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RAND

ON THE HAMILTONIAN GAME
(A Traveling Salesmen Problem)

Julia Robinson

RM-303

5 December 1949

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The RAND Corporation
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PROJECT RAND

RESEARCH MEMORANDUM

ON THE HAMILTONIAN GAME
(A Traveling Salesmen Problem)

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5 December 1949

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1. Introduction

The purpose of this note is to give a method for solving a problem related to the traveling salesman problem. It seems worthwhile to give a description of the original problem.* One formulation is to find the shortest route for a salesman starting from Washington, visiting all the state capitals and then returning to Washington.** More generally, to find the shortest closed curve containing n given points in the plane.

Clearly, it is sufficient to consider curves made up of line segments joining pairs of the given points. Also, unless all the points lie on a straight line, the optimal path will not pass through any point twice. Hence the problem can be stated as follows:

Arrange the n points in a cyclic order so that the sum of the distance between consecutive points is a minimum.

In this statement of the problem, arbitrary real numbers can be assigned as the "distances" between ordered pairs of distinct points. Thus, the "distance" from A to B need not be the same as from B to A. We shall sometimes refer to the "length" of AB instead of the "distance" from A to B.

Since there are only a finite number of paths to consider, the problem consists in finding a method for picking out the optimal path when n is moderately large, say $n = 50$. In this case, there are more than 10^{62} possible paths, so we can not simply try them all. Even for as few as 10 points, some short cuts are desirable.

* Actually, the problem may go back to W. R. Hamilton. See R. W. Ball: Mathematical Recreations on the Hamiltonian game.

** In this paper I shall not be concerned with the various possible applications of the problem solved here.

2. Statement of the problem

An unsuccessful attempt to solve the above problem led to a solution of the following:

Given n points and all the "distances" between ordered pairs of distinct points. The problem is to find a system of ordered circuits such that:

- i. Each point lies on exactly one circuit.
- ii. Each circuit contains at least 2 points.
- iii. No circuit passes through the same point more than once.
- iv. The total "length" of the circuits is a minimum.

However at first glance, it looks more difficult than the traveling salesman problem, for there are obviously many more systems of circuits than circuits. Actually the topological characterization of a system of circuits is much simpler than that of a single circuit and can be used to solve this problem.

The method presented here of handling this problem will enable us to check whether a given system of circuits is optimal or, if not, to find a better one. I believe it would be feasible to apply it to as many as 50 points provided suitable calculating equipment is available.

3. Description of the method.

Number the points $1, 2, \dots, n$. Put $D = \parallel d_{ij} \parallel$, where d_{ij} is the distance from i to j , $d_{ii} = 0$. Let \mathcal{S} be the set of directed segments comprising the proposed system of circuits. We wish to determine if this system is optimal or, if not, to find a better system.

Construct the auxiliary matrix $S = \parallel s_{ij} \parallel$ as follows:

For each i , determine i' so that $ii' \in \mathcal{S}$. Then put,

$$s_{11'} = +\infty$$

and

$$s_{1j} = d_{j1'} - d_{11'} \quad \text{for } j \neq 1'.$$

Now think of the S-matrix as giving new "distances" between the given points and look for a closed circuit of negative S-length. If there is such a circuit, it will have from 2 to n points. Suppose $C = i_0 i_1 \dots i_k$ is a circuit of negative S-length. Then make up a new system of circuits S' by modifying S in the following way:

Remove	Add
$i_0 i_0'$	$i_1 i_0'$
$i_1 i_1'$	$i_2 i_1'$
⋮	⋮
⋮	⋮
$i_k i_k'$	$i_0 i_k'$

The new system of circuits S' , thus obtained, has a shorter total D-length than S . In fact, if we let $L_A(a)$ be the length of a measured by the matrix A , then

$$L_D(S') = L_D(S) + S(C).$$

We then apply the same procedure to S' .

Suppose we can not find a circuit of negative S-length. Then we attempt to show that S is optimal. To do this, enforce the triangle inequality,

$s_{ij} \leq s_{ik} + s_{kj}$ * that is, if $s_{ij} > s_{ik} + s_{kj}$ replace s_{ij} by $s_{ik} + s_{kj}$. These replacements can be carried out in any order.

