TATE'S THESIS

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These are an approximation of what was covered in lecture in 18.786 Number theory II at MIT during the first month or so of Spring 2015. This PDF file is divided into sections; the instructions for viewing the table of contents depend on which PDF viewer you are using.

The syllabus for the semester was

- Tate's thesis
- Galois cohomology
- An introduction to Galois representations (as much as we have time for)

but these notes cover only Tate's thesis. We discuss both the number field and global function field cases, and also the subsequent reformulation by Deligne using L- and ϵ-factors.

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February 3

1. The prototype: analytic continuation and functional equation of the Riemann zeta function

Recall the Riemann zeta function, defined for $\operatorname{Re} s > 1$ by

$$\zeta(s) := \prod_{\text{prime } p} (1 - p^{-s})^{-1} = \sum_{n \ge 1} n^{-s}.$$

Riemann gave two proofs of the following:

Theorem 1.1 (Riemann 1860)**.**

(a) *(Analytic continuation) The function $\zeta(s)$ extends to a meromorphic function on $\mathbb{C}$, holomorphic except for a simple pole at $s = 1$.*

(b) *(Functional equation) The completed zeta function*

$$\xi(s) := \pi^{-s/2} \underbrace{\Gamma\left(\frac{s}{2}\right)}_{\text{defined below}} \zeta(s)$$

satisfies $\xi(s) = \xi(1 - s)$.

Today we will explain one of these proofs, as presented in [Dei05, Appendix A].

Hecke in 1918–1920 generalized Theorem 1.1 to Dedekind zeta functions $\zeta_K(s)$, and then quickly realized that his method applied also to Dirichlet L-functions and even more general L-functions. Tate in his 1950 Ph.D. thesis (reprinted in [Tat67]), building on the 1946 Ph.D.

thesis of Margaret Matchett (both of them were students of Emil Artin), gave an adelic variant of Hecke's proof. The adelic methods provided further insight and became part of the inspiration for the Langlands program.

Before proving Theorem 1.1, we need to recall some definitions and theorems from analysis, which we will assume without proof.

1.1. **The gamma function.** The definition

$$\Gamma(s) := \int_0^\infty e^{-t}\, t^s \,\frac{dt}{t}$$

(convergent for $\operatorname{Re} s > 0$) involves

- an additive character $\mathbb{R} \to \mathbb{C}^\times$,
- a multiplicative character $\mathbb{R}^\times \to \mathbb{C}^\times$, and
- a Haar measure on the group $\mathbb{R}^\times$.

Properties:

- $\Gamma(s+1) = s\,\Gamma(s)$. (Proof: Integration by parts.)
- $\Gamma(s)$ extends to a meromorphic function on $\mathbb{C}$, holomorphic except for simple poles at $0, -1, -2, \ldots$. (Proof: Use the functional equation for Γ above.)
- $\Gamma(s)$ has no zeros.
- $\Gamma(n) = (n-1)!$ for each $n \in \mathbb{Z}_{\geq 1}$. (Proof: Induction.)
- $\Gamma(1/2) = \sqrt{\pi}$. (Proof: Equivalent to $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$).

1.2. **The Fourier transform.** We will consider complex-valued functions on the domain $\mathbb{R}$.

Definition 1.2. Say that a function $f \colon \mathbb{R} \to \mathbb{C}$ tends to 0 rapidly if for every $n \geq 1$ we have $x^n f(x) \to 0$ as $|x| \to \infty$. (Equivalently, for every n we have $|f(x)| = O(1/x^n)$ as $|x| \to \infty$.)

Definition 1.3. Call $f \colon \mathbb{R} \to \mathbb{C}$ a Schwartz function if for every $r \geq 0$ the r^{th} derivative $f^{(r)}$ exists and tends to 0 rapidly. The Schwartz space $\mathscr{S} = \mathscr{S}(\mathbb{R})$ is the set of all Schwartz functions.

Examples of Schwartz functions:

- any C^∞ function of compact support
- e^{-x^2}

Definition 1.4. Given $f \in \mathscr{S}$, define the Fourier transform $\widehat{f}$ by

$$\widehat{f}(y) := \int_{\mathbb{R}} f(x)\, e^{-2\pi i x y}\, dx.$$

It turns out that $\widehat{f} \in \mathscr{S}$.

Example 1.5. The function $f(x) := e^{-\pi x^2}$ is in $\mathscr{S}$. It turns out that $\widehat{f} = f$.

Fourier inversion formula. *If $f \in \mathscr{S}$, then*

$$f(x) := \int_{\mathbb{R}} \widehat{f}(y)\, e^{2\pi i x y}\, dy.$$

In other words, $\hat{\hat{f}}(x) = f(-x)$.

Remark 1.6. With more work, one can define $\widehat{f}$ more generally for $f \in L^2(\mathbb{R})$, and the Fourier inversion formula still holds.

Poisson summation formula. *If $f \in \mathscr{S}$, then*

$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \widehat{f}(n).$$

The Poisson summation theorem can be generalized to a wider class of functions: see Theorem 3.6.1 in [Dei05].

1.3. **A theta function.** For real $t > 0$, define

$$\Theta(t) := \sum_{n \in \mathbb{Z}} e^{-\pi n^2 t}.$$

Theorem 1.7 (Functional equation for Θ)**.** *For every real $t > 0$,*

$$\Theta(t) = t^{-1/2}\, \Theta\left(\frac{1}{t}\right).$$

Proof. If $f \in \mathscr{S}$ and $c \neq 0$, then the Fourier transform of $f(x/c)$ is $c\widehat{f}(cy)$ (integrate by substitution). Apply this to $f(x) := e^{-\pi x^2}$ (which satisfies $\widehat{f} = f$) and $c := t^{-1/2}$: the Fourier transform of $f_t(x) := e^{-\pi t x^2}$ is $\widehat{f_t}(y) = t^{-1/2} e^{-\pi(1/t)y^2}$. Apply the Poisson summation formula to f_t:

$$\sum_{n \in \mathbb{Z}} e^{-\pi n^2 t} = \sum_{n \in \mathbb{Z}} t^{-1/2} e^{-\pi n^2 (1/t)}$$

$$\Theta(t) = t^{-1/2}\, \Theta\left(\frac{1}{t}\right). \qquad \square$$

It is the functional equation for $\Theta(t)$ that will imply the functional equation for $\zeta(s)$.

Remark 1.8. For z in the upper half plane $\mathfrak{h} := \{z \in \mathbb{C} : \operatorname{im} z > 0\}$, define $q := e^{2\pi i z}$. Jacobi's classical theta function

$$\theta(z) := \sum_{n \in \mathbb{Z}} q^{n^2/2} = \sum_{n \in \mathbb{Z}} e^{i\pi n^2 z}$$

converges to a holomorphic function on $\mathfrak{h}$, and $\theta(it) = \Theta(t)$. Theorem 1.7 translates into

$$\theta(-1/z) = (z/i)^{1/2}\, \theta(z),$$

for the branch of the square root defined using the principal branch of the complex logarithm. This, together with the obvious identity $\theta(z+2) = \theta(z)$ (and a technical growth condition), implies that $\theta(z)$ is a modular form of weight $1/2$ for the subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ generated by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.

Remark 1.9. Since $e^{-\pi n^2 t}$ is an even function of n, we have

$$\Theta(t) = 1 + 2\sum_{n\ge 1} e^{-\pi n^2 t}$$

$$\sum_{n\ge 1} e^{-\pi n^2 t} = \frac{\Theta(t)-1}{2}.$$

1.4. **Riemann's proof.** Look at the contribution to $\xi(s)$ coming from one summand in $\zeta(s)$:

$$\begin{aligned}\pi^{-s/2}\,\Gamma(s/2)\,n^{-s} &= \pi^{-s/2}\,n^{-s}\int_0^\infty e^{-x}\,x^{s/2}\,\frac{dx}{x} \\ &= \int_0^\infty e^{-\pi n^2 t}\,t^{s/2}\,\frac{dt}{t} \qquad\qquad (\text{substitute } x = \pi n^2 t).\end{aligned}$$

Summing over integers $n \ge 1$ yields

$$\xi(s) = \int_0^\infty \left(\frac{\Theta(t)-1}{2}\right) t^{s/2}\,\frac{dt}{t} \tag{1}$$

if we can justify the interchange of summation and integration. For $s \in \mathbb{R}_{>1}$, this is OK because everything is nonnegative and the sum on the left converges. In fact, the right hand side converges absolutely for any complex s with $\operatorname{Re} s > 1$, since changing the imaginary part of s does not affect $|t^{s/2}|$. Thus (1) is valid whenever $\operatorname{Re} s > 1$. If s is a negative real number, on the other hand, the integral in formula (1) does not converge, because the integrand grows too quickly as $t \to 0^+$.

The fix will be to split the range of integration $(0,\infty)$ into the good part $(1,\infty)$ and the problematic part $(0,1)$, and then to rewrite the problematic part by using the functional equation of $\Theta(t)$.

- The good part contributes

$$I(s) := \int_1^\infty \left(\frac{\Theta(t)-1}{2}\right) t^{s/2}\,\frac{dt}{t},$$

  which converges to a holomorphic function on all of $\mathbb{C}$ since $\Theta(t) - 1 = O(e^{-\pi t})$ as $t \to \infty$.
- As for the problematic part, the substitution $t \mapsto 1/t$ negates dt/t and leads to

$$\begin{aligned}\int_0^1 \left(\frac{\Theta(t)-1}{2}\right) t^{s/2}\,\frac{dt}{t} &= \int_1^\infty \left(\frac{\Theta\left(\frac{1}{t}\right)-1}{2}\right) t^{-s/2}\,\frac{dt}{t} \\ &= \int_1^\infty \left(\frac{t^{1/2}\,\Theta(t)-1}{2}\right) t^{-s/2}\,\frac{dt}{t} \qquad (\text{by Theorem 1.7}) \\ &= \int_1^\infty \left(\frac{\Theta(t)-1}{2}\right) t^{(1-s)/2}\,\frac{dt}{t} + \frac{1}{2}\int_1^\infty t^{(1-s)/2}\,\frac{dt}{t} - \frac{1}{2}\int_1^\infty t^{-s/2}\,\frac{dt}{t} \\ &= I(1-s) - \frac{1}{1-s} - \frac{1}{s}.\end{aligned}$$

Putting the two parts back together yields

$$\xi(s) = I(s) + I(1-s) - \frac{1}{1-s} - \frac{1}{s}$$

for $\operatorname{Re} s > 1$. But the right hand side is meromorphic on $\mathbb{C}$, symmetric with respect to $s \mapsto 1-s$, and holomorphic except for simple poles at 0 and 1, so it is the desired meromorphic continuation of $\xi(s)$. Dividing by $\pi^{-s/2}\Gamma(s/2)$, which has simple poles at $0, -2, -4, \ldots$ and no zeros, shows that $\zeta(s)$ has a meromorphic continuation to all of $\mathbb{C}$ with a simple pole at $s = 1$ and no other poles.

Remark 1.10. As a bonus, we see that $\zeta(s)$ has zeros at $-2, -4, -6, \ldots$, and that all other zeros lie in the "critical strip" $\{s \in \mathbb{C} : 0 \leq \operatorname{Re} s \leq 1\}$ (exercise).

1.5. **Generalization.** To generalize, we will use a Poisson summation formula in which $\mathbb{Z} \subset \mathbb{R}$ is replaced by $K \subset \mathbb{A}$, where K is a global field and $\mathbb{A}$ is its adèle ring. To carry this out, we must develop integration and Fourier analysis on locally compact abelian groups such as $\mathbb{A}$.

2. Review of integration

This is not supposed to be a course in analysis. In order to get to the number theory as quickly as possible, we will state many analysis facts without proof in Sections 2 and 3. Some references containing the details are [Dei05] and [DE09].

2.1. **Measure.** Let X be a set, and let $\mathscr{M}$ be a collection of subsets of X.

Definition 2.1. Call $\mathscr{M}$ a σ-algebra if $\mathscr{M}$ is closed under complementation and countable unions (including finite unions, even empty unions!)

Example 2.2. Let X be a topological space. The collection of Borel sets is the σ-algebra $\mathscr{B} = \mathscr{B}(X)$ generated by the open subsets of X.

Fix a σ-algebra $\mathscr{M}$. Elements of $\mathscr{M}$ are called measurable sets.

Definition 2.3. A function $f \colon X \to \mathbb{C}$ is called measurable if inverse images of measurable subsets are measurable, i.e., if for every $S \in \mathscr{B}(\mathbb{C})$, we have $f^{-1}(S) \in \mathscr{M}$.

Remark 2.4. It suffices to check the condition for open disks $S \subset \mathbb{C}$. For real-valued f, it suffices to check the condition for $S = (a, \infty)$ for each $a \in \mathbb{R}$.

Definition 2.5. A measure on $(X, \mathscr{M})$ is a function $\mu \colon \mathscr{M} \to [0, \infty]$ such that $\mu(\bigcup A_i) = \sum \mu(A_i)$ for any countable (or finite) collection of disjoint measurable sets (A_i). In the special case in which $\mathscr{M} = \mathscr{B}$, the measure is called a Borel measure.

Think of $\mu(S)$ as the "volume" of S.

A set $N \subseteq X$ is called a null set if N is contained in a measure-0 set. It is easy and convenient to enlarge $\mathscr{M}$ so that all null sets are measurable. Call $f \colon X \to \mathbb{C}$ a null function if $\{x \in X : f(x) \neq 0\}$ is a null set.

2.2. **Integration.** Fix $(X, \mathscr{M}, \mu)$. Given $S \in \mathscr{M}$ with $\mu(S) < \infty$, let 1_S be the function that is 1 on S and 0 outside S, and define $\int 1_S := \mu(S)$. A step function f is a finite $\mathbb{C}$-linear combination of such functions 1_S; define $\int f$ so that it is linear in f. Also, define the L^1-norm of f by $\|f\|_1 := \int |f| \in \mathbb{R}_{\geq 0}$; this leads to a notion of distance and Cauchy sequence. Call $f \colon X \to \mathbb{C}$ integrable if outside a null set it equals the pointwise limit of some L^1-Cauchy sequence of step functions (f_i); then define $\int f := \lim_{i \to \infty} \int f_i \in \mathbb{C}$. (It turns out to be well-defined.) We may also write $\int_X f \, d\mu$ to indicate the dependence on X or μ.

