An Introduction to Surreal Numbers

Gretchen Grimm

May 8, 2012

Acknowledgements

I would like to thank Professor Barry Balof for his guidance through this project. I would also like to thank my classmates for helping me in the editing process.

Introduction

Mathematician John Horton Conway first invented surreal numbers, and Donald Knuth introduced them to the public in 1974 in his mathematical novelette, *Surreal Numbers: How Two Ex-Students Turned on to Pure Mathematics and Found Total Happiness.* In his novel, two college graduates, Bill and Alice, are stranded on a deserted island when they find an ancient stone with the mysterious writing, "In the beginning everything was void, and J.H.W.H. Conway began to create numbers. Conway said, 'Let there be two rules which bring forth all numbers large and small...'" [1]. Bill and Alice then spend their days working through the stone's inscriptions to develop the theory behind surreal numbers. In this paper, I formalize Bill and Alice's results, and investigate some further properties of the surreal numbers.

As well as Conway's two rules, the stone dictates that starting with nothing at all we can create the number zero, which has an empty left set and an empty right set. We call the day that zero is created on day 0. The next day, day 1, we can create more numbers by putting zero in the left set of one number and the right set of another, and we can then use these new numbers to create even more numbers on day 2, and so on. In this way, all surreal numbers are created; to each number, we can associate its "day of creation."

We begin our investigation of the surreals by looking at what numbers are created on the first few days, and then verifying some basic properties of all numbers. We then define addition and multiplication for surreal numbers, and conclude that they form a totally ordered field. Using addition and multiplication we can map out exactly what numbers are created on which days, and we discover that after a finite number of days we get all of the real numbers, plus more. We then look at some structures within the surreals, including integers and real numbers. We end the paper by introducing how surreal numbers can be used to analyze games, in particular the game of Hackenbush.

1 What Are Surreal Numbers?

1.1 Conway's Two Rules

As the stone states, every surreal number is created on a certain day and corresponds to two sets of numbers. For a surreal number, x, we write $x = \{X_L|X_R\}$ and call X_L and X_R the left and right set of x, respectively. In this section we will build the surreal numbers, which we denote S, from the ground up based on two axioms defined by Conway.

Axiom 1. *Every number corresponds to two sets of previously created numbers, such that no member of the left set is greater than or equal to any member of the right set [1].*

So if $x = \{X_L|X_R\}$ then for each $x_L \in X_L$ and $x_R \in X_R$, $x_L \ngeq x_R$. We write this as $X_L \ngeq X_R$.

Axiom 2. *One number is less than or equal to another number if and only if no member of the first number's left set is greater than or equal to the second number, and no member of the second number's right set is less than or equal to the first number [1].*

So $x = \{X_L|X_R\} \leq y = \{Y_L|Y_R\}$ if and only if $X_L \ngeq y$ and $x \ngeq Y_R$.

If every number corresponds to two sets of previously created numbers, then what do we start with on the zeroth day? If we let $\emptyset$ be the empty set then we define zero as $0 = \{\emptyset|\emptyset\}$, which we write more simply as

$$0 = \{|\}.$$

Clearly, 0 is consistent with Axiom 1 since the empty set contains no elements. We also note that $0 \leq 0$, since no element of the empty set is greater than or equal to 0, and 0 is not greater than or equal to any element of the empty set.

Now that we have defined 0, we can create some new numbers on the next day, day 1. We can put 0 in the left set of a number to get $x = \{0|\}$, and we can put 0 in the right set to get $y = \{|0\}$. Note that $\{0|0\}$ is not consistent with Axiom 1, since it has an element in its left set that is greater than or equal to an element in its right set, thus it is not a number.

How do x and y compare with 0? It is not hard to check that $0 \leq x$, since the left set of 0 is empty and the right set of x is empty, but the possibility that $x = 0$ seems rather silly. Using Axiom 2 we can show that $x \leq 0$ is impossible, for if $x \leq 0$ then we must have that $0 \ngeq 0$, a contradiction. Thus 0 is less than but not equal to x. Similarly we can see that $y \ngeq 0$. We call x one and we call y negative one. So on day 1 we get the numbers

$$1 = \{0|\} \text{ and } -1 = \{|0\}.$$

In this case, we know that 0 was created earlier than 1 and -1, but in general we can deduce that if some number x is in either the left or right set of a number y, then x must have been created before y.

Definition 1.1. *We say that a number x is **simpler** than a number y if x was created before y.*

Thus 0 is simpler than 1 and -1.

We can now put any of -1, 0, or 1, into the left of right set of a number to create 18 new numbers on day 2. The numbers created through day 2 are

$$\begin{array}{cccc}
0 = \{|\} & 1 = \{0|\} & -1 = \{|0\} & \{1|\} \\
\{-1|\} & \{0, 1|\} & \{0, -1|\} & \{-1, 1|\} \\
\{-1, 0, 1|\} & \{|1\} & \{| - 1\} & \{|0, 1\} \\
\{|0, -1\} & \{| - 1, 1\} & \{| - 1, 0, 1\} & \{-1|0\} \\
\{-1|1\} & \{-1|0, 1\} & \{0|1\} & \{-1, 0|1\}.
\end{array}$$

Using these numbers we will be able to create even more numbers on the third day, and so on. Since we will eventually have exhaustingly many numbers to work with we would like to check to see if some basic properties that hold for real numbers also hold for surreal numbers.

1.2 Basic Proofs and Properties

Suppose x, y, and z are surreal numbers such that $x \leq y$ and $y \leq z$. Is it necessarily true that $x \leq z$? That is, does the Transitive Law hold?

Theorem 1.1 (Transitive Law). *If $x \leq y$ and $y \leq z$, then $x \leq z$.*

Proof. We will use proof by contradiction. Let $x = \{X_L|X_R\}$, $y = \{Y_L|Y_R\}$, and $z = \{Z_L|Z_R\}$ be the earliest numbers created such that $x \leq y$, $y \leq z$ and $x \not\leq z$. Then either of the following two cases occur:

i.) There exists an $x_L \in X_L$ such that $x_L \geq z$, or

ii.) There exists some $z_R \in Z_R$ such that $z_R \leq x$.

In case (i.) we would have three numbers (y, z, x_L) that do not obey the Transitive Law, since $y \leq z$ and $z \leq x_L$ but $y \nleq x_L$. We note that x_L is a simpler number than x since $x_L \in X_L$. In case (ii.) we also have a different set of three numbers (z_R, x, y) that do not obey the Transitive Law. Similarly, z_R is a simpler number than z. In either case we have a simpler set of three numbers that do not obey the Transitive Law, which is a contradiction since we assumed that x, y, and z were the earliest numbers created that do not obey the Transitive Law. □

Here we see how induction plays a major role in proving properties of surreal numbers. Since the numbers are formed in a somewhat recursive manner we can use induction on the day that a number, or set of numbers, is created to trace properties of arbitrary numbers back to the earliest created numbers, of which we have a solid understanding. In the previous proof, we saw how for any set of numbers, $\{x, y, z\}$, that do not obey the Transitive Law, we get a set of simpler numbers, $\{x', y', z'\}$, that do not obey the Transitive Law. In other words, if we let $d(k)$ be the day that a number, k, was created on, then $d(x') + d(y') + d(z') < d(x) + d(y) + d(z)$. However, we cannot continue to find sets with smaller and smaller day sums that do not obey the Transitive Law since the three simplest numbers (0, 1, and -1) do (this fact is not hard to check and is left for the reader). Therefore, all numbers must obey the Transitive Law. In proofs to come, we will use this day sum argument implicitly.

Earlier we showed that $0 \leq 0$. We can use day induction to extend this result to all numbers.

Theorem 1.2. *$x \leq x$.*

Proof. Let x be the earliest created number such that $x \nleq x$. Then, by Axiom 2, either

i.) there exists some $x_L \in X_L$ such that $x \leq x_L$, or

ii.) there exists some $x_R \in X_R$ such that $x_R \leq x$.

Consider case (i.). Then, by definition we would have that $X_L \ngeq x_L$, which implies that $x_L \ngeq x_L$ since $x_L \in X_L$. But this is impossible since x_L was created before x. Similarly, we arrive at a contradiction if we assume case (ii.). □

How does x compare to its left and right set? Suppose $X_R \leq x$. Then, by Axiom 2, we must have that $X_R \nleq X_R$, which is a contradiction because $x_R \leq x_R$ for each $x_R \in X_R$. Thus $X_R \nleq x$. We can also show that $x \nleq X_L$ in the same way. We can show that every number lies between its left and right set.

Theorem 1.3. *If x is a number, then $X_L \leq x \leq X_R$.*

Proof. Suppose there is some $x_L \in X_L$ such that $x_L \nleq x$. Then either there is a number $x_{LL} \in X_{LL}$ such that $x_{LL} \geq x$, or there is a number $x_R \in X_R$ such that $x_R \leq x_L$. The latter case is clearly impossible by Axiom 1. By induction, we can assume that $x_{LL} \leq x_L$. Thus we have $x \leq x_{LL}$ and $x_{LL} \leq x_L$, meaning that $x \leq x_L$, by the Trasitive Law. But $x \leq x_L$ implies that $X_L \ngeq x_L$, which is a contradiction since $x_L \in X_L$. Thus no such x_L exists, which implies that $X_L \leq x$. We can show that $x \leq X_R$ in an analogous manner. □

We have shown that every number is related to itself, and we can also prove that all numbers are related to each other in some way.

Theorem 1.4. *For all numbers x and y, if $y \nleq x$ then $x \leq y$.*

Proof. We will use proof by contradiction. Suppose there exist numbers x and y such that $y \nleq x$ and $x \nleq y$. Then either

i.) $x_L \geq y$, or

ii.) $x \geq y_R$

must be true. Consider case (i.). We know that $x_L \leq x$, so by the Transitive Law $y \leq x$. But we have assumed that $y \nleq x$, thus we have a contradiction. Similarly, we reach a contradiction with case (ii.). □

Theorem 1.4 means that the the surreal numbers are totally ordered. With this knowledge we can say $x < y$ instead of $x \ngeq y$. Thus we have the following, stronger statement for how a number relates to its left and right sets.

Theorem 1.5. *If x is a number then $X_L < x < X_R$.*

Is it possible to have an x and y such that $x \leq y$ and $y \leq x$? Consider the number $\{-1|1\}$, created on the second day. If we compare this number with 0 we see that $\{-1|1\} \leq 0$ and $\{-1|1\} \geq 0$. Thus it behaves the same as 0, but it is not equal to 0 since its left and right sets are not identical to those of 0. In general, we say that a number x is *like* a number y, written $x \equiv y$, if $x \leq y$ and $y \leq x$. If the left and right sets of two number are like, then those two numbers are also like. That is, if $x_L \equiv y_L$ and $x_R \equiv y_R$, then $x \equiv y$ where $x = \{x_L|x_R\}$ and $y = \{y_L|y_R\}$. We verify that this is indeed true with the following theorem.

