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A shape theorem for the spread of an infection

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Abstract

In [KSb] we studied the following model for the spread of a rumor or infection: There is a "gas" of so-called A-particles, each of which performs a continuous time simple random walk on $\mathbb{Z}^d$, with jump rate D_A. We assume that "just before the start" the number of A-particles at x, $N_A(x, 0-)$, has a mean μ_A Poisson distribution and that the $N_A(x, 0-)$, $x \in \mathbb{Z}^d$, are independent. In addition, there are B-particles which perform continuous time simple random walks with jump rate D_B. We start with a finite number of B-particles in the system at time 0. The positions of these initial B-particles are arbitrary, but they are nonrandom. The B-particles move independently of each other. The only interaction occurs when a B-particle and an A-particle coincide; the latter instantaneously turns into a B-particle. [KSb] gave some basic estimates for the growth of the set $\widetilde{B}(t) := \{x \in \mathbb{Z}^d : \text{a } B\text{-particle visits } x \text{ during } [0, t]\}$. In this article we show that if $D_A = D_B$, then $B(t) := \widetilde{B}(t) + [-\frac{1}{2}, \frac{1}{2}]^d$ grows linearly in time with an asymptotic shape, i.e., there exists a nonrandom set B_0 such that $(1/t)B(t) \to B_0$, in a sense which will be made precise.

1. Introduction

We study the model described in the abstract. One interpretation of this model is that the B-particles represent individuals who are infected, and the A-particles represent susceptible individuals; see [KSb] for another interpretation. $\widetilde{B}(t)$ represents the collection of sites which have been visited by a B-particle during $[0, t]$, and $B(t)$ is a slightly fattened up version of $\widetilde{B}(t)$, obtained by adding a unit cube around each point of $\widetilde{B}(t)$. This fattened up version is introduced merely to simplify the statement of our main result. It is simpler to speak of the shape of the set $(1/t)B(t)$ as a subset of $\mathbb{R}^d$, than of the discrete set $(1/t)\widetilde{B}(t)$.

The aim of this paper is to describe how the infection spreads throughout space as time goes on. In [KSb] we proved a first result in this direction in the case $D_A = D_B$. We proved that under this condition there exist constants

$0 < C_2 \le C_1 < \infty$ such that almost surely

$$\mathcal{C}(C_2 t) \subset B(t) \subset \mathcal{C}(2C_1 t) \text{ for all large } t, \tag{1.1}$$

where

$$\mathcal{C}(r) := [-r, r]^d. \tag{1.2}$$

(1.1) gives upper and lower bounds which are linear in time, for $B(t)$, the region which has been visited by the infection during $[0, t]$. However, the upper and lower bounds in (1.1) are not the same. The principal result of this paper is a so-called shape theorem which gives the first order asymptotic behavior of the region $B(t)$. It shows that $(1/t)B(t)$ converges to a fixed set B_0. Thus, not only is the growth linear in time, but $B(t)$ looks asymptotically like (a scaled version of) B_0. This of course sharpens (1.1) by 'bringing the upper and lower bound together'. However, the result (1.1) is a crucial tool for proving the shape theorem. We do not know of a shortcut which proves the shape theorem without much of the development of [KSb] for (1.1). The precise form of the shape theorem here is as follows:

THEOREM 1. *Consider the model described in the abstract. If $D_A = D_B$, then there exists a nonrandom, compact, convex set B_0 such that for all $\varepsilon > 0$ almost surely*

$$(1-\varepsilon)B_0 \subset \frac{1}{t}B(t) \subset (1+\varepsilon)B_0 \textit{ for all large } t. \tag{1.3}$$

The origin is an interior point of B_0, and B_0 is invariant under reflections in coordinate hyperplanes and under permutations of the coordinates.

Remark 1. It follows immediately from Theorem 1 and Proposition B below that the particle distribution at a large time t looks as follows: The numbers of particles, irrespective of type, that is $N_A(x,t) + N_B(x,t), x \in \mathbb{Z}^d$, is a collection of i.i.d. mean μ_A Poisson variables plus a finite number of particles which started at time zero at fixed locations (these are the particles added as B-particles at the start). For every $\varepsilon > 0$ there are almost surely no A particles in $(1-\varepsilon)tB_0$ and no B-particles outside $(1+\varepsilon)tB_0$ for all large t.

Shape theorems have a fairly long history and have become the first goal of many investigations of stochastic growth models. To the best of our knowledge Eden (see [E]) was the first one to ask for a shape theorem for his celebrated 'Eden model'. The problem turned out to be a stubborn one. The first real progress was due to Richardson, who proved in [Ri] a shape theorem not only for the Eden model, but also for a more general class of models, now called Richardson models. In these models one typically thinks of the sites of $\mathbb{Z}^d$ as cells which can be of two types (for instance B and A or infected and susceptible). Cells can change their type to the type of one of their neighbors according to appropriate rules. One starts with all cells off the origin of type A and a cell of type B at the origin and tries to prove a shape theorem for the set of cells of type B at a large time. An important example of such a model is 'first-passage percolation', which was introduced in [HW] (this includes the Eden model, up to a time change). A quite good shape theorem for first-passage percolation is known (see [Ki], [CD], [Ke]). In more recent first-passage percolation papers even sharper information has been obtained which gives estimates on the rate at which $(1/t)B(t)$ converges to its limit B_0 (see [Ho] for a survey of such results).

Shape theorems for quite a few variations of Richardson's model and first-passage percolation have been proven (see for instance [BG] and [GM]), but as far as we know these are all for models in which the cells do not move over time, with one exception. This exception is the so-called frog model which follows the rules given in our abstract, but which has $D_A = 0$, i.e., the susceptibles or type A cells stand still (see [AMP] and [RS] for this model). The present paper may be the first one which allows both tyes of particles to move.

In nearly all cases shape theorems are proven by means of Kingman's subadditive ergodic theorem (see [Ki]). This is also what is used for the frog model. For this model one can show that the family of random variables $\{T_{x,y}\}$ is subadditive, were $T_{x,y}$ is a version of the first time a particle at y is infected, if one starts with one infected particle at x and one susceptible at each other site. More precisely, the $T_{x,y}$ can all be defined on one probability space such that $T_{x,z} \le T_{x,y} + T_{y,z}$ for all $x, y, z \in \mathbb{Z}^d$, and such that their joint distribution is invariant under translations. Unfortunately this subadditivity property is no longer valid if one allows both types of particles to move. Nevertheless, subadditivity methods are still heavily used in the proof of Theorem 1. However, we now use subadditivity only for certain 'half-space' processes which approximate the true process. Moreover, these half-space processes have only approximate superconvolutive properties (in the terminology of [Ha]). There is no obvious family of random variables with properties like those of the $T_{x,y}$. One only has some relation between the distribution functions of the $H(t, u)$ for a fixed unit vector u, where $H(t, u)$ is basically the maximum of $\langle x, u\rangle$ over all x which have been reached by a B-particle by time t ($\langle x, u\rangle$ is the inner product of x and u; for technical reasons $H(t, u)$ will be calculated in a process in which the starting conditions are slightly different from our original process). These properties are strong enough to show that for each unit vector u there exists a constant $\lambda(u)$ such that almost surely

$$\lim_{n\to\infty} \frac{1}{t} H(t, u) = \lambda(u), \tag{1.4}$$

Thus the B-particles reach in time t half-spaces in a fixed direction u at distances which grow linearly in t. Except in dimension 1, it then still requires a considerable amount of technical work to go from this result about the linear growth of the distances of reached half-spaces to the full asymptotic shape result. We will give more heuristics before some of our lemmas.

Remark 2. Our proof in [KSb] shows that the right-hand inclusion in (1.1) remains valid for arbitrary jump rates of the A and the B-particles. However, it is still not known whether the left-hand inclusion holds in general. The lower bound for $B(t)$ is known only when $D_A = D_B$, or when $D_A = 0$, that is, when the A and B-particles move according to the same random walk (see [KSb]), or in the frog model, when the A-particles stand still (see [AMP], [RS]).

Here is some general notation which will be used throughout the paper: $\|x\|$ without subscript denotes the ℓ^∞-norm of a vector $x = (x(1), \dots, x(d)) \in \mathbb{R}^d$, i.e.,

$$\|x\| = \max_{1 \le i \le d} |x(i)|.$$

We will also use the Euclidean norm of x; this will be denoted by the usual $\|x\|_2$. $\langle x, u\rangle$ denotes the (Euclidean) inner product of two vectors $x, u \in \mathbb{R}^d$, and $\mathbf{0}$ denotes the origin (in $\mathbb{Z}^d$ or $\mathbb{R}^d$). For an event $\mathcal{E}$, $\mathcal{E}^c$ denotes its complement.

$K_1, K_2, \dots$ will denote various strictly positive, finite constants whose precise value is of no importance to us. The same symbol K_i may have different values in different formulae. Further, C_i denotes a strictly positive constant whose value remains the same throughout this paper (a.s. is an abbreviation of almost surely).

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2. Results from [KSb]

Throughout the rest of this paper we assume that

$$D_A = D_B \tag{2.1}$$

and we abbreviate their common value to D. We begin this section with some further facts about the setup. More details can be found in Section 2 of [KSb] which deals with the construction of our particle system. $\{S_t\}_{t \ge 0}$ will be a continuous-time simple random walk on $\mathbb{Z}^d$ with jump rate D and starting at $\mathbf{0}$.

To each initial particle ρ is assigned a path $\{\pi_A(t,\rho)\}_{t\geq 0}$ which is distributed like $\{S_t\}_{t\geq 0}$. The paths $\pi_A(\cdot,\rho)$ for different ρ's are independent and they are all independent of the initial $N_A(x,0-), x\in\mathbb{Z}^d$. The position of ρ at time t equals $\pi(0,\rho)+\pi_A(t,\rho)$, and this can be assigned to ρ without knowing the paths of any of the other particles. The type of ρ at time s is denoted by $\eta(s,\rho)$. This equals A for $0\leq s<\theta(\rho)$ and equals B for $s\geq\theta(\rho)$, where $\theta(\rho)$, the so-called switching time of ρ, is the first time at which ρ coincides with a B-particle. Note that this is simpler than in the construction of [KSb] for the general case which may have $D_A\neq D_B$. In that case we had simple random walks $\{S^\eta\}_{t\geq 0}$ with jump rate D_η for $\eta\in\{A,B\}$, and there were two paths associated with each initial particle $\rho:\pi_\eta(\cdot,\rho),\eta\in\{A,B\}$, with $\{\pi_\eta(t,\rho)\}$ having the same distribution as $\{S_t^\eta\}$. If ρ had initial position z, its position was then equal to $z+\pi_A(t,\rho)$ until ρ first coincided with a B-particle at time $\theta(\rho)$; for $t\geq\theta(\rho)$ the position of ρ was $z+\pi_A(\theta(\rho),\rho)+[\pi_B(t,\rho)-\pi_B(\theta(\rho),\rho)]$. This depends on $\theta(\rho)$ and therefore on the movement of all the other particles.

In the present case we can take $\pi_B=\pi_A$, which has the great advantage that the path of ρ does not depend on the paths of the other particles. This is the reason why the case $D_A=D_B$ is special. We proved in [KSb] that on a certain state space Σ_0 (which we shall not describe here), the collection of positions and types of all particles at time t, with t running from 0 to ∞, is well defined and forms a strong Markov process with respect to the σ-fields $\mathcal{F}_t=\cap_{h>0}\mathcal{F}^0_{t+h},\ t\geq 0$, where $\mathcal{F}^0_t$ is the σ-field generated by the positions and types of all particles during $[0,t]$. The elements of these σ-fields are subsets of $\Sigma^{[0,\infty)}$, where $\Sigma=\prod_{k\geq 1}\big((\mathbb{Z}^d\cup\partial_k)\times\{A,B\}\big)$. $\Sigma^{[0,\infty)}$ is the pathspace for the positions and types of all particles. More explicit definitions are given in [KSb] but are probably not needed for this paper. It was also shown in [KSb] that if one chooses the number of initial A-particles at z, with z varying over $\mathbb{Z}^d$, as i.i.d. mean μ_A Poisson variables, then the process starts off in Σ_0 and stays in Σ_0 forever, almost surely.

We write $N_\eta(z,t)$ for the number of particles of type η at the space-time point (z,t), $z\in\mathbb{Z}^d,\eta\in\{A,B\}$, while $N_A(z,0-)$ denotes the number of A-particles to be put at z 'just before' the system starts evolving. Note that our model always has only particles of one type at each given site, because an A-particle which meets a B-particle changes instantaneously to a B-particle. Thus, if $N_A(z,0-)=N$ for some site z and we add $M(>0)$ B-particles at z at time 0, then we have to say that $N_A(z,0)=0, N_B(z,0)=N+M$. We call a site x *occupied* at time t by a particle of type η if there is at least one particle of type η at x at time t; in this case all particles at (x,t) have type η. Also, x is occupied at time t if there is at least one particle at (x,t), irrespective of the type of that particle.

We shall rely heavily on basic upper and lower bounds for the growth of $B(t)$ which come from Theorems 1 and 2 in [KSb].

THEOREM A. *If $D_A = D_B$, then there exist constants $0 < C_2 \le C_1 < \infty$ such that for every fixed K*

$$P\big\{\mathcal{C}(C_2 t) \subset B(t) \subset \mathcal{C}(2C_1 t)\big\} \ge 1 - \frac{1}{t^K} \tag{2.2}$$

for all sufficiently large t.

We also have some information about the presence of A-particles in the regions which have already been visited by B-particles. The following is Proposition 3 of [KSb].

PROPOSITION B. *If $D_A = D_B$, then for all K there exists a constant $C_3 = C_3(K)$ such that*

(2.3) $P\{$*there are a vertex z and an A-particle at the space-time point (z,t) while there also was a B-particle at z at some time* $\le t - C_3[t \log t]^{1/2}\}$ $\le \dfrac{1}{t^K}$ *for all sufficiently large t.*

Consequently, for large t

(2.4) $P\{$*at time t there is a site in $\mathcal{C}\big(C_2 t/2\big)$ which is occupied by an A-particle*$\} \le \dfrac{2}{t^K}$.

Finally we reproduce here Lemma 15 of [KSb] which gives an important monotonicity property. We repeat that in the present setup, with the $N_A(x, 0-)$ i.i.d. Poisson variables, our process a.s. has values in Σ_0 at all times (see Proposition 5 of [KSb]).

LEMMA C. *Assume $D_A = D_B$ and let $\sigma^{(2)} \in \Sigma_0$. Assume further that $\sigma^{(1)}$ lies below $\sigma^{(2)}$ in the following sense*:

(2.5) *For any site $z \in \mathbb{Z}^d$, all particles present in $\sigma^{(1)}$ at z are also present in $\sigma^{(2)}$ at z,*

and

(2.6) *At any site z at which the particles in $\sigma^{(2)}$ have type A, the particles also have type A in $\sigma^{(1)}$.*

Let $\pi_A(\cdot, \rho)$ be the random-walk paths associated to the various particles and assume that the Markov processes $\{Y_t^{(1)}\}$ and $\{Y_t^{(2)}\}$ are constructed by means of the same set of paths $\pi_A(\cdot, \rho)$ starting with state $\sigma^{(1)}$ and $\sigma^{(2)}$, respectively (*as defined in Section* 2 *of* [KSb], *but with $\pi_A(s, \rho) = \pi_B(s, \rho)$ for all s, ρ; see* (2.6), (2.7) *there*). *Then, almost surely, $\{Y_t^{(1)}\}$ and $\{Y_t^{(2)}\}$ satisfy* (2.5) *and* (2.6) *for all t, with $\sigma^{(i)}$ replaced by $Y_t^{(i)}$, $i = 1, 2$. In particular, $\sigma^{(1)} \in \Sigma_0$.*

In particular, this monotonicity property says that if $\sigma^{(1)}$ is obtained from $\sigma^{(2)}$ by removal of some particles and/or changing some B-particles to A-particles, then the process starting from $\sigma^{(1)}$ has no more B-particles at each space-time point than the process starting from $\sigma^{(2)}$. We note that this monotonicity property holds only under our basic assumption that $D_A = D_B$.

3. A subadditivity relation

In this section we shall prove the basic subadditivity relation of Proposition 3 and deduce from it, in Corollary 5, that the B-particles spread in each fixed direction over a distance which grows asymptotically linearly with time. This statement is ambiguous because we haven't made precise what 'spread in a fixed direction' means. Here this will be measured by

$$\max\{\langle x, u\rangle : x \in \widetilde{B}(t)\}, \tag{3.1}$$

where u is a given unit vector (in the Euclidean norm) in $\mathbb{R}^d$ (see the abstract for $\widetilde{B}$). In addition we will not prove subadditivity (which is an almost sure relation), but only superconvolutivity, in the terminology of [Ha] (which is a relation between distribution functions). The tool of superconvolutivity in other models with no obvious subadditivity in the strict sense goes back to [Ri], and was also used in [BG] and [W].

Actually we prove superconvolutivity only for half-space processes, which we shall introduce now. We define the closed half-space

$$\mathcal{S}(u, c) = \{x \in \mathbb{R}^d : \langle x, u\rangle \geq c\}.$$

Given a $u \in S^{d-1}$ and $r \geq 0$ we consider the *half-space process corresponding to* $(u, -r)$ (also called $(u, -r)$ half-space-process). We define this to be the process whose initial state is obtained by replacing $N_A(x, 0-)$ by 0 for all $x \notin \mathcal{S}(u, -r)$. Thus the initial state of the $(u, -r)$-half-space-process is

$$N_A(x, 0-) \begin{cases} = 0 \text{ if } x \notin \mathcal{S}(u, -r) \\ = \text{ original } N_A(x, 0-) \text{ if } x \in \mathcal{S}(u, -r), \end{cases}$$

where the $N(A, x, -0)$ are i.i.d., mean μ_A Poisson variables. In addition the particles at w_{-r} are turned into B-particles at time 0, where w_{-r} is the site in $\mathcal{S}(u, -r)$ nearest to the origin (in ℓ^∞-norm) with $N_A(w_{-r}, 0-) > 0$. If there are several possible choices for w_{-r}, the tie is broken in the following manner. All vertices of $\mathbb{Z}^d$ are first ordered in some deterministic manner, say lexicographically. Then among all occupied vertices in $\mathcal{S}(u, -r)$ which are nearest to the origin we take w_{-r} to be the first one in this order. There will be many other occasions where ties may occur. These will be broken in the same way as here, but we shall not mention ties or the breaking of them anymore. Note that no extra B-particles are introduced at time 0, but that only the type of the particles at w_{-r} is changed. Thus,

$$N_A(x,0) + N_B(x,0) = N_A(x,0-) \text{ for all } x. \tag{3.2}$$

From time 0 on the particles move and change type as described in the abstract. Note that only the initial state is restricted to $\mathcal{S}(u,-r)$. Once the particles start to move they are free to leave $\mathcal{S}(u,-r)$. The $(u,-r)$ half-space process will often be denoted by $\mathcal{P}^{\text{h}}(u,-r)$.

We further define the $(u,-r)$ *half-space process starting at* (x,t). This process is defined for times $t' \geq t$ only. We define it as follows: at time t let $w_{-r}(x,t)$ be the nearest site to x which is occupied in the $(u,-r)$ half-space process. We then reset the types of the particles at $w_{-r}(x,t)$ to B and the types of all other particles present in the $(u,-r)$ half-space process at time t to A. The particles then move along the same path in the $(u,-r)$ half-space process starting at (x,t) as in $\mathcal{P}^{\text{h}}(u,-r)$ (which starts at $(\mathbf{0},0)$). However, the types of the particles in the $(u,-r)$ half-space process starting at (x,t) are determined on the basis of the reset types at time t. Thus the half-space process starting at (x,t) has at any time only particles which were in $\mathcal{S}(u,-r)$ at time 0.

Moreover, at any site y and time $t' \geq t$, $\mathcal{P}^{\text{h}}(u,-r)$ and the $(u,-r)$ half-space process started at (x,t) contain exactly the same particles. We see from this that the *paths* of the particles in the $(u,-r)$ half-space processes starting at (x,t) and at $(\mathbf{0},0)$ are coupled so that they coincide from time t on, but the types of a particle in these two processes may differ. Lemma C shows that if there is a B-particle in $\mathcal{P}^{\text{h}}(u,-r)$ at x at time t, then in this coupling any B-particle in the $(u,-r)$ half-space process starting at (x,t) also has to have type B in $\mathcal{P}^{\text{h}}(u,-r)$.

The coupling between the two half-space processes clearly relies heavily on the assumption $D_A = D_B$, so that we can assign the same path to a particle in the two processes, even though the types of the particle in the two processes may be different.

It is somewhat unnatural to start the $(u,-r)$ half-space process with B-particles at w_{-r} in case $r < 0$, so that the origin does not lie in the half-space $\mathcal{S}(u,-r)$. We shall avoid that situation. We can, however, use the $(u,-r)$ half-space process starting at (x,t). This is well defined for all r. We merely need to find the site nearest to x which has at time t a particle which started in $\mathcal{S}(u,-r)$ at time 0. We can then reset the type of the particles at this site to B at time t. We shall consider the $(u,-r)$ half-space process starting at (x,t) mostly in cases where we already know that x itself is occupied at time t in the $(u,-r)$ half-space process.

Finally we shall occasionally talk about the *full-space process* and the *full-space process starting at* (x,t). These are defined just as the half-space processes, but with $r = \infty$. In particular, the full-space process starts with

B-particles only at the nearest occupied site to the origin and (3.2) applies. The full-space process starting at (x,t) has B-particles at time t only at the nearest occupied site to x. The type of all particles at other sites are reset to A at time t. Being stationary in time, the full-space process started at (x,t) has the same distribution at the space-time point $(x+y,t+s)$ as the full-space process (started at $(\mathbf{0},0)$) at the point (y,s). Again we shall use the same random walk paths π_A for all the full-state processes and the half-space processes, so that these processes are automatically coupled. We shall denote the full-space process by $\mathcal{P}^{\mathrm{f}}$.

We point out that if $0 \le r_1 \le r_2$, and if $\|w_{-r_2}\| \le r_1/\sqrt{d}$, then $w_{-r_2} \in \mathcal{S}(u,-r_1) \subset \mathcal{S}(u,-r_2)$ and $w_{-r_1} = w_{-r_2}$. In this case, both $\mathcal{P}^{\mathrm{h}}(u,-r_1)$ and $\mathcal{P}^{\mathrm{h}}(u,-r_2)$ start with changing the type to B at the site w_{-r_1} only and all other particles are given by type A. In this situation, by Lemma C, at any time,

$$\text{any } B\text{-particle in } \mathcal{P}^{\mathrm{h}}(u,-r_1) \text{ is also a } B\text{-particle in } \mathcal{P}^{\mathrm{h}}(u,-r_2). \tag{3.3}$$

This comment also applies if $\mathcal{P}^{\mathrm{h}}(u,-r_2)$ is replaced by $\mathcal{P}^{\mathrm{f}}$ (which is the case $r_2=\infty$).

Rather than introduce formal notation for the probability measures governing the many processes here, we shall abuse notation and write $P\{A$ in the process $\mathcal{P}\}$ for the probability of the event A according to the probability measure governing the process $\mathcal{P}$. Neither shall we describe the probability space on which $\mathcal{P}$ lives.

It seems worthwhile to discuss more explicitly the relation of the full-space process to our process as described in the abstract. The latter has some B-particles introduced at time 0 at one or more sites, in addition to the Poisson numbers of particles, $N_A(x,0-), x \in \mathbb{Z}^d$. If exactly one B-particle is added at time 0, and this particle is placed at $\mathbf{0}$, then we shall call the resulting process the *original process.*

Suppose we want to estimate $P\{\mathcal{A}(x_0)\}$ in the full-space process, where

$$x_0 := \text{the nearest occupied site to the origin at time 0 in } \mathcal{P}^{\mathrm{f}}, \tag{3.4}$$

$\mathcal{A}$ is some event and $\mathcal{A}(x)$ is the translation by x of this event (which takes $N_A(\mathbf{0},s)$ to $N_A(x,s)$). Then, for C a subset of $\mathbb{Z}^d$,

$$\begin{aligned} P\{x_0 \in C, \mathcal{A}(x_0) \text{ in } \mathcal{P}^{\mathrm{f}}\} &= \sum_{x\in C} P\{x_0 = x, \mathcal{A}(x)\} \\ &\le \sum_{x\in C} P\{x \text{ is occupied at time } 0, \mathcal{A}(x) \text{ in } \mathcal{P}^{\mathrm{f}}\} \\ &= \sum_{x\in C}\sum_{k=1}^{\infty} e^{-\mu_A}\frac{[\mu_A]^k}{k!} P\{\mathcal{A} | \text{there are } k \text{ } B\text{-particles at } \mathbf{0} \text{ at time } 0\}. \end{aligned} \tag{3.5}$$

(The probability in the last sum is the same in $\mathcal{P}^{\mathrm{f}}$ as in the original process.) On the other hand, in the original process we have

$$(3.6)\quad P\{\mathcal{A} \text{ in the original process}\} = \sum_{k=1}^{\infty} e^{-\mu_A} \frac{[\mu_A]^{k-1}}{(k-1)!} P\{\mathcal{A} | \text{there are } k \ B\text{-particles at } \mathbf{0} \text{ at time } 0\}.$$

Comparison of the right-hand sides in (3.5) and (3.6) yields the crude bound

$$(3.7)\quad P\{x_0 \in C, \mathcal{A}(x_0) \text{ in the full-space process}\} \le (\text{cardinality of } C)\mu_A P\{\mathcal{A} \text{ in original process}\}.$$

We shall repeatedly use a somewhat more general version of this inequality (see for instance (3.25), (3.77), (3.78), (5.35)). Suppose $s \ge 0$ is fixed and X is a random vertex in $\mathbb{Z}^d$, and suppose further that

$$(3.8)\quad P\{\mathcal{A}(X) \text{ but } (X, s) \text{ is not occupied in the full-space process starting at } (X, s)\} = 0.$$

(Note that this is satisfied if (X, s) is occupied almost surely in $\mathcal{P}^{\mathrm{f}}$.) Let $C \subset \mathbb{Z}^d$ as before. Now, given that there are $k \ge 1$ particles at the (nonrandom) space-time point (x, s), the full-space process starting at (x, s) is simply a translation by (x, s) in space-time of the original process, conditioned to start with $k-1$ points at the origin and one B-particle added at the origin. Therefore, essentially for the same reasons as for (3.7),

$$(3.9)\quad P\{X \in C, \mathcal{A}(X) \text{ in the full-space process starting at } (X, s)\} \le (\text{cardinality of } C)\mu_A P\{\mathcal{A} \text{ in original process}\}.$$

For a rather trivial comparison in the other direction we note that if $P\{\mathcal{A} \text{ in } \mathcal{P}^{\mathrm{f}}\} = 0$ for the full-space process, then we certainly have for each $k \ge 1$ that

$$\begin{aligned}(3.10)\qquad 0 &= P\{\mathcal{A} \text{ in } \mathcal{P}^{\mathrm{f}}, x_0 = \mathbf{0}, k \text{ particles at } x_0\} \\ &= P\{\mathcal{A} \text{ in } \mathcal{P}^{\mathrm{f}}, k \text{ particles at } \mathbf{0}\} \\ &= e^{-\mu_A} \frac{[\mu_A]^k}{k!} P\{\mathcal{A} | \text{there are } k \ B\text{-particles at } \mathbf{0} \text{ at time } 0\}.\end{aligned}$$

This implies, via (3.6), that also $P\{\mathcal{A} \text{ in original process}\} = 0$.

It is somewhat more complicated to compare $\mathcal{P}^{\mathrm{f}}$ with the process described in the abstract if more than one B-particle is introduced at time 0. Rather than develop general results in this direction we merely show in our first lemma that it suffices to prove (1.3) for the full-space process.

LEMMA 1. *If* (1.3) *holds in* $\mathcal{P}^{\mathrm{f}}$, *then it also holds in the original process of the abstract with any fixed finite number of* B*-particles added at time* 0.

Proof. The preceding discussion shows that if (1.3) has probability 1 in $\mathcal{P}^{\mathrm{f}}$, then it has probability 1 in the original process (with one particle added at the origin at time 0). By translation invariance (1.3) will then have probability 1 in the process of the abstract with one particle added at any fixed site at time 0.

Lemma C implies that one can couple two processes as in the abstract, with collections of B-particles $A^{(1)} \subset A^{(2)}$ added at time 0, respectively, in such a way that the process corresponding to $A^{(1)}$ always has no more B-particles than the one corresponding to $A^{(2)}$. Therefore, if the left-hand inclusion in (1.3) holds when only one B-particle is added at time 0, then it certainly holds if more than one B-particles are added.

