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SIGMOIDAL FUNCTIONS IN ANTENNA–FEEDER TECHNIQUE

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Abstract: In this note we prove precise upper and lower estimates for the Hausdorff distance of the shifted Heaviside function $h_a(t)$ by means of the sigmoid approximates the amplitude in the transmission waveform. The properties of the amplitude can greatly affect the choice of the waveform. Numerical examples, illustrating our results are given.

AMS Subject Classification: 41A46
Key Words: shifted Heviside function, transmission waveform, Hausdorff distance, upper and lower bounds

1. Introduction

In the case of line source the normalized antenna pattern $F(\omega)$ and its current distribution $i(s)$ are connected as it follows

$$F(\omega) = \int_{-\frac{L}{2\lambda}}^{\frac{L}{2\lambda}} i(s) e^{j2\pi\omega s}\, ds, \tag{1}$$

where L - antenna aperture length, λ - length of the wave, ω - frequency. If we

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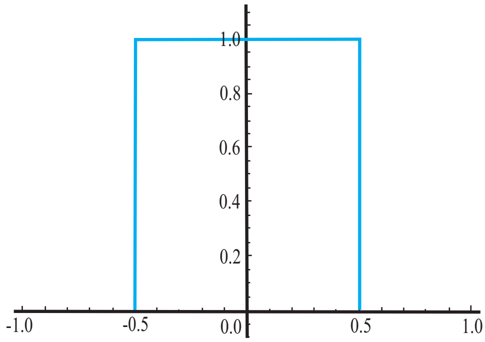

Figure 1: Signal of rectangular type.

suppose that $i(s) = 0$ for $s \notin [-\frac{L}{2\lambda}, \frac{L}{2\lambda}]$ then the equality (1) takes the form

$$F(\omega) = \int_{-\infty}^{\infty} i(s)e^{j2\pi\omega s}\, ds, \tag{2}$$

which is the well known Fourier transform of the function $i(s)$ and the inverse Fourier transform gives

$$i(s) = \int_{-\infty}^{\infty} F(\omega)e^{-j2\pi\omega s}\, d\omega. \tag{3}$$

The function $i(s)$ represents the current distribution in the pattern and it is computed from (3). In antenna-feeder technique most often occurred signals F are of rectangular type as it is shown on Fig.1. This is the reason the notion spectral density of rectangular signal to be used.

Example. If

$$F(\omega) = \begin{cases} 1, & |\omega| \leq c, \\ 0, & |\omega| > c. \end{cases} \tag{4}$$

we get from (3) that $i(s)$ is

$$i(s) = 2c\frac{sin(2\pi cs)}{2\pi cs}.$$

If we truncate the function $i(s)$ and define it as

$$i(s) := \begin{cases} i(s), & |s| \leq \frac{L}{2\lambda}, \\ 0, & |s| > \frac{L}{2\lambda} \end{cases}$$

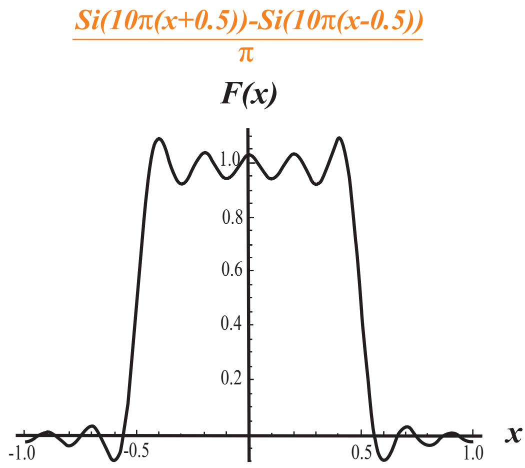

Figure 2: The Fourier analysis

then the expression (2) returns an approximation $\tilde{F}(\omega)$ to the function $F(\omega)$. It follows from (2) that F can be written approximately as

$$F(\omega) \approx \frac{1}{\pi}\left(Si\left(\frac{L}{\lambda}\pi(\omega+c)\right) - Si\left(\frac{L}{\lambda}\pi(\omega-c)\right)\right).$$

Let us fix $c = 0.5$, and $\frac{L}{\lambda} = 10$.

Let us note that Fourier transform is closely connected with Gibbs' phenomena. It is evident on Fig.2. One way to avoid this unpleasant effect is the approximation of function $F(\omega)$ in Hausdorff metric.

