0 as constant simplicial objects in sA_∞ and factor the unit map

$$S^0 \to W_\bullet \to E$$

as a cofibration followed by an acyclic fibration. By the homotopy spectral sequence (which collapses) the map

$$|W_\bullet| \to E$$

is a weak equivalence (of cofibrant-fibrant objects), so the map $A_\infty(E, F) \to A_\infty(|W_\bullet|, F)$ is a weak equivalence, and we can get a spectral sequence by writing the latter as

$$\mathrm{Tot}(A_\infty(W_\bullet, F)).$$

For simplicity of notation, let's write $M^\bullet$ for the cosimplicial space $A_\infty(W_\bullet, F)$.

To define the E_2-term of the spectral sequence, we need to pick an element of $\pi_0 M^0$ which is equalized under the maps $\pi_0 d^0$ and $\pi_0 d^1$. Now by our assumptions, the set $\pi_0 M^k$ is naturally isomorphic to the set of F_*-algebra maps from $F_* W_k$ to F_*. It follows that the equalizer of $\pi_0 d^0$ and $\pi_0 d^1$ is the set of F_*-algebra maps from the coequalizer in F_* algebras of

$$F_* W_1 \underset{F_* d_1}{\overset{F_* d_0}{\rightrightarrows}} F_* W_2.$$

Since this comes from a simplicial diagram this is a reflexive coequalizer and so the F_*-module which underlies the algebra coequalizer coincides with the coequalizer of the underlying diagram of F_*-modules. By part i) of Proposition B.1 this is simply $F_* E$. So in order to define the E_2-term of the spectral sequence, we have to pick an F_*-algebra map from $F_* E$ to F_*.

Having done this we now have a map of simplicial F_*-algebras from $F_* W_\bullet$ to F_*, and we also get a "basepoint" in each M^k. With this choice of basepoint, there is a natural identification

$$\pi_t M^k \approx \mathrm{Der}_{F_*}(F_* W_k, \tilde{F}^0 S^t).$$

The $E_2^{s,t}$-term of the spectral sequence is $\pi^s \pi_t M^\bullet$. Now, by assumption, the map

$$F_* W_\bullet \to F_* E$$

is a weak equivalence of F_*-algebras. Furthermore, each F_*W_k is a free associative F_*-algebra on a projective F_*-module. It follows from the definition of Quillen derived functors, that the E_2-term of the spectral sequence is

$$\mathrm{Der}^s_{F_*}\left(F_*E, \tilde{F}^0(S^t)\right).$$

B.2. The Reedy model category structure. Let $\mathbf{\Delta}$ be the usual category of finite ordered sets, and let $\mathbf{\Delta}_{\leq n}$ be the full subcategory with objects

$$[0], [1], \ldots, [n].$$

If $\mathbf{C}$ is a category with finite limits and colimits, then the restriction functor

$$\mathrm{res}_n^{\mathrm{op}} : \mathbf{C}^{\mathbf{\Delta}^{\mathrm{op}}} \to \mathbf{C}^{\mathbf{\Delta}^{\mathrm{op}}_{\leq n}}$$

has both a left and a right adjoint. Let's denote the left adjoint l_n^{op} and write skel_n for the composite $l_n^{\mathrm{op}}\mathrm{res}_n^{\mathrm{op}}$. The functor skel_n is the n-skeleton functor.

Let $s\mathbf{C}$ be the category $\mathbf{C}^{\mathbf{\Delta}^{\mathrm{op}}}$ of simplicial objects in $\mathbf{C}$.

Definition B.3. Let $X_\bullet$ be an object of $s\mathbf{C}$. The n^{th} *latching object of* $X_\bullet$ is the object $(\mathrm{skel}_{n-1} X_\bullet)_n$.

Now let $c\mathbf{C}$ be the category of cosimplicial objects in $\mathbf{C}$. As above, the restriction functor

$$\mathrm{res}_n : \mathbf{C}^{\mathbf{\Delta}} \to \mathbf{C}^{\mathbf{\Delta}_{\leq n}}$$

has both a left and a right adjoint. Let's call the right adjoint r_n, and write coskel^n for the composite $r_n\mathrm{res}_n$. The functor coskel^n is the n^{th} *coskeleton* functor.

Definition B.4. Let $X^\bullet$ be an object of $c\mathbf{C}$. The n^{th} *matching object of* $X^\bullet$ is the object $\left(\mathrm{coskel}^{n-1} X^\bullet\right)_n$.

These are used to define the Reedy model category structure.

B.3. The P model category structure. This is the variation of the E_2 model category structure described in [**DKS**, §5.9]. Suppose that $\mathbf{C}$ is a closed (pointed) model category, with arbitrary colimits in which every object is fibrant. Suppose also that $M \in \mathrm{ob}\,\mathbf{C}$ is a cofibrant co-grouplike object of $\mathcal{C}$.

Definition B.5. [**DKS**, 5.4–5.6] A map $g_\bullet : X_\bullet \to Y_\bullet$ of $s\mathbf{C}$ is a

(1) *weak M-equivalence* if for each $j \geq 0$ the map

$$\mathrm{ho}\,\mathbf{C}(\Sigma^j M, X_\bullet) \to \mathrm{ho}\,\mathbf{C}(\Sigma^j M, Y_\bullet)$$

is a weak equivalence of simplicial groups

(2) *M-fibration* if it is a Reedy fibration and if for each $j \geq 0$ the map

$$\mathrm{ho}\,\mathbf{C}(\Sigma^j M, X_\bullet) \to \mathrm{ho}\,\mathbf{C}(\Sigma^j M, Y_\bullet)$$

is a fibration of simplicial groups;

(3) *M-cofibration* if it is a retract of an "M-free" map, where a map $X_\bullet \to Y_\bullet$ is M-free if for each $n \geq 0$ there is a cofibrant object $Z_n \in \mathrm{ob}\,\mathbf{C}$ which is weakly equivalent to a coproduct of copies of $\Sigma^j M$ $(j \geq 0)$, and a map $Z_n \to Y_n$ such that the induced map

$$(X_n \amalg_{L_n X_\bullet} L_n Y_\bullet) \amalg Z_n \to Y_n$$

is a trivial cofibration.

Proposition B.6. [**DKS**, 5.3] *The category $s\mathbf{C}$ equipped with M weak equivalences, M fibrations, and M cofibrations, and the natural simplicial structure, is a closed simplicial model category.* $\square$

Remark B.7. (1) It isn't really necessary that $\mathbf{C}$ be pointed. The object M, being co-goup like, comes equipped in $\operatorname{ho}\mathbf{C}$ with a "co-basepoint" map to the initial object, representing the identity element of the corresponding group of maps. One needs to redefine ΣM as the pushout of $CM \leftarrow M \rightarrow CM$, where CM is an object obtained by factoring a representative of this "co-basepoint" into a cofibration followed by an acyclic fibration.

(2) If one has in mind a set P of cofibrant co-grouplike objects, then one can define a P-model category structure by requiring the condition on M to hold for each element of P. Of course this is no gain in generality, since the P-model category structure coincides with the M-model category structure with M taken to be the coproduct of the objects of P.

(3) The way it will come up in the present paper, the set P will be stable under the suspension operator. This means that the clauses "for each $j \geq 0$" reduce to the case $j = 0$ and needn't be considered.

(4) If M and M' are weakly equivalent then the M and M' model category structures coincide.

(5) It follows that the set P doesn't quite need to be a set. It is only necessary that the collection of weak equivalence classes of elements of P be small, since the P model category structure can be built using one cofibrant representative from each weak equivalence class.

B.4. The F^{α} model category structure. Let T be an operad over the linear isometries operad, and T-alg the corresponding category of algebras. The category T-alg is a closed topological model category and is generated by small objects. It has all (small) enriched limits and colimits, and has the property that every object is fibrant.

For a ring spectrum F, let F^{α} be the collection consisting of spectra V which are weakly equivalent to a finite spectrum and for which F_*V is a projective module over F_*.

Definition B.8. The F^{α} *model category structure on* sT-alg is the P model category structure, where

$$P = \{T(V) \mid V \in F^{\alpha}\}.$$

The spectra of the form $T(V)$ are co-grouplike since

$$\operatorname{ho} T\text{-alg}(T(V), Y) = \operatorname{ho}\mathcal{S}(V, Y)$$

is an abelian group.

If F satisfies the condition of Adams then [**Ad**, Lemma 13.8 (page 287)]) shows that given any spectrum X and a homology class in $x \in F_*X$ there is a map $f\colon U \to X$, $U \in F^{\alpha}$, and an element $e \in F_*U$ with the property that

$$x = F_*f(e).$$

More generally suppose that $X : I \to \mathcal{S}$ is a diagram in indexed by a category I which has only finitely many objects and morphisms. Define F_*X to be the diagram of F_*-modules

$$i \mapsto F_*(Xi).$$

Then given any diagram I-diagram V of finite sets, and a map $x : V \to F_* X$ there is an element $U \in F^\alpha$, an element $e \in F_* U$ with the property that x factors through the resulting map

$$\operatorname{ho}\mathcal{S}(U, X) \to F_* X$$
$$\operatorname{ho}\mathcal{S}(U, X) \,=\, i \mapsto \operatorname{ho}\mathcal{S}(U, Xi).$$

The proof follows the argument of [**Ad**, Lemma 13.8] and makes use of the fact that, since I is finite and V takes its values in finite sets, the functor $E \mapsto \mathcal{S}^I(V, E_* X)$ commutes with directed homotopy colimits.

In the situation that arises below, X is a simplicial spectrum and V is a simplicial set with only finitely many non-degenerate simplices. Though in this case the indexing category is not finite, the fact that V has only finitely many non-degenerate simplices means that it is of the form $l_n^{\mathrm{op}} V'$ for some n, and so the diagram hom set can be calculated as a hom set in $\mathcal{S}^{\Delta^{\mathrm{op}}_{\leq n}}$.

Proof of Proposition B.1: Part ii) is easy and follows from the Kunneth spectral sequence (which collapses, since projective modules are flat)

In part i), by factoring the map $X_\bullet \to Y_\bullet$ into an acyclic Reedy cofibration followed by a Reedy fibration, we may in both cases assume that the map is an F^α fibration. In both cases we need to produce a lift in a diagram of the form

$$\begin{tikzcd} V \arrow[r] \arrow[d] & F_t X_\bullet \arrow[d] \\ D \arrow[r] & F_t Y_\bullet. \end{tikzcd}$$

In the "fibration" case, the map $V \to D$ runs through the inclusions $V[n,k] \subset \Delta[n]$ while in the "weak equivalence" case it runs through the inclusions $\dot{\Delta}[n] \subset \Delta[n]$. Since V and D have only finitely many non-degenerate simplices, there is a $U \in F^\alpha$, and an element of $F_* U$ with the property that the diagram factors as

$$\begin{tikzcd} V \arrow[r] \arrow[d] & \operatorname{ho} A_\infty(TU, X_\bullet) \arrow[r] \arrow[d] & F_t X_\bullet \arrow[d] \\ D \arrow[r] & \operatorname{ho} A_\infty(TU, Y_\bullet) \arrow[r] & F_t Y_\bullet, \end{tikzcd}$$

where we have made use of the isomporphism

$$\operatorname{ho} A_\infty(TU, E) \approx \operatorname{ho}\mathcal{S}(U, E).$$

By the definition of the F^α model category structure there is a lift in the left square. Composing this with the rightmost top arrow gives the desired lift. $\square$

Appendix C. A calculation

Notation:

$$
\begin{gathered}
f : E \to F \quad \text{a map of elliptic spectra} \\
\operatorname{spec} \pi_0 E = U \to \mathcal{M}_{\mathrm{Ell}} \\
\operatorname{spec} \pi_0 F = V \to \mathcal{M}_{\mathrm{Ell}} \\
\operatorname{spec} \pi_0 f = s : V \to V \underset{\mathcal{M}_{\mathrm{FGL}}}{\times} U \\
\pi : V^{\mathrm{ord}} \underset{\mathcal{M}_{\mathrm{FGL}}}{\times} U^{\mathrm{ord}} \to V^{\mathrm{ord}} \times U^{\mathrm{ord}}
\end{gathered}
$$

The map s is given to us factored through $V \underset{\mathcal{M}_{\mathrm{Ell}}}{\times} U$.

Now we calculate, twice, in two notations, all reduced mod p^m, but not indicated:

$$
\begin{aligned}
&\operatorname{Rhom}_{\pi_0 F \wedge E}\left(\Lambda_{\pi_0 F \wedge E / \pi_0 F}, \pi_0 F\right) \\
&\qquad = \operatorname{Rhom}_{\pi_0 F \wedge E}\left(\Lambda_{\pi_0 v_1^{-1}(F \wedge E)/\pi_0 v_1^{-1} F}, \pi_0 F\right)
\end{aligned}
$$

$(M = \Sigma^{-1} \pi_0 L_{K(2)} F / F)$

$$
\begin{aligned}
&= \operatorname{Rhom}_{\pi_0 v_1^{-1} F \wedge E}\left(\Lambda_{\pi_0 v_1^{-1}(F \wedge E)/\pi_0 v_1^{-1} F}, M\right) \\
&= \operatorname{Rhom}_{\pi_0 v_1^{-1} F \wedge E}\left(\Lambda_{\pi_0 v_1^{-1}(F \wedge E)/\pi_0 v_1^{-1} F}, M\right) \\
&= \operatorname{Rhom}_{\pi_0 v_1^{-1} F \otimes \pi_0 v_1^{-1} E}\left(\Lambda_{\pi_0 v_1^{-1} F \otimes \pi_0 v_1^{-1} E / \pi_0 v_1^{-1} F}, M\right) \\
&= \operatorname{Rhom}_{\pi_0 v_1^{-1} F \otimes \pi_0 v_1^{-1} E}\left(\pi_0 v_1^{-1} F \otimes \Omega^1\left(\pi_0 v_1^{-1} E\right), M\right) \\
&= \operatorname{Rhom}_{\pi_0 v_1^{-1} E}\left(\Omega^1\left(\pi_0 v_1^{-1} E\right), M\right) \\
&= \operatorname{Rhom}_{\pi_0 v_1^{-1} F}\left(\Omega^1\left(\pi_0 v_1^{-1} F\right), M\right) \\
&= \left(\Omega^1\right)^{-1} \underset{\pi_0 F}{\otimes} M = \left(\omega^2(\text{cusps})\right)^{-1} \underset{\pi_0 F}{\otimes} M \\
&= \omega^{-2}(-\text{cusps}) \otimes M
\end{aligned}
$$

Here's another good one

Lemma C.1. *Suppose that E is the p-adic completion of an étale elliptic spectrum. There is, after p-adic completion, a fibration up to homotopy*

$$
\prod_{\textit{cusps}} \Omega_0^\infty \Sigma^4 KU \to A_\infty \{E\} \to \Omega_0^\infty \Sigma^4 L_{K(2)} E / E
$$

Where Ω_0^∞ means the connected component. In particular, the space $A_\infty\{E\}$ is connected.

Appendix D. Calculation

The substack $\mathcal{M}_{\mathrm{Ell},p}^{\mathrm{ss}}$ is locally defined by the vanishing of a single function. The associated sheaf of ideals is a line bundle over $\mathcal{M}_{\mathrm{Ell}}$, and can naturally be identified with ω^{1-p}. That is to say, there is, (modulo p), an exact sequence $\omega^{1-p} \xrightarrow{v_1} \mathcal{O} \to \mathcal{O}_{\mathcal{M}_{\mathrm{Ell},p}^{\mathrm{ss}}}$. Locally we can find a nowhere vanishing section s of ω^{1-p}. The element

$v_1(s)$ is then a function defining the supersingular locus. Modulo a power of p the above still holds, provided $\neq 2, 3$.

We are interested in computing the derived functors of the inverse limit over $(\mathcal{M}_{\mathrm{Ell}})'_{\mathrm{et}}$ of

$$\begin{aligned}\pi_{2t+1}\,\mathrm{Bhaut}_1\,E_U &= \hom_{\mathcal{O}_U}(\mathcal{O}_{U^{\mathrm{ord}}}, \omega^t \otimes (\Omega^1)^{-1})\\ \pi_{2t}\,\mathrm{Bhaut}_1\,E_U &= \mathrm{Ext}_{\Gamma(\mathcal{O}_U)}(\mathcal{O}_{U^{\mathrm{ord}}}, \omega^t \otimes (\Omega^1)^{-1})\end{aligned}$$

Now the first of these contravariant functors is the restriction of the $(\mathcal{M}_{\mathrm{Ell}})'_{\mathrm{et}}$ sheaf $i_!\mathcal{O}_{\mathcal{M}^{\mathrm{ord}}_{\mathrm{Ell},p}}$. The second is not the restriction of a sheaf, and must be approached differently.

Thus let's restrict our attention to an open U, and let $i : U^{\mathrm{ss}} \to U$ and $j : U^{\mathrm{ord}} \to U$ be the inclusion of the supersingular and ordinary loci. Let's write F for the "hom" sheaf, and G for the "ext" sheaf. Let's also choose a function s which defines the supersingular locus. Then F and G can be identified with the sheaves

$$\begin{aligned}&\varprojlim s^n\omega^t \otimes (\Omega^1)^{-1} \qquad \text{and}\\ &{\varprojlim}^1 s^n\omega^t \otimes (\Omega^1)^{-1}.\end{aligned}$$

There are two spectral sequences for computing the group of sheaf extensions $\mathrm{Ext}^n(\mathcal{O}_{U^{\mathrm{ord}}}, \omega^t \otimes (\Omega^1)^{-1})$. One reduces to the short exact sequence

$${\varprojlim}^1 s^k \cdot H^{n-1}(\omega^t \otimes (\Omega^1)^{-1}) \to \mathrm{Ext}^n \to \varprojlim s^k \cdot H^n(\omega^t \otimes (\Omega^1)^{-1})$$

and the other reduces to the long exact sequence

$$\cdots \to H^{n-2}\,{\varprojlim}^1 \to H^n \varprojlim \to \mathrm{Ext}^n \to H^{n-1}\,{\varprojlim}^1 \to \dots.$$

Since U is affine, the groups H^n are zero for $n > 0$ and so the above data reduces to

$$\begin{gathered}\mathrm{Ext}^0 \approx {\varprojlim}^0 H^0 \approx H^0\,{\varprojlim}^0\\ {\varprojlim}^1 H^0 \approx \mathrm{Ext}^1\\ H^1\,{\varprojlim}^0 \rightarrowtail \mathrm{Ext}^1 \twoheadrightarrow H^0\,{\varprojlim}^1\\ H^i \varprojlim = 0 \qquad i > 1\\ H^i\,{\varprojlim}^1 = 0 \qquad i > 0.\end{gathered}$$

(The last two isomorphisms follow from the Milnor sequences).

We can compute $H^0\,{\varprojlim}^0$ from the exact sequence of a "pair." We can compute ${\varprojlim}^1 H^0$ as well, so we can get $H^0 \lim^1$. The inverse limit coincides with the sheaf cohomology for the lim 1 sheaf, so we get the derived functors of inverse limit in this case. For the lim sheaves use the spectral sequence relating sheaf to Čech cohomology, which in this case reduces to a long exact sequence.

Appendix E. Equivalence of realization spaces

This is an exploration of the possibility of simplifying some of the homological algebra by passing to a refined cover.

First some notation. What I have been calling $(\mathcal{M}_{\mathrm{Ell}})'_{\mathrm{et}}$ I think I will call $\mathrm{Et}'/\mathcal{M}_{\mathrm{Ell}}$, and for the purposes of this appendix I will abbreviate it with an I. We have constructed a diagram $D : I \to \mathrm{ho}\,A_\infty$.

Lemma E.1. *Suppose that $f : J \to I$ is the inclusion of a full subcategory, with the properties that (1) each $U \in \operatorname{ob} I$ is covered by an object of J, and (2) if $V \in \operatorname{ob} J$ and $U \subset V$, then $U \in \operatorname{ob} J$. If X is a fibrant realization of D, then the natural map*

$$X \to f_* f^* X$$

is a weak equivalence.

Proof: Let J/U be the category of J-objects over U. An object of J/U is a pair (V, i) consisting of an object V of J and a map $i : fV \to U$. A map from (V_0, i_0) to (V_1, i_1) is a map $g : V_0 \to V_1$ with the property that $i_1 \circ f(g) = i_0$. There is an obvious functor

$$p : J/U \to J$$

given by $p(V, i) = V$.

For a diagram $Y : J \to A_\infty$, the diagram $f_* Y$ is given by

$$f_* Y(U) = \varprojlim_{J/U} p^* Y.$$

If the diagram Y is fibrant, so is $p^* Y$. This means that the map $\operatorname{ho}\varprojlim p^* Y \to \varprojlim p^* Y$ is a weak equivalence. We can therefore study $f_* Y(U)$ using the homotopy spectral sequence. Now this spectral sequence takes the form

$$\varprojlim_{J/U}{}^{s} \pi_t p^* Y \implies \pi_{t-s} f_* Y(U).$$

Since the diagram X is a realization of D, the groups $\pi_{2t+1} p^* Y(V)$ are zero, and $\pi_{2t} p^* Y(V)$ is the module of global sections of ω^t. Since the funcotr "pullback," described below, is an equivalence of categories, the inverse limit can be computed over $\operatorname{Et}(J)/U$ and so coincides with the Čech cohomology $H^s(\operatorname{ob} J; \omega^t)$. But since U is affine, and ω is quasicoherent, these cohomology groups are zero, unless $s = 0$, in which case it is the module of global sections of ω over U. The spectral sequence therefore degenerates to its edge homomorphism. This completes the proof. □

We have used

Lemma E.2. *The forgetful functor $F : J/U \to \operatorname{Et}(J)/U$, and the functor "pull-back"*

$$P : \operatorname{Et}/U \to J/U$$
$$(i : V \to U) \mapsto i^* A_U$$

are inverse natural equivalences.

Proof: The composite PF is the identity. The composite PF sends the data

$$i : V \to U \qquad t : A_V \xrightarrow{\approx} i^* A_U$$

to the data

$$i : V \to U \qquad 1 : i^* A_U \to i^* A_U.$$

The map t defines the natural isomorphism from PF to the identity functor. □

Remark E.3. The hypotheses on the subcategory J are needed in order that the Čech cohomology and the derived functors of inverse limit coincide.

Now to compare the realization spaces. There are a few realization spaces that need to be considered. First there is the space rD. Next there is the space $\tilde{r}D$ which can be defined either as the nerve of the full subcategory of ${A_\infty}^I$ with objects the diagrams which admit an isomorphism in $\operatorname{ho} A_\infty$ with D, or as the category whose objects are diagrams X together with an isomorphism in $\operatorname{ho} A_\infty$ with D, and in which a morphism consists of a weak equivalence $f : X_0 \to X_1$, and an automorphism $t : D \to D$ such that

$$\begin{array}{ccc} \operatorname{ho} X_0 & \xrightarrow{\operatorname{ho} f} & \operatorname{ho} X_0 \\ \downarrow & & \downarrow \\ D & \xrightarrow[t]{} & D \end{array}$$

commutes. The first description is easier to work with, while the second makes it clear that there is a fibration

$$rX \to \tilde{r}X \to B \operatorname{Aut} D,$$

so it is only the higher homotopy groups of the two spaces that differ.

Lemma E.4. *The map*

$$\tilde{r}D \to \tilde{r}f^*D$$

is a weak equivalence.

Proof: We are interested in comparing certain components of classifying spaces of weak equivalences between the categories ${A_\infty}^I$ and ${A_\infty}^J$. These can be calculated as the classifying spaces of weak equivalences between cofibrant-fibrant objects in any model category structure. The result therefore reduces to the lemma above. □

Lemma E.5. *The map* $\operatorname{Aut} D \to \operatorname{Aut} f^*D$ *is an isomorphism.*

Proof: The diagrams D and f^*D are sections of

$$\operatorname{ho} A_\infty^{\mathrm{Ell}} \to I$$

and of its restriction to J. The groups $\operatorname{Aut} D$ and $\operatorname{Aut} f^*D$ thus sit in exact sequences

$$\operatorname{Aut}^{\mathrm{v}} D \to \operatorname{Aut} D \to \operatorname{Aut} 1_I$$
$$\operatorname{Aut}^{\mathrm{v}} f^*D \to \operatorname{Aut} f^*D \to \operatorname{Aut} 1_J,$$

where the superscript "v" indicates "vertical." Let A be the abelian group valued functor (on D) with

$$A(U) = \operatorname{ho} A_\infty^{\mathrm{Ell}}(E(U), E(U))_1.$$

Then by Proposition A.6, there are natural isomorphisms

$$\operatorname{Aut}^{\mathrm{v}} D \approx \varprojlim A$$
$$\operatorname{Aut}^{\mathrm{v}} f * D \approx \varprojlim f^*A.$$

But the functor A is a sheaf, and J is a cover of the terminal object. Both limits are therefore equal to $H^0(A)$. Finally, the map

$$\operatorname{Aut} 1_I \to \operatorname{Aut} 1_J$$

is an isomorphism, since the functor "isomorphisms between elliptic curves" is a sheaf. This completes the proof. □

Corollary E.6. *The map*

$$rD \to rf^*D$$

is a weak equivalence. □

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From elliptic curves to homotopy theory

Mike Hopkins and Mark Mahowald

Abstract. A surprising connection between elliptic curves over finite fields and homotopy theory has been discovered by Hopkins. In this note we will follow this development for the prime 2 and discuss the homotopy which developed from this.