Theorem 1.6. *Let x and y be numbers whose left and right parts are "like" but not identical. Formally, let*

$$\begin{aligned} f_L &: X_L \to Y_L, \qquad & f_R &: X_R \to Y_R, \\ g_L &: Y_L \to X_L, \qquad & g_R &: Y_R \to X_R \end{aligned}$$

be functions such that $f_L(x_L) \equiv x_L$, $f_R(x_R) \equiv x_R$, $g_L(y_L) \equiv y_L$, and $g_R(y_R) \equiv y_R$. Then $x \equiv y$ [1].

Proof. We want to show that $x \leq y$ and $y \leq x$. For $x \leq y$ to hold we need to show that $X_L < y$ and $x < Y_R$. Consider $x_L \in X_L$. We have that $x_L \equiv f_L(x_L)$. Note that $f_L(x_L) \in Y_L$. Since y is a number $Y_L < y$, meaning that $f_L(x_L) < y$. Thus $x_L < y$, and since x_L is an arbitrary element of X_L we have that $X_L < y$. Now consider $y_R \in Y_R$. We have that $y_R \equiv g_R(y_R)$. Since $g_R(y_R) \in X_R$ and x is a number we have that $x < g_R(y_R)$. Thus $x < y_R$, and since y_R is an arbitrary element of Y_R we can conclude that $x < Y_R$. Therefore $x \leq y$. We can show that $y \leq x$ in an analagous manner. □

2 Developing The Pattern of the Surreal Numbers

If $\{-1|1\} \equiv 0$, then are any of the other 17 numbers created on the second day like 0, 1, or -1? This leads us to explore which of these numbers are truly unique, which will give us an idea of how the surreal numbers form in general.

2.1 Uniqueness

Recall the twenty numbers that we have by the second day.

$$\begin{array}{cccc}
0 = \{|\} & 1 = \{0|\} & -1 = \{|0\} & \{1|\} \\
\{-1|\} & \{0,1|\} & \{-1,0|\} & \{-1,1|\} \\
\{-1,0,1|\} & \{|1\} & \{|-1\} & \{|0,1\} \\
\{|-1,0\} & \{|-1,1\} & \{|-1,0,1\} & \{-1|0\} \\
\{-1|1\} & \{-1|0,1\} & \{0|1\} & \{-1,0|1\}.
\end{array}$$

It would be rather tedious to check every number created on the second day to see if it is both less than or equal to and greater than or equal to 0, 1, or -1 (and this process would get even more tedious by day 3, day 4, etc.). Thus we would like to be able to determine if a given number is like another in a more efficient way. Consider a number $x = \{X_L|X_R\}$. When we compare x to other numbers, we are really only worried about the greatest element of X_L and the smallest element of X_R. We notice that adding numbers to X_L that are smaller than x and adding numbers to X_R that are greater than x do not effectively change x. Thus if Y_L and Y_R are any sets such that $Y_L < x < Y_R$, we would expect that x is like z where $z = \{X_L \cup Y_L|X_R \cup Y_R\}$. For example, $\{1|\} \equiv \{0,1|\} \equiv \{-1,1|\} \equiv \{-1,0,1|\}$. Thich leads us to the Simplicity Theorem [2].

Theorem 2.1 (The Simplicity Theorem). *Given any number $y = \{Y_L|Y_R\}$, if x is the first number created with the property that $Y_L < x$ and $x < Y_R$, then $x \equiv y$.*

Proof. Suppose y is a number and x is the simplest number such that $Y_L < x$ and $x < Y_R$. Consider the number $z = \{X_L \cup Y_L|X_R \cup Y_R\}$. We first prove that $x \equiv z$ by showing that $z \leq x$ and $x \leq z$. To prove $z \leq x$ we need that $X_L \cup Y_L < x$ and $z < X_R$. We know that $X_L < x$ and $Y_L < x$, thus $X_L \cup Y_L < x$. We defined z such that $z < X_R \cup Y_R$, so it must be that $z < X_R$ and thus $z \leq x$. Similarly, we can show that $x \leq z$. Thus $x \equiv z$.

We now prove that $y \equiv z$, and it follows from the Transitive Law that $x \equiv y$. To show that $y \leq z$ we need that $Y_L < z$ and $y < Z_R$. From our hypothesis $Y_L < x$, and we have just shown that $x \leq z$. It follows from the Transitive Law that $Y_L < z$. Again, from our hypothesis $y < Y_R$, and $Y_R \leq Z_R$ since $Y_R \subseteq Z_R$. Thus $y < Z_R$, which means that $y \leq z$. We can show that $z \leq y$ in an analagous manner. □

Using the Simplicity Theorem we can see that $\{-1,0|1\} \equiv \{0|1\}$, $\{|-1,0,1\} \equiv \{|-1,0\} \equiv \{|-1\}$, and so on. The only numbers created by the second day that are essentially different, listed in order, are

$$\{|-1\} < -1 < \{-1|0\} < 0 < \{0|1\} < 1 < \{1|\}.$$

Notice that there seems to be a pattern. On day 0 we had

$$0 = \{|\},$$

and on day 1 the numbers

$$\{|0\} \text{ and } \{0|\}$$

were created. We called these numbers -1 and 1, respecively, so by day 1 we had the numbers

$$-1 < 0 < 1.$$

On day 2 the new numbers that were formed were

$$\{|-1\},\ \{-1|0\},\ \{0|1\},\ \text{and } \{1|\}.$$

In general, the pattern of how numbers form are mapped out in the following theorem.

Theorem 2.2. *Suppose that the different numbers at the end of n days are*

$$x_1 < x_2 < \cdots < x_m.$$

Then the only new numbers that will be created on the $(n+1)$st day are

$$\{|x_1\}, \{x_1|x_2\}, \ldots, \{x_{m-1}|x_m\}, \{x_m|\}.$$

Proof. Using the Simplicity Theorem we know that if either or both of X_L and X_R are sets with more than one element, then $\{X_L|X_R\} \equiv \{\max\{X_L\}|\min\{X_R\}\}$. Thus we want to show that if a number created on the $(n+1)$st day is not of the form $\{|x_1\}$, $\{x_m|\}$, or $\{x_i|x_{i+1}\}$, where $i = 1, 2, \ldots m-1$, then it is like a number already created by the nth day. There are four cases to consider.

Case 1 Consider the number $\{x_{i-1}|x_{i+1}\}$, where $i = 2 \ldots m-1$. If $x_i = \{x_{iL}|x_{iR}\}$ then $x_{iL} \leq x_{i-1}$ and $x_{iR} \geq x_{i+1}$ since $x_{i-1} < x_i < x_{i+1}$. So, by the Simplicity Theorem, $x_i \equiv \{x_{iL}, x_{i-1}|x_{iR}, x_{i+1}\}$. Similarly, $\{x_{i-1}|x_{i+1}\} \equiv \{x_{iL}, x_{i-1}|x_{iR}, x_{i+1}\}$. Thus, by the Transitive Law, $x_i \equiv \{x_{i-1}|x_{i+1}\}$.

Case 2 Consider the number $\{x_{i-1}|x_{j+1}\}$, where $i < j$. If there is a number $x = \{x_L|x_R\}$ such that $x_L \leq x_{i-1}$ and $x_R \geq x_{j+1}$, then $\{x_{i-1}|x_{j+1}\} \equiv x$ by the same reasoning as in case (1). If we let x be the first number created of $x_i, x_{i+1}, \ldots, x_j$, then its left and right sets cannot include any of the other numbers in this list, which means it satisfies our requirements.

Case 3 Consider the number $\{|x_{j+1}\}$. Then, as before, $\{|x_{j+1}\} \equiv x$ where x is the first number created of $x_1, x_2, \ldots, x_j$.

Case 4 Finally, if we have $\{x_{i-1}|\}$, then $\{x_{i-1}|\} \equiv x$ where x is the first number created of $x_i, x_{i+1}, \ldots, x_m$.

□

So from Theorem 2.2 we essentially know all numbers that will be created through a finite number of days. Note that by day 0 we have $1 = 2^1 - 1$ numbers, by day 1 we have $3 = 2^2 - 1$ numbers, and by day 2 we have $7 = 2^3 - 1$ numbers. Suppose we have $2^{n+1} - 1$ numbers after the nth day. Then the next day we will get $2^{n+1} - 1 + 1$ numbers, which means that by the $(n + 1)$th day there will be $2 \cdot 2^{n+1} - 1 = 2^{n+2} - 1$ numbers. Thus the pattern holds for all finite n.

Theorem 2.3. *If n is a finite integer, then by the nth day, $2^{n+1} - 1$ numbers will have been created.*

2.2 Pseudo-Numbers

What would happen if we had something that resembled a surreal number, but did not satisfy Axiom 1? For example $\{0| -1\}$ does not obey Axiom 1 since $0 > -1$, and neither does $\{0|0\}$ since $0 \geq 0$. We call these numbers *pseudo-numbers*. We can verify relationships involving psuedo-numbers using Axiom 2. For example, we can see that $\{0|0\} \leq 1$ since $0 < 1$, and because the right set of $1 = \{0|\}$ is empty.

Now consider the pseudo-number $\{1|0\}$. Using Axiom 2 we see that $\{1|0\} \not\leq 0$, since $1 > 0$, and $0 \not\leq \{1|0\}$, since $0 \leq 0$. Thus 0 and $\{1|0\}$ are not related at all. Here we see that unlike numbers, pseudo-numbers are not completely ordered.

Although pseudo-numbers do not behave quite as nicely as numbers, we can still verify certain properties for them. Since the Transitive Law and Theorem 1.2 do not rely on Axiom 1, they hold for pseudo-numbers as well as numbers.

Note that for the rest of the paper when we are talking about a number, we mean that it obeys Axiom 1. If we are talking about pseudo-numbers then we will explicitly say so.

2.3 Addition

We define the negative of a surreal number, x, as follows.

Definition 2.1. *If $x = \{X_L|X_R\}$, then $-x = \{-X_R| - X_L\}$.*

Theorem 2.4. $-(-x) = x$.

Proof. Using induction we see that for $x = \{X_L|X_R\}$,

$$\begin{aligned} -(-x) &= -(\{-X_R| - X_L\}) \\ &= \{-(-X_L)| - (-X_R)\} \end{aligned}$$

$$
\begin{aligned}
&= \{X_L|X_R\} \\
&= x.
\end{aligned}
$$

□

To check that the negative of a number is well defined we need to make sure that if x is a number then $-x$ is. That is, if $X_L < X_R$ then we must have that $-X_R < -X_L$. Note that this fact follows directly from the following theorem.