If a matrix is eventually obtained for which the triangle inequality holds, then \checkmark is the best system of circuits. If not, there must be some circuit of negative S-length. To find one, we must keep track of the changes made in the S-matrix. For example, under the i, j^{th} entry in the S-matrix, write (ij). Then if s_{ij} is replaced by $s_{ik} + s_{kj}$, replace the (ij) by (ikj). Similarly, if s_{ikj} is replaced by $s_{iHk} + s_{kMj}$, then replace (ikj) by (iHkMj). (Here K, H and M are finite sequences of numbers from 1 to n.) Thus, the entry in the i, j^{th} place will always be the length of the path indicated from i to j. If there is a negative circuit in the S-matrix, then at some stage a negative number can be put on the main diagonal of the modified S-matrix. We can then easily obtain the corresponding circuit in the S-matrix.

4. A numerical example

As an example, we take a set of six points with the following distance matrix:

0	1	4	2	8	7
6	0	5	2	1	9
4	8	0	7	2	6
5	5	5	0	4	8
6	1	5	7	0	4
3	9	1	2	6	0

D

* i, j and k need not be distinct.

As a first trial system of circuits \mathcal{S} take the two circuits 12531 and 464. Then $\mathcal{S} = \{12, 25, 53, 31, 46, 64\}$. Hence $1' = 2, 2' = 5, 3' = 1, 4' = 6, 5' = 3$ and $3' = 1$. Next construct the S-matrix:

0	+∞	+7	+4	0	+8
+7	0	+1	+3	+∞	+5
+∞	+2	0	+1	+2	-1
-1	+1	-2	0	-4	+∞
-1	0	+∞	0	0	-4
0	0	+5	+∞	+5	0

S

We now look for a closed circuit of negative S-length. After a few trials, we find the circuit $C = 456234$ with S-length = -6. We obtain \mathcal{S}' from \mathcal{S} by removing 46, 53, 64, 25 and 31 from \mathcal{S} and adjoining 56, 63, 24, 35 and 41. We then obtain the \mathcal{S}' matrix:

0	+∞	+7	+4	0	+8
0	0	+5	+∞	+5	0
+6	-1	0	+2	+∞	+4
+∞	+1	-1	0	+1	-2
+3	+5	+2	+4	0	+∞
+3	+4	+∞	+4	+4	0

S'

Since we do not find a negative circuit in S' , we try to enforce the triangle inequality, keeping track of the changes we make in case there is a negative circuit. We give one intermediate matrix as an example and the final one in which the triangle inequality holds.

0 (11)	+5 (142)	+2 (153)	+4 (14)	0 (15)	+2 (146)
0 (21)	0 (22)	+2 (2153)	+4 (214)	0 (215)	0 (26)
-1 (321)	-1 (32)	0 (33)	+2 (34)	-1 (3215)	-1 (326)
-2 (4321)	-2 (432)	-1 (43)	0 (44)	-2 (43215)	-2 (46)
+1 (5321)	+1 (532)	+2 (53)	+4 (54)	0 (55)	+1 (5326)
+2 (64321)	+2 (6432)	+3 (643)	+4 (64)	+2 (643215)	0 (66)

Intermediate modified matrix

0 (11)	+1 (1532)	+2 (155)	+4 (14)	0 (15)	+1 (15326)
0 (21)	0 (22)	+2 (2155)	+4 (214)	0 (215)	0 (26)
-1 (321)	-1 (32)	0 (33)	+2 (34)	-1 (3215)	1 (326)
-2 (4321)	-2 (432)	-1 (43)	0 (44)	-2 (43215)	-2 (46)
+1 (5321)	+1 (532)	+2 (53)	+4 (54)	0 (55)	+1 (5326)
+2 (64321)	+2 (6432)	+3 (643)	+4 (64)	+2 (643215)	0 (66)

Final S'-matrix with Δ -inequality holding

Hence \mathcal{S}' is the optimal system of circuits. It consists of the two circuits 1241 and 3563 and has D-length = 15.

5. Justification of the method.

First, notice that a set of n directed segments satisfies 1 - iii of Section 1, if and only if

1. Each of the n points is an initial point of one of the segments;
2. Each of the n points is a terminal point of one of the segments;
3. Each segment is between distinct points.

To see this, think of the terminal points as a permutation of the initial points. This permutation can be expressed as a product of cyclic permutations. These are the circuits.

This insures that, if there is a circuit C of negative S -length and if \mathcal{L}' is obtained from \mathcal{L} by the rule given in Section 3, then \mathcal{L}' will also be an admissible system of circuits. This is clear since, if a segment with initial point a is removed, one is also added and conversely. Similarly, for the terminal points. Hence 1 and 2 remain satisfied. Furthermore, if C is of negative S -length, it can not contain any segments in common with \mathcal{L} , for these have S -length $+\infty$; therefore, the segments added to \mathcal{L} are between distinct points.

