INTRODUCTION TO LIMIT MIXED HODGE STRUCTURES

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1. Hodge theory for a smooth projective curve

Let X be a smooth projective curve over $\mathbb{C}$, then the classical Hodge decomposition yields a decomposition

$$H^1(X,\mathbb{Z})\otimes\mathbb{C} = H^1(X,\mathbb{C}) = H^{10}\oplus H^{01}$$

such that $\overline{H^{10}} = H^{01}$. Here H^{10} can be identified with space of holomorphic 1-forms $H^0(X,\Omega^1_X)$. We refer to a lattice $H_\mathbb{Z}$ equipped with such a decomposition as a Hodge structure of type $\{(1,0),(0,1)\}$, or simply as a Hodge structure. Define the Hodge filtration by the rules $F^0 = H$, $F^1 = H^{10}$ and $F^2 = 0$. This clearly determines the structure, and we prefer to work with for reasons that will be clear later. For many aspects of the theory it is important to note that our geometric example possesses a nondegenerate integer valued symplectic pairing $\langle,\rangle$ given by cup product. Since we can write it as $\int_X \alpha\wedge\beta$, we see that the following Hodge-Riemann relations hold

(HR1) F^1 is isotropic i.e. $\langle F^1, F^1\rangle = 0$
(HR2) if $\alpha\in F^1$ is nonzero, then $-i\langle\alpha,\bar\alpha\rangle > 0$.

In the abstract case, such a pairing is called a polarization.

Fix the lattice $H_\mathbb{Z}$ of rank $2g$ (it has to be even) with a form $\langle,\rangle$. Then the set of polarized Hodge structures on it is a parameterized by a complex manifold D. If we choose symplectic basis of $H_\mathbb{Z}$, F^1 is spanned by the rows of matrix of the form (I,Z) [GH]. The Hodge-Riemann relations will imply that Z is symmetric with positive definite imaginary part, and D can be identified with the space of such matrices (called the Siegel upper half plane). If we drop (HR2), then we get a bigger manifold $D^\vee$ which is compact and contains D as an open submanifold. One rather important fact, which justifies our interest is

Theorem 1.1 (Torelli)**.** *X is determined by the polarized Hodge on $H^1(X)$.*

Torelli's is usually not stated this way. Instead, one forms a torus

$$J(X) = \frac{H^1(X)}{F^1 + H^1(X,\mathbb{Z})}$$

called the Jacobian. This carries an ample line bundle, whose first Chern class (suitably interpreted) is the polarization. The usual statement is that $J(X)$ with its polarization determines X.

2. Hodge theory on a singular projective curve

Let Y be a singular possibly reducible projective curve, and let $\pi:\tilde Y\to Y$ be the normalization. It is not difficult to check that $H_1(\tilde Y,\mathbb{Z})\to H_1(Y,\mathbb{Z})$ is injective, and the dual map $H^1(Y)\to H^1(\tilde Y)$ is surjective. We define the weight filtration by

$W_0 = \ker \pi^*$, $W_1 = H^1(Y)$. To make this more explicit, let us assume that Y has simple normal crossings. This means that the irreducible components $Y_1, Y_2, \ldots$ are smooth and they either meet each other in single point (local analytically) like the x and y-axes in the plane or not all[1]. If Y_i and Y_j intersect in a point (with $i < j$) we label it p_{ij}. Let us write q_{ij} (resp q_{ji}) for a copy of p_{ij} on Y_i (resp Y_j) in the disjoint union $\tilde{Y}$. We can keep track of the combinatorics by forming a graph Γ, called the dual graph. The vertices are components, and they are joined by an edge if the components meet. Let $\Sigma = \{p_{ij}\} \subset Y$ be the set of double points. We have a pair of maps

$$\delta_0, \delta_1 : \Sigma \rightrightarrows \tilde{Y} = \coprod Y_i$$

Where δ_0 (resp δ_1) maps p_{ij} to q_{ij} (resp q_{ji}). We have a Mayer-Vietoris type sequence

$$H^0(\tilde{Y}) \to H^0(\Sigma) \to H^1(Y) \to H^1(\tilde{Y})$$

which shows that $W_0 \cong H^1(\Gamma)$. Now $W_1/W_0 = H^1(Y)$ carries a Hodge structure and therefore subspace F^1. It turns out that there is a natural choice of subspace $F^1H^1(Y) \subset H^1(Y)$ which maps to the previous F^1. We give the construction. We have a short exact sequence

$$0 \to \mathcal{O}_Y \to \pi_*\mathcal{O}_{\tilde{Y}} \to \mathbb{C}_\Sigma \to 0$$

