Branko Grünbaum

Introductory notes for the digital version of my

LECTURES ON LOST MATHEMATICS

presented at the University of Washington during the Fall 1975 quarter

and of the additional comments prepared in collaboration with G. C. Shephard for a Special Session on Rigidity, organized at the 760th meeting of the American Mathematical Society, held in Syracuse, NY, on October 28, 1978.

The original Notes have been dittoed (purple) and distributed in a considerable quantity to the participants in the one hour a week lectures, and to a number of other mathematicians. The supply of the Notes was quickly exhausted. Advances in the technology that have become available in the following years made it possible to reprint the Notes in mimeographed (black) form. The Syracuse Special Session provided an opportunity to make a rather large number of copies available, updated with material that appeared or came to our attention in the intervening years. These copies were also distributed quickly, and I soon have been unable to respond to frequent requests for copies of the notes.

This was the situation when in late 2005 Professor Marjorie Senechal asked whether I agree that she have the notes scanned and distributed to participants of a conference entitled "Structural Topology Revisited", to be held in La Vacquerie (France) in July 2006. I agreed, and the following is the result of turning the Notes into a digital document. Needless to stress, I am greatly obliged to Professor Senechal for this step.

The University of Washington has established a permanent digital depository for lecture notes and similar material, and I am happy to provide a final home for the "Lectures on Lost Mathematics". The reader should be aware that the notes (and the additional comments) where written more than thirty years ago, and that many developments have occurred during that time. I am happy that the material of these notes helped influence some of these developments.

Seattle, April 2010

Branko Grünbaum and G. C. Shephard

LECTURES ON LOST MATHEMATICS *)

Reissued for the

Special Session on Rigidity

at the

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1. INTRODUCTION

During the "Symposium on the Relations Between Infinite-Dimensional and Finite-Dimensional Convexity" arranged by the London Mathematical Society in Durham (England) in Summer 1975, the authors became interested in various geometric topics that have been forced "underground" by the "mainstream of mathematics". Even a very cursory examination of the literature revealed the fact that many questions relating to crystallography, rigidity of structures, etc. lead to challenging problems and attractive results in geometry. Some of these were, at one time, known to some mathematicians, but for the majority of us they have become lost in the overwhelming flood of publications.

On returning to the University of Washington one of us (B.G.) gave in the Fall Quarter of 1975/76 a series of informal talks on this "Lost Mathematics". For the convenience of the listeners dittoed lecture notes were prepared and distributed as the lectures went on. The two main topics of the lectures were tilings and rigidity. The discussion of tilings started the authors on an intensive investigation that led to a series of papers (listed below) and to the monograph Grünbaum-Shephard [1979] that is nearing completion. Notes on these results were distributed to the audience of the lectures (and, later, of the "Geometry and Combinatorics" seminar at the University of Washington) as they were obtained, and were not included in the "Lectures on Lost Mathematics" notes. The material on rigidity was presented in Chapters 2 and 3 of the "Lectures". The reader of the "Lectures" may easily find signs of the haste in which they were prepared, as well as of the fact that they were written as adjuncts to the lectures, and not for publication. The demonstration of many models helped the audience's understanding of the spatial relations; the drawings in the notes were meant more to recall these models than to serve as substitutes for them.

Due to various accidental circumstances a rather large number of copies of the "Lectures" was circulated. The material on rigidity (in particular, the critique of the literature on frameworks, and the concept of "tensed frameworks" that has not been considered previously in the mathematical literature) attracted some attention, and references to the "Lectures" appeared in several papers. This led to additional requests for copies -- but two problems arose. On the one hand, the original stock of the "Lectures" was exhausted; on the other, to just reprint them in the original version would be a disservice to readers, since many of the problems posed there have already been solved.

Since a rewriting of the "Lectures" to account for the new developments is at present not feasible, we decided to use the occasion of the Special Session on Rigidity to solve these problems by reprinting the "Lectures" unchanged except for the inclusion of a number of marginal remarks that provide access to the new results. These marginal remarks, collected in Section 2 below, are keyed to places in the original "Lectures" They are followed by a list of additional references.

It is hoped that this arrangement will do justice to historic fidelity without confusing the reader about the present status of the various questions. For completeness, the additional bibliography lists also recent papers that are not mentioned in the text (although not all were available to the authors), as well as some of the old references for which more precise data have become available.