If f and g are functions $X \to [-\infty, \infty]$, then the notation $f \geq g$ means that $f(x) \geq g(x)$ for all $x \in X$. For measurable $f \geq 0$, the alternative definition

$$\int f := \sup \left\{ \int g : \ g \text{ is a step function and } 0 \leq g \leq f \right\}.$$

agrees with the previous one if f is integrable, and yields ∞ if f is not integrable. (If f takes the value ∞ on a non-null set, then $\int f = \infty$.)

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For a measurable $f \colon X \to \mathbb{C}$, it turns out that f is integrable if and only if $|f|$ is integrable; in this case, $|\int f| \leq \int |f|$.

Here are two important theorems for interchanging limits and integrals:

Monotone convergence theorem. *Suppose that (f_n) is a sequence of measurable functions $X \to [0, \infty]$ such that $0 \leq f_1 \leq f_2 \leq \cdots$. Let f be the pointwise limit of the f_n. Then $\int f_n \to \int f$.*

Dominated convergence theorem. *Let $f_1, f_2, \ldots$ and f be measurable functions $X \to \mathbb{C}$ such that $f_n \to f$ pointwise. If there is an* integrable *function $g \colon X \to \mathbb{C}$ such that $|f_n| \leq |g|$ for all n, then all the f_n and f are integrable, and $\int f_n \to \int f$.*

The dominated convergence theorem applies also when the limit $\lim_{n \to \infty} f_n(x)$ of a sequence is replaced by a limit $\lim_{t \to 0} f(x, t)$ of a family of functions depending on a real parameter t.

2.3. **L^p-spaces.** For $p \in \mathbb{R}_{\geq 1}$, define

$$\mathscr{L}^p(X) := \{\text{measurable } f \colon X \to \mathbb{C} \text{ such that } |f|^p \text{ is integrable}\}.$$

Define the L^p-norm of $f \in \mathscr{L}^p(X)$ by $\|f\|_p := \left(\int |f|^p \right)^{1/p} \in \mathbb{R}_{\geq 0}$. Strictly speaking, $\| \ \|_p$ is not a norm on $\mathscr{L}^p(X)$, since there exist nonzero functions f such that $\|f\|_p = 0$, namely the null functions. But $\| \ \|_p$ induces a norm on

$$L^p(X) := \frac{\mathscr{L}^p(X)}{\{\text{null functions}\}}$$

making $L^p(X)$ a Banach space. We will often pretend that elements of $L^p(X)$ are functions; this is OK as far as integration is concerned.

The space $L^2(X)$ is also a Hilbert space, under the inner product

$$\langle f, g \rangle := \int f\bar{g}.$$

2.4. **Measures and integrals on a locally compact Hausdorff space.**

Definition 2.6. A Hausdorff topological space X is called locally compact if every $x \in X$ has a compact neighborhood.

In the rest of this section, X denotes a locally compact Hausdorff space.

Definition 2.7. An outer Radon measure on X is a Borel measure $\mu \colon \mathscr{B} \to [0, \infty]$ that is

- locally finite: every $x \in X$ has an open neighborhood U such that $\mu(U) < \infty$;
- outer regular: every $S \in \mathscr{B}$ satisfies $\mu(S) = \inf_{\text{open } U \supset S} \mu(U)$; and
- inner regular on open sets: every open $U \subseteq X$ satisfies $\mu(U) = \sup_{\text{compact } K \subseteq U} \mu(K)$.

A function $f \colon X \to \mathbb{C}$ has compact support if the closure of $\{x \in X : f(x) \neq 0\}$ is compact. Define

$$\begin{aligned} C(X) &:= \{\text{continuous } f \colon X \to \mathbb{C}\} \\ C_c(X) &:= \{\text{continuous } f \colon X \to \mathbb{C} \text{ of compact support}\}. \end{aligned}$$

Definition 2.8. A Radon integral on X is a $\mathbb{C}$-linear map $I \colon C_c(X) \to \mathbb{C}$ such that $I(f) \geq 0$ whenever $f \geq 0$.

Given an outer Radon measure μ, define

$$\begin{aligned} I_\mu \colon C_c(X) &\longrightarrow \mathbb{C} \\ f &\longmapsto \textstyle\int f \, d\mu \end{aligned}$$

Riesz representation theorem. *Let X be a locally compact Hausdorff space. Then*

$$\begin{aligned} \{\text{outer Radon measures on } X\} &\xrightarrow{\sim} \{\text{Radon integrals on } X\} \\ \mu &\longmapsto I_\mu \end{aligned}$$

is a bijection.

Example 2.9. Let $X = \mathbb{R}^n$. The map sending $f \in C_c(\mathbb{R}^n)$ to the Riemann integral $\int_{\mathbb{R}^n} f \in \mathbb{C}$ is a Radon integral. Lebesgue measure is defined to be the corresponding outer Radon measure on $\mathbb{R}^n$.

2.5. **Haar measure.**

Definition 2.10. A topological group is a topological space G equipped with a group structure such that multiplication $G \times G \to G$ and inversion $G \to G$ are continuous maps.

Definition 2.11. A Borel measure μ on a topological group G is left-invariant if $\mu(gS) = \mu(S)$ for all $g \in G$ and $S \in \mathscr{B}$. (Right-invariant is defined similarly.)

In the rest of this section, G denotes a locally compact Hausdorff topological group.

Examples:

- $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{A}$ under addition;
- the unit group $A^\times$ of any of the rings A above;
- $\mathrm{GL}_n(A)$ for any of these rings A and any $n \geq 1$; and
- any group equipped with the discrete topology.

Definition 2.12. A left Haar measure on G is a nonzero left-invariant outer Radon measure on G.

Theorem 2.13 (Existence and uniqueness of Haar measure)**.** *Let G be a locally compact Hausdorff topological group.*

(a) *There exists a left Haar measure μ on G (and hence also a corresponding Haar integral).*
(b) *Every other left Haar measure on G is $c\mu$ for some $c \in \mathbb{R}_{>0}$.*

Example 2.14. On $\mathbb{R}^n$, Lebesgue measure is a Haar measure.

Example 2.15. On a discrete group, the counting measure assigning measure 1 to each singleton is a Haar measure.

Warning 2.16. A left Haar measure μ need not be right-invariant. If $g \in G$, then $\mu_g(S) := \mu(Sg)$ defines a possibly different left Haar measure μ_g. By Theorem 2.13(b), $\mu_g = \Delta(g)\mu$ for some $\Delta(g) \in \mathbb{R}_{>0}$. In fact, $\Delta \colon G \to \mathbb{R}_{>0}^\times$ is a homomorphism, called the modular function of G. If G is abelian, then left-invariant implies right-invariant, so $\Delta \equiv 1$. If G is compact, then $\Delta \equiv 1$ since $\Delta(G)$ is a compact subgroup of $\mathbb{R}_{>0}^\times$, and the only such subgroup is $\{1\}$. Thus, if G is abelian or compact, the notions of left Haar measure and right Haar measure coincide. (An example with $\Delta \not\equiv 1$ is the subgroup $\begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix}$ of $\mathrm{GL}_2(\mathbb{R})$.)

Remark 2.17. Let μ be a left Haar measure on G. One can show that G is compact if and only if $\mu(G) < \infty$.

Definition 2.18. Suppose that G is compact. The normalized Haar measure on G is the unique Haar measure μ such that $\mu(G) = 1$.

3. Duality of locally compact abelian groups

3.1. LCA groups.

Definition 3.1. An LCA group is a locally compact abelian Hausdorff topological group.

LCA groups from a category, with continuous homomorphisms as the morphisms.

Example 3.2. Let $\mathbb{T}$ be the unit circle in the complex plane, i.e., the group $\{z \in \mathbb{C} : |z| = 1\}$ under multiplication. So $\mathbb{T} \simeq \mathbb{R}/\mathbb{Z}$ as an LCA group.

3.2. **Short exact sequences.** Call $0 \to A \to B \to C \to 0$ a short exact sequence of LCA groups if A, B, C are LCA groups, the sequence is exact as a sequence of groups without topology, and the topologies of A and C are induced by the topology on B (subspace topology and quotient topology, respectively). In this case, A must be a *closed* subgroup of B, since otherwise B/A would not be Hausdorff. Conversely, given a closed subgroup A of an LCA group B, the sequence $0 \to A \to B \to B/A \to 0$ is a short exact sequence of LCA groups.

Let $0 \to A \to B \to C \to 0$ be a short exact sequence of LCA groups. Given Haar measures on any two of A, B, C, there exists a unique Haar measure on the third such that the three Haar measures da, db, dc are compatible in the sense that

$$\int_B f(b)\, db = \int_C \int_A f(ca)\, da\, dc$$

for every $f \in C_c(B)$ (the inner integral on the right is a function of $c \in B$ that is constant on each coset of A, and hence may be viewed as a function on C).

3.3. Characters.

From now on, G denotes an LCA group.

Definition 3.3. A character of G is a continuous homomorphism $\chi \colon G \to \mathbb{C}^\times$. Let $\mathrm{X}(G)$ be the group of all characters under pointwise multiplication.

Definition 3.4. A unitary character of G is a continuous homomorphism $\chi \colon G \to \mathbb{T}$.

Warning 3.5. Some authors call these quasi-characters and characters, respectively.

Proposition 3.6. *If G is compact, then every character χ of G is unitary.*

Proof. Let $|\chi|$ be the character obtained by taking the complex absolute value of each value. The image of $|\chi|$ is a compact subgroup of $\mathbb{R}^\times_{>0}$, but the only such subgroup is $\{1\}$. □

3.4. The Pontryagin dual.

Definition 3.7. The Pontryagin dual of G is the group

$$\widehat{G} := \mathrm{Hom}_{\mathrm{conts}}(G, \mathbb{T}) = \{\text{unitary characters of } G\}.$$

under pointwise multiplication. Equip $\widehat{G}$ with the compact-open topology, i.e., the topology generated by the sets $\{\chi \in \widehat{G} : \chi(K) \subseteq U\}$ for every compact $K \subseteq G$ and open $U \subseteq \mathbb{T}$.

It turns out that $\widehat{G}$ is another LCA group. Any continuous homomorphism $\phi \colon G \to H$ of LCA groups induces a continuous homomorphism $\widehat{H} \to \widehat{G}$ taking χ to $\chi \circ \phi$. In fact, taking the Pontryagin dual is a contravariant functor from the category of LCA groups to itself.

Theorem 3.8 (Exactness of the Pontryagin dual functor)**.** *If*

$$0 \to A \to B \to C \to 0$$

is a short exact sequence of LCA groups, then so is

$$0 \to \widehat{C} \to \widehat{B} \to \widehat{A} \to 0.$$

Theorem 3.9 (Pontryagin duality)**.** *The canonical homomorphism*

$$\begin{aligned} G &\longrightarrow \hat{\widehat{G}} \\ g &\longmapsto (\chi \mapsto \chi(g)) \end{aligned}$$

is an isomorphism of LCA groups.

There is a continuous bilinear pairing

$$\begin{aligned} G \times \widehat{G} &\to \mathbb{T} \\ (g, \chi) &\mapsto \chi(g). \end{aligned}$$

Pontryagin duality says that the corresponding pairing for $\widehat{G}$ is the same with the roles of G and $\widehat{G}$ reversed. If H is a subgroup of G, let

$$H^{\perp} := \{\chi \in \widehat{G} : \chi(h) = 1 \text{ for all } h \in H\}.$$

Here is a table of some Pontryagin duals (we will prove some of these only later). Let K be a global field, and let $\mathbb{A}$ be its adèle ring.

G	$\widehat{G}$
$\mathbb{R}$	$\mathbb{R}$
$\mathbb{Q}_p$	$\mathbb{Q}_p$
$\mathbb{A}$	$\mathbb{A}$
$\mathbb{Z}$	$\mathbb{R}/\mathbb{Z}$
$\mathbb{Z}_p$	$\mathbb{Q}_p/\mathbb{Z}_p$
K	$\mathbb{A}/K$
finite	finite
discrete	compact
discrete torsion	profinite

Example 3.10. The self-duality of $\mathbb{R}$ is given by the pairing

$$\begin{aligned} \mathbb{R} \times \mathbb{R} &\to \mathbb{T} \\ (x, y) &\mapsto e^{2\pi i x y}. \end{aligned}$$

In other words, we are claiming that the homomorphism $\mathbb{R} \to \widehat{\mathbb{R}}$ sending y to the unitary character $\chi_y(x) := e^{2\pi i x y}$ is an isomorphism. (This will be proved later on.)

3.5. **The Fourier transform.** Recall: If $f \in \mathscr{S}(\mathbb{R})$, then

$$\widehat{f}(y) := \int_{\mathbb{R}} f(x)\, e^{-2\pi i x y}\, dx.$$

Definition 3.11. If $f \in \mathcal{L}^1(G)$, define the Fourier transform $\widehat{f}\colon \widehat{G} \to \mathbb{C}$ by

$$\widehat{f}(\chi) := \int_G f(g)\, \overline{\chi(g)}\, dg.$$

It turns out that $\widehat{f}\colon \widehat{G} \to \mathbb{C}$ is always continuous.

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Cancelled due to snow!

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Fourier inversion formula. *Let G be an LCA group. Let dg be a Haar measure on G. Then there exists a unique Haar measure $d\chi$ on $\widehat{G}$, called the dual measure or Plancherel measure, such that if $f \in \mathcal{L}^1(G)$ is such that $\widehat{f} \in \mathcal{L}^1(\widehat{G})$, then*

$$f(g) := \int_{\widehat{G}} \widehat{f}(\chi)\, \chi(g)\, d\chi. \tag{2}$$

for almost all $g \in G$ (i.e., all g outside a null set). If, moreover, f is continuous, then (2) holds for all *$g \in G$.*

Remark 3.12. Changing the values of f on a null set does not change $\widehat{f}$, so it is impossible to recover f exactly from $\widehat{f}$. This explains why in the theorem it is necessary to exclude a null set, or to assume that f is continuous.

Remark 3.13. Equation 2 is equivalent to the identity $\hat{\hat{f}}(g) = f(-g)$ for almost all g.