It follows that we only have to prove the right-hand inclusion in (1.3) for the process from the abstract with more than one particle added, if we already know it when exactly one particle is added. Assume first that we run this last process with one B-particle ρ_0 added at z_0. We now have to refer the reader to the genealogical paths introduced in the proof of Proposition 5 of [KSb]. The right-hand inclusion in (1.3) then says that for all $\varepsilon > 0$

$$\begin{aligned}(3.11)\quad &P\{\text{there exist genealogical paths from } z_0 \text{ to some point}\\ &\qquad\qquad \text{outside } (1+\varepsilon)tB_0 \text{ for arbitrarily large } t\} = 0.\end{aligned}$$

From the construction of the genealogical paths in Proposition 5 of [KSb] and the fact that a.s. there are only finitely many B-particles at finite times (see (2.18) in [KSb]) it is not hard to deduce that

$$\begin{aligned}(3.12)\quad &\{\widetilde{B}(t) \not\subset (1+\varepsilon)tB_0 \text{ at time } t \text{ if one adds a } B\text{-particle } \rho_i\\ &\qquad \text{at } z_i,\ 1 \le i \le k, \text{ at time } 0\}\\ &= \{\text{there is a genealogical path from some } z_i,\ 1 \le i \le k,\\ &\qquad \text{to the complement of } (1+\varepsilon)tB_0 \text{ at time } t \text{ if one}\\ &\qquad \text{adds a } B\text{-particle } \rho_i \text{ at } z_i,\ 1 \le i \le k, \text{ at time } 0\}\\ &\subset \bigcup_{i=1}^{k} \{\text{there is a genealogical path from } z_i \text{ to the complement of}\\ &\qquad (1+\varepsilon)tB_0 \text{ at time } t \text{ if one adds a } B\text{-particle } \rho_i \text{ at } z_i \text{ at time } 0\}\end{aligned}$$

(the z_i do not have to be distinct here). It follows that

$$\begin{aligned}(3.13)\quad &P\{\widetilde{B}(t) \not\subset (1+\varepsilon)tB_0 \text{ for arbitrarily large times } t \text{ if one}\\ &\qquad \text{adds a } B\text{-particle } \rho_i \text{ at } z_i,\ 1 \le i \le k, \text{ at time } 0\}\\ &\le \sum_{i=1}^{k} P\{\text{there are genealogical paths from } z_i \text{ to the complement}\\ &\qquad \text{of } (1+\varepsilon)tB_0 \text{ at arbitrarily large times } t\\ &\qquad \text{if one adds a } B\text{-particle } \rho_i \text{ at } z_i \text{ at time } 0\}\\ &= 0 \text{ (by (3.11)).}\end{aligned}$$

Thus the right-hand inclusion in (1.3) holds a.s., even if one adds k B-particles at time 0. $\square$

We recall that

$$\mathcal{P}^{\mathrm{h}}(u,-r) \text{ is short for the } (u,-r) \text{ half-space process,}$$
$$\mathcal{P}^{\mathrm{f}} \text{ is short for the full-space process,}$$

and we further introduce

$$\mathcal{B}^{\mathrm{h}}(y,s;u,-r) := \{\text{there is a } B\text{-particle at } (y,s) \text{ in } \mathcal{P}^{\mathrm{h}}(u,-r)\}, \tag{3.14}$$
$$h(t,u,-r) = \max\{\langle x,u\rangle : \mathcal{B}^{\mathrm{h}}(x,t;u,-r) \text{ occurs}\}. \tag{3.15}$$

P^{or} will denote the probability measure for the original process (with one B-particle added at the origin at time 0); E^{or} is expectation with respect to P^{or}. (The superscripts h, f and or are added to various symbols which refer to a half-space process, the full-space process, or the original process, respectively). We use P without superscript if it is clear from the context with which process we are dealing or when we are discussing the probability of an event which is described entirely in terms of the $N_A(x,0-)$ and the paths π_A.

The following technical lemma will be useful. It tells us that, with high probability, $\mathcal{P}^{\mathrm{h}}(u,-r)$ moves out in the direction of u at least at the speed C_4, provided r is large enough (see (3.15) and (3.16)). Its proof would be nicer if we made use of the fact that even the $(u,0)$ half-space-process has, with a probability at least $1-t^{-K}$, B-particles at time t at sites x with $\langle x,u\rangle \ge Ct$, for some constant $C>0$. However, it takes some work to prove this fact and we decided to do without it.

Lemma 2. *Let C_1, C_2 be as in Theorem* A *and let*

$$C_4 = \frac{2\sqrt{d}C_1C_2}{32\sqrt{d}C_1 + C_2}.$$

For all constants $K \ge 0$, there exists a constant $r_0 = r_0(K) \ge 0$ such that for $r \ge r_0$

$$P\Big\{h(t,u,-r) \le C_4 t \text{ for some } t \ge t_1 := \frac{1}{4\sqrt{d}C_1}\Big[1 + \frac{C_2}{32\sqrt{d}C_1}\Big]r\Big\} \le r^{-K}. \tag{3.16}$$

Proof. The lemma is proven in three steps. In the first step we introduce exponentially growing sequences of times $\{t_k\}$ and distances $\{d_k\}$, and prove that we only need a good bound on the probability that there are no B-particles in $\mathcal{P}^{\mathrm{h}}(u,-r)$ at time t_k in $\mathcal{S}(u,d_k) \cap \{x : \|x\| \le 2C_1 d_k\}$. In Step 2 we recursively define further events $\mathcal{E}_{k,1} - \mathcal{E}_{k,5}$ and reduce the lemma to providing a good estimate for the probability that at least one $\mathcal{E}_{k,i}, k \ge 1, 1 \le i \le 5$, fails. The required estimates for these probabilities are derived in Step 3. This last step relies on the left-hand inclusion in (2.2) and on (2.4). Once we know that there is a B-particle far out in the direction u at time t_{k-1}, or more precisely a B-particle at some point x_{k-1} with $\langle x_{k-1}, u\rangle \geq d_{k-1}$, (2.2) and (2.4) allow us to conclude that with high probability there is a B-particle at time t_k at some x_k with $\langle x_k, u\rangle \geq d_k$.

Step 1. For $k \geq 1$ define the times

$$t_k = \frac{1}{4\sqrt{d}C_1}\Big[1 + \frac{C_2}{32\sqrt{d}C_1}\Big]^k r,$$

and the real numbers

$$d_k = \frac{C_2}{32\sqrt{d}C_1}\Big[1 + \frac{C_2}{32\sqrt{d}C_1}\Big]^k r.$$

Also define for each $k \geq 1$ the event

$$\text{(3.17)}\quad \mathcal{D}_k := \big\{\mathcal{B}^{\mathrm{h}}(x_k, t_k; u, -r) \text{ occurs for some } x_k \text{ which satisfies } \langle x_k, u\rangle \geq d_k \text{ and } \|x_k\| \leq 2C_1 t_k\big\}.$$

In this step we shall reduce the lemma to an estimate for the probability that $\mathcal{D}_k$ fails for some $k \geq 1$. Indeed, assume that $\mathcal{D}_k$ occurs for all $k \geq 1$. By definition, there is then a B-particle at (x_k, t_k) in the $(u, -r)$ half-space process (starting at $(\mathbf{0}, 0)$), so that

$$\text{(3.18)}\qquad h(t_k, u, -r) \geq \langle x_k, u\rangle \geq d_k = \frac{C_2}{32\sqrt{d}C_1}\Big[1 + \frac{C_2}{32\sqrt{d}C_1}\Big]^k r, \quad k \geq 1.$$

Recall that $\mathcal{F}_t$ is defined in the beginning of Section 2. In addition to (3.18), we have on the event $\{\langle x_k, u\rangle \geq d_k\}$, for $k \geq 1$,

$$\begin{aligned}\text{(3.19)}\quad & P\{h(t, u, -r) \leq \tfrac{1}{2}d_k \text{ for some } t \in [t_k, t_{k+1})|\mathcal{F}_{t_k}\} \\ &\leq P\{\text{each } B\text{-particle in } \mathcal{P}^{\mathrm{h}}(u, -r) \text{ at } (x_k, t_k) \text{ moves during} \\ &\qquad [t_k, t_{k+1}] \text{ to some site } x \text{ with } \langle x, u\rangle \leq \tfrac{1}{2}d_k\} \\ &\leq P\{\min_{q \leq t_{k+1}-t_k} \langle S_q, u\rangle \leq -\tfrac{1}{2}d_k = -C_4 t_{k+1}\} \\ &\leq K_1 \exp[-K_2 t_{k+1}]\end{aligned}$$

for some constants K_1, K_2 depending on d, D_A only; see (2.42) in [KSa] for the last inequality. It follows that the left-hand side of (3.16) is bounded by

$$\text{(3.20)}\qquad P\{\mathcal{D}_k \text{ fails for some } k \geq 1\} + \sum_{k=1}^{\infty} K_1 \exp[-K_2 t_k].$$

Step 2. We shall now derive a recursive bound for $\cap_{1\le j\le k}\mathcal{D}_j$. Assume that $\cap_{1\le j\le k-1}\mathcal{D}_j$ occurs for some $k\ge 2$. Consider now the full-space process starting at (x_{k-1},t_{k-1}). Define the following events for this process:

$$\begin{aligned}
\mathcal{E}_{k,1} := \{&\text{at time } t_k \text{ all occupied sites in}\\
&x_{k-1}+\mathcal{C}\big((C_2/2)(t_k-t_{k-1})\big) \text{ contain in fact a } B\text{-particle}\},\\
\mathcal{E}_{k,2} := \{&\text{at time } t_k \text{ there is an occupied site in}\\
&x_{k-1}+(C_2/4)(t_k-t_{k-1})u+\mathcal{C}\big([\log t_k]^2\big)\},\\
\mathcal{E}_{k,3} := \{&\text{all particles in } x_{k-1}+\mathcal{C}\big(2C_1(t_k-t_{k-1})\big)\\
&\text{at time } t_{k-1} \text{ started at time 0 in } \mathcal{S}(u,-r)\},\\
\mathcal{E}_{k,4} := \{&\text{there is no } B\text{-particle outside } x_{k-1}+\mathcal{C}\big(C_1(t_k-t_{k-1})\big) \text{ during}\\
&[t_{k-1},t_k] \text{ in the full-space process starting at } (x_{k-1},t_{k-1})\},\\
\mathcal{E}_{k,5} := \{&\text{no particle which is outside } x_{k-1}+\mathcal{C}\big(2C_1(t_k-t_{k-1})\big)\\
&\text{at time } t_{k-1} \text{ enters } x_{k-1}+\mathcal{C}\big(C_1(t_k-t_{k-1})\big) \text{ during } [t_{k-1},t_k]\}.
\end{aligned}$$

We claim that on

$$\mathcal{D}_{k-1}\cap\bigcap_{1\le i\le 5}\mathcal{E}_{k,i} \tag{3.21}$$

also $\mathcal{D}_k$ occurs, provided $r\ge$ some suitable r_1, independent of k, and $k\ge 2$. We merely need to make sure that $\sqrt{d}[\log t_k]^2\le (C_2/8)(t_k-t_{k-1})$ whenever $r\ge r_1$. To prove our claim when $k\ge 2$, observe first that the occurrence of $\mathcal{E}_{k,1}\cap\mathcal{E}_{k,2}$ guarantees that at time t_k there is a B-particle at some x_k in $x_{k-1}+(C_2/4)(t_k-t_{k-1})u+\mathcal{C}\big([\log t_k]^2\big)\subset x_{k-1}+\mathcal{C}\big((C_2/2)(t_k-t_{k-1})\big)$. Such a particle automatically satisfies

(3.22)
$$\langle x_k,u\rangle \ge \langle x_{k-1},u\rangle+\frac{C_2}{4}(t_k-t_{k-1})-\sqrt{d}[\log t_k]^2\ge d_{k-1}+\frac{C_2}{8}(t_k-t_{k-1})=d_k.$$

It also satisfies $\|x_k\|\le 2C_1t_k$, because $\|x_{k-1}\|\le 2C_1t_{k-1}$, and on $\mathcal{E}_{k,2}$, $\|x_k\|\le \|x_{k-1}\|+(C_2/4)(t_k-t_{k-1})+[\log t_k]^2$, while $C_2\le C_1$. This particle at (x_k,t_k) is a B-particle in the full-space process starting at (x_{k-1},t_{k-1}). We are going to show that, in fact, it is also a B-particle in the $(u,-r)$ half-space process starting at (x_{k-1},t_{k-1}). This will prove our claim, because the monotonicity property of Lemma C implies that any B-particle in the $(u,-r)$ half-space process starting at (x_{k-1},t_{k-1}) is also a B-particle in the $(u,-r)$ half-space process (starting at $(\mathbf{0},0)$), provided that there is a B-particle at (x_{k-1},t_{k-1}) in the $(u,-r)$ half-space process. (Note that this proviso is satisfied by the induction assumption that $\mathcal{D}_{k-1}$ occurred.)

We first observe that the particle at (x_k,t_k) must at time t_{k-1} have been in $x_{k-1}+\mathcal{C}\big(2C_1(t_k-t_{k-1})\big)$, because $x_k\in x_{k-1}+\mathcal{C}\big((C_2/2)(t_k-t_{k-1})\big)\subset x_{k-1}+\mathcal{C}\big((C_1/2)(t_k-t_{k-1})\big)$ and $\mathcal{E}_{k,5}$ occurs. By virtue of $\mathcal{E}_{k,3}$ this particle then belongs to $\mathcal{P}^{\mathrm{h}}(u,-r)$ as well as to the $(u,-r)$ half-space process starting at (x_{k-1},t_{k-1}). We still have to show that this particle also has type B in the $(u,-r)$ half-space process starting at (x_{k-1},t_{k-1}). To this end we note that the particles starting outside $x_{k-1}+\mathcal{C}\big(2C_1(t_k-t_{k-1})\big)$ at time t_{k-1} do not influence the type of any particle at time t_k in the full-space process starting at (x_{k-1},t_{k-1}). Indeed, in this process the particles outside $x_{k-1}+\mathcal{C}\big(2C_1(t_k-t_{k-1})\big)$ start at time t_{k-1} as A-particles, and since $\mathcal{E}_{k,4}\cap\mathcal{E}_{k,5}$ occurs, these particles do not meet any B-particle at or before time t_k. Thus, whether the particle at (x_k,t_k) is also a B-particle in the $(u,-r)$ half-space process starting at (x_{k-1},t_{k-1}) depends only on the paths of the particles which were in $x_{k-1}+\mathcal{C}\big(2C_1(t_k-t_{k-1})\big)$ at time t_{k-1} (compare the lines following (2.37) in [KSb]). All these particles were particles in $\mathcal{P}^{\mathrm{h}}(u,-r)$ at time t_{k-1} (on $\mathcal{E}_{k,3}$), and hence also are in this half-space process at time t_k. Thus the type of the particle at (x_k,t_k) is the same in the full-space process starting at (x_{k-1},t_{k-1}) and in the $(u,-r)$ half-space process starting at (x_{k-1},t_{k-1}). This justifies our claim that $\mathcal{D}_k$ occurs for $k\geq 2$. We leave it to the reader to make some simple modifications in the above argument to show that $\mathcal{D}_1$ occurs on

$$\mathcal{D}_0\cap\bigcap_{1\leq i\leq 5}\mathcal{E}_{1,i},$$

where

$$t_0=0 \text{ and } \mathcal{D}_0=\{\|x_0\|\leq K_3\log r\},$$

provided r_1 is chosen large enough; x_0 is defined in (3.4) and K_3 is chosen right after (3.26) and depends on K,d and μ_A only.

We have now shown that on the event (3.21), also, $\mathcal{D}_k$ occurs. If this is the case and also $\cap_{1\leq i\leq 5}\mathcal{E}_{k+1,i}$ occurs, then $\mathcal{D}_{k+1}$ occurs etc. Consequently, for $r\geq r_1$,

(3.23)

$$\begin{aligned}
&P\{\mathcal{D}_0 \text{ occurs, but some } \mathcal{D}_k \text{ fails}\}\\
&\quad\leq\sum_{i=1}^{5}P\{\text{for some } x_0 \text{ with } \|x_0\|\leq K_3\log r,\ x_0 \text{ is occupied but } \mathcal{E}_{1,i} \text{ fails}\}\\
&\qquad+\sum_{k=2}^{\infty}\sum_{i=1}^{5}P\{\text{for some } x_{k-1} \text{ with } \|x_{k-1}\|\leq 2C_1t_{k-1} \text{ and } \langle x_{k-1},u\rangle\geq d_{k-1},\\
&\qquad\qquad\mathcal{B}^{\mathrm{h}}(x_{k-1},t_{k-1};u,-r) \text{ occurs, but } \mathcal{E}_{k,i} \text{ fails}\}.
\end{aligned}$$

Step 3. In this step we shall give most of the estimates for the terms in the right-hand side here for $k\geq 2$. The basic inequalities remain valid for $k=1$ by trivial modifications which we again leave to the reader. For the various estimates we have to take all t_k large. This will automatically be the case if r is large; we shall not explicitly mention this in the estimates below.

We start with the estimate for the failure of $\mathcal{E}_{k,1}$. If $\mathcal{E}_{k,1}$ fails, for a given (x_{k-1}, t_{k-1}), then there must be some $y \in x_{k-1} + \mathcal{C}\big((C_2/2)(t_k - t_{k-1})$ such that y is occupied by an A-particle at time t_k in the full-space process started at (x_{k-1}, t_{k-1}). Recall that if we shift the full-space process starting at (x, t) by $(-x, -t)$ in space-time, then we obtain a copy of the full state process starting at $(\mathbf{0}, 0)$. Moreover, if we condition on the event that x is occupied at time t, then, after the shift by $(-x, -t)$ the $N_A(y, 0)$, $y \neq \mathbf{0}$, are i.i.d. mean μ_A Poisson random variables. Therefore, by summing over the possible values for x_{k-1},

$$\begin{aligned}
&P\{\text{for some } x_{k-1} \text{ with } \|x_{k-1}\| \le 2C_1 t_{k-1} \text{ and } \langle x_{k-1}, u\rangle \ge d_{k-1},\\
&\qquad \mathcal{B}^{\mathrm{h}}(x_{k-1}, t_{k-1}; u, -r) \text{ occurs, but } \mathcal{E}_{k,1} \text{ fails}\}\\
&\le \sum_{\|x\| \le 2C_1 t_{k-1}} P\{\mathcal{B}^{\mathrm{h}}(x, t_{k-1}; u, -r) \text{ occurs and in the full-space process started}\\
&\qquad \text{at } (x, t_{k-1}) \text{ there is an } A\text{-particle in } x + \mathcal{C}\big((C_2/2)(t_k - t_{k-1})\big) \text{ at time } t_k\}\\
&\le \sum_{\|x\| \le 2C_1 t_{k-1}} P\{\mathbf{0} \text{ is occupied at time } 0 \text{ and in } \mathcal{P}^{\mathrm{f}} \text{ there is an } A\text{-particle}\\
&\qquad \text{in } \mathcal{C}\big((C_2/2)(t_k - t_{k-1})\big) \text{ at time } t_k - t_{k-1}\}.
\end{aligned}$$

To the right-hand side here we can apply (3.7) (with $C = \{\mathbf{0}\}$). This shows that the right-hand side is at most

$$\begin{aligned}
&K_4[t_{k-1}]^d \mu_A P^{\mathrm{or}}\{\text{at time } t_k - t_{k-1},\\
&\qquad\qquad\qquad \text{there is an } A\text{-particle in } \mathcal{C}\big((C_2/2)(t_k - t_{k-1})\big)\}.
\end{aligned}$$

The probability in the right-hand side here is calculated for the original process with one particle added at $\mathbf{0}$ at time 0. By (2.4) (with K replaced by $K+d+2$) this probability is at most $2[t_k - t_{k-1}]^{-K-d-2}$. Therefore,

$$\begin{aligned}
&P\{\text{for some } x_{k-1} \text{ with } \|x_{k-1}\| \le 2C_1 t_{k-1} \text{ and } \langle x_{k-1}, u\rangle \ge d_{k-1},\\
&\qquad \mathcal{B}^{\mathrm{h}}(x_{k-1}, t_{k-1}; u, -r) \text{ occurs, but } \mathcal{E}_{k,1} \text{ fails}\}\\
&\qquad\qquad \le 2K_4[t_{k-1}]^d \mu_A [t_k - t_{k-1}]^{-K-d-2} \le K_5 t_k^{-K-2}.
\end{aligned}$$

It turns out that in the estimates for $\mathcal{E}_{k,2}, \mathcal{E}_{k,3}$ and $\mathcal{E}_{k,5}$ we can ignore the type of the particle at (x_{k-1}, t_{k-1}); we just need that this space-time point is occupied. For $\mathcal{E}_{k,2}$ we shift by $(-x_{k-1}, -t_k)$, sum over the possible values of x_{k-1} and apply (3.7). This gives

$$\begin{aligned}
&P\{\text{for some } x_{k-1} \text{ with } \|x_{k-1}\| \le 2C_1 t_{k-1},\\
&\qquad\qquad (x_{k-1}, t_{k-1}) \text{ is occupied but } \mathcal{E}_{k,2} \text{ fails}\}\\
&\le K_4[t_{k-1}]^d \mu_A P\{N_A(y, 0-) = 0 \text{ for all } y \in (C_2/4)(t_k - t_{k-1})u + \mathcal{C}([\log t_k]^2)\}\\
&\le t_k^{-K-2},
\end{aligned}$$

for large r, because the $N_A(y, 0-)$ are independent.

Next, for $\mathcal{E}_{k,3}$ we use that on $\mathcal{D}_{k-1}$, the distance between $x_{k-1} + \mathcal{C}\big(2C_1(t_k - t_{k-1})\big)$ and the complement of $\mathcal{S}(u, -r)$ is at least

$$\begin{aligned}\langle x_{k-1}, u\rangle + r - 2\sqrt{d}C_1(t_k - t_{k-1}) &\geq d_{k-1} + r - 2\sqrt{d}C_1(t_k - t_{k-1})\\ &= \frac{1}{2}d_{k-1} + r.\end{aligned}$$

Thus, if we take the restriction $\langle x_{k-1}, u\rangle \geq d_{k-1}$ into account we find that

$$\begin{aligned}&P\{\text{for some } x_{k-1} \text{ with } \|x_{k-1}\| \leq 2C_1 t_{k-1} \text{ and } \langle x_{k-1}, u\rangle \geq d_{k-1},\\ &\qquad x_{k-1} \text{ is occupied at time } t_{k-1} \text{ but } \mathcal{E}_{k,3} \text{ fails}\}\\ &\qquad \leq \sum_{\substack{\|x\|\leq 2C_1 t_{k-1}\\ \langle x,u\rangle \geq d_{k-1}}} \sum_{y \notin \mathcal{S}(u,-r)} \sum_{z \in x + \mathcal{C}\left(2C_1(t_k - t_{k-1})\right)} P\{S_{t_{k-1}} = z - y\}\\ &\qquad \leq K_6 t_{k-1}^d [t_k - t_{k-1}]^d P\{\|S_{t_{k-1}}\| \geq \frac{1}{2}d_{k-1} + r\}\\ &\qquad \leq K_7 t_k^{2d} \exp\Big[-K_8 \frac{(d_{k-1}+r)^2}{t_{k-1} + d_{k-1} + r}\Big] \text{ (by (2.42) in [KSa])}\\ &\qquad \leq t_k^{-K-2}.\end{aligned}\tag{3.24}$$

The estimate for $\mathcal{E}_{k,4}$ comes from Theorem A, or rather Theorem 1 in [KSb], which is the basis for the right-hand inclusion in Theorem A. Indeed, we have, again by summing over the possible values of x_{k-1} and using (3.9),

(3.25)

$$\begin{aligned}&P\{\text{for some } x_{k-1} \text{ with } \|x_{k-1}\| \leq 2C_1 t_{k-1},\ (x_{k-1}, t_{k-1}) \text{ is occupied}\\ &\quad \text{but } \mathcal{E}_{k,4} \text{ fails in the full-space process starting at } (x_{k-1}, t_{k-1})\}\\ &\qquad \leq K_4[t_{k-1}]^d \mu_A P^{\mathrm{or}}\{\text{there are } B\text{-particles outside } \mathcal{C}\big(C_1(t_k - t_{k-1})\big)\\ &\qquad\qquad \text{at some time } \leq t_k - t_{k-1}\}.\end{aligned}$$

To estimate the probability on the right we argue as in the proof of Theorem 3 of [KSb]. If a particle has type B at some time $s \leq t_k - t_{k-1}$ and is outside the cube $\mathcal{C}\big(C_1(t_k - t_{k-1})\big)$ at that time, then by symmetry of the random walk $\{S_\cdot\}$, the particle has a conditional probability, given $\mathcal{F}_s$, at least $1/2$ of being outside $\mathcal{C}\big(C_1(t_k - t_{k-1})\big)$ at time $t_k - t_{k-1}$. Therefore (with E^{or} denoting expectation with respect to P^{or}),

$$\begin{aligned}&E^{\mathrm{or}}\{\text{number of } B\text{-particles outside } \mathcal{C}\big(C_1(t_k - t_{k-1})\big)\\ &\qquad \text{at some time during } [0, t_k - t_{k-1}]\}\\ &\quad \leq 2E^{\mathrm{or}}\{\text{number of } B\text{-particles outside } \mathcal{C}\big(C_1(t_k - t_{k-1})\big) \text{ at time } t_k - t_{k-1}\}.\end{aligned}$$

The expectation in the right-hand side here is exponentially small in $(t_k - t_{k-1})$ by Theorem 1 of [KSb] and is an upper bound for the probability in the right-hand side of (3.25). Thus the left-hand side of (3.25) is at most $O\big(t_k^{-K-2}\big)$ again.

The probability involving $\mathcal{E}_{k,5}^c$ is also $O(t_k^{-K-2})$. This can be shown by large deviation estimates for random walks, analogously to the terms involving $\mathcal{E}_{k,3}^c$.

This provides the necessary bounds of the summands in (3.23). Finally, we have

$$P\{\mathcal{D}_0 \text{ fails}\} \le P\{N_A(x, 0-) = 0 \text{ for all } x \text{ with } \|x\| < K_3 \log r\} \\ = \exp\big[-\mu_A K_9 [K_3 \log r]^d\big]. \tag{3.26}$$

Thus we can take $K_3 = K_3(K, d, \mu_A)$ so large that the left-hand side of (3.26) is at most r^{-K-1} for all $r \ge 3$. (3.20), (3.26) and (3.23) together now show that the left-hand side of (3.16) is for large r at most

$$r^{-K-1} + \sum_{k=1}^{\infty} K_1 \exp[-K_2 t_k] + \sum_{k=1}^{\infty} K_{10} t_k^{-K-2} \le K_{11} r^{-K-1}. \qquad \square$$

For any vector $v \in \mathbb{R}^d$, we define

$$v^\perp = v^\perp(u) := v - \langle v, u \rangle u.$$

We further introduce the following (semi-infinite) cylinders with axis in the direction of u, for $\alpha, \beta \in \mathbb{R}$ and $\gamma \in \mathbb{R}^d$ a vector orthogonal to u (see Figure 1):

$$\Gamma(\alpha, \beta, \gamma) = \Gamma(\alpha, \beta, \gamma, u) := \{x \in \mathbb{R}^d : \langle x, u \rangle \ge \alpha, \|x^\perp - \gamma\| \le \beta\},$$

and the events

$$\mathcal{G}(\alpha, \beta, \gamma, \mathcal{P}, t) = \mathcal{G}(\alpha, \beta, \gamma, \mathcal{P}, t, u) \\ := \{\text{in the process } \mathcal{P} \text{ there is a } B\text{-particle in } \Gamma(\alpha, \beta, \gamma) \text{ at time } t\}.$$

The last definition will be used with $\mathcal{P}$ taken equal to some half-space, full-space or the original process.

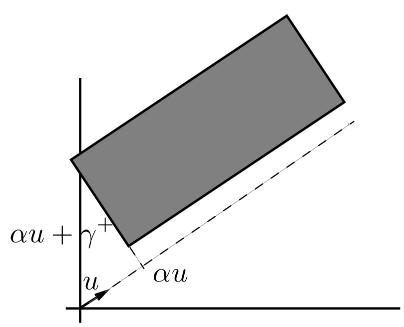

Figure 1: The shaded region represents a cylinder $\Gamma(\alpha, \beta, \gamma)$; it extends to infinity on the upper right.

