Appendix A. Geršgorin's Paper from 1931, and Comments on His Life and Research.

Appendix A.1. Semen Aronovich Geršgorin

It is interesting to first comment on the contents of Geršgorin's original paper from 1931 (in German), on estimating the eigenvalues of a given $n \times n$ complex matrix, which is *reproduced*, for the reader's convenience, at the end of this appendix. There, one can see the originality of Geršgorin pouring forth in this paper! His Satz II corresponds exactly to our Theorem 1.1, his Satz III corresponds to our Theorem 1.6, and his Satz IV, on separated Geršgorin disks, appears in Exercise 4 of Section 1.1. In his final result of Satz V, he uses a positive diagonal similarity transformation, as in our (1.14), which is dependent on a single parameter α, with $0 < \alpha < 1$, to obtain better eigenvalue inclusion results. This approach was subsequently used by Olga Taussky in Taussky (1947) in the practical estimation of eigenvalues in the flutter of airplane wings! However, we must mention that his Satz I is **incorrect**. His statement in Satz I is that if $A = [a_{i,j}] \in \mathbb{C}^{n \times n}$ satisfies

$$|a_{i,i}| \geq r_i(A) := \sum_{j \in N \backslash \{i\}} |a_{i,j}|, \quad \text{for all } i \in N,$$

with strict inequality for at least one i, then A is nonsingular. But, as we have seen in Section 1.2, the matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is a *counterexample*, as A satisfies the above conditions, but is singular. (Olga Taussky was certainly aware of this error, but she was probably just too polite to mention this in print!) As we now know, her assumption of **irreducibility** in Taussky (1949), (cf. Theorem 1.9 in Chapter 1) clears this up nicely, but see also Exercise 1 of Sec. 1.2.

We also mention here the important contribution of Fujino and Fischer (1998) (in German) which provided us with the biographical data below on

Geršgorin, as well as a list of his significant publications. This paper of Fujino and Fischer (1998) also contains pictures, from the Deutsches Museum in Munich, of **ellipsographs**, a mechanical device to draw ellipses, which were built by Geršgorin. There is a very new contribution on the life and works of Geršgorin by Garry Tee (see Tee (2004)).

Semen Aronovich Geršgorin

- Born: 24 August 1901 in Pruzhany (Brest region), Belorussia
- Died: 30 May 1933 in St. Petersburg
- Education: St. Petersburg Technological Institute, 1923
- Professional Experience: Professor 1930-1933, St. Petersburg Machine-Construction Institute

SIGNIFICANT PUBLICATIONS

1. Instrument for the integration of the Laplace equation, Zh. Priklad. Fiz. 2 (1925), 161-7.
2. On a method of integration of ordinary differential equations, Zh. Russkogo Fiz-Khimi. O-va. 27 (1925), 171-178.
3. On the description of an instrument for the integration of the Laplace equation, Zh. Priklad. Fiz. 3(1926), 271-274.
4. On mechanisms for the construction of functions of a complex variable, Zh. Fiz.- Matem. O-va 1 (1926), 102-113.
5. On the approximate integration of the equations of Laplace and Poisson, Izv. Leningrad Polytech. Inst. 20 (1927), 75-95.
6. On the number of zeros of a function and its derivative, Zh. Fiz.- Matem. O-va 1(1927), 248-256.
7. On the mean values of functions on hyper-spheres in n-dimensional space, Mat. Sb. 35 (1928), 123-132.
8. A mechanism for the construction of the function $\xi = \frac{1}{2}(z - \frac{r^2}{z})$, Izv. Leningrad Polytech. Inst. 2 (26) (1928), 17-24.
9. On the electric nets for the approximate solution of the Laplace equation, Zh. Priklad. Fiz. 6 (3-4) (1929), 3-30.
10. Fehlerabschätzung für das Differenzverfahren zur Lösung partieller Differentialgleichungen, J. Angew. Math. Mech. 10 (1930).
11. Über die Abgrenzung der Eigenwerte einer Matrix. Dokl. Akad. Nauk (A), Otd. Fiz.-Mat. Nauk (1931), 749-754.
12. Über einen allgemeinen Mittelwertsatz der mathematischen Physik, Dokl. Akad. Nauk. (A) (1932), 50-53.
13. On the conformal map of a simply connected domain onto a circle, Mat. Sb. 40 (1933), 48-58.

Of the above papers, three papers, 10, 11, and 13, stand out as **seminal contributions**. Paper 10 was the first paper to treat the important topic of the **convergence** of finite-difference approximations to the solution of

Laplace-type equations, and it is quoted in the book by Forsythe and Wasow (1960). Paper 11 was Geršgorin's original result on estimating the eigenvalues of a complex $n \times n$ matrix, from which the material of this book has grown. Paper 13, on numerical conformal maps, is quoted in the book by Gaier (1964). But what is most impressive is that these three papers of Geršgorin are still being referred today in research circles, after more than 70 years!

Next, we have been given permission to give below a translation, from Russian to English, of the following obituary of Geršgorin's passing, as recorded in the journal, Applied Mathematics and Mechanics 1 (1933), no.1, page 4. Then, after this obituary, Geršgorin's original paper (in German) is given in full.

APPLIED MATHEMATICS AND MECHANICS

Volume 1, 1933, No.1

Semen Aronovich Geršgorin has passed away. This news will cause great anguish in everybody who knew the deceased.

The death of a great scientist is always hard to bear, as it always causes a feeling of emptiness that cannot be filled; it is especially sad when a young scientist's life ends suddenly, with his talent in its full strength, when he is still full of unfulfilled research potential.

Semen Aronovich died at the age of 32. Having graduated from the Technological Institute and having defended a brilliant thesis in the Division of Mechanics, he quickly became one of the leading figures in Soviet Mechanics and Applied Mathematics. Numerous works of S.A., in the theory of Elasticity, Theory of Vibrations, Theory of Mechanisms, Methods of Approximate Numerical Integration of Differential Equations and in other parts of Mechanics and Applied Mathematics, attracted attention and brought universal recognition to the author. Already the first works showed him to be a very gifted young scientist; in the last years his talent matured and blossomed. The main features of Geršgorin's individuality are his methods of approach, combined with the power and clarity of analysis. These features are already apparent in his early works (for example, in a very clever idea for constructing the profiles of aeroplane wings), as well as in his last brilliant (and not yet completely published) works in elasticity theory and in theory of vibrations.

S.A. Geršgorin combined a vigorous and active research schedule which, in his last years, centered around the Mathematical and Mechanical Institute at Leningrad State University, as well as around the Turbine Research Institute (NII Kotlo-Turbiny) with wide-ranging teaching activities.