Plancherel theorem. *Let dg and $d\chi$ be dual measures on G and $\widehat{G}$, respectively. If $f \in L^1(G) \cap L^2(G)$, then $\|f\|_2 = \|\widehat{f}\|_2$.*

(The hypothesis $f \in L^1(G)$ ensures that $\widehat{f}$ is defined, and $f \in L^2(G)$ ensures that $\|f\|_2$ is defined.)

Corollary 3.14. *The Fourier transform*

$$\begin{aligned} L^1(G) \cap L^2(G) &\longrightarrow L^2(\widehat{G}) \\ f &\longmapsto \widehat{f} \end{aligned}$$

extends to a map $L^2(G) \to L^2(\widehat{G})$, and this extended Fourier transform is an isomorphism of Hilbert spaces.

Proof. The Plancherel theorem implies that the Fourier transform maps L^2-Cauchy sequences in $L^1(G) \cap L^2(G)$ to L^2-Cauchy sequences, so the Fourier transform can be extended to the completion of $L^1(G) \cap L^2(G)$ with respect to $\| \|_2$. That completion is all of $L^2(G)$, since already the subspace $C_c(G)$ of $L^1(G) \cap L^2(G)$ is dense in $L^2(G)$ (with respect to $\| \|_2$). (In fact, $C_c(G)$ is dense in $L^p(G)$ for each $p \geq 1$.)

The Plancherel theorem implies that the extended Fourier transform preserves L^2-norms. The extended Fourier transform on $\widehat{G}$, combined with composition-with-negation-on-G, provides the inverse map. $\square$

Example 3.15. If G is discrete, the counting measure on G is dual to the normalized Haar measure on the compact group $\widehat{G}$. To see this, check the Fourier inversion formula or the Plancherel theorem for the function $f \colon G \to \mathbb{C}$ that is 1 at the identity and 0 everywhere else.

Remark 3.16. Many results of Section 3 can be extended to locally compact Hausdorff topological groups that are not abelian. But the theory becomes much more difficult, for the same reason that representation theory of finite nonabelian groups is more difficult than representation theory of finite abelian groups: instead of unitary characters $G \to \mathbb{T}$ one must consider representations of dimension greater than 1 (and even infinite-dimensional representations, when the nonabelian group is not compact).

4. Local fields

In this section and the next, we follow Tate's thesis as well as [RV99, Chapter 7].

4.1. **Characterizations of local fields.**

Definition 4.1. A local field is a field F satisfying one of the following equivalent conditions:

1. F is $\mathbb{R}$ or $\mathbb{C}$, or else F is the fraction field of a complete discrete valuation ring with finite residue field.
2. F is a finite separable extension of $\mathbb{R}$, $\mathbb{Q}_p$, or $\mathbb{F}_p((t))$ for some prime p. (Moreover, every finite extension of $\mathbb{F}_p((t))$ is isomorphic to $\mathbb{F}_q((u))$ for some power q of p.)
3. F is the completion of a global field with respect to a nontrivial absolute value.
4. F is a nondiscrete locally compact topological field.

See [RV99, Theorem 4-12] for a proof of the difficult part of the equivalence, namely that nondiscrete locally compact topological fields satisfy the other conditions. We won't use this part — for us, it serves only as extra motivation for considering these particular fields.

$$\boxed{\text{From now on, } F \text{ denotes a local field.}}$$

If F is $\mathbb{R}$ or $\mathbb{C}$, then F is called archimedean; other local fields are called nonarchimedean. In the nonarchimedean case, we use the following notation:

$$\begin{aligned} \mathcal{O}&\colon \text{ the valuation ring} \\ \mathfrak{p}&\colon \text{ the maximal ideal of } \mathcal{O} \\ k&\colon \text{ the residue field } \mathcal{O}/\mathfrak{p} \\ \varpi&\colon \text{ a uniformizer} \\ p&\colon \text{ the characteristic of } k \\ q &:= \#k. \end{aligned}$$

4.2. **The normalized absolute value.** Let F be a local field. The additive group of F is an LCA group. Let dx be a Haar measure on F. If $a \in F^\times$, multiplication-by-a is an isomorphism $F \to F$, under which dx pulls back to another Haar measure, which must be a positive multiple of dx:

$$\begin{aligned} F &\xrightarrow{a} F \\ |a|\, dx &\longleftarrow dx \end{aligned}$$

for some "stretching factor" $|a| \in \mathbb{R}^\times_{>0}$. Replacing dx by a positive real multiple does not change $|a|$, so $|a|$ is independent of the choice of dx. Then

- $|a|$ = the ordinary absolute value if $F = \mathbb{R}$
- $|a|$ = the square of the ordinary absolute value if $F = \mathbb{C}$
- $|a| = \#(\mathcal{O}/a\mathcal{O})^{-1}$ if F is nonarchimedean and $a \in \mathcal{O}$.

Warning 4.2. In the case $F = \mathbb{C}$, we might sometimes write $\| \ \|$ to emphasize that we mean the square of the ordinary absolute value. The function $\| \ \|$ is not literally an absolute value since the triangle inequality does not quite hold.

A subset $S \subseteq F$ is called bounded if there exists $B > 0$ such that $|x| \le B$ for all $x \in S$. The notion extends to subsets of F^n for any $n \ge 0$ by using the sup norm $|(x_1, \ldots, x_n)|_{\text{sup}} := \sup\{|x_i| : 1 \le i \le n\}$.

Theorem 4.3 (Heine–Borel theorem for local fields)**.** *Let S be a subset of a local field F. Then S has compact closure if and only if S is bounded. The same holds for $S \subseteq F^n$ for any $n \ge 0$.*

The nonarchimedean case will be assigned as homework.

4.3. **Additive characters.** By an additive character, we mean a nontrivial unitary character $\psi\colon F \to \mathbb{T}$ of the additive group of F. Fix ψ. Given $a \in F$, define $\psi_a(x) := \psi(ax)$; this is another unitary character of F.

Theorem 4.4. *The map*

$$\Psi\colon F \to \widehat{F}$$
$$a \mapsto \psi_a$$

is an isomorphism of LCA groups.

Proof. Checking that Ψ is an injective homomorphism is easy.

In comparing the original topology on F with the topology corresponding under Ψ to the subspace topology on $\Psi(F)$, it is enough to consider neighborhoods of 0, because F is a topological group under both topologies. For the second topology a basis of neighborhoods of 0 is given by the sets

$$\{a \in F\colon \psi_a(K) \subseteq U\} = \{a \in F : aK \subseteq \psi^{-1}U\}$$

for compact K and open $U \ni 1$. To check:

(1) Given compact K and open $U \ni 1$, does there exist $\delta > 0$ such that

$$|a| < \delta \implies aK \subseteq \psi^{-1}U?$$

(2) Given $\epsilon > 0$, does there exist a compact K and open $U \ni 1$ such that

$$|a| < \epsilon \impliedby aK \subseteq \psi^{-1}U?$$

The answer to (1) is yes, since K is bounded and $\psi^{-1}U$ contains an open disk around 0. The answer to (2) is yes also: choose $b \in F$ such that $\psi(b) \neq 1$, choose $U \ni 1$ so that $\psi(b) \notin U$ and hence $b \notin \psi^{-1}U$, and choose K to be a closed disk centered at 0 of radius at least $|b|/\epsilon$; then $aK \subseteq \psi^{-1}U$ implies $b \notin aK$, so $|b| > |a| \cdot |b|/\epsilon$, so $|a| < \epsilon$. Thus Ψ is a homeomorphism onto its image.

Since F is locally compact, F is complete, so $\Psi(F)$ is complete, so $\Psi(F)$ is closed in $\widehat{F}$.

The remaining claim $\Psi(F) = \widehat{F}$ is equivalent to $\Psi(F)^\perp = \{0\}$, since

$$\perp\colon \{\text{closed subgroups of } \widehat{F}\} \to \{\text{closed subgroups of } F\}$$

is a order-reversing bijection (homework). We prove $\Psi(F)^\perp = \{0\}$: If $x \in \Psi(F)^\perp$, then $\psi_a(x) = 0$ for all $a \in F$, so $\psi(ax) = 0$ for all $a \in F$, so $x = 0$. □

Definition 4.5. Suppose that F is nonarchimedean. Then $\psi|_{\mathfrak{p}^m} = 1$ for some $m \in \mathbb{Z}$; choose the smallest such m. The fractional ideal $\mathfrak{p}^m$ is called the conductor of ψ.

There is a standard ψ on each F:

- If $F = \mathbb{R}$, let $\psi(x) := e^{-2\pi i x}$. (The minus sign is there so that a global product formula later on will hold.)
- If $F = \mathbb{Q}_p$, let ψ be the composition
$$\mathbb{Q}_p \twoheadrightarrow \frac{\mathbb{Q}_p}{\mathbb{Z}_p} \xleftarrow{\sim} \frac{\mathbb{Z}[1/p]}{\mathbb{Z}} \hookrightarrow \frac{\mathbb{R}}{\mathbb{Z}} \simeq \mathbb{T},$$
which is characterized by $\psi|_{\mathbb{Z}_p} = 1$ and $\psi(1/p^n) = e^{2\pi i/p^n}$ for all $n \geq 1$.
- If $F = \mathbb{F}_p((t))$, define $\psi(\sum a_i t^i) := e^{2\pi i a_{-1}/p}$ (lift $a_{-1} \in \mathbb{F}_p$ to $\mathbb{Z}$ to make sense of this).
- Finally, if F_0 is one of the three fields above, and ψ_0 is the additive character chosen for F_0, and F is a finite separable extension of F_0, let ψ be the composition $F \xrightarrow{\operatorname{Tr}_{F/F_0}} F_0 \xrightarrow{\psi_0} \mathbb{T}$.

4.4. **Schwartz–Bruhat functions.** Recall that a Schwartz function $\mathbb{R} \to \mathbb{C}$ is a C^∞ function whose derivatives tend to 0 rapidly. The same definition applies to functions on $\mathbb{R}^n$ for any $n \geq 0$, using partial derivatives. In particular, it applies to functions on $\mathbb{C}$, viewed as $\mathbb{R}^2$.

If F is nonarchimedean, however, one cannot take the derivative of a function $F \to \mathbb{C}$, so instead of requiring that a function be C^∞, we require it to be locally constant.

Definition 4.6. A function $f \colon F \to \mathbb{C}$ is called a Schwartz–Bruhat function if it is
$$\begin{cases} \text{a Schwartz function} & \text{if } F = \mathbb{R} \text{ or } F = \mathbb{C}, \\ \text{a locally constant function of compact support} & \text{if } F \text{ is nonarchimedean.} \end{cases}$$
Let $\mathscr{S} = \mathscr{S}(F)$ be the $\mathbb{C}$-vector space of Schwartz–Bruhat functions.

If F is nonarchimedean and $f \in \mathscr{S}$, then the support of f is covered by finitely many open disks D_i on which f is constant, and by refining the cover we may assume that the D_i are disjoint; then f is a finite $\mathbb{C}$-linear combination of the characteristic functions 1_{D_i}.

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Presidents' Day holiday.

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4.5. **The Fourier transform for local fields.**

Definition 4.7. Fix a local field F, a nontrivial additive character ψ on F, and a Haar measure dx on F. Given $f \in \mathscr{S}$, define the Fourier transform $\widehat{f}$ by
$$\widehat{f}(y) := \int_F f(x)\, \psi(xy)\, dx.$$
(Tate originally took the complex conjugate of the additive character, but many references since then have not done so.) It turns out that $\widehat{f} \in \mathscr{S}$.

Under the isomorphism $F \xrightarrow{\sim} \widehat{F}$ induced by ψ, the measure on $\widehat{F}$ dual to dx might not pull back to dx, but in any case $\hat{\hat{f}}(x) = rf(-x)$ for some $r \in \mathbb{R}_{>0}$. We could arrange $r = 1$ by scaling dx appropriately; then the Fourier inversion formula

$$f(x) = \int_F \widehat{f}(y)\,\overline{\psi(xy)}\,dy$$

holds for all x. (There is no need to exclude a null set, since Schwartz–Bruhat functions are continuous.)

Proposition 4.8. *The unique Haar measure dx that is self-dual (relative to the standard ψ) is described explicitly as follows:*

- *If $F = \mathbb{R}$, then dx is Lebesgue measure.*
- *If $F = \mathbb{C}$, then dx is twice Lebesgue measure.*
- *If F is nonarchimedean, then dx is the Haar measure for which $\mathcal{O}$ gets measure $(N\mathcal{D})^{-1/2}$, where $\mathcal{D}$ is the different of F/F_0, and $N\mathcal{D} := \#(\mathcal{O}/\mathcal{D})$.*

Sketch of proof. It suffices to check the Fourier inversion formula for a single nonzero $f \in \mathscr{S}$.

- For $\mathbb{R}$, take $f(x) := e^{-\pi x^2}$. A calculation shows that $\widehat{f} = f$, so $\hat{\hat{f}} = f$.
- For $\mathbb{C}$, take $f(z) := e^{-2\pi z\bar{z}}$. A calculation shows that $\widehat{f} = f$, so $\hat{\hat{f}} = f$.
- For nonarchimedean F, take $f = 1_{\mathcal{O}}$. Then

$$\widehat{f}(y) := \int_{\mathcal{O}} \psi(xy)\,dx = \begin{cases} \operatorname{Vol}(\mathcal{O}), & \text{if } \psi|_{\mathcal{O}y} = 1, \\ 0, & \text{if } \psi|_{\mathcal{O}y} \text{ is a nontrivial character.} \end{cases}$$

  The first case holds if $\operatorname{Tr}_{F/F_0}(\mathcal{O}y)$ is contained in the valuation ring $\mathcal{O}_0$ of F_0, i.e., if $y \in \mathcal{D}^{-1}$. Thus $\widehat{f} = (N\mathcal{D})^{-1/2}\,1_{\mathcal{D}^{-1}}$. A similar calculation shows that $\hat{\hat{f}} = f$.