Contents

1. Introduction

The path which we wish to follow begins with elliptic curves over finite fields and in particular over $\mathbb{F}_4$. From such a curve we get a formal group which will have height 2. The Lubin-Tate deformation theory constructs a formal group over the ring $\mathbb{W}_{\mathbb{F}_4}[[a]][u, u^{-1}]$. It can be shown that this ring is the homotopy of a spectrum, E_2, which is MU orientable. The group of automorphisms of the formal group over $\mathbb{F}_4$ acts on this ring. The Hopkins-Miller theory constructs a lift of this action to an action on the spectrum E_2. This group is a profinite group, called the Morava stabilizer group S_2. There is a finite subgroup G of S_2 of order 24 which is the automorphism group of the elliptic curve. This finite group acts on E_2 and we define $EO_2 = E_2^{hG}$. It is the torsion homotopy of this spectrum which illuminates much of the homotopy of spheres in the known range.

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We begin with the curve, $x^3+y^2+y=0$ in $\mathbb{P}^2(\mathbb{F}_4)$. In the elliptic curve literature this is called a supersingular curve. It is non-singular and has one point on the line at infinity. If we represent $\mathbb{F}_4$ as the set $\{0,1,\rho,\rho^2\}$ where $1+\rho+\rho^2=0$, then the solution set in the affine plane consists of eight points. If $x=0$ then $y=0$ or 1. If $x\in\mathbb{F}_4^\times$ then $y=\rho^i$ for $i=1,2$. The group of the elliptic curve is $\mathbb{F}_3\oplus\mathbb{F}_3$.

The group of affine transformations of ${\mathbb{F}_4}^2$ consists of matrices

$$\begin{pmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}.$$

Those which leave the equation of the curve alone satisfy

$$\begin{aligned} &a=\alpha\in\mathbb{F}_4^\times \qquad && b=0 \qquad && c^3=f+f^2 \\ &d=ac^2 && e=1 && \end{aligned}$$

It is easy to verify that this group G has order 24 and is $SL(\mathbb{F}_3,2)$. If we include the Galois action we get a $\mathbb{Z}/2$ extension of this group. Let G_{16} be the 2 primary part. We have the following result, which is well known. It will illuminate the latter calculations.

Theorem 1.1. *If we suppress the topological degree, then*

$$H^*(G_{16},\mathbb{Z}/2)\simeq \mathrm{Ext}^*_{A(1)}(\mathbb{Z}/2,\mathbb{Z}/2).$$

We want to calculate $H^*(Q_8\rtimes\mathbb{Z}/2)$. This is equivalent to calculating $H^*(\mathbb{Z}/2,H^*(Q_8))$. Cartan and Eilenberg give an explicit resolution leading to a calculation of $H^*(Q_8)$. In order to use this we need to know the action of $\mathbb{Z}/2$ on Q_8. This is given by: $i\mapsto j, k\mapsto -k$. Then the action is free in dimensions 1 and 2 and trivial in dimensions 3 and 4 modulo 4. This gives the following:

$$H^*(Q_8\rtimes\mathbb{Z}/2)=\mathbb{Z}/2[a_1,a_4]\langle b_0,b_3\rangle\oplus\mathbb{Z}/2[a_4]\langle c_1,c_1^2\rangle$$

which is the conclusion of the theorem.

We should note that there is an extension of D_8 which has the same mod 2 cohomology. Dave Benson has asked if there are groups whose cohomology is related to other sub algebras of the Steenrod algebra.

Our program will be to construct a formal group from the group of this elliptic curve. Then G will be a group of automorphisms of this formal group. We will lift the curve to the ring $\mathbb{W}_{\mathbb{F}_4}[[a]]$ as

$$y^2+a\ xy+y=x^3.$$

We can lift G as a group of automorphisms of this curve. Then the formal group associated to this curve will be the universal formal group given by the Lubin-Tate theory. The E_2 term of the Adams-Novikov spectral sequence to calculate EO_{2*} will be $H^*(G;\mathbb{W}_{\mathbb{F}_4}[[a]][u,u^{-1}])^{Gal}$.

2. The formal group

The material of this section is standard. We will include it for completeness for the homotopy theory reader who might not be familiar with the algebraic theory of elliptic curves.

The formal group constructed from an elliptic curve is constructed by resolving the multiplication on the curve around the point at infinity which is taken as the unit of the group. First we construct a parametric represention in terms of an uniformizer at infinity. Let

$$w = y^{-1}$$
$$z = x/y$$

Then the equation of the curve becomes $w = z^3 + w^2$. We have not noted signs since we are working over $\mathbb{F}_4$.

Proposition 2.1. (i) $w(z) = \Sigma_{i \geq 0} z^{3 \cdot 2^i}$.
(ii) $x(z) = z/w(z) = z^{-2} + z + z^4 + z^{10} + \cdots$
(iii) $y(z) = 1/w(z) = z^{-3} + 1 + z^3 + z^9 + \cdots$

This is an easy calculation. At this point one can follow the discussion in Silverman [**Si**] page 114. This discussion is considerably simplified by the fact that the field has characteristic 2. This gives the following result.

Proposition 2.2. *The formal group constructed from the elliptic curve,*$x^3 + y^2 + y = 0$ *over* $\mathbb{F}_4$ *has as the first few terms*

$$F(u, v) = u + v + u^2v^2 + u^4v^6 + u^6v^4 + u^4v^{12} + u^{12}v^4 + u^8v^8 + \cdots .$$

The next term has degree 22. This is a formal group of height 2 and the 2-series is $z^4(\Sigma_{i \geq 0} z^{12(2^i - 1)})$.

Next we want to lift this formal group to a formal group over the ring $\mathbb{W}_{\mathbb{F}_4}[[a]][u, u^{-1}]$ which gives the above curve under the quotient map to $\mathbb{F}_4$. We will do this by just lifting the elliptic curve. The formal group is then constructed in the usual way as is done in [**Si**]. The equation of the lifted curve is

$$x^3 = y^2 + a\ u\ x\ y + u^3 y.$$

We want to lift our group as a group of affine transformations which leave the curve alone. Thus we want to make the substitutions

$$\begin{aligned} x &\mapsto \alpha x + u^2 r \\ y &\mapsto y + u\ s\ x + u^3 t \end{aligned}$$

In order to preserve the curve we require that the coefficient of x^2, x, and the constant term all be zero. This gives

$$\begin{aligned} 3r &= s^2 + s\ a \\ s &= 3r^2 - 2ast - a(rs + t) \\ t &= r^3 - art - t^2 \end{aligned}$$

The group G is generated by $\alpha \in \mathbb{F}_4{}^+$ and a pair (β, γ) which satisfies the equation $\beta^3 + \gamma + \gamma^2 = 0$. We can take two generators, $\alpha = \rho, (\beta, \gamma) = (0, 0)$ and $\alpha = 1, (\beta, \gamma) = (1, \rho)$ and lift these. The rest of group will be various products of these. It is clear how to lift the first. We will concentrate on the second. We want to find infinite series for r, s and t which reduce to $1, 1$, and ρ modulo the maximum ideal. We begin with these equations and successively substitute into the above equations giving

$$\begin{aligned}
r(a) &= \tfrac{1}{3}\,(1+a)\\
s(a) &= \tfrac{1}{3}\,(1+2a+a^2) - 2a\rho - a((1/3)(1+a)+\rho)\\
t(a) &= \tfrac{1}{3^3}\,(1+a)^3 - (1/3)a(1+a)s(a) - \rho^2\\
r(a) &= \tfrac{1}{3}\,(s(a)^2 + as(a))\\
s(a) &= 3r(a)^2 - 2as(a)t(a) - a(r(a)s(a)+t(a))\\
t(a) &= r(a)^3 - ar(a)t(a) - t(a)^2
\end{aligned}$$

Each time we substitute the formula for the classes on the right hand side from the formulas above. After each process we have correct liftings modulo the maximum ideal raised to one higher power. That this works is just Hensel's Lemma. Compare [**Si**], page 112.

What we have constructed is a map $G \to \mathbb{W}_{\mathbb{F}_4}[[a]][u,u^{-1}]$. This is the beginning of a co-simplical complex

$$\mathbb{W}_{\mathbb{F}_4}[[a]][u,u^{-1}] \Rightarrow \mathrm{Hom}(G, \mathbb{W}_{\mathbb{F}_4}[[a]][u,u^{-1}]) \cdots$$

The action of G on $\mathbb{W}_{\mathbb{F}_4}[[a]][u,u^{-1}]$ is the only additional part to add. That defines

$$\begin{aligned}
a &\mapsto a+2s\\
u^3 &\mapsto u^3+ar+2t.
\end{aligned}$$

The homology of this co-simplical complex is the E_2 term of the Adams-Novikov spectral sequence to calculate the homotopy of the Hopkins-Miller spectrum EO_2. We will do this calculation in several ways but the key will be to show that it is something which is already known.

3. The elliptic curve Hopf algebroid

The Weierstrass form of an elliptic curve is usually written

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

A change of coordinates does not change the curve and so substituting

$$\begin{aligned}
x &= x' + r\\
y &= y' + sx' + t
\end{aligned}$$

gives us the same curve. The coefficients transfer according to the following table. (Compare [**Si**].)

$$\begin{aligned}
a_1' &= a_1 + 2s\\
a_2' &= a_2 - sa_1 + 3r - s^2\\
a_3' &= a_3 + ra_1 + 2t\\
a_4' &= a_4 - sa_3 + 2ra_2 - (t+rs)a_1 + 3r^2 - 2st\\
a_6' &= a_6 + ra_4 + r^2a_2 + r^3 - ta_3 - t^2 - rta_1
\end{aligned}$$

These formulas are very suggestive of the structure formulas which result from MU_* resolutions. Indeed, we can take these formulas to be the definition of η_R and get a Hopf algebroid

$$(A,\Lambda) = (\mathbb{Z}[a_1,a_2,a_3,a_4,a_6], \mathbb{Z}[a_1,a_2,a_3,a_4,a_6,s,r,t]).$$

The two maps from $A \to \Lambda$ are the inclusion and the one given by the table above. In books such as [**Si**] the classes c_4 and c_6 are usually given, and they represent classes in the homology in dimension zero of the simplical complex constructed from the above Hopf algebroid. The formulas for them are

$$\begin{aligned} c_4 &= (a_1^2 + 4a_2)^2 - 24(2a_4 + a_1 a_3) \\ c_6 &= -(a_1^2 + 4a_2)^3 + 36(a_1^2 + 4a_2)(2a_4 + a_1 a_3) - 216(a_3^2 + 4a_6) \end{aligned}$$

Notice that $c_4^3 - c_6^2$ is divisible by 1728. Let $\mathbf{\Delta} = (c_4^3 - c_6^2)/1728$. The zero dimensional homology of the above chain complex is

$$\mathbb{Z}[c_4, c_6, \mathbf{\Delta}]/(c_4^3 - c_6^2 - 1728\mathbf{\Delta}).$$

One of our questions is the computation of the rest of this chain complex. We will do this by getting another interpretation of the chain complex. For this interpretation we will have a complete calculation. Before we do this we want to connect this resolution with the Lubin-Tate theory.

In section 2 we consider the elliptic curve over $\mathbb{W}_{\mathbb{F}_4}[[a]][u, u^{-1}]$ given by the equation $y^2 + a\ u\ xy + u^3 y = x^3$. Thus we have a map $f : A[\rho]/(\rho^2 + \rho + 1) \to \mathbb{W}_{\mathbb{F}_4}[[a]][u, u^{-1}]$ defined by

$$\begin{aligned} a_1 &\mapsto au \\ a_3 &\mapsto u^3 \\ a_i &\mapsto 0, \text{ otherwise} \end{aligned}$$

THEOREM 3.1. *After completing $A[\rho]/(\rho^2 + \rho + 1)$ at the ideal $(2, a_1)$ and inverting Δ, the map f induces an isomorphism between the two chain complexes.*

COROLLARY 3.2. *The E_2 term of the Adams-Novikov spectral sequence to compute the homotopy of EO_2 is the homology of the Hopf algebroid (A, Λ) completed at the ideal $(2, a_1)$ with Δ inverted.*

In the next section we will show that this computation is well known.

4. Ring spectrum resolutions

Using Bott periodicity we have a map $\gamma : \Omega SU(4) \to BU$. Let T be the resulting Thom spectrum. As is the case with any ring spectrum we can construct a resolution $\mathbb{T}$

$$S^0 \to T \rightrightarrows T \wedge T \overset{\to}{\rightrightarrows} T \wedge T \wedge T \cdots .$$

This is acyclic from its definition. The first step in understanding such resolutions is the following version of the Thom isomorphism theorem.

PROPOSITION 4.1. *There is a homotopy equivalence $T \wedge \Omega SU(4)_+ \cong T \wedge T$ This homotopy equivalence is induced by a map between the base spaces*

$$\Omega SU(4) \times \Omega SU(4) \xrightarrow{\Delta, id} \Omega SU(4) \times \Omega SU(4) \times \Omega SU(4) \xrightarrow{id, \mu} \Omega SU(4) \times \Omega SU(4).$$

Here, Δ is the map which sends $x \to (x, -x)$ and μ is the loop space multiplication.

The map in Thom spectra induced by this composite is

$$T \wedge \Omega SU(4)_+ \cong T \wedge T.$$

Let $\bar{T}$ be the cofiber of the unit map. Then

$$T \wedge \bar{T} \cong T \wedge \Omega SU(4).$$

It is the T_* homotopy of $\Omega SU(4)$ which describes the T Hopf algebroid. One of the main results of [**Mo2**] is following.

PROPOSITION 4.2. *The map* $d = \eta_L - \eta_R$ *can be viewed as a map* $T \to T \wedge \Omega SU(4)$ *which is induced by the diagonal*

$$\Delta : \Omega SU(4) \to \Omega SU(4) \times \Omega SU(4).$$

Let $b_i \in H_{2i}(\mathbb{C}P)$ be the homology generators. We will identify these classes with their image in $H_*(\Omega SU(4))$. Thus

$$H_*(\Omega SU(4)) \cong \mathbb{Z}[b_1, b_2, b_3].$$

The homotopy classes in $\pi_*(T)$ which are in the Hurewicz image are multiples of primitive classes. On the other hand the classes b_i are not primitive for $i > 1$. We have:

$$\begin{aligned}
\Delta b_1 &= b_1 \otimes 1 + 1 \otimes b_1 \\
\Delta b_2 &= b_2 \otimes 1 + b_1 \otimes b_1 + 1 \otimes b_2 \\
\Delta b_3 &= b_3 \otimes 1 + b_2 \otimes b_1 + b_1 \otimes b_2 + 1 \otimes b_3
\end{aligned}$$

Thus we can define primitive classes as follows:

$$\begin{aligned}
m_1 &= b_1 \\
m_2 &= 2b_2 - b_1^2 \\
m_3 &= 3(b_3 - b_1 b_2) + b_1^3
\end{aligned}$$

This allows us to define homotopy classes

$$\begin{aligned}
a_1 &= 2m_1 \\
a_2 &= 3m_2 - m_1^2 \\
a_3 &= 2m_3
\end{aligned}$$

We define additional classes

$$\begin{aligned}
a_4 &= 3m_2^2 - 2m_1 m_3 \\
a_6 &= m_2^3 - m_3^2
\end{aligned}$$

Then we calculate da_i by the following rules:

- compute Δa_i
- drop each class of the form $x \otimes 1$.
- classes of the form $x \otimes m_1$ are written as xs
- classes of the form $x \otimes m_2$ are written as xr
- classes of the form $x \otimes m_3$ are written as xt
- classes of the form $x \otimes y$ must have $x \in \mathbb{Z}[a_1, a_2, a_3]$. We write them as xy.

If $\Omega SU(4)$ stably split as a wedge of spheres, then $T \wedge \Omega SU(4)$ would give the free splitting of $T \wedge T$ into a wedge of T's. This is what $A[s, r, t]$ represents. But $\Omega SU(4)$ does not split in this manner. It would be enough if the pieces into which $\Omega SU(4)$ does split were trivial T_* modules, but that is not true either. The further splitting produces an extra term, $a_1 r$ in the expression for $\eta_R a_3$. This gives us:

$$\begin{aligned}
\eta_R a_1 &= a_1 + 2s \\
\eta_R a_2 &= a_2 + 3r - a_1 s - s^2 \\
\eta_R a_3 &= a_3 + 2t + a_1 r \\
\Delta a_4 &= 3(m_2 \otimes 1 + 1 \otimes m_2)^2 \\
&\quad - 2(m_3 \otimes 1 + 1 \otimes m_3 + m_1 \otimes m_2)(m_1 \otimes 1 + 1 \otimes m_1) \\
\eta_R a_4 &= a_4 + 2a_2 r + 3r^2 - a_3 s - st - a_1 t - a_1 sr \\
\eta_R a_6 &= a_6 + a_4 r + a_2 r^2 + r^3 - a_3 t - t^2 - a_1 rt
\end{aligned}$$

Thus, we have reproduced the formulas constructed in the previous section from the change of variables formulas. We still need to get the setting where the polynomial algebra $\mathbb{Z}[a_1, a_2, a_3, a_4, a_6]$ does represent the homotopy of something.

Let E be any spectrum. If we smash the resolution $\mathbb{T}$ with E, we still have an acyclic complex with augmentation E. If we apply homotopy, we get a complex whose homology is the E_2 term of a spectral sequence to compute the homotopy of E. We need a spectrum E so that $\pi_*(E \wedge T) \cong \mathbb{Z}[a_1, a_2, a_3, a_4, a_6]$. Hopkins and Miller [**HM**] have constructed a spectrum which almost works. In a latter section a connected version of the Hopkins-Miller spectrum eo_2 is constructed. It has the key properties:

THEOREM 4.3. *Let $D(A_1)$ be a spectrum whose cohomology, as a module over the Steenrod algebra is free on Sq^2 and Sq^4. Then localized at 2, $eo_2 \wedge D(A_1) \cong BP\langle 2\rangle$. Let X be the spectrum whose cohomology, as a module over the mod 3 Steenrod algebra is free on P^1, the localized at 3, $eo_2 \wedge X \cong BP\langle 2\rangle \wedge (S^0 \vee S^8)$*

COROLLARY 4.4. *$\pi_*(eo_2 \wedge T) \cong \mathbb{Z}[a_1, a_2, a_3, a_4, a_6]$ and*

$$\pi_*(eo_2 \wedge T \wedge T) \cong \mathbb{Z}[a_1, a_2, a_3, a_4, a_6] \otimes \mathbb{Z}[s, r, t].$$

An easy calculation gives us the following.

THEOREM 4.5. *If we apply the functor $\mathrm{Ext}_{A(2)}(\_, \mathbb{Z}/2)$ to the resolution $\mathbb{T}$, we get*

$$\mathbb{Z}/2[v_0, v_1, a_2, v_3, a_4, a_6] \to \mathbb{Z}/2[v_0, v_1, a_2, v_3, a_4, a_6, s, r, t] \to \cdots$$

where a_i has filtration 0. This chain complex will compute $\mathrm{Ext}_{A(2)}(\mathbb{Z}/2, \mathbb{Z}/2)$. In particular, this implies $H^*(eo_2, \mathbb{Z}/2) \simeq A \otimes_{A(2)} \mathbb{Z}/2$.

5. An outline of the calculation

In the rest of this paper, we will discuss the homotopy of the spectrum constructed by Hopkins and Miller [**HM**] which they labeled EO_2. We will also be quite interested in the Hurewicz image in $\pi_*(S^0)$.

THEOREM 5.1. *The action of S_2 on E_{2*} lifts to an E_∞ ring action of S_2 on E_2. Furthermore, S_2 has a subgroup of order 24, $GL(\mathbb{F}_3, 2)$. This group can be extended by the Galois group, $\mathbb{Z}/2$. The group of order 48 acts on EO_2 and the homotopy fixed point set of this action defines EO_2. In addition,*

$$EO_2 \wedge D(A_1) = E_2.$$

We will take this result as an axiom for the rest of this paper. We will calculate the homotopy of EO_2 in two ways. First we will construct a spectral sequence which untangles the formula $EO_2 \wedge D(A_1) = E_2$. This is done in the next section. Next we will consider the connected cover of EO_2 and show that it essentially has $A \otimes_{A(2)} \mathbb{Z}/2$ as its cohomology. We then have an Adams spectral sequence calculation which has been known for about twenty years. This approach allows one to have available a rather large collection of spaces whose EO_{2*} homology has been computed. See for example [**DM1**].

These results also give a counter example to the main result of [**DM2**] which asserted that $A \otimes_{A(2)} \mathbb{Z}/2$ could not be the cohomology of a spectrum. The error in that paper can be traced to a homotopy calculation in [**DM3**] which was in error. The correction of the appropriate homotopy calculation is done in [**KM**].

In the last section we discuss homotopy classes in the spheres which can be constructed by this spectrum.

6. The homotopy of EO_2

Our first calculation of EO_{2*} uses the formula

$$EO_2 \wedge D(A_1) = E_2$$

The CW complex $D(A_1)$ is constructed by the following lemma where we use the notation $M_\alpha = S^0 \cup e^{|\alpha|+1}$.

LEMMA 6.1. *There is a map*

$$\gamma : \Sigma^5 M_\eta \wedge M_\nu \to M_\eta \wedge M_\nu$$

PROOF. This is a straightforward calculation in $\pi_*(S^0)$. □

We can use the definition of $D(A_1)$ and the formula in (2.1) to construct a spectral sequence. Abstractly, we think of $D(A_1)$ as constructed out of three mapping cones, M_η, M_ν and M_γ where γ is defined in the Lemma. Thus we have a contracting homomorphism in $P(h_1, h_2, h_{2,0}) \otimes H_*(D(A_1))$ with $d\ h_1 = e_\eta$, $d\ h_2 = e_\nu$ and $d\ h_{2,0} = e_\gamma$. We will use the defining equation (2.1) to give us a free E_2 resolution. For the moment we want to think of this as an unfiltered but graded object. There is a total differential whose homology is EO_{2*}. If we assign filtration 0 to h_1 and v_1 and filtration one to each of $h_2, h_{2,0}, v_2$ then the corresponding E_1 will be $bo_*[h_2, h_{2,0}, v_2]$. If we recognize the bo structure of the set $< h_2, h_{2,0}, v_2 >$ then the corresponding resolution is just the Kozul resolution of [**DM1**], section 5 (page 319ff). Of course, v_2 should be inverted and $2 = v_0$. This gives the following result.

THEOREM 6.2. *There is a spectral sequence with*

$$E_2^{s,t} = v_2^{-1}\mathrm{Ext}_{A(2)}^{s,t}(\mathbb{Z}/2, \mathbb{Z}/2)$$

which converges to $E_0(EO_{2*})$.

We will explore this in more detail in latter sections. In particular we will want to understand the differentials.

Using the Adams Novikov differentials from the same starting point we get another spectral sequence. In this case we assign filtration 0 to v_1 and v_2, filtration 1 to h_1, h_2 and $h_{2,0}$. We will also work over the integers.

In order to state the answer in a compact form we introduce several homotopy patterns.

FIGURE 6.3. *The following diagram defines A. The solid circles represent $\mathbb{Z}/2$'s and the open circles represent a $\mathbb{Z}/8$ ($\mathbb{Z}/4$) in stem 0 (stem 3). In stem 3 there is an extension to the $\mathbb{Z}/2$ giving a $\mathbb{Z}/8$ in this case too.*

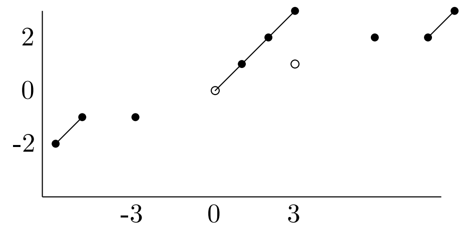
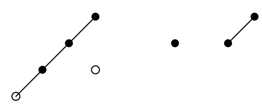


FIGURE 6.4. *The following diagram defines B. The diagram starts in filtration $(0,0)$. The starting circle represents a $\mathbb{Z}$ and the circle in dimension 3 represents a $\mathbb{Z}/4$ with an extension to the $\mathbb{Z}/2$ giving a $\mathbb{Z}/8$.*



THEOREM 6.5. *There is a spectral sequence which converges to EO_{2*} and the E_2 is given by*

$$h_{2,0}^4 P(h_{2,0}^4, v_2^4, v_2^{-4}) \otimes A \oplus (P(v_2^4, v_2^{-4}) \otimes B) \oplus v_1^4 bo[v_2^4, v_2^{-4}]$$

where A is the module of Figure 6.3 and B is the module of Figure 6.4.