The last map takes a collection of functions f_i on Y_i to $f_i(p_{ij}) - f_j(p_{ij})$. We can form a sort of de Rham complex

$$\mathcal{O}_Y \to \pi_*\Omega^1_{\tilde{Y}}$$

Lemma 2.1. *This resolves $\mathbb{C}_Y$*

Proof. It comes down to a diagram chase using

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \mathbb{C}_Y \arrow[r] \arrow[d] & \mathcal{O}_Y \arrow[d] \arrow[dr] & & \\
0 \arrow[r] & \pi_*\mathbb{C}_{\tilde{Y}} \arrow[r] \arrow[d] & \pi_*\mathcal{O}_{\tilde{Y}} \arrow[r] \arrow[d] & \pi_*\Omega^1_{\tilde{Y}} \arrow[r] & 0 \\
 & \mathbb{C}_\Sigma \arrow[r] \arrow[d] & \mathbb{C}_\Sigma \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

where the middle row and two columns are exact. □

The lemma implies

$$H^1(Y, \mathbb{C}) \cong H^1(Y, \mathcal{O}_Y \to \pi_*\Omega^1_{\tilde{Y}})$$

The image

$$F^1H^1(Y) = \operatorname{im} H^0(Y, \pi_*\Omega^1_{\tilde{Y}}) = H^1(Y, \mathcal{O}_Y \to \pi_*\Omega^1_{\tilde{Y}})$$

[1]There was some discussion about what "simple normal crossings" means, for now I use the most restrictive sense, although this will be relaxed later.

gives the desired space. The data of $H^1(Y)$ and the filtrations W, F is called the mixed Hodge structure. More formally, a mixed Hodge structure of type $\{(1,0),(0,1),(0,0)\}$ is lattice $H_{\mathbb{Z}}$, a subspace $W_1 \subset H_{\mathbb{Q}}$, and a subpsace $F^1 \subset H_{\mathbb{C}}$ such that $F^1 \cap W_1 = 0$ and H/W_1 with $\operatorname{im} F^1$ is a Hodge structure in the previous sense. A very nice alternative way of thinking about this will be given later.

Another consequence of the lemma is that $H^1(Y)$ is the hypercohomology of the total complex associated to the double complex

$$\begin{tikzcd} & F^0 & F^1 \\ W_0 & \pi_*\mathcal{O}_{\tilde{Y}} \arrow[r] \arrow[d] & \pi_*\Omega^1_{\tilde{Y}} \\ W_{-1} & \mathbb{C}_\Sigma & \end{tikzcd}$$

This bigger complex allows us to see both F and W at level of complexes as indicated (the indexing for W should be ignored for now).

3. 1-motives

Let Y be as in the previous section. One may ask what the mixed Hodge structure on $H^1(Y)$. To answer this, form the (generalized) Jacobian

$$J(Y) = \frac{H^1(Y)}{F^1H^1(Y) + H^1(Y,\mathbb{Z})}$$

Since $F^1 \cap W_1 = 0$, this fits into an exact sequence

$$0 \to W_1/W_1 \cap H^1(Y,\mathbb{Z}) \to J(Y) \to J(\tilde{Y}) \to 0$$

The group on the right is an abelian variety, and on the left is a product of $\mathbb{C}^*$'s. Such a group is called a semiabelian variety. Using the exponential sequence, we can identify $J(Y) = Pic^0(Y)$. We can reverse the above construction, and recover the MHS on $H^1(Y)$ from $J(Y)$. So in this sense, they are equivalent. More or less the same reasoning shows that

Theorem 3.1 (Deligne)**.** *There is an equivalence between (the categories of) polarizable mixed Hodge structures of type* $\{(1,0),(0,1),(0,0)\}$ *and semiabelian varieties.*