* * *

We would also like to use this occasion to correct the "Sample conjecture" mentioned in our abstract for the Special Session on Rigidity (Abstract 760-D3, Notices Amer. Math. Soc. 25(1978), A-642). We inadvertently omitted from its wording the requirement that the framework F is <u>convex</u>, that is, that each element of F (cable or rod) be contained in the <u>boundary</u> of the convex hull of F .

2. REMARKS ON THE "LECTURES"

1. (page 1.3, line -10) For information on the current status of this problem see Schattschneider [1978]. Other aspects of the theory of planar tilings are investigated in Grünbaum-Shephard [1977], [1977a], [1977b], [1977c], [1978], [1978a], [1978b], [1978c], [1978d], [1978e], [1978f], and a systematic treatment will be presented in Grünbaum-Shephard [1979].

2. (page 1.4, line 6) For a survey of this area and for references to the literature see Grünbaum-Shephard [1978].

3. (page 1.4, line -4) For results concerning convex polyhedral tiles for the 3-dimensional space that are far better than anything mentioned in the mathematical literature see Fischer-Koch [1973] and the references given there.

4. (page 1.5, line 5) More details concerning such networks can be found in A. F. Wells [1977].

5. (page 1.5, line 11) For a survey of related material see A. F. Wells [1977], as well as Grünbaum [1977] and Grünbaum-Shephard [1978], where additional references may be found.

6. (page 1.6, line 1) The investigation of such "aperiodic tilings" underwent an explosive development in the last three years, though very little has been published on the topic. Some information on the "Penrose tiles" may be found in Gardner [1977]; a detailed exposition is given in Chapters 10 and 11 of Grünbaum-Shephard [1979].

7. (page 1.6, line 15) It is not clear whether the results in the literature mean that arbitrarily large parts of each algebraic

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curve can be traced. Precisely what is true in case of algebraic surfaces, and under what conditions, is even less clear.

8. (page 1.6, line -3) The various regidity questions are discussed in great detail in Chapters 2 and 3 of the "Lectures".

9. (page 1.7, line 5) See also Otto [1973] and Pugh [1976].

10. (page 2.3, line 9) A different approach is presented in lecture notes by H. Hopf [1946].

11. (page 2.3, line 10) The proof in Gluck [1975] -- although interesting from several points of view -- established Cauchy's rigidity theorem <u>only</u> for simplicial convex polyhedra and not for <u>all</u> convex polyhedra as claimed.

12. (page 2.3, line 14) For a recent discussion of such examples see Goldberg [1978].

13. (page 2.4, line -6) The Bricard polyhedra are <u>not</u> immersions of the 2-sphere in 3-space, as implied by the text.

14. (page 2.5, line -1) The questions of terminology are exceedingly vexing. The fact that different authors use the same words to describe distinct notions (often without giving precise definitions) is probably one of the causes of many of the errors in the literature. It also contributes to the difficulties we have in understanding the claims and proofs presented by various authors. This may in some cases be our fault; but there are sufficiently many instances in which the authors misled themselves to make an agreement on terminology (and an insistence on clear and complete definitions) most highly desirable. It is to be hoped that the Special Session on Rigidity will produce suggestions concerning the

terminology.

As an example of the problems in communications that arise (and for which it is not clear to us whether they are due to our misunderstanding of the writers' intentions, or to an error on their part) we mention the following:

Whiteley [1978, p.12] (see also Abstracts 760-D4 and 760-D5, Notices Amer. Math. Soc. 25(1978), A-642) quotes from Crapo-Whiteley [1978]: "...For a bar and joint structure with an underlying planar graph the result is very explicit: the set of bars is dependent iff it contains the projection of some non-degenerate plane-faced polyhedron from 3-space. ..." The example in Figure 1 - in which rods under compression are shown by solid lines, those under tension by dotted lines -- seems to us to contradict this statement, for it clearly does not arise from any "plane-faced polyhedron from 3-space".

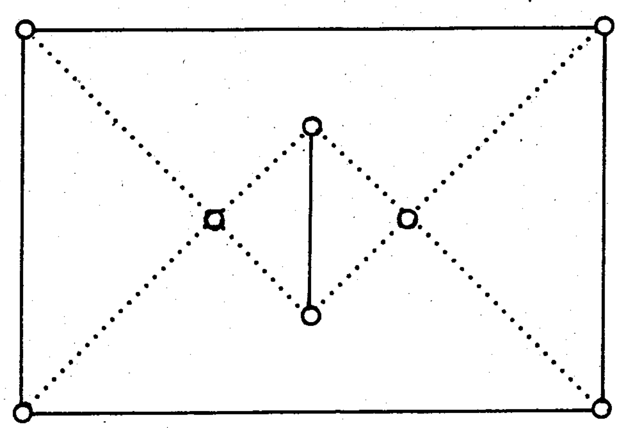

Figure 1

15. (page 2.6, line 15) Connelly [1978b] introduced the concept of "second order rigidity" and used it to obtain very interesting results.