We remind the reader that $\mathcal{P}^{\mathrm{h}}(u,-r)$, the $(u,-r)$ half-space process, only uses particles which at time 0 are in the half-space $\mathcal{S}(u,-r) = \{x : \langle x,u\rangle \ge -r\}$. We shall work a great deal with the process $\mathcal{P}^{\mathrm{h}}\big(u,-C_5\kappa(s)\big)$, where

$$\kappa(s) = [(s+1)\log(s+1)]^{1/2}$$

and C_5 is a constant to be determined below (see the lines following (3.63)). We make several more definitions:

$$\begin{aligned}(3.27)\quad h^*(s,u) := h(s,u,-C_5\kappa(s)) &= \max\{\langle x,u\rangle : \mathcal{B}^{\mathrm{h}}\big(x,s;u,-C_5\kappa(s)\big) \text{ occurs}\}\\ &= \max\{\langle x,u\rangle : x \text{ is occupied by a } B\text{-particle in}\\ &\qquad \mathcal{P}^{\mathrm{h}}\big(u,-C_5\kappa(s)\big) \text{ at time } s\}.\end{aligned}$$

We order the sites of $\mathbb{Z}^d$ in some deterministic way, say lexicographically, and take

(3.28)
$$\begin{aligned}\ell^*(s,u) :=\ &\text{the first } x \text{ in this order for which } \mathcal{B}^{\mathrm{h}}\big(x,s;u,-C_5\kappa(s)\big) \text{ occurs}\\ &\text{and which satisfies } \langle x,u\rangle = h^*(s,u).\end{aligned}$$

Thus, $h^*(s,u)$ is the furthest displacement in the direction of u among the B-particles in the process $\mathcal{P}^{\mathrm{h}}(u,-C_5\kappa(s))$ at time s, and $\ell^*(s,u)$ is the first site occupied by a B-particle in this process at time s on which this maximal displacement is reached. We shall write $m^*(s,u)$ for $[l^*(s,u)]^{\perp}$ so that we have the orthogonal decomposition

$$\ell^*(s,u) = h^*(s,u)u + m^*(s,u). \tag{3.29}$$

The following proposition contains our principal "subadditivity" property. If we take $\beta = \infty$, that is, if we only look at its statement about displacements in the direction of u, then the proposition says that (up to error terms) the maximal displacement in the direction u at time $s+t+C_6\kappa(t)$ in the process $\mathcal{P}^{\mathrm{h}}\big(u,-C_5\kappa(s+t+C_6\kappa(t))\big)$ is stochastically larger than the sum of two independent such displacements, which are distributed like the maximal displacement in $\mathcal{P}^{\mathrm{h}}\big(u,-C_5\kappa(s)\big)$ at time s and the maximal displacement in $\mathcal{P}^{\mathrm{h}}\big(u,-C_5\kappa(t)\big)$ at time t, respectively (see Corollary 5 for more details). The basic idea of the proof (for any value of β) is that if ℓ^* is a point where $\mathcal{P}^{\mathrm{h}}\big(u,-C_5\kappa(s)\big)$ achieves its maximum displacement in the direction u at time s, then we can start a new half-space process at time $s+C_6\kappa(t)$ 'near' ℓ^* which is 'nearly' a copy of $\mathcal{P}^{\mathrm{h}}\big(u,-C_5\kappa(t)\big)$ and which is 'nearly' independent of the first process $\mathcal{P}^{\mathrm{h}}\big(u,-C_5\kappa(s)\big)$. If we run the second process for t units of time the sum of the displacements in the direction of u in the first and second process is 'nearly' a lower bound for the displacement of the original process at time $s+t+C_6\kappa(t)$.

PROPOSITION 3. *Let $u \in S^{d-1}, \alpha \in \mathbb{R}, \beta \geq 0$ and $\gamma_s, \gamma_t \in \mathbb{R}^d$ orthogonal to u. For any $K > 0$ there exist constants $0 < C_5 - C_8, s_0 < \infty$, which depend on K, but are independent of $u \in S^{d-1}$ and of $\alpha, \beta, \gamma_s, \gamma_t$, such that for*

$$s_0 \leq s \leq t \text{ and } t \log t \leq C_7 s^2, \tag{3.30}$$

$$\begin{aligned} P\{&\mathcal{G}(\alpha, \beta, \gamma_s + \gamma_t, \mathcal{P}^{\mathrm{h}}(u, -C_5\kappa(s + t + C_6\kappa(t))), s + t + C_6\kappa(t))\} \\ &\geq \int_{h\in\mathbb{R}} \int_{m\in\mathbb{R}^d} P\{h^*(s, u) \in dh, m^*(s, u) - \gamma_s \in dm\} \\ &\qquad\qquad \times P\{\mathcal{G}(\alpha - h, \beta - d, \gamma_t - m, \mathcal{P}^{\mathrm{h}}(u, -C_5\kappa(t)), t)\} \\ &\quad -C_8 s^{-K-1}. \end{aligned} \tag{3.31}$$

Proof. The constants C_i and s_0 will be fixed later. K_i will be used to denote other auxiliary constants. For the time being we only do manipulations which do not depend on the specific value of the C_i, K_i.

We break the proof up into four steps, the last one of which is formulated as a separate lemma which will also be used in the next section. The left-hand side of (3.31) is the probability that there is a B-particle in a certain semi-infinite cylinder in the process $\mathcal{P}^{\mathrm{h}}(u, -C_5\kappa(s + t + C_6\kappa(t)))$ at time $s + t + C_6\kappa(t)$. In the first step we introduce the set $A_1(s, t)$ of sites which actually have a B-particle in the process $\mathcal{P}^{\mathrm{h}}(u, -C_5\kappa(s + t + C_6\kappa(t)))$ at time $s + t + C_6\kappa(t)$. (3.31) then claims a lower bound on the probability that A_1 intersects $\Gamma(\alpha, \beta, \gamma_s + \gamma_t)$. To prove this lower bound we further introduce in Step 1 a collection of sites $A_2(s, t, v)$, for $v \in \mathbb{Z}^d$, and show that $A_1(s, t) \supset A_2(s, t, \ell^*(s, u))$ (outside the event (3.36)) and such that $A_1(s, t) - \overline{\ell^*(s, u)}$ is 'at least as large' as

$$\begin{aligned} A_3(t) := \{x :\ & x \text{ is occupied by one or more } B\text{-particles at time } t \text{ in} \\ & \text{an independent copy of the process } \mathcal{P}^{\mathrm{h}}(u, -C_5\kappa(t))\}, \end{aligned}$$

outside an event of probability at most s^{-K-1}. The vector $\overline{\ell^*(s, u)}$ is defined in the beginning of Step 1, and Step 2 formulates the meaning of 'at least as large' here as a precise probability estimate. Step 3 and Lemma 4 then prove that this probability estimate indeed holds. It is for this estimate that the collection $A_2(s, t, \ell^*(s, u))$ is used. As we indicated above, we try to approximate the collection of B-particles in $\mathcal{P}^{\mathrm{h}}(u, -C_5\kappa(s + t + C_6\kappa(t)))$ by the sum of $\ell^*(s, u)$ and displacements of a second proces which starts near $\overline{\ell^*(s, u)}$ at time $s + C_6\kappa(t)$. Thus for our approximation to work, there should be a B-particle at $\ell^*(s, u)$ at time s which produces a B-particle essentially at $\overline{\ell^*(s, u)}$ at time $s + C_6\kappa(t)$, at which we can start the second process (more precisely, (3.36) has to hold with high probability). Lemma 4 is used to show that such B-particles exist with high probability. (Note that $\ell^*(s, u)$ has a B-particle of $\mathcal{P}(u, -C_5\kappa(s))$ at time s.)

Step 1. Run $\mathcal{P}^{\mathrm{h}}(u, -C_5\kappa(s))$ till time s. Let $h^*(s,u) = h \in \mathbb{R}$, $\ell^*(s,u) = y \in \mathbb{Z}^d$. Set $\overline{y} := \lfloor y + 4C_5\kappa(t)u \rfloor$ (the meaning of this last notation is that we take the largest integer for each coordinate separately). Next we run the $(u, \langle y,u\rangle + 2C_5\kappa(t))$ half-space process starting at the space-time point $\big(\overline{y}, s + C_6\kappa(t)\big)$ for t units of time. This latter half-space process will be shown to be almost an independent copy of the translate by $\big(\overline{y}, s + C_6\kappa(t)\big)$ of $\mathcal{P}^{\mathrm{h}}\big(u, -C_5\kappa(t)\big)$. Define $z(s,t)$ to be the nearest site in $\mathbb{Z}^d$ to $\overline{y}$ which is occupied at time $s + C_6\kappa(t)$ by a particle which started at time 0 in the half-space $\mathcal{S}\big(u, \langle y,u\rangle + 2C_5\kappa(t)\big)$. It will be useful to define for general $v \in \mathbb{Z}^d$

$$z_v = \begin{array}{l}\text{the nearest site on } \mathbb{Z}^d \text{ to } \overline{v} := \lfloor v + 4C_5\kappa(t)u\rfloor \text{ which is} \\ \text{occupied at time } s + C_6\kappa(t) \text{ by a particle which started} \\ \text{at time 0 in } \mathcal{S}\big(u, \langle v,u\rangle + 2C_5\kappa(t)\big).\end{array} \tag{3.32}$$

Thus, z_v has the same relation to v as $z(s,t)$ has to y. In particular, $z_y = z(s,t)$. We can now define, still for any $v \in \mathbb{Z}^d$, the sets

$$\begin{aligned} A_1(s,t) = \{x :\ & x \text{ is occupied by one or more } B\text{-particles at time} \\ & s + t + C_6\kappa(t) \text{ in the process } \mathcal{P}^{\mathrm{h}}\big(u, -C_5\kappa(s + t + C_6\kappa(t))\big)\}, \\ A_2(s,t,v) = \{x :\ & x \text{ is occupied by one or more } B\text{-particles at time} \\ & s + t + C_6\kappa(t) \text{ in the } \big(u, \langle v,u\rangle + 2C_5\kappa(t)\big) \text{ half-space} \\ & \text{process starting at } (\overline{v}, s + C_6\kappa(t)\}. \end{aligned} \tag{3.33}$$

In addition $A_3(t)$ was defined just before the start of this step. We stress that A_3 is defined by means of a new copy of all initial data and paths. It is independent of the processes we have worked with so far.

Our aim is to prove the following two statements, and to show that they imply (3.31). The first statement is that outside an event of probability at most s^{-K-1} it is the case that

$$A_1(s,t) \supset A_2(s,t,y). \tag{3.34}$$

The second statment is that

$$\begin{array}{l} A_1(s,t) - \overline{y} \text{ is at least as large as } A_3(t), \text{ outside} \\ \qquad\qquad\qquad\qquad \text{an event of probability at most } s^{-K-1} \end{array} \tag{3.35}$$

(still $y = \ell^*(s,u)$ in these relations). The relation (3.35) is stated somewhat imprecisely, but a precise version will be given below (see (3.51)). In this first step we shall reduce the proofs of (3.34) and (3.35) to a number of probability estimates.

To begin with the inclusion (3.34), we claim that this holds on the intersection of the event

$$\begin{array}{l}\{\langle y,u\rangle \ge 0\} \cap \{z(s,t) \text{ is occupied by a } B\text{-particle at time} \\ \qquad\qquad\qquad\qquad s + C_6\kappa(t) \text{ in } \mathcal{P}^{\mathrm{h}}\big(u, -C_5\kappa(s)\big)\}\end{array} \tag{3.36}$$

with the event (see (3.4) for x_0)

$$\{\|x_0\| \le K_3 \log s\}. \tag{3.37}$$

This follows from two applications of the monotonicity property in Lemma C. Indeed, under (3.37) (and $s \ge s_1$ for a large enough s_1), both the $(u, -C_5\kappa(s))$ and the $\big(u, -C_5\kappa(s+t+C_6\kappa(t))\big)$ half-space processes begin with B-particles at x_0. One application of the monotonicity property therefore gives us that (under (3.37)) $\mathcal{P}^{\rm h}\big(u, -C_5\kappa(t+s+C_6\kappa(t))\big)$ has more B-particles than $\mathcal{P}^{\rm h}\big(u, -C_5\kappa(s)\big)$ at each space-time point, and therefore

$$A_1(s,t) \supset \{x:\ x \text{ is occupied by one or more } B\text{-particles at time } s+t+C_6\kappa(t) \text{ in the process } \mathcal{P}^{\rm h}\big(u, -C_5\kappa(s)\big)\}. \tag{3.38}$$

For the second application we recall that (by definition) $z(s,t)$ is occupied at time $s + C_6\kappa(t)$ by a particle which started in $\mathcal{S}\big(u, \langle y, u\rangle + 2C_5\kappa(t)\big)$, and in fact is the closest occupied site to $\overline{y}$ with this property. To run the $\big(u, \langle y, u\rangle + 2C_5\kappa(t)\big)$ half-space process starting at $\big(\overline{y}, s + C_6\kappa(t)\big)$ and to find $A_2(s,t,y)$ we first remove all particles which at time 0 were in the half-space $\{x : \langle x, u\rangle < \langle y, u\rangle + 2C_5\kappa(t)\}$. After that, at time $s + C_6\kappa(t)$, we reset to A the types of all particles not at $z(s,t)$ and give all particles at $z(s,t)$ type B. Note that in the first step all particles which do not belong to $\mathcal{P}^{\rm h}\big(u, -C_5\kappa(s)\big)$ are removed, since

$$-C_5\kappa(s) \le 2C_5\kappa(t) \le \langle y, u\rangle + 2C_5\kappa(t) \text{ (on (3.36))}.$$

Thus, at time $s + C_6\kappa(t)$ after both steps, all remaining particles are also in $\mathcal{P}^{\rm h}\big(u, -C_5\kappa(s)\big)$, and the particles which have type B, i.e., only the particles at $z(s,t)$, also have type B in $\mathcal{P}^{\rm h}\big(u, -C_5\kappa(s)\big)$ (still on the event (3.36)). By virtue of the monotonicity property of Lemma C, at time $s + t + C_6\kappa(t)$, any B-particle present in the $\big(u, \langle y, u\rangle + 2C_5\kappa(t)\big)$ half-space process starting at $\big(\overline{y}, s+C_6\kappa(t)\big)$ is also a B-particle in $\mathcal{P}^{\rm h}\big(u, -C_5\kappa(s)\big)$. Therefore, on the event (3.36),

$$A_2(s,t,y) \subset \{x:\ x \text{ is occupied by one or more } B\text{-particles at time } s+t+C_6\kappa(t) \text{ in the process } \mathcal{P}^{\rm h}\big(u, -C_5\kappa(s)\big)\}. \tag{3.39}$$

Combining (3.38) and (3.39) gives (3.34) on the intersection of the events (3.36) and (3.37). We postpone the proof that this intersection indeed has probability at least $1 - s^{-K-1}$ to Step 3.

To prepare for the desired precise form of (3.35) we shall prove that there exist constants K_1 and s_2 such that for $t \ge s \ge s_2$, Λ any nonrandom subset of $\mathbb{Z}^d$, and any fixed $v \in \mathbb{Z}^d$,

$$P\{A_2(s,t,v) \text{ intersects } \Lambda\} \ge P\big\{\big(\overline{v} + A_3(t)\big) \text{ intersects } \Lambda\big\} - K_1 t^{-K-d-1}. \tag{3.40}$$

To prove this inequality we remind the reader that $A_2(s,t,v)$ is the collection of sites where B-particles are present at time $s+t+C_6\kappa(t)$, if one starts at time $s+C_6\kappa(t)$ in the state obtained by removing the particles which started outside $\mathcal{S}\big(u,\langle v,u\rangle+2C_5\kappa(t)\big)$ at time 0, and by resetting all particles not at z_v to type A, while setting the type of the particles at z_v to B. To find the distribution of $A_2(s,t,v)$ we must first describe the state at time $s+C_6\kappa(t)$ (after the removal of particles and resetting of types) in more detail. First let us look how many particles there are at the various sites, irrespective of their type. We began at time 0 with $N_A(w,0-)$ particles at w, for $w\in\mathcal{S}\big(u,\langle v,u\rangle+2C_5\kappa(t)\big)$ and with 0 particles at any w outside $\mathcal{S}\big(u,\langle v,u\rangle+2C_5\kappa(t)\big)$. The $N_A(w,0-)$ are i.i.d. mean μ_A Poisson random variables. We let these particles perform their random walks till time $s+C_6\kappa(t)$. Let us write $\widehat{N}\big(w,s+C_6\kappa(t)\big)$ for the number of particles (of either type) at w at this time. By properties of the Poisson distribution, all the $\widehat{N}\big(\overline{v}+w,s+C_6\kappa(t)\big)$, $w\in\mathbb{Z}^d$, are still independent Poisson variables, but

$$
\begin{aligned}
(3.41)\quad & E\widehat{N}\big(\overline{v}+w,s+C_6\kappa(t)\big)\\
&= \sum_{w'\in\mathcal{S}\left(u,\langle v,u\rangle+2C_5\kappa(t)\right)} \mu_A P\{S_{s+C_6\kappa(t)}=\overline{v}+w-w'\} =: \nu(v,w,s,t).
\end{aligned}
$$

Now, z_v is the nearest lattice point to $\overline{v}$ which is occupied by some particle at time $s+C_6\kappa(t)$. We then reset all particles not at z_v to type A, and the ones at z_v to type B. If we shift everything by $\big(-\overline{v},-s-C_6\kappa(t)\big)$ (i.e., move (w,r) to $\big(w-\overline{v},r-s-C_6\kappa(t)\big)$), then, at $(w,0)$ we have $M(w):=\widehat{N}\big(\overline{v}+w,s+C_6\kappa(t)\big)$ particles, all of which will be reset to type A, except those at $a_0:=$ the nearest lattice site to the origin with $M(w)>0$. In fact, $a_0=z_v-\overline{v}$. The $M(w)$ are independent Poisson variables, and $M(w)$ has mean $\nu(v,w,s,t)$. It follows from the definition of $A_2(s,t,v)$ and of the $\big(u,\langle v,u\rangle+2C_5\kappa(t)\big)$ half-space process started at $\big(\overline{v},s+C_6\kappa(t)\big)$ that $A_2(s,t,v)-\overline{v}$ has the distribution of

$$
(3.42)\qquad \{x:\ \text{there is a } B\text{-particle at } x \text{ at time } t \text{ in this shifted system}\}.
$$

For the $w\in\mathcal{S}\big(u,-C_5\kappa(t)\big)$, the means $\nu(v,w,s,t)$ are close to μ_A. In fact, it follows from (3.41) that for $t\ge t_0\vee s$, for some t_0 (independent of v,w,u), and for $w\in\mathcal{S}\big(u,-C_5\kappa(t)\big)$,

$$
\begin{aligned}
(3.43)\quad \mu_A\ge\nu(v,w,s,t)&\ge\mu_A\Big[1-\sum_{\widetilde{w}:\langle\widetilde{w},u\rangle\ge C_5\kappa(t)-d} P\{S_{s+C_6\kappa(t)}=\widetilde{w}\}\Big]\\
&=\mu_A\big[1-P\{\langle S_{s+C_6\kappa(t)},u\rangle\ge C_5\kappa(t)-d\}\big]\\
&\ge\mu_A\big[1-K_2\exp[-K_3C_5^2\log t]\big].
\end{aligned}
$$

for some constants K_2,K_3 that depend on d,D only (see (2.42) in [KSa] and the definition of κ and recall that we assume (3.30)). From now on we take C_5

so large that for large t

(3.44) $$\mu_A[1 - t^{-K-2d-1}] \le \nu(v, w, s, t) \le \mu_A \text{ for all } w \in \mathcal{S}\big(u, -C_5\kappa(t)\big).$$

It suffices for this that $K_3 C_5^2 \ge K+2d+2$. We may have to raise C_5 in the proof of (3.54) and (3.65) in Step 3, but that can only improve the present estimates. With such a choice of C_5 the distribution of the particle numbers $\{M(w) : w \in \mathcal{S}\big(u, -C_5\kappa(t)\big) \cap \mathcal{C}(3C_1 t)\}$ differs in total variation from the distribution of an i.i.d. collection of mean μ_A Poisson variables on $\mathcal{S}\big(u, -C_5\kappa(t)\big) \cap \mathcal{C}(3C_1 t)$ by at most

(3.45) $$\sum_{w \in \mathcal{S}\left(u, -C_5\kappa(t)\right) \cap\, \mathcal{C}(3C_1 t)} \mu_A t^{-K-2d-1} \le K_4 t^{-K-d-1}$$

for some constant $K_4 = K_4(\mu_A, d)$.

Now consider an auxiliary process which starts at time 0 with $N_A(w, 0-)$ particles only at the vertices $w \in \mathcal{S}\big(u, -C_5\kappa(t)\big) \cap \mathcal{C}(3C_1 t)$, and with no particles outside this set. Let b_0 be the nearest vertex in $\mathcal{S}\big(u, -C_5\kappa(t)\big)$ to the origin with $N_A(b_0, 0-) > 0$. (In the beginning of this section this vertex was denoted by $w_{-C_5\kappa(t)}(\mathbf{0}, 0)$, but for the present argument the simpler notation b_0 is preferable.) Take the type of all particles not at b_0 equal to A and the type of the particles at b_0 equal to B. If b_0 lies outside $\mathcal{S}\big(u, -C_5\kappa(t)\big) \cap \mathcal{C}(3C_1 t)$, then this auxiliary process has never any B-particles. On the other hand, if $b_0 \in \mathcal{S}\big(u, -C_5\kappa(t)\big) \cap \mathcal{C}(3C_1 t)$, then the auxiliary process is obtained from $\mathcal{P}^{\mathrm{h}}\big(u, -C_5\kappa(t)\big)$ by removing at time 0 all particles in $\mathcal{S}\big(u, -C_5\kappa(t)\big) \setminus \mathcal{C}(3C_1 t)$. Finally, let

$$A_4(t) = \{x : \text{ there is a } B\text{-particle at } x \text{ at time } t \text{ in this auxiliary system}\}.$$

From our considerations above (in particular (3.42), (3.45)) we have that

(3.46)
$$P\{A_2(s, t, v) \text{ intersects } \Lambda\} \ge P\{\overline{v} + A_4(t) \text{ intersects } \Lambda\} - K_4 t^{-K-d-1}.$$

Indeed, were it not for the fact that $N_A(w, 0-)$ is a Poisson variable of mean μ_A instead of $\nu(v, w, s, t)$, the auxiliary system would be obtained from the system in which $A_2(s, t, v)$ is computed by translation by $\big(-\overline{v}, -s - C_6\kappa(t)\big)$ and by removing the particles outside $\mathcal{S}\big(u, -C_5\kappa(t)\big) \cap \mathcal{C}(3C_1 t)$. The term $-K_4 t^{-K-d-1}$ corrects for increasing the mean from $\nu(v, w, s, t)$ to μ_A.

To come to (3.40) we still want to prove the inequality

(3.47)
$$P\{\overline{v} + A_4(t) \text{ intersects } \Lambda\} \ge P\{\overline{v} + A_3(t) \text{ intersects } \Lambda\} - K_5 t^{-K-d-1}.$$

This follows from the fact that if, in $\mathcal{P}^{\mathrm{h}}\big(u, -C_5\kappa(t)\big)$, all B-particles stay inside $\mathcal{C}(2C_1 t)$ during $[0, t]$, and no particle which starts outside $\mathcal{C}(3C_1 t)$ at time 0 enters $\mathcal{C}(2C_1 t)$ during $[0, t]$, then the particles which start outside $\mathcal{C}(3C_1 t)$ do not interact with any particle, and do not cause the creation of any B-particles during $[0,t]$ (compare the argument for (2.36) in [KSb]). In these circumstances $\mathcal{P}^{\rm h}\big(u,-C_5\kappa(t)\big)$ has no more B-particles at time t than the auxiliary process, which is obtained by removing the particles which start outside $\mathcal{C}(3C_1t)$ at time 0, as described above. Therefore

$$
\begin{aligned}
&\big|P\{\overline{v}+A_4(t) \text{ intersects } \Lambda\} - P\{\overline{v}+A_3(t) \text{ intersects } \Lambda\big|\\
&\quad\le P\{b_0 \notin \mathcal{S}\big(u,-C_5\kappa(t)\big)\cap\mathcal{C}(3C_1t)\}\\
&\qquad +P\{\text{in } \mathcal{P}^{\rm h}\big(u,-C_5\kappa(t)\big) \text{ some } B\text{-particles leave } \mathcal{C}(2C_1t) \text{ during } [0,t]\}\\
&\qquad +P\{\text{in } \mathcal{P}^{\rm h}\big(u,-C_5\kappa(t)\big) \text{ some particles which start outside } \mathcal{C}(3C_1t)\\
&\qquad\qquad \text{enter } \mathcal{C}(2C_1t) \text{ during } [0,t]\}.
\end{aligned}
\tag{3.48}
$$

The first term in the right-hand side here is trivially $o(t^{-K-d-1})$ (compare (3.26)).

To estimate the second term in the right-hand side of (3.48) we shall derive the more general bound

$$
\begin{aligned}
&P\{\text{in } \mathcal{P}^{\rm h}\big(u,-C_5\kappa(t)\big) \text{ some } B\text{-particles leave } \mathcal{C}(\alpha) \text{ during } [0,\alpha/(2C_1)]\}\\
&\qquad\qquad\qquad\qquad \le K_6\exp[-\alpha/(2C_1)],\ \alpha\ge 0.
\end{aligned}
\tag{3.49}
$$

To this end, we remind the reader that b_0 is the nearest vertex to the origin in $\mathcal{S}\big(u,-C_5\kappa(t)\big)$ which is occupied at time 0. By the monotonicity property of Lemma C, on the event $\{b_0=w\}$, all B-particles in $\mathcal{P}^{\rm h}\big(u,-C_5\kappa(t)\big)$ are also B-particles in the full-space process started at $(w,0)$. Therefore, the left-hand side of (3.49) is bounded by

$$
\begin{aligned}
&P\{b_0\notin\mathcal{C}(\alpha/2)\}\\
&\quad+\sum_{w\in\mathcal{C}(\alpha/2)} P\big\{w \text{ is occupied at time 0 and in the full-space process starting}\\
&\qquad\qquad \text{at } (w,0) \text{ there is a } B\text{-particle outside } \mathcal{C}(\alpha/2)\\
&\qquad\qquad \text{at some time during } \big[0,\alpha/(2C_1)\big]\big\}\\
&\quad\le P\{b_0\notin\mathcal{C}(\alpha/2)\}+K_7(\alpha+1)^d\mu_A P^{\rm or}\{\text{there is a } B\text{-particle}\\
&\qquad\qquad\qquad \text{outside } \mathcal{C}(\alpha/2) \text{ during } \big[0,\alpha/(2C_1)\big]\big\}\\
&\qquad (\text{by } (3.9)).
\end{aligned}
$$

In turn, by the argument following (3.25), the far right side here is bounded by

$$
\begin{aligned}
&P\{b_0\notin\mathcal{C}(\alpha/2)\}\\
&\quad+2K_7(\alpha+1)^d\mu_A E^{\rm or}\{\text{number of } B\text{-particles outside } \mathcal{C}\big(\alpha/2\big)\\
&\qquad\qquad \text{at time } \alpha/(2C_1)\big]\}.
\end{aligned}
$$

The first term in the right-hand side here is at most $\exp[-K_8\alpha^d]$, and the second term is at most $2K_7(\alpha+1)^d\mu_A\exp[-\alpha/(2C_1)]$, by virtue of Theorem 1 in [KSb]. Thus (3.49) holds.

The third term in the right-hand side of (3.48) is at most

$$\sum_{w\notin\mathcal{C}(3C_1t)} E\{N_A(w,0-)\}P\{\sup_{r\le t}\|S_r\|\ge\|w\|-2C_1t\} \\ \le \sum_{w\notin\mathcal{C}(3C_1t)} 8d\mu_A\exp[-K_7\|w\|]\le K_9t^{d-1}\exp[-K_{10}C_1t] \tag{3.50}$$

(see (2.42) in [KSa]). Thus (3.47) and (3.40) hold.