In 1930 he became a Professor at the Institute of Mechanical Engineering (Mashinostroitelnyi); he then became head of the Division of Mechanics at the Turbine Institute. He also taught very important courses at Leningrad State University and at the Physical-Mechanical Institute of Physics and Mechanics.

A vigorous, stressful job weakened S.A.'s health; he succumbed to an accidental illness, and a brilliant and successful young life has ended abruptly.

S.A. Geršgorin's death is a great and irreplaceable loss to Soviet Science. He occupied a unique place in the Soviet science - this place is now empty.

A careful collection and examination of everything S.A. has done, has been made, so that none of his ideas are lost - this is the duty of Soviet science in honor of one of its best representatives.

ИЗВЕСТИЯ АКАДЕМИИ НАУК СССР. 1931

BULLETIN DE L'ACADÉMIE DES SCIENCES DE L'URSS

Classe des sciences mathématiques et naturelles

Отделение математических и естественных наук

ÜBER DIE ABGRENZUNG DER EIGENWERTE EINER MATRIX

Von S. GERSCHGORIN

(Présenté par A. Krylov, *membre de l'Académie des Sciences)*

§ 1. Haben wir eine Matrix

$$(1) \qquad A = \begin{Vmatrix} a_{11}, & a_{12}, & \ldots & a_{1n} \\ a_{21}, & a_{22}, & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1}, & a_{n2}, & \ldots & a_{nn} \end{Vmatrix}$$

wo die Elemente a_{ik} beliebige komplexe Zahlen sein dürfen, und bezeichnen wir durch z_k $(k = 1, 2, \ldots, n)$ ihre Eigenwerte, d. h. die Wurzeln der Gleichung

$$(2) \qquad \begin{Vmatrix} a_{11} - z, & a_{12}, & \ldots & a_{1n} \\ a_{21}, & a_{22} - z, & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1}, & a_{n2}, & \ldots & a_{nn} - z \end{Vmatrix} = 0,$$

so gilt nach Bendixson und Hirsch * die Ungleichung

$$|z_k| \leq na,$$

wo a den Maximalwert aller Zahlen $|a_{ik}|$ bedeutet.

Wir wollen im folgenden zeigen, dass man im allgemeinen viel schärfere Aussagen über die Lage der Eigenwerte machen kann.

* Sur les racines d'une équation fondamentale. Acta Mathematica, t. 25 (1900).

Wir beweisen zunächst den folgenden Satz, der einem von L. Lévy* über Matrizen mit reellen Elementen ausgesprochenen völlig analog ist.

Satz I. Sind in der Matrix (1) die Bedingungen

$$|a_{ii}| \geq \sum_k{}' |a_{ik}|,^{**} \qquad (3) \qquad (i = 1, \ldots n$$

erfüllt (wobei das Ungleichheitszeichen mindestens für einen Wert von i gilt), so ist die Determinante Δ dieser Matrix gewiss von 0 verschieden.

Zum Beweis betrachten wir das zu der Matrix (1) zugehörige homogene Gleichungssystem

$$(4) \quad \begin{cases} a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n = 0, \\ a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n1} x_1 + a_{n2} x_2 + \cdots + a_{nn} x_n = 0. \end{cases}$$

Sollte entgegen der gemachten Annahme $\Delta = 0$ sein, so hat das System (4) eine nichtverschwindende Lösung $x_1^0, x_2^0, \ldots x_n^0$ (wobei diese Werte auch nicht alle einander gleich sein können). Sei $|x_\mu^0|$ die grösste unter den Zahlen $|x_i^0|$, so dass

$$(5) \qquad |x_i^0| \leq |x_\mu^0| \qquad (i = 1, \ldots, n).$$

Wir betrachten nun die μ-te der Gleichungen (4), welche lautet

$$(6) \qquad a_{\mu\mu} x_\mu^0 = - \sum_k{}' a_{\mu k} x_k^0.$$

Aus den Ungleichungen (3) und (5) folgt aber

$$|a_{\mu\mu}| \, |x_\mu^0| > \sum_k{}' |a_{\mu k}| \, |x_k^0|,$$

was mit der Gleichung (6) unvereinbar ist. Damit ist der Satz bewiesen.***

* Sur la possibilité de l'équilibre électrique. C. R. de l'Académie des Sciences, t. XCIII (1881).

** $\sum_k{}'$ bedeutet die Summation über alle Werte von k, ausser $k = i$.

*** Eine analoge Überlegung wurde schon früher von R. Kusmin zum Beweis des L. Lévy'schen Satzes verwendet.

§ 2. Verwenden wir den oben gefundenen Satz zur Matrix

$$
\left\| \begin{matrix} a_{11}-z, & a_{12}, & \ldots & a_{1n} \\ a_{21}, & a_{22}-z, & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1}, & a_{n2}, & \ldots & a_{nn}-z \end{matrix} \right\| \tag{7}
$$

so finden wir, dass die zugehörige Determinante von 0 verschieden ist, falls die Bedingungen

$$
|a_{ii}-z| \geq \sum_k{}' |a_{ik}| \qquad (i=1,\ldots n) \tag{8}
$$

(wo das Ungleichheitszeichen mindestens für ein i gilt) erfüllt sind.

Die geometrische Interpretation dieses Resultates führt uns auf den folgenden Satz.

Satz II. Die Eigenwerte $z_1, \ldots z_n$ der Matrix (1) liegen nur innerhalb des abgeschlossenen Gebietes G, das aus allen Kreisen K_i $(i=1, \ldots n)$ der z-Ebene mit den Mittelpunkten a_{ii} und zugehörigen Radien

$$
R_i = \sum_k{}' |a_{ik}|
$$

besteht.

Es kann vorkommen, dass m von den Kreisen K_i $(m=1, \ldots n)$ zu einem zusammenhängenden Gebiet $H_{(m)}$ zusammenfallen, wobei alle übrigen Kreise ausserhalb dieses Gebietes liegen. Über die Verteilung der Eigenwerte unter verschiedenen so definierten Gebieten $H_{(m)}$ kann der folgende Satz ausgesprochen werden.

Satz III. In jedem Gebiet $H_{(m)}$ liegen genau m Eigenwerte der Matrix (1).