16. (page 2.7, line 4) The conjecture has been disproved, even for the case in which the manifold is a sphere, by the spectacularly ingenious examples found by R. Connelly; see Connelly [1978], [1978a], Kuiper [1978].

17. (page 2.7, line -7) These "rings of tetrahedra" are now available as Schattschneider-Walker [1977].

18. (page 2.7, line -2) For a related result see Poznyak [1960].

19. (page 2.8, line -8) An affirmative solution is given in Asimow-Roth [1978].

20. (page 2.9, line -5) See, for example, Timoshenko-Young [1945, p. 189] .

21. (page 2.10, line 6) Conjecture 3 has been established by Aleksandrov [1950] (see also Asimow-Roth [1978a]) in case no vertex of the framework is a relatively interior point of any face of the convex hull of the framework. Without any such restriction Conjecture 3 has been proved by Connelly [1978b]; it appears to be contained also among the results of Whiteley [1976, Corollary 3.5 and Remark 2], though we were unable to follow the proof.

22. (page 2.10, line -8) The term "cabled frameworks", introduced by R. Connelly, is much more appropriate that the "tensed frameworks" used in the text.

23. (page 2.12, line 2) For the case of the cube see also D. Wells [1975].

24. (page 2.12, line 5) The conjecture made in Grünbaum - Shephard [1975, p. 31] concerning the minimal number of cables

needed to insure stiffness is disproved by the example of the dodecahedral cabled framework in Figure 12 of the text taken together with Connelly's result about Conjecture 4 (see below).

25. (page 2.12, line 8) The conjecture is invalid, even if restricted to convex frameworks (with no cables), as was shown by Connelly [1978b, Figure 19]. Possibly it becomes valid if it is required that each edge of the convex hull is a rod. A beautiful theorem of Connelly [1978b] establishes just that, under the added assumptions that the cabled framework has no nodes other than the vertices of its convex hull, and that for each face of the convex hull the planar cabled framework contained in the face is planarly infinitesimally rigid. A somewhat weaker result may be found in Whiteley [1976, Corollary 3.6]; however, we were unable to verify the proof (see Remark 14, above).

26. (page 2.12, line -3) To avoid misunderstandings, it should be stressed that each edge of a convex polygon is the intersection of the polygon with one of its supporting lines. (This notion is called "strictly convex polygon" by some authors.)

27. (page 2.13, line 11) B. Roth (private communication) has recently established a negative solution of the problem.

28. (page 2.13, line -9) Conjecture 5 has been established by R. Connelly (private communication).

29. (page 2.14, line 1) A variant of Conjecture 6 has been established by B. Roth (private communication).

30. (page 2.15, line -5) The finiteness of the number $f(r)$, together with several related results, has been established by Kahn [1978].

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Lectures on

LOST MATHEMATICS

1. Introduction.

Through a concatenation of events I was led, during the past several months, to the realization of the following fact, which I find rather disturbing:

In a considerable number of fields and professions, people are engaged in research of purely mathematical questions; we, as mathematicians, not only failed to answer those questions previously -- we are even post factum unaware that anything has been happening. No trace of this sometimes sizable literature is found in the Mathematical Reviews or in the other mathematical survey journals.

I find this situation distressing for many reasons, - but the most important one is that we are missing out on much beautiful mathematics. One aspect of the loss is that we are (generally) ignorant of many facts and theories - just because they do not fit into our (rather harebrained) curricula and the "mainstream of mathematics". Another aspect is even worse: Interesting, challenging and important mathematical problems are not considered at all, in many cases because the workers in other disciplines have neither the motivation nor the training to do so.

Let me illustrate these contentions with just a few examples. While I have no doubts that there are many other instances, my examples will (naturally) have a geometric context, since geometry happens to be a very exciting part of mathematics which I find altogether overly neglected.