Step 2. We wish to prove the following precise version of (3.35): for $t\ge s\ge s_0, t\log t\le C_7s^2$ and for some constant K_{11}, independent of s,t,u,

$$P\{A_1(s,t)\text{ intersects }\Lambda\} \\ \ge P\big\{\big(\overline{\ell^*(s,u)}+A_3(t)\big)\text{ intersects }\Lambda\big\}-K_{11}s^{-K-1}. \tag{3.51}$$

To this end we define the following events for any vector $v\in\mathbb{Z}^d$:

$$\mathcal{I}_1(v):=\Big\{\text{during }[0,s]\text{ in the process }\mathcal{P}^{\mathrm{h}}\big(u,-C_5\kappa(s)\big)\text{ all the }B\text{-particles} \\ \text{stay in the set }\mathcal{C}(2C_1s)\cap\{x:\langle x,u\rangle<\langle v,u\rangle+C_5\kappa(t)\}\Big\},$$

$$\mathcal{I}_2(v):=\Big\{\text{none of the particles which were at time 0 in the half-space} \\ \mathcal{S}\big(u,\langle v,u\rangle+2C_5\kappa(t)\big)=\{x:\langle x,u\rangle\ge\langle v,u\rangle+2C_5\kappa(t)\}\text{ enters} \\ \text{the set }\mathcal{C}(2C_1s)\cap\{x:\langle x,u\rangle<\langle v,u\rangle+C_5\kappa(t)\}\text{ during }[0,s]\Big\}.$$

The following independence property is crucial for our argument: Let $\mathcal{J}(v)$ be an event which depends only on $v\in\mathbb{Z}^d$ and the particles which start in $\mathcal{S}\big(u,\langle v,u\rangle+2C_5\kappa(t)\big)$ at time 0, and the *paths* of these particles. Then

$$P\{\ell^*(s,u)=v,\mathcal{I}_1(v),\mathcal{I}_2(v),\mathcal{J}(v)\} \\ =P\{\ell^*(s,u)=v,\mathcal{I}_1(v),\mathcal{I}_2(v)\}P\{\mathcal{J}(v)|\mathcal{I}_2(v)\}. \tag{3.52}$$

The important feature here is that in the last conditional probability v is a constant, without relation to $\ell^*(s,u)$. To see (3.52) we note that in the event $\mathcal{I}_1\cap\mathcal{I}_2$ none of the particles which start in $\mathcal{S}\big(u,\langle v,u\rangle+2C_5\kappa(t)\big)$ coincides with any B-particle during $[0,s]$. Therefore, changing the paths of any of the particles which start in $\mathcal{S}\big(u,\langle v,u\rangle+2C_5\kappa(t)\big)$ has no influence on the types of any of the other particles during $[0,s]$ (and of course no influence on the paths of these other particles), as long as we stay on $\mathcal{I}_1\cap\mathcal{I}_2$ (compare the argument for (2.36) in [KSb]). In particular,

$$P\{\ell^*(s,u)=v,\mathcal{I}_1(v)|\mathcal{I}_2(v),\mathcal{J}(v)\}=P\{\ell^*(s,u)=v,\mathcal{I}_1(v)|\mathcal{I}_2(v)\}.$$

This is clearly equivalent to (3.52).

We take

$$\mathcal{J}(v) = \{A_2(s,t,v) \text{ intersects } \Lambda\},$$

where Λ is some nonrandom set in $\mathbb{Z}^d$. By definition, $A_2(s,t,v)$ depends only on v and the particles which start in the half-space $\mathcal{S}\big(u, \langle v,u\rangle + 2C_5\kappa(t)\big)$. Thus also $\mathcal{J}(v)$ depends only on v and this last collection of particles and their paths. (This is true despite the fact that we talk about B-particles in the definition (3.33). Indeed, these are B-particles in $\big(u, \langle v,u\rangle + 2C_5\kappa(t)\big)$ half-space process, started at $\big(\overline{v}, s + C_6\kappa(t)\big)$, and the types of these particles are reset at time $s + C_6\kappa(t)$ and after that do not depend on particles which started outside $\mathcal{S}\big(u, \langle v,u\rangle + 2C_5\kappa(t)\big)$.) With this choice of $\mathcal{J}$ we obtain from (3.52) for every fixed v,

$$\begin{aligned}
(3.53)\qquad P\{\ell^*(s,u) = v, \mathcal{I}_1(v), \mathcal{I}_2(v), A_2(s,t,v) \text{ intersects } \Lambda\}& \\
\geq P\{\ell^*(s,u) = v, \mathcal{I}_1(v), \mathcal{I}_2(v)\}& \\
\times \big[P\{A_2(s,t,v) \text{ intersects } \Lambda\} - P\{\mathcal{I}_2^c(v)\}\big]^+&.
\end{aligned}$$

We shall show in Step 3 that for suitable choice of constants $0 < K_i = K_i(K,d) < \infty$, independent of s, u and v, it is the case that for the process $\mathcal{P}^{\mathrm{h}}\big(u, -C_5\kappa(s)\big)$

$$(3.54)\qquad P\{\ell^*(s,u) \in \mathcal{C}(2C_1 s),\ \mathcal{I}_1^c(\ell^*(s,u))\} \leq K_{12} s^{-K-1},$$

$$(3.55)\qquad P\{\mathcal{I}_2^c(v)\} \leq K_{12} s^{-K-d-1},$$

and

$$(3.56)\qquad P\{(3.36) \text{ (with } y = \ell^*(s,u)) \text{ fails or } (3.37) \text{ fails}\} \leq K_{12} s^{-K-1}.$$

In the remainder of this step we only show how to complete the proof of (3.51) and the proposition from the estimates (3.54)–(3.56). To this end we apply (3.53). By using (3.53), (3.55), (3.40) and $t \geq s$, we get

$$\begin{aligned}
(3.57)\quad & P\{\ell^*(s,u) = v, A_2(s,t,\ell^*(s,u)) \text{ intersects } \Lambda\} \\
&\geq P\{\ell^*(s,u) = v, \mathcal{I}_1(v), \mathcal{I}_2(v), A_2(s,t,v) \text{ intersects } \Lambda\} \\
&\geq \Big[P\{\ell^*(s,u) = v\} - P\{\ell^*(s,u) = v, \mathcal{I}_1^c(v)\} - P\{\ell^*(s,u) = v, \mathcal{I}_2^c(v)\}\Big] \\
&\quad \times \big[P\{A_2(s,t,v) \text{ intersects } \Lambda\} - P\{\mathcal{I}_2^c(v)\}\big]^+ \\
&\geq P\{\ell^*(s,u) = v\} P\{A_2(s,t,v) \text{ intersects } \Lambda\} \\
&\quad - P\{\ell^*(s,u) = v, \mathcal{I}_1^c(v)\} - 2K_{12} s^{-K-d-1} \\
&\geq P\{\ell^*(s,u) = v\} P\{\overline{v} + A_3(t) \text{ intersects } \Lambda\} \\
&\quad - P\{\ell^*(s,u) = v, \mathcal{I}_1^c(v)\} - (2K_{12} + K_1) s^{-K-d-1}.
\end{aligned}$$

Now recall that $A_1(s,t) \supset A_2\big(s,t,\ell^*(s,u)\big)$ in the intersection of (3.36) and (3.37). Summing (3.57) over all $v \in \mathcal{C}(2C_1 s)$, and using (3.54) and (3.56), therefore give

$$(3.58)\quad P\{A_1(s,t) \text{ intersects } \Lambda \text{ and (3.36), (3.37) occur}\}$$
$$\geq P\{A_2(s,t,\ell^*(s,u)) \text{ intersects } \Lambda\} - P\{\text{(3.36) or (3.37) fails}\}$$
$$\geq \sum_{v\in\mathcal{C}(2C_1s)} P\{\ell^*(s,u)=v\}P\{\overline{v}+A_3(t) \text{ intersects } \Lambda\} - K_{13}s^{-K-1}.$$

Finally, since $\ell^*(s,u)$ is the location of a B-particle at time s in $\mathcal{P}^{\mathrm{h}}\big(u,-C_5\kappa(s)\big)$, we have, essentially as in the estimate for $P\{\mathcal{E}^c_{k,4}\}$ in (3.25) and the lines following it, or the estimate of the second term in the right-hand side of (3.48)

$$(3.59)\quad P\{\ell^*(s,u)\notin\mathcal{C}(2C_1s)\} \leq P\{\|x_0\| > C_1s\wedge C_5\kappa(s)/\sqrt{d}\}$$
$$+ \sum_{\|x\|\leq C_1s} \mu_A P^{\mathrm{or}}\{\text{there is a } B\text{-particle outside } \mathcal{C}(C_1s) \text{ at time } s\}$$
$$\leq K_{14}s^{-K-1}.$$

Consequently

$$(3.60)\quad P\{A_1(s,t) \text{ intersects } \Lambda\}$$
$$\geq \sum_{v\in\mathbb{Z}^d} P\{\ell^*(s,u)=v\}P\{\overline{v}+A_3(t) \text{ intersects } \Lambda\} - (K_{13}+K_{14})s^{-K-1}.$$

This is the desired (3.51).

(3.31) is just the special case of (3.60) with $\Lambda=\Gamma(\alpha,\beta,\gamma_s+\gamma_t)$. Indeed, $\{A_1(s,t) \text{ intersects } \Lambda\}$ is the event that there is a B-particle in Λ at time $s+t+C_6\kappa(t)$ in the process $\mathcal{P}^{\mathrm{h}}\big(u,-C_5\kappa(s+t+C_6\kappa(t))\big)$. For $\Lambda=\Gamma(\alpha,\beta,\gamma_s+\gamma_t)$ this event is also denoted by $\mathcal{G}\big(\alpha,\beta,\gamma_s+\gamma_t,\mathcal{P}^{\mathrm{h}}\big(u,-C_5\kappa(s+t+C_6\kappa(t))\big),s+t+C_6\kappa(t)\big)$. Thus, the left-hand sides of (3.31) and (3.60) are the same for this choice of Λ. We leave it to the reader to check that the right-hand side of (3.60) is at least as large as the right-hand side of (3.31), provided we choose $C_8\geq K_{13}+K_{14}$.

Step 3. Here we prove the relations (3.54)–(3.56). Note that (3.56) also supplies the missing estimates for (3.34), to wit, $P\{\text{(3.36) and (3.37) hold}\} \geq 1-s^{-K-1}$.

Now we start on (3.54). First

$$(3.61)\quad P\{\text{in } \mathcal{P}^{\mathrm{h}}\big(u,-C_5\kappa(s)\big) \text{ some } B\text{-particle leaves } \mathcal{C}(2C_1s) \text{ during } [0,s]\}$$
$$= O\big(s^{-K-d-1}\big)$$

(see (3.49)). In addition, by definition of $\ell^*(s,u)$, $\langle\ell^*(s,u),u\rangle = h\big(s,u,-C_5\kappa(s)\big)$. Thus, if we also take into account that $s\leq t$,

(3.62)

$$\begin{aligned}
&P\bigl\{\text{in } \mathcal{P}^{\mathrm{h}}\bigl(u,-C_5\kappa(s)\bigr), \text{ during } [0,s] \text{ all } B\text{-particles stay in } \mathcal{C}(2C_1s), \text{ but some}\\
&\qquad \text{of them leave } \{x : \langle x,u\rangle < \langle \ell^*(s,u),u\rangle + C_5\kappa(t)\}\ \bigr\}\\
&\quad \le P\bigl\{\text{in } \mathcal{P}^{\mathrm{h}}\bigl(u,-C_5\kappa(s)\bigr), \text{ at some time } r\le s \text{ there are}\\
&\qquad\quad B\text{-particles at some } v\in\mathcal{C}(2C_1s) \text{ with}\\
&\qquad\quad \langle v,u\rangle \ge \max\{\langle x,u\rangle :\ \text{there is a } B\text{-particle at } x \text{ at time } s\} + C_5\kappa(s)\bigr\}.
\end{aligned}$$

This last event can happen only if some B-particle reaches a vertex $v \in \mathcal{C}(2C_1s)$ before time s and then this particle moves to some x at time s with $\langle x,u\rangle < \langle v,u\rangle - C_5\kappa(s)$. The probability that such a particle started outside $\mathcal{C}(3C_1s)$ is bounded by the third term in the right-hand side of (3.48), with t replaced by s. Therefore, the right-hand side of (3.62) is at most

$$\begin{aligned}
&\text{(third term in right-hand side of (3.48) with } t \text{ replaced by } s)\\
&\qquad + \sum_{w\in\mathcal{C}(3C_1s)} E\{N_A(w,0-)P\{\sup_{0\le r_1,r_2\le s}\|S_{r_1}-S_{r_2}\| \ge C_5\kappa(s)/\sqrt{d}\}\\
&\qquad \le K_9 s^{d-1}\exp[-K_{10}C_1s] + K_{15}(3C_1s)^d \exp[-K_{16}C_5^2\log s],
\end{aligned} \tag{3.63}$$

by (3.50) and by (2.42) in [KSa]. Together with (3.61) this proves that we can take C_5 so large that (3.54) holds. As observed after (3.44) we can even choose C_5 so that (3.44) is also valid. Once we have chosen C_5 we fix

$$C_6 = \frac{16C_5}{C_2}. \tag{3.64}$$

As for (3.55), we have

$$\begin{aligned}
P\{\mathcal{I}_2^c(v)\} \le &\sum_{w\in\mathcal{S}(u,\langle v,u\rangle+2C_5\kappa(t))} E\{N_A(w,0-\}\\
&\times P\{\sup_{r\le s}\|S_r\| \ge C_5\kappa(t)\vee\bigl(\|w\|-2C_1s\bigr)\}.
\end{aligned} \tag{3.65}$$

We leave it to the reader to show that this is $O\bigl(s^{-K-d-1}\bigr)$ for $t\ge s$ and large enough C_5 (again by (2.42) in [KSa]).

Finally, to prove (3.56), we note first that $P\{(3.37) \text{ fails}\} = O(s^{-K-1})$, provided $K_3 = K_3(\mu_A,d)$ is taken large enough, just as in (3.26). Next,

$$\begin{aligned}
P\{\langle \ell^*(s,u),u\rangle < 0\} &= P\{h\bigl(s,u,-C_5\kappa(s)\bigr) < 0\}\\
&\le P\{h\bigl(s,u,-C_5\kappa(s)\bigr) \le C_4s\}\\
&\le [C_5\kappa(s)]^{-2K-2} \le s^{-K-1}
\end{aligned} \tag{3.66}$$

for large s, by virtue of Lemma 2 with K replaced by $2K+2$. Lastly, we have to show that for the choice of C_6 in (3.64)

$$\begin{aligned}(3.67)\qquad & P\{z(s,t) \text{ is not occupied by a } B\text{-particle in} \\ & \quad \mathcal{P}^{\mathrm{h}}\big(u, -C_5\kappa(s)\big) \text{ at time } s + C_6\kappa(t)\} \\ & \le P\{z(s,t) \text{ is not occupied by a } B\text{-particle in } \mathcal{P}^{\mathrm{h}}\big(u, -C_5\kappa(s)\big) \\ & \qquad \text{at time } s + C_6\kappa(t), \text{ but } z(s,t) \in \ell^*(s,u) + \mathcal{C}(C_2C_6\kappa(t)/2\} \\ & \quad + P\{z(s,t) \notin \ell^*(s,u) + \mathcal{C}(C_2C_6\kappa(t)/2\} \\ & = O\big(s^{-K-1}\big).\end{aligned}$$

The first inequality here is obvious. The bound $O(s^{-K-1})$ for the middle member of (3.67) is formulated as a separate lemma, because the same argument will be needed once more in the next section. To see that (3.67) follows from Lemma 4 below, recall that $z(s,t)$ is occupied at time $s + C_6\kappa(t)$ by some particle which started at time 0 in $\mathcal{S}\big(u, \langle \ell^*(s,u), u\rangle + 2C_5\kappa(t)\big)$ (see a few lines before (3.32)). In particular there is some particle at $z(s,t)$ at time $s+C_6\kappa(t)$, so that $z(s,t)$ is occupied in $\mathcal{P}^{\mathrm{f}}$ at time $s + C_6\kappa(t)$. Also, $\ell^*(s,u)$ is occupied by at least one B-particle in $\mathcal{P}^{\mathrm{h}}\big(u, -C_5\kappa(s)\big)$ at time s. So Lemma 4 with $\widetilde{s} = s + C_6\kappa(t)$ and $y(s) = \ell^*(s,u)$ (and C_6 as in (3.64)) shows that the middle member of (3.67) is at most

$$\begin{aligned}(3.68)\quad P\{\ell^*(s,u) \notin \mathcal{C}(2C_1s)\} + P\{\langle \ell^*(s,u), u)\rangle < C_4s/2\} + 5s^{-K-1} \\ + P\{z(s,t) \notin \ell^*(s,u) + \mathcal{C}\big(C_2C_6\kappa(t)/2\big)\}.\end{aligned}$$

Note that we used the second part of condition (3.30) here; we have to choose C_7 small enough to make sure that (3.71) holds for $\widetilde{s} - s = C_6\kappa(t)$. The first two terms in (3.68) are $O(s^{-K-1})$, by virtue of (3.61) and (3.66). The fourth term is bounded by

(3.69)

$$\begin{aligned}& P\{z(s,t) \notin \ell^*(s,u) + \mathcal{C}(C_2C_6\kappa(t)/2)\} \le P\{\|z(s,t) - \overline{\ell^*(s,u)}\| > 4C_5\kappa(t) - 1\} \\ & (\text{because } C_2C_6/2 = 8C_5 \text{ and } \|\overline{\ell^*(s,u)} - \ell^*(s,u)\| \le 4C_5\kappa(t)) + 1) \\ & \le P\{\ell^*(s,u) \notin \mathcal{C}(2C_1s)\} + P\{\ell^*(s,u) \in \mathcal{C}(2C_1s), \text{ and none of the sites} \\ & \qquad \text{in } \overline{\ell^*(s,u)} + \mathcal{C}\big(4C_5\kappa(t) - 1\big) \text{ are occupied at time } s + C_6\kappa(t) \text{ by} \\ & \qquad \text{a particle which started in } \mathcal{S}(u, \langle \ell^*(s,u), u\rangle + 2C_5\kappa(t))\}.\end{aligned}$$

We already saw in (3.59) that the first term in the right-hand side is $O\big(s^{-K-1}\big)$. As for the second term in the right-hand side, this is by a decomposition with respect to the possible values of $\ell^*(s,u)$, analogously to (3.9), at most

(3.70)

$$\begin{aligned}K_{17} \sum_{v \in \mathcal{C}(2C_1s)} & P\{\text{none of the sites in } \overline{v} + \mathcal{C}\big(4C_5\kappa(t) - 1\big) \text{ is occupied at time} \\ & s + C_6\kappa(t) \text{ by a particle which started in } \mathcal{S}\big(u, \langle v, u\rangle + 2C_5\kappa(t)\big)\}.\end{aligned}$$

However, the numbers of particles at sites $\overline{v} + w$ at time $s + C_6\kappa(t)$ which started in $\mathcal{S}(u, \langle v, u\rangle + 2C_5\kappa(t))$ are independent Poisson variables with means

$\nu(v,w,s,t)$ given in (3.41). By the estimate (3.43) we have $\nu(v,w,s,t) \ge \mu_A/2$ for $\langle w,u\rangle \ge 0$ and all v (and t large enough). Therefore (3.70) is at most $K_{18}s^d \exp[-K_{19}\kappa^d(t)\mu_A]$. This proves the bound $O\big(s^{-K-1}\big)$ in (3.67), and therefore (3.56) is reduced to the next lemma.

Roughly speaking, the next lemma guarantees that if a certain vertex $y(s)$ has a B-particle in the half-space process $\mathcal{P}^{\mathrm{h}}\big(u,-C_5\kappa(s)\big)$ at a time s, then a little later all occupied sites 'near' $y(s)$ will actually have a B-particle in $\mathcal{P}^{\mathrm{h}}\big(u,-C_5\kappa(s)\big)$.

LEMMA 4. *Let $s,\widetilde{s}$ be such that*

$$\frac{16C_5}{C_2}\kappa(s) \le \widetilde{s} - s \le \frac{C_4}{8C_1}s. \tag{3.71}$$

Let $u \in S^{d-1}$ be fixed and let $y(s) \in \mathbb{Z}^d$ be a random point (that is, $y(s)$ may depend on the sample point σ). Define the event $\mathcal{K}(y)$ by

$$\begin{aligned}\mathcal{K}(y) := \{&\textit{there exists a site } z \in y + \mathcal{C}\big(C_2(\widetilde{s}-s)/2\big) \textit{ such that}\\ &\textit{at time } \widetilde{s}, z \textit{ is occupied in } \mathcal{P}^{\mathrm{f}}, \textit{ but is not occupied}\\ &\textit{by a } B\textit{-particle in } \mathcal{P}^{\mathrm{h}}\big(u,-C_5\kappa(s)\big)\}.\end{aligned} \tag{3.72}$$

Then for each $K > 0$ there exists an $s_1 = s_1(K)$ (independent of u) such that

$$\begin{aligned}&P\{\mathcal{B}^{\mathrm{h}}\big(y(s),s;u,-C_5\kappa(s)\big) \cap \mathcal{K}(y(s))\}\\ &\qquad \le P\{y(s) \notin \mathcal{C}(2C_1s)\} + P\{\langle y(s),u\rangle < \tfrac{1}{2}C_4 s\} + 5s^{-K-1}\end{aligned} \tag{3.73}$$

for $s \ge s_1$ (see (3.14) for $\mathcal{B}^{\mathrm{h}}$).

Proof. Assume that the space-time point (y,s) is occupied by some particle in $\mathcal{P}^{\mathrm{h}}\big(u,-C_5\kappa(s)\big)$. We can then define the following auxiliary events:

$$\begin{aligned}\mathcal{K}_1(y) := \{&\text{there exists a site } z \in y + \mathcal{C}\big(C_2(\widetilde{s}-s)/2\big) \text{ such that } (z,\widetilde{s})\\ &\text{is occupied in } \mathcal{P}^{\mathrm{f}}, \text{ but is not occupied in } \mathcal{P}^{\mathrm{h}}(u,-C_5\kappa(s))\},\\ \mathcal{K}_2(y) := \{&\text{there exists a site } z \in y + \mathcal{C}\big(C_2(\widetilde{s}-s)/2\big) \text{ such that } (z,\widetilde{s})\\ &\text{is occupied by an } A\text{-particle in the full-space process}\\ &\text{starting at } (y,s)\},\\ \mathcal{K}_3(y) := \{&\text{there exists a site } z \in y + \mathcal{C}\big(C_2(\widetilde{s}-s)/2\big) \text{ such that } (z,\widetilde{s})\\ &\text{is occupied by a } B\text{-particle in the full-space process starting}\\ &\text{at } (y,s), \text{ but occupied by an } A\text{-particle in the } (u,-C_5\kappa(s))\\ &\text{half-space process starting at } (y,s)\},\\ \mathcal{K}_4(y) := \{&\text{in the full-space process starting at } (y,s) \text{ some } B\text{-particles}\\ &\text{leave } y + \mathcal{C}\big(2C_1(\widetilde{s}-s)\big) \text{ during } [s,\widetilde{s}]\},\end{aligned}$$

$$\mathcal{K}_5(y) := \{\text{some particles which start outside } \mathcal{S}\big(u, -C_5\kappa(s)\big) \text{ at time } 0 \\ \text{enter } y + \mathcal{C}\big(2C_1(\widetilde{s} - s)\big) \text{ during } [s, \widetilde{s}]\}.$$

We shall first show that

$$\mathcal{B}^{\mathrm{h}}\big(y, s; u, -C_5\kappa(s)\big) \cap \mathcal{K}(y) \subset \bigcup_{i=1}^{3} \mathcal{K}_i(y) \text{ and } \mathcal{K}_3(y) \subset \mathcal{K}_4(y) \cup \mathcal{K}_5(y), \tag{3.74}$$

and then estimate $P\{y(s) \in \mathcal{C}(2C_1 s), \langle y(s), u\rangle \geq C_4 s/2, \mathcal{K}_i(y(s))\}$ for $1 \leq i \leq 5$. To prove the first part of (3.74), consider a sample point for which $\mathcal{B}^{\mathrm{h}}\big(y, s; u, -C_5\kappa(s)\big) \cap \mathcal{K}(y)$ occurs and let z be a site in $y + \mathcal{C}\big(C_2(\widetilde{s} - s)/2\big)$ such that $(z, \widetilde{s})$ is occupied in $\mathcal{P}^{\mathrm{f}}$, but is not occupied by a B-particle in $\mathcal{P}^{\mathrm{h}}\big(u, -C_5\kappa(s)\big)$. Then it may be that $(z, \widetilde{s})$ is not occupied at all in $\mathcal{P}^{\mathrm{h}}\big(u, -C_5\kappa(s)\big)$. This would mean that $\mathcal{K}_1(y)$ occurs. If this fails, then $(z, \widetilde{s})$ is occupied in $\mathcal{P}^{\mathrm{h}}\big(u, -C_5\kappa(s)\big)$, necessarily by an A-particle. We claim that $(z, \widetilde{s})$ is then also occupied by an A-particle in the $\big(u, -C_5\kappa(s)\big)$ half-space process starting at (y, s). This is so, because starting at (y, s) does not remove any particles, but it may change some types. But on $\mathcal{B}^{\mathrm{h}}\big(y, s; u, -C_5\kappa(s)\big)$, y has already at least one B-particle at time s in $\mathcal{P}^{\mathrm{h}}\big(u, -C_5\kappa(s)\big)$. Thus the resetting at time s only changes some types from B to A, and since z already has type A at time $\widetilde{s}$ in $\mathcal{P}^{\mathrm{h}}\big(u, -C_5\kappa(s)\big)$, it will (by Lemma C) also have type A at time $\widetilde{s}$ in the $\big(u, -C_5\kappa(s)\big)$ half-space process started at (y, s), as claimed. Also, $(z, \widetilde{s})$ is occupied in the full-space process starting at (y, s) (since it is occupied in the full-space process, starting at $(\mathbf{0}, 0)$). The type at $(z, \widetilde{s})$ in this process may be A, in which case $\mathcal{K}_2(y)$ occurs, or B, in which case $\mathcal{K}_3(y)$ occurs. This proves the first inclusion in (3.74).

The second part of (3.74) follows from the argument given for (3.47). $\mathcal{K}_3(y)$ requires that at time $\widetilde{s}$ there are particles in $y + \mathcal{C}\big(C_2(\widetilde{s} - s)/2\big)$ which have different types in the full-space and in the $(u, -C_5\kappa(s))$ half-space process, both starting at (y, s). This means that in the full-space process starting at (y, s) the type of some particle which is in $y + \mathcal{C}\big(C_2(\widetilde{s} - s)/2\big)$ at time $\widetilde{s}$ is influenced by particles which started outside $\mathcal{S}\big(u, -C_5\kappa(s)\big)$ at time 0. However, this can happen only if in the full-space process starting at (y, s), these particles meet some B-particles during $[s, \widetilde{s}]$. In turn, this can happen only if $\mathcal{K}_4(y)$ or $\mathcal{K}_5(y)$ occurs. This proves the second inclusion in (3.74).

Our next task is to find bounds for

$$P\{y(s) \in \mathcal{C}(2C_1 s), \langle y(s), u\rangle \geq C_4 s/2, \mathcal{B}^{\mathrm{h}}\big(y(s), s; u, -C_5\kappa(s)\big), \mathcal{K}_i\big(y(s)\big)\},$$

when $i = 1, 2, 4, 5$. For $i = 1$ we have

$$\begin{aligned} &P\{y(s) \in \mathcal{C}(2C_1 s), \langle y(s), u\rangle \geq C_4 s/2, \mathcal{K}_1(y(s))\} \\ &\quad \leq \sum_{w \notin \mathcal{S}\left(u, -C_5\kappa(s)\right)} \sum_{\substack{\langle z, u\rangle \geq C_4 s/2 - C_2(\widetilde{s} - s)/2 \\ z \in \mathcal{C}\left((2C_1 + C_4)s\right)}} EN_A(w, 0-)P\{w + S_{\widetilde{s}} = z\} \end{aligned} \tag{3.75}$$

$$\leq \sum_{w \in \mathcal{C}\left((4C_1+2C_4)s\right)} \mu_A P\{\|S_{\widetilde{s}}\| \geq C_4 s/(4\sqrt{d})\}$$
$$+ \sum_{w \notin \mathcal{C}\left((4C_1+2C_4)s\right)} \mu_A P\{\|S_{\widetilde{s}}\| \geq \|w\|/2\} \leq s^{-K-1}$$

for all $s \geq$ some $s_1 = s_1(K)$. In the first inequality we used $\|z\| \leq \|z - y(s)\| + \|y(s)\| \leq C_2(\widetilde{s} - s)/2 + 2C_1 s \leq (2C_1 + C_4)s$, by virtue of (3.71) and the inequality $C_2 \leq C_1$ (see Theorem A). In the second inequality we used that for the summands here $\|w - z\| \geq \langle (z-w), u\rangle/\sqrt{d} \geq [C_4 s/2 - C_2(\widetilde{s} - s)/2 + C_5\kappa(s)]/\sqrt{d} \geq C_4 s/(4\sqrt{d})$. For the third inequality we use $\widetilde{s} \leq \big(1 + C_4/(8C_1)\big)s$ plus (2.42) in [KSa]; compare (3.24).