Moreover, in all three cases, $f(x) = f(-x)$, so the Fourier inversion formula is verified. □

4.6. **Multiplicative characters.** Define

$$U := \{x \in F^\times : |x| = 1\} = \begin{cases} \{\pm 1\}, & \text{if } F = \mathbb{R}, \\ \mathbb{T}, & \text{if } F = \mathbb{C}, \\ \mathcal{O}^\times, & \text{if } F \text{ is nonarchimedean,} \end{cases}$$

$$\underbrace{|F^\times|}_{\text{value group}} := \{|x| : x \in F^\times\} = \begin{cases} \mathbb{R}^\times_{>0}, & \text{if } F \text{ is archimedean,} \\ q^{\mathbb{Z}} & \text{if } F \text{ is nonarchimedean.} \end{cases}$$

We have a short exact sequence of LCA groups

$$1 \longrightarrow U \longrightarrow F^\times \xrightarrow{|\,|} |F^\times| \longrightarrow 1 \tag{3}$$

Call $\chi \in \mathrm{X}(F^\times)$ unramified if $\chi|_U = 1$.

Remark 4.9. Let's explain the reason for this terminology. Suppose that F is nonarchimedean. A character η of $\mathrm{Gal}(F^{\mathrm{ab}}/F)$ factors as $\mathrm{Gal}(F^{\mathrm{ab}}/F) \twoheadrightarrow \mathrm{Gal}(L/F) \hookrightarrow \mathbb{C}^\times$ for some finite Galois extension L of F. Let I be the inertia group of $\mathrm{Gal}(F^{\mathrm{ab}}/F)$, which maps onto the inertia group $I_{L/F}$ of $\mathrm{Gal}(L/F)$. The following are equivalent:

- L/F is unramified.
- $I_{L/F} = 1$.
- $\eta|_I = 1$.

Therefore, call η unramified if $\eta|_I = 1$.

Let $\theta \colon F^\times \to \mathrm{Gal}(F^{\mathrm{ab}}/F)$ be the local Artin homomorphism, which maps U isomorphically to I. Let χ be the composition $F^\times \xrightarrow{\theta} \mathrm{Gal}(F^{\mathrm{ab}}/F) \xrightarrow{\eta} \mathbb{C}^\times$. Then the three conditions above are equivalent also to

- $\chi|_U = 1$.

Proposition 4.10. *For a character $\chi \colon F^\times \to \mathbb{C}^\times$, the following are equivalent:*

(i) *χ is unramified (i.e., $\chi|_U = 1$);*
(ii) *χ factors through $|F^\times|$;*
(iii) *$\chi = |\ |^s$ for some $s \in \mathbb{C}$.*

Proof.
(i)$\Leftrightarrow$(ii): Use (3).
(iii)$\Rightarrow$(ii): Trivial.
(ii)$\Rightarrow$(iii): We must show that every character χ of $|F^\times|$ is $x \mapsto x^s$ for some $s \in \mathbb{C}$. If $|F^\times| = q^{\mathbb{Z}}$, it is enough to choose s so that $q^s = \chi(q)$. If $|F^\times| = \mathbb{R}^\times_{>0}$, we use the following observations:

- $\mathbb{R}^\times_{>0}$ is isomorphic to $\mathbb{R}$;
- $\mathbb{R}$ is simply connected, so every character $\mathbb{R} \to \mathbb{C}^\times$ factors through the universal cover $\mathbb{C} \overset{\exp}{\twoheadrightarrow} \mathbb{C}^\times$;
- Every continuous homomorphism $\eta \colon \mathbb{R} \to \mathbb{C}$ is $r \mapsto sr$ for some $s \in \mathbb{C}$: namely, $\eta(1)$ determines $\eta|_{\mathbb{Z}}$, which determines $\eta|_{\mathbb{Q}}$ (since $\mathbb{C}$ is divisible and torsion-free), which determines η on $\mathbb{R}$ by continuity. $\square$

Corollary 4.11. *The unramified characters of $F^\times$ form a subgroup of $\mathrm{X}(F^\times)$ isomorphic to $\mathbb{C}$ (if F is archimedean) or $\mathbb{C}/\left(\mathbb{Z} \cdot 2\pi i/\log q\right)$ (if F is nonarchimedean).*

Thus the subgroup of unramified characters has the structure of a Riemann surface, and so does each coset. Viewing $\mathrm{X}(F^\times)$ as the disjoint union of these cosets makes $\mathrm{X}(F^\times)$ a Riemann surface with infinitely many connected components.

Proposition 4.12. *Every character χ of $F^\times$ is $\eta|\ |^s$ for some unitary character η and some $s \in \mathbb{C}$.*

Proof. Factor χ as $\frac{\chi}{|\chi|} \cdot |\chi|$. The first factor $\frac{\chi}{|\chi|}$ is unitary. The second factor $|\chi|$ is unramified since χ restricted to the compact group U is automatically unitary, so by Proposition 4.10(i)$\Rightarrow$(iii) $|\chi|$ is $|\ |^s$ for some $s \in \mathbb{C}$. $\square$

Corollary 4.13. *If $\chi \in \mathrm{X}(F^\times)$, then $|\chi| = |\ |^\sigma$ for a uniquely determined $\sigma \in \mathbb{R}$.*

Proof. If $\chi = \eta |\ |^s$ with η unitary, then $|\chi| = |\ |^\sigma$ with $\sigma := \operatorname{Re} s$. □

The number σ in Corollary 4.13 is called the exponent of χ.

Definition 4.14. Suppose that F is nonarchimedean. Then $\chi|_{1+\mathfrak{p}^m} = 1$ for some $m \in \mathbb{Z}_{\geq 0}$ (we interpret $1 + \mathfrak{p}^0$ as $U = \mathcal{O}^\times$); choose the smallest such m. The ideal $\mathfrak{p}^m$ is called the conductor of χ. The conductor of χ is the unit ideal if and only if χ is unramified, so the conductor measures the extent to which χ is ramified.

In the archimedean cases, we can classify the characters of $F^\times$ in an even more explicit way:

Corollary 4.15.

(a) *Every character of $\mathbb{R}^\times$ has the form $\chi_{a,s}(x) := x^{-a}|x|^s$ for some $a \in \{0,1\}$ and $s \in \mathbb{C}$.*
(b) *Every character of $\mathbb{C}^\times$ has the form $\chi_{a,b,s}(z) := z^{-a}\bar{z}^{-b}\|z\|^s$ for some $a, b \in \mathbb{Z}$ with $\min(a,b) = 0$ and some $s \in \mathbb{C}$. (Here $\bar{z}$ is the complex conjugate of z.)*

Proof. If $\chi \in \mathrm{X}(\mathbb{R}^\times)$, then $\chi|_U$ is given by x^{-a} for some $a \in \{0,1\}$, so $x^a\chi(x)$ is unramified, and hence equals $|x|^s$ for some $s \in \mathbb{C}$. A similar argument works for $\mathbb{C}^\times$. □

Remark 4.16. The character $x \mapsto x^{-1}$ of $\mathbb{R}^\times$ is not unitary, but the sign character $\operatorname{sgn}(x) := x^{-1}|x|$ is. The reason for writing $x^{-a}|x|^s$ instead of just the cases $|\ |^s$ and $\operatorname{sgn} \cdot |\ |^s$ is to simplify the definition of local L-factors later on. A similar comment applies to characters of $\mathbb{C}^\times$.

In what follows, L-factors and zeta integrals will be not functions of $s \in \mathbb{C}$, but functions of $\chi \in \mathrm{X}(F^\times)$.

The Riemann zeta function had a functional equation relating its values at s and $1-s$. What operation on characters sends $|\ |^s$ to $|\ |^{1-s}$? Answer:

Definition 4.17. If $\chi \in \mathrm{X}(F^\times)$, define its twisted dual as $\chi^\vee := \chi^{-1}|\ |$.

4.7. **Local L-factors.** Each factor in

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1}$$

is an example of a local L-factor. If $F = \mathbb{Q}_p$ and $\chi = |\ |^s$, then $p^{-s} = \chi(\varpi)$. This partially motivates the following definition.

If F is nonarchimedean, and $\chi \in \mathrm{X}(F^\times)$, then

$$L(\chi) := \begin{cases} (1 - \chi(\varpi))^{-1} & \text{if } \chi \text{ is unramified,} \\ 1 & \text{if } \chi \text{ is ramified.} \end{cases}$$

We need to define $L(\chi)$ also in the archimedean cases:

$$L(\chi_{a,s}) := \Gamma_{\mathbb{R}}(s) := \pi^{-s/2}\Gamma(s/2),$$
$$L(\chi_{a,b,s}) := \Gamma_{\mathbb{C}}(s) := \Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s+1) = 2(2\pi)^{-s}\Gamma(s).$$

(Recall that $\pi^{-s/2}\Gamma(s/2)$ was the archimedean factor used to complete the Riemann zeta function. The definition over $\mathbb{C}$ can be explained in terms of induced representations when the definition is extended to higher-dimensional representations of Weil groups.) Call $L(\chi)$ the local L-factor of χ.

In all cases, $L(\chi)$ is a meromorphic function of $\chi \in \mathrm{X}(F^\times)$ with no zeros.

4.8. **Multiplicative Haar measure.** Fix a Haar measure $d^\times x$ on $F^\times$. Then $d^\times x = c\frac{dx}{|x|}$ for some $c \in \mathbb{R}_{>0}$, by which we mean that the Radon integrals agree: for all $f \in C_c(F^\times)$,

$$\int_{F^\times} f(x)\, d^\times x = \int_F f(x)\, \frac{c}{|x|}\, dx.$$

4.9. **Local zeta integrals.** Fix $d^\times x$. Given $f \in \mathscr{S}$ and $\chi \in \mathrm{X}(F^\times)$, define the local zeta integral

$$Z(f,\chi) := \int_{F^\times} f(x)\,\chi(x)\, d^\times x.$$

We are now ready for the main theorem in the local setting:

Theorem 4.18 (Meromorphic continuation and functional equation of local zeta integrals)**.**

(a) *For every $f \in \mathscr{S}$, the integral $Z(f,\chi)$ converges for χ of exponent $\sigma > 0$.*

(b) *For every $f \in \mathscr{S}$, the function $Z(f,\chi)$ of χ extends to a meromorphic function on $\mathrm{X}(F^\times)$.*

(c) *For every $f \in \mathscr{S}$, the meromorphic function $Z(f,\chi)/L(\chi)$ on $\mathrm{X}(F^\times)$ is* holomorphic.

(d) *For each component X_0 of $\mathrm{X}(F^\times)$, there exists $f \in \mathscr{S}$ such that the holomorphic function $Z(f,\chi)/L(\chi)$ is nonvanishing on X_0. Moreover, if F is nonarchimedean, $\mathrm{X}_0 = \{|\ |^s : s \in \mathbb{C}\}$, and $f = 1_{\mathcal{O}}$, then $Z(f,\chi)/L(\chi)$ is 1 on X_0.*

(e) *Fix choices of ψ and dx (to define Fourier transforms). There exists a nonvanishing holomorphic function $\epsilon(\chi,\psi,dx)$ of $\chi \in \mathrm{X}(F^\times)$ such that*

$$\frac{Z(\widehat{f},\chi^\vee)}{L(\chi^\vee)} = \epsilon(\chi,\psi,dx)\frac{Z(f,\chi)}{L(\chi)} \tag{4}$$

for all $f \in \mathscr{S}$. For $\chi = \eta\,|\ |^s$ for fixed η (i.e., on one component of $\mathrm{X}(F^\times)$), the function $\epsilon(\chi,\psi,dx)$ has the form Ae^{Bs} for some $A, B \in \mathbb{C}$. Moreover, if F is nonarchimedean, ψ has conductor $\mathfrak{p}^0$, and $\int_{\mathcal{O}} dx = 1$ (so dx is self-dual relative to ψ), then $\epsilon(|\ |^s,\psi,dx) = 1$ for all $s \in \mathbb{C}$.

Parts (c) and (d) can be viewed as saying that the function $L(\chi)$ is the "greatest common divisor" of the functions $Z(f,\chi)$ as f varies. Part (e) implies that $Z(\widehat{f},\chi^\vee)$ is related to $Z(f,\chi)$ by a meromorphic factor that is independent of f. In (e), the function $\epsilon(\chi,\psi,dx)$ is called a local ϵ-factor; it is independent of the choice of $d^\times x$, but it depends on ψ and dx since these appear in the definition of $\widehat{f}$.

Remark 4.19. In his thesis, Tate had a single function measuring the "multiplicative error" in the functional equation relating $Z(f,\chi)$ and $Z(\widehat{f},\chi^\vee)$. Only later, in [Del73, §3], did Deligne separate this function into L-factors and an ϵ-factor.

Proof of convergence for $\sigma > 0$. We will prove absolute convergence. Since $|\chi(x)| = |x|^\sigma$, the question is whether $\int_{F^\times} |f(x)|\,|x|^\sigma\, d^\times x < \infty$. Since $f \in \mathscr{S}$, the integrand decays rapidly as $x \to \infty$, so the integral over the region $|x| > 1$ is finite. On the other hand, f is bounded in the region $|x| \le 1$. Thus it suffices to prove finiteness of $I := \int_{0<|x|\le 1} |x|^\sigma\, d^\times x$. Choose $a \in F^\times$ with $|a| < 1$. For $n \in \mathbb{Z}_{\ge 0}$, let A_n be the annulus $|a|^{n+1} < |x| \le |a|^n$, and let $I_n := \int_{A_n} |x|^\sigma\, d^\times x$, so $I = \sum_{n \ge 0} I_n$. Since A_n has compact closure in $F^\times$ and $|x|^\sigma$ is bounded on A_n, we have $I_n < \infty$. Also, $I_n = |a^n|^\sigma I_0$, by translation invariance of $d^\times x$. Thus $\sum_{n\ge 0} I_n$ is a convergent geometric series if $\sigma > 0$. $\square$

This proves part (a) of Theorem 4.18.

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Proof that $Z(f,\chi)$ and $Z(f,\chi)/L(\chi)$ are holomorphic for $\sigma > 0$. Write $\chi = \eta\,|\;|^s$ with η unitary and $s \in \mathbb{C}$. For $\operatorname{Re} s > 0$, the derivative of $Z(f,\chi) = \int_{F^\times} f(x)\,\eta(x)\,|x|^s\, d^\times x$ with respect to s exists (differentiate under the integral sign, by using absolute convergence). Since $L(\chi)$ has no zeros, $Z(f,\chi)/L(\chi)$ is holomorphic in this region too. $\square$

We next prove the functional equation for *one* f for each X_0. Each X_0 is $\{\eta\,|\;|^s : s \in \mathbb{C}\}$ for some unitary character $\eta \in \widehat{F^\times}$ (determined up to multiplication by $|\;|^{ir}$ for $r \in \mathbb{R}$). A function of $\chi = \eta\,|\;|^s \in \mathrm{X}_0$ may be viewed as a function of $s \in \mathbb{C}$.