In [4] the following basic problems are considered – approximation of functions and point sets by algebraic and trigonometric polynomials in Hausdorff metric [1] as well as their applications in the field of antenna-feeder technique, analysis and synthesis of antenna patterns and filters, noise minimization by suitable approximation of impulse functions.

The advantage of Hausdorff metric is the removal of "Gibbs' phenomena".

For other results, see [6]–[14].

In antenna–feeder techniques most often occurred signal F (or, electric stage) are of rectangular type as it is shown on Fig. 1.

Let $\mathbf{\Delta}_\alpha^{\mathbf{M}}$ denote the point set

$$\Delta_\alpha^M = \{(x,y) : (-1 \le x \le 1,\ y = 0) \cup (x = \alpha,\ 0 \le y \le M)\}.$$

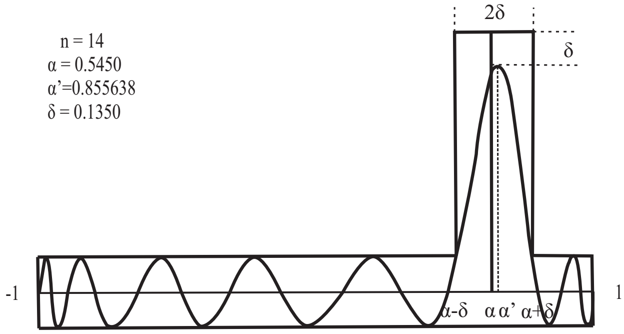

Figure 3: The polynomial of the best Hausdorff approximation of $\boldsymbol{\Delta}_\alpha^{\mathbf{M}}$.

$\boldsymbol{\Delta}_\alpha^{\mathbf{M}}$ is the completed graph of the function (it is called "**translated Δ function**")

$$\Delta_\alpha^M(x) = \begin{cases} 0, \ x \in [-1,\alpha) \cup (\alpha, 1] \\ M, \ x = \alpha, \ M > 0. \end{cases} \tag{5}$$

This function is one of the mostly used impulse functions in antenna-feeder technique.

Therefore it is important to find a convenient numerical method for determination of its n-degree algebraic polynomial of the best Hausdorff approximation.

In [3] the problem is considered for polynomial Hausdorff approximation of $\boldsymbol{\Delta}_\alpha^{\mathbf{M}}$ and the application of these polynomials for antenna synthesis.

The polynomial P satisfies the following differential equation

$$\frac{P'(x)}{\sqrt{\delta^2 - P^2(x)}} = n\frac{\alpha' - x}{\sqrt{(1-x^2)(\alpha-\delta-x)(\alpha+\delta-x)}}. \tag{6}$$

where the meanings of α, α' and δ can be seen on Fig.3.

The asymptotic estimation takes the form (see Kyurkchiev and Sendov [3]):

$$\delta = \sqrt{1-\alpha^2}\frac{\ln n}{n} + 0(n^{-1}), \ M > 0, \ \alpha \in (0,1).$$

Evidently the task of great difficulty is to determine the value of δ.

The polynomials $P_{n\alpha}$ play an important role in approximation of antenna factor $Q(t)$ for scanning of directed chart.

Let the transmission waveform is given by [5]

$$S(t) = C(t)e^{j2\pi b\xi\left(\frac{t}{t_r}\right)}, \tag{7}$$

where $C(t)$ is the amplitude function.

Definition 1. [5] Define the sigmoid function as

$$C(t) = p\left(\frac{1}{1+e^{-qt}} - \frac{1}{1+e^{-q(t-\lambda)}}\right), \tag{8}$$

where the parameter p is chosen such that $S(t)$ in (7) has unit energy, λ is the duration of the waveform, and q is a design parameter.

Definition 2. [1], [2] The Hausdorff distance (the H–distance) [1] $\rho(f,g)$ between two interval functions f, g on $\Omega \subseteq \mathbb{R}$, is the distance between their completed graphs $F(f)$ and $F(g)$ considered as closed subsets of $\Omega \times \mathbb{R}$. More precisely,

$$\rho(f,g) = \max\{\sup_{A\in F(f)} \inf_{B\in F(g)} ||A-B||, \sup_{B\in F(g)} \inf_{A\in F(f)} ||A-B||\}, \tag{9}$$

wherein $||.||$ is any norm in $\mathbb{R}^2$, e. g. the maximum norm $||(t,x)|| = \max\{|t|,|x|\}$; hence the distance between the points $A = (t_A, x_A)$, $B = (t_B, x_B)$ in $\mathbb{R}^2$ is $||A-B|| = \max(|t_A - t_B|, |x_A - x_B|)$.