7. The Bockstein spectral sequence

In this section we will give a proof of Theorem 6.5. We start with a resolution $\mathbb{Z}[v_1, v_2, h_1, h_2, h_{2,0}]$. The filtrations of the generators are given by the following table.

$$\begin{array}{rcl} v_i & \mapsto & (0, 2(2^i - 1) \\ h_i & \mapsto & (1, 2^i) \\ h_{2,0} & \mapsto & (1, 6) \end{array}$$

The Adams-Novikov spectral sequence or the elliptic curve Hopf Algebroid give the following differentials.

$$\begin{array}{rcl} v_2 & \mapsto & v_1 h_2 + 2h_{2,0} + v_1^2 h_1 \\ h_{2,0} & \mapsto & h_1 h_2 \\ h_{2,0}^2 & \mapsto & h_2^3 + v_2 h_1^3 \end{array}$$

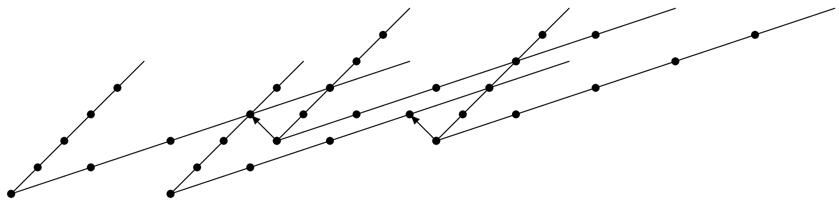
First we will do the calculation modulo the ideal $I_2 = (2, v_1)$. This gives

$$(\mathbb{Z}/2[h_1] \oplus \mathbb{Z}/2\langle h_2, h_2^2 \rangle) \otimes \mathbb{Z}/2[v_2, h_{2,0}^4]/v_2 h_1^4$$

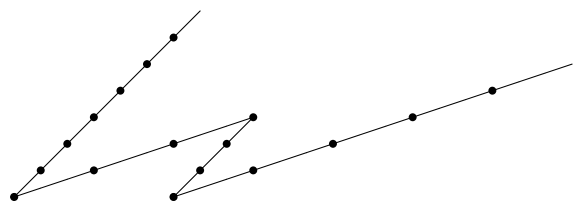
To see this first consider $P[h_1, h_2, h_{2,0}]$. The differential on $h_{2,0}$ leaves

$$(P[h_1, h_2]/h_1 h_2) \otimes P[h_{2,0}^2]$$

If we add v_2 and use the differential on $h_{2,0}^2$ we get the above result. The following picture illustrates this.



The result of this calculation is the following pattern.



The next step is to do the calculation modulo I_1. The calculation modulo v_1^2 has the following differentials:

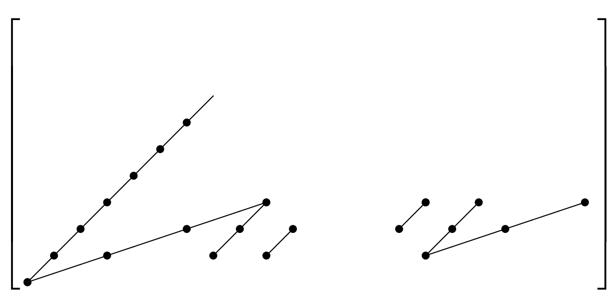
$$\begin{array}{rcl} v_2 & \mapsto & v_1 h_2 \\ v_2 h_2^2 & \mapsto & v_1 h_1^3 v_2 \end{array}$$

The calculation modulo v_1^4 has the following differentials:

$$\begin{array}{rcl} v_2^2 & \mapsto & v_1^2 h_1 v_2 \\ a & \mapsto & v_1^2 v_2^2 h_1^2 \end{array}$$

The class a is $h_1 v_2^3$.

After this step we have the following picture where the polynomial algebra in h_1 starting in filtration $(0,0)$ is really $\mathbb{Z}/2[h_1, v_1]$.



$[h_{2,0}^4, v_2^4]/(v_2^4 h_1^4 - h_{2,0}^4 v_1^4)$

This picture is an associated graded form. We have two extensions:

$$\begin{array}{rcl} h_1 v_2^2 h_2 & = & v_1 h_1^2 v_2 \\ v_2^4 h_1^4 & = & v_1^4 h_{2,0}^4 \end{array}$$

The first follows easily from the bracket $\langle h_1, h_2, h_1 \rangle = h_2^2$. The final step is to determine the torsion Bocksteins.

8. The Adams-Novikov spectral sequence

In this section we will compute the Adams-Novikov differentials and thus calculate the associated graded homotopy of EO_2. The starting point is the following. We will show latter that it is a d_1 in the usual Adams spectral sequence.

Proposition 8.1. *In the Adams-Novikov spectral sequence for* EO_2 *we have*

$$d_5(v_2^4) = h_2 h_{2,0}^4.$$

Proof. We begin with a calculation in stable homotopy.

Lemma 8.2. $\nu^2\bar{\kappa} \in \langle \eta_4\sigma, \eta, 2\iota\rangle$.

We will first use this lemma to complete the proof of the proposition. We note that $h_2^2 h_{2,0}^4$ is the Adams-Novikov name for $\nu^2\bar{\kappa}$. By checking the above calculation, we see that $\eta_4\sigma = 0$ in EO_{2*}. Thus the bracket of the lemma must go to zero in EO_2. Hence the class of $h_2^2 h_{2,0}^4$ must be in the indeterminacy of the bracket. It is easy to see that only zero is in the indeterminacy and so $h_2^2 h_{2,0}^4$ must project to the zero class. The only way this can happen is for

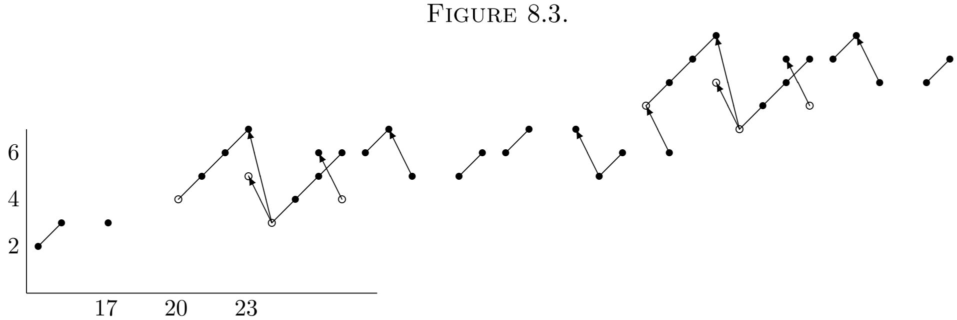
$$d_5(v_2^4 h_2) = h_2^2 h_{2,0}^4.$$

dividing by h_2 gives the proposition. □

Now we will prove the Lemma. This is essentially an Adams d_1. First recall that $\eta_4\sigma$ is represented by $h_1 h_4 c_0$ in the Adams E_2. In order to form a bracket such as $\langle \eta_4\sigma, \eta, 2\iota\rangle$ we need to know why $h_1^2 h_4 c_0 = 0$. The easiest approach is to use the lambda algebra and the calculations of [**Ta**]. We see that $\lambda_2\lambda_3\lambda_5\lambda_7\lambda_7 = h_1 h_4 c_0$. Then from [**Ta**] we see that $\lambda_8\lambda_9\lambda_3\lambda_3\lambda_3$ hits $\lambda_1\lambda_2\lambda_3\lambda_5\lambda_7\lambda_7$. Thus $\lambda_0\lambda_8\lambda_9\lambda_3\lambda_3\lambda_3 \in \langle h_1 h_4 c_0, h_1, h_0\rangle$. Up to the addition of some boundaries, this is just $\lambda_6\lambda_6\lambda_5\lambda_3\lambda_3\lambda_3$. This is equivalent to the leading term name of $\nu^2\bar{\kappa}$. This completes the proof.

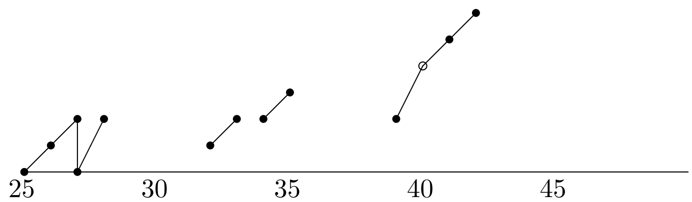
The following figure illustrates this first differential. We place the chart for v_2^4 in filtration 3 so it is easier to see just what is happening.

Figure 8.3.



We can collect the result of this computation in the following chart. We have listed some exotic multiplications which we will prove in the rest of this section.

Figure 8.4. *The class in dimension 4 is a* $\mathbb{Z}/4$. *Lines which connect adjacent elements but are of length 2 represent exotic extensions. There are three such. One is multiplication by 2 in stem 27. The other two are multiplications by* η, *one in stem 27 and the other in stem 39. The complete calculation has this chart multiplied by* $\mathbb{Z}[\bar{\kappa}]$.



We need to establish some of the compositions which are non-zero in this homotopy module. We introduce some notation. We let ι, η, ν, to represent the generators of the 0, 1, and 3 stems. This is consistent with the traditional names of these classes in the homotopy of spheres. The elements in the 8, 14 and 20 stem we will label $\epsilon, \kappa, \bar{\kappa}$ respectively. For other classes, we will use the symbol a_i for an element in the ith stem. The exotic extensions referred to above then are covered by the following proposition.

Proposition 8.5. *The following compositions are non-zero:* $\eta a_{27}, 2a_{27}, \eta a_{39}$.

Proof. First note that $2a_{27}$ is just the standard extension which comes from the 3 stem where $4\nu = \eta^3$. Next, the class $a_{28} = \epsilon\bar{\kappa}$. This is a filtration preserving calculation. The definition of ϵ forces $\epsilon \in \langle \nu, 2\nu, \eta\rangle$. When we multiply this bracket by $\bar{\kappa}$ we see that $\epsilon\bar{\kappa} = \langle \bar{\kappa}, \nu, 2\nu\rangle\eta$. This bracket clearly represents a_{27}. Notice that we can not form this latter bracket in spheres but need the differential on v_2^4 in order to form the bracket.

In the homotopy of spheres we have the bracket relation $\langle \nu, \kappa\eta, \eta\rangle = 2\bar{\kappa}$. This follows easily from the Adams spectral sequence where there is a d_2 which makes $\kappa\eta^2 = 0$. In the usual naming, we have $d_2 e_0 = h_1^2 d_0$. We also have $h_2 e_0 = h_0 g$. This establishes this relationship. Now if we multiply both sides by $\bar{\kappa}$ we have $\bar{\kappa}\langle \nu, \kappa\eta, \eta\rangle = 2\bar{\kappa}^2$. But $\bar{\kappa}\langle \nu, \kappa\eta, \eta\rangle = \langle \bar{\kappa}, \nu, \eta\kappa\rangle\eta$. This is the relation we wanted. □

Next we want to establish the special ν multiplications.

Proposition 8.6. *We have the following compositions:* $\nu a_{25} = a_{28}, \nu a_{32} = a_{35}, \nu a_{39} = a_{42}$.

Proof. A bracket construction for a_{25} is $a_{25} = \langle \bar{\kappa}, \nu, \eta\rangle$. If we multiply this on the right by ν we have $\langle \bar{\kappa}, \nu, \eta\rangle\nu = \bar{\kappa}\langle \nu, \eta, \nu\rangle = \bar{\kappa}\epsilon = a_{28}$. In a similar way we see that $a_{32} = \langle \bar{\kappa}, \nu, \epsilon\rangle$. If we multiply both sides by ν we get $\langle \bar{\kappa}, \nu, \epsilon\rangle\nu = \bar{\kappa}\langle \nu, \epsilon, \nu\rangle$. But in spheres $\langle \nu, \epsilon, \nu\rangle = \eta\kappa$ and this gives the relation. In the above proposition we showed $a_{39} = \langle \bar{\kappa}, \nu, \eta\kappa\rangle$. Multiplying this by ν we have $a_{39}\nu = \bar{\kappa}\langle \nu, \eta\kappa, \nu\rangle = \bar{\kappa}^2\eta^2$ and this is the relationship we wanted. □

In a very similar fashion we establish the following. We will skip the proof.

Proposition 8.7. *We have the following ϵ compositions.* $\epsilon a_{25} = a_{33}, \epsilon a_{27} = a_{35}, \epsilon a_{32} = 2a_{40}, \epsilon a_{34} = a_{42}, \epsilon a_{39} = a_{47} = \bar{\kappa}a_{25}, \epsilon a_{40} = a_{48}$.

With these extensions established, the rest of the spectral sequence is quite easy. We have the following theorem.

THEOREM 8.8. *We have the following differentials:*

$$d_5(v_2^8) = \bar{\kappa}a_{27} \ (= 2\nu\bar{\kappa}v_2^4 = 2d_5(v_2^4)\cdot v_2^4)$$

and

$$d_7(v_2^{16}) = \eta^2 a_{25}\bar{\kappa}v_2^8 \ (= 2v_2^8 d_5(v_2^8))$$

THEOREM 8.9. v_2^{32} *is a permanent cycle.*

9. The connected cover of EO_2

In this section we will construct the connected cover of EO_2 and get some of its properties. We begin with the following which is proved in [**HM**].

THEOREM 9.1. $v_1^{-1}EO_2 = KO[[v_2/v_1^3]][v_2^4, v_2^{-4}]$

Our strategy to construct the the connected cover of EO_2 will be to construct the following map.

$$f : bo[v_2^4/v_1^{12}] \to v_1^{-1}EO_2[0, \cdots, \infty].$$

With this map we will consider the pull back square as defining the spectrum Y

$$\begin{array}{ccc} Y & \longrightarrow & bo[v_2^4/v_1^{12}] \\ \downarrow & & \downarrow \\ EO_2[0,\cdots,\infty] & \longrightarrow & v_1^{-1}EO_2[0,\cdots,\infty] \end{array}$$

We will show:

THEOREM 9.2. *The cohomology of* Y *is* $H^*(Y) = A \otimes_{A(2)} \mathbb{Z}$ *and the Adams spectral sequence to calculated* $\pi_*(Y)$ *is that given by Theorem 2.2.*

The first step is the following Lemma.

LEMMA 9.3. *There is a map* $g : bo \to v_1^{-1}EO_2[0, \cdots, \infty]$ *such that* $g_*(\iota) = \iota$, *the unit in* $v_1^{-1}EO_2[0, \cdots, \infty]$.

PROOF. We begin with $\iota : S^0 \to v_1^{-1}EO_2[0, \cdots, \infty]$. We recall that there is a short exact sequence

$$\Sigma^{4k-1}B(k) \to bo_k \to bo_{k+1}$$

where $B(k)$ is the integral Brown Gitler spectrum [**CDGM**] and bo_k is the bo Brown Gitler spectrum. This sequence is constructed in [**GJM**]. The K theory of $B(k)$ is easily computed and it is zero in dimensions of the form $4k-1$. Thus we can proceed by induction starting with the map ι. This constructs one copy of bo into $v_1^{-1}EO_2[0, \cdots, \infty]$. □

To continue with the proof of the theorem, we next construct a map

$$\Sigma^\infty(\Omega S^{25}) \simeq \bigvee_{k \geq 0} S^{24k} \to v_1^{-1}EO_2[0, \cdots, \infty]$$

which gives the polynomial algebra on v_2^4. Using the ring structure we have the desired map f of the diagram. This completes the construction of Y.

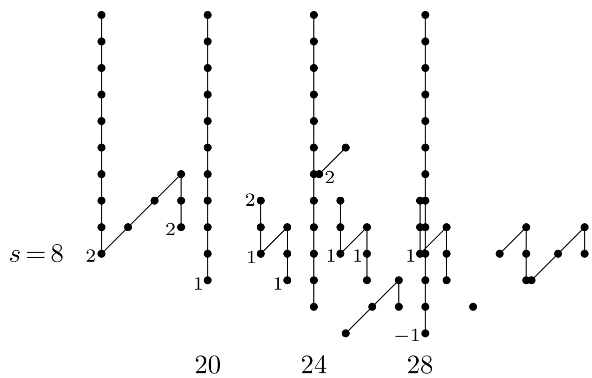
Next we want to compute the homotopy of Y. The E_2 term of the Adams spectral sequence for Y is $\mathrm{Ext}_{A(2)}(\mathbb{Z}/2, \mathbb{Z}/2)$. This has been calculated by many people. The first calculation is due to Iwai and Shimada [**IS**]. Extensive $\mathrm{Ext}_{A(2)}(M, \mathbb{Z}/2)$ calculations are given in [**DM1**]. We refer the reader there to find the details of the calculation. The answers given there are in a compact form which is quite useful. It is based on the following definition.

Definition 9.4. *An indexed chart is a chart in which some elements are labeled with integers. An unlabeled x receives the label*

$$\max\{\text{label }(y) : x = h_0^i y \text{ or } x = h_1^i y,\ \text{ some } i \geq 1\}$$

or 0 if this set is empty. If C is a labeled chart then $\langle C \rangle$ denotes the chart consisting of all elements $v_1^{4i}x$ such that $i + \text{label}(x) \geq 0$.

The following is an example of an indexed chart.



Let this chart be called E_0. Then the following is proved in [**DM1**]. (Actually, the chart in [**DM1**] has a dot missing in dimension (30,6).)

Theorem 9.5. $\mathrm{Ext}_{A(2)}(\mathbb{Z}/2, \mathbb{Z}/2)$ *is free over* $\mathbb{Z}/2[v_2^8]$ *on*

$$\langle E_0 \rangle \oplus \mathbb{Z}/2[v_1, w] \cdot g_{35,7}.$$

We have the following differentials in the chart E_0. We use the notation $g_{t-s,s}$ to refer to the dot in position $(t-s, s)$.

Proposition 9.6. $d_2(g_{20,7}) = g_{19,9}$.

Proof. When translated to more familiar notation this is a consequence of the following Lemma.

Lemma 9.7. *In the Adams spectral sequence of Theorem 2.1 the first differential occurs in dimension 12 and hits the class $v_1^4 h_2$.*

Proof. First we need to construct the element. Using the above formulas which are filtration preserving we see that

$$v_0 v_2^2 + v_2 h_2^2 + v_1 h_{2,0}^2$$

is a cycle and it generates a v_0 tower in the 12 stem. When we use the filtration increasing part of the differentials we see this class is not a cycle but its boundary is

$$v_1^2 v_0 h_1 v_2 + v_0^2 v_2 h_{2,0} + v_0 v_1^4 h_2 + v_1^2 h_1 h_2^2$$

We can begin to try to complete this into a cycle. The first class we would add is $v_1^3 v_2$. The boundary on this class is

$$v_1^2 v_0 h_1 v_2 + v_1^4 h_2 + v_0 v_1^3 h_{2,0} + v_1^5 h_1$$

There is nothing we can add to get rid of the $v_1^4 h_2$ class and this gives the differential of the Lemma. □

□

Using h_2 multiplications we have the following additional differentials.

$$d_2(g_{23,7}) = g_{22,9}$$
$$d_2(g_{26,7}) = g_{25,9}$$
$$d_2(g_{29,7}) = g_{28,9}$$
$$d_2(g_{28,5}) = g_{27,7}$$

We have the following d_3.

PROPOSITION 9.8. $d_3(g_{24,6}) = g_{23,9}$.

This differential implies the following in addition.

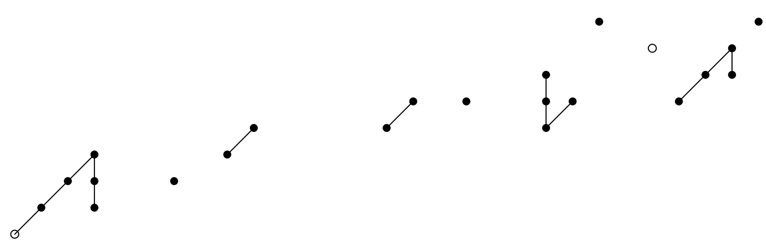
$$d_3(g_{25,8}) = g_{24,11}$$
$$d_3(g_{30,6}) = g_{29,9}$$

We have the following d_4.

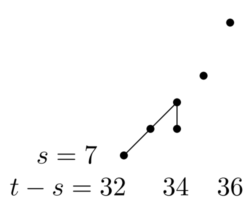
PROPOSITION 9.9. $d_4(g_{31,8}) = v_1^4 g_{22,8}$.

We wish to collect the result of these differentials. The pattern which is left from the upper left corner of the figure generates a copy of bo starting in dimension $(8,4)$. The second pair of $\mathbb{Z}$ towers generates a bo in dimension $(32,8)$. This second bo uses $v_1^4 g_{25,5}$ and h_1 times this class and the class in dimension $(32,7)$ which has h_0 nonzero on it.

This leaves a copy of bo, after some extensions, which starts in filtration $(32,7)$. It represents $v_1^4 v_2^4$. There is an extra class in filtration $(35,10)$ which we still have to account for. The following chart lists everything which is left because the source of a differential is not present.



In addition to this part we have the polynomial algebra on the two generators and v_1^4 free on the following.

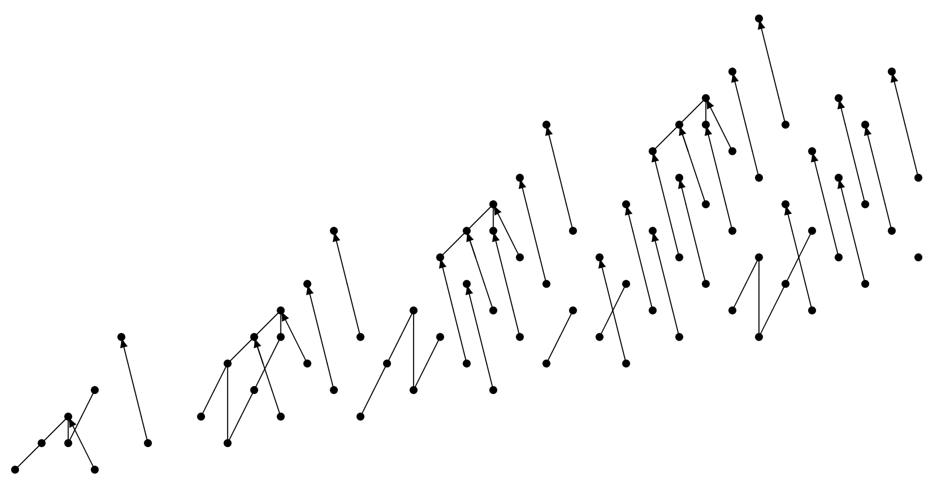


The following result gives the differentials for this part of the picture.

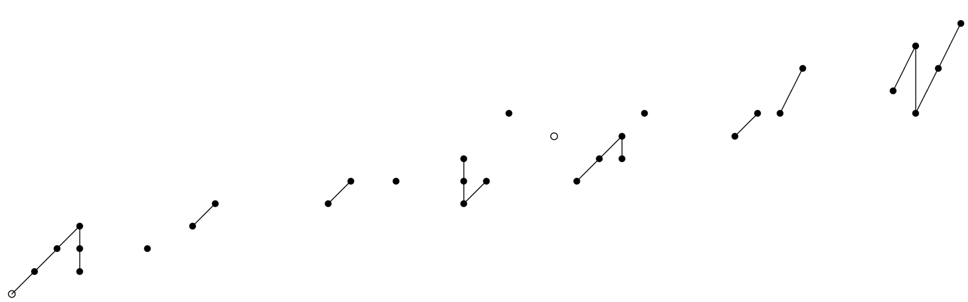
PROPOSITION 9.10. *Among classes in $\mathbb{Z}/2[v_1, w] \cdot g_{35,7}$ and between this polynomial algebra and classes in the above diagram we have the following differentials:*

$$\begin{aligned}
d_2(g_{35,7}) &= g_{36,9}\\
d_4(v_1 g_{35,7}) &= g_{38,12}\\
d_4(v_1 w g_{35,7}) &= v_1^4 g_{35,8}\\
d_4(v_1^2 w g_{35,7}) &= v_1^4 g_{35,10}\\
d_4(v_1^2 w^2 g_{35,7}) &= v_1^8 g_{32,7}\\
d_4(v_1^3 w^2 g_{35,7}) &= v_1^8 g_{34,8}\\
d_4(w^3 g_{35,7}) &= v_1^7 g_{35,7}\\
d_4(v_1^3 w^3 g_{35,7}) &= v_1^{10} g_{35,7}
\end{aligned}$$

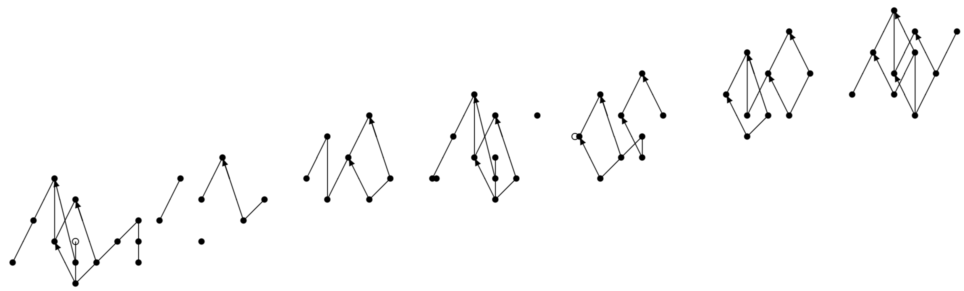
If we combine the above diagram, the polynomial algebra and the differentials above we have the following figure.