This can be extended slightly. We start with a motivating example, suppose that $U \subset Y$ is obtained by removing a finite set of smooth points $D = \{y_0, \ldots, y_n\}$. Let γ_i be a loop around y_i. Then we have an exact sequence

$$\langle \gamma_0, \ldots, \gamma_n \rangle \to H_1(U,\mathbb{Z}) \to H_1(Y,\mathbb{Z}) \to 0$$

Note that the kernel of first map is the subgroup generated by the relation $\sum \gamma_i = 0$. Dualizing gives

$$0 \to \underbrace{H^1(Y,\mathbb{Z})}_{W_1} \to \underbrace{H^1(U,\mathbb{Z})}_{W_2} \to \mathbb{Z}^{n-1} \to 0$$

We extend the weight filtration as indicated. F^1 can also be extended. To simplify the discussion, suppose that Y is smooth. Let $\Omega^1_Y(\log D)$ be the sheaf of meromorphic 1-forms with simple poles at the y_i, and no other singularities. Then $F^1 = H^1(Y, \Omega_Y(\log Y))$. In the more general case, where Y has singularities,

$$F^1H^1(Y) = \mathrm{im}\, H^0(Y, \pi_*\Omega^1_{\tilde{Y}}(\log \tilde{D})) = H^1(Y, \mathcal{O}_Y \to \pi_*\Omega^1_{\tilde{Y}}(\log \tilde{D}))$$

where $\tilde{D} = \pi^{-1}D$. What we have is an integral two step filtration $W_\bullet$, a complex subspace F^1 such that $F^1 \cap W_1$ satisfies earlier conditions, and $F^1 \mod W_1$ is everything. Such a thing is called an MHS of type $\{(0,0),(1,0),(0,1),(1,1)\}$. Given a such a polarized MHS H, we get a semiabelian variety $J(W_1H) = W_1H/(F^1 \cap H + W_1H_{\mathbb{Z}})$ as before. We have an isomorphism $W_1/F^1 \cap W_1 \cong H/F^1$. Thus we get a projection $H_{\mathbb{Z}} \to J(W_1H)$, which factors through the lattice $\Lambda = W_2H_{\mathbb{Z}}/W_1H_{\mathbb{Z}}$. A semiabelian variety A, together with a homomorphism from a lattice $\alpha : \Lambda \to A$, is called a 1-motive.

Theorem 3.2 (Deligne)**.** *There is an equivalence between (the categories of) polarizable mixed Hodge structures of the type and 1-motives.*

In our geometric example, $A = J(Y)$, Λ is the free abelian group generated by differences $y_i - y_0$, and α is simply the Abel-Jacobi map restricted to this group.

4. Limit MHS

Let us suppose that $f : X \to \Delta$ is a (flat) family of projective curves with exactly one singular fibre $X_0 = Y$. We assume that this is reduced with (not necessarily strict) normal crossings. After shrinking Δ, we can assume that $Y \subset X$ is a homotopy equivalence. Therefore we have a map

$$sp : H^1(Y) \cong H^1(X) \to H^1(X_t)$$

for any $t \neq 0$, called specialization. This is almost never a morphism of mixed Hodge structures, when $H^1(X_t)$ is given the standard Hodge structure. We can give it a nonstandard mixed Hodge structure called the limit mixed Hodge structure, where *sp* does become a morphism. To be clear, we point out we don't actually work with a particular $H^1(X_t)$, but a sort of idealized version of it, which is the cohomology of the nearby fibre $H^1(\mathbb{R}\psi\mathbb{Q})$ (which can be understood as $H^1(X \times_\Delta \tilde{\Delta}^*, \mathbb{Q})$, where $\tilde{\Delta}^*$ is the universal cover of the punctured disk). These are isomorphic but not canonically. For the moment however, we won't worry about this.