Definition 3. The (basic) step function is:

$$h_a(t) = \begin{cases} 0, & \text{if} \quad t < a, \\ 1/2, & \text{if} \quad t = a, \\ 1, & \text{if} \quad t > a, \end{cases} \tag{10}$$

usually known as *shifted Heaviside function.*

2. Main Results

The properties of the amplitude $C(t)$ in (7) can greatly affect the choice of the waveform.

Important characteristics for analysis of such differential charts is the parameter μ - curvature of the differential chart in equisignal direction, defined by

$$\mu = \frac{\partial}{\partial t}\left\{\frac{|S(t)|}{|S_{max}|}\right\}_{t=const}.$$

As a rule the computation of μ is a difficult task.

Let the rectangular part (see Fig. 1) is defined on interval (a,b).

In this section we prove upper and lower estimates for the Hausdorff approximation of the shifted Heaviside function $h_a(t)$ (10) by means of the sigmoid (8).

Numerical examples, illustrating our results are given.

Let $p = \frac{1}{2\left(\frac{1}{2}-\frac{1}{1+e^{q\lambda}}\right)}$, i.e. $C(0) = \frac{1}{2}$.

The Hausdorff distance d between the $h_a(t)$ and the sigmoid (8) satisfies the relation:

$$C(d) = p\left(\frac{1}{1+e^{-qd}} - \frac{1}{1+e^{-q(d-\lambda)}}\right) = 1 - d. \tag{11}$$

The following Theorem gives upper and lower bounds for d

Theorem 2.1. *For the Hausdorff distance d between the $h_a(t)$ and the function (4) the following inequalities hold for $B > \frac{2}{5}e$ and $A + \frac{2}{5} < 0$:*

$$d_l = \frac{1}{2.5B} < d < \frac{\ln(2.5B)}{2.5B} = d_r \tag{12}$$

where

$$A = -1 + \frac{p}{2} - \frac{p}{1+e^{q\lambda}}$$

and

$$B = 1 + \frac{pq}{4} - \frac{e^{q\lambda}pq}{(1+e^{q\lambda})^2}.$$

*Proof.*We define the functions

$$F(d) = p\left(\frac{1}{1+e^{-qd}} - \frac{1}{1+e^{-q(d-\lambda)}}\right) - 1 + d \tag{13}$$

$$G(d) = A + Bd. \tag{14}$$

From Taylor expansion

$$F(d) - G(d) = O(d_0^2)$$

we see that $G(d)$ approximates $F(d)$ with $d \to 0$ as $O(d^2)$ (cf. Fig.4).

In addition $G'(d) > 0$ and

$$G(d_l) = A + \frac{2}{5} < 0$$

$$G(d_r) = A + \frac{2}{5}\ln(2.5B) > 0.$$

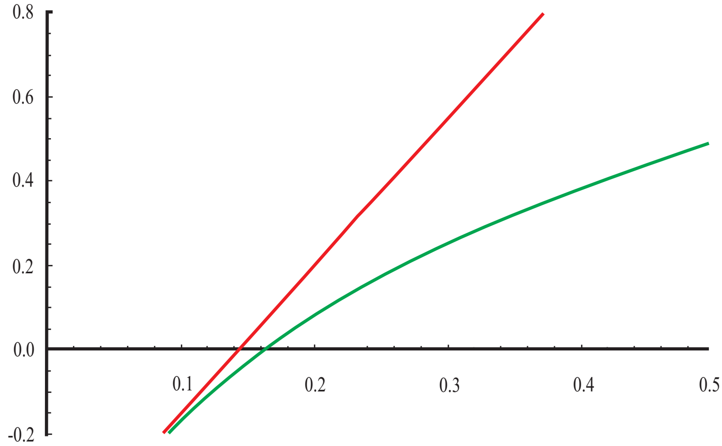

Figure 4: The functions $F(d)$ and $G(d)$.