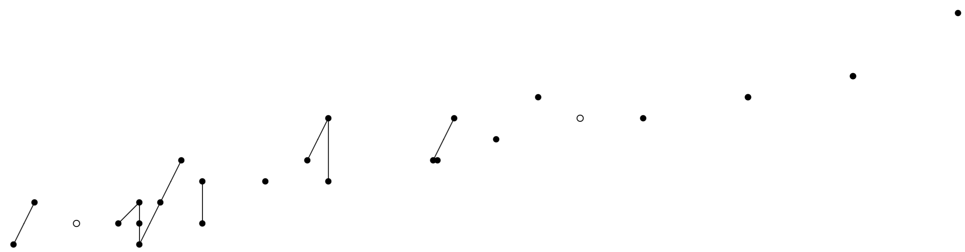
This allows us to write the v_1 torsion part of the answer out though the 42 stem. The following is the correct chart.



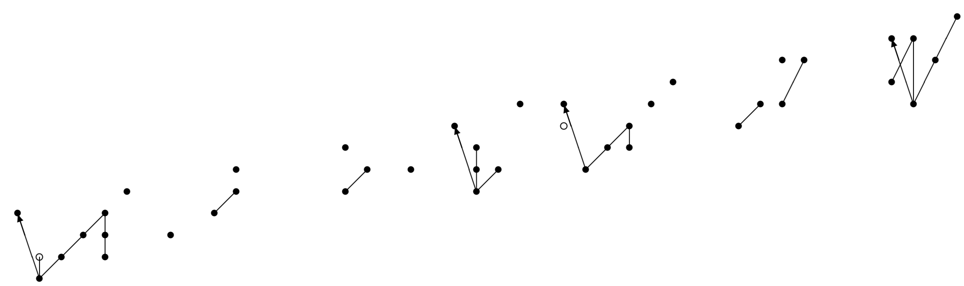
To compute the next 48 groups we need to put the earlier calculation together with the first 42 groups above multiplied by v_2^8. This gives the following chart.



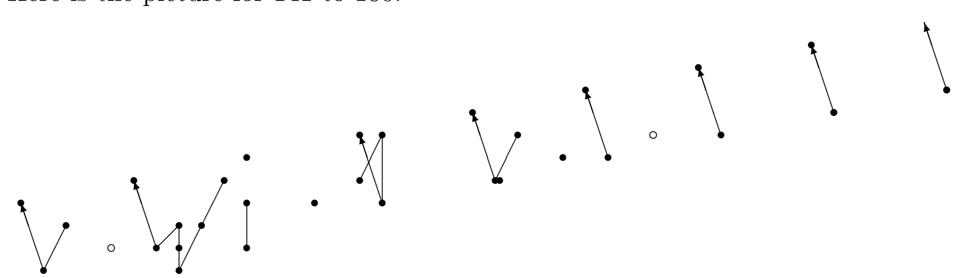
This gives the following homotopy starting in dimension 45.



Beyond 95 this differential pattern leaves a class every 5 dimensions. Because the differential on v_2^8 is a d_2, the polynomial algebra $v_2^8\mathbb{Z}/2[v_1, w]$ is mapped monomorphically into $\mathbb{Z}/2[v_1, w]$ leaving just $w^9 g_{35,7}\mathbb{Z}/2[w]$. To complete the calculation we need to take into account v_2^{16}. We do this by putting our calculation so far together with this pattern and writing in the new differentials. This gives the following pattern. The first chart calculates the homotopy from 95 to 140.

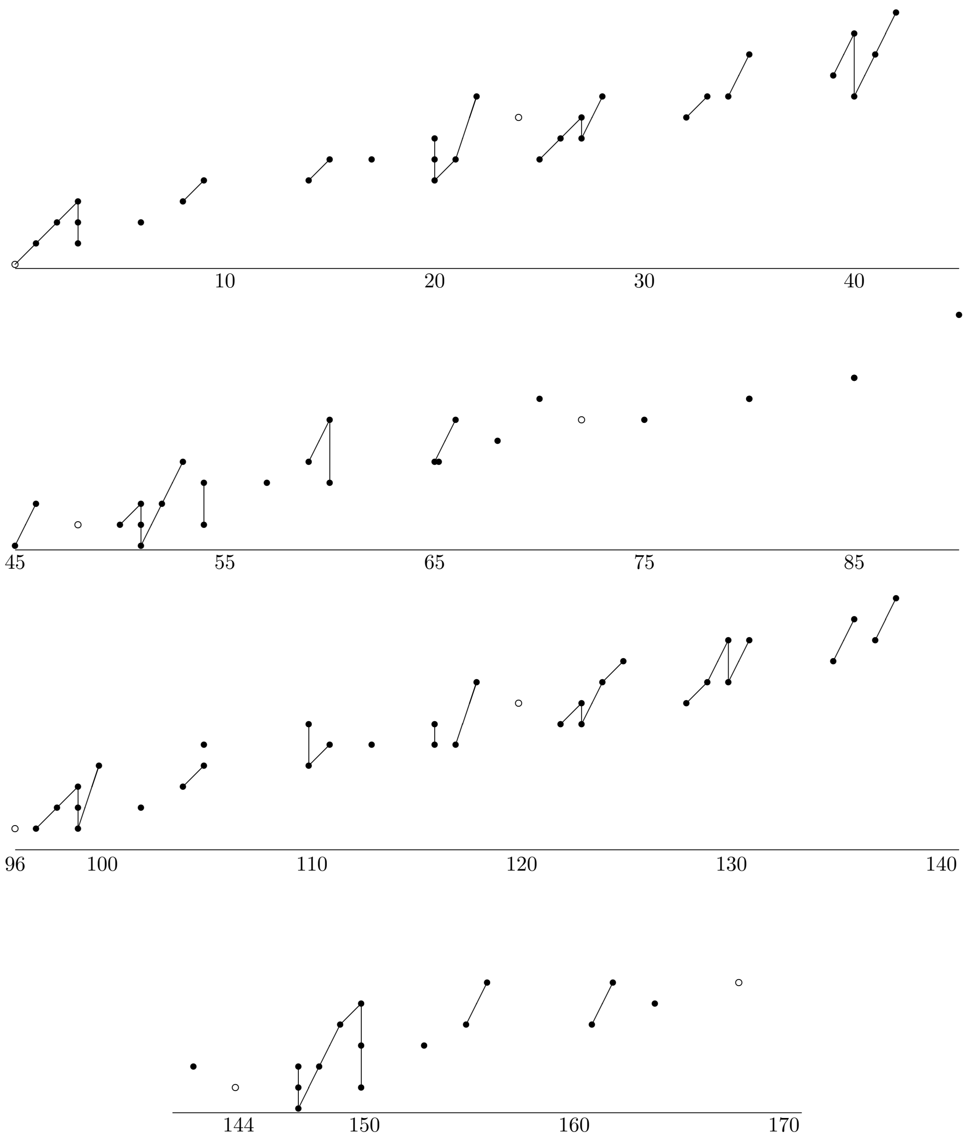


Here is the picture for 141 to 180.



We can now collect the final charts and write in one place the v_1 torsion homotopy. Dots correspond to $\mathbb{Z}/2$'s and circles correspond to $\mathbb{Z}$'s. Vertical lines indicate multiplication by 2 and slanting lines to the right indicate multiplication by η.

There are a large number of multiplications by ν but they are not indicated on these charts.



THEOREM 9.11. *The homotopy of eo_2 is given by the following:*

$$v_1^4 bo[v_2^4] \oplus E[v_2^{32}]$$

where E is the homotopy described in the above charts.

10. Some self maps

Let A_1 be the suspension spectrum of one of the complexes whose cohomology is free over $A(1)$, the sub algebra of A generated by Sq^1 and Sq^2. Let $M(i_0, i_1)$ be the mapping cone of $\Sigma^{2i_1} M(2^{i_0}) \to M(2^{i_0})$ which induces an isomorphism in K-theory. In [**DM3**] it is claimed that A_1 and $M(1,4)$ admitted a self map raising dimension by 48 and inducing an isomorphism in $K(2)_*$. This result is false as the results here have shown. The argument in [**DM3**] is correct in showing the following.

THEOREM 10.1. *There is a class representing* $v_2^8 \in \mathrm{Ext}_A^{8,56}(A(1), A(1))$.

Consider a resolution by Eilenberg-Mac Lane spaces constructed as follows for any suspension spectrum X with the property that there is only one class $\alpha \in \pi_*(X)$ which is non-zero in mod 2 homology. We begin with a map f_0 so that the composite

$$S^0 \xrightarrow{\alpha} X \xrightarrow{f_0} K(\mathbb{Z}/2)$$

is non-zero. Now we construct a tower inductively. Suppose we have

$$X \xrightarrow{f_s} X_s \xrightarrow{g_{s-1}} \cdots \xrightarrow{g_0} K(\mathbb{Z}/2)$$

with $g_0 \cdots g_{s-1} f_s = f_0$. Let $h_s : X_s \to K(\mathbf{ker}(H^*(f_s)))$ and let X_{s+1} be the fiber. Clearly, f_s lifts to give f_{s+1}.

Each such resolution defines a spectral sequence with

$$E_1^{s,t} = \mathbf{ker}(H^*(f_s))^{t-s+1}$$

As with the usual Adams spectral sequence we have convergence to the 2-adic completion and we can define $E_2(X, X)$ as equivalence classes of maps between these resolutions.

In this language [**DM3**] showed the following result for $M(1, 4)$. A very similar argument works for $M(2, 4)$.

THEOREM 10.2. *For* $X = M(1, 4)$ *or* $X = M(2, 4)$ *there is a class*

$$v_2^8 \in E_2^{8,56}(X, X)$$

.

Using an eo_2 resolution we see that v_2^8 commutes with the possible targets of the differentials on v_2^8. Thus in each case if $d_2(v_2^8) \neq 0$, which is the case, then v_2^{32} will be a class in E_4 and for dimensional reasons, $d_4(v_2^{32}) = 0$. This proves the following theorem.

THEOREM 10.3. *For* $X = A_1$, $X = M(1, 4)$ *or* $X = M(2, 4)$, *there is a map*

$$v_2^{32} : \Sigma^{192} X \to X$$

which is detected in $K(2)_*$.

Thus the results discussed in [**Rav**], [**Mo1**] and possibly other places which used the maps of [**DM3**] are established in this modified form. We will discuss some of these classes, particularly those of [**Mo1**] in the next section.

11. The Hurewicz image and some homotopy constructed from EO_2

Using the results of the last section we can construct many v_2-families.

THEOREM 11.1. *Let* $\alpha \in \pi_*(S^0)$ *be such that* $4\alpha = 0$ *and such that* v_1^4 *kills an extension* $M^{|\alpha|+1}(2) \to M^0(2)$ *of* α.

Assume that α *is represented by* $a \in Ext_A(\mathbb{Z}/2, \mathbb{Z}/2)$ *and that under the map* $Ext_A(\mathbb{Z}/2, \mathbb{Z}/2) \to Ext_{A(2)}(\mathbb{Z}/2, \mathbb{Z}/2)$, *a maps to a non-zero cycle. Then* $v_2^{32k}\alpha \neq 0$.

PROOF. This is the standard Greek letter proof. There is an $\alpha^{\#}$ so that the following composite is α.

$$S^{|\alpha|} \to M^{|\alpha|+10}(2,4) \xrightarrow{\alpha^{\#}} M^0(2,4) \to S^0$$

Then

$$\begin{tikzcd}
S^{|\alpha|+192} \arrow[r] & M^{|\alpha|+202} \arrow[r, "v"] \arrow[d, "\Sigma\alpha^{\#}"'] & M^{|\alpha|+10} \arrow[d, "\alpha^{\#}"] \arrow[dr] & & \\
 & & & S^0 \arrow[r] & EO_2 \\
 & M^{192} \arrow[r, "v"'] & M^0 \arrow[ur] & &
\end{tikzcd}$$

is non-zero in $\pi_* EO_2$. $\square$

COROLLARY 11.2. *We list the classes which this is known to apply to*

stem	6	8	9	14	15	17	28	32	33	34	35	39	40	41	42	45	46
name	ν^2	ϵ	$\eta\epsilon$	κ	$\eta\kappa$	$\nu\kappa$	$\epsilon\kappa$	q	ηq	e_0^2	ηe_0^2	u	$\bar\kappa^2, 2\bar\kappa$	$\eta\bar\kappa^2$	$\eta^2\kappa^2$	w	ηw

stem	52	53	59	60	65	66	80	85
name	$\bar\kappa q$	$\bar\kappa\eta q$	$\bar\kappa u$	$\bar\kappa^3, 2\bar\kappa^3$	$\bar\kappa w$	$\eta\bar\kappa w$	$\bar\kappa^4$	$\bar\kappa^2 w$

Next consider $\bar\kappa$. The only problem is that it has order 8 and so the argument does not quite work.

PROPOSITION 11.3. *$\bar\kappa$ is v_2^{32}-periodic.*

PROOF. We look at map $\bar\kappa^{\#}$ which makes this diagram commute.

$$\begin{tikzcd}
M^{29}(2,4) \arrow[r, "\bar\kappa^{\#}"] & M^0(2,4) \arrow[d] \\
M^{20}(2) \arrow[u] \arrow[r, "\bar\kappa\wedge 1"'] & M^0(2)
\end{tikzcd}$$

We do the Greek letter construction and get

$$\begin{tikzcd}
M^{20+192}(2,4) \arrow[r] & M^{29+192}(2,4) \arrow[r] \arrow[d] & M^{29}(2,4) \arrow[r] \arrow[d] & M^0(2) \arrow[d] \\
* & M^{192} \arrow[r] & M^0(2,4) \arrow[r] & M^0(2)\wedge EO_2
\end{tikzcd}$$

This composite is non-zero. We can break apart the calculation and get that $\bar\kappa$ as an element of order 8 is v_2^{32}-periodic. Of course we get easily $4\bar\kappa = v^2\kappa$ is v_2^{32}-periodic. $\square$

The next problem class is $\nu\bar\kappa$.

PROPOSITION 11.4. *$\nu\bar\kappa$ is v_2^{32} periodic.*

PROOF. Suppose not, that is, suppose $\nu\bar{\kappa}(v_2^{32}) = 0$. Then we have a map f

$$\begin{tikzcd} S^{20+192} \cup e^{24+192} \arrow[r, "f"] \arrow[dd] & S^0 \arrow[d] \\ & EO_2 \arrow[d] \\ S^{24+192} \arrow[r, "g"'] & E_2 \end{tikzcd}$$

in the diagram, the map g is v_2^{36}. Thus f would have ∞ order. This contradiction completes the proof. □

Next we have classes in EO_2 which do not come from the sphere. Since there should be nothing extra in homotopy we use these to detect something also.

PROPOSITION 11.5. *The classes η_4, $\eta\eta_4$, $\eta^2\eta_4$ and $\frac{1}{2}\eta^2\eta_4$ are v_2-periodic.*

PROOF. It is an easy *Ext* calculation to show that in the following diagram

$$\begin{tikzcd} & M^{26}(1,4) \arrow[r, "\eta_4^\#"] \arrow[dd] & M^0(1,4) \arrow[d] \\ & & S^0 \arrow[d] \\ S^{25} \arrow[r] & M^{26}(1) \arrow[r] & EO_2 \end{tikzcd}$$

the composite along the bottom row is non-zero. $M(1,4)$ also has a v_2^{32} self map and this gives the diagram

$$\begin{tikzcd} S^{208} \arrow[r] & M^{26+192}(1,4) \arrow[r] \arrow[d] & M^{26}(1,4) \arrow[d] & \\ & M^{192}(1,4) \arrow[r] & S^0 \arrow[r] & EO_2 \end{tikzcd}$$

The map $M^{26+192} \to EO_2$ is essential. Suppose the composite

$$S^{16+192} \to M^{26+192} \to M^{26}(1,4) \to S^0$$

was zero. Then M^{26+192}/S^{16+192} factors through S^0 giving

$$M^{26+192}/S^{16+192} \to S^0 \to EO_2.$$

This map is v_1-periodic and this contradicts the v_1-structure of S^0. Thus $v_2^{32}\eta_4$ is essential. $\eta\eta_4$ works the same way and also $\eta^2\eta_4/2$. □

It is interesting that each extra class in EO_2 detected something in the sphere.

The next place to study is the 47 stem. Here the differential on v_2^8 eliminates homotopy classes which are in the sphere. Similar to Proposition 11.4 we have

PROPOSITION 11.6. *The classes $\{e_0 r\}$ and $\eta\{e_0 r\}$ are v_2-periodic.*

Proof. We have established that $2v_2^{32}\bar{\kappa}^2 = \eta v_2^{32}\{u\}$ by Theorem 11.1. Thus we can form the bracket $\langle v_2^{32}\bar{\kappa}^2, 2\nu, \nu\rangle$. Suppose this bracket contains zero. Then

$$\begin{tikzcd} S^{40+192} \cup e^{45+192} \cup e^{48+192} \arrow[r] \arrow[d] & S^0 \arrow[d] \\ S^{48+192} \arrow[r, "v_2^{40}"'] & BP\langle 2\rangle \end{tikzcd}$$

is a commutative diagram. This gives a class of ∞-order in $I\pi_* S^0$, a contradiction. Thus the bracket is essential and defines $v_2^{32}\{e_0 r\}$. Now $\eta\{e_0 r\} = \nu\{\bar{w}\}$. Suppose $v_2^{32}\nu w = 0$. Then we could form the bracket $\langle v_2^{32}\{w\}, \nu, \eta\rangle \in \pi_{192+49}(S^0)$ and this class maps to $v_2^{40}\eta^2$ in EO_2, which is a v_1 periodic class. This contradiction establishes the result. $\square$

We also get some mileage out of the failure of the v_2^8 self map.

Proposition 11.7. *There is a family of classes of order 4 in* $38 + k192$ *stem detected by* v_2^{8+32k} *in* BP.

Proof. Here we use the centrality of our self map. We have established $\{e_0 r\}$ is v_2^{32} periodic and the failure of the proof of v_2^8 self map shows that the composite $M^{47}(2,4) \xrightarrow{g} M^{47}(4) \to S^0$ is null. This gives the following diagram

$$\begin{tikzcd} M^{47+192}(2,4) \arrow[r] \arrow[d] & M^{58+192}(2,4) \arrow[d] & \\ M^{47}(2,4) \arrow[r] & M^{58}(2,4) \arrow[r] & M^0(2) \end{tikzcd}$$

Going around the top is the same, by centrality, as going around bottom. The top represents $v_2^{32}g$ and so is null which implies that there is a class β such that

$$M^{47+192}(2) \xrightarrow{v_1^4} M^{39+192}(2) \to S^0$$

is $v_2^{32}\{e_0 r\}$. As before, β must be on the bottom class or we would get a v_1-periodic class. $\square$

Following closely the proof of Proposition 11.5 we get,

Proposition 11.8. $v_1^4\eta_5$, $\mu\eta_5$, $\eta\mu n_5/4$ *are* v_2^{32} *periodic. The class* θ_4 *is* v_2 *to some power periodic but we don't know the power.*

Proof. First we look at $\eta_5\mu$. We have $M^{51}(1,4) \to S^0$ extending $\eta_5\mu$. The map is detected in EO_2 by $\{v_2^8 h_1^2\}$. Thus we have an essential composite

$$M^{51+192}(1,4) \to S^0 \to EO_2$$

detected by $v_2^{40}h_1^2$.

Suppose the composite $S^{41+192} \to M^{51+192}(1,4) \to S^0$ is zero. Then it factors through $M^{51+192}(1,4)/S^{41+192} \to S^0 \to EO_2$. This map would be v_1 periodic. This completes the proof. The argument is similar for the other cases. We look at θ_4. We have the following

$$M^{47}(1) \xrightarrow{v_1^8} M^{31}(1) \xrightarrow{\theta_4} S^0 \text{ is essential and detected by } \{e_0 r\}.$$

Thus we can consider the composite $M^{47}(1,8) \to M^{56}(1,4) \to S^0$. This is null. We have

$$\begin{tikzcd} M^{47+2^i\cdot 6}(1,8) \arrow[r] \arrow[d] & M^{56+2^i\cdot 6}(1,4) \arrow[d] & \\ M^{47}(1,8) \arrow[r] & M^{56}(1,4) \arrow[r] & S^0 \end{tikzcd}$$

The map from $M^{47+2^i\cdot 6}(1,8) \to S^0$ is null so the map $M^{47+2^i\cdot 6}(1)$ factors through $M^{31+2^i\cdot 6}$. As before, it must be on the bottom cell or we get a v_1-periodic class. $\square$

Some of these homotopy classes were discussed in [1]. The proofs there are valid for $32k$ replacing $8k$. In that note $\rho_k\eta_j$ was also studied and shown to be non-zero. We first discuss $\eta_4\sigma$. In the sphere we are looking at the classes $\{h_4h_2\}$, h_3^3, h_3c_0, $h_3h_1c_0$ and $\nu^2\bar{\kappa}$.

Proposition 11.9. *The classes $\{h_4h_3\}$, h_3^3, h_3c_0,$h_3h_1c_0$ and $\nu^2\bar{\kappa}$ are v_2^{32} periodic.*

Proof. We begin with $\nu^2\bar{\kappa}$. The $M^{27}(1) \xrightarrow{\nu^2\bar{\kappa}\#} S^0 \to EO_2$ is detected by $M^{27}(1) \to S^{27} \to EO_2$. Thus the map of $M^{27}(1) \to S^0$ is v_2-periodic. Suppose $S^{26+192} \to M^{27+192}(1) \to S^0$ is null. Then $S^{27+192} \to EO_2$ factors through S^0. This class has finite order and so extends to $M^{28+192}(2^i) \to S^0 \to EO_2$. The composite is v_1-periodic, a contradiction. Next note that the composite

$$M^{26}(1,4) \to M^{26}(1) \to S^{26} \xrightarrow{\nu^1\kappa} S^0$$

is null since $M^{26}(1) \to S^0$ factors through

$$M^{26}(1) \to M^{18}(1) \to S^{18} \xrightarrow{\{h_2h_4\}} S^0.$$

By centrality

$$M^{26+192}(1,4) \to M^{26+192}(1) \xrightarrow{\nu} S^{26+192} \xrightarrow{v_2^{32}\nu^2\bar{\kappa}} S^0$$

is null. This gives $v_2^{32}\{h_2h_4\}$. $\nu \circ \{h_4h_2\} = \sigma^3$. This gives that

$$S^{26+192} \xrightarrow{n^2\bar{n}} M^{22+192}(1) \to S^0$$

is $\nu^2\bar{\kappa}v_2^{32}$, and so all the classes in between are non-zero too. $\square$

We start with $\{\rho^1h_2h_5\}$. There is an extension of $M^{42}(1) \to S^{42} \to \rho^1h_1h_5\ \ S^0$ to $M^{51}(1,4) \to S^0$ and the composite is detected by $v_2^8\nu$ and so this gives a v_2^{32} family. As before, this class must live in the $S^{42+k192}$ stem. Composing with ν to get $M^{55}(1,4) \xrightarrow{\{h_2\rho^1h_2h_5\}} S^0$ and this is detected by $\nu^2v_2^8$. This gives $v_2\cdot\{\rho^1h_2h_5\}$, ρn_5, $\rho\wedge n_5$, as v_2^{32k} periodic classes. The same family of arguments works for n_5 and related classes. The approach in [1] is different since EO_2 was not available but complimentary.

The remaining task is to show all the classes in EO_2 come from the sphere in the above sense. This requires constructing new homotopy classes. These classes are covered by the following propositions.

Proposition 11.10. *The class of order 4 represented by $\{h_5h_0i\}$ in the 54 stem is v_2^{32} periodic and $2\{h_5h_0i\} = \bar{\kappa}\{e_0^2\}$.*

Proof. We first note that $\bar{\kappa}\{e_0^2\}$ is $\{e_0^2 g\}$ and this class fits 11.1. This completes the proof since $\{h_5 h_0 i\}$ must map essentially to EO_{2*}. □

Proposition 11.11. *The class corresponding to $\{Ph_5 h_1 e_0\}$ is v_2^{32} periodic.*

Proof. In EO_{2*} we have $\nu\{h_5 h_0 i\} \neq 0$. Since $h_2 h_5 h_0 i = h_1 h_5 P e_0$ in *Ext* we are done. □

Proposition 11.12. *The class in the 65 stem with Adams spectral sequence name $Ph_5 j$ maps to a non zero class in EO_{2*}. Thus it represents a homotopy class which is v_2^{32} periodic.*

We remark that this is the first class beyond Kochman's calculations that was not covered by 11.1.