Lemma 4.20. *For each $\eta \in \widehat{F^\times}$, there exists $f \in \mathscr{S}$ such that for $\chi = \eta\,|\;|^s$,*

(a) *The holomorphic function $Z(f,\chi)/L(\chi)$ on the half-plane $\operatorname{Re} s > 0$ is nonvanishing (and equal to 1 if F is nonarchimedean, $\eta = 1$, and $f = 1_{\mathcal{O}}$).*
(b) *The holomorphic function $Z(\widehat{f},\chi^\vee)/L(\chi^\vee)$ on the half-plane $\operatorname{Re} s < 1$ is nonvanishing.*
(c) *There exists a nonvanishing holomorphic function $\epsilon(s) = \epsilon(\chi,\psi,dx)$ of the form Ae^{Bs} such that the functional equation (4) holds on the strip $0 < \operatorname{Re} s < 1$ (and $\epsilon = 1$ if F is nonarchimedean, ψ has conductor $\mathfrak{p}^0$, and $\int_{\mathcal{O}} dx = 1$).*

Proof. Changing ψ or dx changes $Z(\widehat{f},\chi^\vee)$ by an easy factor, so it suffices to prove this for one choice of ψ and dx. We may also choose $d^\times x$.

Case 1: $F = \mathbb{R}$. Let $\psi(x) := e^{-2\pi i x}$, let dx be Lebesgue measure, and let $d^\times x = dx/|x|$. By Corollary 4.15(a), it suffices to consider $\eta = 1$ and $\eta = \operatorname{sgn}$.

- Suppose that $\eta = 1$. Let $f(x) := e^{-\pi x^2}$. Then for $\operatorname{Re} s > 0$,
$$\begin{aligned} Z(f,\chi) &= \int_{\mathbb{R}^\times} e^{-\pi x^2}\,|x|^s\,d^\times x \\ &= 2\int_0^\infty e^{-\pi x^2}\,x^s\,\frac{dx}{x} \\ &= \pi^{-s/2}\int_0^\infty e^{-u}\,u^{s/2}\,\frac{du}{u} \qquad \text{(we let } u = \pi x^2\text{)} \\ &= \pi^{-s/2}\Gamma(s/2) \\ &= \Gamma_{\mathbb{R}}(s) \\ &= L(\chi). \end{aligned}$$
Also $\widehat{f} = f$, so $Z(\widehat{f},\chi^\vee) = L(\chi^\vee)$ for $\operatorname{Re} s < 1$. Set $\epsilon := 1$.
- Suppose that $\eta = \operatorname{sgn}$, i.e., $\eta(x) = x^{-1}|x|$. Choosing $f(x) := e^{-\pi x^2}$ would give $Z(f,\chi) = 0$ by symmetry, so instead let $f(x) := xe^{-\pi x^2}$. A calculation similar to that above leads to $\epsilon := -i$.

Case 2: $F = \mathbb{C}$. Let $\psi(z) := e^{-2\pi i(z+\bar{z})}$, let dx be twice Lebesgue measure, and let $d^\times x = dx/\|x\|$. By Corollary 4.15(b), it suffices to consider the cases where η is $z^{-a}\|z\|^{a/2}$ or $\bar{z}^{-b}\|z\|^{b/2}$ for some $a, b \in \mathbb{Z}_{\ge 0}$. By symmetry (changing the identification of F with $\mathbb{C}$), it suffices to consider $z^{-a}\|z\|^{a/2}$. Let $f(z) := \pi^{-1}z^a e^{-2\pi z\bar{z}}$. A calculation in polar coordinates leads to
$$\begin{aligned} Z(f,\chi) &= \Gamma_{\mathbb{C}}(s + a/2), \\ L(\chi) &= \Gamma_{\mathbb{C}}(s + a/2), \\ \widehat{f}(z) &= (-i\bar{z})^a e^{-2\pi z\bar{z}}, \\ \chi^\vee &= \bar{z}^{-a}\|z\|^{1-s+a/2}, \\ Z(\widehat{f},\chi^\vee) &= (-i)^a\Gamma_{\mathbb{C}}(1 - s + a/2), \\ L(\chi^\vee) &= \Gamma_{\mathbb{C}}(1 - s + a/2), \\ \epsilon &:= (-i)^a. \end{aligned}$$

Case 3: F is nonarchimedean.

Choose dx so that $\int_{\mathcal{O}} dx = 1$; then $\int_{\mathfrak{p}^k} dx = q^{-k}$ for all $k \in \mathbb{Z}$. Choose $d^\times x = dx/|x|$; then $\int_{\mathcal{O}^\times} d^\times x = 1 - q^{-1}$ and $\int_{1+\mathfrak{p}^k} d^\times x = q^{-k}$ for each $k \ge 1$.

The calculations for ramified η will depend on the following properties of Gauss sums:

Proposition 4.21. *Given unitary characters $\omega\colon \mathcal{O}^\times \to \mathbb{T}$ and $\psi\colon \mathcal{O} \to \mathbb{T}$, define the Gauss sum*
$$g(\omega,\psi) := \int_{\mathcal{O}^\times} \omega(x)\,\psi(x)\,d^\times x.$$
Suppose ω is of conductor $\mathfrak{p}^n$ with $n > 0$, and ψ is of conductor $\mathfrak{p}^m$.

(i) *If $m \neq n$, then $g(\omega, \psi) = 0$.*
(ii) *If $m = n$, then $|g(\omega, \psi)|^2 = q^{-m}$. (Here $|\ |$ is the usual absolute value on $\mathbb{C}$, so $|z|^2 = z\bar{z}$.)*

Proof.

(i) If $m > n$, then the integral over each coset of $1 + \mathfrak{p}^n$ is 0 since ω is constant and ψ is a nontrivial character on $\mathfrak{p}^n$. If $m < n$, then the integral over each coset of $1 + \mathfrak{p}^m$ is 0 since ψ is constant and ω is a nontrivial character on $1 + \mathfrak{p}^m$ (interpret $1 + \mathfrak{p}^m$ as $\mathcal{O}^\times$ if $m = 0$).
(ii) If $m = n > 0$, then

$$\begin{aligned}
|g(\omega, \psi)|^2 &= \int_{\mathcal{O}^\times} \omega(x)\, \psi(x)\, d^\times x \quad \overline{\int_{\mathcal{O}^\times} \omega(y)\, \psi(y)\, d^\times y} \\
&= \int_{\mathcal{O}^\times} \int_{\mathcal{O}^\times} \omega(xy^{-1})\, \psi(x - y)\, d^\times x\, d^\times y \\
&= \int_{\mathcal{O}^\times} \int_{\mathcal{O}^\times} \omega(z)\, \psi(yz - y)\, d^\times y\, d^\times z \qquad (\text{let } x = yz) \\
&= \int_{\mathcal{O}^\times} \omega(z)\, h(z)\, d^\times z
\end{aligned}$$

where

$$\begin{aligned}
h(z) &= \int_{\mathcal{O}^\times} \psi(yz - y)\, d^\times y \\
&= \int_{\mathcal{O}^\times} \psi(y(z-1))\, dy \qquad (\text{since } |y| = 1 \text{ on } \mathcal{O}^\times) \\
&= \int_{\mathcal{O}} \psi(y(z-1))\, dy - \int_{\mathfrak{p}} \psi(y(z-1))\, dy \\
&= \begin{cases} 1 - q^{-1} & \text{if } v(z-1) \geq m \qquad (\text{both integrands are } 1), \\ -q^{-1} & \text{if } v(z-1) = m - 1 \qquad (\text{second integrand is } 1), \\ 0 & \text{if } v(z-1) < m - 1 \qquad (\text{neither integrand is constant}), \end{cases} \\
&= 1_{1+\mathfrak{p}^m}(z) - q^{-1}\, 1_{1+\mathfrak{p}^{m-1}}(z).
\end{aligned}$$

Thus

$$|g(\omega, \psi)|^2 = \int_{1+\mathfrak{p}^m} \omega(z)\, d^\times z - q^{-1} \int_{1+\mathfrak{p}^{m-1}} \omega(z)\, d^\times z = q^{-m} - 0 = q^{-m}. \qquad \square$$

Now we return to Case 3 of the zeta integral calculations. Choose $\psi \colon F \to \mathbb{T}$ to be of conductor $\mathfrak{p}^0$. Every χ can be written $\eta |\ |^s$ where η is a unitary character of $\mathcal{O}^\times$ extended to $F^\times$ by setting $\eta(\varpi) = 1$, so assume that η has this form. Let $\mathfrak{p}^n$ be the conductor of η. We will assume $n > 0$, and leave the easier case $n = 0$ as homework. We choose $f = 1_{1+\mathfrak{p}^n}$, since this makes

$$Z(f,\chi) = \int_{1+\mathfrak{p}^n} \eta(x)|x|^s \, d^\times x = \int_{1+\mathfrak{p}^n} dx = q^{-n} \neq 0.$$

Next,

$$\widehat{f}(y) = \int_{1+\mathfrak{p}^n} \psi(xy)\, dx = \int_{\mathfrak{p}^n} \psi((1+z)y)\, dz = \psi(y) \int_{\mathfrak{p}^n} \psi(zy)\, dz = q^{-n}\, \psi(y)\, 1_{\mathfrak{p}^{-n}}(y),$$

since $\psi|_{\mathfrak{p}^n y} = 1$ if and only if $y \in \mathfrak{p}^{-n}$. For $\operatorname{Re} s < 1$,

$$\begin{aligned}
Z(\widehat{f},\chi^\vee) &= \int_{F^\times} \widehat{f}(x)\, \chi^\vee(x)\, d^\times x \\
&= q^{-n} \int_{\mathfrak{p}^{-n}} \psi(x)\, \eta(x)^{-1}\, |x|^{1-s}\, d^\times x \\
&= q^{-n} \sum_{k \geq -n} \int_{\mathfrak{p}^k - \mathfrak{p}^{k+1}} \psi(x)\, \eta(x)^{-1}\, (q^{-k})^{1-s}\, d^\times x \\
&= q^{-n} \sum_{k \geq -n} q^{-k(1-s)} \int_{\mathcal{O}^\times} \psi(\varpi^k z)\, \eta(\varpi^k z)^{-1}\, d^\times z \qquad \text{(substitute } x = \varpi^k z) \\
&= q^{-n} \sum_{k \geq -n} q^{n(1-s)} g(\eta^{-1}, \psi_{\varpi^k}) \\
&= q^{-n} q^{n(1-s)} g(\eta^{-1}, \psi_{\varpi^{-n}}),
\end{aligned}$$

by Proposition 4.21, since the conductor $\mathfrak{p}^{-k}$ of ψ_{ϖ^k} equals the conductor $\mathfrak{p}^n$ of η^{-1} if and only if $k = -n$. Also, since $n > 0$, the characters χ and $\chi^\vee$ are ramified, so $L(\chi) = 1$ and $L(\chi^\vee) = 1$. Hence the functional equation holds with

$$\epsilon := q^{n(1-s)} g(\eta^{-1}, \psi_{\varpi^{-n}}). \qquad \square$$

To pass from knowing the functional equation (4) for one f to knowing it for an arbitrary $g \in \mathscr{S}$, Tate used Fubini's theorem to prove the following, which measures the "difference" between (4) for f and (4) for g.

Lemma 4.22. *Let $f, g \in \mathscr{S}$. For χ of exponent σ satisfying $0 < \sigma < 1$, we have*

$$Z(f,\chi)\, Z(\widehat{g},\chi^\vee) = Z(g,\chi)\, Z(\widehat{f},\chi^\vee).$$

Proof. We may assume that $d^\times x = dx/|x|$. By definition,

$$\begin{aligned}
Z(f,\chi)\, Z(\widehat{g},\chi^\vee) &= \int_{F^\times} f(x)\, \chi(x)\, d^\times x \int_{F^\times} \underbrace{\left(\int_F g(z)\, \psi(yz)\, dz \right)}_{\widehat{g}(y)} \underbrace{\chi(y)^{-1}|y|}_{\chi^\vee(y)}\, d^\times y \\
&= \int_{(F^\times)^3} f(x)\, g(z)\, \chi(xy^{-1})\, \psi(yz)\, |yz|\, d^\times x\, d^\times y\, d^\times z \qquad \text{(Fubini's theorem)} \\
&= \int_{(F^\times)^3} f(x)\, g(z)\, \chi(t^{-1})\, \psi(txz)\, |txz|\, d^\times x\, d^\times t\, d^\times z \qquad \text{(set } y = xt),
\end{aligned}$$

which is symmetric in f and g (the dummy variables x and z can be renamed z and x). $\square$

Proof of Theorem 4.18. We already proved (a). Let f and ϵ be as in Lemma 4.20, so

$$\frac{Z(\widehat{f},\chi^\vee)}{L(\chi^\vee)} = \epsilon(\chi,\psi,dx)\frac{Z(f,\chi)}{L(\chi)}$$

for $0<\sigma<1$. Multiply by the identity of Lemma 4.22, and cancel $Z(f,\chi)Z(\widehat{f},\chi^\vee)$ to obtain an identity of meromorphic functions

$$\frac{Z(\widehat{g},\chi^\vee)}{L(\chi^\vee)} = \epsilon(\chi,\psi,dx)\frac{Z(g,\chi)}{L(\chi)},$$

again for $0<\sigma<1$. The two sides define holomorphic functions in $\sigma<1$ and $\sigma>0$, respectively, so they glue to give a holomorphic extension of $Z(\widehat{g},\chi^\vee)/L(\chi^\vee)$ on all of $\mathrm{X}(F^\times)$. Dividing by $\epsilon(\chi,\psi,dx)$ gives the corresponding extension of $Z(g,\chi)/L(\chi)$, which proves (c). Multiplying by the meromorphic function $L(\chi)$ yields (b). Finally, (d) and (e) follow from Lemma 4.20. $\square$

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5. Adèle ring of a global field

5.1. **Global fields.** A number field is a finite extension of $\mathbb{Q}$. A global function field is a finite extension of $\mathbb{F}_p(t)$ for some prime p, or equivalently is the function field of a geometrically integral curve over a finite field $\mathbb{F}_q$, where q is a power of some prime p. Notation:

K: global field (number field or global function field)

K_v: the completion of K at v, a local field

$\mathcal{O}_v$: the valuation ring of K_v (set $\mathcal{O}_v := K_v$ if v is archimedean)

$\mathfrak{p}_v$: the maximal ideal of $\mathcal{O}_v$

k_v: the residue field $\mathcal{O}_v/\mathfrak{p}_v$

$|\ |_v$: normalized "absolute value" on K_v

$\mathbb{A} := \prod'_v (K_v,\mathcal{O}_v)$ adèle ring

$\mathbb{A}^\times = \prod'_v (K_v^\times,\mathcal{O}_v^\times)$ idèle group.