Theorem 2.5. *$x \leq y$ if and only if $-y \leq -x$.*

Proof. By our definition of negation, for any number, x, $-X_L = (-X)_R$ and $-X_R = (-X)_L$. Suppose that $x \leq y$. By Axiom 2, $X_L < y$ and $x < Y_R$ and thus $-(-X)_R < y$ and $x < -(-Y)_L$. Since both pairs of numbers $\{(-X)_R, y\}$ and $\{x, (-Y)_L\}$ have a smaller day sum than $\{x, y\}$, we can assume that $-y < (-X)_R$ and $(-Y)_L < -x$ by induction. Therefore, by Axiom 2, $-y \leq -x$. We can prove the converse in the same manner. □

Using our definition for negation we can see that

$$-\{1|\} = \{|-1\} \qquad \text{and} \qquad -\{0|1\} = \{-1|0\}.$$

Thus we can re-write the seven numbers created by the second day as

$$-a < -1 < -b < 0 < b < 1 < a,$$

where $a = \{1|\}$ and $b = \{0|1\}$.

How would we define addition for surreal numbers? In Knuth's book the stone states, "The left set of the sum of two numbers shall be the sums of all left parts of each number with the other; and in like manner the right set shall be from the right parts, each according to its kind," [1]. In symbols, we define addition between two surreal numbers x and y as follows.

Definition 2.2. $x + y = \{X_L + y, x + Y_L | X_R + y, x + Y_R\}$.

To check that addition is well defined we need to make sure that if x and y are numbers then $(x + y)_L < (x + y)_R$. The following inequalities must hold:

$$X_L + y < X_R + y$$

$$X_L + y < x + Y_R$$

$$x + Y_L < X_R + y$$

$$x + Y_L < x + Y_R.$$

All of the inequalities above would hold if addition is transitive. For example in proving the second inequality we could suppose that $X_L + y \geq x + Y_R$. Since $X_L < x$ we could use that addition is transitive to deduce that $X_L + y < x + y$. Similarly, since $y < Y_R$, $x + y < x + Y_R$. Thus, by the transitive law we have that $X_L + y < x + Y_R$, a contradiction. Note that even though we don't know yet that $X_L + y$, $x + y$, and $x + Y_R$ are numbers, we know that even pseudo-numbers obey the transitive law. We would like to prove the following theorem.

Theorem 2.6 (Transitive Law for Addition)**.** *$x \leq y$ if and only if $x + z \leq y + z$.*

Proof. We assume that addition is commutative, which we will prove shortly. We begin the proof as introduced by Knuth in *Surreal Numbers*. let $\mathrm{I}(x, y, z)$ denote the statement: if $x \leq y$ then $x + z \leq y + z$, and let $\mathrm{II}(x, y, z)$ denote its converse: if $x + z \leq y + z$ then $x \leq y$ [1]. If $x \leq y$, then for $\mathrm{I}(x, y, z)$ to be true we must have that $\{X_L + z\} \cup \{x + Z_L\} < y + z$ and $x + z < \{Y_R + z\} \cup \{y + Z_R\}$. Thus the following inequalites must hold.

$$X_L + z < y + z \tag{1}$$
$$x + Z_L < y + z \tag{2}$$
$$x + z < Y_R + z \tag{3}$$
$$x + z < y + Z_R. \tag{4}$$

Suppose there exists some $x_L \in X_L$ such that $y + z \leq x_L + z$. Since $\{y, x_L, z\}$ has a smaller day sum than $\{x, y, z\}$, by induction we can use $\mathrm{II}(y, x_L, z)$ to conclude that $y \leq x_L$. But $x_L < x \leq y$; thus we have a contradiction, which means that inequality (1) holds. Similarly, we can deduce inequality (3) from $\mathrm{II}(y_R, x, z)$. To prove inequality (2), suppose there exists some $z_L \in Z_L$ such that $y + z \leq x + z_L$. Since $x \leq y$ and $\{x, y, z_L\}$ has a smaller day sum than $\{x, y, z\}$, by induction we can use $\mathrm{I}(x, y, z_L)$ to conclude that $x + z_L \leq y + z_L$. Thus, by the transitive law $y + z \leq y + z_L$. Recall that $y + z = \{Y_L + z, y + Z_L | Y_R + z, y + Z_R\}$. By Axiom 2, $y + z \leq y + z_L$ means that $y + Z_L < y + z_L$, a clear contradiction since $z_L \in Z_L$ and each number and pseudo-number is less than or equal to itself. Therefore, inequality (2) holds. Similarly, we can deduce inequality (4) using $\mathrm{II}(x, y, z_R)$.

To prove $\mathrm{II}(x, y, z)$ we must show that if $x + z \leq y + z$, then $x \leq y$. The following inequalities must hold.

$$X_L < y \tag{5}$$
$$x < Y_R. \tag{6}$$

Suppose there exists some $x_L \in X_L$ such that $y \leq x_L$. Since $\{y, x_L, z\}$ has a smaller day sum than $\{x, y, z\}$, by induction we can use $\mathrm{I}(y, x_L, z)$ to conclude that $y + z \leq x_L + z$. However this is a contradiction since $x + z \leq y + z$ implies that $\{X_L + z\} \cup \{x + Z_L\} < y + z$ by Axiom 2, meaning $x_L + z < y + z$. Thus inequality (5) holds. Similarly we can deduce inequality (6) using $\mathrm{I}(y_R, x, z)$. □

Now that we have proved the Transitive Law for addition we conclude that addition is well defined. Verifying that addition is also commutative and associative is relatively straightforward.

Theorem 2.7 (Commutative Law for Addition)**.** *$x + y = y + x$.*

Proof. Using the definition of addition and induction we see that

$$x + y = \{X_L + y, x + Y_L | X_R + y, x + Y_R\} = \{Y_L + x, y + X_L | Y_R + x, y + X_R\} = y + x.$$

$\square$

Theorem 2.8 (Associative Law for Addition)**.** *$(x + y) + z = x + (y + z)$.*

Proof. Using induction and the fact that addition is commutative we can conclude

$$\begin{aligned}
(x+y)+z &= \{X_L + y, x + Y_L | X_R + y, x + Y_R\} + z \\
&= \{(X_L + y) + z, (x + Y_L) + z, (x + y) + Z_L | (X_R + y) + z, (x + Y_R) + z, (x + y) + Z_R\} \\
&= \{X_L + (y + z), x + (Y_L + z), x + (y + Z_L) | X_R + (y + z), x + (Y_R + z), x + (y + Z_R)\} \\
&= x + \{Y_L + z, y + Z_L | Y_R + z, y + Z_R\} \\
&= x + (y + z).
\end{aligned}$$

$\square$

We would like for there to be an additive identity, and indeed, we can verify that $x + 0 = x$ for all x. Consider $x + 0 = \{X_L + 0 | X_R + 0\}$. Since each $x_L \in X_L$ is simpler than x, we can assume that $X_L + 0 = X_L$ by induction. Similarly, $X_R + 0 = X_R$, thus we have the following theorem.

Theorem 2.9. *$x + 0 = x$.*

We can also show that each x has an additive inverse, $-x$.

Theorem 2.10. *$x + (-x) \equiv 0$.*

Proof. We have

$$\begin{aligned}
x + (-x) &= \{X_L + (-x), x + (-X)_L | X_R + (-x), x + (-X)_R\} \\
&= \{X_L + (-x), x + (-X_R) | X_R + (-x), x + (-X_L)\}.
\end{aligned}$$

To show that $x + (-x) \equiv 0$ we need to prove that $x + (-x) \leq 0$ and $x + (-x) \geq 0$. From Axiom 2 we deduce that

$$x + (-x) \leq 0 \text{ if and only if } X_L + (-x) < 0 \text{ and } x + (-X_R) < 0, \text{ and}$$
$$x + (-x) \geq 0 \text{ if and only if } X_R + (-x) > 0 \text{ and } x + (-X_L) > 0.$$

Suppose that $X_L + (-x) \geq 0$, then by Axiom 2, $\{X_{LR} + (-x)\} \cup \{X_L + (-X)_R\} > 0$, meaning $x_L + (-X)_R > 0$ for some $x_L \in X_L$. Furthermore, $(-X)_R = -(X_L)$, thus we have that $x_L + -(X_L) > 0$. Since $-x_L \in -(X_L)$, this implies that $x_L + (-x_L) > 0$. But x_L is simpler than x, so we can assume that $x_L + (-x_L) \equiv 0$ by induction. Thus we have a contradiction, meaning $X_L + (-x) < 0$. We can prove the other three inequalities analogously. $\square$

In general we define subtraction as follows.

Definition 2.3. $x - y = x + (-y)$.

In conclusion we have shown that

- Addition is associative
- Addition is commutative
- Each surreal number x has an additive inverse, which is $-x$, and
- There exists an additive identity, which is 0.

Thus the surreals form an abelian group under addition!

2.4 Multiplication

The stone states, "Let part of one number be multiplied by another and added to the product of the first number by part of the other, and let the product of the parts be subtracted. This shall be done in all possible ways, yielding a number in the left set of the product when the parts are of the same kind, but in the right set when they are of opposite kinds." [1] In other words we need a definition for xy such that elements its left set are less than $x \cdot y$ and elements in its right set are greater than $x \cdot y$. We define mulitplication as follows.

Definition 2.4.

$$x \cdot y = \{X_L \cdot y + x \cdot Y_L - X_L \cdot Y_L, X_R \cdot y + x \cdot Y_R - X_R \cdot Y_R | X_L \cdot y + x \cdot Y_R - X_L \cdot Y_R, X_R \cdot y + x \cdot Y_L - X_R \cdot Y_L\}.$$

Like we had to do with addition, we must verify some basic properties of multiplication before we can actually prove that it is well defined. By the definition of multiplication we can easily check that $x \cdot 0 = 0$ and $0 \cdot x = 0$ for any x, since the left and right set of 0 is the empty set and anything added to or multiplied by the empty set is the empty set. Thus we have the following theorem.

Theorem 2.11. *$x \cdot 0 = 0$ and $0 \cdot x = 0$.*

We would also like for there to be a multiplicative identity. As we expect, we can check that the identity is 1.

Theorem 2.12. *$1 \cdot y = y$ and $y \cdot 1 = y$.*

Proof. Using induction we see that for all y,

$$1 \cdot y = \{0 \cdot y + 1 \cdot Y_L - 0 \cdot Y_L, 1 \cdot Y_R | 0 \cdot y + 1 \cdot Y_R - 0 \cdot Y_R, 1 \cdot Y_L\} = \{Y_L, Y_R | Y_R, Y_L\} \equiv \{Y_L | Y_R\} = y.$$

Similarly we can show that $y \cdot 1 = y$ for all y. □

We can also prove that multiplication is commutative, associative, that it distributes over addition, and that the product of two positive, or negative, numbers is positive. These proofs all rely on induction and are straightforward to check, thus they are omitted. They can be found in *On Numbers and Games* [2].