As $t \to 0$, X_t may have several cycles δ_i, called vanishing cycles, which shrink to points. By Poincaré duality, we view this as cohomology classes. If T denotes the monodromy on $H = H^1(X_t)$, then the Picard-Lefschetz formula gives

$$T\alpha = \alpha \pm \sum (\alpha \cdot \delta_i)\delta_i$$

This implies that $N = T - I$ satisfies $N^2 = 0$. Let

$$M_0 = \mathrm{im}\, N, M_1 = \ker N, M_2 = H$$

This called the monodromy weight filtration. A fact, which is more or less obvious, is that N induces an isomorphism $M_2/M_1 \cong M_0$. In higher dimensions, M is characterized by a generalization of this property.

Theorem 4.1 (Schmid [S])**.** $H^1(X_t)$ *(or more correctly $H^1(\mathbb{R}\psi\mathbb{Q})$) carries a mixed Hodge structure such that the weight filtration is the monodromy filtration, and the specialization map is a morphism.*

My attribution is bit misleading, since the main point of Schmid's theorem is to extend this to higher dimensions. The curve case was known before and is comparatively easy [G, §13]. In outline, one can choose a symplectic basis $\gamma_1, \ldots, \gamma_{2g}$ for $H^1(X_t, \mathbb{Q})$, so that

$$T\gamma_i = \begin{cases} \gamma_i + \sum_{j=1}^h S_{ij}\gamma_{g+j} & \text{if } i \leq h \\ \gamma_i & \text{otherwise} \end{cases}$$

for some $h \leq g$ and S a positive definite symmetric $h \times h$ matrix. Moreover, $F^1H^1(X_t)$ is spanned by

$$\alpha_i(t) = \gamma_i + \sum_{j=1}^{g} Z_{ij}(t)\gamma_{g+j}$$

such that the period matrix has a block decomposition

$$Z(t) = \begin{pmatrix} \frac{\log t}{2\pi\sqrt{-1}}S + A(t) & B(t) \\ B(t)^T & C(t) \end{pmatrix}$$

The matrices $A(t), B(t), C(t)$ are holomorphic at $t = 0$, and $C(t)$ is symmetric with positive definite imaginary part. Thus $C(t)$ lies in the Siegel upper half plane of $(g-h) \times (g-h)$ matrices. One finds that on

$$M_1/M_0 = \langle \gamma_{h+1}, \ldots, \gamma_g, \gamma_{g+h+1}, \ldots, \gamma_{2g} \rangle$$

the restriction of $F^1H^1(X_t)$ is the row space of the matrix $(I, C(t))$. This gives a Hodge structure for all t, and in particular in the limit $t = 0$.

I will outline Steenbrink's construction [St], again for curves. This is more geometric. As first step, write

$$H^1(X_t, \mathbb{C}) = H^1(X_t, \mathcal{O}_{X_t}) \oplus H^1(X_t, \Omega^1_{X_t})$$

Both factors are g dimensional, where g is the genus. By flatness and Serre duality

$$\dim H^0(Y, \omega_Y) = \dim H^1(Y, \mathcal{O}_Y) = g$$

where ω_Y is the dualizing sheaf. Thus

$$H^1(Y, \mathcal{O}_Y) \oplus H^0(Y, \omega_Y)$$

has the correct dimension and perhaps the right "feel", but it is still far from what we want. To go further, we need to use the geometry of X. By adjunction, $\omega_Y \cong \Omega^2_X(Y)|_Y$. We can write $\Omega^2_X = \Omega^2_X(\log Y)$, which is locally given by $\Omega^2_Y\langle dz_1 \wedge dz_2/z_1z_2\rangle$ if $Y = \{z_1z_2 = 0\}$. We filter this by $W_k\Omega^\bullet_X(\log Y)$ by allowing at most k z_i's in the denominator. In particular, $W_0\Omega^\bullet_X(\log Y) = \Omega^\bullet_X$. Then the adjunction isomorphism can be written as $\Omega^2_X(\log Y)/W_0 \cong \omega_Y$. The remaining sheaf $\mathcal{O}_Y$ can be resolved by logarithmic differentials

$$0 \to \mathcal{O}_Y \xrightarrow{dt/t} \Omega^1_X(\log Y)/W_0 \xrightarrow{dt/t} \Omega^2_X(\log Y)/W_1 \to 0$$

Putting these together yields Steenbrink's double complex $A^{\bullet\bullet}$:

$$\begin{tikzcd} & & F^1 \\ 0 \arrow[r] & \Omega^1(\log Y)/W_0 \arrow[r, "d"] \arrow[d, "\frac{dt}{t}"] \arrow[dl, dashed, "N"] & \Omega^2(\log Y)/W_0 \arrow[dl, dashed, "N"] \\ 0 \arrow[r] & \Omega^2(\log Y)/W_1 & \end{tikzcd}$$

where the differentials are marked with solid arrows. The operators N are given by the projections up to sign.