Parenthetically I would like to add that it is not very surprising that geometry, and the mathematics of interest to various sciences, have been jointly neglected by "mainstream" mathematics. Sociologists of science will probably discover that in human activities there is the following analogy to the sometimes fatal concentration of pollutants in the ascending links of the food chain. In many of our endeavors, as the torch of inquiry is passed on to successive generations — a process with a time-span of just 5 or so years — the motivation and context of the questions we are investigating are being rapidly lost, and their objects assume an independent form of life, complete with procreation of derived questions. Naturally, after very few generations the process leads to deeply inbred questions — possibly very hard but rarely of interest to anybody but the most devoted followers of some particular cult. In contrast, problems arising from other sciences, and frequently also problems from geometry (which, after all, is almost a physical science) often lead to questions which have no "elegant" answer and hence tend to leave us with a feeling of frustration unless we are able to appreciate their inherent beauty. It is probably time to realize that "elegance" - or should we say "glibness" - is not always the most desirable goal in mathematics. Possibly a better attitude for many of us would be to try to make some headway with the "messy" questions; the unsolved parts would continue to prod us on and would help fight the incipient sterility of many mathematical disciplines.

I should like to stress that I am not proposing that we all turn into "applied" mathematicians overnight. I am not expecting us to go (or to even attempt to go) and solve problems in other areas, but - as has been happening throughout history - to obtain from the outside inspiration and motivation for purely mathematical investigations; it would seem reasonable to expect that if any worthwhile mathematics results, other disciplines would derive benefits as well.

Note that a similar interaction occured (and is still happening) in connection with computers. Many purely mathematical problems originated there, and mathematical insights proved to be useful in very "applied" settings.

Let's return to the examples I promised; several ofthem will be examined in future talks in more detail, and in some depth. Right now I only wish to transmit a feeling for the type of problems and interactions.

In chemistry, crystals of elements and of many organic compounds lead to purely geometric questions about tilings or packings by congruent objects, sometimes with certain side-conditions. The importance of questions of that nature was quite clear to Hilbert at the turn of the century, who included it among his famous problems.

(1) Despite that, we still do not even know what convex polygons are tiles for the plane ! (A polygon P is a tile if the plane can be covered by copies of P with disjoint interiors.) In contrast, chemists are interested in much more complicated possibilities - such as tilings with squares <u>and</u> heptagons (or other kinds of polygons), that possess a high degree of symmetry. The related questions concerning the crystallization of ionic or metallic compounds are (largely) not at all known to mathematicians. Not to mention one other aspect - the systematics and classification of crystals - which once occupied people like Schoenflies, but which has since then

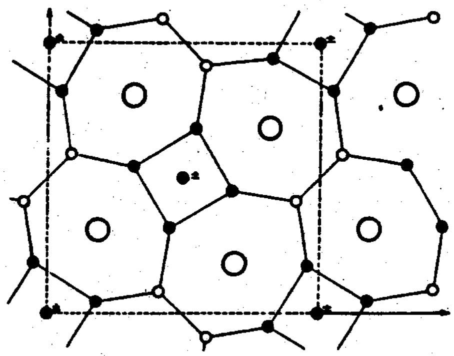
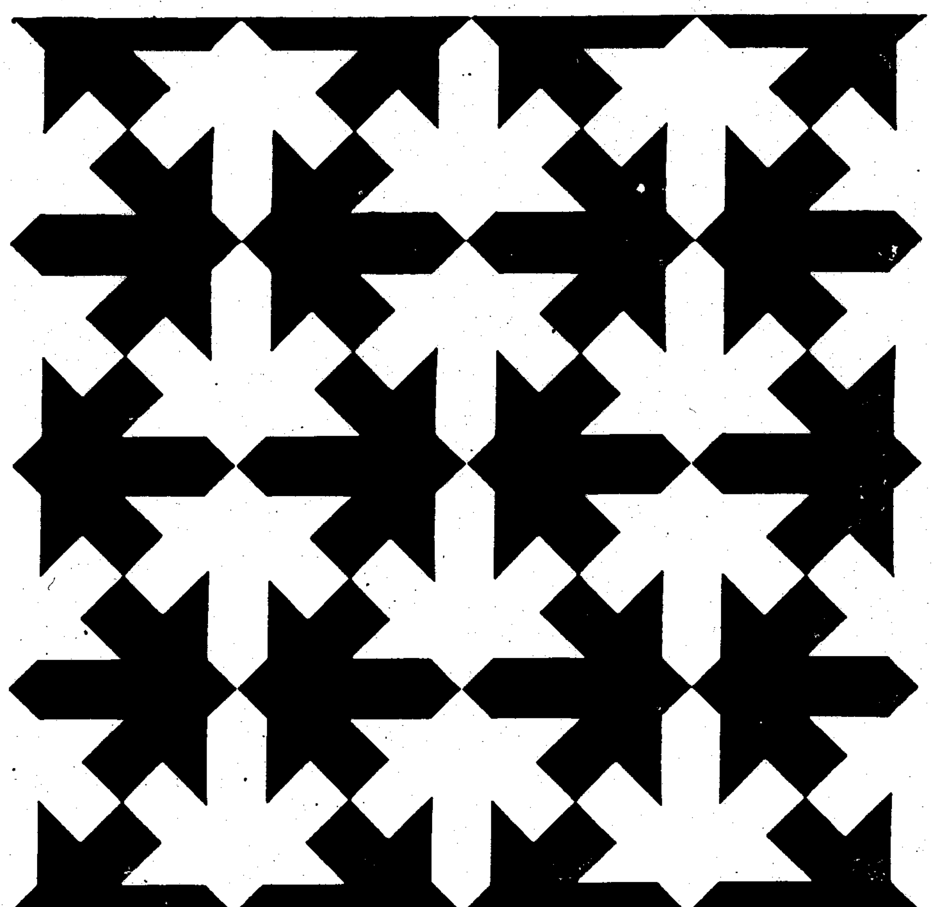