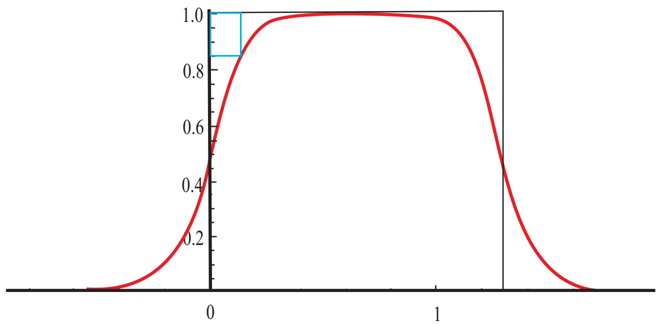

Figure 5: Approximation by sigmoidal function. Rectangular window (blue) and sigmoidal window (red) in the time–domain; $a = 0$, $b = 1.5$, $p = 1$, $q = 10$, $\lambda = 1.5$; Hausdorff distance $d = 0.163351$, $d_l = 0.114286$, $d_r = 0.247892$.

This completes the proof of the inequalities (12).

Let us point out that the Hausdorff distance is a natural measuring criteria for the approximation of bounded discontinuous functions.

Some computational examples using relations (12) are presented in Table 1.

The last column of Table 1 contains the values of d computed by solving the nonlinear equation (11).

The sigmoidal functions for various λ and q are visualized on Fig.5–Fig.8.

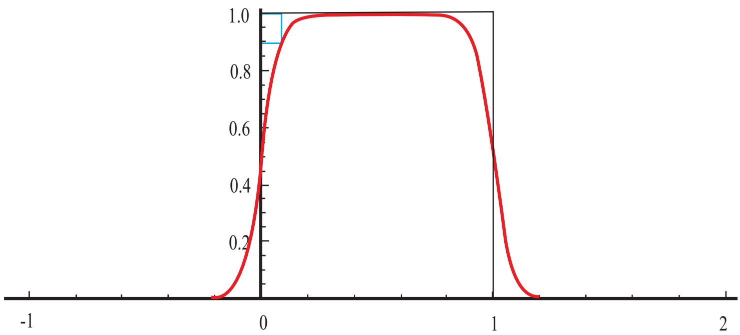

Figure 6: Approximation by sigmoidal function. Rectangular window (blue) and sigmoidal window (red) in the time–domain; $a = 0$, $b = 1$, $p = 1$, $q = 25$, $\lambda = 1$; Hausdorff distance $d = 0.0917149$, $d_l = 0.0551724$, $d_r = 0.159851$.

q	λ	d_l	d_r	d from (11)
10	1.5	0.114286	0.247892	0.163351
15	1.3	0.0842105	0.208374	0.127948
25	1	0.0551724	0.159851	0.0917149
35	1.1	0.0410256	0.131018	0.0727288
50	0.9	0.0296296	0.104266	0.0563598
60	0.8	0.025	0.092222	0.0493157
75	0.6	0.0202532	0.0789761	0.0417716

Table 1: Bounds for d computed by (12) for various λ and q.

3. Appendix

In this Section we construct a family of recurrence generated sigmoidal functions based on $C(t)$.

We consider the following family:

$$C_{i+1}(t) = p\left(\frac{1}{1+e^{-q(t-\frac{1}{2}+C_i(t))}} - \frac{1}{1+e^{-q(t-\frac{1}{2}-\lambda+C_i(t))}}\right), \quad i = 0, 1, 2, \ldots \quad (15)$$