We can re-write our definition of multiplication in the following form.

$$xy = \{xy-(x-X_L)(y-Y_L), xy-(X_R-x)(Y_R-y)|xy+(x-X_L)(Y_R-y), xy+(X_R-x)(y-Y_L)\}$$

Although this form may not look much simpler, it is easier to see that multiplication is well defined. We need to verify that xy is greater than its left set and less than its right set. Thus the following inequalities should hold:

$$xy > xy - (x - X_L)(y - Y_L)$$
$$xy > xy - (X_R - x)(Y_R - y)$$
$$xy < xy + (x - X_L)(Y_R - y)$$
$$xy < xy + (X_R - x)(y - Y_L).$$

Simplifying, we have

$$(x - X_L)(y - Y_L) > 0$$
$$(X_R - x)(Y_R - y) > 0$$
$$(x - X_L)(Y_R - y) > 0$$
$$(X_R - x)(y - Y_L) > 0.$$

Since the product in each inequality is either a product of two negative numbers or two positive numbers, it must be greater than zero. Therefore we can conclude that xy is a well defined number.

In Conway's *On Numbers and Games* he gives a definition of division for surreal numbers, and shows that each number x has a multiplicative inverse $\frac{1}{x}$ such that $x \cdot \frac{1}{x} = 1$. The notation for division is rather tedious, thus we won't go into it in this paper, but with this knowledge we conclude that the surreals form a totally ordered field.

2.5 Day Two and Beyond

Recall that by Day 2 we have the numbers

$$-a < -1 < -b < 0 < b < 1 < a,$$

where $a = \{1|\}$ and $b = \{0|1\}$.

Using our well-behaved definition of addition we can investigate sums of these numbers. For example, $1 + 1 = \{0 + 1, 0 + 1|\} = \{1|\} = a$. Thus a must be 2. We would like for b to be $\frac{1}{2}$, which we can check by evaluating $b + b$. We have $b + b = \{b + 0|b + 1\} = \{b|b + 1\}$, but this number does not resemble anything in our list, meaning it has not yet been created. However, using

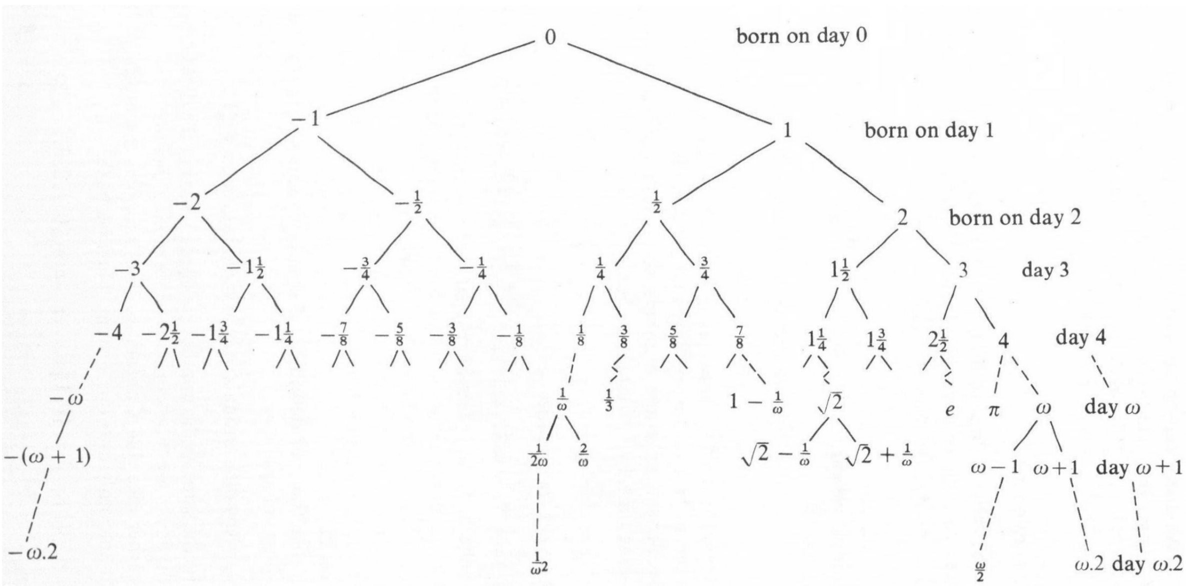


Figure 1: Day 0 to infinity [2]

the Simplicity Theorem we see that 1 is the first number created between b and $b+1$, therefore $b+b=\{b|b+1\}\equiv 1$, as we hoped. So the seven numbers created by the second day are

$$-2<-1<-\frac{1}{2}<0<\frac{1}{2}<1<2.$$

From Theorem 2.2 we know that on day 3 we will get the numbers

$$\{|-2\},\ \{-2|-1\},\ \{-1|-\tfrac{1}{2}\},\ \{-\tfrac{1}{2}|0\},\ \{0|\tfrac{1}{2}\},\ \{\tfrac{1}{2}|1\},\ \{1|2\},\text{ and }\{2|\}.$$

As with the new numbers from day 2, we can find the value of these numbers by adding them to themselves and other numbers. Since $1+2=\{2|\}$, we see that $\{2|\}$ is 3. If we let $x=\{1|2\}$, then $x+x=\{x+1|x+2\}$. Since x is a number we know that $1<x<2$, thus $x+1>2$. The only number created by day 3 that is bigger than 2 is 3, and $3=1+2<x+2$; therefore 3 is the simplest number between $x+1$ and $x+2$. Thus $x+x=3$ and thus $x=\{1|2\}\equiv\frac{3}{2}$. Similarly we find that $\{0|\frac{1}{2}\}$ is $\frac{1}{4}$ and $\{\frac{1}{2}|1\}$ is $\frac{3}{4}$. So the numbers created after four days are

$$-3<-2<-\frac{3}{2}<-1<-\frac{3}{4}<-\frac{1}{2}<-\frac{1}{4}<0<\frac{1}{4}<\frac{1}{2}<\frac{3}{4}<1<\frac{3}{2}<2<3.$$

Note that the value of a number created on day 3 is the arithmetic mean of its left and right set, which are adjacent numbers from day 2. Similarly, the value of a number created on day 4 is the mean of two adjacent numbers from day 3. In general, this pattern continues into infinite days and is pictured in Figure 1.

Theorem 2.13. *If the numbers at the end of n days are*

$$-x_m<-x_{m-1}<\cdots<-x_2<-x_1<x_0=0<x_1<x_2<\cdots<x_{m-1}<x_m,$$

then the new numbers created on the $(n+1)$*st day are*

$$\{|{-x_m}\} < \{-x_m|{-x_{m-1}}\} < \cdots < \{-x_2|{-x_1}\} < \{-x_1|0\} < \{0|x_1\} < \{x_1|x_2\} < \cdots < \{x_{m-1}|x_m\} < \{x_m|\}.$$

Furthermore, for positive numbers,

$$\{x_m|\} \equiv x_m + 1 \textit{ and}$$
$$\{x_i|x_{i+1}\} \equiv \tfrac{x_i+x_{i+1}}{2}, \textit{ for } i = 0, \ldots m-1.$$

Thus for negative numbers,

$$\{|-x_m\} \equiv -(x_m+1) \textit{ and}$$
$$\{-x_{i+1}|-x_i\} \equiv -\tfrac{x_i+x_{i+1}}{2}, \textit{ for } i = 0, \ldots m-1.$$

Proof. From Theorem 2.2 we know that the numbers created on the $(n+1)$st day are

$$\{|-x_m\}, \{-x_m|-x_{m-1}\}, \ldots, \{-x_2|-x_1\}, \{-x_1|0\}, \{0|x_1\}, \{x_1|x_2\}, \ldots, \{x_{m-1}|x_m\}, \{x_m|\}.$$

To prove the order of the positive numbers, suppose that there exist some $y = \{x_{i-1}|x_i\}$ and $z = \{x_i|x_{i+1}\}$ such that $y \geq z$. Then by Axiom 2, $x_i < y$. But this is a contradiction because we know that $x_{i-1} < y < x_i$, since y is a number. Thus $\{x_{i-1}|x_i\} < \{x_i|x_{i+1}\}$. Now let $x = \{x_m|\}$ and $y = \{x_{m-1}|x_m\}$, and suppose that $x \leq y$. Then by Axiom 2, $x_m < y$, but this is a contradiction since $x_{m-1} < y < x_m$. Thus $\{x_m|\}$ is the largest number created after $n+1$ days. The order of the negative numbers follows in the same way.

To prove that $\{x_m|\}$ is the integer x_m+1, Consider the sum $x_m+1 = \{x_{mL}+1, x_m+0|\}$. By induction we can assume that $x_m = \{x_{mL}|\}$ is an integer where $x_{mL} = x_m-1$. Thus $x_{mL}+1 = x_m-1+1 = x_m$ and it follows that $x_m+1 = \{x_m|\}$.

Now consider a number, x, of the form $x = \{x_i|x_{i+1}\}$ created on the $(n+1)$st day. Doubling x we see that

$$\{x_i|x_{i+1}\} + \{x_i|x_{i+1}\} = \{x_i + x|x_{i+1} + x\}.$$

Since x is a number we know that $x_i < x < x_{i+1}$. Thus $x_i + x < x_i + x_{i+1} < x_{i+1} + x$. If $x_i + x_{i+1}$ is the simplest number between $x_i + x$ and $x_{i+1} + x$, then $\{x_i + x|x_{i+1} + x\} \equiv x_i + x_{i+1}$ by the Simplicity Theorem. Suppose there exists some number z simpler than $x_i + x_{i+1}$ such that $x_i + x < z < x_{i+1} + x$. Then $z = x + y$ for some y, which implies that $x_i < y < x_{i+1}$. But x_i and x_{i+1} are adjacent numbers, thus no such y exists. Therefore

$$\{x_i|x_{i+1}\} + \{x_i|x_{i+1}\} = \{x_i + x|x_{i+1} + x\} \equiv x_i + x_{i+1},$$

which means that $\{x_i|x_{i+1}\} \equiv \frac{x_i+x_{i+1}}{2}$.

The result for negative numbers immediately follows by the definition of negation. □

2.6 Day ω and Beyond

We now see more generally the pattern of how numbers are formed: on the nth day the extreme numbers created are the integers $-n$ and n, and each other number created is the arithmetic mean of the two previously created numbers that are closest to it in value. However, when will numbers such as $\frac{1}{3}$ be formed? So far, all the numbers that we have looked at have been *dyadic rational* numbers, meaning that they are of the form $\frac{m}{2^n}$, where m and n are integers. In other words, they have terminating decimal expansions. So when would $\frac{1}{3} = 0.333333\ldots$ be formed? If we let the left and right set of a surreal number contain infinitely many numbers, then we can form numbers other than the dyadic rationals.