Es sei $H_{(m)}$ aus den Kreisen

$$
K_{i_1}, K_{i_2}, \ldots K_{i_m}
$$

gebildet. Wir betrachten neben der Matrix A eine andere Matrix A', bei welcher alle nicht in der Diagonale stehende Elemente der Zeilen

$$
i_1, i_2, \ldots i_m
$$

verschwinden, die übrigen aber denjenigen der Matrix A gleich sind. Die Matrix A' hat sicher die Eigenwerte

$$a_{i_1 i_1}, a_{i_2 i_2}, \dots a_{i_m i_m}.$$

Nun fangen wir an die oben erwähnten verschwindenden Elemente der Matrix A' von 0 bis zu ihren Werten in der Matrix A so stetig zu verändern, dass ihre absoluten Beträge monoton wachsen. Die Kreise

$$K_{i_1}, K_{i_2}, \dots K_{i_m}$$

wachsen dabei stetig, bleiben jedoch immer von den übrigen festen Kreisen K_i der z-Ebene getrennt. Da die Eigenwerte der Matrix stetig von ihren Elementen abhängen, folgt daraus, dass in den Kreisen

$$K_{i_1}, K_{i_2}, \dots K_{i_m}$$

immer m Eigenwerte liegen müssen. Die Zahl der Eigenwerte in $H_{(m)}$ kann nicht m überschreiten, da ihre gesamte Anzahl in allen Gebieten $H_{(m)}$ genau n gleich sein muss. Damit ist unser Satz bewiesen.*

Liegen alle Kreise K_i getrennt voneinander, was durch die Bedingungen

$$(9) \qquad |a_{ii} - a_{jj}| \geq \sum_k{}' |a_{ik}| + \sum_k{}' |a_{jk}| \quad (i = 1, \dots, n;\ j = 2, \dots, n;\ j > i)$$

ausgedrückt werden kann, so sind alle Eigenwerte voneinander abgegrenzt. Da eine Gleichung mit reellen Koeffizienten nur paarweise konjugierte komplexe Wurzeln besitzen kann, folgt daraus unter anderen der folgende Satz.

Satz IV. Sind alle Elemente der Matrix (1) reel und bestehen die Relationen (9), so sind die sämtlichen Eigenwerte dieser Matrix reel.

§ 3. In allen vorstehenden Sätzen kann man statt der Zeilen die Spalten heranziehen. Wir gelangen in dieser Weise im allgemeinen zu einem neuen System G' von Kreisen K_i', welche auch zur Abgrenzung der Wurzeln dienen können. Wir können auch mehrere solche Kreissysteme bekommen, indem wir unsere Matrix verschiedenen Transformationen unterwerfen, bei

* Der Satz bleibt auch dann richtig, wenn sich $H_{(m)}$ mit den übrigen Kreisen von aussen berührt, so dass man bei Bestimmung der Gebiete $H_{(m)}$ solche Berührungen ausser acht lassen kann.

denen das Spektrum sich nicht ändert. Man gelangt dabei im allgemeinen zu einer besseren Abgrenzung der Eigenwerte, da die letzteren nur in denjenigen Punkten liegen dürfen, welche sämtlichen Kreissystemen gehören. Genauer: es seien die Kreissysteme G_λ ($\lambda = 1, \ldots l$) vorhanden, von denen jedes aus den Kreisen $K_i^{(\lambda)}$ ($i = 1, \ldots n$) besteht. Wir stellen uns vor, dass die Kreise von G_λ in n_λ ($n_\lambda \leqq n$) voneinander getrennte zusammenhängende Gebiete

$$H_1^{(\lambda)}, H_2^{(\lambda)}, \ldots H_{n_\lambda}^{(\lambda)}$$

zerfallen. Zu jedem Gebiet $H_j^{(\lambda)}$ ($j = 1, \ldots n_\lambda$) soll $m_j^{(\lambda)}$ von den Kreisen $K_i^{(\lambda)}$ gehören. Wir bezeichnen weiter durch $S_{j_1, \ldots j_l}$ ein Gebiet, welches allen Gebieten

$$H_{j_1}^{(1)}, H_{j_2}^{(2)}, \ldots H_{j_l}^{(l)}$$

gemeinsam ist (wo j_λ bestimmte Zahlen $\leqq n_\lambda$ bedeuten). Dann liegen im Gebiet $S_{j_1, \ldots j_l}$ (es kann auch nicht zusammenhängend sein) genau $m_{j_1, \ldots j_l}$ Eigenwerte, wo $m_{j_1 \ldots j_l}$ die kleinste der Zahlen

$$m_{j_1}^{(1)}, m_{j_2}^{(2)}, \ldots m_{j_l}^{(l)}$$

ist.

Wir können diese Überlegung in folgender Weise verwenden. Es sei $H_{(m)}$ ein aus den Kreisen

$$K_{i_1}, K_{i_2}, \ldots K_{i_m} \quad (m < n)$$

bestehendes zusammenhängendes Gebiet, welches von den anderen Kreisen K_i getrennt liegt. Wir unterwerfen unsere Matrix A einer Transformation mit Hilfe der Matrix $S = ||s_{ik}||$, wo

$$s_{ik} = 0 \qquad (i \neq k)$$

$$s_{ii} = \begin{cases} \alpha & (i = i_1, i_2, \ldots i_m) \\ 1 & (i \neq i_1, i_2, \ldots i_m) \end{cases}$$

Die Zahl $0 < \alpha < 1$ ist noch später genauer zu definieren. Die transformierte Matrix $B = SAS^{-1}$ entsteht aus A durch Multiplikation der Reihen $i_1, i_2, \ldots i_m$ mit α und Division der entsprechenden Spalten durch α. Wir können α so wählen, dass die Kreise

$$K_{i_1}, K_{i_2}, \ldots K_{i_m}$$

des Bereiches $H_{(m)}$ verkleinert werden, ohne die übrigen Kreise K_t, welche sich dabei vergrössern, zu schneiden (es darf höchstens eine Berührung von aussen eintreten). Damit erreichen wir eine bessere Abgrenzung der in $H_{(m)}$ liegenden Eigenwerte.

Wir wollen näher auf den Fall $m = 1$ eingehen. Es sei K_i ein isoliert liegender Kreis. Die Bedingungen für α lauten dann

$$(10) \qquad |a_{ii} - a_{jj}| \geq \alpha \sum_k{}' |a_{ik}| + \frac{1}{\alpha} |a_{ji}| + \sum_k{}'' |a_{jk}|, \qquad (j = 1, \ldots n;\ j \neq i)$$

wobei $\sum\limits_k{}''$ die Summation über alle k mit Ausnahme $k = i$ und $k = j$ bedeutet. Man kann, wie leicht zu ersehen ist, allen über α gestellten Bedingungen genügen, indem wir setzen*

$$\alpha = \max \frac{|a_{ii} - a_{jj}| - \sum\limits_k{}'' |a_{jk}| - \sqrt{\left(|a_{ii} - a_{jj}| - \sum\limits_k{}'' |a_{jk}|\right)^2 - 4\,|a_{ji}| \sum\limits_k{}' |a_{ik}|}}{2 \sum\limits_k{}' |a_{ik}|}.$$

Wir kommen damit zum folgenden Resultat.