Let $C = i_0 i_1 \dots i_k$. Then

$$\begin{aligned} \ell_S(C) &= s_{i_0 i_1} + s_{i_1 i_2} + \dots + s_{i_k i_0} \\ &= (d_{i_1 i_0'} - d_{i_0 i_0'}) + (d_{i_2 i_1'} - d_{i_1 i_1'}) + \dots + (d_{i_0 i_k'} - d_{i_k i_k'}) \\ &= (d_{i_1 i_0'} + d_{i_2 i_1'} + \dots + d_{i_0 i_k'}) - (d_{i_0 i_0'} + d_{i_1 i_1'} + \dots + d_{i_k i_k'}). \end{aligned}$$

Hence $\ell_D(\mathcal{L}') - \ell_D(\mathcal{L}) = \ell_S(C)$. Thus, we see that if there is a circuit C of negative S -length, then \mathcal{L} is not an optimal system and we can construct a system \mathcal{L}' of shorter total length.

Conversely, if \mathcal{L} is not optimal, then we will show that there is a circuit

\mathcal{C} of negative S-length. Let \mathcal{S}' be a system of circuits of shorter total D-length than \mathcal{S} . Let \mathcal{A} be the set of segments in \mathcal{S} but not in \mathcal{S}' and \mathcal{B} be the set of segments in \mathcal{S}' but not in \mathcal{S} . Let

$\mathcal{A} = \{i_0 i_0', i_1 i_1', \dots, i_k i_k'\}$. Then \mathcal{B} must consist of a set of segments with the same initial points as in \mathcal{A} , with the same terminal points and the same number of segments. Hence let $\mathcal{B} = \{j_0 i_0', \dots, j_k i_k'\}$, where j_0, j_1, \dots, j_k is a permutation of i_0, i_1, \dots, i_k . Then

$$\begin{aligned} \ell_D(\mathcal{S}') - \ell_D(\mathcal{S}) &= (d_{j_0 i_0'} - d_{i_0 i_0'}) + (d_{j_1 i_1'} - d_{i_1 i_1'}) + (d_{j_k i_k'} - d_{i_k i_k'}) \\ &= s_{i_0 j_0} + s_{i_1 j_1} + \dots + s_{i_k j_k}. \end{aligned}$$

Express the permutation $\begin{pmatrix} i_0 i_1 \dots i_k \\ j_0 j_1 \dots j_k \end{pmatrix}$ as the product of cycles, say

$\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_t$. Then by rearranging and collecting terms of

$$s_{i_0 j_0} + s_{i_1 j_1} + \dots + s_{i_k j_k},$$

we see that this sum is just

$$\ell_S(\mathcal{C}_1) + \dots + \ell_S(\mathcal{C}_t)$$

where $\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_t$ are the circuits corresponding to the cycles of the permutation. Hence

$$\ell_D(\mathcal{S}') - \ell_D(\mathcal{S}) = \ell_S(\mathcal{C}_1) + \dots + \ell_S(\mathcal{C}_t).$$

Since \mathcal{L}' is shorter than \mathcal{L} , one of the circuits C_1, \dots, C_t must have negative S-length. Therefore \mathcal{L} is optimal if and only if there is no closed circuit of negative S-length.

It remains to show that the non-existence of a circuit of negative S-length is equivalent to the existence of a modified S-matrix for which the triangle inequality holds. Assume first that A is a modified S-matrix and that the triangle inequality holds in A. Let C be a circuit of negative S-length. It corresponds to a circuit C' of negative A-length. Let $C' = i_0 i_1 i_2 \dots i_k$. Then $C'' = i_0 i_2 \dots i_k$ is also of negative A-length since $a_{i_0 i_2} \leq a_{i_0 i_1} + a_{i_1 i_2}$. Hence, if there is any circuit of negative A-length we can find a one-point circuit of negative length i.e. for some i, $a_{ii} < 0$. But this is impossible since then $a_{ii} + a_{ii} < a_{ii}$ contrary to the assumption that the triangle inequality holds.

On the other hand, if there is no circuit of negative S-length we can enforce the triangle inequality. The resulting matrix will give in the i, jth place the S-length of the shortest path from i to j. If there is no circuit of negative length, there clearly is a shortest path between any two points.

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