To form $\frac{1}{3}$ we need a sequence of numbers that approaches $\frac{1}{3}$ from below as its left set, and a sequence of numbers that approaches it from above as its right set. Thus it seems probable that the number $x = \{\frac{1}{4}, \frac{5}{16}, \frac{21}{64}, \ldots | \frac{1}{2}, \frac{3}{8}, \frac{11}{32}, \ldots\}$ would be $\frac{1}{3}$. To verify this claim, we calculate $x+x+x$. First, note that

$$x + x = \{\frac{1}{4} + x, \frac{5}{16} + x, \frac{21}{64} + x, \ldots | \frac{1}{2} + x, \frac{3}{8} + x, \frac{11}{32} + x, \ldots\},$$

and thus

$$(x+x)+x = \{\frac{1}{4} + x + x, \frac{5}{16} + x + x, \frac{21}{64} + x + x, \ldots | \frac{1}{2} + x + x, \frac{3}{8} + x + x, \frac{11}{32} + x + x, \ldots\}.$$

Every element of the left set of $x+x+x$ is a positive number less than 1 and every element of the right set of $x+x+x$ is a positive number greater than 1. For example, $\frac{1}{2}+x+x$ is in the right set of $x+x+x$ and we have that $\frac{1}{2}+x+x > \frac{1}{2}+\frac{1}{4}+\frac{1}{4} = 1$, since $\frac{1}{4}$ is in the left set of x. Thus $x+x+x \equiv 1$, which means that x is $\frac{1}{3}$.

We call the day that $\frac{1}{3}$ and the rest of the reals are created on day ω, where ω is the earliest number greater than all the finite counting numbers [2]. Another familiar number created on day ω is π. Noting that the binary representation of π is $\pi = 11.00100100001111\ldots$, we can see that

$$\pi = \{11.001, 11.001001, 11.00100100001, \ldots | 11.1, 11.01, 11.0011, 11.00101, \ldots\}.$$

We get Π_L by stopping at every 1 in the binary expansion of π, and we get Π_R by stopping at every 0 and chaging it to a 1 [1].

As well as the reals, we also get some interesting infinite numbers on the ω day. One number is ω itself, which is $\omega = \{1, 2, 3, 4, \ldots |\}$. We can show that it is larger than all the other numbers created through day ω. Suppose there exists some $x = \{X_L|X_R\}$ such that $x \geq \omega$. Then by Axiom 2, for all $n \in \{1, 2, 3, 4, \ldots\}$, $n < x$. But this is clearly a contradiction since the set $\{1, 2, 3, 4, \ldots\}$ has no upper bound. Thus $x < \omega$ for all x. Note that ω has many other forms such as $\omega = \{1, 2, 4, 8, 16, \ldots |\}$, or $\omega = \{\text{all dyadic numbers}|\}$ [2]. Similarly, $-\omega = \{| -1, -2, -3, \ldots\}$ is the most negative number created by day ω.

In addition, $\epsilon = \{0|1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\}$ is the smallest positive number created on or before day ω. Using Axiom 2, it is not hard to see that $\epsilon > 0$ since the left set of ϵ is 0, and $0 \leq 0$. Now, suppose there exists a nonzero real number $x = \{X_L|X_R\}$ such that $x \leq \epsilon$. Then $x < \{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\}$. But this is impossible since x is a nonzero real number. Thus $\epsilon < x$.

Consider the product

$$\begin{aligned}\epsilon\omega &= \{\epsilon\omega - \epsilon(\omega - \{1,2,4,\ldots\}) | \epsilon\omega + (\{1, \frac{1}{2}, \frac{1}{4}, \ldots\} - \epsilon)(\omega - \{1,2,4,\ldots\})\} \\ &= \{\epsilon, 2\epsilon, 4\epsilon, \ldots | \epsilon\omega + (\{1-\epsilon, \frac{1}{2} - \epsilon, \frac{1}{4} - \epsilon, \ldots\})(\{\omega - 1, \omega - 2, \omega - 4, \ldots\})\}.\end{aligned}$$

Since ϵ is positive, yet smaller than any real number, the left set of $\epsilon\omega$ is greater than 0, but less than 1. We also have that

$$\epsilon\omega + (\{1-\epsilon, \frac{1}{2} - \epsilon, \frac{1}{4} - \epsilon, \ldots\})(\{\omega-1, \omega-2, \omega-4, \ldots\}) > \epsilon\omega + (\{\omega-1, \omega-2, \omega-4, \ldots\}) > \epsilon\omega + 1$$

since ω is greater than any finite counting number. Thus the right set of $\epsilon\omega$ is greater than 1. Therefore, by the Simplicity Theorem, $\epsilon\omega = 1$, which means that ϵ is the reciprocal of ω.

We also have numbers such as $\{1 | 1\frac{1}{2}, 1\frac{1}{4}, 1\frac{1}{8}, 1\frac{1}{16}, \ldots\}$, which is just barely bigger than 1. Thus surreal numbers fill in the tiny gaps between the reals, and they even continue to fill in the gaps between themselves. For example, from Theorem 2.13 we know that on the $\omega + 1$ day we will get the number $\frac{\epsilon}{2} = \{0|\epsilon\}$ between ϵ and 0, and the next day we will get $\frac{\epsilon}{4} = \{0|\frac{\epsilon}{2}\}$, which lies between $\frac{\epsilon}{2}$ and 0, and so on and so on.

From Theorem 2.13 we also see that we will get bigger and bigger infinites as the days go on. For example on the $\omega + 1$ day we can create $\{\omega|\}$, which turns out to be $\omega + 1$, since $\omega + 1 = \{2,3,4,5,\ldots,\omega|\} \equiv \{\omega|\}$. On the $\omega + 2$ day we have $\omega + 2 = \{3,4,5,\ldots,\omega+1|\} \equiv \{\omega+1|\}$, on the $\omega + 3$ day we have $\omega + 3 \equiv \{\omega + 2|\}$, and so on. Since $\omega + n$ is created on the $(\omega + n)$th day for each integer n, on the 2ω day we have the number $\{\omega+1, \omega+2, \omega+3, \omega+4, \ldots|\}$. Note that $\omega + \omega = \{\omega+1, \omega+2, \omega+3, \omega+4, \ldots|\}$, thus this number must be 2ω. Similarly, $3\omega = \{2\omega+1, 2\omega+2, 2\omega+3, \ldots|\}$ is created on the 3ω day, and so on. Eventually, we can even create $\omega^2 = \{\omega, 2\omega, 3\omega, \ldots|\}$ and $\omega^\omega = \{\omega, \omega^2, \omega^3, \ldots\}$.

Note that we will also get infinite numbers between infinities. For example on the $\omega + 2$ day we have $\omega + \frac{1}{2} = \{1\frac{1}{2}, 2\frac{1}{2}, 3\frac{1}{2}, \ldots, \omega|\omega+1\} \equiv \{\omega|\omega+1\}$, which is between ω and $\omega + 1$.

What happens when we subtract 1 from ω? We get $\omega - 1 = \{0,1,2,3,4,\ldots|\omega\}$, which must be created on the $\omega + 1$ day and is larger than all the integers, yet less than ω. Similarly, on the $\omega + 2$ day we have $\omega - 2 = \{0,1,2,\ldots|\omega, \omega-1\}$, which is larger than all the integers yet less than $\omega - 1$. On the $\omega + 3$ day we have $\omega - 3 = \{0,1,2,|\omega,\omega-1,\omega-2\}$, and so on. Consider the number $z = \{0,1,2,\ldots|\omega,\omega-1,\omega-2,\ldots\}$ [2]. If we calculate $z + z$ then we find that $z + z = \{z, z+1, z+2, \ldots | z+\omega, z+\omega-1, z+\omega-2, \ldots\}$. Note that for any positive integer, n, $z + n < \omega - n + n = \omega$, and $z + \omega - n > z + \omega > \omega$. So ω lies in between the left and right set of $z + z$. Furthermore, by defintion, z is greater than any finite number, and thus the left set of $z + z$ is greater than any finite number. Since ω is the simplest number greater than any finite number, we conclude that $z + z \equiv \omega$, by the Simplicity Theorem. Thus z must be $\frac{\omega}{2}$.

2.7 Induction with Infinite Day Sums

Since we now know that a set of numbers could have an infinite day sum, we need to check that the proofs we have completed using day induction still hold. In the induction process we noted that if a theorem fails for some x then it must fail for some $x_L \in X_L$, meaning that it fails for some $x_{LL} \in X_{LL}$, and so on. In each proof we have completed, every such sequence is eventually finite, meaning we reach a case in which $X_{LLL...L}$ is empty and we have the desired contradiction. For example, in proving that $x + 0 = x$, we have $x + 0 = \{X_L + 0|X_R + 0\}$ and we need to assume that $x_L + 0 = x_L$ for each $x_L \in X_L$ by induction. If this assumption is false then $X_L + 0 = \{X_{LL} + 0|X_{LR} + 0\} \neq X_L$, meaning that either $X_{LL} + 0 \neq X_{LL}$ or $X_{LR} + 0 \neq X_{LR}$, which would mean that either $X_{LLL} + 0 \neq X_{LLL}$, $X_{LLR} + 0 \neq X_{LLR}$, $X_{LRL} + 0 \neq X_{LRL}$, or $X_{LRR} + 0 \neq X_{LRR}$, and so on.

Notice that any counterexample would imply an infinite sequence of counterexamples if there existed a sequence of numbers $x_1, x_2, x_3, x_4, \ldots$ such that $x_{i+1} \in X_{iL} \cup X_{iR}$. We need to show that no such sequence exists. We can do this by using Axiom 1, which states that each number is created using previously created numbers. Whenever we create a number x we can prove that there does not exist an infinite sequence starting with $x = x_1$, since we would need to find an x_2 in either the left or right set of x that is also the start of an infinite sequence. But because $x_2 \in X_L \cup X_R$, it must have been created before x, meaning that we have already proved that no such sequence exists. Thus our proofs by induction still hold for numbers with infinite day sums because any sequence of counterexamples is eventually finite. Also, we do not need to prove a base case as with traditional mathematical induction because we eventually get to a case involving the empty set.

3 Structures within S

3.1 Generalized Integers

Call x a *generalized integer* if

$$x \equiv \{x - 1|x + 1\} \text{ [1]}.$$

For example, 0 is a generalized integer since $0 \equiv \{-1|1\}$, by the Simplicity Theorem. Since $-2 \equiv \{-3| - 1\}$, -2 is a generalized integer. However, $\frac{1}{4}$ is not a generalized integer since the simplest number between $-\frac{3}{4}$ and $\frac{5}{4}$ is clearly 0, meaning that $\frac{1}{4} \not\equiv \{-\frac{3}{4}|\frac{5}{4}\}$.