Next, we remind the reader that P^{or} is the probability measure governing the original model, in which one B-particle is added at the origin at time 0. In this notation we have, by (3.9) and (2.4),

$$\begin{aligned}(3.76)\qquad & P\{y(s) \in \mathcal{C}(2C_1 s), \mathcal{B}^{\mathrm{h}}\big(y(s), s; u, -C_5\kappa(s)\big), \mathcal{K}_2(y(s))\} \\ &\quad \leq K_{20} s^d P^{\mathrm{or}}\{\text{there exists a } z \in \mathcal{C}\big(C_2(\widetilde{s} - s)/2\big) \text{ which is} \\ &\qquad\qquad \text{occupied by an } A\text{-particle at time } \widetilde{s} - s\} \\ &\quad \leq 2s^{-K-1}.\end{aligned}$$

Again by (3.9)

(3.77)
$$\begin{aligned}& P\{y(s) \in \mathcal{C}(2C_1 s), \mathcal{B}^{\mathrm{h}}\big((y(s), s; u, -C_5\kappa(s)\big), \mathcal{K}_4(y(s))\} \\ &\quad \leq K_{20} s^d P^{\mathrm{or}}\{\text{some } B\text{-particles leave } \mathcal{C}\big(2C_1(\widetilde{s} - s)\big) \text{ during } [0, \widetilde{s} - s]\} \\ &\quad \leq s^{-K-1} \text{ (by the argument for (3.49) or Theorem 1 in [KSb]).}\end{aligned}$$

Finally,

(3.78)
$$\begin{aligned}& P\{y(s) \in \mathcal{C}(2C_1 s), \langle y(s), u\rangle \geq C_4 s/2, \mathcal{B}^{\mathrm{h}}\big((y(s), s; u, -C_5\kappa(s)\big), \mathcal{K}_5(y(s))\} \\ & \leq \sum_{w: \langle w, u\rangle < -C_5\kappa(s)} \sum_{\substack{v \in \mathcal{C}\left((2C_1+C_4)s\right) \\ \langle v, u\rangle \geq C_4 s/4}} EN_A(w, 0-) P\{w + S_r = v \text{ for some } r \leq \widetilde{s}\} \\ & \leq \mu_A \sum_{w \in \mathcal{C}\left((4C_1+2C_4)s\right)} P\{\sup_{r \leq \left(1+C_4/(8C_1)\right)s} \|S_r\| \geq C_4 s/(4\sqrt{d})\} \\ & \quad + \mu_A \sum_{w \notin \mathcal{C}\left((4C_1+2C_4)s\right)} \sum_{v \in \mathcal{C}\left((2C_1+C_4)s\right)} P\{\sup_{r \leq \left(1+C_4/(8C_1)\right)s} \|S_r\| \geq \|w/2\|\} \\ & \leq s^{-K-1} \text{ (by (2.42) in [KSa])).}\end{aligned}$$

Together with (3.74) these estimates prove (3.73). $\square$

Corollary 5. *For every unit vector u there exists a constant $\lambda(u) \in [C_4, 2\sqrt{d}C_1]$ such that*

$$\lim_{t\to\infty} \frac{1}{t} h^*(t,u) = \lambda(u) \text{ almost surely and in } L^p \text{ for all } p > 0 \tag{3.79}$$

(t runs through the reals here). Moreover, for each $\eta > 0$ there exist an exponentially increasing sequence $\{n_1 < n_2 < \dots\} = \{n_1(\eta) < n_2(\eta) < \dots\}$ (independent of u) and a constant $\zeta = \zeta(\eta) > 0$ such that

$$1 + \zeta < \frac{n_{j+1}}{n_j} \le 1 + \eta, \quad j \ge 1, \tag{3.80}$$

and such that for every $\varepsilon > 0$,

$$\sum_{k=0}^{\infty} P\Big\{\Big|\frac{1}{n_k} h^*(n_k,u) - \lambda(u)\Big| > \varepsilon\Big\} < \infty. \tag{3.81}$$

Proof. The basis for this proof is (3.31) with $\beta = \infty$. Since $\Gamma(\alpha,\infty,\gamma,u) = \{x \in \mathbb{R}^d : \langle x,u\rangle \ge \alpha\} = \mathcal{S}(u,\alpha)$, we have

$$\begin{aligned}&\mathcal{G}(\alpha,\infty,\gamma,\mathcal{P},t)\\ &\qquad = \{\text{in } \mathcal{P}, \text{ at time } t, \text{ there is a } B\text{-particle at some } x \text{ with } \langle x,u\rangle \ge \alpha\}.\end{aligned}$$

In particular

$$\begin{aligned}&\mathcal{G}\big(\alpha,\infty,\gamma,\mathcal{P}^{\mathrm{h}}(u,-C_5\kappa(s+t+C_6\kappa(t)),s+t+C_6\kappa(t)\big)\\ &\quad = \{h\big(s+t+C_6\kappa(t),u,-C_5\kappa(s+t+C_6\kappa(t))\big) \ge \alpha\}\\ &\quad = \{h^*(s+t+C_6\kappa(t),u) \ge \alpha\}.\end{aligned}$$

Similarly,

$$\mathcal{G}\big(\alpha,\infty,\gamma,\mathcal{P}^{\mathrm{h}}\big(u,-C_5\kappa(t)\big),t\big) = \{h^*(t,u) \ge \alpha\}.$$

Thus, (3.31) with $\beta = \infty$ says that, under (3.30),

$$P\{h^*(s+t+C_6\kappa(t),u) \ge \alpha\} \ge P\{h_1^*(s,u) + h_2^*(t,u) \ge \alpha\} - C_8 s^{-K-1}, \tag{3.82}$$

where $h_1^*(s,u)$ and $h_2^*(t,u)$, are independent copies of $h^*(s,u)$ and $h^*(t,u)$, respectively.

The corollary will be derived from this relation by more or less standard subadditivity techniques. To apply these techniques we first derive some simple properties of $h^*(s,u)$. Note that these properties hold as soon as $s,t \ge s_0$; the rest of the condition (3.30) is not needed. The first is the following tail estimate:

$$P\{h^*(s,u) \ge \alpha\} + P\{\|m^*(s,u)\| \ge \alpha\} \le \exp[-K_1\alpha] \text{ for } \alpha \ge 2\sqrt{d}C_1 s. \tag{3.83}$$

The second property is an estimate for the negative tail of $h^*(s,u)$:

$$P\{h^*(s,u) \le -\alpha\} \le K_2 \exp\Big[-\frac{K_3\alpha^2}{s+\alpha}\Big] \text{ for } \alpha \ge 0. \tag{3.84}$$

We remind the reader that we also have the bound (3.66) for $P\{h^*(s,u) \le C_4 s\}$, which for small α is better than (3.84). The third and fourth property are semi-continuity properties in s, namely

$$(3.85)\quad P\{\inf_{r\le t} h^*(s+r,u) - h^*(s,u) \le -\alpha\} \\ \le K_4 s^{-K} + P\{\|\sup_{r\le t} S_r\| \ge \alpha\} \le K_4 s^{-K} + 8d\exp\Big[-\frac{K_3\alpha^2}{t+\alpha}\Big],\ \alpha \ge 0,$$

and

$$(3.86)\quad P\{\sup_{r\le t} h^*(s+r,u) - h^*(s+t,u) \ge \alpha\} \\ \le K_4 s^{-K} + K_5(s+t)^d \exp\Big[-\frac{K_3\alpha^2}{t+\alpha}\Big],\ \alpha \ge 0.$$

To prove (3.83) take $\alpha \ge 2\sqrt{d}C_1 s$. Since $\langle x,u\rangle \le \|x\|_2 \le \sqrt{d}\|x\|$, as well as $\|x^\perp\| \le \|x\|_2 \le \sqrt{d}\|x\|$, the left-hand side of (3.83) is bounded by

$$(3.87)\quad 2P\{\text{in } \mathcal{P}^{\rm h}\big(u,-C_5\kappa(s)\big) \text{ there is a } B\text{-particle outside } \mathcal{C}(\alpha/\sqrt{d}) \\ \text{at some time during } [0,s] \subset \big[0,[2\sqrt{d}C_1]^{-1}\alpha\big]\}.$$

The inequality (3.83) now follows from (3.49).

To prove (3.84), let ρ be any particle at $w_{-C_5\kappa(s)}$ at time 0 (see a few lines before (3.2) for w_{-r}). In $\mathcal{P}^{\rm h}\big(u,-C_5\kappa(s)\big)$, ρ is given type B at time 0, and ρ remains a B-particle in $\mathcal{P}^{\rm h}\big(u,-C_5\kappa(s)\big)$ at all times, and in particular at time s. The distribution at time s of the position of ρ is that of $w_{-C_5\kappa(s)} + S_s$ with S_s independent of $w_{-C_5\kappa(s)}$. For $h^*(s,u) \le -\alpha$ to occur, ρ must lie in the half space $\{x : \langle x,u\rangle \le -\alpha\}$ at time s. Thus

$$P\{h^*(s,u) \le -\alpha\} \le P\{\langle w_{-C_5\kappa(s)},u\rangle + \langle S_s,u\rangle \le -\alpha\} \\ \le P\{\|w_{-C_5\kappa(s)}\| \ge \alpha/(2\sqrt{d})\} + P\{\|S_s\| \ge \alpha/(2\sqrt{d})\}.$$

As before, the first term in the right-hand side is at most $\exp[-K_6\alpha^d]$ and the second one is at most $8d\exp\big[-K_3\alpha^2/(s+\alpha)\big]$ (by (2.42) in [KSa]). (3.84) follows.

The argument for (3.85) is basically already given in (3.19) and in the preceding paragraph. Moreover, it is similar to, but simpler than, the proof of (3.86) and so we only prove the latter. If $h^*(s+t,u) = h$, then all B-particles in $\mathcal{P}^{\rm h}\big(u,-C_5\kappa(s+t)\big)$ are located in $\{x : \langle x,u\rangle \le h\}$ at time $s+t$. If further, for some $0 \le r \le t, h^*(s+r,u) \ge h+\alpha$, then there is some B-particle ρ in $\mathcal{P}^{\rm h}\big(u,-C_5\kappa(s+r)\big)$ in $\{x : \langle x,u\rangle \ge h+\alpha\}$ at time $s+r$. This ρ is also a particle present in $\mathcal{P}^{\rm h}\big(u,-C_5\kappa(s+t)\big)$ and even of type B in $\mathcal{P}^{\rm h}\big(u,-C_5\kappa(s+t)\big)$ at time $s+t$, provided $\|x_0\| \le C_5\kappa(s)/\sqrt{d}$ (see (3.3)). Thus in this case ρ moved over a distance at least $\alpha/\sqrt{d}$ during $[s+r,s+t]$. Similarly, ρ is a B-particle in $\mathcal{P}^{\rm f}$ at time $s+t$. Therefore, the left-hand side of (3.86) is at most

$$\begin{aligned}
&P\{\|x_0\| > C_5\kappa(s)/\sqrt{d}\}\\
&\quad + P\{\text{some particle which starts outside } \mathcal{C}\big(3C_1(s+t)\big) \text{ becomes}\\
&\qquad \text{a } B\text{-particle in } \mathcal{P}^{\rm f} \text{ before time } s+t\}\\
&\quad + \sum_{x\in\mathcal{C}\left(3C_1(s+t)\right)} \mu_A P\{\sup_{r\le t}\|S_r - S_t\| \ge \alpha/\sqrt{d}\}\\
&\le K_7 s^{-K} + K_8(s+t)^d \exp\Big[-\frac{K_9\alpha^2}{d(t+\alpha)}\Big]
\end{aligned}$$

(see (3.48)–(3.50), as well as (2.42) in [KSa]). Thus (3.86) holds.

We can now proceed with subadditivity arguments. We introduce the random variables

$$X(s) = [2\sqrt{d}C_1 s - h^*(s,u)]^+$$

and the deterministic quantities $Y(t) = 2\sqrt{d}C_1C_6\kappa(t)$, and let $X'(t)$ be a copy of $X(t)$ which is independent of $X(s)$. Then (3.82) shows that, under (3.30), these random variables satisfy

$$\begin{aligned}
(3.88)\quad & P\{X\big(s+t+C_6\kappa(t)\big) \le \beta\} \ge P\{X(s) + X'(t) + Y(t) \le \beta\}\\
&\qquad - C_8 s^{-K-1} - \exp[-2K_1\sqrt{d}C_1 s] - \exp[-2K_1\sqrt{d}C_1 t] \le 2C_8 s^{-K-1}
\end{aligned}$$

for $\beta \ge 0$, $s \ge$ some constant s_3. Here we used that.

$$P\{X(s) \neq [2\sqrt{d}C_1 s - h^*(s,u)]\} = P\{h^*(s,u) > 2\sqrt{d}C_1 s\} \le \exp -[2K_1\sqrt{d}C_1 s]$$

(see (3.83)). Of course (3.88) also holds trivially for $\beta < 0$. This is very close to the principal hypothesis of the lemma on p. 674 of [Ha] but we have to do some extra work because of the $C_6\kappa(t)$ which appears in the argument on the left-hand side of (3.88). From now on we take $K = 4$. We first derive a simple approximation for moments of $X(s)$. Fix $K_{10} \ge 2\sqrt{d}C_1$. Then for $p > 0$

$$EX^p(s) = p\int_0^{K_{10}s} \alpha^{p-1}P\{X(s)\ge\alpha\}d\alpha + p\int_{K_{10}s}^\infty \alpha^{p-1}P\{X(s)\ge\alpha\}d\alpha.$$

By virtue of (3.66) with K replaced by $K+p$ and (3.84) the last integral is for $s \ge$ some s_4 and a suitable constant $K_{11} = K_{11}(p)$ bounded by

$$\begin{aligned}
(3.89)\qquad & p\int_{K_{10}s}^{(K_{10}+C_4)s} \big[(K_{10}+C_4)s\big]^{p-1} P\{h^*(s,u)\le 0\}d\alpha\\
&\qquad + p\int_{C_4 s}^\infty \big[K_{10}s+\alpha\big]^{p-1} P\{h^*(s,u)\le -\alpha\}d\alpha\\
&\quad \le p\big[(K_{10}+C_4)s\big]^{p-1} s^{-K-p-1} C_4 s\\
&\qquad + p\int_{C_4 s}^\infty \big[K_{10}s+\alpha\big]^{p-1} K_2 \exp\Big[-\frac{K_3\alpha^2}{s+\alpha}\Big]d\alpha\\
&\quad \le K_{11}s^{-K-1}.
\end{aligned}$$

Thus we have

$$\big|EX^p(s) - p\int_0^{K_{10}s} \alpha^{p-1}P\{X(s) \geq \alpha\}d\alpha\big| \leq K_{11}s^{-K-1} \tag{3.90}$$

for $s \geq s_4$. We note in passing that this shows that all moments of $X(s)$ are finite.

We now apply the relation (3.90) with s replaced by $s + t + C_6\kappa(t)$ for $s \geq s_4$. In combination with (3.88) this gives

$$\begin{aligned} &EX^p(s+t+C_6\kappa(t)) - K_{11}[s+t+C_6\kappa(t)]^{-K-1} \\ &\quad \leq p\int_0^{K_{10}\left(s+t+C_6\kappa(t)\right)} \alpha^{p-1}P\{X\big(s+t+C_6\kappa(t)\big) \geq \alpha\}d\alpha \\ &\quad \leq p\int_0^{\infty} \alpha^{p-1}P\{X(s)+X'(t)+Y(t) \geq \alpha\}d\alpha \\ &\qquad +2C_8s^{-K-1}[K_{10}\big(s+t+C_6\kappa(t)\big)]^p. \end{aligned}$$

In particular the cases $p = 1$ and $p = 2$ show that there exists a constant C_9 such that under (3.30) and $s \geq s_4$

$$\begin{aligned} EX\big(s+t+C_6\kappa(t)\big) &\leq EX(s) + EX(t) + 2\sqrt{d}C_1C_6\kappa(t) \\ &\qquad +2K_{10}C_8\big(s+t+C_6\kappa(t)\big)s^{-K-1} + K_{11}s^{-K-1} \\ &\leq EX(s) + EX(t) + C_9\kappa(t) \end{aligned} \tag{3.91}$$

and

$$\begin{aligned} EX^2\big(2s+C_6\kappa(s)\big) &\leq E\big[X(s)+X'(s)+Y(s)\big]^2 + K_{12}[s+C_6\kappa(s)]^2s^{-K-1} \\ &\leq E\big[X(s)+X'(s)+Y(s)\big]^2 + 4K_{12}s^{-K+1}. \end{aligned} \tag{3.92}$$

Without loss of generality we may take s_0 so large that(3.91) holds under (3.30). For (3.92) we took $t = s$, so that this holds as soon as $s \geq s_0$. Fortunately there is a simple replacement for (3.91) that holds as soon as $s_0 \leq s \leq t$. Indeed, assume that $s_0 \leq s \leq t$, but $t \log t > C_7s^2$. It then follows from the simple inequality

$$[a+b-c]^+ - [a-d]^+ \leq |b| + [a-c]^+ - [a-d]^+ \leq |b| + [c-d]^- \tag{3.93}$$

that

$$X\big(s+t+C_6\kappa(t)\big) - X(t) \leq 2\sqrt{d}C_1[s+C_6\kappa(t)] + [h^*(s+t+C_6\kappa(t),u) - h^*(t,u)]^-.$$

Consequently

$$\begin{aligned} &EX\big(s+t+C_6\kappa(t)\big) - EX(t) \\ &\quad \leq 2\sqrt{d}C_1[s+C_6\kappa(t)] + E\big\{[h^*(s+t+C_6\kappa(t),u) - h^*(t,u)]^-\big\} \\ &\quad \leq 2\sqrt{d}C_1[s+C_6\kappa(t)] + \int_0^{\infty} P\{h^*\big(s+t+C_6\kappa(t),u\big) \leq h^*(t,u) - \alpha\}d\alpha. \end{aligned} \tag{3.94}$$

We break the last integral up into the integrals over $[0, 4\sqrt{d}C_1 t]$ and over $[4\sqrt{d}C_1 t, \infty)$. The first piece is by (3.85) with s and t replaced by t and $s + C_6\kappa(t)$, respectively, and K taken as 1, at most

$$\int_0^{4\sqrt{d}C_1 t} \Big[\frac{K_4}{t} + 8d\exp\Big[-\frac{K_3\alpha^2}{s + C_6\kappa(t) + \alpha}\Big]\Big] d\alpha \le K_{13}[s + \kappa(t)]^{1/2} \le K_{14}\kappa(t).$$

For the second piece we use that

$$\begin{aligned} &P\{h^*\big(s + t + C_6\kappa(t), u\big) \le h^*(t, u) - \alpha\} \\ &\quad \le P\{h^*\big(s + t + C_6\kappa(t), u\big) \le -\alpha/2\} + P\{h^*(t, u) \ge \alpha/2\} \\ &\quad \le K_2 \exp\Big[-\frac{K_3\alpha^2/4}{s + t + C_6\kappa(t) + \alpha}\Big]\Big] + \exp[-K_1\alpha/2] \text{ (by (3.84) and (3.83))} \\ &\quad \le (K_2 + 1)\exp[-K_{15}\alpha] \end{aligned}$$

on $\{\alpha \ge 4\sqrt{d}C_1 t\}$ and $t \ge s \ge s_0$. Thus the second piece of the integral is bounded by

$$\int_{4\sqrt{d}C_1 t}^{\infty} (K_2 + 1)\exp[-K_{15}\alpha] d\alpha \le K_{16}.$$

Returning to (3.94) we now find that for $s_0 \le s \le t$ but $t \log t > C_7 s^2$,

$$\begin{aligned} EX\big(s + t + C_6\kappa(t)\big) - EX(t) \le{}& 2\sqrt{d}C_1[s + C_6\kappa(t)] \\ &+ K_{14}\kappa(t) + K_{16} \le K_{17}\kappa(t). \end{aligned}$$

Therefore, by raising C_9, if necessary, we obtain that (3.91) holds for all $s_0 \le s \le t$.

We shall next use a small variation on the argument of [Ha] to show that (3.91) implies

$$\lambda(u) := \lim_{t\to\infty} \frac{1}{t} Eh^*(t, u)) \text{ exists.} \tag{3.95}$$

It suffices for (3.95) to show that

$$\lim_{t\to\infty} \frac{1}{t} EX(t) = 2\sqrt{d}C_1 - \lambda(u), \tag{3.96}$$

because

$$\lim_{t\to\infty} P\{h^*(t, u) \ge 2\sqrt{d}C_1 t\} = 0 \text{ and } \lim_{t\to\infty} \frac{1}{t} E\{h^*(t, u); h^*(t, u) \ge 2\sqrt{d}C_1 t\} = 0,$$

by virtue of (3.83). Now define for any $M \ge e$,

$$t_0(M) = M,\ t_{k+1}(M) = 2t_k(M) + C_6\kappa(t_k(M)).$$

Note that $t_{k+1}/t_k > 2$, and hence $t_k(M) \ge 2^k M$, and for large k

$$1 < \frac{t_{k+1}(M)}{2t_k(M)} \le 1 + C_6\Big(\frac{k\log 2 + \log M}{2^k M}\Big)^{1/2},$$

and for some K_{18}, independent of $k \geq 0$,

$$1 \leq \prod_{j=0}^{k-1} \frac{t_{j+1}(M)}{2t_j(M)} = \frac{t_k(M)}{M2^k} \leq 1 + K_{18}\Big[\frac{\log M}{M}\Big]^{1/2}. \tag{3.97}$$

Also, by (3.91), for all $M \geq s_0 + e$,

$$EX(t_k(M)) \leq 2EX(t_{k-1}(M)) + C_9\kappa\big(t_{k-1}(M)\big), \; k \geq 1.$$

Consequently, by induction on k,

$$\begin{aligned}\frac{EX(t_k(M))}{t_k(M)} &\leq \frac{EX(M)}{M} \prod_{j=1}^{k} \frac{2t_{j-1}(M)}{t_j(M)} \\ &\quad + \frac{C_9}{2} \sum_{\ell=1}^{k} \frac{\kappa\big(t_{k-\ell}(M)\big)}{t_{k-\ell}(M)} \prod_{j=k-\ell+1}^{k} \frac{2t_{j-1}(M)}{t_j(M)} \\ &\leq \frac{EX(M)}{M} + K_{19}\frac{[\log M]^{1/2}}{M^{1/2}}, \quad k \geq 0.\end{aligned}$$

In particular, $\liminf_{s\to\infty} EX(s)/s < \infty$. Moreover, for given $\varepsilon > 0$ we can choose $M \geq s_0 + e$ so large, that

$$K_{19}[\log M]^{1/2}M^{-1/2} < \varepsilon \text{ and } EX(M)/M \leq \liminf_{s\to\infty} EX(s)/s + \varepsilon.$$

Then

$$\frac{EX(t_k(M))}{t_k(M)} \leq \liminf_{s\to\infty} \frac{EX(s)}{s} + 2\varepsilon, \quad k \geq 0. \tag{3.98}$$

Now let $q_0 \geq s_0 + M$ be large. We shall expand q_0 as a sum of the form $\sum t_{k(i)}$ plus some error terms (see (3.100)) and obtain a corresponding bound for $EX(q_0)$ in (3.99). We define $k(1)$ as the unique integer k for which $t_k \leq q_0 < t_{k+1}$. We distinguish two cases. We are in the first case if $q_0 \geq t_{k(1)} + C_6\kappa\big(t_{k(1)}\big) + s_0 + M$. In this case we set $q_1 = q_0 - t_{k(1)} - C_6\kappa\big(t_{k(1)}\big) < t_{k(1)+1} - t_{k(1)} - C_6\kappa\big(t_{k(1)}\big) = t_{k(1)}$. Then $s_0 + M \leq q_1 < t_{k(1)}$ and

$$EX(q_0) \leq EX\big(t_{k(1)}\big) + EX(q_1) + C_9\kappa(q_0),$$

by virtue of (3.91). If $t_{k(1)} \leq q_0 < t_{k(1)} + C_6\kappa\big(t_{k(1)}\big) + s_0 + M$, then, as in (3.93), (3.94), and the estimates following (3.94)

$$\begin{aligned}EX(q_0) &\leq EX\big(t_{k(1)}\big) + 2\sqrt{d}C_1[q_0 - t_{k(1)}] \\ &\qquad + \int_0^\infty P\{h^*(q_0,u) - h^*\big(t_{k(1)},u\big) \leq -\alpha\}d\alpha \\ &\leq EX\big(t_{k(1)}\big) + 2\sqrt{d}C_1[q_0 - t_{k(1)}] + K_{20}[q_0]^{1/2} \\ &\leq EX\big(t_{k(1)}\big) + K_{21}\kappa(q_0)\end{aligned}$$

for suitable large constants K_{20}, K_{21}. If we are in the first case, we repeat the above procedure with q_0 replaced by q_1. That is, we find $k(2)$ such that

$t_{k(2)} \le q_1 < t_{k(2)+1}$ etc. We continue to determine $k(i)$ and q_i until for the first time q_i is in the second case, i.e., $t_{k(i+1)} \le q_i < t_{k(i+1)} + C_6\kappa\big(t_{k(i+1)}\big) + s_0 + M$. Suppose this first happens at the index i_0. We then have

$$\begin{aligned}
EX(q_0) &\le EX\big(t_{k(1)}\big) + EX(q_1) + C_9\kappa(q_0) \le \cdots \\
&\le \sum_{i=1}^{i_0+1} EX\big(t_{k(i)}\big) + (C_9 + K_{21}) \sum_{i=0}^{i_0} \kappa(q_i) \\
&\le \sum_{i=1}^{i_0+1} EX\big(t_{k(i)}\big) + (C_9 + K_{21}) \Big[\kappa(q_0) + \sum_{i=1}^{i_0} \kappa\big(t_{k(i)}\big)\Big].
\end{aligned} \tag{3.99}$$

Note that by construction, $q_i < t_{k(i)}$ for $1 \le i \le i_0$, and consequently, $k(i+1) < k(i)$ for $i < i_0$. Therefore the above procedure ends at a finite i_0, and

$$\begin{aligned}
&(C_9 + K_{21}) \Big[\kappa(q_0) + \sum_{i=1}^{i_0} \kappa\big(t_{k(i)}\big)\Big] \\
&\qquad \le (C_9 + K_{21}) \Big[\kappa(q_0) + \sum_{k: t_k \le q_0} \kappa(t_k)\Big] \le K_{22} \big[q_0 \log q_0\big]^{1/2}.
\end{aligned}$$

In addition we have either $i_0 = 0$ and $q_0 \ge t_{k(1)}$, or $i_0 \ge 1$ and

$$q_0 = t_{k(1)} + C_6\kappa\big(t_{k(1)}\big) + q_1 = \cdots = \sum_{i=1}^{i_0} \big[t_{k(i)} + C_6\kappa\big(t_{k(i)}\big)\big] + q_{i_0} \ge \sum_{i=1}^{i_0+1} t_{k(i)}. \tag{3.100}$$

Finally, we note that by definition of i_0, $q_{i-1} \ge s_0 + M$, and therefore $t_{k(i)} \ge M$, for $i \le i_0$. (3.99) and (3.98) now show that

$$\begin{aligned}
\frac{EX(q_0)}{q_0} &\le \frac{1}{q_0} \sum_{i=1}^{i_0} t_{k(i)} \Big[\liminf_{s\to\infty} \frac{EX(s)}{s} + 2\varepsilon\Big] \\
&\quad + K_{22}\Big[\frac{\log q_0}{q_0}\Big]^{1/2} + I[t_{k(i_0+1)} < M] \frac{\max_{j<M} EX(j)}{q_0},
\end{aligned}$$

whence

$$\limsup_{q\to\infty} \frac{EX(q)}{q} \le \liminf_{s\to\infty} \frac{EX(s)}{s} + 3\varepsilon.$$

Thus the limit in (3.96) exists and we can use (3.96) to define $\lambda(u)$.