Figure 1. A periodic tiling of the plane by squares and heptagons. (From P. Blum and F. Bertaut, Acta Cryst. 7(1954), 81-86.)



REVERSAL AND ROTATION occur simultaneously in this ingenious design. When the stylized maple-leaf pattern alternates between black and white, it also rotates 90 degrees.

Figure 2. Some colored symmetries.
(a) 2 colors, from F. Attneave, Scientific Amer., Dec. 1971,62-71.

been completely excised from our collective mathematical consciousness. That's why, for example, the crystallography in higher dimensions is being actively pursued - but almost exclusively by physicists, chemists, etc. Similar is the situation concerning the so-called <u>colored</u> <u>symmetries</u> - a topic in which hard sciences make contact
(2) with art on a geometric and group-theoretic ground, of which mathematicians are totally ignorant.

I shall return to crystallography a bit later, but let me make here one more observation. All crystals, and similar structures periodic in space, are well known to have symmetries based on 2, 3 and their multiples. But recent experimental evidence indicates that pentagonal symmetry does occur, even if exceptionally. The explanation is that this phenomenon may be due to impurities - in other words to local perturbations of an otherwise regular tiling or packing. Now this is a question that leads at once to a whole class of mathematical problems of the following general type: If we inquire about tilings of the "punched" plane - that is, the plane from which a small (say finite) part was deleted - are there any new possibilities ? Here one should recall that already Kepler tried to tile the plane by various combinations of pentagons, decagons, pentagrams, etc. - and it turns out that this is indeed possible.

In another way of looking at certain types of crystals, chemists have been led to consider various polyhedra, associated with atoms or groups of atoms. Many of their results concerning packings
(3) of such polyhedra and similar topics were news to me. But an even greater mathematical challenge is presented by the "networks" considered by crystallographers. The question is to classify "regular networks" - that is, infinite 3-connected graphs with rectilinear

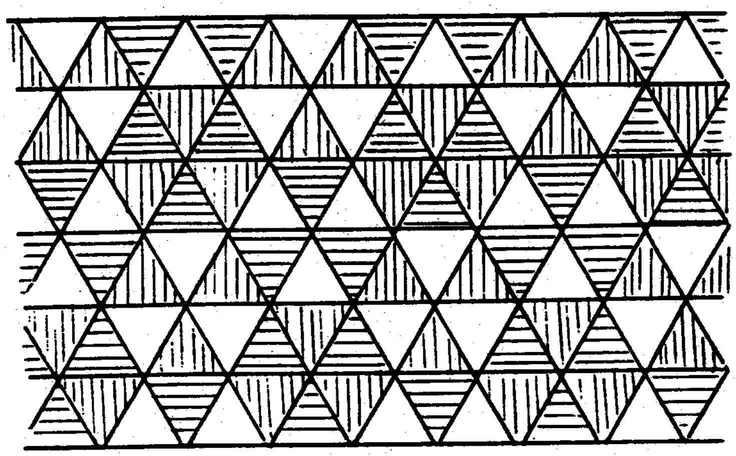


Figure 2(b). 3 colors, following A. L. Loeb, 'Color and Symmetry', Wiley-Interscience 1971.