Theorem 4.2 (Steenbrink). *$H^1(Tot(A^{\bullet\bullet})) \cong H^1(\mathbb{R}\psi\mathbb{C})$, and the action of N on the complex induces N on the cohomology on the right.*

To build an MHS, we filter the complex by F^1 as indicated above, and

$$M_k = W_{2q+k+1}\Omega_X^{p+q+1}(\log Y)/W_q$$

The filtrations for the limit MHS are induced by F^1 in the obvious way, and by $M_\bullet$ with shift

$$M_{k+1}H^1(\mathbb{R}\psi\mathbb{C}) = H^1(\operatorname{im} M_k)$$

To get some sense of what's happening, let us compute M_0. By definition it is

$$\begin{tikzcd} \Omega^1(\log Y)/W_0 \arrow[r] \arrow[d] & W_1\Omega^2(\log Y)/W_0 \\ \Omega^2(\log Y)/W_1 & \end{tikzcd}$$

Using Poincaré residue isomorphisms (which amounts to formally integrating out the logarithmic terms) we can identify M_0 with

$$\begin{tikzcd} \pi_*\mathcal{O}_{\tilde{Y}} \arrow[r] \arrow[d] & \pi_*\Omega^1_{\tilde{Y}} \\ \mathbb{C}_\Sigma & \end{tikzcd}$$

This is exactly the complex we had before for computing the mixed Hodge structure of Y. Moreover the F^1's match, and M_{-1} maps to W_{-1} from before. Thus we really are getting $H^1(Y) = M_1H^1(\mathbb{R}\psi\mathbb{C})$ as mixed Hodge structures.

Finally, I should add that in order for the machinery[2] to work, the underlying rational structure needs to be constructed at the complex level as well. Steenbrink's original arguments were incomplete. But the correct construction was given in a follow up paper [St2]. The idea is $\mathbb{Q}_{X-Y}$ is quasi-isomorphic to the complex

$$(\mathcal{O}_{X-Y} \stackrel{e^{2\pi i}}{\to} \mathcal{O}^*_{X-Y}) \otimes \mathbb{Q}$$

To extend this across Y, let $j : X - Y \to X$ be the inclusion, and $\mathcal{M} = \mathcal{O}_X \cap j_*\mathcal{O}^*_X$ the sheaf of multiplicative monoids. The sheaf $\mathcal{M}$ is basically a log structure in the sense of Fontaine-Illusie-Kato [K]. Given a two term complex $C^0 \to C^1$ of $\mathbb{Q}$-vector

[2]The name of the machine is "cohomological mixed Hodge complex" [D, §8]

spaces, the second symmetric power is the subcomplex of symmetric elements of $C^\bullet \otimes C^\bullet$ or more explicitly

$$S^2(C^\bullet) = S^2C^0 \to C^0 \otimes C^1 \to \wedge^2 C^1$$

If $\mathcal{M}^{gp}$ is the group completion, then the symmetric power

$$K^\bullet = S^2(\mathcal{O}_X \stackrel{e^{2\pi i}}{\to} \mathcal{M}^{gp}) \otimes \mathbb{Q}$$

gives a complex such that $K \otimes \mathbb{C} \cong \Omega^\bullet_X(\log Y)$ in the derived category. One can put a compatible filtration W on K. Then proceeding as above, we get a filtered complex $(A^\bullet, M)$ defined over $\mathbb{Q}$ which becomes filtered quasi-isomorphic to $(Tot(A^{\bullet\bullet}), M)$ after tensoring with $\mathbb{C}$. Let me remark that with a bit more care, we can work integrally. Also one doesn't need the whole of X just the restriction of the log structure to Y; this is essentially first order information.