Proof. In EO_{2*} the Toda bracket $\langle \bar{\kappa}^3, \eta, \nu\rangle \neq 0$. we can form the bracket in the sphere too and so it must be non-zero there. Since $d_4 PG = gz$ and $\bar{\kappa}^2\eta = \{z\}$ we see that $\langle \bar{\kappa}^3, \eta, \nu\rangle \in \{h_2 PG\} = 0$. Thus it has filtration ≥ 12. The only other class of higher filtration not in J_* is $\bar{\kappa}\{w\}$ and this is also present in EO_{2*}. This gives the result. □

Corollary 11.13. *The class $Ph_5 h_0 k$ in the 63 stem maps to a class in EO_{2*} and thus is v_2^{32} periodic.*

Proof. In EO_{2*} we have $\nu\langle \bar{\kappa}^3, \eta, \nu\rangle \neq 0$. We also have $h_2 P h_5 j = Ph_5 h_0 k$ in Ext. This completes the proof. □

We remark that in the E_∞ term of the May spectral sequence the class $Ph_5 h_0 k$ is divided by only h_0. Brunner's calculation of the Ext shows that it is actually divided by h_0^5. In particular, we have $h_0^5 G_{21} = Ph_5 h_0 k$.

Proposition 11.14. *The bracket $\langle \epsilon, \bar{\kappa}^3, \eta\rangle$ is detected in EO_{2*}. In the Adams spectral sequence it is in the coset $\{P^2 G\}$. In addition η on this class is divisible by 4 which is represented by the bracket $\langle \bar{\kappa}^3, \nu, 2\nu, \nu\rangle$ and is in the coset $\{Q_5\}$. We also have $\eta^2\langle \bar{\kappa}^3, \nu, 2\nu, \nu\rangle \neq 0$.*

Proof. In EO_{2*} it is easy to see that $\langle \epsilon, \bar{\kappa}^3, \eta\rangle \neq 0$. It is also straightforward to see that it is in $\{P^2 G\}$. In Ext we have $h_1 P^2 G = h_0^2 Q_5$. This is verified by Brunner's calculations. Now Q_5 is a permanent cycle because there is nothing of higher filtration for it to hit. It has order eight for the same reason. It is in the four fold bracket $\langle \bar{\kappa}^3, \nu, 2\nu, \nu\rangle$ by construction. The other claims follow by easy arguments except we need to show that Q_5 is non-zero. To this end consider the following diagram.

$$\begin{tikzcd}
S^{72} \arrow[rr, "f"] & & EO_2 \\
M^{72}(8) \arrow[u] \arrow[rr, "g"] & & S^0 \arrow[u] \\
& S^{71} \arrow[ul, "i"] \arrow[ur, "\{Q_5\}"'] &
\end{tikzcd}$$

Since $\eta\langle\epsilon, \bar\kappa^3, \eta\rangle = 4\{Q_5\}$, $f_*(i) = \{h_0^3 v_2^{12}$, the generator of Z_2 in the the 72 stem. Thus Q_5 must be a non-zero cycle. $\square$

PROPOSITION 11.15. *The composition $\nu\{Q_5\}$ is a class of order 4 with the generator having Adams spectral sequence name $\{PD'\}$. This class is also $\bar\kappa\{Ph_5 ih_0\}$.*

PROOF.

$$\begin{tikzcd}
S^{75} \arrow[rr, "f"] & & EO_2 \\
M^{75}(2) \arrow[u] \arrow[rr, "g"] & & S^0 \arrow[u] \\
& S^{71} \arrow[ul, "i"] \arrow[ur, "\kappa\bar\kappa^3"'] &
\end{tikzcd}$$

The map f is $v_2^8\eta^2 b$ where b is the EO_{2*} class of order 2 in dimension 27. Thus the map g can not be null but f can not factor through S^0 since it would have Adams filtration at least 16 and there is nothing there. Thus $\kappa\bar\kappa^3 \neq 0$ in $\pi_*(S^0)$ and we have $\bar\kappa 2$ $\{h_1 Ph_5 e_0\} = \kappa\bar\kappa^3$ by 11.11. Thus $\bar\kappa\{Ph_5 ih_0\} \neq 0$ and generates a $Z/4$. We need to show that it is $\nu\{Q_5\}$. This follows from the bracket constructed for $\{Q_5\}$ in 11.14. $\square$

PROPOSITION 11.16. *The remaining classes through the 95 stem detected by EO_{2*} are $\bar\kappa^4$, $\bar\kappa^3\{w\}$ and $\langle\bar\kappa^3\{w\}, \nu, \eta\rangle = \{v_2^8 P^1 d_0 g\}$.*

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$K(1)$-local E_∞ ring spectra

M. J. Hopkins

Contents

1. Certain $K(1)$-local spectra

Let $\mathcal{C}$ be the topological model category of $K(1)$-local spectra. Some useful examples of objects of $\mathcal{C}$ are p-adic K-theory, K, the Adams "summand" B of K, and the sphere S. At the prime 2, the spectrum B is KO, and is not a summand of K.

The group $\mathbb{Z}_p^\times$ of p-adic units acts on K via the Adams operations. If λ is a p-adic unit we will denote $\psi_\lambda : K \to K$ the corresponding Adams operation. Let $\mu \subset \mathbb{Z}_p^\times$ be the maximal finite subgroup. When p is odd, μ is the group of $(p-1)^{\mathrm{st}}$ roots of unity, and when p is 2 it is the group $\{\pm 1\}$. The spectrum B is the homotopy fixed point spectrum of the action of μ on K. The action of the Adams operations on K restricts to an action of $\mathbb{Z}_p^\times/\mu \approx \mathbb{Z}_p$ on B.

2. Homotopy groups of $K(1)$-local spectra

Let g be a topological generator of $\mathbb{Z}_p^\times/\mu$, and

$$\psi_g : B \to B$$

the corresponding map. For any object X of $\mathcal{C}$, there is a fibration

$$X \to B \wedge X \xrightarrow{(\psi_g - 1)\wedge 1} B \wedge X.$$

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This makes it easy to compute the homotopy groups of X in terms of the homotopy groups of $B \wedge X$. Take for example X to be the sphere spectrum. The action of ψ_g on $\pi_0 B = \mathbb{Z}_p$ is trivial. This means that the element 1 in the rightmost $\pi_0 B$ comes around to give a non-trivial element

$$\zeta \in \pi_{-1} S^0.$$

We will see that ζ plays an important role in things to come. Though not mentioned in the notation, the element ζ depends on the choice of generator g.

Given an element $f \in \pi_0(K \wedge K)$ and $\lambda \in \mathbb{Z}_p^\times$, let $f(\lambda)$ be the element of $\pi_0 K = \mathbb{Z}_p$ which is the image of f under the map induced by the composite

$$K \wedge K \xrightarrow{\psi_\lambda \wedge 1} K \wedge K \to K.$$

Thinking of λ as a variable, this defines a map

$$\pi_0(K \wedge K) \to \mathrm{Hom}_{\mathrm{cts}}(\mathbb{Z}_p^\times, \mathbb{Z}_p).$$

Proposition 2.1. *The above map is an isomorphism. It gives rise to isomorphisms*

$$\begin{aligned} \pi_*(K \wedge K) &\approx \mathrm{Hom}_{cts}(\mathbb{Z}_p^\times, \pi_* K) \\ \pi_*(K \wedge B) &\approx \mathrm{Hom}_{cts}(\mathbb{Z}_p^\times/\mu, \pi_* K) \\ \pi_*(B \wedge B) &\approx \mathrm{Hom}_{cts}(\mathbb{Z}_p^\times/\mu, \pi_* B). \end{aligned}$$

□

With respect to the above isomorphism, the actions of $\psi_g \wedge 1$ and $1 \wedge \psi_g$ are given by

$$\begin{aligned} (\psi_g \wedge 1\, f)(\lambda) &= f(\lambda g) \\ (1 \wedge \psi_g\, f)(\lambda) &= \psi_q f(g^{-1}\lambda). \end{aligned}$$

Let

$$M_\zeta = S^0 \underset{\zeta}{\cup} e^0$$

be the mapping cone of ζ, and fix a generator g of $\mathbb{Z}_p^\times/\mu$. By definition, we have a diagram

$$\begin{tikzcd} S^0 \arrow[r] \arrow[d, equal] & M_\zeta \arrow[r, "\delta"] \arrow[d, "\iota"'] & S^0 \arrow[d, "1"] \\ S^0 \arrow[r] & B \arrow[r, "\psi_g - 1"'] & B \end{tikzcd}$$

The maps

$$1 \circ \delta,\ \iota : M_\zeta \to B$$

form a basis of $\mathrm{ho}\,\mathcal{C}(M_\zeta, B)$. From the above diagram it follows that

$$\psi_g \iota = \iota + 1 \circ \delta.$$

This will be more useful when written in homology. Thus define

$$\{a, b\} \subset \pi_0\big(B \wedge M_\zeta\big)$$

by

$$\begin{aligned} \langle \iota, a \rangle &= 1 & \langle \iota, b \rangle &= 0 \\ \langle 1 \circ \delta, a \rangle &= 0 & \langle 1 \circ \delta, b \rangle &= 1. \end{aligned} \tag{2.1}$$

One easily checks that

LEMMA 2.2. *Under the map*

$$\pi_0\bigl(B \wedge M_\zeta\bigr) \to \pi_0(B \wedge B) \to \mathrm{Hom}_{cts}(\mathbb{Z}_p^\times/\mu, \mathbb{Z}_p)$$

the element a goes to the constant function 1, and the element b goes to the unique homomorphism sending g to 1. □

3. The category of E_∞ ring spectra

The topological model category of E_∞ ring spectra in $\mathcal{C}$ will be denoted $\mathcal{C}^{E_\infty}$. The spectra K have unique E_∞ structures, and the Adams operations act by E_∞ maps. This gives the spectrum B an E_∞ structure as well.

Let $B\Sigma_{p+}$ be the image in $\mathcal{C}$ of the unreduced suspension spectrum of the classifying space of Σ_p. There are two natural maps

$$B\Sigma_{p+} \to S^0.$$

One is derived from the map $\Sigma_p \to \{e\}$ and will be denoted ϵ. The other is the transfer map

$$B\Sigma_{p+} \to S^0$$

and will be denoted Tr.

LEMMA 3.1. *The map*

$$B\Sigma_{p+} \xrightarrow{(\epsilon,\mathrm{Tr})} S^0 \times S^0$$

is a weak equivalence in $\mathcal{C}$. □

Define maps in $\mathrm{ho}\,\mathcal{C}$

$$\theta, \psi : S^0 \to B\Sigma_{p+}$$

by requiring

$$\begin{aligned} \mathrm{Tr}(\theta) &= -(p-1)! \qquad & \mathrm{Tr}(\psi) &= 0 \\ \epsilon(\theta) &= 0 & \epsilon(\psi) &= 1. \end{aligned}$$

The map $B\{e\} \to B\Sigma_p$ gives rise to a map

$$e : S^0 \approx B\{e\}_+ \to B\Sigma_{p+}.$$

It follows from the definition that

$$\epsilon \circ e = 1,$$

and from the double coset formula that

$$\mathrm{Tr} \circ e = p!.$$

It follows that

$$e = \psi - p\theta \tag{3.2}$$

Let $E \in \mathcal{C}^{E_\infty}$, and $x \in \pi_0 E$. The E_∞ structure associates to x a map

$$P(x) : B\Sigma_{p+} \to X$$

with the property that

$$P(x) \circ e = x^p.$$

We define operations

$$\theta, \psi : \pi_0 E \to E$$

by

$$\theta(x) = P(x) \circ \theta \qquad \psi(x) = P(x) \circ \psi.$$

In view of (3.2) we have

$$\psi(x) - x^p = p\,\theta(x).$$

Thus the operation ψ is determined by θ. One easily checks that ψ is a ring homomorphism, and that θ does what it has to so that the above equation will remain true:

$$\theta(x+y) = \theta(x) + \theta(y) - \sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} x^i y^{p-i}$$
$$\theta(x\,y) = p\theta(x)\theta(y) + \theta(x)y^p + \theta(y)x^p.$$

Since the Adams operations are E_∞ maps, they commute with the operations ψ and θ when acting on

$$\pi_0(K \wedge R).$$

4. Some algebra

We now work in the category of p-complete abelian groups, and we want to consider comutative algebras with operations θ and ψ as described above. Let's call them *Frobenius algebras* (even though this collides with another use of the phrase).

There is a free Frobenius algebra on one generator. The underlying ring is

$$\mathbb{Z}_p[x, x_1, x_2, \dots].$$

One defines θ by setting $\theta(x_i) = x_{i+1}$ ($\theta(x) = x_1$), and extending it to the whole ring by requiring that $p\theta(x) + x^p$ be a ring homomorphism. We'll call the free Frobenius algebra on one generator x, $T\{x\}$. Similarly, if X is a spectrum, we will call the free $K(1)$-local commutative ring spectrum on X $T\{X\}$. The following result plays an important role in everything we do.

Theorem 4.1 (McClure). *For any $K(1)$-local E_∞ ring spectrum E, the natural map of Frobenius algebras*

$$E_* \otimes T\{x\} \to \pi_*\big(E \wedge T\{S^0\}\big)$$

is an isomorphisms.

There is another, perhaps more useful description of this free algebra. Let $\mathbb{W} = \mathbb{Z}_p[a_0, a_1, \dots]$ be the p-completion of the Witt Hopf algebra. The classical result says that if one defines elements $w_i \in \mathbb{W}$ by

$$w_n = a_0^{p^n} + p a_1^{p^{n-1}} + \cdots + p^n a_n,$$

then $\mathbb{W}$ has a unique Hopf algebra structure for which the w_i are primitive. One defines ψ by $\psi(w_i) = w_{i+1}$, and checks that this extends uniquely to a Frobenius structure on $\mathbb{W}$. This is the map which co-represents the classical Frobenius map.

Define a ring homomorphism

$$\mathbb{Z}_p[w_0, w_1, \dots] \to \mathbb{Z}_p[x_0, x_0, \dots]$$

by sending w_0 to $x = x_0$, and requiring that it be compatible with the map ψ.

LEMMA 4.2. *The above map extends uniquely to an isomorphism of Frobenius algebras*

$$\mathbb{W} \to \mathbb{Z}_p[x_0, x_1, \ldots].$$

□

The proof of the above lemma makes use of the following result of Dwork:

LEMMA 4.3. *Let A be a ring with a ring homomorphism $\phi : A \to A$ satisfying*

$$\phi(a) \equiv a^p \mod p$$

(thus if A is torsion free, then A is a Frobenius algebra). Let $w_0, w_1, \ldots$ be a sequence of elements of A. In order that the system of equations

$$\begin{aligned} a_0 &= w_0 \\ a_0^p + pa_1 &= w_1 \\ a_0^{p^2} + pa_1^p + p^2 a_2 &= w_2 \\ a_0^{p^n} + \cdots + p^n a_n &= w_n \end{aligned}$$

have a solution, it is necessary and sufficient that for each n

$$w_n \equiv \phi(w_{n-1}) \mod p^n.$$

This gives the map. The isomorphism follows easily from the fact that, modulo decomposables,

$$\begin{aligned} w_n &= p^n a_n \\ \psi^n x &= p^n x_n. \end{aligned}$$

This gives the free Frobenius algebra on one generator the structure of a Hopf algebra. We knew it had one anyway, since it was free.

5. Continuous functions on $\mathbb{Z}_p$

Another important example of a Frobenius algebra is the ring C of continuous functions

$$\mathbb{Z}_p \to \mathbb{Z}_p,$$

with $\psi(f) = f$. The point of this section is to describe a small projective resolution of C.

Each p-adic number λ can be written uniquely in the form

$$\lambda = \sum_{i \geq 0} \alpha_i p^i$$

with each $\alpha_i = \alpha_i(\lambda)$ equal to 0 or a $(p-1)^{\mathrm{st}}$ root of unity. The α_i are continuous functions from $\mathbb{Z}_p$ to $\mathbb{Z}_p$, and satisfy

$$\alpha_i^p = \alpha_i.$$

For $i < n$, the α_i can be regarded as functions from $\mathbb{Z}/p^n \to \mathbb{Z}_p$.

PROPOSITION 5.1. *For each m, n, the map*

$$\mathbb{Z}/p^n[\alpha_0, \alpha_1, \ldots, \alpha_{m-1}]/(\alpha_i^p - \alpha_i) \to \mathrm{Hom}_{cts}(\mathbb{Z}/p^m, \mathbb{Z}/p^n)$$

is an isomorphism.

Corollary 5.2. *The map*

$$\mathbb{Z}_p[\alpha_0, \alpha_1, \dots]/(\alpha_i^p - \alpha_i) \to \operatorname{Hom}_{cts}(\mathbb{Z}_p, \mathbb{Z}_p)$$

is an isomorphism. □

Proof of Proposition 5.1: When $m = 1$ the injectivity follows from the linear independence of characters (of μ_{p-1}), and surjectivity follows from the fact that both groups are finite of the same order. For the general case, note that for finite (discrete) sets S, and T, the natural map

$$\operatorname{Hom}_{\mathrm{cts}}(S, \mathbb{Z}/p^n) \otimes \operatorname{Hom}_{\mathrm{cts}}(T, \mathbb{Z}/p^n) \to \operatorname{Hom}_{\mathrm{cts}}(S \times T, \mathbb{Z}/p^n)$$

is an isomorphism. Apply this to the (set-theoretic) isomorphism

$$\mathbb{Z}/p^m \xrightarrow{\prod \alpha_i} \prod \left(\{0\} \cup \mu_{p-1}\right).$$

□

Lemma 5.3. *Under the map of Frobenius algebras*

$$\iota : \mathbb{W} \to \operatorname{Hom}_{cts}(\mathbb{Z}_p, \mathbb{Z}_p)$$

sending a_0 to the identity map, one has

$$\iota(a_i) \equiv \alpha_i \mod p.$$

Proof: This map sends each w_i to the identity function. To work out the values of $a_i \mod p$ we work by induction on i. Suppose we have proved the result for $i < n$. The inductive step is provided by the congruence

$$w_n = a_0^{p^n} + pa_1^{p^{n-1}} + \cdots + p^n a_n \mod p^{n+1}.$$

Since for $i < n$, we have

$$a_i \equiv \alpha_i \mod p$$

it follows that

$$a_i^{p^i} \equiv \alpha_i^{p^i} = \alpha_i \mod p^{i+1},$$

and so

$$p^{n-i} a_i^{p^i} \equiv p^{n-i} \alpha_i \mod p^{n+1}.$$

Solving for a_n then gives the result. □

Map

$$T\{x\} \to C$$

by sending x to the identity map $\mathbb{Z}_p$. The element $\psi(x) - x$ then goes to zero.

Lemma 5.4. *The commutative diagram*

$$\begin{tikzcd}
T\{y\} \arrow[r, "y\mapsto 0"] \arrow[d, "y\mapsto\psi(x)-x"'] & \mathbb{Z}_p \arrow[d] \\
T\{x\} \arrow[r] & C
\end{tikzcd}$$

is a pushout square in the category of Frobenius algebras. The left vertical map is étale. □.

6. Structure of B

Define T_ζ by the pushout (in $K(1)$-local E_∞ ring spectra)

$$\begin{tikzcd} T\{S^{-1}\} \arrow[r, "*"] \arrow[d, "\zeta"'] & S^0 \arrow[d] \\ S^0 \arrow[r] & T_\zeta \end{tikzcd}$$

By definition of ζ and the fact that B is E_∞, there is a canonical map

$$T_\zeta \to B.$$

Since $B \wedge \zeta$ is null, we have

$$\pi_0\big(B \wedge T_\zeta\big) = T\{b\},$$

where b was defined in Equation 2.1. The same thing holds with B replace by K, only in that case, the odd homotopy groups are zero. Under the map of Frobenius algebras

$$\pi_0\big(K \wedge T_\zeta\big) = T\{b\} \to \pi_0\big(K \wedge B\big) = \mathrm{Hom}_{\mathrm{cts}}(\mathbb{Z}_p^\times/\mu, \mathbb{Z}_p)$$

the element b goes to the unique abelian group homomorphism sending g to 1. Let's use this homomorphism to identify $\mathbb{Z}_p^\times/\mu$ with $\mathbb{Z}_p$ and hence $\pi_0(K \wedge B)$ with $\mathrm{Hom}_{\mathrm{cts}}(\mathbb{Z}_p, \mathbb{Z}_p)$ After doing this, we find that b maps to the identity map of $\mathbb{Z}_p$.

Now consider the element $\psi(b) - b$. This is fixed under ψ_g, since

$$\begin{aligned} \psi_g(\psi(b) - b)) &= \psi_g\psi(b) - \psi_g(b) \\ &= \psi\psi_g(b) - \psi_g(b) \\ &= \psi(b+1) - (b+1) = \psi(b) - b. \end{aligned}$$

Lemma 6.1. *The maps*

$$\pi_*\big(K \wedge T_\zeta\big) \xrightarrow{(\psi_g-1)\wedge 1} \pi_*\big(K \wedge T_\zeta\big)$$
$$\pi_*\big(B \wedge T_\zeta\big) \xrightarrow{(\psi_g-1)\wedge 1} \pi_*\big(B \wedge T_\zeta\big)$$

are surjective. The map

$$\pi_* T_\zeta \to \pi_*\big(B \wedge T_\zeta\big)$$

is therefore a monomorphism, with image the invariants under ψ_g.

We therefore have a unique element f in $\pi_0 T_\zeta$ whose image in $\pi_0\big(B \wedge T_\zeta\big) \approx \pi_0\big(K \wedge T_\zeta\big)$ is $\psi(b) - b$. This leads to the diagram

$$\begin{tikzcd} T\{S^0\} \arrow[r, "*"] \arrow[d, "f"'] & S^0 \arrow[d] \\ T_\zeta \arrow[r] & B \end{tikzcd}$$

Our main result is

Proposition 6.2. *The map K_*f is étale. The above diagram is a pushout in $\mathcal{C}^{E_\infty}$.* □

7. $K(1)$-local E_∞-elliptic spectra

We first recall the geometric interpretation of the operation ψ. In general, an E_∞ structure on a complex oriented cohomology theory E gives the following structure. Given a map

$$f : \pi_0 E \to R$$

and a closed finite subgroup $H \subset f^* G$, one gets a new map

$$\psi_H : \pi_0 E \to R$$

and an isogeny $f^*G \to \psi^*G$ with kernel H. When the formal group is isomorphic, locally in the flat topology, to the formal multiplicative group, one can take f to be the identity map, and H the "canonical subgroup" of order p. Since it is so canonical, it is invariant under all automorphisms of G, and one doesn't even need a formal group in this case.

An *elliptic spectrum* is a ring spectrum E with $\pi_{\mathrm{odd}} E = 0$, and for which there exists a unit in $\pi_2 E$, (hence E is complex orientable and we get a canonical formal group G over $\pi_0 E$), together with an elliptic curve over $\pi_0 E$ and an isomorphism of the formal completion of this elliptic curve with the formal group G.

An E_∞ *elliptic spectrum* is an E_∞ spectrum E which is elliptic, and for which the isogenies described above come equipped with extensions to isogenies of the elliptic curve.

Suppose that E is a $K(1)$-local E_∞ elliptic spectrum with

$$\pi_1(K \wedge E) = 0$$

(this is automatic if $\pi_0 E$ is torsion free). Then the sequence

$$\pi_0 E \to \pi_0(B \wedge E) \xrightarrow{(\psi_g - 1)\wedge 1} \pi_0 B \wedge E$$

is short exact. There is therefore an element b in $\pi_0 B \wedge E$ with $\psi_g b = b + 1$. Such a b is well-defined up to translation by an element in the image of $\pi_0 E$. Here is another description. Choose an extension of the unit to a map

$$\iota : M_\zeta \to E.$$

The map ι is unique up to translation by an element in $\pi_0 E$. Now look at the image of the element $b \in \pi_0\big(K \wedge M_\zeta\big)$ in $\pi_0(K \wedge E)$. This class, which we shall also call b is well-defined up to translation by an element in the image of $\pi_0 E$. The fact that it satisfies $\psi_g b = b + 1$ follows from the commutative diagram

$$\begin{tikzcd}
\pi_0\big(B \wedge M_\zeta\big) \arrow[r, "(\psi_g - 1)\wedge 1"] \arrow[d] & \pi_0\big(B \wedge M_\zeta\big) \arrow[d] \\
\pi_0(B \wedge E) \arrow[r, "(\psi_g - 1)\wedge 1"'] & \pi_0(B \wedge E)
\end{tikzcd}$$

Suppose we have chosen such a b. Since ψ and ψ_g commute (ψ_g is an E_∞ map) one easily checks that $\psi(b)$ is another such element. It follows that $\psi(b) - b$ lies in

$$\pi_0 E \subset \pi_0(B \wedge E).$$

We are interested in choosing such a b as canonically as possible, and looking at the element $\psi(b) - b$. This will be the obstruction to making an E_∞ map from

B to E, but it also tells us quite a bit about the E_∞ structure of E. Since the map

$$\pi_0(B \wedge E) \to \pi_0(K \wedge E)$$

is an isomorphism, it suffices to make this calculation in $\pi_0(K \wedge E)$.