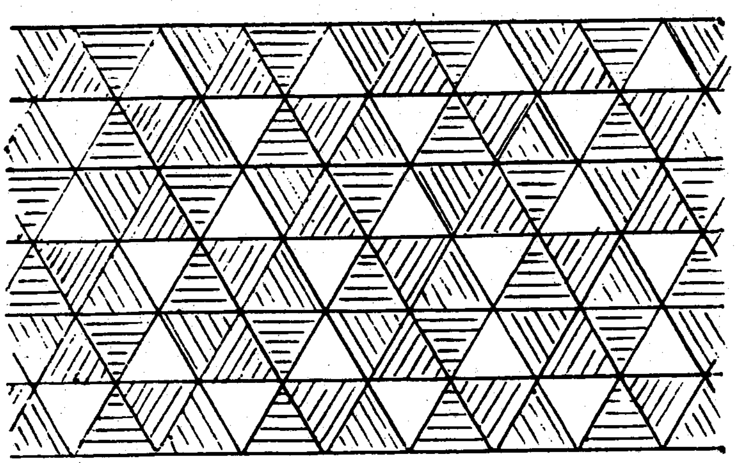


Figure 2(c). 4 colors.

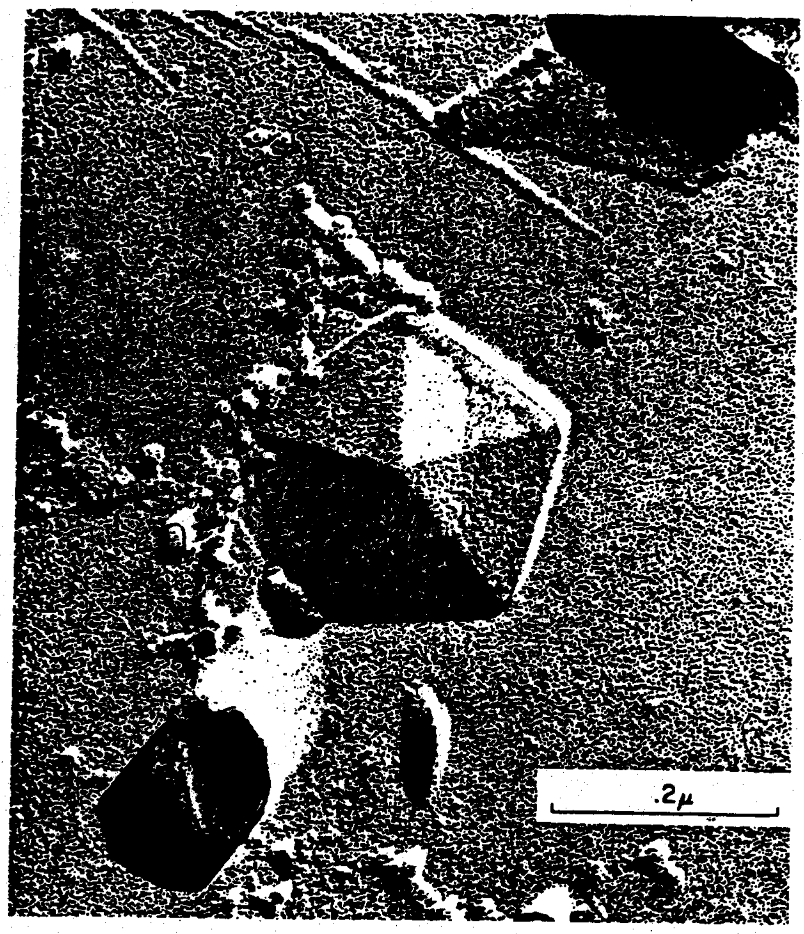


Figure 3. An anomalous structure with 5-fold rotational symmetry on the surface of a single crystal of gold. (From R. L. Schwoebel, J. Appl. Phys. 37(1966), 2515-2516.)