Satz V. Ist K_i ein isoliert liegender Kreis des Gebietes G, so liegt der zugehörige Eigenwert innerhalb des zu K_i konzentrischen kleineren Kreises K_i' mit dem Radius

$$R_i' = \alpha R_i =$$

$$= \max \frac{1}{2} \left[|a_{ii} - a_{jj}| - \sum_k{}'' |a_{jk}| - \sqrt{\left(|a_{ii} - a_{jj}| - \sum_k{}'' |a_{jk}|\right)^2 - 4\,|a_{ji}| \sum_k{}' |a_{ik}|} \right].$$

* Das Zeichen max bedeutet das Maximum der nachstehenden Grösse für alle Werte von j ausser $j = i$.

Appendix B. Vector Norms and Induced Operator Norms.

With $\mathbb{C}^n$ denoting, for any positive integer n, the complex n-dimensional vector space of all column vectors $\mathbf{v} = [v_1, v_2, \cdots, v_n]^T$, where each v_i is a complex number, we have

Definition B.1. Let $\varphi : \mathbb{C}^n \to \mathbb{R}$. Then, φ is a **norm** on $\mathbb{C}^n$ if

$$\text{(B.1)}\quad \begin{array}{rl} i) & \varphi(\mathbf{x}) \geq 0 \quad (\text{all } \mathbf{x} \in \mathbb{C}^n); \\ ii) & \varphi(\mathbf{x}) = 0 \text{ if and only if } \mathbf{x} = \mathbf{0}; \\ iii) & \varphi(\gamma\mathbf{x}) = |\gamma|\varphi(\mathbf{x}) \quad (\text{any scalar } \gamma, \text{ any } \mathbf{x} \in \mathbb{C}^n); \\ iv) & \varphi(\mathbf{x}+\mathbf{y}) \leq \varphi(\mathbf{x}) + \varphi(\mathbf{y}) \quad (\text{all } \mathbf{x}, \mathbf{y} \in \mathbb{C}^n). \end{array}$$

Next, given a norm φ on $\mathbb{C}^n$, consider any matrix $B = [b_{i,j}] \in \mathbb{C}^{n\times n}$, so that B maps $\mathbb{C}^n$ into $\mathbb{C}^n$. Then,

$$\text{(B.2)}\qquad ||B||_\varphi := \sup_{\mathbf{x}\neq\mathbf{0}} \frac{\varphi(B\mathbf{x})}{\varphi(\mathbf{x})} = \sup_{\varphi(\mathbf{x})=1} \varphi(B\mathbf{x})$$

is called the **induced operator norm** of B, with respect to φ.

Proposition B.2. *Given any $A = [a_{i,j}] \in \mathbb{C}^{n\times n}$, let $\sigma(A)$ denote its* **spectrum**, *i.e.,*

$$\sigma(A) := \{\lambda \in \mathbb{C} : \det(\lambda I - A) = 0\},$$

and let $\rho(A)$ denote its **spectral radius**, *i.e.,*

$$\rho(A) := \max\{|\lambda| : \lambda \in \sigma(A)\}.$$

Then, for any norm φ on $\mathbb{C}^n$,

$$\text{(B.3)}\qquad \rho(A) \leq ||A||_\phi.$$

Proof. For any $\lambda \in \sigma(A)$, there is an $\mathbf{x} \neq \mathbf{0}$ in $\mathbb{C}^n$ with $\lambda\mathbf{x} = A\mathbf{x}$. Then, given any norm φ on $\mathbb{C}^n$, we normalize $\mathbf{x}$ so that $\varphi(\mathbf{x}) = 1$. Thus, from (B.1*iii*), (B.2), and our normalization, we have

$$\varphi(\lambda\mathbf{x}) = |\lambda|\varphi(\mathbf{x}) = |\lambda| = \varphi(A\mathbf{x}) \leq ||A||_\phi \cdot \varphi(\mathbf{x}) = ||A||_\varphi,$$

i.e., $|\lambda| \leq ||A||_\varphi$. As this is true for each $\lambda \in \sigma(A)$, then $\rho(A) \leq ||A||_\varphi$. ∎

Proposition B.3. *Let A and B be any matrices in $\mathbb{C}^{n\times n}$, and let φ be any norm on $\mathbb{C}^n$. Then, the induced operator norms of $A+B$, and $A\cdot B$ satisfy*

$$||A+B||_\varphi \le ||A||_\varphi + ||B||_\varphi \text{ and } ||A\cdot B||_\varphi \le ||A||_\varphi \cdot ||B||_\varphi. \tag{B.4}$$

Proof. From (B.1) and (B.2), we have

$$\begin{aligned} ||A+B||_\varphi = \sup_{\varphi(\mathbf{x})=1} \varphi((A+B)\mathbf{x}) &= \sup_{\varphi(\mathbf{x})=1} \varphi(A\mathbf{x}+B\mathbf{x}) \\ &\le \sup_{\varphi(\mathbf{x})=1} \{\varphi(A\mathbf{x}) + \varphi(B\mathbf{x})\} \\ &\le \sup_{\varphi(\mathbf{x})=1} \varphi(A\mathbf{x}) + \sup_{\varphi(\mathbf{x})=1} \varphi(B\mathbf{x}) \\ &= ||A||_\varphi + ||B||_\varphi. \end{aligned}$$

Similarly, from (B.2)

$$||A\cdot B||_\varphi = \sup_{\mathbf{x}\neq\mathbf{0}} \frac{\varphi(A(B\mathbf{x}))}{\varphi(\mathbf{x})} \le \sup_{\mathbf{x}\neq\mathbf{0}} \left\{ ||A||_\varphi \cdot \frac{\varphi(B\mathbf{x})}{\varphi(\mathbf{x})} \right\} \le ||A||\cdot||B||.$$