In general, consider an integer $n \geq 1$. From Theorem 2.13 we know that $n - 1$ was created on the $(n - 1)$st day and that the only number greater than $n - 1$ created through the nth day is n. On the $(n + 1)$st day the only two numbers greater than $n - 1$ that are formed are $\frac{(n-1)+n}{2}$ and $n + 1$. Since n is simpler than $\frac{(n-1)+n}{2}$, it is the simplest number between $n - 1$ and $n + 1$, meaning that $n \equiv \{n - 1|n + 1\}$. Thus all positive integers are generalized integers. We can prove that all negative integers are generalized integers in the same way.

On the other hand, consider the irreducible fraction $\frac{a}{b}$, where a and b are non-generalized integers and $b > 1$. From Theorem 2.13 we know that the numbers between any integers n and $n + 1$ are formed on days after the $(n + 1)$st day. Thus for any two fractions that differ by at least 1, the simplest number between them will be an integer. Since $\frac{a}{b} - 1$ and $\frac{a}{b} + 1$ differ by 2, $\{\frac{a}{b} - 1|\frac{a}{b} + 1\} \equiv n$

for some integer n. Therefore $\{\frac{a}{b}-1|\frac{a}{b}+1\} \not\equiv \frac{a}{b}$, meaning that no proper fraction is a generalized integer.

Along with the familiar integers, generalized integers also include some infinite numbers. Recall that on day ω we have the number $\omega = \{0, 1, 2, \ldots |\}$, and that on the $(\omega+1)$st day we have the numbers $\omega-1 = \{0, 1, 2, \ldots |\omega\}$ and $\omega+1 = \{\omega|\}$. Since ω is the only number between between $\omega-1$ and $\omega+1$ by the $(\omega+1)$st day, $\omega \equiv \{\omega-1|\omega+1\}$. Consider $\omega+n$ for any integer $n \geq 1$. Similar to our investigation of finite generalized integers, $\omega+n-1$ was created on the $(\omega+n-1)$st day, $\omega+n$ was created on the $(\omega+n)$th day, and the only two numbers greater than $\omega+n-1$ created on the $(\omega+n+1)$st day are $\frac{(\omega+n-1)+(\omega+n)}{2} = \omega+n-\frac{1}{2}$ and $\omega+n+1$. Thus $\omega+n \equiv \{\omega+n-1|\omega+n+1\}$. We can show that $\omega-n$ is a generalized integer for $n \geq 1$ in the same way.

What about $\frac{\omega}{2} = \{1, 2, 3, \ldots |\omega, \omega-1, \omega-2, \ldots\}$, created on day 2ω? Calculating $\frac{\omega}{2}-1 = \{0, 1, 2, \ldots |\frac{\omega}{2}\}$ and $\frac{\omega}{2}+1 = \{\frac{\omega}{2}|\omega+1, \omega, \omega-1\}$, we see that both of these numbers must have been created on the $2\omega+1$ day and that $\frac{\omega}{2}$ is the only number between $\frac{\omega}{2}-1$ and $\frac{\omega}{2}+1$ by the $2\omega+1$ day. Thus $\frac{\omega}{2} \equiv \{\frac{\omega}{2}-1|\frac{\omega}{2}+1\}$. So fractions involving ω are also generalized integers.

Suppose x and y are generalized integers. Then $x \equiv \{x-1|x+1\}$ and $y \equiv \{y-1|y+1\}$. It is not hard to check that the generalized integers are closed under addition,

$$x+y = \{x-1+y, y-1+x|x+1+y, y+1+x\} = \{x+y-1|x+y+1\}$$

subtraction,

$$x-y = \{x-1-y, -y-1+x|x+1-y, -y+1+x\} = \{x-y-1|x-y+1\}$$

and multiplication

$$\begin{aligned} xy &= \{(x-1)y + x(y-1) - (x-1)(y-1), \ldots |(x-1)y + x(y+1) - (x-1)(y+1), \ldots\} \\ &= \{xy-1|xy+1\}. \end{aligned}$$

Thus the generalized integers form a subring of the surreals, just as the integers form a subring of the real numbers.

3.2 Real Numbers

Call x a **real number* if $-n < x < n$ for some (nongeneralized) integer n, and if

$$x \equiv \{x-1, x-\tfrac{1}{2}, x-\tfrac{1}{4}, \ldots |x+1, x+\tfrac{1}{2}, x+\tfrac{1}{4}, \ldots\} \text{ [2]}.$$

It is clear that ω and other infinities are not *real since they are not bounded, but what about $\epsilon = \{0|1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\}$, since $-1 < \epsilon < 1$? Consider

$$x = \{\epsilon-1, \epsilon-\frac{1}{2}, \epsilon-\frac{1}{4}, \ldots |\epsilon+1, \epsilon+\frac{1}{2}, \epsilon+\frac{1}{4}, \ldots\}.$$

Since ϵ is smaller than every positive number, the left set of x is less than 0. Since ϵ is positive, the right set of x is greater than 0. Thus

$$\{\epsilon - 1, \epsilon - \frac{1}{2}, \epsilon - \frac{1}{4}, \ldots | \epsilon + 1, \epsilon + \frac{1}{2}, \epsilon + \frac{1}{4}, \ldots\} \equiv 0,$$

which means that ϵ is not a *real number, as we expected.

Suppose x and y are *real numbers. Then we have the following three equalities:

$$x+y = \{x-1+y+x-\frac{1}{2}+y, \ldots, x+y-1, x+y-\frac{1}{2}, \ldots | x+1+y, x+\frac{1}{2}+y, \ldots, x+y+1, x+y+\frac{1}{2}, \ldots\}$$
$$= \{x+y-1, x+y-\frac{1}{2}, \ldots | x+y+1, x+y+\frac{1}{2}, \ldots\},$$

$$x-y = \{x-1-y+x-\frac{1}{2}-y, \ldots, x-y-1, x-y-\frac{1}{2}, \ldots | x+1-y, x+\frac{1}{2}-y, \ldots, x-y+1, x-y+\frac{1}{2}, \ldots\}$$
$$= \{x-y-1, x-y-\frac{1}{2}, \ldots | x-y+1, x-y+\frac{1}{2}, \ldots\},$$

$$xy = \{\{(x-1)y, (x-\frac{1}{2})y, \ldots\} + \{x(y-1), x(y-\frac{1}{2}), \ldots\} - \{x-1, x-\frac{1}{2}, \ldots\}\{y-1, y-\frac{1}{2}, \ldots\}, \ldots |$$
$$|\{(x-1)y, (x-\frac{1}{2})y, \ldots\} + \{x(y+1), x(y+\frac{1}{2}), \ldots\} - \{x-1, x-\frac{1}{2}, \ldots\}\{y+1, y+\frac{1}{2}, \ldots\}, \ldots\}$$
$$= \{xy-1, xy-\frac{1}{2}, \ldots | xy+1, xy+\frac{1}{2}, \ldots\}.$$

Since the *real numbers are closed under addition, subtraction, and multiplication, they form a subring of the surreals.

It appears that the *real numbers of the surreals are essentially the same as the real numbers that we are more familiar with, which leads us to look for an isomorphism between the two. Consider the function f which maps $\mathbb{R}$ to the *real numbers of the surreals, defined by

$$f(x) = \{x-1, x-\frac{1}{2}, x-\frac{1}{4}, \ldots | x+1, x+\frac{1}{2}, x+\frac{1}{4}, \ldots\}.$$

We would like to show that f is bijective. Suppose that for real numbers x and y, $f(x) = f(y)$. Then

$$\{X_L|X_R\} = \{x-1, x-\frac{1}{2}, \ldots | x+1, x+\frac{1}{2}, x\ldots\} = \{y-1, y-\frac{1}{2}, \ldots | y+1, y+\frac{1}{2}, \ldots\} = \{Y_L|Y_R\}.$$

If we take an $x_L \in X_L$, then $x_L < x - \frac{1}{n}$ for some n. Thus $x_L + \frac{1}{2^n} < x - \frac{1}{n} + \frac{1}{2^n} < x$. Thus $x_L + \frac{1}{2^n} \in X_L$, which means that X_L has no greatest element. Similarly Y_L has no greatest element. If we take an $x_R \in X_R$, then $x_R > x + \frac{1}{n}$ for some n. Thus $x_R - \frac{1}{2^n} > x + \frac{1}{n} - \frac{1}{2^n} > x$. Since $x_R - \frac{1}{2^n} \in X_R$, we see that X_R has no least element. Similarly Y_R has no least element. Thus

$\{x-1, x-\frac{1}{2}, \ldots | x+1, x+\frac{1}{2}, x \ldots\} \equiv x$ and $\{y-1, y-\frac{1}{2}, \ldots | y+1, y+\frac{1}{2}, \ldots\} \equiv y$, which means that $x = y$. Therefore f is injective. We see that f is surjective since every *real number is of the form $\{x-1, x-\frac{1}{2}, x-\frac{1}{4}, \ldots | x+1, x+\frac{1}{2}, x+\frac{1}{4}, \ldots\}$, which is clearly the image of x under f.

In addition, we note that for real numbers x and y,

$$f(x+y) = \{x+y-1, x+y-\frac{1}{2}, \ldots | x+y+1, x+y+\frac{1}{2}, \ldots\} = f(x) + f(y),$$

and

$$f(xy) = \{xy-1, xy-\frac{1}{2}, \ldots | xy+1, xy+\frac{1}{2}, \ldots\} = f(x)f(y).$$

Therefore f is a ring isomorphism from the reals to the *reals. Thus the real numbers of the surreals are an isomorphic copy of the real numbers that we are more familar with, meaning that we no longer have to distinguish them with a pesky asterisk! Note that we have also shown that the real numbers are contained within the surreal numbers.

3.3 A FUNction Defined on S

We now investigate a function defined for surreal numbers. Assume g is a function from numbers to numbers such that $x \leq y$ implies $g(x) \leq g(y)$, and define

$$f(x) = \{f(X_L) \cup \{g(x)\} | f(X_R)\} \text{ [1].}$$

We can show that f is monotonically increasing, which means that $f(x) \leq f(y)$ if and only if $x \leq y$. Suppose that $f(x) \leq f(y)$. Then by Axiom 2, $f(X_L) < f(y)$, $\{g(x)\} < f(y)$, and $f(x) < f(Y_R)$. To prove $x \leq y$ we need that

i.) $X_L < y$, and

ii.) $x < Y_R$.

For (i.) we take an arbitrary $x_L \in X_L$ and note that $f(x_L) < f(y)$. Since x_L and y have a smaller day sum than x and y we can use induction to conclude that $x_L < y$. We can prove (ii.) in the same way.

Now suppose that $x \leq y$. Then by Axiom 2, $X_L < y$ and $x < Y_R$. To prove $f(x) \leq f(y)$ we need that

i.) $f(X_L) < f(y)$,

ii.) $\{g(x)\} < f(y)$, and

iii.) $f(x) < f(Y_R)$.