We next turn our attention to the second moments. We shall denote the variance of a random variable Z by $\sigma^2(Z)$. By taking $s = t_k$ and $K = 4$ in (3.92), expanding the square in the right-hand side and a little algebra we obtain

$$\begin{aligned}
\sigma^2\big(X(t_{k+1})\big) &\le 2\sigma^2\big(X(t_k)\big) + Y^2(t_k) + 4Y(t_k)EX(t_k) + 2E\big[X(t_k)X'(t_k)\big] \\
&\quad + 2\big[EX(t_k)\big]^2 - \big[EX(t_{k+1})\big]^2 + 4K_{12}[t_k]^{-K+1}
\end{aligned} \tag{3.101}$$

$$\le 2\sigma^2\big(X(t_k)\big) - \Big[\big[EX(t_{k+1})\big]^2 - \big[2EX(t_k)\big]^2\Big] + K_{23}[t_k]^{3/2}[\log t_k]^{1/2}$$

(see [Ha, p. 676] or [SW, pp. 21, 22]). For the last inequality we used that $X(t_k)$ and $X'(t_k)$ are i.i.d., that $Y(t_k)$ has the constant value $2\sqrt{d}C_1C_6\kappa(t_k)$, and that $EX(t_k)$ is bounded by a multiple of t_k (by virtue of (3.96)). As shown in [Ha, p. 676] or [SW, pp. 21, 22], (3.101), (3.97) and the boundedness of $EX(t)/t$ immediately give for any $M \ge$ some s_5

$$\sum_{k=0}^{\infty} \frac{\sigma^2\big(X(t_k(M))\big)}{(M2^k)^2} < \infty.$$

Since $t_k(M)/(M2^k) \ge 1$ (see (3.97)) we even have

$$\sum_{k=0}^{\infty} \frac{\sigma^2\big(X(t_k(M))\big)}{[t_k(M)]^2} < \infty, \tag{3.102}$$

and hence for any $\varepsilon > 0$,

$$\sum_{k=0}^{\infty} P\Big\{\frac{1}{t_k(M)}\Big|X\big(t_k(M)\big) - (2\sqrt{d}C_1 - \lambda(u)\big)t_k(M)\Big| \ge \varepsilon\Big\} < \infty$$

(see (3.96)). By (3.83) also

$$\sum_{k=0}^{\infty} P\{X\big(t_k(M)\big) \ne 2\sqrt{d}C_1t_k(M) - h^*\big(t_k(M), u\big)\} < \infty,$$

so that for each fixed $M \ge s_5$ and $u \in S^{d-1}$,

$$\sum_{k=0}^{\infty} P\Big\{\Big|\frac{h^*(t_k(M), u)}{t_k(M)} - \lambda(u)\Big| \ge \varepsilon\Big\} < \infty. \tag{3.103}$$

Of course (3.103) implies $h^*(t_k(M), u)/t_k(M) \to \lambda(u)$, almost surely. Since $X(s) \ge 0$ by definition, $2\sqrt{d}C_1 - \lambda(u) \ge 0$ in (3.96), and hence $\lambda(u) \le 2\sqrt{d}C_1$, as claimed. Finally, $\lambda(u) \ge C_4$ follows from Lemma 2 and the almost sure convergence of $h^*(t_k(M), u)/t_k(M)$ to $\lambda(u)$. In fact, (3.16) shows that almost surely $h^*(t_k(M), u) = h\big(t_k(M), u, -C_5\kappa(t_k(M)\big) \ge C_4t_k(M)$ for all large k.

Now choose a large $M_0 \ge s_5$ and for some large integer r take $M_i = M_02^{i/r}, i = 0, 1, \dots, r-1$. Further take $M_r = t_1(M_0)$ and note that $M_{i+1}/M_i \to 2^{1/r}$ as $M_0 \to \infty$ for fixed r and $0 \le i \le r-1$ (since $t_1(M)/M \to 2$ as $M \to \infty$). For given $\eta > 0$ we can therefore first choose r large, such that $1 < 2^{4/r} < 1+\eta$, and then M_0 so large that

$$2^{1/(2r)} \le \frac{M_{i+1}}{M_i} \le 2^{2/r},\ 0 \le i \le r-1.$$

By (3.97) we may further take M_0 so large that

$$2^{-1/(4r)}\frac{M}{M'} \le \frac{t_k(M)}{t_k(M')} \le 2^{1/r}\frac{M}{M'}, \text{ for } M \ge M' \ge M_0,\ k \ge 0.$$

Once these choices have been made we take for $\{n_j\}_{j\ge 0}$ the collection of all distinct $t_k(M_i), 0 \le i \le r-1, k \ge 0$, arranged in increasing order. Note that i only runs to $r-1$ here. We claim that the collection $\{n_j\}$ in increasing order is $\{M_0, M_1, \dots, M_{r-1}, t_1(M_0), \dots, t_1(M_{r-1}), t_2(M_0), \dots\}$. To verify this we merely need to check that $t_k(M_0) > t_{k-1}(M_{r-1})$, since the other orderings are obvious from the monotonicity of $t_j(\cdot)$. However, $t_k(M_0) = t_{k-1}(t_1(M_0)) > t_{k-1}(M_{r-1})$ is also easy from $t_1(M_0) \ge 2M_0 > M_{r-1}$. This proves our claim.

By construction we now have for all $j \ge 0$,

$$\begin{aligned}(3.104)\quad 2^{1/(4r)} &\le 2^{-1/(4r)} \inf\Big\{\frac{t_k(M_{i+1})}{t_k(M_i)} : k \ge 0,\ 0 \le i \le r-1\Big\} \le \frac{n_{j+1}}{n_j}\\ &\le 2^{1/r} \sup\Big\{\frac{t_k(M_{i+1})}{t_k(M_i)} : k \ge 0,\ 0 \le i \le r-1\Big\} \le 2^{4/r} \le 1+\eta.\end{aligned}$$

These inequalities show (3.80) holds, so that n_j increases exponentially with j.

Next, (3.81) holds, because by (3.103)

$$\begin{aligned}(3.105)\quad &\sum_{k=0}^{\infty} P\Big\{\Big|\frac{1}{n_k}h^*(n_k,u) - \lambda(u)\Big| > \varepsilon\Big\}\\ &\qquad = \sum_{i=0}^{r-1}\sum_{k=0}^{\infty} P\Big\{\Big|\frac{1}{t_k(M_i)}h^*(t_k(M_i),u) - \lambda(u)\Big| > \varepsilon\Big\} < \infty.\end{aligned}$$

Thus also

$$(3.106)\qquad \lim_{k\to\infty} \frac{1}{n_k} h^*(n_k,u) = \lambda(u) \text{ a.s.}$$

Now let $0 < \varepsilon \le C_4/4 \le \lambda(u)/4$ and $2\eta\lambda(u) < \varepsilon/2$. Also, let $\{n_k\}$ be a sequence satisfying $1 < n_{j+1}/n_j \le 1+\eta \le 2$ for $j \ge 1$ as well as (3.81). Let $0 < K_{24} < \infty$ be a constant and assume further that

$$(3.107)\qquad \lambda(u) - 2\varepsilon \le \frac{h^*(n_k,u)}{n_k} - \varepsilon \le \frac{1}{n_k(1+\eta)}\big[h^*(n_k,u) - K_{24}\kappa(n_k)\big]$$

and

$$(3.108)\qquad \frac{1+\eta}{n_{k+1}}\big[h^*(n_{k+1},u) + K_{24}\kappa(n_{k+1})\big] \le \frac{h^*(n_{k+1},u)}{n_{k+1}} + \varepsilon \le \lambda(u) + 2\varepsilon.$$

If further

$$\begin{aligned}(3.109)\quad 0 &\le h^*(n_k,u) - K_{24}\kappa(n_k)\\ &\le h^*(t,u) \le h^*(n_{k+1},u) + K_{24}\kappa(n_{k+1}) \quad \text{for all } n_k \le t \le n_{k+1},\end{aligned}$$

then, for these t

$$
\begin{aligned}
(3.110)\quad \lambda(u)-2\varepsilon &\le \frac{h^*(n_k,u)}{n_k}-\varepsilon \le \frac{1}{n_k(1+\eta)}\big[h^*(n_k,u)-K_{24}\kappa(n_k)\big]\\
&\le \frac{h^*(t,u)}{t} \le \frac{1}{n_k}\big[h^*(n_{k+1},u)+K_{24}\kappa(n_{k+1})\big]\\
&\le \frac{1+\eta}{n_{k+1}}\big[h^*(n_{k+1},u)+K_{24}\kappa(n_{k+1})\big]\\
&\le \frac{h^*(n_{k+1},u)}{n_{k+1}}+\varepsilon \le \lambda(u)+2\varepsilon.
\end{aligned}
$$

Now we know already that (3.107) and (3.108) hold a.s. for all large k and for our choice of η. Moreover, by (3.85) and (3.86)

$$
\begin{aligned}
&P\{(3.109) \text{ fails for some } n_k \le t \le n_{k+1}\}\\
&\quad\le P\{\inf_{r\le n_{k+1}-n_k} h^*(n_k+r,u)-h^*(n_k,u)\le -K_{24}\kappa(n_k)\}\\
&\qquad + P\{\sup_{r\le n_{k+1}-n_k} h^*(n_k+r,u)-h^*(n_{k+1},u)\ge K_{24}\kappa(n_{k+1})\}\\
&\quad\le 2\frac{K_4}{n_k^K}+\big(16d+K_5n_{k+1}\big)^d\exp\Big[-\frac{K_3[K_{24}\kappa(n_k)]^2}{4n_k+4K_{24}\kappa(n_k)}\Big].
\end{aligned}
$$

Since the n_k grow exponentially we can choose K_{24} so large that

$$
\sum_{k=0}^{\infty} P\{(3.109) \text{ fails for some } n_k \le t \le n_{k+1}\} < \infty.
$$

Thus almost surely (3.109) and (3.110) fail only for finitely many k, and $\lambda(u)-2\varepsilon \le h^*(t,u)/t \le \lambda(u)+2\varepsilon$ holds for all large t. Since $\varepsilon>0$ was arbitrary this proves the almost sure convergence in (3.79). The L^p convergence along all reals in (3.79) follows from the almost sure convergence and the tail estimates (3.83) and (3.84). □

4. From half-space to full-space processes

The goal for this section is to prove that the B-particles in the full-space process do not spread faster than in the appropriate half-space process (see Corollary 8 for a precise statement). The first lemma establishes that for every $u\in S^{d-1}$ there are deterministic vectors V_k such that for all $\eta>0$ there is, with a probability close to 1, a B-particle in $\mathcal{P}^{\mathrm{h}}\big(u,-C_5\kappa((1+\eta)n_k)\big)$ 'near' V_k at time n_k, for all large k. Here n_k is the $n_k(\eta)$ of Corollary 5 and $\langle V_k,u\rangle$ has to grow essentially like $h^*(n_k,u)\sim n_k\lambda(u)$ (see (4.1)). Apart from this growth condition the behavior of V_k as a function of k,u is unimportant for us. The only important aspect is that it is nonrandom, so that we can find, with high probability, a B-particle in a nonrandom location at which $h^*(n_k, u)$ is (almost) achieved. This will be used in the second lemma to concatenate $\mathcal{P}^{\mathrm{h}}\big(u, -C_5\kappa(n_k)\big)$ with another process which runs from time $(1+\eta)n_k$ to $(1+\eta)n_k + r_k$ with r_k also of order n_k. By starting the second process at the space-time point $\big(V_k, (1+\eta)n_k\big)$ we will be able to assure that a B-particle at time $(1+\eta)n_k + r_k$ in the second process is also a B-particle in

$$\mathcal{P}^{\mathrm{h}}\big(u, -C_5\kappa((1+\eta)n_k + r_k)\big).$$

LEMMA 6. *Let $u \in S^{d-1}$ be fixed, and let $n_k = n_k(\eta)$ be as in Corollary* 5. *Then, for all $0 < \eta < C_4/(8C_1)$ there exists a deterministic sequence of vectors $\{V_k\} = \{V_k(\eta, u)\}$ such that*

$$\langle V_k(\eta, u), u\rangle = n_k(\eta)\lambda(u), \tag{4.1}$$

and such that

(4.2)

$$\sum_{k=0}^{\infty} P\{\textit{in } \mathcal{P}^{\mathrm{h}}\big(u, -C_5\kappa(n_k)\big) \textit{ there is at time } (1+\eta)n_k \textit{ either no particle at all}$$
$$\textit{in } V_k + \mathcal{C}(C_2 n_k \eta/4) \textit{ or there is an A-particle in } V_k + \mathcal{C}(C_2 n_k\eta/4)\}$$
$$< \infty.$$

Proof. Fix $u \in S^{d-1}$ and $\varepsilon > 0$. Let σ be a time which is so large that $\sigma \geq s_0$ (with s_0 as in Proposition 3) and such that

$$\Big|\frac{1}{\sigma} E h^*(\sigma, u) - \lambda(u)\Big| \leq \frac{1}{4}\varepsilon \tag{4.3}$$

(see (3.27) for h^*). Define the further times

$$\sigma_1 = \sigma, \sigma_{j+1} = \sigma + \sigma_j + C_6\kappa(\sigma_j),\ j \geq 1.$$

Now apply (3.31) with the following choices: $s = \sigma, t = \sigma_j, \gamma = \gamma_s = Em^*(\sigma, u)$ (see (3.29) for m^*) and $\gamma_t = jEm^*(\sigma, u)$. This yields

$$\begin{aligned}
&P\{\mathcal{G}\big(\alpha, \beta, (j+1)Em^*(\sigma, u), \mathcal{P}^{\mathrm{h}}\big(u, -C_5\kappa(\sigma_{j+1})\big), \sigma_{j+1}\big)\\
&\quad\geq \int_{h\in\mathbb{R}}\int_{m\in\mathbb{R}^d} P\{h^*(\sigma, u) \in dh, m^*(\sigma, u) \in \gamma + dm\}\\
&\qquad\times P\{\mathcal{G}\big(\alpha - h, \beta - d, j\gamma - m, \mathcal{P}^{\mathrm{h}}\big(u, -C_5\kappa(\sigma_j)\big), \sigma_j\big)\} - C_8\sigma^{-K-1},
\end{aligned}$$

provided (3.30) holds, that is, provided $(\sigma_j + 1)\log(\sigma_j + 1) \leq C_7\sigma^2$. We start with $j = r-1$, then use the case $j = r-2$ with α, β replaced by $\alpha - h$ and $\beta - d$, respectively, etc., all the way down to $j = 1$. With $(h_j^*, m_j^*),\ j \geq 1$, i.i.d. copies of $\big(h^*(\sigma, u), m^*(\sigma, u)\big)$ we obtain

(4.4)

$$
\begin{aligned}
&P\{\mathcal{G}\big(\alpha,\beta,rEm^*(\sigma,u),\mathcal{P}^{\mathrm{h}}\big(u,-C_5\kappa(\sigma_r)\big),\sigma_r\big)\}\\
&\quad\ge \int_{\substack{h_j\in\mathbb{R},\\ 1\le j\le r-1}}\int_{\substack{m_j\in\mathbb{R}^d\\ 1\le j\le r-1}}\prod_{j=1}^{r-1}P\{h^*(\sigma,u)\in dh_j, m^*(\sigma,u)\in\gamma+dm_j\}\\
&\qquad\times P\Big\{\mathcal{G}\big(\alpha-\sum_{j=1}^{r-1}h_j,\beta-(r-1)d,\gamma-\sum_{j=1}^{r-1}m_j,P^{\mathrm{h}}\big(u,-C_5\kappa(\sigma)\big),\sigma\big)\Big\}\\
&\qquad-(r-1)C_8\sigma^{-K-1}\\
&\quad= P\Big\{\text{in }\mathcal{P}^{\mathrm{h}}\big(u,-C_5\kappa(\sigma)\big)\text{ there is at time }\sigma\text{ a }B\text{-particle at some }x\text{ with}\\
&\qquad\qquad\langle x,u\rangle+\sum_{j=1}^{r-1}h_j^*\ge\alpha\text{ and }\Big\|x^\perp+\sum_{j=1}^{r-1}m_j^*-r\gamma\Big\|\le\beta-(r-1)d\Big\}\\
&\qquad-(r-1)C_8\sigma^{-K-1}\\
&\quad\ge P\Big\{\sum_{j=1}^{r}h_j^*\ge\alpha,\Big\|\sum_{j=1}^{r}(m_j^*-\gamma)\Big\|\le\beta-(r-1)d\Big\}-(r-1)C_8\sigma^{-K-1},
\end{aligned}
$$

provided

$$(\sigma_r+1)\log(\sigma_r+1)\le C_7\sigma^2. \tag{4.5}$$

It is easy to see by induction that each σ_j is a continuous, increasing function of σ on $[0,\infty)$. We further see by induction that $\sigma_k\ge k\sigma$ and σ_j increases with j. Finally, we can for any fixed $\sigma\ge 1$ find a $K_1=K_1(\sigma)$ such that

$$\sigma K_1 2k(\log k+1)\ge C_6\kappa\big(\sigma K_1k^2(\log k+1)\big),\quad k\ge 1,$$

and $\sigma_1\le\sigma K_1\log 2$. One more induction argument then shows that for all $k\ge 1$, $\sigma_k\le\sigma K_1k^2(\log k+1)$. Now let $s\ge s_0$ be large and take $r=\lfloor s^{1/3}\rfloor$. The preceding argument shows that we can fix σ such that $\sigma_r=s$. Thus for $j-1\le r$ we have $\sigma_{j-1}\le\sigma_r=s$ and $\sigma_j\le\sigma+\sigma_{j-1}+C_6\kappa(s)$. Consequently, $r\sigma\le s=\sigma_r\le r\sigma+rC_6\kappa(s)=r\sigma+\lfloor s^{1/3}\rfloor C_6\kappa(s)=r\sigma+o(s)$, and necessarily $\sigma\sim s/r\sim s^{2/3}$ for large s. (4.5) is therefore automatically satisfied. Now let $h_j^*,j\ge 1$, be i.i.d. copies of $h^*(\sigma,u)$. If we further take

$$\alpha=r\sigma[\lambda(u)-\frac{1}{2}\varepsilon],$$

then, by (4.3) and the fact that Variance $(h_j^*)\le K_2\sigma^2$ (by (3.90)),

$$P\Big\{\sum_{j=1}^{r}h_j^*\le\alpha\Big\}\le P\Big\{\sum_{j=1}^{r}\big(h_j^*-Eh_j^*\big)\le -r\sigma\varepsilon/4\Big\}\le\frac{K_3}{r\varepsilon^2}. \tag{4.6}$$

Further, fix s_6 so large that $2s\varepsilon \geq r\sigma\varepsilon \geq (1/2)s\varepsilon \geq 2rd \sim 2s^{1/3}d$ for $s \geq s_6$. Then we have similarly to (4.6), for $s \geq s_6, \beta = s\varepsilon$ and $\gamma = Em^*(\sigma, u)$,

(4.7)
$$P\big\{\big\|\sum_{j=1}^{r}\big(m_j^* - \gamma\big)\big\| > \beta - (r-1)d\big\} \leq P\big\{\big\|\sum_{j=1}^{r}\big(m_j^* - \gamma\big)\big\| > r\sigma\varepsilon/4\big\} \leq \frac{K_4}{r\varepsilon^2}.$$

The last two inequalities provide us with a lower bound for the right-hand side of (4.4). We conclude that for $s \geq s_6$

$$P\{\mathcal{G}\big(\alpha, \beta,\ rEm^*(\sigma, u), \mathcal{P}^{\mathrm{h}}\big(u, -C_5\kappa(\sigma_r)\big), \sigma_r\big)\} \geq 1 - \frac{(K_3 + K_4)}{r\varepsilon^2} - (r-1)C_8\sigma^{-K-1} \geq 1 - \frac{K_5}{s^{1/3}\varepsilon^2} \tag{4.8}$$

(use any $K \geq 1$). Let $n_j(\eta)$ be as in Corollary 5, and take $s = n_k = n_k(\eta)$. In agreement with our previous choice for r, σ we then take $r = \lfloor n_k^{1/3}(\eta)\rfloor$ and σ such that $\sigma_r = n_k(\eta)$. Then, by going over to the complementary event in (4.8), we find for any $\eta > 0$, that

$$\begin{aligned}\sum_{k=0}^{\infty} P\big\{&\text{in } \mathcal{P}^{\mathrm{h}}\big(u, -C_5\kappa(n_k)\big) \text{ there is at time } n_k \text{ no } B\text{-particle} \\ &\text{in } \Gamma\big(n_k[\lambda(u) - \tfrac{1}{2}\varepsilon], n_k\varepsilon, rEm^*(\sigma, u)\big)\big\} \\ &\leq \sum_{k=0}^{\infty}\frac{K_5}{n_k^{1/3}\varepsilon^2} < \infty\end{aligned} \tag{4.9}$$

(recall that the n_j grow exponentially). But (3.81) says in particular that

$$\begin{aligned}\sum_{k=0}^{\infty} P\big\{&\text{in } \mathcal{P}^{\mathrm{h}}\big(u, -C_5\kappa(n_k)\big) \text{ there is at time } n_k \text{ a } B\text{-particle} \\ &\text{in } \Gamma\big(n_k[\lambda(u) + \tfrac{1}{2}\varepsilon], n_k\varepsilon, rEm^*(\sigma, u)\big)\big\} \\ &< \infty.\end{aligned} \tag{4.10}$$

We now take

$$V_k = V_k(\eta, u) = n_k(\eta)\lambda(u)u + rEm^*(\sigma, u). \tag{4.11}$$

Note that r and σ are determined by k and η, so that V_k really is a function of k, η, u. Since m^* is orthogonal to u (by definition (3.29)), this choice of V_k satisfies (4.1). Moreover, (4.9) and (4.10) together give

(4.12)
$$\begin{aligned}\sum_{k=0}^{\infty} P\{&\text{in } \mathcal{P}^{\mathrm{h}}\big(u, -C_5\kappa(n_k)\big) \text{ there is at time } n_k \text{ no } B\text{-particle} \\ &\text{at any site } x \in V_k + \mathcal{C}(2\sqrt{d}n_k\varepsilon)\}\end{aligned}$$

$$\leq \sum_{k=0}^{\infty} P\{\text{in } \mathcal{P}^{\mathrm{h}}\big(u, -C_5\kappa(n_k)\big) \text{ there is at time } n_k \text{ no } B\text{-particle at any site } x$$
$$\text{with } \langle x, u\rangle \in \Big[n_k[\lambda(u) - \frac{\varepsilon}{2}], n_k[\lambda(u) + \frac{\varepsilon}{2}]\Big], \|x^{\perp} - rEm^*(\sigma, u)\| \leq n_k\varepsilon\}$$
$$< \infty.$$

The convergence of the sums in (4.12) shows that almost surely, for all large $n_k(\eta)$, there is in $\mathcal{P}^{\mathrm{h}}\big(u, -C_5\kappa(n_k)\big)$ a B-particle in $V_k + \mathcal{C}(2\sqrt{d}n_k\varepsilon)$ at time $n_k(\eta)$. We claim that this implies that if we take $\varepsilon = C_2\eta/(16d)$, then, in $\mathcal{P}^{\mathrm{h}}\big(u, -C_5\kappa(n_k)\big)$ at time $(1+\eta)n_k$, all occupied sites in $V_k + \mathcal{C}(C_2 n_k\eta/4)$ are occupied by B-particles (and there are such occupied sites). More precisely, we claim that (4.2) holds. To see this we apply Lemma 4 with the following choices: $s = n_k, \widetilde{s} = (1+\eta)n_k$, and finally $y(n_k)$ is the location of any B-particle in $\mathcal{P}^{\mathrm{h}}\big(u, -C_5\kappa(n_k)\big)$ at time n_k in the set $V_k + \mathcal{C}\big(C_2 n_k\eta/(8\sqrt{d})\big)$, if such a B-particle exists. If several such B-particles exist we pick the location of one of them according to some deterministic rule chosen in advance. On the event that no such B-particle exists we cannot apply Lemma 4, but this does not cause any problems, because (4.12) already tells us that

$$\sum_{k=0}^{\infty} P\{\text{no choice for } y(n_k) \text{ exists}\} < \infty. \tag{4.13}$$

If $y(n_k)$ exists, then there is automatically a particle in $\mathcal{P}^{\mathrm{h}}\big(u, -C_5\kappa(n_k)\big)$ at time n_k at $y(n_k) \in V_k + \mathcal{C}\big(C_2 n_k\eta/(8\sqrt{d})\big)$. If this particle does not move a distance $> C_2 n_k\eta/8$ during $[n_k, (1+\eta)n_k]$, then it is in $y(n_k) + \mathcal{C}(C_2 n_k\eta/8) \subset V_k + \mathcal{C}(C_2 n_k\eta/4)$ at time $(1+\eta)n_k$. We recall further that all particles in $\mathcal{P}^{\mathrm{h}}\big(u, -C_5\kappa(n_k)\big)$ are also particles in $\mathcal{P}^{\mathrm{f}}$. It follows that the k-th summand in (4.2) is bounded by the k-th summand in (4.13) plus

$$P\{\|S_{n_k\eta}\| > C_2 n_k\eta/8\} + P\{\mathcal{B}^{\mathrm{h}}\big(y(n_k), n_k; u, -C_5\kappa(n_k)\big) \cap \mathcal{K}\big(y(n_k)\big)\} \tag{4.14}$$

(see (3.14) for $\mathcal{B}^{\mathrm{h}}$ and (3.72) for $\mathcal{K}(y)$). The first probability in (4.14) is at most $K_6\exp[-K_7 n_k\eta]$ by (2.42) in [KSa]. The last probability in (4.14) is by Lemma 4 at most

$$P\{y(n_k) \notin \mathcal{C}(2C_1 n_k)\} + P\{\langle y(n_k), u\rangle < \frac{1}{2}C_4 n_k\} + 5n_k^{-K-1}. \tag{4.15}$$

The first probability in (4.15) is $O\big(n_k^{-K-1}\big)$ by the estimates (3.49). The second probability in (4.15) is zero, because, by construction, $y(n_k) \in V_k + \mathcal{C}\big(C_2 n_k\eta/(8\sqrt{d})\big)$, so that

$$\begin{aligned}\langle y(n_k), u\rangle &\geq \langle V_k, u\rangle - C_2 n_k\eta/8\\ &= n_k\lambda(u) - C_2 n_k\eta/8 \geq n_k(C_4 - C_2\eta/8) \text{ (see Corollary 5) } \geq \frac{1}{2}C_4 n_k.\end{aligned}$$

It follows that the sum of (4.15) over k is also finite, and this proves (4.2). □

The method of proof of the next lemma will be used again in Lemma 10; it shows how to concatenate two processes, as outlined before the last lemma.

LEMMA 7. *Define*

(4.16)
$$H(t,u) = h(t,u,-\infty) \\ = \max\{\langle x,u\rangle : x \textit{ is occupied by a } B\textit{-particle in } \mathcal{P}^{\mathrm{f}} \textit{ at time } t\}.$$

Assume that for some fixed $u \in S^{d-1}$ *and* $\mu \geq 0$

$$P\{\limsup_{t\to\infty} \frac{1}{t} H(t,u) \geq \mu\} > 0. \tag{4.17}$$

Then

$$\lambda(u) \geq \mu. \tag{4.18}$$

Proof. We divide the proof into four steps. We shall introduce events $\mathcal{L}'_k$ which occur if the full-space process started at $(V_k, (1+\eta)n_k)$ has a B-particle in a certain half-space at time $\big(1+\eta+K_2)n_k(\eta)\big)$. The assumption (4.17) says that the full-space process visits certain half-spaces infinitely often. In Step 1 we show that (4.17) and a kind of maximal inequality imply that slightly larger half-spaces must be visited infinitely many times from the sequence $\{(1+\eta+K_2)n_k\}$. Borel-Cantelli and a translation in space-time immediately deduce from this that $\sum_k P\{\mathcal{L}'_k\} = \infty$.

We also introduce events $\mathcal{M}_k$ which are almost the same as the $\mathcal{L}'_k$, except that they depend only on particles which start in certain 'slabs'. These are chosen in such a way that $\mathcal{M}_k$ and $\mathcal{M}_\ell$ depend only on disjoint collections of particles, and are therefore independent, if $|k-\ell| \geq K_5$ for some constant $K_5(\eta)$. We then show in Step 2 that $\sum_k P\{\mathcal{L}'_k\} = \infty$ implies that also $\sum_k P\{\mathcal{M}_k\} = \infty$. By the independence of $\mathcal{M}_k$ and $\mathcal{M}_\ell$ for $|k-\ell| \geq K_5$ this implies that a.s., $\mathcal{M}_k$ occurs infinitely often (see Step 3). In the last step we show that a.s., for all large k, $\mathcal{M}_k$ implies that $h^*\big((1+\eta+K_2)n_k, u\big) \geq [\lambda(u) + K_2(\mu-\varepsilon)]n_k$. Since Corollary 5 tells us that $h^*\big((1+\eta+K_2)n_k, u\big)/(1+\eta+K_2)n_k \to \lambda(u)$ a.s., one concludes that $\lambda(u) \geq K_2(\mu-\varepsilon)/(\eta+K_2)$. As ε and η tend to 0 one obtains the desired (4.18).

Since $\lambda(u) \geq C_4$ we assume without loss of generality that $\mu > 0$. Throughout this proof ε is a small strictly positive number.