A basis for the topology on $\mathbb{A}$ consists of the subsets $\prod_v U_v$, where U_v is open in K_v, and $U_v=\mathcal{O}_v$ for all but finitely many v. A basis for the topology on $\mathbb{A}^\times$ consists of the subsets $\prod_v U_v$, where U_v is open in $K_v^\times$, and $U_v=\mathcal{O}_v^\times$ for all but finitely many v. Both the additive group $\mathbb{A}$ and the multiplicative group $\mathbb{A}^\times$ are LCA groups.

Recall that the diagonally embedded $K \subset \mathbb{A}$ is analogous to $\mathbb{Z} \subset \mathbb{R}$: for example, K is discrete and $\mathbb{A}/K$ is compact.

5.2. **Additive characters.** As on a local field, additive characters of $\mathbb{A}$ are assumed to be unitary.

Proposition 5.1. *We have an isomorphism*

$$\begin{aligned} \widehat{\mathbb{A}} &\longrightarrow \prod\nolimits_v' (\widehat{K}_v, \widehat{K_v/\mathcal{O}_v}) \\ \psi &\longmapsto (\psi|_{K_v}) \\ \prod \psi_v &\longleftarrow (\psi_v). \end{aligned}$$

In other words, to give an additive character ψ of $\mathbb{A}$ is the same as giving a collection (ψ_v) such that each ψ_v is an additive character of K_v, and $\psi_v|_{\mathcal{O}_v} = 1$ for all but finitely many v.

Proof. This is a formal property that holds for any restricted direct product $\prod_v'(G_v, H_v)$ where G_v is an LCA group and $H_v \leq G_v$ is an open compact subgroup for all but finitely many v. $\square$

- If K is a number field, choose the standard ψ_v on each K_v, and choose $\prod_v \psi_v$ as the standard ψ.
- Now suppose that K is the function field of a curve X over $\mathbb{F}_q$. Let Ω_X be its sheaf of 1-forms. Let Ω_K be the stalk at the generic point, so Ω_K is the 1-dimensional K-vector space of "meromorphic" 1-forms on X. Define $\Omega_v := \Omega_K \underset{K}{\otimes} K_v$. There is a residue map $\operatorname{Res}_v \colon \Omega_v \to k_v$, which can be described in terms of Laurent expansions as follows: if u is a uniformizer at the closed point v, then the residue map is

$$\begin{aligned} \operatorname{Res}_v \colon \Omega_v = k_v((u))\, du &\longrightarrow k_v \\ \sum_{i \in \mathbb{Z}} a_i u^i \, du &\longmapsto a_{-1} \end{aligned}$$

  (this is independent of the choice of uniformizer). A choice of nonzero *global* meromorphic 1-form $\omega \in \Omega_K$ gives rise to

$$\psi_v(x) := \exp\left(\frac{2\pi i}{p} \operatorname{Tr}_{k_v/\mathbb{F}_p} \operatorname{Res}_v(x\omega)\right)$$

  for each v. Let κ_v be the "order of vanishing of ω at v", i.e., the largest integer such that $\omega \in \mathfrak{p}_v^{\kappa_v} \Omega_{X,v}$; then the conductor of ψ_v is $\mathfrak{p}_v^{-\kappa_v}$. Let $\prod \psi_v$ serve as a "standard" ψ.

For all but finitely many v,

- (number field case) v is nonarchimedean and $K/\mathbb{Q}$ is unramified at v
- (function field case) ω generates the $\mathscr{O}_{X,v}$-module $\Omega_{X,v}$ (i.e., ω is regular and nonvanishing at v),

in which case ψ_v has conductor $\mathfrak{p}_v^0$.

From now on, ψ is a standard additive character as above. For $a \in \mathbb{A}$, define $\psi_a(x) := \psi(ax)$.

Corollary 5.2. *The map*

$$\begin{aligned} \Psi \colon \mathbb{A} &\to \widehat{\mathbb{A}} \\ a &\mapsto \psi_a \end{aligned}$$

is an isomorphism of LCA groups.

Proof. Each ψ_v gives rise to an isomorphism $\Psi_v \colon K_v \xrightarrow{\sim} \widehat{K}_v$. For all but finitely many v, the character ψ_v has conductor $\mathfrak{p}_v^0$, so $\mathcal{O}_v^\perp = \mathcal{O}_v$, and Ψ_v identifies $\mathcal{O}_v$ with $\widehat{K_v/\mathcal{O}_v}$. Now Ψ is the isomorphism

$$\prod_v{}'(K_v, \mathcal{O}_v) \xrightarrow{\prod \Psi_v} \prod_v{}'(\widehat{K}_v, \widehat{K_v/\mathcal{O}_v}).$$

□

Proposition 5.3. *We have* $\psi|_K = 1$.

Sketch of proof. The number field case is left as an exercise. In the function field case, it reduces to the fact that for a meromorphic 1-form on a smooth projective integral curve over an algebraically closed field (e.g., $\overline{\mathbb{F}}_q$), the sum of the residues is 0. □

Corollary 5.4. *The map*

$$\begin{aligned} K &\to \widehat{\mathbb{A}/K} = K^\perp \\ a &\mapsto \psi_a \end{aligned}$$

is an isomorphism.

Proof. We have

(1) $K^\perp$ is discrete. (Proof: It is the Pontryagin dual of the compact group $\mathbb{A}/K$.)

(2) $K^\perp$ is a K-subspace of $\widehat{\mathbb{A}} \simeq \mathbb{A}$. (Proof: If $\psi_a|_K = 1$ and $\kappa \in K$, then $\psi_{\kappa a}|_K = 1$.)

By (1), $K^\perp/K$ is a discrete subgroup of the compact group $\mathbb{A}/K$, so $K^\perp/K$ is finite. By (2), $K^\perp/K$ is also a K-vector space. But K is infinite! Thus $K^\perp/K = 0$, so $K^\perp = K$. □

5.3. **The Tamagawa measure.** Let dx_v be the self-dual measure on K_v with respect to the standard ψ_v. The Tamagawa measure $dx := \prod dx_v$ on $\mathbb{A}$, is the Haar measure such that for each basic open set $\prod U_v$, we have $\int_{\prod U_v} dx = \prod \int_{U_v} dx_v$.

The counting measure on K and the measure dx on $\mathbb{A}$ induce a Haar measure on $\mathbb{A}/K$ compatible with respect to

$$0 \to K \to \mathbb{A} \to \mathbb{A}/K \to 0.$$

Proposition 5.5. *The volume of* $\mathbb{A}/K$ *is* 1.

Proof. We'll give the proof in the number field case. Later we'll explain a calculation-free way of proving this for any global field.

Let K be a number field. Let D be a fundamental domain for the quotient $\mathbb{A}/K$, i.e., a measurable subset of $\mathbb{A}$ such that $\mathbb{A}$ is the disjoint union $\coprod_{\kappa\in K}(D+\kappa)$. For any fundamental domain D, we have $\mathrm{Vol}(\mathbb{A}/K) = \mathrm{Vol}(D)$. By strong approximation,

$$\underbrace{\prod_{v|\infty} K_v}_{K_{\mathbb{R}}} \times \prod_{v \text{ finite}} \mathcal{O}_v \longrightarrow \mathbb{A}/K$$

is surjective; the kernel is $\mathcal{O}_K$ (ring of integers) embedded diagonally. Therefore, let D_∞ be a fundamental domain for $K_{\mathbb{R}}/\mathcal{O}_K$, and let $D = D_\infty \times \prod_{v \text{ finite}} \mathcal{O}_v$. Then

$$\begin{aligned}\mathrm{Vol}(D) &= \mathrm{Vol}(D_\infty) \prod_{v \text{ finite}} \mathrm{Vol}(\mathcal{O}_v) \\ &= (\operatorname{disc}\mathcal{O}_K)^{1/2} \prod_{v \text{ finite}} (N\mathcal{D}_v)^{-1/2}\end{aligned}$$

(the first volume was calculated last semester). Since $\prod_{v \text{ finite}} N\mathcal{D}_v = \operatorname{disc}\mathcal{O}_K$, we get $\mathrm{Vol}(D) = 1$. $\square$

5.4. **The adelic Fourier transform.** If for each v we have $f_v \in \mathscr{S}(K_v)$, and $f_v = 1_{\mathcal{O}_v}$ for all but finitely many v, then $\prod f_v \colon \mathbb{A} \to \mathbb{C}$ is defined. A Schwartz–Bruhat function $f\colon \mathbb{A} \to \mathbb{C}$ is a finite $\mathbb{C}$-linear combination of such functions. Let $\mathscr{S} = \mathscr{S}(\mathbb{A})$ be the space of all Schwartz–Bruhat functions.

Fix the standard ψ and the self-dual measure dx. Given $f \in \mathscr{S}$, the Fourier transform $\widehat{f}$ is defined as usual,

$$\widehat{f}(y) := \int_{\mathbb{A}} f(x)\,\psi(xy)\,dx,$$

and it is again in $\mathscr{S}$, since the Fourier transform of $\prod f_v$ as above is $\prod \widehat{f}_v$, where $\widehat{f}_v = 1_{\mathcal{O}_v}$ for all v for which $\mathcal{O}_v^\perp = \mathcal{O}_v$ (all but finitely many).

There is also a Fourier transform for $\mathbb{A}/K$ and its Pontryagin dual K. Functions on $\mathbb{A}/K$ will be identified with functions on $\mathbb{A}$ that are K-periodic (i.e., $f(x+\kappa) = f(x)$ for all $\kappa \in K$). Given $f \in \mathcal{L}^1(\mathbb{A}/K)$, define $\widehat{f}\colon K \to \mathbb{C}$ by

$$\widehat{f}(\kappa) := \int_D f(x)\,\psi(\kappa x)\,dx.$$

(If we do not take for granted that $\mathrm{Vol}(D) = 1$, then the right side should be multiplied by $\frac{1}{\mathrm{Vol}(D)}$ since the dual of the counting measure on K is the normalized Haar measure $\frac{dx}{\mathrm{Vol}(D)}$ on $\mathbb{A}/K$.) For continuous $f \in \mathcal{L}^1(\mathbb{A}/K)$, if $\widehat{f} \in \mathcal{L}^1(K)$, then the Fourier inversion formula

$$f(x) = \sum_{\kappa\in K} \widehat{f}(k)\,\overline{\psi(\kappa x)}$$

holds (an integral with respect to counting measure is a sum).

5.5. **The adelic Poisson summation formula.**

Lemma 5.6. *If $f \in \mathscr{S}(\mathbb{A})$, then $\sum_{\kappa \in K} f(x+\kappa)$ converges, absolutely and uniformly on compact sets, to a K-periodic function.*

Proof. We may assume that $f = \prod f_v$. Every compact subset of $\mathbb{A}$ is contained in one of the form $S = \prod S_v$ with $S_v \subseteq K_v$ compact and $S_v = \mathcal{O}_v$ for all but finitely many v, so it suffices to consider convergence on S. At each nonarchimedean v, the set of κ such that $f_v(S_v + \kappa)$ is not identically 0 is v-adically bounded, and the bound is $|\kappa|_v \le 1$ for all but finitely many v. In the function field case, this means that the sum becomes a finite sum. In the number field case, the sum is effectively a sum over κ in a fractional ideal I of K; since I is a lattice in $\prod_{v|\infty} K_v = K \otimes \mathbb{R}$, and since $f_v \in \mathscr{S}(K_v)$ tends to 0 rapidly, $\sum_{\kappa \in I} f(x+\kappa)$ converges absolutely and uniformly on S. Translating the function $\sum_{\kappa \in K} f(x+\kappa)$ by an element of K just permutes the terms. □

Theorem 5.7 (Poisson summation formula for $K \subset \mathbb{A}$)**.** *If $f \in \mathscr{S}(\mathbb{A})$, then*

$$\sum_{\kappa \in K} f(\kappa) = \sum_{\kappa \in K} \widehat{f}(\kappa).$$

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Proof. The idea is to apply the Fourier inversion formula to the function

$$F(x) := \sum_{\kappa \in K} f(x+\kappa)$$

on $\mathbb{A}/K$. Let $\widehat{F}\colon K \to \mathbb{C}$ be the Fourier transform of F. (Although the notation for $\widehat{f}$ and $\widehat{F}$ look the same, $\widehat{f}$ is a function on the Pontryagin dual $\mathbb{A}$ of $\mathbb{A}$, while $\widehat{F}$ is a function on the Pontryagin dual K of $\mathbb{A}/K$.) For $\kappa \in K$,

$$\begin{aligned}
\widehat{F}(\kappa) &= \int_D \sum_{\ell \in K} f(x+\ell)\,\psi(\kappa x)\,dx \\
&= \sum_{\ell \in K} \int_D f(x+\ell)\,\psi(\kappa x)\,dx \\
&= \sum_{\ell \in K} \int_{D+\ell} f(z)\,\psi(\kappa(z-\ell))\,dz \quad \text{(substitute } z = x+\ell) \\
&= \int_{\mathbb{A}} f(z)\,\psi(\kappa z)\,dz \\
&= \widehat{f}(\kappa).
\end{aligned}$$

The Fourier inversion formula then gives

$$F(x) = \sum_{\kappa \in K} \widehat{F}(\kappa)\, \overline{\psi(\kappa x)},$$

which may be rewritten as

$$\sum_{\kappa \in K} f(x+\kappa) = \sum_{\kappa \in K} \widehat{f}(\kappa)\, \overline{\psi(\kappa x)}.$$

Set $x = 0$ to obtain

$$\sum_{\kappa \in K} f(\kappa) = \sum_{\kappa \in K} \widehat{f}(\kappa). \qquad \square$$

Remark 5.8. If we had not assumed $\operatorname{Vol}(D) = 1$, we would have obtained

$$\sum_{\kappa \in K} f(\kappa) = \frac{1}{\operatorname{Vol}(D)} \sum_{\kappa \in K} \widehat{f}(\kappa)$$

for all $f \in \mathscr{S}(\mathbb{A})$, but then applying this to $\widehat{f}$ would yield

$$\sum_{\kappa \in K} \widehat{f}(\kappa) = \frac{1}{\operatorname{Vol}(D)} \sum_{\kappa \in K} f(\kappa).$$

Together, these prove that $\operatorname{Vol}(D)^2 = 1$, so $\operatorname{Vol}(D) = 1$!