5. 1-motive of limit MHS

Keep the same notation as in the last section. The limit MHS is of type $\{(0,0),\ldots,(1,1)\}$, so it corresponds to 1-motive. I want to explain a nice description of thus due to Hoffman [H]. For the semiabelian variety, we have no choice but to take the generalized Jacobian $J(Y) = Pic^0(Y)$. For the lattice Λ, we take subgroup of divisors $D = \sum n_q q$ on $\tilde{Y}$ supported on $\tilde{\Sigma} = \pi^{-1}\Sigma$ such that D has degree zero on every component, and such that $n_q + n_{q'} = 0$ whenever $\pi(q) = \pi(q')$. Note that $\Lambda \cong H^1(\Gamma, \mathbb{Z})$. The part that is a bit tricky is the homomorphism $\alpha : \Lambda \to J(Y)$. $\alpha(D)$ should be a line bundle L_D on Y. As a first step, L_D would pull back to the line bundle $\tilde{L}_D = \mathcal{O}_{\tilde{Y}}(D)$ on $\tilde{Y}$. To get L_D, we need to glue the restrictions of $\tilde{L}_D|_{\tilde{Y}_i}$ along the double points. So for each $p \in \Sigma$, we need to choose isomorphisms between the fibres of $\tilde{L}_D$ at q and q', whenever $\pi(q) = \pi(q')$. Thus the set of gluing data is noncanonically a product of $\mathbb{C}^*$'s. For single pair of points q, q' such that $\pi(q) = \pi(q')$, we have

Proposition 5.1 (Hoffman)**.** *For any singular point p, $\dim Ext^1(\Omega^1_{Y,p}, \mathcal{O}_p) = 1$. If q, q' are distinct points such that $\pi(q) = \pi(q') = p$, the set of nonzero elements in the dual $Ext^1(\Omega^1_{Y,p}, \mathcal{O}_p)^*$ can be canonically identified with set of isomorphisms of the fibre of $\mathcal{O}(q)$ at q with the fibre of $\mathcal{O}(-q')$ at q'.*

An element of $Ext^1(\Omega^1_{Y,p}, \mathcal{O}_p)$ gives a rule for gluing $\mathcal{O}(q)$ with $\mathcal{O}(-q')$ by taking identities at the other pairs. This Ext also parameterizes the first order deformations of the singularities $Spec\, \mathcal{O}_p$ [DM]. The first order deformations of Y are given by $Ext^1(\Omega^1_Y, \mathcal{O}_Y)$. The local to global spectral sequence gives a surjection

$$Ext^1(\Omega^1_Y, \mathcal{O}_Y) \to \bigoplus_{p\in\Sigma} Ext^1(\Omega^1_{Y,p}, \mathcal{O}_p)$$

Since our family $X \to \Delta$ gives a deformation of Y, the tangent vector $\frac{\partial}{\partial t}$ on Δ gives an element of $Ext^1(\Omega^1_Y, \mathcal{O}_Y)$. Its images in the local Ext's are all nonzero because all the singularities are smoothed. Therefore we get preferred bases for these groups, and hence for their duals. This gives a rule gluing $\mathcal{O}(q)$ and $\mathcal{O}(-q')$ to obtain a line bundle which we define to be $L_{q-q'}$ on Y. Since any divisor $D \in \Lambda$ is a linear combination of divisors of the form $q - q'$. We can define L_D by linearity.

5.1. **Comments/Questions.**

(1) I haven't really gone through Hoffman's arguments, but I suspect that they can be simplified, perhaps using later developments such log geometry.
(2) One reason for doing this is to extend the picture. Given a family $X \to \Delta$, as before, with n sections σ_i in general position, one can ask whether there is a limit for the mixed Hodge structures $H^1(X_t - \{\sigma_1(t), \ldots\})$. Work of Steenbrink and Zucker [SZ] shows that this can be done. This should be governed by 1-motive as well, which should be constructed explicitly.
(3) The nice thing about 1-motives is that they are algebraic, so they make sense over any field. In general, 1-motives posses ℓ-adic realizations. Given a family of curves over the spectrum of a Henselian DVR, this *should* tie in with the existing of theory [Gr].

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