Step 1. We choose

$$K_1 > 2\sqrt{d}C_1 \geq \lambda(u), \quad K_1 > \frac{1}{C_4} \geq \frac{1}{\lambda(u)}. \tag{4.19}$$

For each small $\eta > 0$ we then define

$$m_k = m_k(\eta) = K_2 n_k(\eta), \tag{4.20}$$

where $n_k = n_k(\eta)$ has the properties (3.80) and (3.81) of Corollary 5. We take $\eta_0 = \eta_0(\varepsilon) > 0$ so small that

$$1 + \eta_0 \leq \frac{\mu - \varepsilon/2}{\mu - 3\varepsilon/4}.$$

Note that these definitions imply that for $\eta \leq \eta_0$,

$$\frac{m_{k+1}}{m_k} = \frac{n_{k+1}}{n_k} \leq 1 + \eta \leq \frac{\mu - \varepsilon/2}{\mu - 3\varepsilon/4}. \tag{4.21}$$

Further, for small $\varepsilon > 0$, define the events

$$\begin{aligned}
&\mathcal{L}_k(\eta, \mu - \varepsilon) \\
&= \{\text{in } \mathcal{P}^{\mathrm{f}} \text{ there is a } B\text{-particle in the half-space } \mathcal{S}\big(u, m_k(\mu - \varepsilon)\big) \text{ at time } m_k\} \\
&= \{H(m_k, u) \geq m_k(\mu - \varepsilon)\}.
\end{aligned}$$

In this step we shall show that for fixed $\varepsilon > 0$ and all $0 < \eta \leq \eta_0(\varepsilon)$,

$$\sum_{k=0}^{\infty} P\{\mathcal{L}_k(\eta, \mu - \varepsilon)\} = \infty. \tag{4.22}$$

To prove this we shall show that

$$P\{\mathcal{L}_k(\eta, \mu - \varepsilon) \text{ occurs for infinitely many } k\} > 0. \tag{4.23}$$

(4.22) then follows from the Borel-Cantelli lemma. Now, (4.17) implies that for every $\varepsilon > 0$

(4.24)
$$P\{\text{for infinitely many } k, H(t, u) > (\mu - \varepsilon/2)t \text{ for some } t \in [m_k, m_{k+1}]\} > 0.$$

However, by (3.86) with h^* replaced by H (this amounts to taking $C_5 = \infty$, which does not influence the estimate (3.86); see (3.27) for h^*) and with $\alpha = (\varepsilon/4)m_{k+1} \leq (\mu - \varepsilon/2)m_k - (\mu - \varepsilon)m_{k+1}$ (see (4.21)),

$$\begin{aligned}
&P\{H(t, u) \\
&\qquad > (\mu - \varepsilon/2)t \text{ for some } t \in [m_k, m_{k+1}] \text{ but } H(m_{k+1}, u) \leq (\mu - \varepsilon)m_{k+1}\} \\
&\qquad \leq P\Big\{\sup_{r \in [m_k, m_{k+1}]} \big[H(r, u) - H(m_{k+1}, u)\big] \geq (\mu - \varepsilon/2)m_k - (\mu - \varepsilon)m_{k+1}\Big\} \\
&\qquad \leq K_3(\varepsilon, \eta)[m_k]^{-K}.
\end{aligned}$$

In particular, by Borel-Cantelli, the event in the left-hand side here occurs almost surely only finitely often. Together with (4.24) this shows that

$$P\{\text{for infinitely many } k, H(m_{k+1}, u) \geq (\mu - \varepsilon)m_{k+1}\} > 0.$$

This is the required (4.23).

Step 2. The remaining steps are based on (4.22) only; (4.17) itself is not needed. With $V_k = V_k(\eta, u)$ as in (4.11) we define an auxiliary process

$\mathcal{Q}_k = \mathcal{Q}_k(\eta, u)$ which is more or less the full-space process started at the deterministic space-time point $(V_k, (1+\eta)n_k)$. The only difference is that $\mathcal{Q}_k$ only uses the particles which are *at time* 0 in the 'slab'

$$\{x : -n_k/K_1 \le \langle x, u\rangle - n_k\lambda(u) < K_1 n_k\}, \tag{4.25}$$

with K_1 satisfying (4.19). Thus $\mathcal{Q}_k$ is defined only from time $(1+\eta)n_k$ on. At time $(1+\eta)n_k$ it has at any x only the particles which started at time 0 in the set (4.25). If no such particles exist, then there never are any particles in the process $\mathcal{Q}_k$. Otherwise, let z_k be the nearest site to V_k which is occupied at time $(1+\eta)n_k$ by some particle, which at time 0 was in (4.25). The types of all particles in $\mathcal{Q}_k$ at time $(1+\eta)n_k$ are reset to type A, except for the particles at z_k, which are reset to type B. From time $(1+\eta)n_k$ the process then develops by our standard rules. Even though the process $\mathcal{Q}_k$ is defined for all times in $[(1+\eta)n_k, \infty)$ we are only interested in what happens during $[(1+\eta)n_k, (1+\eta)n_k + m_k]$. Specifically, we define the events

(4.26)

$$\begin{aligned}\mathcal{M}_k = \mathcal{M}_k(\eta, \mu-\varepsilon) = \{&\text{in } \mathcal{Q}_k \text{ there is a } B\text{-particle in the half-space}\\ &\mathcal{S}\big(u, n_k\lambda(u) + m_k(\mu-\varepsilon)\big) \text{ at time } (1+\eta)n_k + m_k\}.\end{aligned}$$

In this step we shall prove that

$$\sum_{k=0}^{\infty} P\{\mathcal{M}_k\} = \infty. \tag{4.27}$$

To this end let us shift the event $\mathcal{L}_k$ by $(1+\eta)n_k$ in time and by V_k in space. Then $\mathcal{L}_k$ goes over into the event

$$\begin{aligned}\mathcal{L}'_k := \{&\text{in the full-space process started at } (V_k, (1+\eta)n_k) \text{ there}\\ &\text{is a } B\text{-particle in the half-space } \mathcal{S}\big(u, n_k\lambda(u) + m_k(\mu-\varepsilon)\big)\\ &\text{at time } (1+\eta)n_k + m_k\}\end{aligned}$$

(recall (4.1)). $\mathcal{L}'_k \setminus \mathcal{M}_k$ can occur only if one of the following two events occurs:

$$\begin{aligned}\{&\text{at time } (1+\eta)n_k, \text{ some particle at the nearest occupied site to}\\ &V_k \text{ in the full-space process started at time 0 outside the set (4.25)}\},\end{aligned} \tag{4.28}$$

or

$$\begin{aligned}\{&\text{in the full-space process started at } \big(V_k, (1+\eta)n_k\big) \text{ there is a}\\ &\text{particle which starts at time 0 outside the set (4.25) and}\\ &\text{which coincides with a } B\text{-particle during } [(1+\eta)n_k, (1+\eta)n_k + m_k]\}\end{aligned} \tag{4.29}$$

(compare the argument for (3.47)). It follows that

$$P\{\mathcal{L}'_k \setminus M_k\} \le P\{(4.28) \text{ or } (4.29) \text{ occurs}\}.$$

But

$$(4.30)\quad \begin{aligned} &P\{(4.28) \text{ occurs}\}\\ &\quad\le P\{\text{nearest occupied site to } V_k \text{ in } \mathcal{P}^{\mathrm{f}} \text{ at time } (1+\eta)n_k \text{ has}\\ &\qquad\quad \text{distance more than } K_4 \log k \text{ from } V_k\}\\ &\qquad +P\{\text{some particle which starts at time 0 outside the set (4.25)}\\ &\qquad\quad\ \text{is in } V_k+\mathcal{C}(K_4\log k) \text{ at time } (1+\eta)n_k\}. \end{aligned}$$

Also,

$$(4.31)\quad \begin{aligned} &P\{(4.29) \text{ occurs}\}\\ &\quad\le P\{\text{in the full-space process started at } (V_k,(1+\eta)n_k)\\ &\qquad\quad \text{there are } B\text{-particles outside } V_k+\mathcal{C}(2C_1m_k) \text{ at}\\ &\qquad\quad \text{some time during } [(1+\eta)n_k,(1+\eta)n_k+m_k]\}\\ &\qquad +P\{\text{some particle which starts at time 0 outside the set (4.25)}\\ &\qquad\quad\ \text{visits } V_k+\mathcal{C}(2C_1m_k) \text{ during } [0,(1+\eta)n_k+m_k]\}. \end{aligned}$$

The first probability in the right-hand side of (4.30)) can be made $O\bigl(k^{-K}\bigr)$ for any given K, by choosing K_4 large (compare (3.26)). The second probability in the right-hand side of (4.30) is for large k no more than the second probability in the right-hand side of (4.31). To estimate the latter, we merely point out that a particle which starts at some z outside the set (4.25) and visits $V_k+\mathcal{C}(2C_1m_k)$ during $[0,(1+\eta)n_k+m_k]$ has to move over a distance of at least

$$\begin{aligned} &\|z-V_k\|-2C_1m_k \ge d^{-1/2}|\langle(z-V_k),u\rangle|-2C_1m_k\\ &= d^{-1/2}|\langle z,u\rangle-n_k\lambda(u)|-2C_1m_k \ge n_k/(\sqrt{d}K_1)-2C_1m_k = n_k/(2\sqrt{d}K_1), \end{aligned}$$

by virtue of our choice of m_k. We leave it to the reader to use this to check that the last probability in (4.31) is $O\bigl([n_k]^{-K}\bigr)$ (see also the estimates in (3.24) and (3.50) or (3.75)). Finally, the first probability in the right-hand side of (4.31) equals

$$(4.32)\qquad P\{\text{in } \mathcal{P}^{\mathrm{f}} \text{ there are } B\text{-particles outside } \mathcal{C}(2C_1m_k) \text{ during } [0,m_k]\},$$

and this is $O\bigl([m_k]^{-K-2}\bigr)$, as in (3.49) or (3.25) and the lines following it. It follows from these estimates that $\sum_k P\{\mathcal{L}'_k\setminus\mathcal{M}_k\}<\infty$. From (4.22) and the fact that $P\{\mathcal{L}'_k\}=P\{\mathcal{L}_k\}$, this implies (4.27).

Step 3. In this step we show that

$$(4.33)\qquad P\{\mathcal{M}_k \text{ occurs for infinitely many } k\}=1.$$

This is an easy application of Borel-Cantelli, because $\mathcal{M}_k$ and $\mathcal{M}_\ell$ depend on particles which start at disjoint sets of sites (and are therefore independent) as soon as the set (4.25) and the corresponding set with k replaced by ℓ are disjoint. If $\ell > k$, this is the case if $n_k(\lambda(u) + K_1) < n_\ell(\lambda(u) - 1/K_1)$ and similarly if $k > \ell$. In particular, by (3.80), there is some integer $K_5 = K_5(\eta)$ such that $\mathcal{M}_k$ and $\mathcal{M}_\ell$ are independent as soon as $|k - \ell| \geq K_5$. Moreover, by (4.27), there is some integer $j \in [0, K_5 - 1]$ such that

$$\sum_{k \equiv j \pmod{K_5}} P\{\mathcal{M}_k\} = \infty.$$

Thus (4.33) is true.

Step 4. We now complete the proof of the lemma by showing that, almost surely, for all large k for which $\mathcal{M}_k$ occurs, also

(4.34)
$$\begin{aligned}&\bigl\{\text{in } \mathcal{P}^{\mathrm{h}}\bigl(u, -C_5\kappa((1+\eta)n_k + m_k)\bigr) \text{ there is a } B\text{-particle in the half-space}\\ &\quad \mathcal{S}\bigl(u, n_k\lambda(u) + m_k(\mu - \varepsilon)\bigr) \text{ at time } (1+\eta)n_k + m_k\bigr\}\\ &= \bigl\{h^*\bigl((1+\eta)n_k + m_k, u\bigr) \geq n_k\lambda(u) + m_k(\mu - \varepsilon)\bigr\}\end{aligned}$$

occurs. This will indeed complete the proof, since we already know from Corollary 5 that $\bigl((1+\eta)n_k + m_k\bigr)^{-1} h^*\bigl((1+\eta)n_k + m_k, u\bigr) \to \lambda(u)$. Thus (4.33) and (4.34) will imply, for all $\varepsilon > 0, 0 < \eta < \eta_0(\varepsilon)$,

$$\begin{aligned}\lambda(u) &\geq \liminf_{k\to\infty} \Bigl[\frac{n_k}{(1+\eta)n_k + m_k}\lambda(u) + \frac{m_k}{(1+\eta)n_k + m_k}(\mu - \varepsilon)\Bigr]\\ &= \frac{1}{1+\eta+K_2}\lambda(u) + \frac{K_2}{1+\eta+K_2}(\mu - \varepsilon),\end{aligned}$$

and hence

$$\lambda(u) \geq \frac{K_2}{\eta + K_2}(\mu - \varepsilon). \tag{4.35}$$

Now to prove (4.34), we write, as in the lines following (4.25), z_k for the nearest site to V_k at time $(1+\eta)n_k$ which is occupied by a particle which started at time 0 in (4.25). We already proved that, almost surely, (4.28) occurs only finitely often. Thus, except for finitely many k, z_k actually equals the nearest occupied site to V_k at time $(1+\eta)n_k$ in $\mathcal{P}^{\mathrm{f}}$. Since the set (4.25) is contained in $\mathcal{S}(u, 0) \subset \mathcal{S}\bigl(u, -C_5\kappa(n_k)\bigr)$, z_k is also the nearest occupied site to V_k at time $(1+\eta)n_k$ in $\mathcal{P}^{\mathrm{h}}\bigl(u, -C_5\kappa(n_k)\bigr)$. By virtue of Lemma 6, we further know that, a.s. for all large k, z_k is occupied by B-particles at time $(1+\eta)n_k$ in $\mathcal{P}^{\mathrm{h}}\bigl(u, -C_5\kappa(n_k)\bigr)$. By using the monotonicity property of Lemma C we conclude that, almost surely, for all large k all the B-particles in $\mathcal{Q}_k$ at time $(1+\eta)n_k + m_k$ are also B-particles in $\mathcal{P}^{\mathrm{h}}\bigl(u, -C_5\kappa(n_k)\bigr)$ at time $(1+\eta)n_k + m_k$. In turn, these particles are a.s. B-particles in $\mathcal{P}^{\mathrm{h}}\bigl(u, -C_5\kappa((1+\eta)n_k + m_k)\bigr)$ at time $(1+\eta)n_k + m_k$, by another application of Lemma C and (3.3) (recall that a.s $x_0 = w_{-C_5\kappa(n_k)} = w_{-C_5\kappa\left((1+\eta)n_k+m_k\right)}$ for all large k). In particular,

$$h^*\big((1+\eta)n_k + m_k, u\big) \geq n_k\lambda(u) + m_k(\mu - \varepsilon)$$

for all large k for which $\mathcal{M}_k$ occurs. This is the required (4.34). □

COROLLARY 8. *For every unit vector u*

$$\lim_{t\to\infty} \frac{1}{t} H(t,u) = \lambda(u) \text{ almost surely and in } L^p \text{ for all } p > 0. \tag{4.36}$$

(t runs through the reals here). Moreover, for $n_k = n_k(\eta)$ as in Corollary 5, for any $\delta > 0$ and $\eta > 0$,

$$\sum_{k=0}^{\infty} P\Big\{\Big|\frac{1}{n_k} H(n_k, u) - \lambda(u)\Big| > \delta\Big\} < \infty. \tag{4.37}$$

Proof. By the monotonicity property of Lemma C

$$H(t,u) \geq h^*(t,u) \text{ on the event } \{\|x_0\| \leq C_5\kappa(t)/\sqrt{d}\} \tag{4.38}$$

(see (3.3)). Thus, by the estimate (3.26)

$$\liminf_{t\to\infty} \frac{1}{t} H(t,u) \geq \lim_{t\to\infty} \frac{1}{t} h^*(t,u) = \lambda(u)$$

(see Corollary 5). In the other direction, we have from Lemma 7 that

$$P\Big\{\limsup_{t\to\infty} \frac{1}{t} H(t,u) \geq \mu\Big\} = 0 \text{ for all } \mu > \lambda(u).$$

This proves the almost sure convergence in (4.36). The L^p convergence follows from the almost sure convergence and the tail estimate

$$P\{|H(s,u)| \geq \alpha\} \leq \exp[-K_1\alpha] \text{ for } \alpha \geq 2\sqrt{d}C_1 s, \tag{4.39}$$

which can be proven in the same way as (3.83), (3.84) (or we can take $C_5 = \infty$ in (3.83), (3.84)).

As for (4.37), we have by (4.38), (3.81) and an estimate like (3.26) that

$$\sum_{k=0}^{\infty} P\Big\{\frac{1}{n_k} H(n_k, u) < \lambda(u) - \delta\Big\} < \infty. \tag{4.40}$$

For the other direction, we begin with an indirect argument. Assume, to derive a contradiction, that for some $\delta > 0$ and $0 < \eta \leq C_4/(8C_1)$

$$\sum_{k=0}^{\infty} P\Big\{\frac{1}{m_k} H(m_k, u) > \lambda(u) + \delta/2\Big\} = \infty,$$

with $m_k = m_k(\eta)$ as in (4.20). This is just (4.22) with $\mu - \varepsilon$ replaced by $\lambda(u) + \delta/2$. By Steps 2–4 of the proof of Lemma 7 we then have that (4.35), again with $\mu - \varepsilon$ replaced by $\lambda(u) + \delta/2$, holds. This is impossible for $\eta < K_2\delta/(2\lambda(u))$. Thus for all $\delta > 0, 0 < \eta < C_4/(8C_1) \wedge K_2\delta/(2\lambda(u))$, it is the case that

$$\sum_{k=0}^{\infty} P\Big\{\frac{1}{m_k} H(m_k, u) > \lambda(u) + \delta/2\Big\} < \infty. \tag{4.41}$$

Finally, for given k, let $\ell = \ell(k)$ be determined by $m_\ell < n_k \le m_{\ell+1}$. We now use that

$$\begin{aligned}P\{H(n_k, u) > n_k\big(\lambda(u) + \delta\big)\} &\le P\{H(m_{\ell+1}, u) > m_{\ell+1}\big(\lambda(u) + \delta/2\big)\} \\ &+ P\{H(m_{\ell+1}, u) - H(n_k, u) \le m_{\ell+1}\big(\lambda(u) + \delta/2\big) - n_k\big(\lambda(u) + \delta\big)\}.\end{aligned} \tag{4.42}$$

But, by (3.85) (with C_5 taken to be infinity) we have

$$P\{\inf_{r \le t} H(s + r, u) - H(s, u) \le -\alpha\} \le K_4 s^{-K} + 8d \exp\Big[-\frac{K_3\alpha^2}{t + \alpha}\Big],\ \alpha \ge 0. \tag{4.43}$$

Moreover, $m_{\ell+1} \le (1 + \eta)m_\ell \le (1 + \eta)n_k$ (see (4.21)). Therefore the second term in the right-hand side of (4.42) is at most

$$\begin{aligned}P\{H(m_{\ell+1}, u) - H(n_k, u) \le n_k[(1 + \eta)\big(\lambda(u) + \delta/2\big) - \big(\lambda(u) + \delta\big)] \le -n_k\delta/4\} \\ \le K_4 n_k^{-K} + 8d \exp[-K_6 n_k \delta^2/(\eta + \delta)],\end{aligned}$$

provided

$$\eta < \min\Big\{\frac{C_4}{8C_1}, \frac{K_2\delta}{2\lambda(u)}, \frac{\delta}{4(\lambda(u) + \delta/2)}\Big\}. \tag{4.44}$$

It follows that under this last condition

$$\begin{aligned}\sum_{k=0}^{\infty} &P\{H(n_k, u) > n_k(\lambda(u) + \delta)\} \\ &\le \sum_{k=0}^{\infty} P\{H(m_{\ell(k)+1}, u) > m_{\ell(k)+1}(\lambda(u) + \delta/2)\} + O(1).\end{aligned}$$

The right-hand side here is finite by virtue of (4.41), because $m_{\ell(k)} = K_2 n_{\ell(k)} < n_k \le K_2 n_{\ell(k)+1}$ forces $|\ell(k) - k| \le K_7$ for some K_7 which is independent of k (see (3.80)). Thus

$$\sum_{k=0}^{\infty} P\Big\{\frac{1}{n_k} H(n_k, u) > \lambda(u) + \delta\Big\} < \infty. \tag{4.45}$$

Finally, we may drop the condition (4.44), because if η does not satisfy this condition, then we can choose an η' such that

$$0 < \eta' < \min\Big\{\frac{C_4}{8C_1}, \frac{K_2\delta}{4\lambda(u)}, \frac{\delta}{8(\lambda(u) + \delta/2)}, \frac{\zeta(\eta)}{2}\Big\},$$

with $\zeta(\eta)$ the quantity in (3.80). Let us write r_k for $n_k(\eta')$, where $n_k(\eta')$ satisfies (3.80) with η replaced by η'. By what we proved so far we then have

$$\sum_{k=0}^{\infty} P\big\{\big|\frac{1}{r_k}H(r_k,u)-\lambda(u)\big| > \frac{\delta}{2}\big\} < \infty. \tag{4.46}$$

Furthermore, if s_k is the unique index for which $r_{s_k} < n_k \le r_{s_k+1}$, then also

$$\sum_{k=0}^{\infty} P\big\{\frac{1}{n_k}H(n_k,u) - \frac{1}{r_{s_k+1}}H(r_{s_k+1},u) > \frac{\delta}{2}\big\} < \infty, \tag{4.47}$$

by virtue of (3.86) (with h^* replaced by H and $s = r_{s_k}, t = r_{s_k+1} - r_{s_k}, \alpha = \frac{1}{4}\delta r_{s_k}$). To conclude, $\eta' < \zeta(\eta)$ implies that $n_{k+1}/n_k \ge 1+\zeta(\eta) > 1+2\eta' \ge r_{s_k+1}/r_{s_k}$, so that $s_{k+1} > s_k$ and there is at most one n_k between two successive s's. This, together with (4.46) and (4.47), implies (4.45). □

5. Proof of the shape theorem

Now that we have shown that the spread of the B-particles in the full space process has a definite speed in each direction, the half-space processes are no longer of importance. In fact *Corollary* 8 *contains Theorem* 1 *in the one-dimensional case* (with $B_0 = [-\lambda(e_1), \lambda(e_1)]$). For the higher dimensional case, we shall show in this section how to go from the existence of $\lim_{t\to\infty}(1/t)H(t,u)$ for all $u \in S^{d-1}$ to the full shape theorem. This should work for a fairly general class of processes. The idea to derive the shape theorem via results on the propagation of half-spaces we learned from [GG]. However, the details in our case differ from those in [GG].

The remaining problem in dimension $d > 1$ is that even if we know that $H(t,u)$ grows at rate $\lambda(u)$, it only tells us that there exist B-particles at time t at some random site x_t for which $\langle x_t, u\rangle \sim t\lambda(u)$. It does not tell us where the points x_t near the hyperplane $\{x : \langle x,u\rangle = t\lambda(u)\}$ are. In particular, it does not guarantee that we can find x_t which converge in direction to a prescribed unit vector, i.e., for given $v \in S^{d-1}$ we do not know whether we can choose x_t such that $x_t/\|x_t\|_2 \to v$.

To attack this problem we first write down the conjectured limiting shape B_0 in terms of the function $\lambda(\cdot)$ on S^{d-1}. This conjectured B_0 is convex (for trivial reasons). We then show that we can guarantee $x_t/\|x_t\|_2 \to v$ if v corresponds to a so-called exposed point of the convex set B_0. Using some further properties of convex sets, as well as approximate convexity properties of the set of points which can be reached by the B-particles in a large time, we can then show that the limiting shape result (1.3) holds.

The convergence result (4.36) suggests that the limit set B_0 in (1.3) should be given by

$$B_0 = \{z \in \mathbb{R}^d : \langle z,u\rangle \le \lambda(u) \text{ for all } u \in S^{d-1}\}. \tag{5.1}$$

Clearly this set B_0 is a closed convex set. In fact it is also bounded and hence compact, because $\lambda(u) \le 2\sqrt{d}C_1$ for all u. The origin is an interior point of B_0 because $\lambda(u) \ge C_4$. We call a point $w \in \partial B_0$ an *exposed point* of B_0 if there exists a supporting hyperplane $\{z \in \mathbb{R}^d : \langle a, z\rangle = b\}$ of B_0 which contains w, but no other point of B_0. Thus

$$\langle a, w\rangle = b \text{ but } \langle a, z\rangle < b \text{ for all } z \in B_0 \setminus \{w\}. \tag{5.2}$$

Note that this forces $a \neq \mathbf{0}$. If $\langle a, z\rangle > b$ for $z \in B_0$ we can replace (a, b) by $(-a, -b)$ to make the inequality go in the indicated direction. We now show that $\mathcal{P}^{\mathrm{f}}$ indeed grows in the direction of an exposed point at the rate which is necessary for (1.3).

LEMMA 9. *Let w be an exposed point of B_0 and let $(a, b) \in \mathbb{R}^d \times \mathbb{R}$ satisfy* (5.2). *Let $u = a/\|a\|_2$. Then, there exists a sequence $\varepsilon_n \downarrow 0$ such that*

$$P\{\mathcal{N}_n(w, \varepsilon_n) \textit{ occurs for all large integers } n\} = 1, \tag{5.3}$$

where

$$\begin{aligned} \mathcal{N}_n(w, \varepsilon) := \{&\textit{in } \mathcal{P}^{\mathrm{f}} \textit{ there are at time } \big(1 + 8\varepsilon/C_2\big)n \textit{ occupied} \\ &\textit{sites in } nw + \mathcal{C}(2\varepsilon n) \textit{ and all these sites are in fact} \\ &\textit{occupied by } B\textit{-particles at time } \big(1 + 8\varepsilon/C_2\big)n\}. \end{aligned} \tag{5.4}$$

Also, define

$$\mathcal{O}_n(w, \delta) = \big\{\textit{in } \mathcal{P}^{\mathrm{f}} \textit{ there is at time } n \textit{ a } B\textit{-particle in } nw + \mathcal{C}(\delta n)\big\}.$$

Finally, let $n_k = n_k(\eta)$ be as in Corollary 5. *Then for all $\delta, \eta > 0$*

$$\sum_{k=0}^{\infty} \big[1 - P\{\mathcal{O}_{n_k(\eta)}(w, \delta)\}\big] < \infty. \tag{5.5}$$

Proof. Let B_0 be given by (5.1) and fix an exposed point w of B_0. Order the vertices of $\mathbb{Z}^d$ in some deterministic way, for instance in the lexicographic way. Let x_t be the first vertex x in this order which is occupied by a B-particle in $\mathcal{P}^{\mathrm{f}}$ at time t and with $\langle x, u\rangle = H(t, u)$. By (4.36), almost surely,

$$\frac{1}{t}\langle x_t, u\rangle \to \lambda(u) = \lim_{t\to\infty} \frac{1}{t} H(t, u) \tag{5.6}$$

as $t \to \infty$. Moreover, by (4.37), for each $\delta > 0, \eta > 0$,

$$\sum_{k=0}^{\infty} P\big\{\big|\frac{1}{n_k}\langle x_{n_k}, u\rangle - \lambda(u)\big| > \delta\big\} < \infty. \tag{5.7}$$

We want to show that for each $\delta > 0$

$$P\Big\{\|\frac{1}{n} x_n - w\| \le \delta \text{ for all large integers } n\Big\} = 1. \tag{5.8}$$

Note that $w \in B_0$ implies

$$\langle w, u\rangle \le \lambda(u). \tag{5.9}$$

Recall next that $P\{x_n \notin \mathcal{C}(2C_1 n)\} \le K_6 n^{-K-d-1}$, by virtue of (3.49) or the estimates for (3.25). So,

$$P\{x_n \in \mathcal{C}(2C_1 n) \text{ for all large } n\} = 1. \tag{5.10}$$

Also

$$\sum_{k=0}^{\infty} P\{x_{n_k} \notin \mathcal{C}(2C_1 n_k)\} < \infty. \tag{5.11}$$

So, we can ignore the events $\{x_n \notin \mathcal{C}(2C_1 n)\}$. Next, let $v \in S^{d-1}$ be a unit vector which is not a multiple of w. We claim that there exists some $\delta = \delta(v) > 0$ such that

$$P\Big\{\Big\|\frac{x_n}{\|x_n\|_2} - v\Big\| < \delta \text{ i.o.}\Big\} = 0 \tag{5.12}$$

and

$$\sum_{k=0}^{\infty} P\Big\{\Big\|\frac{x_{n_k}}{\|x_{n_k}\|_2} - v\Big\| < \delta\Big\} < \infty \tag{5.13}$$

(i.o. stands for infinitely often). To prove this, note first that (5.12) holds if $\langle v, u\rangle = 0$, because

$$\liminf_{n\to\infty} \langle \frac{x_n}{\|x_n\|_2}, u\rangle \ge \frac{\lambda(u)}{\limsup_{n\to\infty} \|x_n\|_2/n} \ge \frac{C_4}{2\sqrt{d}C_1} \quad \text{a.s.}$$

by virtue of (5.6), (5.10) and the fact the $\lambda(u) \in [C_4, 2\sqrt{d}C_1]$. Similarly, (5.13) holds if $\langle v, u\rangle = 0$, by virtue of (5.7) and (5.11). To take care of other vectors v, define for any $y \in \mathbb{R}^d \setminus \{\mathbf{0}\}$, with $\langle y, u\rangle \ne 0$,

$$\widetilde{y} = \text{ the unique multiple of } y \text{ which satisfies } \langle \widetilde{y}, u\rangle = b/\|a\|_2.$$

In particular, $\widetilde{y}$ lies in the in the supporting hyperplane $\{z : \langle a, z\rangle = b\}$ (recall that $u = \frac{a}{\|a\|_2}$). Now, by assumption $\widetilde{v} \ne w$, so that $\widetilde{v} \notin B_0$. By definition of B_0 this means that there exists some $u' \in S^{d-1}$ such that $\langle \widetilde{v}, u'\rangle > \lambda(u')$. We can then find $\delta > 0$ and $\eta > 0$ such that $\langle \widetilde{z}, u'\rangle > (1+\eta)\lambda(u')$ for all $z \in S^{d-1}$ with $\|z - v\| < \delta$. Thus, if

$$\Big\|\frac{x_n}{\|x_n\|_2} - v\Big\| < \delta, \tag{5.14}$$

then

$$\langle \widetilde{x}_n, u'\rangle = \langle \widetilde{\big(x_n/\|x_n\|_2\big)}, u'\rangle > (1+\eta)\lambda(u'). \tag{5.15}$$

In addition, by (5.6) and (5.9),

$$\lim_{n\to\infty}\frac{1}{n}\langle x_n,u\rangle = \lambda(u) \geq \langle w,u\rangle = \frac{b}{\|a\|_2} \text{ (see (5.2))},$$

while, by definition of $\widetilde{y}$,

$$\langle \widetilde{x}_n,u\rangle = \frac{b}{\|a\|_2}.$$

Moreover, we must have

$$\|a\|_2\langle w,u\rangle = b > 0 \tag{5.16}$$

by (5.2) and the fact that $\mathbf{0}\in B_0$. Consequently, $x_n=\gamma_n\widetilde{x}_n$ for some reals γ_n which satisfy $\gamma_n/n\to 1$. Thus (5.14) together with (5.15) implies

$$\langle x_n,u'\rangle = \gamma_n\langle\tilde{x}_n,u'\rangle > n(1+\eta/2)\lambda(u')$$

for large n. But, $P\{\langle x_n,u'\rangle > n(1+\eta/2)\lambda(u') \text{ i.o.}\}=0$, by virtue of (4.36) with u replaced by u' and by the fact that $H(n,u')\geq\langle x_n,u'\rangle$ (by definition of H). Thus (5.12) holds for the chosen δ. Similarly, (5.13) follows by means of (5.7) with u' instead of u and (4.37).