■

For $\mathbf{x} := [x_1, x_2, \cdots, x_n]^T \in \mathbb{C}^n$, perhaps the three most widely used norms on $\mathbb{C}^n$ are $\ell_1, \ell_2,$ *and* ℓ_∞, where

$$\begin{cases} ||\mathbf{x}||_{\ell_1} := \sum_{j=1}^n |x_j|, \quad ||\mathbf{x}||_{\ell_2} := \left(\sum_{i=1}^n |x_i|^2\right)^{\frac{1}{2}}, \\ \text{and} \\ ||\mathbf{x}||_{\ell_\infty} := \max_{1\le i\le n} |x_i|. \end{cases} \tag{B.5}$$

Given any matrix $C = [c_{i,j}] \in \mathbb{C}^{n\times n}$, the associated induced operator norms of C for the norms of (B.5) are easily shown to be

$$\begin{cases} ||C||_{\ell_1} = \max_{1\le j\le n} \left(\sum_{i=1}^n |a_{i,j}|\right); ||C||_{\ell_2} = [\rho(CC^*)]^{\frac{1}{2}}, \\ \text{and} \\ ||C||_{\ell_\infty} = \max_{1\le i\le n} \left(\sum_{j=1}^n |a_{i,j}|\right), \end{cases} \tag{B.6}$$

where $C^* := [\overline{c}_{j,i}] \in \mathbb{C}^{n\times n}$.

Appendix C. The Perron-Frobenius Theory of Nonnegative Matrices, *M*-Matrices, and *H*-Matrices.

To begin, if $B = [b_{i,j}] \in \mathbb{R}^{n \times n}$ is such that $b_{i,j} \geq 0$ for all $1 \leq i, j \leq n$, we write $B \geq O$. Similarly, if $\mathbf{x} = [x_1, x_2, \cdots, x_n]^T \in \mathbb{R}^n$ is such that $x_i > 0$ ($x_i \geq 0$) for all $1 \leq i \leq n$, we write $\mathbf{x} > \mathbf{0}$ ($\mathbf{x} \geq \mathbf{0}$). We also recall Definition 1.7 from Chapter 1, where **irreducible** and **reducible** matrices in $\mathbb{C}^{n \times n}$ are defined. Then, we state the following strong form of the **Perron-Frobenius Theorem** for irreducible matrices $A \geq O$ in $\mathbb{C}^{n \times n}$. Its complete proof can be found, for example, in Horn and Johnson (1985), Section 8.4, Meyer (2000), Chapter 8, or Varga (2000), Chapter 2. For notation, we again have $N := \{1, 2, \cdots, n\}$.

Theorem C.1. *(Perron-Frobenius Theorem) Given any $A = [a_{i,j}] \in \mathbb{R}^{n \times n}$, with $A \geq O$ and with A irreducible, then:*

- *i)* *A has a positive real eigenvalue equal to its spectral radius $\rho(A)$;*
- *ii)* *to $\rho(A)$, there corresponds an eigenvector $\mathbf{x} = [x_1, x_2, \cdots, x_n]^T > \mathbf{0}$;*
- *iii)* *$\rho(A)$ increases when any entry of A increases;*
- *iv)* *$\rho(A)$ is a simple eigenvalue of A;*
- *v)* *the eigenvalue $\rho(A)$ of A satisfies*

$$\text{(C.1)} \quad \sup_{\mathbf{x} > \mathbf{0}} \left\{ \min_{i \in N} \left[\frac{\sum_{j \in N} a_{i,j} x_j}{x_i} \right] \right\} = \rho(A) = \inf_{\mathbf{x} > \mathbf{0}} \left\{ \max_{i \in N} \left[\frac{\sum_{j \in N} a_{i,j} x_j}{x_i} \right] \right\}.$$

In the case that $A \geq O$ but is not necessarily irreducible, then the analog of Theorem C.1 is

Theorem C.2. *Given any $A = [a_{i,j}] \in \mathbb{R}^{n \times n}$ with $A \geq O$, then:*

- *i)* *A has a nonnegative eigenvalue equal to its spectral radius $\rho(A)$;*
- *ii)* *to $\rho(A)$, there corresponds an eigenvector $\mathbf{x} \geq \mathbf{0}$ with $\mathbf{x} \neq \mathbf{0}$;*
- *iii)* *$\rho(A)$ does not decrease when any entry of A increases;*
- *iv)* *$\rho(A)$ may be a multiple eigenvalue of A;*
- *v)* *the eigenvalue of $\rho(A)$ of A satisfies*

$$\rho(A) = \inf_{\mathbf{x}>\mathbf{0}} \left\{ \max_{i \in N} \left[\frac{\sum_{j \in N} a_{i,j} x_j}{x_i} \right] \right\}. \tag{C.2}$$

Next, given $A = [a_{i,j}] \in \mathbb{R}^{n \times n}$, then A is said (cf. Birkhoff and Varga (1958)) to be **essentially nonnegative** if $a_{i,j} \geq 0$ for all $i \neq j$, $(i, j \in N)$, and **essentially positive** if, in addition, A is irreducible. Similarly, we use the notation

$$\mathbb{Z}^{n \times n} := \left\{ A = [a_{i,j}] \in \mathbb{R}^{n \times n} : a_{i,j} \leq 0 \text{ for all } i \neq j \ (i, j \in N) \right\}, \tag{C.3}$$

which also is given in equation (5.5) of Chapter 5. We see immediately that A is essentially nonnegative if and only if $-A \in \mathbb{Z}^{n \times n}$.

For additional notation, consider any $A = [a_{i,j}] \in \mathbb{C}^{n \times n}$. We say that $\mathcal{M}(A) := [\alpha_{i,j}] \in \mathbb{R}^{n \times n}$ is the **comparison matrix** of A if $\alpha_{i,i} := |a_{i,i}|$, and $\alpha_{i,j} := -|a_{i,j}|$ for $i \neq j$ $(i, j \in N)$, i.e.,

$$\mathcal{M}(A) := \begin{bmatrix} +|a_{1,1}| & -|a_{1,2}| & \cdots & -|a_{1,n}| \\ -|a_{2,1}| & +|a_{2,2}| & \cdots & -|a_{2,n}| \\ \vdots & & & \vdots \\ -|a_{n,1}| & -|a_{n,2}| & \cdots & +|a_{n,n}| \end{bmatrix}, \tag{C.4}$$

where we note that $\mathcal{M}(A) \in \mathbb{Z}^{n \times n}$, for any $A \in \mathbb{C}^{n \times n}$. This brings us to our next important topic of M-matrices.