For (i.) we take an arbitrary $x_L \in X_L$. Since x is a number we have that $x_L < x \leq y$, and since x_L and y have a smaller day sum than x and y, we can conclude that $f(x_L) < f(y)$ by induction. We can prove (iii.) in the same way. To prove (ii.) we note that $g(x) \leq g(y)$. We can conclude that $g(x) \leq g(y) < f(y)$ if $f(y)$ is a number, thus we must prove that $f(x)$ is a number for all x. To show that $f(x)$ is a number, we must show that its left set is strictly less than its right set. The following equalities must hold.

iv.) $f(X_L) < f(X_R)$, and

v.) $\{g(x)\} < f(X_R)$.

Inequality (iv.) follows from induction since $x_L < x_R$ for all x_L and x_R. To prove (v.), assume the contrary: that there exists some $g(x)$ and $f(x_R)$ such that $g(x) \geq f(x_R)$. Then, by definition, we have that $g(x) < g(x)$, which is a contradiction since $x \leq x$ implies that $g(x) \leq g(x)$. Thus inequality (v.) holds, and it follows that $f(x)$ is a number for all x. Furthermore, we can conclude that inequality (ii.) holds, completing our proof that f is monotonically increasing.

What does this function actually look like? We can get an idea by computing values for a set function, g. Suppose we let $g(x) = 0$ for each surreal number x. Then $f(x)$ is defined by

$$f(x) = \{f(X_L), 0 | f(X_R)\}.$$

Since $f(x)$ is a number and 0 will be in the left set of every output of f, we can immediately see that $f(x) > 0$ for all x. If we assume that $f(\emptyset) = \emptyset$ we find that

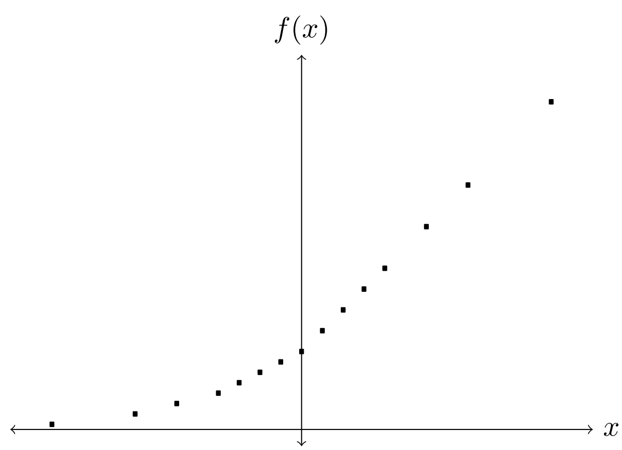
$$f(0) = \{f(\emptyset), 0 | f(\emptyset)\} = \{0|\} = 1.$$

We can now compute

$$f(-1) = \{f(\emptyset), 0 | f(0)\} = \{0|1\} = \frac{1}{2}$$
$$f(1) = \{f(0), 0 | f(\emptyset)\} = \{1, 0|\} \equiv 2,$$

and eventually we get the values of $f(x)$ for all x created through the third day, shown in the table and plot below.

x	$f(x)$
-3	$\frac{1}{8}$
-2	$\frac{1}{4}$
$-\frac{3}{2}$	$\frac{3}{8}$
-1	$\frac{1}{2}$
$-\frac{3}{4}$	$\frac{5}{8}$
$-\frac{1}{2}$	$\frac{3}{4}$
$-\frac{1}{4}$	$\frac{7}{8}$
0	1
$\frac{1}{4}$	$\frac{5}{4}$
$\frac{1}{2}$	$\frac{3}{2}$
$\frac{3}{4}$	$\frac{7}{4}$
1	2
$\frac{3}{2}$	$\frac{5}{2}$
2	3
3	4



We see that f is somewhat like the exponential fuction defined for real numbers in the domain $(-\infty, 0)$ of the surreals. However, in $[0, \infty)$, f is the simple linear function $f(x) = x + 1$.

4 Surreal Numbers and Games

We can think of any number $g = \{a, b, c, \ldots | d, e, f, \ldots\}$[2] as a game where the elements of the left set represent moves that one player can make, and the elements in the right set represent the moves that another player can make. For example, if g was a game between players Left and Right, then Left could move from some starting point, g, to any of $a, b, c, \ldots$, and Right could move from g to any of $d, e, f, \ldots$. If Left starts the game and moves to a, then the representation of the game is changed to $a = \{A, B, C, \ldots | D, E, F, \ldots\}$. Thus Right can now move to any of $D, E, F, \ldots$. If she moves to $E = \{\alpha, \beta, \gamma, \ldots | \epsilon, \delta, \zeta, \ldots\}$, then Left can then move to any of $\epsilon, \delta, \zeta, \ldots$, and so on. The last person to make a move wins the game.

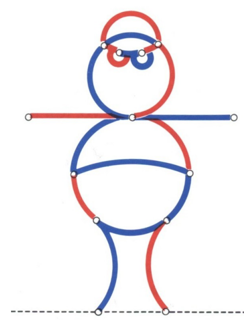


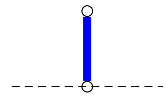
Figure 2: A Hackenbush game [3]

One specific game that we can consider is Hackenbush (Figure 2 provides a fancy example). Hackenbush is a two-player game played with a picture of nodes joined by edges that are colored with two different colors (we will use red and blue). The picture must be constructed so that you can reach the ground (which is the dotted line in Figure 2) from any node by travelling along a series of adjacent edges. The two players, Left and Right, take turns alternately. Left can delete only blue edges and Right can delete only red edges. After one edge is deleted, any edges no longer connected to the ground are also deleted. The last player to delete an edge wins.

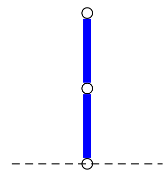
4.1 Basic Games

We will now analyze some simple games.

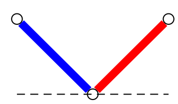
If there are no red or blue edges then neither player has any moves, meaning that the game would be $\{\emptyset|\emptyset\} = 0$. We call this state *endgame* [2]. Note that the first person to move automatically loses.



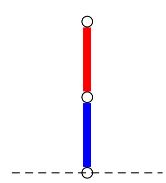
If there is just one blue edge, then Left can move to 0 while Right has no moves; thus the game would be $\{0|\} = 1$. In this case, Left automatically wins since there are no legal moves for Right. If there were just one red edge, then Left would have no moves while Right could move to 0. Thus the game would be $\{|0\} = -1$, and Right would win.



If there are two blue edges stacked on top of one another, then Left can pull the bottom edge to form game 0, or the top edge to form 1. Thus the game would be $\{0, 1|\} \equiv \{1|\} = 2$. Again, Left automaticallly wins. If we had two red edges stacked on top of each other then we would have the game $\{|0, -1\} \equiv \{| -1\} = -2$. In general, if we have a chain n blue edges then the game will have a value of n, and if we have a chain of n red edges then the game will have a value of $-n$, where n is a positive integer.



If there is one red edge and one blue edge coming from one node, then the game would be $\{-1|1\}$. Since the second player to move will cause an endgame, we see that the first player to move will lose. Thus this game is equivalent to 0, meaning the game 0 has multiple forms.



If there is one red edge on top of one blue edge, then Left can delete the bottom edge to form game 0, while Right can delete the top edge to form game 1. Thus the game would be $\{0|1\} = \frac{1}{2}$. We note that Left will win regardless of who goes first. If we had one blue edge stacked on top of one red edge, then Left could delete the top edge to form -1 while Right could delete the bottom to form 0. Thus we would have $\{-1|0\} = -\frac{1}{2}$, and Right would win regardless of who goes first.

In general we use the following notations from *On Numbers and Games* [2]:

$G > 0$ (G is positive) if there is a winning strategy for Left

$G < 0$ (G is negative) if there is a winning strategy for Right

$G \equiv 0$ (G is like 0) if there is a winning strategy for the second person to move

$G \parallel 0$ (G is *fuzzy*) if there is a winning stategy for the first person to move

$G \geq 0$ if $G > 0$ or $G \equiv 0$, which means that if Right starts there is a winning strategy for Left, since Left would then be the second to move.

$G \leq 0$ if $G < 0$ or $G \equiv 0$, which means that if Left starts there is a winning strategy for Right.

$G \mathrel{\shortparallel\!\!>} 0$ if $G > 0$ or $G \parallel 0$, which means if Left starts then there is a winning strategy for Left, since they would be the first to move.

$G \mathrel{<\!\!\shortparallel} 0$ if $G < 0$ or $G \parallel 0$, which means that if Right starts then there is a winning strategy for Right.

4.2 Are All Games Numbers?

Every Hackenbush game is essentially built up from the zero game. In other words, for every move that a player makes the game will be reduced to a simpler game, until the the players reach endgame. Thus the left and right sets of a game g are composed of simpler games than g, which leads us to expect that all two-color Hackenbush games will indeed be numbers. Suppose we have a two-color Hackenbush game, $g = \{G_L|G_R\}$. If Left deletes a blue edge then we have some game g_L, where $g_L \in G_L$. By induction, g_L is a number. Suppose that $g \leq g_L$. Then, by Axiom 2, $g_L < g_L$, a contradiction. Thus the value of g stirctly increases when Left deletes a blue edge. Similarly, the value of g strictly decreases when Right deletes an edge. Thus $g_L < g < g_R$, which means that g is a number.

Recall that we defined a game g to be fuzzy, denoted $g \parallel 0$, if there was a winning strategy for the first player to move. However, we know that all surreal numbers are either equal to, less than, or greater than zero. Since all two-color Hackenbush games are numbers, none of them can be fuzzy! Only games that are represented by pseudo-numbers can be fuzzy.

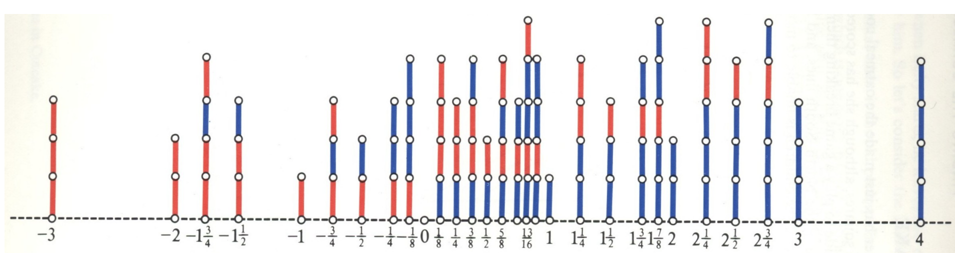
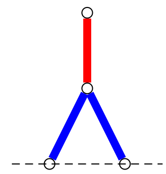


Figure 3: Some Hackenbush numbers [3]

We can also ask, are all numbers games? Indeed, we can build any number if we consider its binary representation. The first pair of edges of opposite colors is treated as a binary point, and the blue and red edges above the pair are read as digits 1 and 0 after the binary point. We then add an extra 1 if the chain is finite [3]. The Hackenbush representations of several numbers are shown in Figure 3.