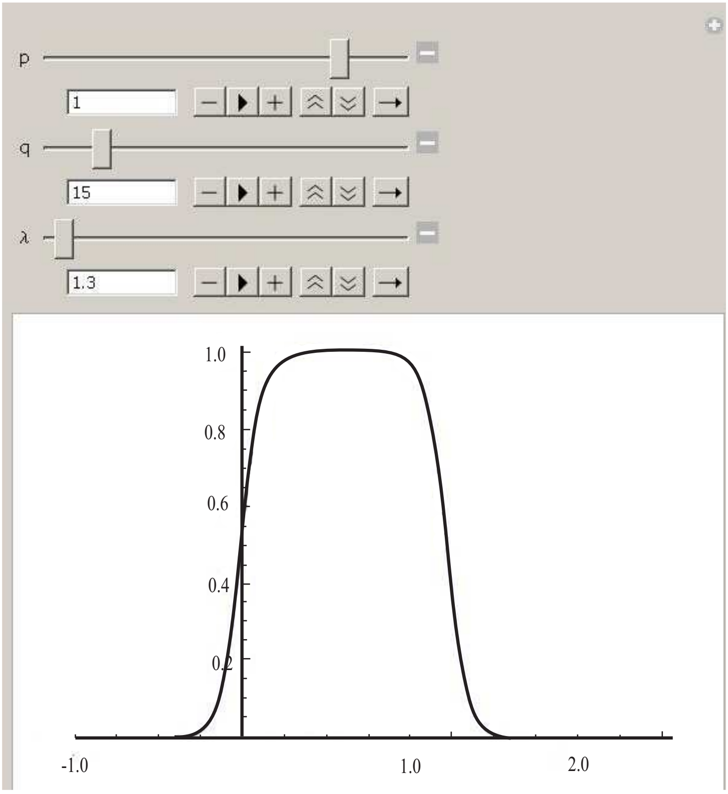

Figure 7: Software tools in CAS Mathematica. $a = 0$, $b = 1.3$, $p = 1$, $q = 15$, $\lambda = 1.3$; Hausdorff distance $d = 0.127948$, $d_l = 0.0842105$, $d_r = 0.208374$.

with

$$C_0(t) = p\left(\frac{1}{1+e^{-qt}} - \frac{1}{1+e^{-q(t-\lambda)}}\right);\ p = \frac{1}{2\left(\frac{1}{2} - \frac{1}{1+e^{q\lambda}}\right)};\quad C_0(0) = \frac{1}{2}. \quad (16)$$

Evidently, $C_{i+1}(0) = \frac{1}{2}$ for $i = 0, 1, 2, \ldots,$.

The recurrence generated functions $C_0(t)$, $C_1(t)$, $C_2(t)$ and $C_3(t)$ for $\lambda = 1.5$ and $q = 10$ are visualized on Fig.9.

We will explicitly say that the results have independent significance in the study of issues related to neural networks.

Some family of recurrence generated parametric activation functions is dis-

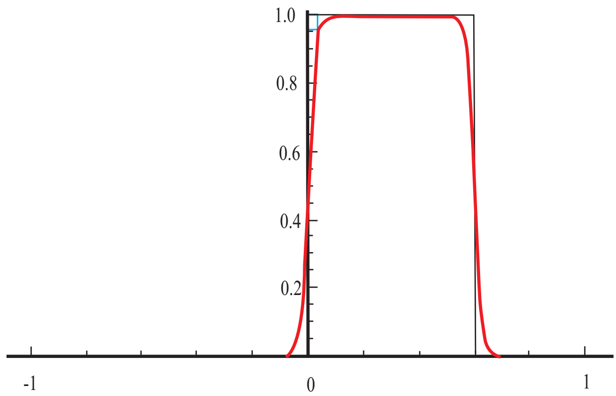

Figure 8: Approximation by sigmoidal function. Rectangular window (blue) and sigmoidal window (red) in the time–domain; $a = 0$, $b = 0.6$, $p = 1$, $q = 75$, $\lambda = 0.6$; Hausdorff distance $d = 0.0417716$, $d_l = 0.0202532$, $d_r = 0.0789761$.

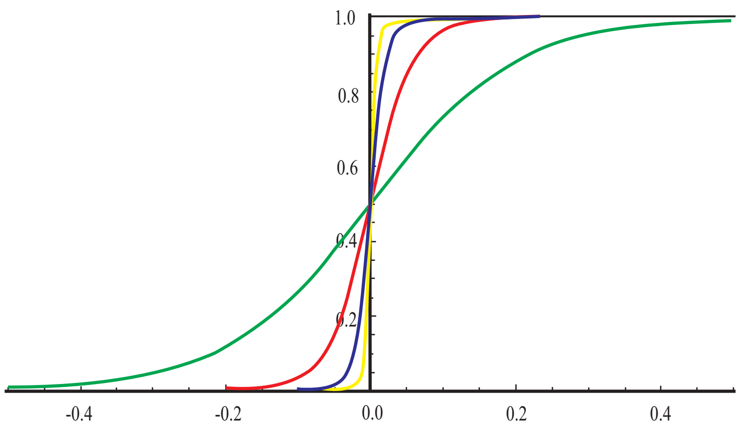

Figure 9: Approximation of the $h_a(t)$ by family (15) for $\lambda = 1.5$, $q = 10$; The graphics of recurrence generated functions: C_0 (green), C_1 (red), C_2 (blue) and C_3 (orange); Hausdorff distance: $d_0 = 0.163351$, $d_1 = 0.0743641$, $d_2 = 0.0375973$, $d_3 = 0.0201154$.

cussed from various approximation and modeling aspects in [15]–[19].

The computer realization for sigmoidal function studies will be adapted to a numerical methods study module as a part of the Distributed Platform for eLearning — DisPeL [20]–[23].

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