Let's first do this at the prime 2. Let $c_4 \in \pi_8 E$ be the normalized modular form of weight 4. The q-expansion of c_4 is given by

$$c_4 = 1 + 240 \sum_{n \geq 1} \sigma_3(n) q^n,$$

where

$$\sigma_3(n) = \sum_{d|n} d^3.$$

Let $g \in \mathbb{Z}_2^\times$ be an element which projects to a topological generator of $\mathbb{Z}_2^\times/\{\pm 1\}$. Let $u \in \pi_2 K$ be the Bott class. We have

$$g^4 \equiv 1 \mod 16,$$

and checking the q-expansion gives

$$u^4 \equiv c_4 \mod 16.$$

Now consider the element

$$b = -\frac{\log(c_4/u^4)}{\log g^4},$$

where

$$\log(1+x) = \sum_{n \geq 0} (-1)^n \frac{x^{n+1}}{n+1}$$

is regarded as a 2-adic analytic function. Then, since $\psi_g c_4 = c_4$ (it is in the image of $\pi_8 E$), and $\psi_g u^4 = g^4 u^4$, we have

$$\begin{aligned}\psi_g b &= -\frac{\log(c_4/g^4 u^4)}{\log g^4} \\ &= -\frac{\log(c_4/u^4)}{\log g^4} + \frac{\log g^4}{\log g^4} \\ &= b + 1,\end{aligned}$$

so b is an *Artin-Schrier* element.

Now let $f = \psi(b) - b$. The element f is a a modular function (since $\psi_g f = f$), and so is an element of

$$\mathbb{Z}_2[j^{-1}],$$

where $j = \frac{c_4^3}{\Delta}$ is the modular function.

LEMMA 7.1. *The map*

$$\mathbb{Z}_2[f] \to \mathbb{Z}_2[j^{-1}]$$

is an isomorphism.

Proof: It clearly suffices to do this modulo 2.
Working mod 2 we have

$$b \equiv \sum_{n \geq 1} \sigma_3(n) q^n \mod 2$$

Writing

$$n = 2^{m_0} p_1^{m_1} \dots p_k^{m_k}$$

One easily checks that the number of divisors of n is

$$(1 + m_0) \times \cdots \times (1 + m_k),$$

and that the number of *odd* divisors of n is

$$(1 + m_1) \times \cdots \times (1 + m_k).$$

It follows that $\sigma_3(n)$ is even unless n is the product of a power of 2 and an odd square. This gives

$$b \equiv \sum_{m,d \geq 0} q^{2^m (2d+1)^2} \mod 2,$$

and so

$$\begin{aligned} \psi(b) - b &\equiv \sum_{d \geq 0} q^{(2d+1)^2} \\ &= q \sum_{d \geq 0} q^{8(d(d+1)/2)} \end{aligned}$$

since the operation ψ is given in terms of q-expansions by

$$\psi(q) = q^2.$$

As for j^{-1} we have

$$j^{-1} = \frac{\Delta}{c_4^3} \equiv \Delta \mod 2,$$

and

$$\Delta = q \prod (1 - q^n)^{24} \equiv q \prod (1 - q^{8n})^3 \mod 2.$$

The congruence

$$\psi(b) - b \equiv j^{-1} \mod 2$$

is then a consequence of the following special case of the Jacobi triple product identity

$$\sum_{d \geq 0} (-1)^d (2d+1) z^{d(d+1)/2} = \prod_{k \geq 0} (1 - z^k)^3.$$

□

At the prime 3 we can do the analogous thing with c_6 (which represents v_1^3)

$$c_6 = 1 - 504 \sum_{n \geq 1} \sigma_5(n) q^n.$$

The following identity holds

$$\frac{\log c_6}{9} \equiv j^{-1} \mod 3.$$

Oddly, the analogous result for $p > 3$ seems not to hold, though it is not really clear what "analogous" means.

REMARK 7.3. *Fred Diamond and Kevin Buzzard explained that both follow from known congruences for the Ramanujan τ function as described in Serre's course in arithmetic.*

Proof: Note that

$$b = -\frac{\log(c_4/u^4)}{\log g^4} \equiv -\sum_{n\geq 1} \sigma_3(n) q^n \mod 8,$$

and so

$$\psi(b) - b \equiv \sum_{n\geq 1} (\sigma_3(n) - \sigma_3(n/2)) q^n \mod 8,$$

where we adopt the convention that $\sigma_3(n) = 0$ if n is not an integer. Now it's easy to check that

$$\sigma_3(2n) \equiv \sigma_3(n) \mod 8,$$

so that

$$\psi(b) - b \equiv \sum_{n \mathrm{odd}} (\sigma_3(n)) q^n \mod 8.$$

On the other hand,

$$\frac{1}{j} = \frac{\Delta}{c_4^3} \equiv q \prod_{n\geq 1} (1-q^n)^{24} \equiv \sum \tau(n) q^n \mod 8.$$

It follows from a congruence of Ramanujan (see [**Sw**, page 4]) that

$$\tau(n) \equiv \begin{cases} 0 \mod 8 & \text{if } n \text{ is even} \\ \sigma_3(n) \mod 8 & \text{if } n \text{ is odd.} \end{cases}$$

This means that we have the congruence

$$\psi(b) - b \equiv j^{-1} \mod 8.$$

□

Returning to the prime 2 we will now build a canonical $K(1)$-local E_∞ ring spectrum mapping to any $K(1)$-local elliptic spectrum. Since ψ_g and θ commute, the element

$$\theta(f) \in \pi_0 E$$

is a modular function, and hence is can be written as a 2-adically convergent power series in j^{-1}. By Lemma 7.1 there is a 2-adically convergent power series h with

$$\theta(f) = h(f).$$

This gives a universal relation in the homotopy groups of any $K(1)$-local E_∞ ring spectrum.

Returning to T_ζ, let $b \in \pi_0\big(B \wedge T_\zeta\big)$ once again denote the universal "b," let $f \in \pi_0 T_\zeta$ be the unique element whose image in $\pi_0\big(B \wedge T_\zeta\big)$ is

$$f = \psi(b) - b,$$

and, finally, set

$$y = \theta(f) - h(f) \in \pi_0 T_\zeta.$$

This gives the vertical map in the following diagram. The requirement that it be a pushout defines the $K(1)$-local E_∞ ring spectrum M.

$$\begin{tikzcd} T\{S^0\} \arrow[r, "*"] \arrow[d, "y"'] & S^0 \arrow[d] \\ T_\zeta \arrow[r] & M \end{tikzcd}$$

by construction it is clear that there is a canonical map from M to any $K(1)$-local elliptic E_∞-ring spectrum.

Proposition 7.2. *The map*

$$K_* y : K_* T\{S^0\} \to K_* T_\zeta$$

is smooth of relative dimension 1. Therefore

$$K_* M = K_* T_\zeta \underset{K_* T\{S^0\}, y}{\otimes} \mathbb{Z}_2$$

and

$$\pi_* M = KO_*[j^{-1}].$$

Proof: The map $T\{S^0\} \to T_\zeta$ comes about as the composite

$$T\{S^0\} \xrightarrow{\theta(x)-h(x)} T\{S^0\} \xrightarrow{f} T_\zeta.$$

Since $K_* f$ is etale (Lemma 5.4), it suffices to show that the map $K_*(\theta(x) - h(x))$ is smooth of relative dimension 1. If we write out the rings, we are looking at the map of Frobenius algebras

$$\begin{aligned} \mathbb{Z}_2[y_0, y_1, \dots] &\to \mathbb{Z}_2[x_0, x_1, \dots] \\ y_0 &\mapsto x_1 - h(x_0). \end{aligned}$$

It is easy to check that $h(x_0) = x_0^2 + \dots$, so that our map is of the form

$$y_n \mapsto x_{n+1} \quad \text{mod decomposables.}$$

This probably shows that the map

$$\begin{aligned} \mathbb{Z}_2[y_0, y_1, \dots][x_0] &\to \mathbb{Z}_2[x_0, x_1, \dots] \\ y_0 &\mapsto x_1 - h(x_0) \dots \\ x_0 &\mapsto x_0 \end{aligned}$$

is an isomorphism.

Consider the increasing filtration by the x_i. One easily checks that the map is of the form

$$y_i \mapsto x_{i+1} + t(x_0, \dots, x_i),$$

which gives an easy inductive proof of surjectivity. Injectivity follows from this being an "isomorphism mod decomposables" since the intersection of the powers of the obvious "augmentation ideal" is zero. □

8. Homotopy groups of T_ζ

Recall that $g \in \mathbb{Z}_p^\times$ is chosen so that g projects to a topological generator of $\mathbb{Z}_p^\times/\mu$. Define $h \in \mathbb{Z}_p$ by

$$1 + h = \begin{cases} g^{p-1} & p > 2 \\ g^2 & p = 2. \end{cases}$$

Then for p odd, $h \equiv 0 \mod p$ and for $p = 2$, $h \equiv 0 \mod 8$. In both cases

$$(1+h)^{(-b)} = \sum \binom{-b}{n} h^n.$$

defines an element of $\mathbb{Z}_p[b]$ (recall that this ring is p-complete).

Define a multiplicative map

$$i : B_* \to B_* T_\zeta = B_* \otimes T\{b\},$$

by

$$i(v_1) = v_1 g^{(p-1)^{(-b)}} = v_1(1+h)^{(-b)},$$

for p odd, and

$$i(2v_1^2) = 2v_2^2 g^{2^{(-b)}} = 2v_1^2(1+h)^{(-b)}$$
$$i(v_1^4) = v_1^4 g^{4^{(-b)}} = v_1^4(1+h)^{(-2b)}.$$

At the prime 2, the image of the element $\eta \in \pi_1 KO$ is forced, since it is in the image of the homotopy groups of spheres. Note that in all cases, $i(v) \equiv v \mod 2$. This makes it easy to check that the map at the prime 2 is multiplicative (where one needs, perhaps, to worry about the elements in dimension $8k+1$ and $8k+2$, since they are in the image of the homotopy groups of spheres).

We have a surjective map

$$B_* T_\zeta \to B_* B$$

of Frobenius algebras with an action of $\mathbb{Z}_p^\times$. We are going to define an additive section which is compatible with the action of $\mathbb{Z}_p^\times$. For this we need to refer to the "big Witt vectors."

Consider the algebra

$$A = \mathbb{Z}[a_1, a_2, \dots]$$

and define, for each $n \geq 1$

$$w_n^B = \sum_{d|n} d a_d^{n/d}.$$

The algebra has a unique Hopf algebra structure for which the w_n^B are primitive. In fact, the group-valued functor represented by A is the functor

$$R \mapsto (1 + x\, R[\![x]\!])^\times.$$

The universal series in A is the series

$$a(x) = \prod (1 - a_n x^n),$$

and one easily checks that

$$x d \log a(x) = -\sum w_n^B x^n.$$

There is the following an analogue of Dwork's lemma for the big Witt vectors.

It is helpful to write $a = (a_1, \dots)$ and define

$$(1-x)^a = a(x).$$

The "group law" is then defined so that

$$(1-x)^{a+b} = (1-x)^a (1-x)^b.$$

Lemma 8.1. *Suppose that A is a ring with endomorphisms*

$$\phi_p : A \to A \qquad p \text{ prime}$$

satisfying

$$\phi_p(a) \equiv a^p \mod p.$$

Then, given a sequence $w_1^B, w_2^B, \dots \in A$, then the system

$$w_n^B = \sum_{d|n} d a_d^{n/d}$$

has a solution $a_1, a_2, \dots \in A$ if and only if for each prime p, each $k \geq 1$, and each $(m,p)=1$, one has

$$w_{p^k m}^B \equiv \phi_p(w_m^B) \mod p^k.$$

If A is torsion free then such a solution is unique.

The proof is very similar to the p-typical one.

One application of this is that there is a unique map from the ring of big Witt vectors to the ring of functions from $\mathbb{Z}$ to $\mathbb{Z}$ sending each w_n^B to the identity map. One easily checks that this maps sends the series $(1-x)^a$ to the map

$$n \mapsto (1-x)^n.$$

One useful consequence of this is that if we define elements $c_k \in A$

$$c_k = (-1)^k a_k + \text{monomials in } a_i,\ i < k$$

by writing

$$(1-x)^a = \sum c_k (-x)^k,$$

then under the map described above, c_k is sent to the binomial function

$$n \mapsto \binom{n}{k}.$$

In fact, the image of A in the ring of functions on $\mathbb{Z}$ is the ring of "numerical polynomials," and has basis these binomial functions.

The section we are describing results from lifting the "Pascal's triangle" identity

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

to A.

LEMMA 8.2. *There is a unique map of Hopf algebras*

$$T : A \to A$$

with the property that for all n,

$$T w_n^B = w_n^B + 1.$$

With respect to the map to the ring of functions on $\mathbb{Z}$, we have

$$T f(n) = f(n+1).$$

Finally, the following "Pascal's triangle" identity holds:

$$T c_k = c_k + c_{k-1}.$$

Proof: Let's define the elements $T a_n$ by writing

$$(1-x)(1-x)^a = \prod (1 - T a_n \, x^n).$$

The effect of T on the w_i is easily checked by taking the log of both sides. The rest of the lemma also follows easily. □

We now map the ring A to the ring $\mathbb{W}$.

LEMMA 8.3. *There is a unique map of Hopf algebras*

$$f : A \to \mathbb{W}$$

with

$$f(w^B_{p^k\, m}) = w_k \qquad (m,p) = 1.$$

This map is compatible with the action of T on $\mathbb{W}$ and on A.

Proof: We define ring homomorphisms

$$\phi_l : \mathbb{W} \to \mathbb{W} \qquad l \text{ prime},$$

by setting $\phi_l = 0$ if $l \neq p$, and by setting

$$\phi_p a_n = a_n^p.$$

The result then follows easily from Dwork's lemma. The compatibility with T follows easily from the effect on primitives. □

The entire point of all of this was to define the binomial classes in the Witt world. We abuse the heck out of the notation to do this.

DEFINITION 8.4. *Let $c_n \in \mathbb{W}$ be the image of the classes $c_n \in A$ under the map from Lemma 8.3.*

We can finally define our ψ_g-equivariant section.

LEMMA 8.5. *The map*

$$s : \mathrm{Hom}_{cts}(\mathbb{Z}_p^\times/\mu, \mathbb{Z}_p) \to \pi_0\big(B \wedge T_\zeta\big)$$

defined by

$$\begin{tikzcd}
\mathrm{Hom}_{cts}(\mathbb{Z}_p^\times/\mu, \mathbb{Z}_p) \arrow[r] \arrow[d, "g"'] & \pi_0\big(B \wedge T_\zeta\big) \arrow[d, "\approx"] \\
\mathrm{Hom}_{cts}(\mathbb{Z}_p, \mathbb{Z}_p) \arrow[r, "\binom{n}{k} \mapsto c_k"] & \mathbb{W}
\end{tikzcd}$$

is a ψ_g-equivariant map of co-algebras, where the binomial functions have the "Cartan" coproduct.

Proof: This follows from the above when one notes that under the vertical isomorphisms in the diagram, the map ψ_g is sent to T. □

Finally, we can return to our computation of the homotopy groups of T_ζ. We have defined a ring homomorphism

$$i : \pi_* B \otimes T\{f\} \to \pi_*\big(B \wedge T_\zeta\big)$$

which is compatible with the action of ψ, and whose image is fixed under the action of ψ_g.

LEMMA 8.6. *The sequence*

$$0 \to \pi_* B \otimes T\{f\} \to \pi_*\big(B \wedge T_\zeta\big) \xrightarrow{\psi_g - 1} \pi_*\big(B \wedge T_\zeta\big) \to 0$$

is exact.

Proof: Consider the *additive* map

$$\mathrm{Hom}_{\mathrm{cts}}(\mathbb{Z}_p^\times/\mu, \mathbb{Z}_p) \otimes \pi_* B \otimes T\{f\} \xrightarrow{\mu\circ s\otimes i} \pi_*\big(B \wedge T_\zeta\big). \tag{8.4}$$

We will see below that it is an isomorphism. Granting this, the lemma then reduces to showing that the sequence

$$0 \to \mathbb{Z}_p \xrightarrow{\text{constants}} \mathrm{Hom}_{\mathrm{cts}}(\mathbb{Z}_p^\times/\mu, \mathbb{Z}_p) \xrightarrow{\psi_g - 1} \mathrm{Hom}_{\mathrm{cts}}(\mathbb{Z}_p^\times/\mu, \mathbb{Z}_p) \to 0$$

is exact. But this is easy to check: just use the basis of binomial functions, and write down the map. □

We have used

LEMMA 8.7. *The map* (8.4) *is an isomorphism.*

Proof: This is the standard Milnor–Moore argument. We have an exact sequence of (p-complete) Hopf-algebras

$$T\{y\} \xrightarrow{\psi(b)-b} T\{b\} \xrightarrow{\pi} \mathrm{Hom}_{\mathrm{cts}}(\mathbb{Z}_p, \mathbb{Z}_p)$$

and a section

$$s : \mathrm{Hom}_{\mathrm{cts}}(\mathbb{Z}_p, \mathbb{Z}_p) \to T\{b\},$$

which is a map of co-algebras. It is formal to check that the maps

$$\mu \circ s \otimes \iota : \mathrm{Hom}_{\mathrm{cts}}(\mathbb{Z}_p, \mathbb{Z}_p) \otimes T\{y\} \to T\{b\}$$

and

$$T\{b\} \xrightarrow{\text{coproduct}} T\{b\} \otimes T\{b\} \xrightarrow{\pi\otimes(1-s\circ\pi)} \mathrm{Hom}_{\mathrm{cts}}(\mathbb{Z}_p, \mathbb{Z}_p) \otimes T\{y\}$$

are inverses. □

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GLOSSARY

1. Spectra, ring spectra

A_∞-ring spectrum: An A_∞-ring spectrum is a ring spectrum E whose multiplication is associative up to all higher homotopies. Classically this has been described as an action of an A_∞-operad on E; in more modern terms, an A_∞-ring spectrum can be realized by an associative $\mathbb{S}$-algebra, symmetric, or orthogonal ring spectrum.

E_∞-ring spectrum: An E_∞-ring spectrum is a ring spectrum E whose multiplication is associative and commutative up to all higher homotopies. Classically this has been described as an action of an E_∞-operad on E; in more modern terms, an E_∞-ring spectrum can be realized by an associative, commutative $\mathbb{S}$-algebra, symmetric, or orthogonal ring spectrum.

Even periodic cohomology theory; weakly even periodic theory: A multiplicative cohomology E theory is even periodic if $\pi_*(E)$ is concentrated in even degrees and $\pi_2(E)$ contains a unit. It is weakly periodic if $\pi_2(E)$ is an invertible $\pi_0(E)$ module, and $\pi_{2k}(E) \cong \pi_2(E)^{\otimes k}$ for all $k \in \mathbb{Z}$.

$\mathbb{S}$-module; (commutative) $\mathbb{S}$-algebra: The category of $\mathbb{S}$-modules is one of various point-set models of spectra with a strictly symmetric monoidal smash product, and historically the first. An $\mathbb{S}$-module is indexed not by the natural numbers, but by the finite-dimensional sub-vector spaces of a universe, i.e. an infinite-dimensional real inner product space. It has more structure, though: it is also a module over the sphere spectrum $\mathbb{S}$ with respect to a smash product of so-called $\mathbb{L}$-spectra, which in turn are algebras for a monad derived from the linear isometries operad. The category of $\mathbb{S}$-modules very easily generalizes to equivariant spectra, but its definition is arguably more complicated than symmetric or orthogonal spectra.

Symmetric ((commutative) ring) spectrum: Symmetric spectra are one of various point-set models of spectra with a strictly symmetric monoidal smash product. A symmetric spectrum consists of a sequence of spaces X_n with operations of the symmetric group Σ_n and maps $\sigma\colon X_n \wedge S^1 \to X_{n+1}$ such that the composite $X_n \wedge S^m \xrightarrow{\sigma^m} X_{n+m}$ is $\Sigma_n \times \Sigma_n$-equivariant. An advantage of the symmetric spectrum model of homotopy theory is its simplicity, however they are difficult to use in an equivariant context. A symmetric ring spectrum is a monoid with respect to the smash product; a symmetric commutative ring spectrum a commutative monoid.

Orthogonal ((commutative) ring) spectrum: Orthogonal spectra are one of various point-set models of spectra with a strictly symmetric monoidal smash product. It is a middle ground between $\mathbb{S}$-modules and symmetric spectra. Their definition is identical to symmetric spectra, but the symmetric groups Σ_n are replaced by the orthogonal groups $O(n)$. An advantage over symmetric spectra is the possibility to use this setup in an equivariant context. Orthogonal (commutative) ring spectra are definied in an analogous way to symmetric (commutative) ring spectra.

Units, space of, $GL_1(R)$: If R is an A_∞-ring spectrum, the space of units of R is defined to be the subspace $GL_1(R) \subset \Omega^\infty R$ which is the union of those components that represent an invertible element in the ring $\pi_0 R$.

Units, spectrum of, $gl_1(R)$: If R is an E_∞-ring spectrum, the space $GL_1(R)$ of units of R is naturally the zeroth space of a connective spectrum $gl_1(R)$, the spectrum of units of R.

2. Localization

Arithmetic square: Sullivan's arithmetic square is a way to recover a spectrum X from its p-completions and its $\mathbb{Q}$-localization. That is, X is equivalent to the homotopy limit of the diagram $\prod_p L_p X \to L_{\mathbb{Q}}(\prod_p L_p X) \leftarrow L_{\mathbb{Q}} X$. There is a similar arithmetic square when X is a space, in which X can be recovered as the homotopy limit of this diagram given the additional condition that X be nilpotent. More generally, the pullback of the arithmetic square is the $H\mathbb{Z}$-localization of a space.

Bousfield localization; E-localization: Bousfield localization of model categories is a homotopy-theoretic analogue of the usual localization of a category $\mathcal{C}$, with respect to a collection of morphisms I. The localization $C[I^{-1}]$ is the universal category receiving a functor from $\mathcal{C}$ and such that the image of the morphisms in I are all isomorphisms. Similarly, the Bousfield localization of a model category $\mathcal{C}$, with respect to a collection of morphisms I, is the universal model category receiving a left Quillen functor from $\mathcal{C}$, and such that image of the morphisms in I are all weak equivalences. For a given spectrum E, the Bousfield localization of spectra or spaces with respect to the collection of E-equivalences (that is, morphisms f such $E_*(f)$ is an isomorphism), is referred to as Bousfield localization with respect to E. The fibrant replacement in the resulting model category $X \to L_E X$ is then called the Bousfield localization of X with respect to E.

Bousfield-Kuhn functor: A functor Φ_n from spaces to spectra that factors $K(n)$-localization as $L_{K(n)} = \Phi_n \circ \Omega^\infty$. The existence of Φ_n implies that if two spectra have equivalent k-connected covers for some $k \in \mathbb{N}$, then their $K(n)$ localizations agree.

E-acyclic: A spectrum (or space) X is E-acyclic if the E-homology of X is zero. By definition, the Bousfield localization $L_E X$ of an E-acyclic spectrum X is contractible.

E-equivalence: A map of spectra (of spaces) $f\colon X \to Y$ is an E-equivalence if it induces an isomorphism in E-homology.

E-local: An E-local spectrum (or space) is a fibrant object of the Bousfield localized model category. A fibrant spectrum F is E-local iff for any E-acyclic spectrum X, the F-cohomology of X is zero. The Bousfield localization $L_E X$ of a spectrum X is initial (up to homotopy) amongst all E-local spectra receiving a map from X. Is is also terminal (up to homotopy) amongst all spectra Y equipped with an E-equivalence $X \to Y$.

E-nilpotent completion: For a ring spectrum E, the E-nilpotent completion of a spectrum X is the totalization of the cosimplicial spectrum $E^\bullet X$. In general, the E-based Adams spectral sequence for X converges to the homotopy of the E-nilpotent completion of X. If X is connective and $E = H\mathbb{F}_p$, then the E-nilpotent completion is the same as the p-completion.