5.6. **The Riemann–Roch theorem.** Let K be the function field of a curve X over $\mathbb{F}_q$. Places v of K may be identified with closed points of X.

The divisor group of X is the free abelian group having as basis the set of all places v. We write a divisor D as $\sum_v d_v v$ with $d_v \in \mathbb{Z}$ and $d_v = 0$ for all but finitely many v. Define $\deg D := \sum_v d_v [k_v : \mathbb{F}_q] \in \mathbb{Z}$. Each $f \in K^\times$ gives rise to a divisor $\sum_v v(f)\, v$. Similarly, our chosen nonzero $\omega \in \Omega_K$ gives rise to a canonical divisor $\mathcal{K} = \sum_v \kappa_v v$, where κ_v is the "order of vanishing of ω at v".

Define $\mathcal{O}_{\mathbb{A}} := \prod \mathcal{O}_v \subset \mathbb{A}$. Given $D = \sum d_v v$, define

$$\mathcal{O}_{\mathbb{A}}(D) := \prod_v \mathfrak{p}_v^{-d_v} \subset \mathbb{A}.$$

(The product is a direct product, not an ideal product!) The $\mathbb{F}_q$-vector space

$$L(D) := K \cap \mathcal{O}_{\mathbb{A}}(D)$$

consists of rational functions f on X having at worst a pole of order d_v at v for each v. Let $\ell(D) := \dim_{\mathbb{F}_q} L(D)$.

Example 5.9. We claim that $L(0) = \mathbb{F}_q$, and hence $\ell(0) = 1$. If $t \in \mathbb{F}_q$, then $t \in K \cap \mathcal{O}_{\mathbb{A}} = L(0)$. If $t \in K - \mathbb{F}_q$, then the $\frac{1}{t}$-adic valuation on $\mathbb{F}_q(t)$ extends to a place v of K for which $v(t) < 0$, so $t \notin L(0)$.

Define the genus of X (or of K) by $g := \ell(\mathcal{K}) \in \mathbb{Z}_{\geq 0}$.

Theorem 5.10 (Riemann–Roch theorem)**.** *For any divisor D on X,*

$$\ell(D) - \ell(\mathcal{K} - D) = \deg D + 1 - g.$$

Sketch of proof. First calculate some local and adelic Fourier transforms:

$$\begin{aligned} \widehat{1}_{\mathfrak{p}_v^{-d_v}} &= q^{d_v - \kappa_v/2}\, 1_{\mathfrak{p}_v^{d_v - \kappa_v}} && \text{(local calculation)} \\ \widehat{1}_{\mathcal{O}_{\mathbb{A}}(D)} &= q^{\deg D - \deg \mathcal{K}/2}\, 1_{\mathcal{O}_{\mathbb{A}}(\mathcal{K}-D)} && \text{(product over } v\text{)}. \end{aligned}$$

(An alternative way to get the right powers of q is to compare L^2-norms, by the Plancherel theorem.)
Applying the Poisson summation formula to $f = 1_{\mathcal{O}_{\mathbb{A}}(D)}$ yields

$$\begin{aligned} \sum_{x \in L(D)} 1 &= q^{\deg D - \deg \mathcal{K}/2} \sum_{x \in L(\mathcal{K}-D)} 1 \\ \ell(D) &= \deg D - \frac{\deg \mathcal{K}}{2} + \ell(\mathcal{K} - D) \\ \ell(D) - \ell(\mathcal{K} - D) &= \deg D + 1 - h \end{aligned}$$

for some number h independent of D. Setting $D = 0$ shows that $h = g$. □

5.7. **Norm of an idèle.** If $a = (a_v) \in \mathbb{A}^\times$, multiplication-by-$a$ is an isomorphism $\mathbb{A} \to \mathbb{A}$, under which dx corresponds to another Haar measure:

$$\begin{gathered} \mathbb{A} \xrightarrow{a} \mathbb{A} \\ |a|\, dx \longleftarrow dx \end{gathered}$$

for some "stretching factor" $|a| \in \mathbb{R}_{>0}^\times$; in fact, then $|a| = \prod_v |a_v|_v$ (note that $|a_v|_v = 1$ for all but finitely many v).

We can reprove the following result from last semester in a conceptual way:

Product formula. *If $a \in K^\times$, then $|a| = 1$.*

Sneaky proof. Multiplication-by-a is an isomorphism $\mathbb{A}/K \to \mathbb{A}/K$, but it scales volume by $|a|$, so

$$\operatorname{Vol}(\mathbb{A}/K) = |a| \operatorname{Vol}(\mathbb{A}/K).$$

Since $\operatorname{Vol}(\mathbb{A}/K)$ is finite and positive, we get $|a| = 1$! □

Let $|\mathbb{A}^\times|$ be the group $\{|x| : x \in \mathbb{A}^\times\}$. For $t \in |\mathbb{A}^\times|$, let $\mathbb{A}_t^\times := \{x \in \mathbb{A}^\times : |x| = t\}$.

5.8. Idèle-class characters.

> "We can do nothing really significant with the idele group until we imbed the multiplicative group $K^\times$ in it". — John Tate, in his thesis.

Not all characters of $\mathbb{A}^\times$ are of interest to us.

Definition 5.11. An idèle-class character or Hecke character or Größencharakter is a continuous homomorphism $\chi\colon \mathbb{A}^\times \to \mathbb{C}^\times$ such that $\chi|_{K^\times} = 1$.

Idèle-class characters can be identified with characters of the idèle class group $\mathbb{A}^\times/K^\times$.

Example 5.12. If $s \in \mathbb{C}$, then $|\ |^s$ is an idèle-class character.

Example 5.13. Take $K = \mathbb{Q}$. Let χ be a Dirichlet character, i.e., a homomorphism $(\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ for some $N \geq 1$. Define the ring $\widehat{\mathbb{Z}} := \varprojlim \mathbb{Z}/NZ \simeq \prod_p \mathbb{Z}_p$, in which the inverse limit is over positive integers N ordered by divisibility (the hat on the $\mathbb{Z}$ signifies *completion*, not Pontryagin dual). Then the composition

$$\mathbb{A}^\times = \mathbb{Q}^\times \times \mathbb{R}^\times_{>0} \times \widehat{\mathbb{Z}}^\times \twoheadrightarrow \widehat{\mathbb{Z}}^\times \twoheadrightarrow (\mathbb{Z}/N\mathbb{Z})^\times \xrightarrow{\chi} \mathbb{C}^\times$$

is an idèle-class character (which by abuse of terminology might again be called a Dirichlet character). One can show that every finite-order idèle-class character arises in this way. Over $\mathbb{Q}$, Dirichlet characters look simpler than the corresponding idèle-class characters, but if one is working over higher number fields K, then the class group and unit group prevent such an easy decomposition of $\mathbb{A}^\times$ into three factors. For such K, it is the idèle-theoretic definition of character that is the useful definition, the one that lets us avoid worrying about class groups and unit groups.

Example 5.14. Let K^{s} be a separable closure of K. Let $K^{\mathrm{ab}} \subset K^{\mathrm{s}}$ be the maximal abelian extension of K. Let $\theta\colon \mathbb{A}^\times/K^\times \to \operatorname{Gal}(K^{\mathrm{ab}}/K)$ be the global Artin homomorphism of class field theory. Any continuous character $\rho\colon \operatorname{Gal}(K^{\mathrm{s}}/K) \to \mathbb{C}^\times$ factors through $\operatorname{Gal}(K^{\mathrm{ab}}/K)$, and then the composition along the bottom row of

$$\begin{tikzcd}
 & & \operatorname{Gal}(K^s/K) \arrow[d] \arrow[dr, bend left, "\rho"] & \\
\mathbb{A}^\times \arrow[r, two heads] & \mathbb{A}^\times/K^\times \arrow[r, "\theta"] & \operatorname{Gal}(K^{\mathrm{ab}}/K) \arrow[r, dashed] & \mathbb{C}^\times
\end{tikzcd}$$

is an idèle-class character.

Warning 5.15. Because θ is not quite an isomorphism, it is not quite true that all idèle-class characters arise from continuous characters $\operatorname{Gal}(K^s/K) \to \mathbb{C}^\times$. In fact, since $\operatorname{Gal}(K^s/K)$ is profinite and since a sufficiently small open neighborhood of 1 in $\mathbb{C}^\times$ contains no nontrivial subgroups, any continuous character $\operatorname{Gal}(K^s/K) \to \mathbb{C}^\times$ has finite image, but idèle-class characters can have infinite image. One can fix this by replacing $\operatorname{Gal}(K^s/K)$ by a closely related group called a Weil group W_K.

Using the compactness of $\mathbb{A}_1^\times/K^\times$ and the exact sequence

$$1 \longrightarrow \mathbb{A}_1^\times/K^\times \longrightarrow \mathbb{A}^\times/K^\times \xrightarrow{|\ |} |\mathbb{A}^\times| \longrightarrow 1$$

analogous to (3), we see that every idèle-class character χ is $\eta|\ |^s$ for some unitary η and some $s \in \mathbb{C}$. Define its exponent σ and twisted dual $\chi^\vee$ as in the local case, and let $\chi_v := \chi|_{K_v^\times}$.

Let $\mathscr{X}$ be the group of all idèle-class characters χ. Then $\mathscr{X}$ has the structure of a Riemann surface, just as $\mathrm{X}(F^\times)$ did for a local field F.

5.9. **Multiplicative Haar measure.**

First attempt: Define $d^\times x_v := \frac{dx_v}{|x_v|_v}$, and define $d^\times x := \prod_v d^\times x_v$ so that $\int_{\prod U_v} d^\times x = \prod \int_{U_v} d^\times x_v$ for each basic open subset $\prod U_v$ of $\mathbb{A}^\times$. Problem: For almost all v, we have $U_v = \mathcal{O}_v^\times$ and $\int_{\mathcal{O}_v} dx_v = 1$, so

$$\int_{U_v} d^\times x_v = \int_{\mathcal{O}_v^\times} d^\times x_v = \int_{\mathcal{O}_v^\times} dx_v = 1 - q_v^{-1},$$

but $\prod_{\text{finite } v}(1-q_v^{-1})$ diverges to 0 (for example, when $K = \mathbb{Q}$, this is $\prod_p(1-p^{-1})$, whose inverse is the divergent Euler product for $1 + \frac{1}{2} + \frac{1}{3} + \cdots$).

Second attempt (the good one): Define

$$d^\times x_v := \begin{cases} \dfrac{dx_v}{|x_v|_v}, & \text{if } v \text{ is archimedean} \\ (1-q_v^{-1})^{-1}\dfrac{dx_v}{|x_v|_v}, & \text{if } v \text{ is nonarchimedean.} \end{cases}$$

Then $\int_{\mathcal{O}_v^\times} d^\times x_v = 1$ for all but finitely many v, so we can define $d^\times x := \prod d^\times x_v$.

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If K is a number field, then $|\mathbb{A}^\times| = \mathbb{R}_{>0}^\times$, which we give the Haar measure dt/t. If K is a function field, then there exists $q > 1$ such that $|\mathbb{A}^\times| = q^{\mathbb{Z}}$, which we give the counting measure *multiplied by* $\log q$ (the "logarithmic spacing" between powers of q) and denote dt/t again!

Equip $\mathbb{A}_1^\times$ with the Haar measure d^*x compatible with the other two measures $d^\times x$ and dt/t in the short exact sequence

$$1 \longrightarrow \mathbb{A}_1^\times \longrightarrow \mathbb{A}^\times \xrightarrow{|\ |} |\mathbb{A}^\times| \longrightarrow 1.$$

For $t \in |\mathbb{A}^\times|$, multiplication by any idèle a of norm t is a homeomorphism $\mathbb{A}_1^\times \to \mathbb{A}_t^\times$ so d^*x gives rise to a measure on $\mathbb{A}_t^\times$ (independent of the choice of a).

Finally, the measure on $\mathbb{A}_t^\times$ induces a measure on the orbit space $\mathbb{A}_t^\times/K^\times$; any of these measures will be denoted d^*x. Let V be the volume of the compact group $\mathbb{A}_1^\times/K^\times$.