4.3 The Value of Games

Recall how the game $\{-1|1\}$ turned out to be 0. Now that we know that all games are numbers, this is not too surprising. In fact, it is just the Simplicity Theorem at work! Consider the following game, g.



If Left deletes either of the blue edges, then the game will be $\frac{1}{2}$. Alternatively, if Right deletes the top edge then it is not too hard to see that the game will be 2. Thus $g = \{\frac{1}{2}|2\} \equiv 1$, by the Simplicity Theorem. In general, we use the same strategy to find the values of more complicated games, although it can be more tedious. In Figure 4 we see that the value of the horse is $\frac{1}{2}$.

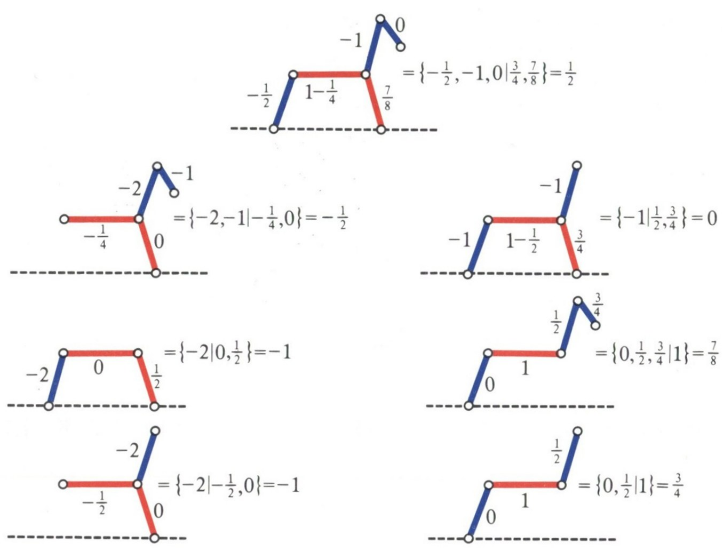


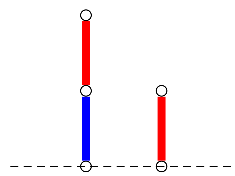
Figure 4: Finding the value of a horse [3]

What does it mean for one game to be greater than, or less than another? For games g and h we have

> $g = h$ means that g and h are equally favorable to Left,
>
> $g > h$ means that g is more favorable to Left than h,
>
> $g < h$ means that g is less favorable to Left than h, and
>
> $g \parallel h$ means that g is only more favorable to Left if Left is the first to move.

From our studies of numbers we can conclude that all of the usual order relations hold. For example, if we have three games g, h, and k such that $g < h$ and $h < k$, then $g < k$.

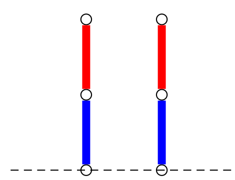
We know that $1 > \frac{1}{2}$, so the game 1 is more favorable to Left than $\frac{1}{2}$. However, recall how in both games Left won regardless, so how could one be more favorable than the other? Observe what happens if we give Right an extra move in $\frac{1}{2}$.



If Left goes first, then Right will surely win. If Right goes first, she can win by first deleting the red edge in $\frac{1}{2}$. Since adding -1 to $\frac{1}{2}$ turned the game to Right's favor, Left had less than a one-move advantage in $\frac{1}{2}$; whereas Left clearly has a one-move advantage in 1. Thus $1 > \frac{1}{2}$.

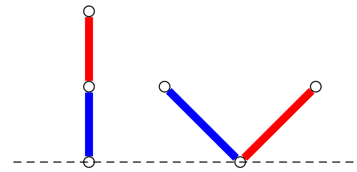
4.4 Sums of Games

If we have two games, g and h, then the sum $g + h$ is the compound game that we get by playing both g and h simultaneously side by side, as we did with $\frac{1}{2}$ and 1 in the previous section. For example consider the game $\frac{1}{2} + \frac{1}{2}$.

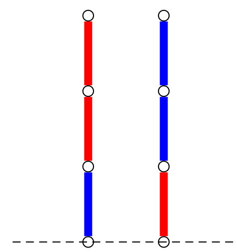


We see that Left can move to $0 + \frac{1}{2}$ or $\frac{1}{2} + 0$, and Right can move to $1 + \frac{1}{2}$ or $\frac{1}{2} + 1$. Thus $\frac{1}{2} + \frac{1}{2} = \{\frac{1}{2}|1 + \frac{1}{2}\} \equiv 1$. We can also verify that the compound game $\frac{1}{2} + \frac{1}{2}$ is 1 by adding -1 to it. Although we leave it to the interested reader, it is not hard to deduce that in $\frac{1}{2} + \frac{1}{2} + (-1)$, the second player to move will win, and thus $\frac{1}{2} + \frac{1}{2} + (-1) = 0$.

Suppose we have the sum of any two games $g = \{G_L|G_R\}$ and $h = \{H_L|H_R\}$. Then the legal moves for Left will be $G_L + h \cup g + H_L$ and the legal moves for Right will be $G_R + h \cup g + H_R$. Thus $g + h = \{G_L + h, g + H_L|G_R + h, g + H_R\}$, and we see that addition for games is consistent with our definition of addition for numbers. This means that we can apply the properites of addition that we have proved for numbers to two-color Hackenbush games. For example it is not hard to see that the game below is really just $\frac{1}{2}$, since adding a zero game to any other game will not change its outcome.

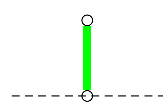


If we have any Hackenbush game, then we get the negative of that game by interchanging red and blue edges. When we negate $g = \{G_L|G_R\}$, the possible moves in G_R become moves for Left, and the moves in G_L become moves for Right. Thus $-g = \{-G_R|-G_L\}$, and we see that our definition is consistent with our definition for numbers. Thus we can conclude that $g - g$ is a zero game for any two-color Hackenbush game g. For example, the game $\frac{1}{4} - \frac{1}{4}$ pictured below is a zero game.



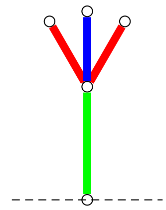
4.5 Hackenbush Hotchpotch

So far we have been looking at two-color Hackenbush games in which each player can only delete edges of a certain color. Consider the following simple game, in which either player could delete the single green edge.

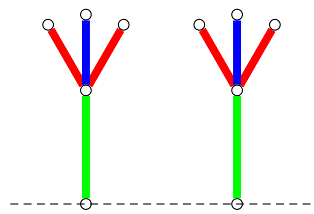


We see that both players can move to 0. Thus the game has value $\{0|0\}$. By Axiom 1, since $0 \leq 0$, $\{0|0\}$ is not a number, but a pseudo-number.

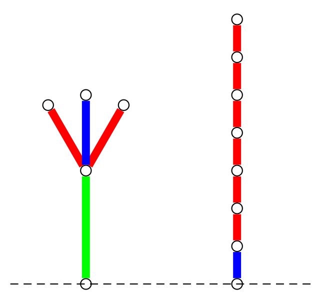
Hackenbush Hotchpotch is just like Hackenbush except that there can be green edges in a picture, which either player can delete. Since every green edge represents a move for both Left and Right, we see that in every Hackenbush Hotchpotch game, $g = \{G_L|G_R\}$, there will be some x corresponding to the resulting position from deleting a green edge such that $x \in G_L$ and $x \in G_R$. Thus $G_L \not< G_R$, which means that G is not a number by Axiom 1. Thus every Hackenbush Hotchpotch game is a pseudo-number. Consider the game, g, below.



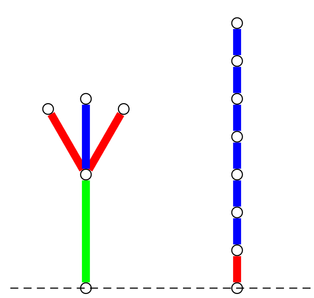
Since the first player to move can delete the green edge, causing endgame, we see that whoever moves first will win. Thus $g \parallel 0$. Since there is no particular advantage to Left or Right, why are these games not equal to zero? Consider the sum $g + g$ below.



Whether Left or Right starts, Right has enough edges so that she can force Left to delete one of the green edges first. Right can then win by taking the second green edge. Thus the sum $g + g$ is negative. Clearly, a zero game plus a zero game cannot be negative. In fact, fuzzy games are neither equal to, greater than, or less than zero, but rather "confused" with zero [3]. Consider what happens when we add a small positive number to g, like $\frac{1}{64}$.

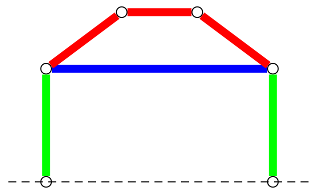


The game $g + \frac{1}{64}$ is positive, since Left will win either by deleting the green edge or, if Right deletes the green edge, by deleting the blue edge from the component $\frac{1}{64}$. If we add a small negative number, like $-\frac{1}{64}$, to g then we get the game pictured below.



This game is negative since Right will always win by either taking the green edge or her red edge of $-\frac{1}{64}$. Thus $-\frac{1}{64} < g < \frac{1}{64}$. It is not hard to see that this argument will hold for smaller and smaller fractions, thus g will be greater than all negative numbers and less than all positive numbers.

In fact, any Hackenbush picture in which all of the edges directly connected to the ground are green has a value that lies strictly between all negative and positive numbers [3]. However, such a game can still have a value that is positive or negative. Consider the house below.



The first person forced to delete one of the green edges will lose, since the other player can then end the game by deleting the second green edge. Since Right can force Left to delete the first green edge by deleting her red edges on her first one or two turns (depending on who goes first), Right can always win. Thus the game has a negative value. However, as before, if we add $\frac{1}{64}$, Left will win. Thus $house + \frac{1}{64} > 0$, which means that the house is greater than $-\frac{1}{64}$. Again, this will hold for any negative number, which means that the house has a negative value, but is greater than any negative number.

Conclusion

Although we have just scratched the surface of game theory, we get an idea of how surreal numbers can be used to analyze games. In addition to Hackenbush, they can be applied to all sorts of two-player games, examples of which can be found in *On Numbers and Games* and *Winning Ways*. As well as looking deeper into game theory, interested readers can look into more advanced topics involving surreal numbers, such as number theory, algebra, and analysis. For example, how would polynomials with surreal coefficients behave? Or what would the field $S[i]$ look like, where $i = \sqrt{-1}$? Conway explores all of these topics, and more, in *On Numbers and Games*.

References

[1] Donald E. Knuth. *Surreal Numbers.* Reading, MA: Addison-Wesley Pub., 1974. Print.

[2] John H. Conway. *On Numbers and Games.* London: Academic, 1976. Print.

[3] Elwyn R. Berlekamp, John H. Conway, and Richard K. Guy. *Winning Ways, for Your Mathematical Plays.* London: Academic, 1982. Print.