Hasse square: Similar to the arithmetic square. It recovers the p-completion of an $E(2)$-local spectrum (e.g. and elliptic spectrum) from its $K(1)$- and $K(2)$-localizations, as the homotopy limit of the diagram $L_{K(2)}E \to L_{K(1)}L_{K(2)}E \leftarrow L_{K(1)}E$.

p-localization: The localization of spectra (or spaces) at a prime p is a particular case of Bousfield localization (in this case, with respect to the Eilenberg-MacLane spectrum $\mathrm{H}\mathbb{Z}_{(p)}$). If X is a spectrum (or a simply connected space), then $\pi_*(X_{(p)})$ is isomorphic to $\pi_* X \otimes \mathbb{Z}_{(p)}$.

p-completion: The p-completion – also called $\mathbb{Z}/p$-localization – of spectra (or spaces) at a prime p is a particular case of Bousfield localization (in this case, with respect to the mod p Moore spectrum $M(p)$). If X is a spectrum (or a simply connected space) with finitely presented homotopy groups, then $\pi_*(X_p)$ is isomorphic to $\pi_* X \otimes \mathbb{Z}_p$.

3. Orienations

$\widehat{A}$-genus: The $\widehat{A}$ genus is a $\mathbb{Q}$-valued genus of oriented manifolds given by $\int_M \widehat{A}(M)$, where $\widehat{A}(M) = \prod \frac{\sqrt{x_i}/2}{\sinh(\sqrt{x_i}/2)} \in H^*(M, \mathbb{Q})$ and the total Pontryagin class of M is factored as $\sum p_i(M) = \prod(1 + x_i)$ in some algebraic extension of $H^*(M, \mathbb{Q})$. If M is a spin manifold, then the $\widehat{A}$ is an integer. The $\widehat{A}$-genus of a spin manifold M is the image (mod torsoin) of $[M] \in \pi_n MSpin$ under the Atiyah-Bott-Shapiro orientation $MSpin \to ko$.

Complex oriented cohomology theory: Informally, a multiplicative cohomology theory E^* is called complex oriented if it admits a theory of Chern classes. More precisely, a complex orientation of E^* is a class $z \in E^2(\mathbb{C}P^\infty)$ whose restriction to $E^2(S^2) \cong E^0(S^0)$ along the standard inclusion $S^2 \cong \mathbb{C}P^1 \to \mathbb{C}P^\infty$ maps to the unit element. If E is 2-periodic, the orientation is often taken in degree 0 instead of degree 2. The multiplication on the topological group $\mathbb{C}P^\infty$ gives rise to a map

$$E^*[[z]] \cong E^*(\mathbb{C}P^\infty) \to E^*(\mathbb{C}P^\infty)\hat{\otimes}_{E_*} E^*(\mathbb{C}P^\infty) \cong E^*[[x, y]],$$

and the image of z under this map becomes a formal group law. Examples of complex oriented cohomology theories include singular cohomology, complex K-theory, MU, BP, $E(n)$, Morava K-theory, and several versions of elliptic cohomology.

Elliptic genus: Historically, the elliptic genus was introduced by Ochanine as a genus for oriented manifolds, taking values in the ring $\mathbb{Z}[1/2, \delta, \epsilon]$ (the ring of modular forms for $\Gamma_0(2)$), whose associated complex genus has logarithm $\int \frac{dt}{\sqrt{1-2\delta t^2+\epsilon t^4}}$. Landweber, Ravenel, and Stong subsequently showed that, after inverting Δ, the associated complex genus satisfies the criteria of the Landweber exact functor theorem, yielding a cohomology theory which they denoted *Ell*, and named "Elliptic Cohomology' (also known as $TMF_0(2)$, topological modular forms for $\Gamma_0(2)$). Today, a variety of genera associated with elliptic curves are called elliptic genera, and a variety of cohomology theories similarly associated to elliptic curves are referred to as elliptic (c.f. "Elliptic spectra").

Genus: A genus φ with values in a graded ring R_* is a map of graded rings $\varphi : MG_* \to R_*$, where MG_* denotes the bordism ring of manifolds with a G-structure on their stable normal bundle. In the case of $G = U$, the genus is called a complex genus.

Landweber exact; Landweber exact functor theorem: A p-typical formal group law over a ring R is classified by a map $BP_* \to R$. The formal group law is called Landweber exact if $X \mapsto BP_*(X) \otimes_{BP_*} R$ is a homology theory (the long exact sequence being the crucial point). Landweber's exact functor theorem characterizes Landweber exact formal group laws as those for which the images $(p, v_1, v_2, \ldots, v_n)$ form a regular sequence for all $n \in \mathbb{N}_0$.

Orientation of a cohomology theory: A G-orientation of a multiplicative cohomology theory E with respect to a given a topological structure group G over the infinite-dimensional orthogonal group O is a compatible orientation of all G-vector bundles on any space with respect to E. This gives rise to a theory of characteristic classes for G-bundles in E-cohomology. Such an orientation can be described as a map of ring spectra $MG \to E$, where MG denotes the Thom spectrum associated to $BG \to BO$. Important examples of orientations include the SO-orientation and U-orientations of singular cohomology giving rise to Pontryagin resp. Chern classes, the *Spin*-orientation of real K-theory, and the *String*-orientation of *TMF*. Note that a *complex oriented cohomology theory* is the same thing as a cohomology theory with a U-orientation.

Orientation of a vector bundle for a group G: Given a topological group G with a morphism $G \to O(n)$, a G-orientation on an n-dimensional vector bundle V on a space X is a homotopy lift of the map $X \to BO(n)$ classifying the bundle V to $X \to BG$. Of particular importance are the groups $U(n)$, $SU(n)$, $SU\langle 6\rangle(n)$, $SO(n)$, $Spin(n)$, $String(n)$. Similarly, a stable G-orientation (where $G \to O$ is a group homomorphism to the infinite orthogonal group) is a lift of the map $X \to BO$ classifying the stabilization of V to $X \to BG$.

Orientation of a vector bundle with respect to a cohomology theory: An orientation of a vector bundle V on a space X with respect to a multiplicative cohomology theory E is a class $u \in E^n(X^V)$, the Thom class, whose restriction to any fiber is a unit in $E^n(\mathbb{R}^n) = \pi_0 E$. Multiplication by u yields the Thom isomorphism $E_{*+n}(X^V) \cong E_*(X)$. The latter can be described as a homotopy equivalence $E \wedge X^V \to \Sigma^n E \wedge X$ between the E-homology of the Thom space X^V and the shifted E-homology of X, at the spectrum level.

String group: The string group $String(n)$ is a group model of the 6-connected cover of the orthogonal group $O(n)$. Unlike $Spin(n)$, it is necessarily infinite-dimensional.

σ-orientation: The σ-orientation is the $String$-orientation of tmf. It is a map of E_∞ ring spectra $MString \to tmf$ that realizes the Witten genus at the level of homotopy groups.

Witten genus: The Witten genus of a string manifold M is the image of $[M] \in \pi_n MString$ under the σ-orientation $MString \to tmf$. The q-expansion of the corresponding modular form can be computed as $\prod_{i\geq 0}(1-q^i)^n \cdot \int_M \widehat{A}(M) ch(\bigotimes_{i\geq 1} Sym_{q^i} T_{\mathbb{C}})$, where ch is the Chern character, $T_{\mathbb{C}}$ is the complexified tangent bundle of M, and, given a vector bundle E, the expression $Sym_t E$ stands for $\sum_{i\geq 0} t^i Sym^i E$, a vector bundle valued formal power series. At a physical level of rigor, the Witten genus can be described as the S^1-equivariant index of the Dirac operator on the free loop space of M.

4. Misc. tools in stable homotopy

Adams condition: A technical hypothesis on a homotopy associative ring spectrum E which guarantees the existence of a universal coefficient spectral sequence. The condition is that E is a filtered colimit of finite spectra E_α, such that the E-cohomology of the Spanier-Whitehead dual E^*DE_α is projective over E_*, and for every E-module M, the map

$$M^*DE_\alpha \to \mathrm{Hom}_{E_*}(E_*DE_\alpha, M_*)$$

is an isomorphism.

Adams spectral sequence: The Adams spectral sequence is a spectral sequence which computes the homotopy groups of the p-nilpotent completion of a spectrum X from its cohomology:

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathcal{A}_p}(H^*(X;\mathbb{F}_p),\mathbb{F}_p) \Longrightarrow \pi_{t-s}(X_{H\mathbb{F}_p}),$$

where $\mathcal{A}_p$ denotes the mod-p Steenrod algebra. The Adams spectral sequences converges conditionally in the sense of Boardman, implying that it converges strongly whenever the derived E_∞-term vanishes.

Adams-Novikov spectral sequence (generalized): A (generalized) Adams-Novikov spectral sequence is a variation of the classical Adams spectral sequence where mod-p cohomology is replaced by another (co-)homology theory. If E is a flat homotopy commutative ring spectrum which is either A_∞, or satisfies the Adams condition, then (E_*, E_*E) is a Hopf algebroid, and the E-based ANSS takes the form

$$E^2_{s,t} = \mathrm{Ext}^{s,t}_{E_*E}(E_*, E_*(X)) \Longrightarrow \pi_{t-s}(X_E),$$

where X_E denotes the E-completion of X and Ext is the derived functor of homomorphisms of E_*E-comodules. The Adams-Novikov spectral sequence can be seen as a Bousfield-Kan spectral sequence of the tower of spectra $\mathrm{Tot}[E^\bullet X]$ whose totalization is the E-completion.

Descent spectral sequence: The descent spectral sequence associated to a sheaf of spectra $\mathcal{F}$ over some space (or Grothendieck site) X computes the homotopy groups of $\mathcal{F}(X)$. Its E_2 page is the sheaf cohomology of X with coefficient in the homotopy sheaves of $\mathcal{F}$. In the case $X = \mathcal{M}_{ell}$ or

$\overline{\mathcal{M}}_{ell}$ and $\mathcal{F} = \mathcal{O}^{top}$, the structure sheaf for TMF, this spectral sequence is also called the elliptic spectral sequence.

Dyer-Lashof algebra: (a.k.a. the big Steenrod algebra). At a prime p, the algebra of Dyer-Lashof algebra is the algebra that acts on the homotopy groups of any E_∞-$\mathrm{H}\mathbb{F}_p$-ring spectrum.

Flat ring spectrum: A homotopy commutative ring spectrum E is said to be flat if E_*E is flat over E_*.

Goerss-Hopkins obstruction theory: Given a flat homotopy commutative ring spectrum E which satisfies the Adams condition, a homotopy commutative E-complete ring spectrum A, and a simplicial resolution $\mathcal{O}_\bullet$ of the commutative operad, the Goerss-Hopkins obstruction defines a sequence of obstructions in the Quillen cohomology of simplicial $E_*\mathcal{O}_\bullet$-algebras in E_*E-comodules to refining the homotopy commutative ring structure on E to an E_∞-structure. More generally, it gives a framework to compute the homotopy groups of the moduli space of E_∞-structures in terms of the aforementioned Quillen cohomology.

Homotopy limit, homotopy colimit: The right (resp. left) derived functors of limit (resp. colimit) on the category of diagrams in a model category, with respect to objectwise weak equivalence.

Hopkins-Miller theorem, Goerss-Hopkins-Miller theorem: The original Hopkins-Miller theorem states that Morava E-theory E_n admits an A_∞ structure, and a point-set level action by the Morava stabilizer group. Subsequently the A_∞ obstruction theory utilized by Hopkins and Miller was refined by Goerss and Hopkins to an E_∞ obstruction theory, resulting in an E_∞ version of the Hopkins-Miller theorem commonly referred to as the Goess-Hopkins-Miller theorem.

Hypercover: A hypercover is a generalization of the Čech nerve

$$\coprod U_\alpha \leftleftarrows \coprod U_\alpha \times_X U_\beta \Lleftarrow \coprod U_\alpha \times_X U_\beta \times_X U_\gamma \cdots$$

of a covering family $\{U_\alpha \to X\}$. It can be defined as a simpicial sheaf all of whose stalks are contractible Kan complexes.

Hyperdescent: A presheaf $\mathcal{F}$ satisfies hyperdescent if for any hypercover $U_\bullet$ of X, the value of $\mathcal{F}$ on X can be recovered as the homotopy limit of the cosimplicial object $\mathcal{F}(U_\bullet)$.

Injective model structure: For a model category $\mathcal{C}$ and a small category I, a model structure of the diagram category $\mathcal{C}^I$ can often (e.g. for combinatorial model categories $\mathcal{C}$) be defined by defining weak equivalences and cofibrations levelwise. This model structure is referred to as the injective model structure.

Jardine model structure: For a model category $\mathcal{C}$ and a Grothendiek site S, the Jardine model structure is a model structure on the category of $\mathcal{C}$-valued presheaves on S. It is the Bousfield localization of the injective model structure, where the weak equivalences are those morphisms that induce isomorphisms on homotopy sheaves. In this model structure, the fibrant objects satisfy hyperdescent.

Morava stabilizer group: The Morava stabilizer group $\mathbb{G}_n$ is the automorphism group of the unique formal group law of height n over $\overline{\mathbb{F}}_p$. It is a pro-finite group, and it is isomorphic to the maximal order in the central division algebra over $\mathbb{Q}_p$ with Hasse-invariant $1/n$.

Projective model structure: For a model category $\mathcal{C}$ and a small category I, a model structure of the diagram category $\mathcal{C}^I$ can often (e.g. for cofibrantly generated $\mathcal{C}$) be defined by defining weak equivalences and fibrations levelwise. This model structure is referred to as the projective model structure.

Quillen cohomology: Quillen cohomology is a generalization André-Quillen cohomology to model categories. If R is an object of a model category $\mathcal{C}$, and M is an abelian group in the overcategory $\mathcal{C}_{/R}$, then the Quillen cohomology of R with coefficients in M is given by the derived maps $\mathbb{R}\operatorname{Hom}_{\mathcal{C}_{/R}}(R, M)$. If $\mathcal{C}$ is the category of simplicial commutative rings, Quillen cohomology reduces to André-Quillen cohomology. If $\mathcal{C}$ is the category of spaces, Quillen cohomology is equivalent to usual singular cohomology (potentially with twisted coefficients).

θ-algebra: A θ-algebra is a $\mathbb{Z}_p$-algebra equipped with operators ψ^k for all $k \in \mathbb{Z}_p^\times$, ψ^p, and θ. The operators ψ^k and ψ^p are ring homomorphisms, and the operations ψ^k give a continuous action of the profinite group $\mathbb{Z}_p^\times$. The operator ψ^p is a lift of Frobenius, and the operator θ satisfies $\psi^p(x) = x^p + p\theta(x)$. The p-adic K-theory of an E_∞ algebra has the structure of a θ-algebra.

Type-n spectrum: A (usually finite) spectrum X is said to be of type n (at some given prime p) if its nth Morava K-theory, $K(n)_*(X)$, is nonzero while all smaller Morava K-theories are trivial. Every finite spectrum is of type n for some $0 \leq n < \infty$.

5. Important examples of spectra

***bo*, *bso*, *bspin*, *bstring* — connective covers of real K-theory:** The spectrum *bo* (also written *ko*) denotes connective real K-theory, i.e. the (-1)-connected cover of BO. The spectra *bso*, *bspin*, and *bstring* denote the covers of *bo* that are obtained by consecutively killing the next non-zero homotopy groups: π_1, π_2, and π_4.

BP — Brown–Peterson spectrum: When localized at a prime p, the complex cobordism spectrum MU splits as a wedge of spectra by the so-called Quillen idempotents. The summand containing the unit is called BP. It is a commutative ring spectrum itself (up to homotopy). The coefficient ring of BP is $BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \dots]$ with $|v_i| = 2p^i - 2$. The ring BP_* classifies *p-typical* formal group laws; over a torsion-free ring, a formal group law is p-typical if its logarithm series is of the form $x + \sum_{i=1}^\infty m_i x^{p^i}$. By a theorem of Cartier, any formal group law over a p-local ring is isomorphic to a p-typical one.

$BP\langle n\rangle$ — n-truncated Brown-Peterson spectrum: The notation $BP\langle n\rangle$ stands for any BP-module spectrum with $\pi_* BP\langle n\rangle \cong BP_*/(v_{n+1}, v_{n+2}, \dots)$. They are complex oriented and classify p-typical formal group laws of height bounded by n or infinity. The spectrum $BP\langle 1\rangle$ is the Adams summand of connective K-theory.

$E(n)$ — Johnson-Wilson spectrum: For a (fixed, not notated) prime p, the ring spectrum $E(n)$ is a Landweber exact spectrum with coefficients

$E(n)_* = \mathbb{Z}_{(p)}[v_1, \ldots, v_{n-1}, v_n, v_n^{-1}]$ with $|v_i| = 2p^i - 2$. Explicitly, $E(n)$-homology can be defined by $E(n)_*(X) = BP_*(X) \otimes_{BP_*} E(n)_*$ for the map $BP_* \to E(n)_*$ that sends v_i to the class of the same name for $i \leq n$ and to 0 for $i > n$. The Lubin-Tate spectrum E_n can be obtained from $E(n)$ by completing and performing a ring extension; those two spectra belong to the same Bousfield class.

E_n **or** $E(k, \Gamma)$ **— Morava E-theory, aka Lubin-Tate spectrum:** For k a field of characteristic p and Γ a 1-dimensional formal group of height n over k, the Morava E-theory spectrum $E(k, \Gamma)$ is an E_∞-ring spectrum such that $\pi_0 E(k, \Gamma)$ is isomorphic to the universal deformation ring $A(k, \Gamma) \cong W(k)[[u_1, \ldots, u_{n-1}]]$ constructed by Lubin-Tate. Morava E-theory is complex-orientable, even-periodic, and Landweber exact; its associated formal group is the universal deformation of Γ. In the case where k is the algebraic closure of the field with p elements, the Morava E-theory spectrum $E(\overline{\mathbb{F}}_p, \Gamma)$ is often abbreviated E_n. (cf entry on Universal deformation, and on Witt vectors.) Morava E-theory is closely related to $L_{K(2)} TMF$, and to the restriction of the sheaf $\mathcal{O}^{top}$ to the locus $\mathcal{M}_{ell}^{ss}$ of supersingular elliptic curves.

EO_n **— higher real K-theory:** The higher real K-theory spectra are the homotopy fixed points of the action on the nth Morava E-theory E_n of a maximal finite subgroup of the Morava stabilizer group. This construction is a consequence of the Hopkins-Miller theorem. At the primes 2 and 3, there is an equivalence $L_{K(2)} TMF \simeq EO_2$.

eo_2 **— p-local topological modular forms:** This is an older name for the p-localization of connective topological modular forms, *tmf* ($p = 2, 3$). Its notation is in analogy with BO and bo (BO being the periodization of the connective real K-theory spectrum bo).

ko**,** ku**,** KO**,** KU**,** K **— K-theory spectra:** The spectra ko and ku are the connective (i.e. (-1)-connected) covers of KO and $KU = K$. The spectrum KU of complex K-theory is 2-periodic with $(KU)_{2n} = \mathbb{Z} \times BU$ and $(KU)_{2n+1} \cong U$, with one structure map $U \to \Omega(\mathbb{Z} \times BU)$ being the standard equivalence and the other $\mathbb{Z} \times BU \to \Omega U$ given by the Bott periodicity theorem. If X is compact, the group $KU^0(X)$ can be geometrically interpreted as the Grothendieck group of complex vector bundles on X. The spectrum KO is the real equivalent of KU. It is 8-periodic with coefficients $KO_* = \mathbb{Z}[\eta, \mu, \sigma, \sigma^{-1}]$ with $|\eta| = 1$, $|\mu| = 4$, $|\sigma| = 8$, and $2\eta = \eta^3 = \mu\eta = 0$ and $\mu^2 = 4\sigma$.

$K(n)$ **— Morava K-theory:** The nth Morava K-theory at a prime p (p is not included in the notation). $K(n)$ is a complex orientable cohomology theory whose associated formal group is the height n Honda formal group. The coefficient ring $K(n)_* \cong \mathbb{F}_p[v_n^{\pm 1}]$ is a Laurent polynomial algebra on a single invertible generator in degree $2(p^n - 1)$. The generator $v_n \in \pi_{2(p^n-1)} K(n)$ is the image of an element with same name in $\pi_{2(p^n-1)} MU$.

MU **— complex bordism:** MU is the spectrum representing complex cobordism, the cobordism theory defined by manifolds with almost complex structures and bordisms between them, with compatible almost complex structures. MU is a Thom spectrum. MU_* is the Lazard ring, which carries the universal formal group law.

MP — **periodic complex bordism:** $MP = \bigvee_{n\in\mathbb{Z}} \Sigma^{2n} MU$ is the periodic version of MU, in which we add an invertible element of degree 2. This represents periodic complex cobordism.

$MU\langle 6\rangle$**:** The ring spectrum $MU\langle 6\rangle$ represent cobordism of manifolds with trivializations of the first and second Chern classes. As a spectrum, it can be constructed as the Thom spectrum over $BU\langle 6\rangle$, the 6-connected cover of BU. The spectrum *tmf* of topological modular forms is oriented with respect to $MU\langle 6\rangle$; this orientation corresponds to the unique cubical structure on every elliptic curve.

$MO\langle 8\rangle$ — **string cobordism:** The ring spectrum $MO\langle 8\rangle$ represents cobordism of string manifolds, which are spin manifolds equipped with a trivialization of $\frac{1}{2}p_1 \in H^4(M, \mathbb{Z})$, the latter being the pullback of the generator $\frac{1}{2}p_1 \in H^4(BSpin) = \mathbb{Z}$. As a spectrum, it can be constructed as the Thom spectrum over $BO\langle 8\rangle$, the 8-connected cover of BO. The spectrum *tmf* of topological modular forms is oriented with respect to $MO\langle 8\rangle$ (this refines the $MU\langle 6\rangle$-orientation of *tmf*), and that orientation is a topological incarnation of the Witten genus.

$X(n)$ — **Ravenel spectrum:** The spectra $X(n)$ are defined as Thom spectra of $\Omega SU(n) \to \Omega SU \to BU$, where the second map is the Bott isomorphism. They play a role in the proof of the nilpotence theorem and in constructing a complex oriented theory $A = TMF \wedge X(4)$ classifying elliptic curves with a parameter modulo degree 5, or equivalently, Weierstrass parameterized elliptic curves.

6. Commutative algebra

Adic rings: An adic Noetherian ring A is a topological Noetherian ring with a given ideal $I \subset A$, the ideal of definition, such that the map $A \to \varprojlim A/I^n$ is an isomorphism, and the topology on A is the I-adic topology. Adic rings are the local buildings blocks for formal schemes just as rings are the local buildings blocks for schemes. The functor that assings to an adic ring A, with ideal of definition I, the pro-ring $\{A/I^n\}$ embeds the category of adic rings, with continuous rings maps as morphisms, as a full subcategory of pro-rings: $\mathrm{Hom}_{\mathrm{Adic}}(A, B) \cong \varprojlim_m \varinjlim_n \mathrm{Hom}(A/I^n, B/J^m)$.

André-Quillen cohomology: The André-Quillen cohomology of a commutative ring R with coefficients in an R-module M, is defined as a derived functor of derivations of R into M, $\mathbb{R}\,\mathrm{Der}(R, M)$. This can also be expressed in terms of the cotangent complex of R, L_R, in that there is a natural equivalence $\mathbb{R}\,\mathrm{Hom}_R(L_R, M) \simeq \mathbb{R}\,\mathrm{Der}(R, M)$.

Cotangent complex: (Also called André-Quillen homology.) The cotangent complex L_R of a commutative ring R is the left derived functor of Kähler differentials, Ω^1. If the ring R is smooth, then there is an equivalence $L_R \simeq \Omega^1_R$.

Étale morphism: A map of commutative rings $S \to R$ is étale if it is flat and unramified. Equivalently, the map is étale if and only if the relative cotangent complex $L_{R|S}$ is trivial and R is a finitely presented S-algebra. The conditions “flat and unramified” essentially mean that the map is a local isomorphism (perhaps after base-change), and we should think of étale maps as finite covering maps.

Flat morphism; faithfully flat morphism: A map of commutative rings $S \to R$ is flat if R is a flat S-module, i.e., the functor $R \otimes_S (-)$ is exact. Localizations at ideals are flat. The map is faithfully flat if, additionally, the functor is conservative, meaning that $R \otimes_S M$ is zero if and only if M is zero. Adjoining roots of monic polynomials is a faithfully flat operation.

Kähler differentials: For R a commutative k-algebra, the R-module $\Omega^1_{R|k}$ of Kähler differentials (a.k.a. 1-forms) can be presented as the free R-module on symbols da, $a \in R$, subject to the relations that $d(ab) = adb + bda$ and $da = 0$ for $a \in k$. There is a natural isomorphism $\Omega^1_{R|k} \simeq I/I^2$, where I is the kernel of the multiplication map $R \otimes_k R \to R$. $\Omega^1_{R|k}$ has the important property that it corepresents the functor of derivations; there is a universal derivation $R \to \Omega^1_{R|k}$ that induces an isomorphism $\mathrm{Der}_k(R, M) \cong \mathrm{Hom}_R(\Omega^1_{R|k}, M)$ for any R-module M.