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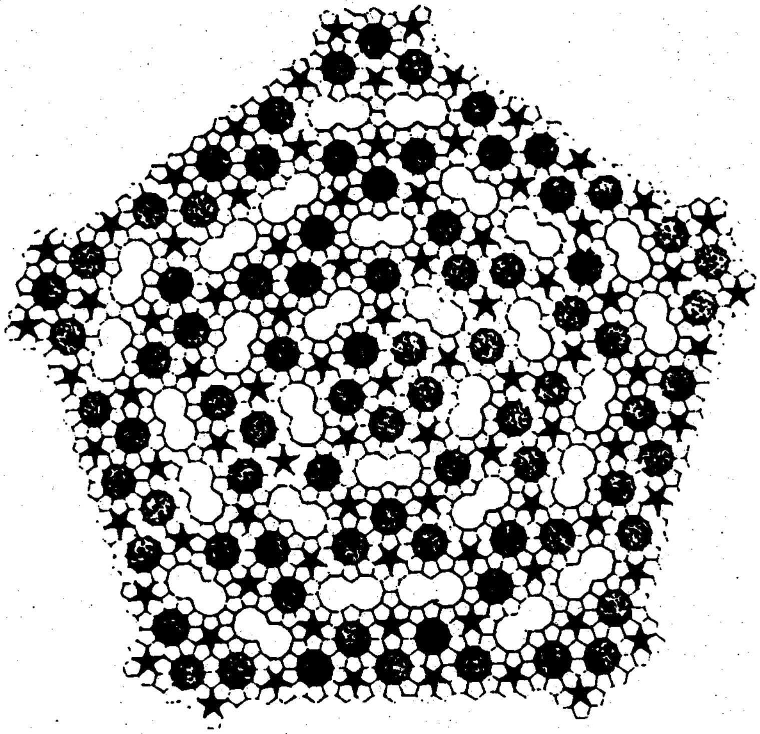
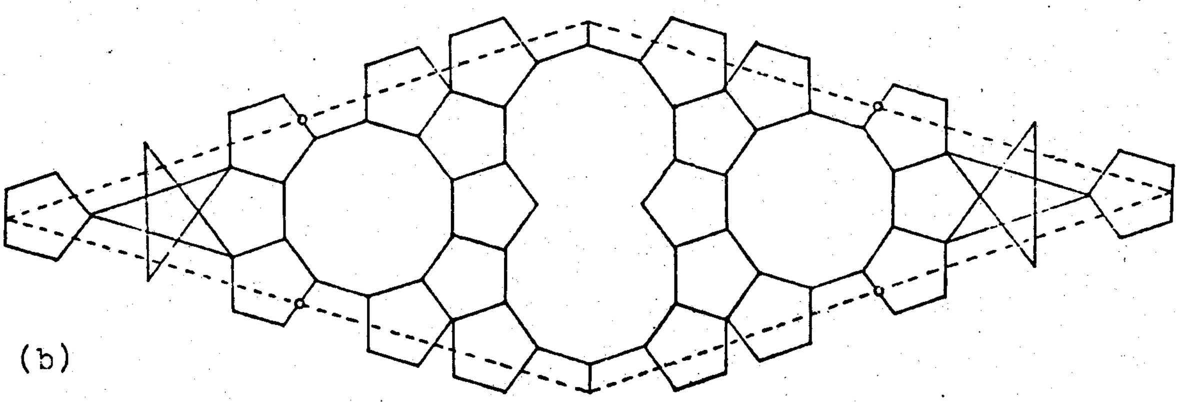


(b)



(c)