Given any $A = [a_{i,j}] \in \mathbb{Z}^{n \times n}$, let $\mu := \max_{i \in N} a_{i,i}$, so that $A = \mu I - B$, where the entries of $B = [b_{i,j}] \in \mathbb{R}^{n \times n}$ satisfy $b_{i,i} = \mu - a_{i,i} \geq 0$ and $b_{i,j} = -a_{i,j} \geq 0$ for all $i \neq j$. Thus, $b_{i,j} \geq 0$ for all $1 \leq i, j \leq n$, i.e., $B \geq O$. Then, as in Definition 5.4, we have

Definition C.3. Given any $A = [a_{i.j}] \in \mathbb{Z}^{n \times n}$, let $A = \mu I - B$ be as described above, where $B \geq O$. Then, A is an M**-matrix** if $\mu \geq \rho(B)$. More precisely, A is a **nonsingular** M-matrix if $\mu > \rho(B)$, and a singular M-matrix if $\mu = \rho(B)$.

With Definition C.3, we come to

Proposition C.4. *Given any $A = [a_{i,j}] \in \mathbb{R}^{n \times n}$ which is a nonsingular M-matrix (i.e., $A = \mu I - B$ where $B \geq O$ with $\mu > \rho(B)$), then $A^{-1} \geq O$.*

Proof. Since $A = \mu I - B$ where $B \geq O$ with $\mu > \rho(B)$, we can write that $A = \mu\{I - (B/\mu)\}$, where $\rho(B/\mu) < 1$. Then $I - (B/\mu)$ is also nonsingular, with its known convergent matrix expansion of

$$\{I - (B/\mu)\}^{-1} = I + (B/\mu) + (B/\mu)^2 + \cdots. \tag{C.5}$$

Since B/μ is a nonnegative matrix, so are all powers of (B/μ), and it follows from (C.5) that

$$\{I - (B/\mu)\}^{-1} \geq O; \text{ whence, } A^{-1} = \frac{1}{\mu}\{I - (B/\mu)\}^{-1} \geq O.$$

■

In a similar way (cf. Berman and Plemmons (1994), (A_3) of 4.6 Theorem), Proposition C.4 can be extended to

Proposition C.5. *Given any* $A = [a_{i,j}] \in \mathbb{R}^{n \times n}$ *which is a (possible singular) M-matrix (i.e.,* $A = \mu I - B$ *with* $B \geq O$ *and* $\mu \geq \rho(B)$*), then, for any* $\mathbf{x} = [x_1, x_2, \cdots, x_n]^T > \mathbf{0}$, $A+$ *diag*$[x_1, \cdots, x_n]$ *is a nonsingular M-matrix.*

Now, we come to the associated topic of H-matrices. Given $A = [a_{i,j}] \in \mathbb{C}^{n \times n}$, let $\mathcal{M}(A)$ be its comparison matrix of (C.4).

Definition C.6. Given $A = [a_{i,j}] \in \mathbb{C}^{n \times n}$, then A is an H**-matrix** if $\mathcal{M}(A)$ of (C.4) is an M-matrix.

Proposition C.7. *Given any* $A = [a_{i,j}] \in \mathbb{C}^{n \times n}$ *for which* $\mathcal{M}(A)$ *is a nonsingular M-matrix, then A is a nonsingular H-matrix.*

Proof. By Definition C.6, A is certainly an H-matrix, so it remains to show that A is nonsingular. As in the proof of Theorem 5.5 in Chapter 5, given any $\mathbf{u} = [u_i, u_2, \cdots, u_n]^T \in \mathbb{C}^n$, then the particular **vectorial norm** $\mathbf{p}(\mathbf{u})$ on $\mathbb{C}^n$ is defined by

(C.6) $$\mathbf{p}(\mathbf{u}) := [|u_1|, |u_2|, \cdots, |u_n|]^T \quad (\text{any } \mathbf{u} = [u_1, u_2, \cdots, u_n]^T \in \mathbb{C}^n).$$

Now, it follows by the reverse triangle inequality that, for any $\mathbf{y} = [y_1, y_2, \cdots, y_n]^T$ in $\mathbb{C}^n$,

(C.7) $$|(A\mathbf{y})_i| = \left| \sum_{j \in N} a_{i,j} y_i \right| \geq |a_{i,i}| \cdot |y_i| - \sum_{j \in N \setminus \{i\}} |a_{i,j}| \cdot |y_j| \qquad (\text{any } i \in N).$$

Recalling the definitions of $\mathcal{M}(A)$ of (C.4) and $\mathbf{p}(\mathbf{u})$ in (C.6), the inequalities of (C.7) nicely reduce to

(C.8) $$\mathbf{p}(A\mathbf{y}) \geq \mathcal{M}(A)\mathbf{p}(\mathbf{y}) \quad (\text{any } \mathbf{y} \in \mathbb{C}^n),$$

and we say that $\mathcal{M}(A)$ is a **lower bound matrix** for A. But as $\mathcal{M}(A)$ is, by hypothesis, a nonsingular M-matrix, then $(\mathcal{M}(A))^{-1} \geq O$, from Proposition C.4. As multiplying (on the left) by $(\mathcal{M}(A))^{-1}$ preserves the inequalities of (C.8), we have

(C.9) $$(\mathcal{M}(A))^{-1}\mathbf{p}(A\mathbf{y}) \geq \mathbf{p}(\mathbf{y}) \qquad (\text{any } \mathbf{y} \in \mathbb{C}^n).$$

But, the inequalities of (C.9) give us that A is nonsingular, for if A were singular, we could find a $\mathbf{y} \neq \mathbf{0}$ in $\mathbb{C}^n$ with $A\mathbf{y} = \mathbf{0}$, so that $\mathbf{p}(\mathbf{y}) \neq \mathbf{0}$ and $\mathbf{p}(A\mathbf{y}) = \mathbf{0}$. But this contradicts the inequalities of (C.9). ■

It is important to mention that the terminology of H- and M- matrices was introduced in the seminal paper of Ostrowski (1937b). Here, A.M. Ostrowski paid homage to his teacher, H. Minkowski, and to J. Hadamard, men who had inspired Ostrowski's work in this area. By naming these two classes of matrices after them, their names are forever honored and remembered in mathematics.

The theory of M-matrices and H-matrices has proved to be an incredibly useful tool in linear algebra, and it is as fundamental to linear algebra as topology is to analysis. For example, one finds 50 *equivalent* formulations of a nonsingular M-matrix in Berman and Plemmons (1994). Some additional equivalent formulations can be found in Varga (1976), and it is plausible that there are now over 70 such equivalent formulations of a nonsingular M-matrix.

Appendix D. Matlab 6 Programs.