Now, for any $\varepsilon>0$, the compact set

$$\begin{aligned}W(\varepsilon) := \{z\in S^{d-1} : z = \frac{x}{\|x\|_2} &\text{ for some } x\in\mathcal{C}(2C_1n) \text{ with}\\ &\langle x,u\rangle \geq n\lambda(u)/2, \|z-\frac{w}{\|w\|_2}\|\geq\varepsilon\}\\ = \{z\in S^{d-1} : z = \frac{x}{\|x\|_2} &\text{ for some } x\in\mathcal{C}(2C_1) \text{ with}\\ &\langle x,u\rangle \geq \lambda(u)/2, \|z-\frac{w}{\|w\|_2}\|\geq\varepsilon\}\end{aligned}$$

is independent of n and is covered by finitely many neighborhoods $U_1,\dots,U_N$ of the form $U_i=\{z\in S^{d-1}:\|z-v_i\|<\delta(v_i)\}$ with $v_i\in S^{d-1}$. Thus, by (5.12), $P\{x_n/\|x_n\|_2\in W(\varepsilon) \text{ i.o.}\}=0$. This holds for all $\varepsilon>0$. In view of (5.6) and (5.10), this implies

$$P\Big\{\frac{x_n}{\|x_n\|_2}\to\frac{w}{\|w\|_2}\Big\}=1. \tag{5.17}$$

In turn, this together with (5.6) implies

$$\lim_{n\to\infty}\frac{n\lambda(u)}{\|x_n\|_2} = \lim_{n\to\infty}\frac{\langle x_n,u\rangle}{\|x_n\|_2} = \frac{\langle w,u\rangle}{\|w\|_2} \text{ a.s.}$$

Since $\langle w,u\rangle\neq 0$ (see (5.16)), $\|x_n\|_2\sim n\|w\|_2\lambda(u)/\langle w,u\rangle$ and

$$\lim_{n\to\infty}\frac{1}{n}x_n = \frac{\lambda(u)}{\langle w,u\rangle}w \text{ a.s.} \tag{5.18}$$

To complete the proof of (5.8) we show that

$$\lambda(u)=\langle w,u\rangle. \tag{5.19}$$

Indeed, we already saw that $\langle \widetilde{x}_n, u\rangle = b/\|a\|_2 = \langle w, u\rangle$. We also saw that $x_n = \gamma_n \widetilde{x}_n$ with $\gamma_n \sim n$. Therefore $\langle x_n/n, u\rangle \sim \langle \widetilde{x}_n, u\rangle = \langle w, u\rangle$. On the other hand, (5.18) implies that $\lim_{n\to\infty}\langle x_n/n, u\rangle = \lambda(u)$. Thus (5.19) and (5.8) hold.

We now also obtain (5.5). Indeed, essentially the same argument as for (5.17), but now using (5.13) instead of (5.12) gives, for any $\delta > 0$,

$$\sum_{k=0}^{\infty} P\Big\{\Big\|\frac{x_{n_k}}{\|x_{n_k}\|_2} - \frac{w}{\|w\|_2}\Big\| > \delta\Big\} < \infty. \tag{5.20}$$

Consequently also

$$\sum_{k=0}^{\infty} P\big\{\big|\frac{1}{n_k}\langle x_{n_k}, u\rangle - \frac{\|x_{n_k}\|_2}{n_k\|w\|_2}\langle w, u\rangle\big| > \frac{\delta}{n_k}\|x_{n_k}\|_2\big\} < \infty.$$

Together with (5.7), (5.19) and (5.11) this last relation yields

$$\sum_{k=0}^{\infty} P\big\{\big|\lambda(u) - \frac{\|x_{n_k}\|_2}{n_k\|w\|_2}\lambda(u)\big| > (1 + 2C_1\sqrt{d})\delta\big\} < \infty.$$

Thus, for suitable constants K_9, K_{10}

$$\sum_{k=0}^{\infty} P\Big\{\Big|\frac{\|x_{n_k}\|_2}{n_k} - \|w\|_2\Big| > K_9\delta\Big\} < \infty$$

and then, by (5.20),

$$\sum_{k=0}^{\infty} P\Big\{\Big\|\frac{x_{n_k}}{\|x_{n_k}\|_2} - \frac{n_k w}{\|x_{n_k}\|_2}\Big\| > K_{10}\delta\Big\} < \infty.$$

This finally gives for some other constant K_{11}

$$\sum_{k=0}^{\infty} \big[1 - P\{\mathcal{O}_{n_k(\eta)}(w, K_{11}\delta)\}\big] \le \sum_{k=0}^{\infty} P\big\{\big\|\frac{x_{n_k}}{n_k} - w\big\| > K_{11}\delta\big\} < \infty.$$

Since this holds for any $\delta > 0$, this is equivalent to (5.5).

The preceding (see (5.8)) shows that there exists a sequence $\varepsilon_n \to 0$, and random vertices x_n such that with probability 1, for all large n,

$$x_n \in nw + \mathcal{C}(\varepsilon_n n) \text{ and } \mathcal{B}^{\mathrm{f}}(x_n, n) \text{ occurs,} \tag{5.21}$$

where

$$\mathcal{B}^{\mathrm{f}}(x, s) := \{\text{there is } B\text{-particle at } x \text{ at time } s \text{ in } \mathcal{P}^{\mathrm{f}}\}.$$

Now take

$$\widetilde{n} := n\big(1 + \frac{8\varepsilon_n}{C_2}\big)$$

and define the event

$$\begin{aligned}\mathcal{R}(x, n) = \{&\text{at time } \widetilde{n} \text{ there is some particle in } \mathcal{P}^{\mathrm{f}} \text{ which lies in} \\ &x + \mathcal{C}\big(C_2(\widetilde{n} - n)/2\big) = x + \mathcal{C}(4\varepsilon_n n) \text{ but is of type } A\}.\end{aligned}$$

We shall complete the proof of the lemma by proving that the event

$$\{\text{for infinitely many } n \text{ there exists an } x_n \text{ for which } \mathcal{B}^{\mathrm{f}}(x_n, n) \cap \mathcal{R}(x_n, n) \text{ occurs}\} \tag{5.22}$$

has probability 0. First we show that this will indeed prove the lemma. The probability that any particle which is in $nw + \mathcal{C}(\varepsilon_n n)$ at time n is outside $nw + \mathcal{C}(2\varepsilon_n n)$ at time $\widetilde{n}$ is bounded by

$$\begin{aligned} &E\{(\text{number of particles in } \mathcal{P}^{\mathrm{f}} \text{ in } nw + \mathcal{C}(\varepsilon_n n))\} \cdot P\{\sup_{r \le \widetilde{n} - n} \|S_r\| \ge \varepsilon_n n\} \\ &\qquad \le K_{12}[\varepsilon_n n]^d P\{\sup_{r \le \widetilde{n} - n} \|S_r\| \ge \varepsilon_n n\}. \end{aligned} \tag{5.23}$$

Without loss of generality we can let ε_n go to 0 so slowly that for large n this expression is no more than n^{-K-1} (by (2.42) in [KSa]) and such that

$$\varepsilon_n \ge n^{-1/2}. \tag{5.24}$$

From this and the fact that the event (5.21) occurs for all large n, we conclude via the Borel-Cantelli lemma that almost surely, for all large n there are particles in $\mathcal{P}^{\mathrm{f}}$ in the set $nw + \mathcal{C}(2\varepsilon_n n)$ at time $\widetilde{n}$. Further, the fact that (5.22) has probability 0 implies that $\mathcal{R}(x_n, n)$ must fail for all large n. But this implies that a.s. there are particles in $\mathcal{P}^{\mathrm{f}}$ which lie in $nw + \mathcal{C}(2\varepsilon_n n) \subset x_n + \mathcal{C}(4\varepsilon_n n)$ at time $\widetilde{n}$, and all of these particles must have type B. This is the desired result (5.3).

It remains to prove that (5.22) has probability 0. But this is almost immediate from Proposition B. Indeed,

$$\begin{aligned} &P\{\mathcal{B}^{\mathrm{f}}(x_n, n) \cap \mathcal{R}(x_n, n)\} \\ &\le P\{x_n \notin \mathcal{C}(2C_1 n)\} + \sum_{x \in \mathcal{C}(2C_1 n)} P\{\mathcal{B}^{\mathrm{f}}(x, n) \text{ but at time } \widetilde{n} \text{ there is a particle} \\ &\qquad\qquad \text{in } \mathcal{P}^{\mathrm{f}} \text{ of type } A \text{ at some } z \in x + \mathcal{C}(4\varepsilon_n n)\} \\ &\le K_4 n^{-K-d-1} + \sum_{x \in \mathcal{C}(2C_1 n)} P\{x \text{ is occupied at time } n \text{ in } \mathcal{P}^{\mathrm{f}} \text{ and in the} \\ &\qquad\qquad \text{full-space process started at } (x, n) \text{ there is an} \\ &\qquad\qquad A\text{-particle at some } z \in x + \mathcal{C}(4\varepsilon_n n) \text{ at time } \widetilde{n}\}, \end{aligned}$$

where we used Lemma C for the last inequality. As in the estimate for $\mathcal{K}_2$ in (3.76), by (3.9) and (2.4) with K replaced by $2K + 2d$, the last sum here is at most

$$K_{10} n^d (\text{left-hand side of (2.4) with } t = \widetilde{n} - n = 8\varepsilon_n n / C_2) \le K_{11} n^{-K}$$

(see (5.24)). □

The preceding lemma shows that the set $B(t)$ grows in the direction of the exposed points of B_0 in ∂B_0 at the "right" speed. More specifically, if w is such a point, then almost surely, for all large t, there exist points $w(t) \in (1/t)B(t)$ such that $w(t) \to w$. We merely have to choose n in Lemma 9 such that $n(1 + 8\varepsilon_n/C_2) \le t$ but $n/t \to 1$, and then choose $w(t)$ a point in $\widetilde{B}\big(n(1 + 8\varepsilon_n/C_2)\big) \cap [nw + \mathcal{C}(2\varepsilon_n n)]$. Lemma 9 guarantees that this last intersection is nonempty for large n. The next two lemmas will show that the same is true for any point $w \in \partial B_0$. This is basically done by concatenating a number of paths which produce B-particles at $\alpha_i n w_{n,i}$ for exposed points $w_{n,i}$ with $\sum_{i=1}^k \alpha_i w_{n,i} \to w, \alpha_i \ge 0, \sum_{i=1}^k \alpha_i = 1$. Lemma 10 contains the basic technical step. It explains how the concatenation works; this is basically the same construction as in the proof of Lemma 7.

LEMMA 10. *Let $w_1, w_2 \in \partial B_0$. Assume that there exist $\varepsilon_n > 0$ such that $\varepsilon_n \to 0$ and such that* (5.3) *holds with w replaced by w_1; that is,*

$$P\{\mathcal{N}_n(w_1, \varepsilon_n) \text{ occurs for all large integers } n\} = 1. \tag{5.25}$$

(*We are not assuming that w_1 is an exposed point of B_0.*) *In addition, assume that for all $\delta, \eta > 0$*

$$\sum_{k=0}^{\infty} \big[1 - P\{\mathcal{O}_{n_k(\eta)}(w_2, \delta)\}\big] < \infty \tag{5.26}$$

(*see Corollary* 5 *for $n_k = n_k(\eta)$*). *Let $0 < \alpha < 1$ and $\eta > 0$. Then there exists $\delta_n > 0$ such that $\delta_n \to 0$ and such that*

$$P\{\mathcal{N}_n(\alpha w_1 + (1-\alpha) w_2, \delta_n) \text{ occurs for all large } n\} = 1. \tag{5.27}$$

Proof. Fix $0 < \alpha < 1$. Also fix

$$\delta > 0 \text{ and } 0 < \eta < \delta/2$$

for the time being. Take

$$p_k = p_k(\eta) = \Big\lfloor \frac{\alpha}{1-\alpha} n_k(\eta) \Big\rfloor$$

and

$$q_k = q_k(\eta) = \big(1 + 8\varepsilon_{p_k(\eta)}/C_2\big) p_k.$$

Define $\mathcal{O}'_{n_k}(w_2, \delta)$ as the translate by $\big(p_k(\eta) w_1, q_k(\eta)\big)$ (in space-time) of $\mathcal{O}_{n_k}(w_2, \delta)$. Explicitly,

$$\begin{aligned}\mathcal{O}'_{n_k}(w_2, \delta) = \{&\text{in the full-space process started at } \big(p_k(\eta) w_1, q_k(\eta)\big) \text{ there} \\ &\text{is at time } q_k + n_k \text{ a } B\text{-particle in } p_k w_1 + n_k w_2 + \mathcal{C}(\delta n_k)\}.\end{aligned}$$

(We suppress the dependence on w_1 and η in this notation). Also let

$$z_k = \text{nearest occupied site to } p_k w_1 \text{ in } \mathcal{P}^{\mathrm{f}} \text{ at time } q_k.$$

Since $P\{\mathcal{O}'_{n_k}(w_2,\delta)\} = P\{\mathcal{O}_{n_k}(w_2,\delta)\}$, assumption (5.26) implies that almost surely,

$$\mathcal{O}'_{n_k}(w_2,\delta) \text{ occurs for all large } k. \tag{5.28}$$

Also, by assumption (5.25), almost surely,

$$\mathcal{N}_{p_k}(w_1,\varepsilon_{p_k}) \text{ occurs for all large } k. \tag{5.29}$$

Now consider a k for which $\mathcal{N}_{p_k}(w_1,\varepsilon_{p_k}) \cap \mathcal{O}'_{n_k}(w_2,\delta)$ occurs. By the definition (5.4) of $\mathcal{N}_{p_k}$ this implies that z_k lies in $p_k w_1 + \mathcal{C}(2\varepsilon_{p_k} p_k)$ and that the particles at z_k at time q_k have type B in $\mathcal{P}^{\mathrm{f}}$. Therefore the resetting of the types to start the full-space process at $(p_k w_1, q_k)$ does not change the type at z_k. By the monotonicity property of Lemma C, $\mathcal{P}^{\mathrm{f}}$ therefore has at least as many B-particles at any space-time point (x,t) with $t \ge q_k$ as the full state process started at $(p_k w_1, q_k)$. Since $\mathcal{O}'_{n_k}(w_2,\delta)$ occurs this implies that in $\mathcal{P}^{\mathrm{f}}$ there is a B-particle in $p_k w_1 + n_k w_2 + \mathcal{C}(\delta n_k)$ at time $q_k + n_k$.

Let the nearest B-particle to $p_k w_1 + n_k w_2$ in $\mathcal{P}^{\mathrm{f}}$ at time $q_k + n_k$ be at the position y_k, so that $\mathcal{B}^{\mathrm{f}}(y_k, q_k + n_k)$ occurs. The last paragraph gives us that $\|y_k - p_k w_1 - n_k w_2\| \le \delta n_k$. These are only statements for the times $q_k + n_k$. Since (5.25) requires that certain events happen for all large n we now first show how to go from the $q_k + n_k$ to general integers n. For any large n let $k(n)$ be such that $q_k + n_k \le n < q_{k+1} + n_{k+1}$. Then for large n

$$q_k + n_k \le n \le (q_k + n_k)(1+2\eta) \le (q_k+n_k)(1+\delta),$$

since $n_{k+1}/n_k \le 1+\eta$ and $q_{k+1}/q_k \sim p_{k+1}/p_k \sim n_{k+1}/n_k$. Also by our choice of p_k, q_k

$$\alpha n \sim \alpha(q_k + n_k) + \mathrm{O}(\eta n) = \alpha(p_k + n_k) + \mathrm{O}((\eta + \varepsilon_k)n) = p_k + \mathrm{O}(\delta n)$$

and

$$\|p_k w_1 + n_k w_2 - n[\alpha w_1 + (1-\alpha) w_2]\| \le K_{12}\delta n$$

for large n. Thus, on $\mathcal{B}^{\mathrm{f}}(y_k, q_k + n_k)$, there is a B-particle at $y_k \in n[\alpha w_1 + (1-\alpha)w_2] + \mathcal{C}\big((K_{12}+1)\delta n\big)$ at time $q_k + n_k$. Moreover, as in (5.23) we have

$$\begin{aligned} P\{&\text{in } \mathcal{P}^{\mathrm{f}} \text{ there is a } B\text{-particle in } n[\alpha w_1 + (1-\alpha)w_2] + \mathcal{C}\big((K_{12}+1)\delta n\big) \\ &\text{at time } q_k + n_k \text{ which is no longer in } n[\alpha w_1 + (1-\alpha) w_2] \\ &+ \mathcal{C}\big((K_{12}+2)\delta n\big) \text{ at time } n\} = O\big(n^{-K}\big). \end{aligned}$$

Thus, almost surely, there is in $\mathcal{P}^{\mathrm{f}}$ for all large n a B-particle in

$$n[\alpha w_1 + (1-\alpha)w_2] + \mathcal{C}\big((K_{12}+2)\delta n\big)$$

at time n. We can now proceed as in Lemma 9. Essentially as in (5.22) and in the lines following it we now have that almost surely

(5.30)

$$\{\text{there is some } y \in n[\alpha w_1 + (1-\alpha)w_2] + \mathcal{C}\big((K_{12}+2)\delta n\big) \text{ for which } \mathcal{B}^{\mathrm{f}}(y,n) \text{ occurs, but in } \mathcal{P}^{\mathrm{f}} \text{ there are either no particles or an } A\text{-particle in } n[\alpha w_1 + (1-\alpha)w_2] + \mathcal{C}\big(2(K_{12}+2)\delta n\big) \text{ at time } \big(1+8(K_{12}+2)\delta/C_2\big)n\}$$

occurs only for finitely many n. This shows that

$$P\{\mathcal{N}_n\big(\alpha w_1 + (1-\alpha)w_2, (K_{12}+2)\delta\big) \text{ occurs for all large } n\} = 1. \tag{5.31}$$

This holds for all $\delta > 0$ and $\eta < \delta/2$. However, (5.31) is already independent of η, so that it holds for all $\delta > 0$. There then also exists a sequence $\delta_n \to 0$ such that almost surely $\mathcal{N}_n\big(\alpha w_1 + (1-\alpha)w_2, \delta_n\big)$ occurs for all large n. $\square$

Proof of Theorem 1. We shall prove (1.3) with the B_0 defined in (5.1). For the right-hand inclusion in (1.3) we note that for any $\varepsilon > 0$ there exists finitely many half-spaces $\{z \in \mathbb{R}^d : \langle z, u_i\rangle \le \lambda(u_i)\}$, $1 \le i \le N$, with $u_i \in S^{d-1}$ such that

$$\bigcap_{i=1}^{N} \{z \in \mathbb{R}^d : \langle z, u_i\rangle \le \lambda(u_i)\} \subset (1+\varepsilon/3)B_0. \tag{5.32}$$

Indeed, B_0 is contained in the cube $\widetilde{\mathcal{C}} := \cap_{i=1}^N \{z \in \mathbb{R}^d : -\lambda(e_i) \le \langle z, e_i\rangle \le \lambda(e_i)\}$ (with $e_i = i$-th coordinate vector), and by compactness, $\widetilde{\mathcal{C}} \setminus$(interior of $(1+\varepsilon/3)B_0$ is covered by finitely many relatively open subsets of $\widetilde{\mathcal{C}}$ of the form $\widetilde{\mathcal{C}} \cap \{z \in \mathbb{R}^d : \langle z, u\rangle > \lambda(u)\}$. Thus (5.32) holds. In addition to (5.32) we know from (4.36) that, almost surely, $H(t, u_i) < t(1+\varepsilon/3)\lambda(u_i)$ for all large t and $i = 1, \dots, N$. Consequently, almost surely

$$\widetilde{B}(t) \subset t(1+\varepsilon/3) \bigcap_{i=1}^{N} \{z \in \mathbb{R}^d : \langle z, u_i\rangle \le \lambda(u_i)\} \subset (1+\varepsilon/3)^2 t B_0$$

for all large t. Thus the right-hand inclusion in (1.3) holds.

For the left-hand inclusion in (1.3) we first observe that by Lemma 9, the hypotheses (5.25) and (5.26) of Lemma 10 hold for all exposed points $w_1, w_2 \in \partial B_0$. It then follows from Lemma 10 that (5.27) holds. In turn, (5.27) states that the hypothesis (5.25) with w_1 replaced by $\alpha w_1 + (1-\alpha)w_2$ is satisfied. Therefore, if $w_3 \in \partial B_0$ is also an exposed point of B_0 and $0 < \beta < 1$, then we get from Lemma 10 that there exist $\delta_n' \to 0$ such that

$$P\{\mathcal{N}_n(\beta\alpha w_1 + \beta(1-\alpha)w_2 + (1-\beta)w_3, \delta_n') \text{ occurs for all large } n\} = 1.$$

But as α and β vary over $(0,1)$, $\beta\alpha w_1 + \beta(1-\alpha)w_2 + (1-\beta)w_3$ varies over the convex combinations $\alpha_1 w_1 + \alpha_2 w_2 + \alpha_3 w_3$ with $\alpha_i > 0, \sum_{i=1}^3 \alpha_i = 1$. We can repeat this procedure to obtain that for each convex combination $\sum_{i=1}^k \alpha_i w_i$

with $\alpha_i \geq 0, \sum_{i=1}^k \alpha_i = 1$ and $w_i \in \partial B_0$ exposed points of B_0, there exist $\delta_n \to 0$ such that

$$P\{\mathcal{N}_n\big(\sum_{i=1}^k \alpha_i w_i, \delta_n\big) \text{ for all large integers } n\} = 1.$$

In particular (see (5.4)), for each such $\sum_{i=1}^k \alpha_i w_i$ and each fixed $\eta > 0$

$$\begin{aligned}P\{&\text{in } \mathcal{P}^{\mathrm{f}} \text{ there are at time } (1+8\delta_n/C_2)n \text{ } B\text{-particles in}\\ &n\sum_{i=1}^k \alpha_i w_i + \mathcal{C}(2\eta n) \text{ for all large integers } n\} = 1.\end{aligned}$$

In turn, this means that if for a given vector v and $\eta > 0$ we can find α_i, w_i as above such that $\|v - \sum_{i=1}^k \alpha_i w_i\| \leq \eta$, then also

$$\begin{aligned}P\{&\text{in } \mathcal{P}^{\mathrm{f}} \text{ there are at time } (1+8\delta_n/C_2)n \text{ } B\text{-particles in}\\ &nv + \mathcal{C}(3\eta n) \text{ for all large integers } n\} = 1.\end{aligned} \tag{5.33}$$

If v is such that there exist $k^{(r)} < \infty, \alpha_i^r \geq 0$ and $w_i^{(r)} \in \partial B_0$ exposed points of B_0 such that $\sum_{i=1}^{k^{(r)}} \alpha_i = 1$ and $\|v - \sum_{i=1}^{k^{(r)}} \alpha_i^{(r)} w_i^{(r)}\| \to 0$ (as $r \to \infty$), then (5.33) holds for each $\eta > 0$. For such v there then exist $\eta_n \to 0$ such that almost surely, for all large n there exist B-particles within distance $4\eta_n n$ of nv at time $(1+8\delta_n/C_2)n$, for some $\delta_n \to 0$ (δ_n and η_n may depend on v).

The last statement applies to each $v \in B_0$, because each such v is a convex combination of at most $(d+1)$ extreme points of B_0 (see [Ru, Th. 3.22 and the lemma following Th. 3.25]) and the exposed points of B_0 are dense in the extreme points (Strascewicz' theorem; see Theorem 18.6 in [Ro]). Thus, by applying the last result to a fixed $v \in B_0$ with $n = \lfloor (1-\varepsilon)t \rfloor$ and $0 < \varepsilon < 1$, we find that almost surely for all large t,

$$\begin{aligned}&\text{at time } (1+8\delta_n/C_2)n \text{ there exists a site } v_n \text{ with}\\ &\|v_n - nv\| \leq 4\eta_n n, \text{ which is occupied in } \mathcal{P}^{\mathrm{f}} \text{ by } B\text{-particles.}\end{aligned} \tag{5.34}$$

We claim that

$$\begin{aligned}P\{&(5.34) \text{ holds, but not all sites in } (1-\varepsilon)tv + \mathcal{C}(C_2\varepsilon t/4) \text{ belong to } \widetilde{B}(t)\}\\ &\leq K_{13}t^{-K}.\end{aligned} \tag{5.35}$$

This is an easy consequence of (3.9) and Theorem A. Indeed, from (3.9) with (X, s) taken to be $\big(v_n, (1+8\delta_n/C_2)n\big)$, $s = (1+8\delta_n/C_2)n$, $X = v_n =$ the nearest site to v which is occupied in $\mathcal{P}^{\mathrm{f}}$ at time s by some B-particle and

$$\begin{aligned}\mathcal{A} &= \{\text{not all vertices in } \mathcal{C}(C_2\varepsilon n/2) \text{ have been}\\ &\qquad \text{visited by a } B\text{-particle by time } \varepsilon n/2\}\\ &= \{\mathcal{C}(C_2\varepsilon n/2) \not\subset B(\varepsilon n/2)\},\end{aligned}$$

we see that the probability in (5.35) is for large t at most

$$K_{14}n^d P^{\text{or}}\{\mathcal{C}(C_2\varepsilon n/2) \not\subset B(\varepsilon n/2)\} \le K_{15}n^{-K} \le K_{13}t^{-K}$$

(for the first inequality here we used Theorem A with $K + d$ in the place of K). This establishes the claim (5.35).

To obtain Theorem 1 we now choose for a given ε a finite number of vectors $v^{(1)}, \dots, v^{(N)}$ in B_0 such that each $v \in B_0$ satisfies $\|v - v^{(r)}\| < C_2\varepsilon/4$ for at least one r. This means that

$$B_0 \subset \bigcup_{1\le r\le N} \big[v^{(r)} + \mathcal{C}(C_2\varepsilon/4)\big].$$

Moreover, by (5.34) and (5.35) it holds almost surely for all large t that

$$\bigcup_{1\le r\le N} \big[(1-\varepsilon)tv^{(r)} + \mathcal{C}(C_2\varepsilon t/4)\big] \subset \widetilde{B}(t).$$

Together, these last two inclusions imply that almost surely the left-hand inclusion in (1.3) holds for all large t.

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