Witt vectors: The Witt vector functor associates to a ring R a new ring $W(R)$ which has R as a quotient and acts as a universal deformation in many cases. In particular, the Witt vectors of a finite field k of characteristic p are a complete local ring with residue field k; for instance, $W(\mathbb{F}_p) \cong \mathbb{Z}_p$.

7. Algebraic geometry, sheaves and stacks

Additive formal group $\mathbb{G}_a$: The additive formal group is the affine 1-dimensional formal group scheme $\mathbb{G}_a = \mathrm{Spf}(\mathbb{Z}[[t]])$ with comultiplication given by $t \mapsto t \otimes 1 + 1 \otimes t$. It is the completion at 0 of the additive group scheme denoted by the same symbol. Topologically, the additive formal group (over a field) arises as the formal group associated with singular cohomology with coefficients in that field.

Deligne-Mumford compactification: The Deligne-Mumford compactification of the stack $\mathcal{M}_g$ of smooth curves of genus g is the stack $\overline{\mathcal{M}}_g$ obtained by allowing certain singularities in those curves: those with at most nodal singularities, and finite automorphism group. The latter are known as *stable curves*.

Étale topology: This is the Grothendieck topology on the category of schemes in which a family $\{f_\alpha : X_\alpha \to X\}$ is covering if the maps f_α are étale and if $\coprod_\alpha X_\alpha(k) \to X(k)$ is surjective for every algebraically closed field k.

étale site (small): Given a scheme (or stack) X, the small étale site of X is the full subcategory of schemes (or stacks) over X whose reference map to X is étale, equipped with the étale Grothendieck topology.

Finite morphism: A morphism of stacks (or schemes) $X \to Y$ is finite if there is a étale cover $\mathrm{Spec}\, S \to Y$ such that $\mathrm{Spec}\, S \times_Y X = \mathrm{Spec}\, R$ is an affine scheme with R finitely generated as an S-module. A morphism is finite iff it is representable, affine, and proper.

Finite type: A morphism of stacks (or schemes) $X \to Y$ is of finite type if there is a cover $\mathrm{Spec}\, S \to Y$ and a cover $\mathrm{Spec}\, R \to \mathrm{Spec}\, S \times_Y X$ such that R is finitely generated as an S-algebra.

Formal spectrum Spf: The formal spectrum $\mathrm{Spf}\, A$ of an I-adic Noetherian ring A consists of a topological space with a sheaf of topological rings $(\mathrm{Spf}\, A, \mathcal{O})$. The topological space has points given by prime ideals that contain I, with generating opens $U_x \subset \mathrm{Spf}\, A$ the set of prime ideals not containing an element x of A. The value of the sheaf $\mathcal{O}$ on these opens is $\mathcal{O}(U_x) = A[x^{-1}]^{\wedge}_{I}$, the completion of the ring of fractions $A[x^{-1}]$ at the ideal $I[x^{-1}]$.

Formal scheme: A formal scheme is topological space with a sheaf of topological rings, that is locally equivalent to Spf A for some adic Noetherian ring A. The category of affine formal schemes is equivalent to the opposite category of adic Noetherian rings. Formal schemes often arise as the completions, or formal neighborhoods, of a subscheme $Y \subset X$ inside an ambient scheme, just as the completion of a Noetherian ring with respect to an ideal has the structure of an adic ring. Formal schemes embed as a full subcategory of ind-schemes by globalizing the functor that assigns to an adic ring the associated pro-ring.

Formal group: A formal group is a group object in the category of formal schemes. An affine formal group being the same as a cogroup in the category of adic rings, it is thus a certain type of topological Hopf algebra. A 1-dimensional (commutative) formal group over a ring R is a (commutative) formal group whose underlying formal schemes is equivalent to Spf $R[[t]]$ – sometimes this last condition only étale locally in R.

Formal group law: A 1-dimensional formal group law over a commutative ring R is a cocommutative cogroup structure on $R[[t]]$ in the category of adic R-algebras. I.e., it has a commutative comultiplication $R[[t]] \to R[[t]]\widehat{\otimes}_R R[[t]] \cong R[[x, y]]$. This comultiplication is determined by the formal power series that is the image of the element t, so formal group laws are often specified by this single formal power series. A 1-dimensional formal group law is equivalent to the data of a formal group G together with a specified isomorphism $G \cong \mathrm{Spf}\, R[[t]]$, i.e., a choice of coordinate t on G.

Grothendieck site: A category with a Grothendieck topology.

Grothendieck topology: A Grothendieck topology on a category $\mathcal{C}$ – sometimes also called a Grothendieck pretopology – consists of a distinguished class of families of morphisms $\{X_\alpha \to X\}$, called a covering families, subject to the following conditions: 1. base changing a covering family along any map $Y \to X$ should remain a covering family, 2. if $\{X_\alpha \to X\}$ is a covering family and for every α, $\{X_{\beta\alpha} \to X_\alpha\}$ is a covering family, then the family of composites $\{X_{\beta\alpha} \to X\}$ should also be a covering family. A primary example is the étale topology on the category of schemes.

Group scheme: A group scheme is group object in the category of schemes. Algebraic groups, such as GL_n or SL_n, form a particular class of group schemes. Elliptic curves and, more generally, abelian varieties are also group schemes.

Height: For a homomorphism of formal groups defined over a field of characteristic $p > 0$, say $f : G \to G'$, f may be factorized $G \xrightarrow{(-)^{p^n}} G \to G'$ through a p^n power map, i.e. the n-fold iteration of the Frobenius endomorphism of G. The height of the map f is the maximum n for which such a factorization exists, and is ∞ exactly when $f = 0$. The height of a formal group G is defined as the height of the multiplication by p-map $[p] : G \to G$.

Hopf algebroid: A (commutative) Hopf algebroid A is a cogroupoid object in the category of commutative rings, just as a commutative Hopf algebra is a cogroup object in the category of commutative rings. In other words, if A_0 is a commutative ring, the additional structure of a Hopf algebroid on A_0 is a choice of lift of the functor $\mathrm{Hom}(A_0, -) : \mathrm{Rings} \to \mathrm{Sets}$ to the category of groupoids: A_0 corepresents the objects of the groupoid, and the extra structure provided by the lift amounts to having another ring A_1 that corepresents the morphisms, together with the data of various maps between A_0 and A_1. For a ring spectrum E, the pair (E_*, E_*E) frequently defines a Hopf algebroid. Every Hopf algebroid $A = (A_0, A_1)$ has an associated stack, $\mathcal{M}_A$, defined by forcing the groupoid-valued functor to satisfy descent. In the example of $A = (MP_0, MP_0MP)$, where MP is periodic complex bordism, the associated stack $\mathcal{M}_A$ is the moduli stack of formal groups, $\mathcal{M}_{FG}$.

Hopf algebroid comodule: For a Hopf algebroid A, there is a notion of an A-comodule M, which is roughly a left A_1-comodule in the category of A_0-modules. The category of A-comodules is equivalent to the category of quasicoherent sheaves on the associated stack $\mathcal{M}_A$.

Hopf algebroid cohomology: The nth cohomology of a Hopf algebroid $A = (A_0, A_1)$ with coefficients in an A-comodule M is the nth derived functor of the functor that sends M to $\mathrm{Hom}_{(A_0,A_1)}(A_0, M)$.

Moduli stack of formal groups, $\mathcal{M}_{FG}$: The R-valued points of the stack $\mathcal{M}_{FG}$ are the groupoid of formal groups over R and their isomorphisms. The stack $\mathcal{M}_{FG}$ is the stack associated to the Lazard Hopf algebroid $(L, \Gamma) = (MP_0, MP_0MP)$. The invariant differential on a formal group defines a line bundle ω on $\mathcal{M}_{FG}$, and the E_2-term of the Adams-Novikov spectral sequence can be understood as the stack cohomology group $E_2^{s,2t} = H^s(\mathcal{M}_{FG}, \omega^t)$.

Moduli space of formal group laws, $\mathcal{M}_{FGL}$: A formal group law is a formal group along with a choice of coordinate. The moduli space of formal group laws is the scheme $\mathrm{Spec}(L)$, where $L = MP_0$ is the Lazard ring.

Multiplicative formal group $\mathbb{G}_m$: The multiplicative formal group is the affine 1-dimensional formal group scheme $\mathbb{G}_m = \mathrm{Spf}(\mathbb{Z}[[t]])$ with comultiplication given by $t \mapsto t \otimes 1 + 1 \otimes t + t \otimes t$. It is the completion at 1 of the multiplicative group scheme $\mathrm{Spec}(\mathbb{Z}[u, u^{-1}])$. Topologically, the multiplicative formal group arises as the formal group associated with complex K-theory.

p-divisible group: (Also called Barsotti-Tate groups.) An algebraic group G is a p-divisible group of height n if: the multiplication map $p^i : G \to G$ is surjective; the group $G[p^i] : \mathrm{Ker}(G \xrightarrow{p^i} G)$ is commutative, finite, and flat of rank p^{ni}; the natural map $\varinjlim G[p^i] \to G$ is an isomorphism. Every elliptic curve C defines an associated p-divisible group $C[p^\infty] := \varinjlim C[p^i]$, where $C[p^i]$ is the kernel of $p^i : C \to C$. The Serre-Tate theorem relates the deformation theory of the elliptic curve to that of its p-divisible group.

Proper morphism: A morphism of stacks $X \to Y$ is proper if the following two conditions hold. 1. (separated): For any complete discrete valuation ring V and fraction field K and any morphism $f : \mathrm{Spec}\, V \to Y$ with lifts

$g_1, g_2 : \operatorname{Spec} V \to X$ which are isomorphic when restricted to $\operatorname{Spec} K$, then the isomorphism can be extended to an isomorphism between g_1 and g_2. 2. (proper): for any map $\operatorname{Spec} V \to Y$ which lifts over $\operatorname{Spec} K$ to a map to X, there is a finite separable extension K' of K such that the lift extends to all of $\operatorname{Spec} V'$ where V' is the integral closure of V in K'.

Relative-dimension-zero morphism: A representable morphism $X \to Y$ of stacks (or schemes) has relative dimension zero if all of its fibers have Krull dimension 0.

Relative Frobenius: If S is a scheme over $\mathbb{F}_p$ and $X \to S$ a map of schemes, there are compatible absolute Frobenius maps $F\colon S \to S$ and $F\colon X \to X$ obtained locally by the pth power map. The relative Frobenius $F_{S/X}\colon X \to X(1) = X \times_S S$ is the corresponding map into the fiber product, which is taken using the projection $X \to S$ and the absolute Frobenius $S \to S$.

Representable morphism: A morphism of stacks $f : X \to Y$ is representable if for any map $\operatorname{Spec} R \to Y$, the fiber product $\operatorname{Spec} R \times_Y X$ is representable, i.e., it is equivalent to a scheme. It is called representable and affine, if in addition all the schemes $\operatorname{Spec} R \times_Y X$ are affine.

Stack: A stack is a groupoid-valued functor $\mathcal{F}$ on the category of commutative rings that satisfies descent. The descent property is a generalization of the sheaf property. It says that whenever $\operatorname{Spec}(R) \to \operatorname{Spec}(S)$ is an étale cover, the diagram

$$\mathcal{F}(R \otimes_S R \otimes_S R) \overset{\leftarrow}{\underset{\leftarrow}{\leftarrow}} \mathcal{F}(R \otimes_S R) \leftleftarrows \mathcal{F}(R) \leftarrow \mathcal{F}(S)$$

exhibits $\mathcal{F}(S)$ as a 2-categorical limit. A standard example is the stack BG, that assigns R the groupoid of principal G-bundles over $\operatorname{Spec}(R)$. Another standard example is the stack $\mathcal{M}_{ell}$, than assigns R the groupoid of elliptic curves over R.

Stack, Deligne-Mumford: A Deligne-Mumford stack is a stack that is locally affine in the étale topology. That is, it is a stack X for which there exists an étale cover $\operatorname{Spec} R \to X$ by an affine scheme. One often also imposes a quasicompactness condition. Deligne-Mumford stacks are the most gentle kinds of stacks and almost all notions that make sense for schemes also make sense for Deligne-Mumford stacks.

Universal deformation (of a formal group): Fix a perfect field k of positive characteristic p, e.g. k could be any finite or algebraically closed field, and a formal group Γ over k of finite height $1 \le n < \infty$. For every Artin local Ring R with residue field k, a deformation of Γ to R is a formal group $\tilde{\Gamma}$ over R together with an isomorphism $\tilde{(\Gamma)} \otimes_R k \simeq \Gamma$. Lubin and Tate determine the deformation theory of Γ by showing that there is a complete local ring R^{univ} such that for every R as above, the set of isomorphism calsses of defomations of Γ to R naturally biject with continous ring homomorphisms from R^{univ} to R. Even more, they prove that R^{univ} is non-canonically isomorphis to a power series ring $R^{univ} \simeq W(k)[[u_1, \ldots, u_{n-1}]]$ over the ring of Witt vectors of k.

8. Elliptic curves and their moduli

Discriminant: The discriminant of the elliptic curve $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$ is given by $\Delta = -b_2^2b_8 - 8b_4^3 - 27b_6^2 + 9b_2b_4b_6$, where $b_2 = a_1^2 + 4a_2$, $b_4 = 2a_4 + a_1a_3$, $b_6 = a_3^2 + 4a_6$, $b_8 = a_1^2a_6 + 4a_2a_6 - a_1a_3a_4 + a_2a_3^2 - a_4^2$. Over a field, the discriminant vanishes if and only if the elliptic curve is singular.

Elliptic curve: An elliptic curve C over a ring R is a smooth, projective curve of genus one together with a marked point, i.e., a map $\operatorname{Spec} R \to C$. An elliptic curve has a natural group structure, which can be completed to give a formal group over R.

Elliptic curve, generalized: An algebraic geometer would mean a curve locally given by a Weierstrass equation with one of Δ and c_4 not vanishing at any given point of the base; this is the notion origianlly coined by Deligne and Rapoport and these curves assemble into a proper and smooth Deligne-Mumford stack $\overline{\mathcal{M}}_{ell}$ of relative dimension one over $\mathbb{Z}$. In topology, we consider even more generalized curves: all those given locally by Weierstrass-equation. The resulting stack over $\mathbb{Z}$ is not a Deligne-Mumford stack anymore but an Artin-stack, because the additional curve $y^2 = x^3$ admits non-trivial infinitesimal automorphisms. This point is the only one carrying an additive formal group law which makes it of outstanding topological interest since it is the only point which "knows" about singular (mod p) cohomology.

Elliptic curve, ordinary: An ordinary curve is an elliptic curve over a field of characteristic $p > 0$ whose associated formal group law has height 1.

Elliptic curve, supersingular: A supersingular curve is an elliptic curve over a field of characteristic $p > 0$ whose associated formal group law has height 2.

Hasse invariant: For a fixed prime p, the Hasse invariant is a global section $H \in H^0(\overline{\mathcal{M}}_{ell} \otimes_{\mathbb{Z}} \mathbb{F}_p, \omega^{p-1})$, i.e. a mod p modular form of weight $p-1$. It admits a lift to characteristic 0 (as an Eisenstein series) exactly if $p \neq 2, 3$. The vanishing locus of H are the super-singular points (all with multiplicity 1).

Invariant differentials/canonical bundle: The canonical bundle ω over $\mathcal{M}_{ell}$ (or $\overline{\mathcal{M}}_{ell}$, or $\overline{\mathcal{M}}_{ell}^{+}$) is the sheaf of (translation invariant) relative differentials for the universal elliptic curve over $\mathcal{M}_{ell}$. The stalk of ω at an elliptic curve $C \in \mathcal{M}_{ell}$ is the 1-dimensional vector space of Kähler differentials on C. The sections of $\omega^{\otimes 2k}$ over $\overline{\mathcal{M}}_{ell}$ are modular forms of weight k (for odd n, the line bundle $\omega^{\otimes n}$ has no sections).

j-invariant: The j-invariant of the elliptic curve $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$ is given by $(b_2^2 - 24b_4)^3/\Delta$, where b_2, b_4, and Δ are as above. Over an algebraically closed field, the j-invariant is a complete isomorphism invariant of the elliptic curve. More geometrically, the j-invariant is a map $\overline{\mathcal{M}}_{ell} \to \mathbb{P}^1_{\mathbb{Z}}$ which exhibits the projective line as the coarse moduli space of the Deligne-Mumford compactification of $\mathcal{M}_{ell}$.

Level structure: A level structure on an elliptic curve C can refer to either: 1. (a $\Gamma(N)$-structure) an isomorphism between $(\mathbb{Z}/N)^2$ and the group $C[N]$ of N-torsion points of C. 2. (a $\Gamma_1(N)$-structure) an injective homomorphism $\mathbb{Z}/N \to C[N]$. 3. (a $\Gamma_0(N)$-structure) a choice of subgroup of $C[N]$ that is isomoprhic to $\mathbb{Z}/N$. Moduli spaces of elliptic curves with level structures provide examples of stacks over $\mathcal{M}_{ell}$ (or $\overline{\mathcal{M}}_{ell}$) on which one can evaluate $\mathcal{O}^{top}$, the structure sheaf for TMF.

Modular form; weight: A modular form of weight k is a section of $\omega^{\otimes 2k}$ over $\overline{\mathcal{M}}_{ell}$. When restricted to a formal neighborhood of the multiplicative curve $\mathbb{G}_m \in \overline{\mathcal{M}}_{ell}$, the canonical bundle ω trivializes, and one can identify a modular form with an element of $\mathbb{Z}[[q]]$.

Moduli stacks of elliptic curves:

$\mathcal{M}_{ell}$**:** Also denoted $\mathcal{M}_{1,1}$ in the algebraic geometry literature. The moduli stack of smooth elliptic curves.

$\mathcal{M}_{ell}^{ord}$**:** The substack of the moduli stack of elliptic curves $\mathcal{M}_{ell}$ over $\mathbb{F}_p$ consisting of ordinary elliptic curves, whose associated formal group has height one. The coarse moduli space $\mathcal{M}_{ell}^{ord}$ at a prime p is a disk with punctures corresponding to the number of supersingular elliptic curves at p.

$\mathcal{M}_{ell}^{ss}$**:** The substack of the moduli stack $\mathcal{M}_{ell}$ over $\mathbb{F}_p$ consisting of supersingular elliptic curves, whose associated formal group has height two. At a prime p, $\mathcal{M}_{ell}^{ss}$ is a disjoint union of stacks of the form $BG = */G$, where G is the group of automorphisms of a supersingular elliptic curve. Thus, the associated coarse moduli space is a disjoint union of points.

$\overline{\mathcal{M}}_{ell}$**:** The moduli stack of elliptic curves, possibly with nodal singularities. This is the Deligne-Mumford compactification of the moduli stack of smooth elliptic curves $\mathcal{M}_{ell}$.

$\mathcal{M}_{Weier}$**:** (also denoted $\overline{\mathcal{M}}_{ell}^{+}$) The moduli stack of Weierstrass elliptic curves, associated to the Weierstrass Hopf algebroid. Includes curves with both nodal and cuspidal singularities.

Serre-Tate theorem: The Serre-Tate theorem relates the deformation theory of an elliptic curve to that of its associated p-divisible group. As a consequence, if C is a supersingular elliptic curve, then the map $\mathcal{M}_{ell} \to \mathcal{M}_{FG}$ induces an isomorphism of the formal neighborhood of C in $\mathcal{M}_{ell}$ with the formal neighborhood of the associated formal group $\widehat{C}$ in $\mathcal{M}_{FG}$. More generally, there is a map from $\mathcal{M}_{ell}$ to the moduli stack of p-divisible groups, and this induces an isomorphism of a formal neighborhood of any elliptic curve C with the formal neighborhood of the point given by the associated p-divisible group $C[p^\infty]$. Remark that the p-divisible group $C[p^\infty]$ (which governs the deformation theory of C) is formal if and only if C is supersingular.

Weierstrass curve, Weierstrass form: A Weierstrass curve (or a curve in Weierstrass form) is an affine curve with a parametrization of the form

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6,$$

or its projective equivalent. A Weierstrass curve defines an elliptic curve if and only if its discriminant, a polynomial of the a_i, is invertible. Over a field, any elliptic curve can be expressed in Weierstrass form by the Riemann-Roch theorem. More generally, this is true Zariski-locally over any ring, i.e. if R is a ring and C/R is an elliptic curve, there exist elements $r_1, \ldots, r_n \in R$ such that $r_1 + \ldots + r_n = 1$ and for every i, the elliptic curve $C \otimes_R R[\frac{1}{r_i}]$ admits a Weiserstrass equation.

9. Spectra of topological modular forms

Elliptic spectrum: A triple (E, C, α) where E is a weakly even periodic ring spectrum, C is an elliptic curve over $\pi_0 E$, and $\alpha : \mathbb{G}_E \xrightarrow{\cong} \widehat{C}$ is an isomorphism between the formal group of E and the formal group of C.

Elliptic spectral sequence: This can refer to any one of the following spectral sequences: $H^q(\mathcal{M}_{ell}, \omega^p) \Rightarrow \pi_{2p-q}(TMF)$, $H^q(\overline{\mathcal{M}}_{ell}, \omega^p) \Rightarrow \pi_{2p-q}(Tmf)$, and $H^q(\overline{\mathcal{M}}^+_{ell}, \omega^p) \Rightarrow \pi_{2p-q}(tmf)$. The first two are examples of descent spectral sequences. The last one is the Adams-Novikov spectral sequence for tmf, and it is not a descent spectral sequence.

$\mathcal{O}^{top}$**, the structure sheaf for** TMF**:** is a sheaf of E_∞-ring spectra on the small étale site of $\overline{\mathcal{M}}_{ell}$, i.e., the site whose objects are stacks equipped with an étale map $\overline{\mathcal{M}}_{ell}$, and whose covering families are étale covers (strictly speaking, this is a 2-category, by it is actually equivalent to a 1-category). The corresponding sheaf over the stack $\overline{\mathcal{M}}^+_{ell}$ does not seem to exist, but if it existed its value on $\overline{\mathcal{M}}^+_{ell}$ would be tmf.

TMF**, periodic topological modular forms:** The spectrum TMF is the global sections of the sheaf $\mathcal{O}^{top}$ of E_∞-ring spectra over $\mathcal{M}_{ell}$. In other words, it is the value of $\mathcal{O}^{top}$ on $\mathcal{M}_{ell}$. Since $\mathcal{M}_{ell}$ is the open substack of $\overline{\mathcal{M}}_{ell}$ where the discriminant is invertible, there is a natural equivalence $TMF \simeq Tmf[\Delta^{-1}]$ (TMF is also equivalent to $tmf[\Delta^{-1}]$). Note that this is a slight abuse of notation: it is better to write $Tmf[(\Delta^{24})^{-1}]$ (and $tmf[(\Delta^{24})^{-1}]$), as only $\Delta^{24} \in \pi_{576}(Tmf)$ survives the descent spectral sequence.

Tmf**:** This is the global section spectrum of the sheaf $\mathcal{O}^{top}$ on $\overline{\mathcal{M}}_{ell}$. In positive degrees, its homotopy groups are rationally equivalent to the ring $\mathbb{Z}[c_4, c_6, \Delta]/(c_4^3 - c_6^2 - 1728\Delta)$ of classical modular forms. The negative homotopy groups of Tmf are related to those in positive degree by $\pi_{-n} Tmf \cong \mathrm{Free}(\pi_{n-21} Tmf) \oplus \mathrm{Tors}(\pi_{n-22} Tmf)$.

tmf**, connective topological modular forms:** This is the connective cover of the spectrum Tmf of global sections of $\mathcal{O}^{top}$ on $\overline{\mathcal{M}}_{ell}$. Its homotopy groups are rationally equivalent to the ring $\mathbb{Z}[c_4, c_6, \Delta]/(c_4^3 - c_6^2 - 1728\Delta)$ of classical modular forms. Apart from its $\mathbb{Z}$-free part, $\pi_*(tmf)$ also contains intricate patters of 2- and 3-torsion, that approximate rather well the $K(2)$-localizations of the sphere spectrum at those primes.

TMF**, localizations of:** The $K(1)$-localization $L_{K(1)} TMF$ is the spectrum of sections of $\mathcal{O}^{top}$ over the ordinary locus $\mathcal{M}^{ord}_{ell}$, while the $K(2)$-localization $L_{K(2)} TMF$ is the spectrum of sections of $\mathcal{O}^{top}$ over the supersingular locus $\mathcal{M}^{ss}_{ell}$. The latter is a product of various quotients of the Lubin-Tate spectra E_2, indexed by the finite set of isomorphism classes of supersingular elliptic curves. At the primes 2 and 3, there is only one supersingular elliptic curve, and $L_{K(2)} TMF \simeq EO_2$.