In this appendix, Professor Arden Ruttan of Kent State University has kindly gathered several of the various Matlab 6 programs for figures generated in this book, so the interested readers can study these programs and alter them, as needed, for their own purposes.

Programs are listed on the following pages according to their figure numbers.

Fig. 2.1

```
x=[-2.5:0.05:2.5];
y=[-2.5:0.05:2.5];;
[X,Y]=meshgrid(x,y);
hold on
plot([1],[0],'Marker','o','MarkerSize',2)
plot([-1],[0],'Marker','o','MarkerSize',2)
axis equal
colormap([.7,.7,.7;1,1,1])
caxis([-1 1])
Z=abs(X+i*Y-1).*abs(X+i*Y+1)-2.0^2;
contourf(X,Y,-Z-1,[-1 -1],'k')
Z=abs(X+i*Y-1).*abs(X+i*Y+1)-1.41^2;
contourf(X,Y,-Z-1,[-1 -1],'k')
Z=abs(X+i*Y-1).*abs(X+i*Y+1)-1.2^2;
contourf(X,Y,-Z-1,[-1 -1],'k')
Z=abs(X+i*Y-1).*abs(X+i*Y+1)-1.0^2;
contourf(X,Y,-Z,[0 0],'k')
Z=abs(X+i*Y-1).*abs(X+i*Y+1)-0.9^2;
contourf(X,Y,-Z,[0 0],'k')
Z=abs(X+i*Y-1).*abs(X+i*Y+1)-0.5^2;
contourf(X,Y,-Z-1,[-1 -1],'k')
plot([-1],[0],'.k')
plot([1],[0],'.k')

title('Figure 2.1')
```

Fig. 2.2

```
hold on
x=[-.5:0.05:2.5];
y=[-1.5:0.05:1.5];;
[X,Y]=meshgrid(x,y);
Z=abs(X+i*Y-1)-1;
contour(X,Y,-Z,[0 0],'k')
y=[-.5:0.05:2.5];
x=[-1.5:0.05:1.5];;
[X,Y]=meshgrid(x,y);
Z=abs(X+i*Y-i)-1;
contourf(X,Y,-Z,[0 0],'k')
x=[-2.5:0.05:0.5];
y=[-1.5:0.05:1.5];;
[X,Y]=meshgrid(x,y);
Z=abs(X+i*Y+1)-1;
contourf(X,Y,-Z,[0 0],'k')
y=[-2.5:0.05:0.5];
x=[-1.5:0.05:1.5];;
[X,Y]=meshgrid(x,y);
Z=abs(X+i*Y+i)-1;
contourf(X,Y,-Z,[0 0],'k')

x=[-2.5:0.05:2.5];
y=[-2.5:0.05:2.5];;
[X,Y]=meshgrid(x,y);
Z=abs(X+i*Y-1).*abs(X+i*Y-i)-1;
contourf(X,Y,-Z-1,[-1 -1],'k')
axis equal
colormap([.7,.7,.7;1,1,1])
axis([-2.2,2.2,-2.2,2.2])
Z=abs(X+i*Y-1).*abs(X+i*Y+1)-1;
contourf(X,Y,-Z-1,[-1 -1],'k')
Z=abs(X+i*Y-1).*abs(X+i*Y+i)-1;
contourf(X,Y,-Z-1,[-1 -1],'k')
Z=abs(X+i*Y+1).*abs(X+i*Y-i)-1;
contourf(X,Y,-Z-1,[-1 -1],'k')
Z=abs(X+i*Y-i).*abs(X+i*Y+i)-1;
contourf(X,Y,-Z-1,[-1 -1],'k')
```

```
Z=abs(X+i*Y+1).*abs(X+i*Y+i)-1;
contourf(X,Y,-Z-1,[-1 -1],'k')
plot([0],[1],'.k')
plot([0],[-1],'.k')
plot([1],[0],'.k')
plot([-1],[0],'.k')
text(0,.8,'i')
text(0,-1.2,'i')
text(1,-.2,'1')
text(-1,-.2,'-1')

title('Figure 2.2')
a='Set Transparency of grey part to .5'
```

Fig. 2.7

```
x=[-2.5:0.05:2.5];
y=[-2.5:0.05:2.5];;
[X,Y]=meshgrid(x,y);
hold on
axis equal
caxis([-1,0])
colormap([.7,.7,.7;1,1,1])
axis([-2,2,-2,2])
Z=abs((X+i*Y).^2-1)-1;
contourf(X,Y,-Z,[0 0],'k')
Z=(abs(X+i*Y-1).^2).*abs(X+i*Y+1)-1/2.0;
contourf(X,Y,-Z-1,[-1 -1],'k')
plot([1],[0],'.k','MarkerSize',10)
plot([-1],[0],'.k','MarkerSize',10)
text(-1.08,-.075,'-1')
text(1,-.1,'1')
text(0,-.2,'0')
text(-.3,.5,'|z-1| |z+1|=1/2')
text(-.3,-.6,'|z -1|=1')

title('Figure 2.7')
```

Fig. 2.9

```
x=[-2.5:0.05:2.5];
y=[-2.5:0.05:2.5];;
[X,Y]=meshgrid(x,y);
hold on
axis equal
caxis([-1,0])
colormap([.7,.7,.7;1,1,1])
axis([-2,2,-2,2])
Z=abs((X+i*Y).^4-1)-1;
contourf(X,Y,-Z-1,[-1 -1],'k')
Z=abs(X+i*Y-1).*abs(X+i*Y-i)-1.0;
contourf(X,Y,-Z-1,[-1 -1],'k')
%plot([1],[0],'Marker','+','MarkerSize',10)
%plot([-1],[0],'Marker','+','MarkerSize',10)
a='Set transparency to 0.5'
title('Figure 2.9')
```

Fig. 3.2

```
x=[-.5:0.05:2.5];
y=[-1.5:0.05:1.5];;
[X,Y]=meshgrid(x,y);
Z=abs(X+i*Y-1)-1;
contour(X,Y,Z,[0 0],'k')
hold on
axis equal
colormap([.7,.7,.7;1,1,1])
caxis([-1 0])
axis([-2.2,2.2,-2.2,2.2])
y=[-.5:0.05:2.5];
x=[-1.5:0.05:1.5];;
[X,Y]=meshgrid(x,y);
Z=abs(X+i*Y-i)-1;
contour(X,Y,Z,[0 0],'k')
x=[-2.5:0.05:0.5];
y=[-1.5:0.05:1.5];;
[X,Y]=meshgrid(x,y);
Z=abs(X+i*Y+1)-1;
contour(X,Y,Z,[0 0],'k')
y=[-2.5:0.05:0.5];
x=[-1.5:0.05:1.5];;
[X,Y]=meshgrid(x,y);
Z=abs(X+i*Y+i)-1;
contour(X,Y,Z,[0 0],'k')

x=[-2.5:0.05:2.5];
y=[-2.5:0.05:2.5];;
[X,Y]=meshgrid(x,y);
Z=abs((X+i*Y).^4-1)-1;
contourf(X,Y,-Z-1,[-1 -1],'k')
%plot([1],[0],'Marker','+','MarkerSize',10)
%plot([2],[0],'Marker','+','MarkerSize',10)
title('Figure 3.2')
```