5.10. **Global zeta integrals.** Given $f \in \mathscr{S}(\mathbb{A})$ and an idèle-class character χ, define the global zeta integral

$$Z(f,\chi) := \int_{\mathbb{A}^\times} f(x)\,\chi(x)\,d^\times x; \tag{5}$$

this is a generalization of the *completed* Riemann zeta function $\xi(s)$. We are now ready for the main theorem in the global setting:

Theorem 5.16 (Meromorphic continuation and functional equation of global zeta integrals)**.** *Fix $f \in \mathscr{S}(\mathbb{A})$.*

(a) *The integral $Z(f,\chi)$ converges for idèle-class characters χ of exponent $\sigma > 1$.*

(b) *The function $Z(f,\chi)$ of χ extends to a meromorphic function on $\mathscr{X}$. In fact, it is* holomorphic *except for*
- *a simple pole at $|\ |^0$ with residue $-V f(0)$, and*
- *a simple pole at $|\ |^1$ with residue $V\widehat{f}(0)$.*

(c) *We have $Z(f,\chi) = Z(\widehat{f},\chi^\vee)$ as meromorphic functions of $\chi \in \mathscr{X}$.*

Proof of convergence for $\sigma > 1$. We may assume that $f = \prod f_v$. We may replace f and χ by their absolute values, so χ becomes $|\ |^\sigma$. Now

$$Z(f,\chi) = \prod_v Z(f_v, |\ |_v^\sigma).$$

For all but finitely many v, the function f_v is $1_{\mathcal{O}_v}$ and $Z(f_v, |\ |_v^\sigma) = (1 - q_v^{-\sigma})^{-1}$, so we need to prove convergence of

$$\prod_{\text{nonarchimedean } v} (1 - q_v^{-\sigma})^{-1}.$$

If K is a number field of degree d, this is the Dedekind zeta function ζ_K evaluated at σ, which converges for $\sigma > 1$ — the argument given last semester was that there are at most d places of K above each prime p of $\mathbb{Q}$, and each has norm q_v at least p, so

$$\prod_{\text{nonarchimedean } v} (1 - q_v^{-\sigma})^{-1} \le \prod_{\text{prime p}} (1 - p^{-\sigma})^{-d} = \left(\sum_{n \ge 1} n^{-\sigma} \right)^d < \infty.$$

If K is a function field, we can similarly reduce to the case of $\mathbb{F}_q(t)$, in which every v except one corresponds to a maximal ideal of $\mathbb{F}_q[t]$, so if N_n denotes the number of monic irreducible polynomials in $\mathbb{F}_q[t]$ of degree n, we need convergence of

$$\prod_{v \ne \infty} (1 - q_v^{-\sigma}) = \prod_{n \ge 1} (1 - q^{-n\sigma})^{-N_n},$$

or equivalently, of

$$\sum_{n \ge 1} N_n q^{-n\sigma} \le \sum_{n \ge 1} q^n q^{-n\sigma} < \infty$$

if $\sigma > 1$ (geometric series). $\square$

Slice $\mathbb{A}^\times$ according to norm. Then $K^\times$ is a discrete co-compact subgroup in the slice $\mathbb{A}_1^\times$. For $\sigma > 1$, performing the integration in (5) slice-by-slice yields

$$Z(f,\chi) = \int_{|\mathbb{A}|^\times} Z_t(f,\chi)\,\frac{dt}{t}, \tag{6}$$

where the integral over the norm t slice, defined for χ of arbitrary exponent, is

$$Z_t(f,\chi) := \int_{\mathbb{A}_t^\times} f(x)\,\chi(x)\,d^*x.$$

Remark 5.17. Equation (6) in the case $\chi = |\;|^s$ is the analogue of the equation

$$\xi(s) = \int_0^\infty \left(\frac{\Theta(t)-1}{2}\right) t^{s/2}\,\frac{dt}{t}$$

that appeared in Riemann's proof. In particular, the next lemma shows that $Z_t(f,\chi)$ has its own functional equation, analogous to the functional equation of $\Theta(t)$, or more closely, analogous to the functional equation of $\left(\frac{\Theta(t)-1}{2}\right) t^{s/2}$.

Lemma 5.18. *For any idèle-class character* χ,

$$Z_t(f,\chi) + f(0)\int_{\mathbb{A}_t^\times/K^\times} \chi(x)\,d^*x = Z_{1/t}(\widehat{f},\chi^\vee) + \widehat{f}(0)\int_{\mathbb{A}_{1/t}^\times/K^\times} \chi^\vee(x)\,d^*x.$$

Proof. By tiling $\mathbb{A}_t^\times$ into translates of a fundamental domain for the action of $K^\times$, we get

$$\begin{aligned} Z_t(f,\chi) &= \int_{\mathbb{A}_t^\times/K^\times} \sum_{a\in K^\times} f(ax)\,\chi(ax)\,d^*x \\ &= \int_{\mathbb{A}_t^\times/K^\times} \left(\sum_{a\in K^\times} f(ax)\right) \chi(x)\,d^*x \quad \text{(since } \chi \text{ is 1 on } K^\times\text{)}. \end{aligned}$$

It would be better to have a sum over K instead of $K^\times$, in order to apply the Poisson summation formula/Riemann–Roch theorem, so we add an $a=0$ term to both sides:

$$\begin{aligned}
Z_t(f,\chi) + f(0)\int_{\mathbb{A}_t^\times/K^\times} \chi(x)\,d^*x &= \int_{\mathbb{A}_t^\times/K^\times} \left(\sum_{a\in K} f(ax)\right)\chi(x)\,d^*x \\
&= \int_{\mathbb{A}_t^\times/K^\times} \left(\frac{1}{|x|}\sum_{a\in K} \widehat{f}(a/x)\right)\chi(x)\,d^*x \\
&\qquad \text{(by the Riemann–Roch theorem)} \\
&= \int_{\mathbb{A}_{1/t}^\times/K^\times} \left(|y|\sum_{a\in K} \widehat{f}(ay)\right)\chi(y^{-1})\,d^*y \qquad (\text{set } x=y^{-1}) \\
&= \int_{\mathbb{A}_{1/t}^\times/K^\times} \left(\sum_{a\in K} \widehat{f}(ay)\right)\chi^\vee(y)\,d^*y \\
&= Z_{1/t}(\widehat{f},\chi^\vee) + \widehat{f}(0)\int_{\mathbb{A}_{1/t}^\times/K^\times} \chi^\vee(y)\,d^*y,
\end{aligned}$$

by the same argument in reverse. $\square$

Lemma 5.19. *For $t\in|\mathbb{A}^\times|$ and an idèle-class character χ,*

$$\int_{\mathbb{A}_t^\times/K^\times} \chi(x)\,d^*x = \begin{cases} Vt^s & \text{if } \chi = |\ |^s \text{ for some } s\in\mathbb{C}, \\ 0 & \text{otherwise.}\end{cases}$$

Proof. If $\chi=|\ |^s$ for some $s\in\mathbb{C}$, then the integrand is the constant t^s, and $\operatorname{Vol}(\mathbb{A}_t^\times/K^\times) = \operatorname{Vol}(\mathbb{A}_1^\times/K^\times) = V$. Otherwise χ does not factor through $\mathbb{A}^\times/\mathbb{A}_1^\times$, so χ is not constant on the compact group $\mathbb{A}_1^\times/K^\times$, so its integral on any coset of $\mathbb{A}_1^\times/K^\times$ is 0. $\square$

Proof of Theorem 5.16. Divide the range of integration in (6) in half:

$$Z(f,\chi) = \underbrace{\int_0^1 Z_t(f,\chi)\frac{dt}{t}}_{J(f,\chi)} + \underbrace{\int_1^\infty Z_t(f,\chi)\frac{dt}{t}}_{I(f,\chi)};$$

in the function field case, these integrals are really ($\log q$ times) sums over $q^{\mathbb{Z}}$ for some $q>1$, and by convention we assign half of the q^0 term to each of $I(f,\chi)$ and $J(f,\chi)$.

In the region $\sigma>1$, both integrals converge to a holomorphic function. In fact, $I(f,\chi)$ converges to a holomorphic function everywhere and its convergence only gets better as σ decreases, since the $\chi(x)$ in $Z_t(f,\chi)$ has absolute value t^σ, which for $t\geq 1$ decreases as σ decreases.

We handle the problematic half $J(f,\chi)$ by using the functional equation for $Z_t(f,\chi)$ and flipping the integral with $t \mapsto 1/t$, as in Riemann's proof: for $\sigma > 1$,

$$\begin{aligned}
J(f,\chi) &= \int_0^1 Z_t(f,\chi)\,\frac{dt}{t} \\
&= \int_0^1 Z_{1/t}(\widehat{f},\chi^\vee)\,\frac{dt}{t} + \underbrace{\int_0^1 \left(V\widehat{f}(0)\left(\frac{1}{t}\right)^{1-s} - Vf(0)\,t^s\right)\frac{dt}{t}}_{\text{if } \chi = |\,|^s} \\
&\qquad \text{(by Lemmas 5.18 and 5.19)} \\
&= \int_1^\infty Z_u(\widehat{f},\chi^\vee)\,\frac{du}{u} + \underbrace{\int_0^1 \left(V\widehat{f}(0)\,t^{s-1} - Vf(0)\,t^s\right)\frac{dt}{t}}_{\text{if } \chi = |\,|^s} \\
&= I(\widehat{f},\chi^\vee) + \underbrace{V\widehat{f}(0)\int_0^1 t^{s-1}\,\frac{dt}{t} - Vf(0)\int_0^1 t^s\,\frac{dt}{t}}_{\text{if } \chi = |\,|^s}.
\end{aligned}$$

Adding $I(f,\chi)$ to both sides yields

$$Z(f,\chi) = I(f,\chi) + I(\widehat{f},\chi^\vee) + \underbrace{V\widehat{f}(0)\int_0^1 t^{s-1}\,\frac{dt}{t} - Vf(0)\int_0^1 t^s\,\frac{dt}{t}}_{\text{if } \chi = |\,|^s}.$$

In the region where χ is not of the form $|\,|^s$, the formula on the right hand side provides a holomorphic extension of $Z(f,\chi)$ that is symmetric with respect to $(f,\chi) \leftrightarrow (\widehat{f},\chi^\vee)$. (Note that $Z(\hat{\hat{f}},\chi) = Z(f(-x),\chi) = Z(f,\chi)$ by the substitution $x \mapsto -x$, since $\chi(-x) = \chi(-1)\chi(x) = \chi(x)$.)

Now consider the region where $\chi = |\,|^s$. In the number field case, the extra terms contribute

$$\frac{V\widehat{f}(0)}{s-1} - \frac{Vf(0)}{s},$$

which is meromorphic on $\mathbb{C}$ and symmetric with respect to $(f,s) \leftrightarrow (\widehat{f},1-s)$, with residues $-Vf(0)$ at $s=0$ and $V\widehat{f}(0)$ at $s=1$, as claimed. In the function field case, the extra terms contribute

$$\begin{aligned}
&V\widehat{f}(0)\log q\left(-\frac{1}{2} + \sum_{n=0}^\infty (q^{-n})^{s-1}\right) - Vf(0)\log q\left(-\frac{1}{2} + \sum_{n=0}^\infty (q^{-n})^{s}\right) \\
&= \frac{V\log q}{2}\left(\widehat{f}(0)\,\frac{1+q^{1-s}}{1-q^{1-s}} + f(0)\,\frac{1+q^s}{1-q^s}\right),
\end{aligned}$$

which again is meromorphic and symmetric, again with the claimed residues when χ is $|\,|^0$ or $|\,|^1$, that is, when q^s is q^0 or q^1. ∎

5.11. **Hecke L-functions.** By comparing the local and global functional equations, we will get functional equations for global L-functions.

Definition 5.20. For each idèle-class character χ, define

$$\epsilon(\chi) := \prod_v \epsilon(\chi_v)$$
$$L(\chi) := \prod_v L(\chi_v)$$

(when the products converge).

Remark 5.21. In defining $\epsilon(\chi_v)$ we are implicitly using the standard ψ and the Tamagawa measure dx. If we change ψ to some other additive character ψ_a trivial on K (so $a \in K^\times$), then each $\epsilon(\chi_v)$ is multiplied by $\chi_v(a)\,|a_v|_v^{-1}$, and the product is unchanged since $\prod_v \chi_v(a) = \chi(a) = 1$ and $\prod_v |a_v|_v^{-1} = |a|^{-1} = 1$. Thus it is reasonable to omit ψ and the canonical dx from the notation for global ϵ-factors.

Theorem 5.22.

(a) *The product $\epsilon(\chi)$ converges to a nonvanishing holomorphic function on all of $\mathscr{X}$.*
(b) *The product $L(\chi)$ converges for $\sigma > 1$, and extends to a meromorphic function on all of $\mathscr{X}$, holomorphic everywhere outside $|\;|^0$ and $|\;|^1$.*
(c) *We have $L(\chi) = \epsilon(\chi)L(\chi^\vee)$ as meromorphic functions on $\mathscr{X}$.*

Proof. We restrict attention to one component $\mathscr{X}_0$ of $\mathscr{X}$. The local calculations for Theorem 4.18 showed that

(i) $\epsilon_v(\chi_v)$ is a nonvanishing holomorphic function on $\mathscr{X}_0$, equal to 1 for all but finitely many v;

and that we can find $f_v \in \mathscr{S}(K_v)$ with $f_v = 1_{\mathcal{O}_v}$ for all but finitely many v such that

(ii) $Z(f_v, \chi_v)/L(\chi_v)$ is a nonvanishing holomorphic function on $\mathscr{X}_0$, equal to 1 for all but finitely many v; and
(iii) the local functional equation

$$\frac{Z(\widehat{f_v}, \chi_v^\vee)}{L(\chi_v^\vee)} = \epsilon(\chi_v)\frac{Z(f_v, \chi_v)}{L(\chi_v)}$$

of Theorem 4.18 holds for all v.

Taking the product over v and setting $f := \prod f_v \in \mathscr{S}(\mathbb{A})$ shows that

(i) $\epsilon(\chi)$ is a nonvanishing holomorphic function on $\mathscr{X}_0$;
(ii) $\prod_v (Z(f_v, \chi_v)/L(\chi_v))$ is a nonvanishing holomorphic function on $\mathscr{X}_0$, so the properties of $L(\chi)$ in (b) follow from those of $Z(f, \chi)$ in Theorem 5.16; and

(iii)
$$\frac{Z(\widehat{f},\chi^\vee)}{L(\chi^\vee)} = \epsilon(\chi)\frac{Z(f,\chi)}{L(\chi)},$$
so that dividing by the global functional equation
$$Z(\widehat{f},\chi^\vee) = Z(f,\chi)$$
of Theorem 5.16 gives
$$\frac{1}{L(\chi^\vee)} = \epsilon(\chi)\frac{1}{L(\chi)}$$
$$L(\chi) = \epsilon(\chi)\, L(\chi^\vee). \qquad \square$$

To get meromorphic functions of $s \in \mathbb{C}$, fix an idèle-class character χ, and define
$$\epsilon(s,\chi) := \epsilon(\chi|\;|^s)$$
$$L(s,\chi) := L(\chi|\;|^s).$$
Functions of the form $L(s,\chi)$ are called Hecke L-functions; these include (completed) Dedekind zeta functions and Dirichlet L-functions as special cases. In these terms, the functional equation is
$$L(s,\chi) = \epsilon(s,\chi)\, L(1-s,\chi^{-1}).$$
For homework, you will specialize this to obtain the functional equation of the Dedekind zeta function.

Remark 5.23. In Warning 5.15, we alluded to the fact that idèle-class characters are in bijection with continuous characters $W_K \to \mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^\times$ of the Weil group W_K. One can generalize the whole theory by considering continuous representations $W_K \to \mathrm{GL}_n(\mathbb{C})$ for larger n. For an introduction to this generalization, see [Tat79].

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