Fig. 3.4

```
x=[0:0.05:5];
y=[-2:0.05:2];
[X,Y]=meshgrid(x,y);
caxis([-1 0])
colormap([.7,.7,.7;1,1,1])
Z=(abs(X+i*Y-2).^2).*abs(X+i*Y-1)
   -abs(X+i*Y-1)-abs(X+i*Y-2);
contourf(X,Y,-Z-1,[-1 -1],'k')
axis equal
hold on
Z=(abs(X+i*Y-2).^2).*abs(X+i*Y-1)
   -abs(X+i*Y-1)+abs(X+i*Y-2);
contourf(X,Y,-Z,[0 0],'k')
Z=(abs(X+i*Y-2).^2).*abs(X+i*Y-1)
   +abs(X+i*Y-1)-abs(X+i*Y-2);
contourf(X,Y,-Z,[0 0],'k')
text(2,.4,'(13)(2)')
text(.7,.25,'(1)(23)')
text(1.5,1.5,'(1)(2)(3)')
plot([1],[0],'.k')
plot([2],[0],'.k')
title('Figure 3.4')
text(1,-.2,'1')
text(2,-.2,'2')
text(2,-.2,'0')
```

Fig. 6.1

```
x=[-20:0.1:40];
y=[-20:0.1:20];
[X,Y]=meshgrid(x,y);
hold on
axis equal
axis([.5 7.5 -2 2])
colormap bone
brighten(.9)
Z=-100*bc1(X,Y);% 0.059759, 5.831406
contourf(X,Y,Z,[0 0],'k.')
Z=-bc2(X,Y); % 0.063666, 4.693469
contour(X,Y,Z,[0 0],'k')
     Z=-bc3(X,Y); %3.617060, 32.247282
contour(X,Y,Z,[0 0],'k')
plot([2.2679],[0],'kx')
plot([4],[-1],'kx')
plot([4],[1],'kx')
plot([5.7321],[0],'kx')

with files bc1, bc2, and bc3, respectively:
function mm=bc1(x,y)
z=x+i*y;
mm=abs(z-2).*(abs(z-4).^2).*abs(z-6)
   -(abs(z-3)+1).*(abs(z-5)+1);

function mm=bc2(x,y)
z=x+i*y;
mm=abs(z-2).*abs(z-4)-(abs(z-3)+1);

function mm=bc3(x,y)
z=x+i*y;
mm=abs(z-4).*abs(z-6)-(abs(z-5)+1);
```

Fig. 6.2, and 6.3

```
x=[-20:0.1:40];
y=[-20:0.1:20];
[X,Y]=meshgrid(x,y);
hold on
Z=bb1(X,Y);
contour(X,Y,Z,[0 0], 'b--')
Z=bb2(X,Y);
axis equal
axis([-15 35 -16 16])
contour(X,Y,Z,[0 0], 'b--')
     Z=bb3(X,Y);
contour(X,Y,Z,[0 0], 'b')
     Z=bb4(X,Y);
contour(X,Y,Z,[0 0], 'b--')
W=bb(X,Y);
contour(X,Y,W,[102.96 102.96])
x=[0.03:.0005:0.12];
y=[-.04:.0005:0.04];
[X,Y]=meshgrid(x,y);
Z=bb1(X,Y);
contour(X,Y,Z,[0 0])
Z=bb2(X,Y);
contour(X,Y,Z,[0 0])
Z=bb3(X,Y);
contour(X,Y,Z,[0 0])
Z=bb4(X,Y);
contour(X,Y,Z,[0 0])
figure
hold on
Z=bb1(X,Y);
contourf(X,Y,Z,[0 0])
Z=bb2(X,Y);
contourf(X,Y,Z,[0 0],'k-')
Z=bb3(X,Y);
contourf(X,Y,Z,[0 0],'k-')
Z=bb4(X,Y);
contourf(X,Y,Z,[0 0],'k-')
```

Fig. 6.5

```
x=[-20:0.1:40];
y=[-20:0.1:20];
hold on
axis equal
axis([-15 35 -15 15])
caxis([-1 0])
colormap([.7,.7,.7;1 1 1])
[X,Y]=meshgrid(x,y);
Z=b1(X,Y)-1;
contour(X,Y,Z,[0 0],'k')
Z=b2(X,Y)-1;
contour(X,Y,Z,[0 0],'k')
Z=b3(X,Y)-1;
contour(X,Y,Z,[0 0],'k')
Z=b4(X,Y)-1;
contour(X,Y,Z,[0 0],'k')
Z=bb(X,Y)-102.96;
contour(X,Y,Z,[0 0],'k')
Z=bb1(X,Y).*bb3(X,Y)-1;
contourf(X,Y,Z-1,[-1 -1],'k')
plot([15],[0],'.k')
plot([-14 35],[0 0],'-k')
text(0,-1,'0')
text(15,-1,'15')
```

Fig. 6.6

```
x=[0.03:0.001:.12];
y=[-0.04:0.001:0.04];
[X,Y]=meshgrid(x,y);
hold on
axis equal
axis([-.005 .12 -.04 .04])
colormap([1,1,1;.7,.7,.7])
Z=b1(X,Y)-1;
contourf(X,Y,Z-1,[-1 -1],'k')
Z=b2(X,Y)-1;
contourf(X,Y,Z-1,[-1 -1],'k')
Z=b3(X,Y)-1;
contourf(X,Y,Z-1,[-1 -1],'k')
Z=b4(X,Y)-1;
contourf(X,Y,Z-1,[-1 -1],'k')
Z=bb1(X,Y).*bb3(X,Y)-1;
contourf(X,Y,Z,[0 0],'k')
plot([-.005 .12],[0 0],'-k')
plot([.0482],[0],'.k')
plot([.0882],[0],'.k')
plot([0],[0],'.k')
text(.09,-.004,'0.0882')
text(.05,-.004,'0.0482')
text(0,-.004,'0')
```

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