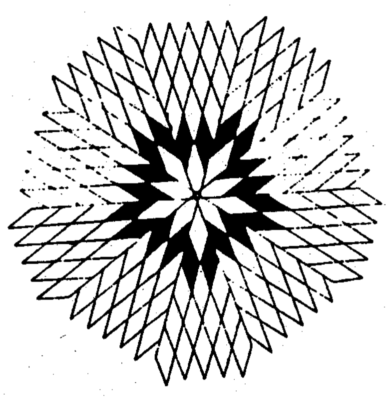
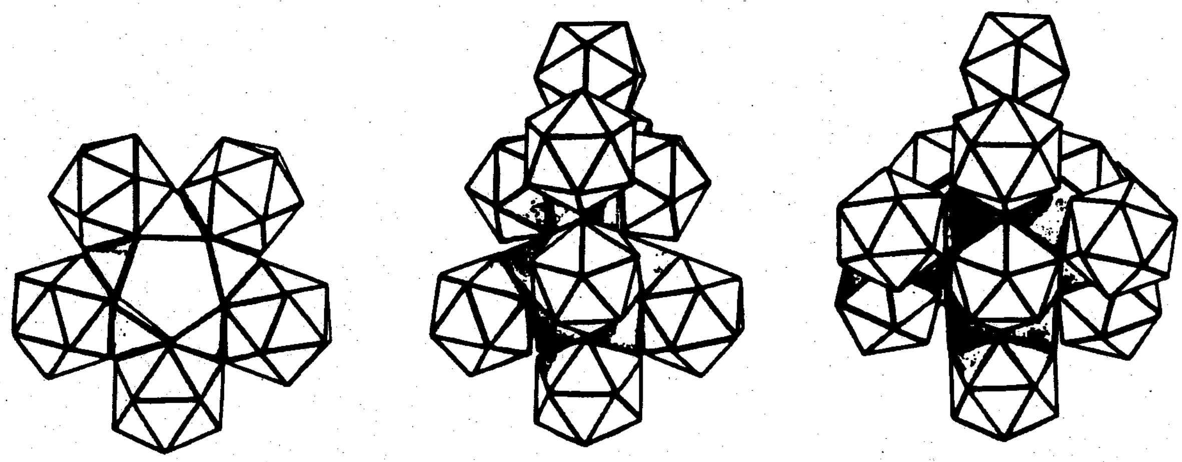
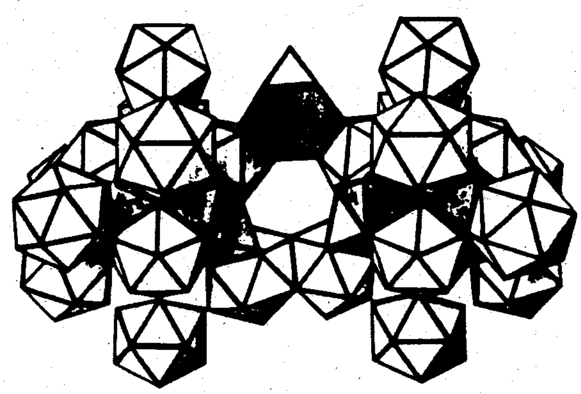


Figure 4. (From "Mathematical-Physical Correspondence", No. 12 (1975), edited by Stephen Eberhart.) (a) Kepler's tiling of the plane by pentagons, pentagrams, decagons, and "fused decagon pairs". (b),(c) Explanation of Kepler's tiling given by Wolfgang Dessecker in 1964; each small rhombus in (c) is a copy of (b).



(a) (b) (c)

Fig. 5 (a) The basic building block of Na_6Tl consisting of an aggregate of five icosahedra arranged approximately about a fivefold axis of symmetry. (b) Two such fivefold rings interpenetrating at right angles in such a way that the central pentagonal prism in (a) is shared by two icosahedra. (c) Six interpenetrating fivefold rings forming a complex of 14 icosahedra and 42 centered pentagonal prisms.



(d Two icosahedral complexes of the kind shown in Fig. 5(c) sharing one equilateral triangle each with a Laves–Friauf polyhedron (dark), the center of which is a fourfold inversion center. Hence, each Laves–Friauf polyhedron is shared between four such complexes which are tetrahedrally arranged around the $\bar{4}$ center. For the sake of perspicuity one icosahedron has been removed from each 98-atom complex.

Figure 5. Packings of icosahedra. (From S. Samson and D. A. Hansen, Acta Cryst. B28(1972), 930 - 935.)

edges, all of the same length equal to the minimal distance between vertices, such that geometric symmetries of the network act transitively on the vertices. A particular question, typical for problems in this area, is: How large can be the girth (=length of
(4) minimal circuit) of 3-valent regular networks ? The maximal known girth is 12 (in a network found by a chemist, A. F. Wells, in 1954), - but it is not known even whether there is an upper bound on girth !

A similar problem (which exhibits an analogous gap between the range of the empirically found examples and the total lack of theoretical bounds) was investigated mainly by architects: What polyhedral surfaces (that is, 2-manifolds) can be formed by regular
(5) convex polygons in 3-space, if one insists that (geometric) symmetries of the surface act transitively on the vertices. If only one type of polygons (n-gons) is allowed, and if k of them meet at each vertex, the following pairs (n,k) are known to be possible in non-planar and non-spherical surfaces of this kind: (3,6), (3,7), (3,8), (3,9), (3,10), (3,12), (4,4), (4,5), (4,6), (5,5), (6,4), (6,6). However, no other pair with $n \geqslant 3$ and $k \geqslant 4$ (except (3,4) and (3,5)) has been ruled out as a possibility.

A large area of unexplored ground is suggested by various <u>patterns</u> that occur in art, in nature, - and even in mathematics. Almost without exception mathematicians tend to equate geometric patterns with symmetries, and those with groups of transformations. But many simple, attractive, and important patterns fail to be expressible by groups in any reasonable way. Many such patterns can be found in the works of J. Albers, B. Riley and other painters. A different type of pattern underlies the tilings constructed by

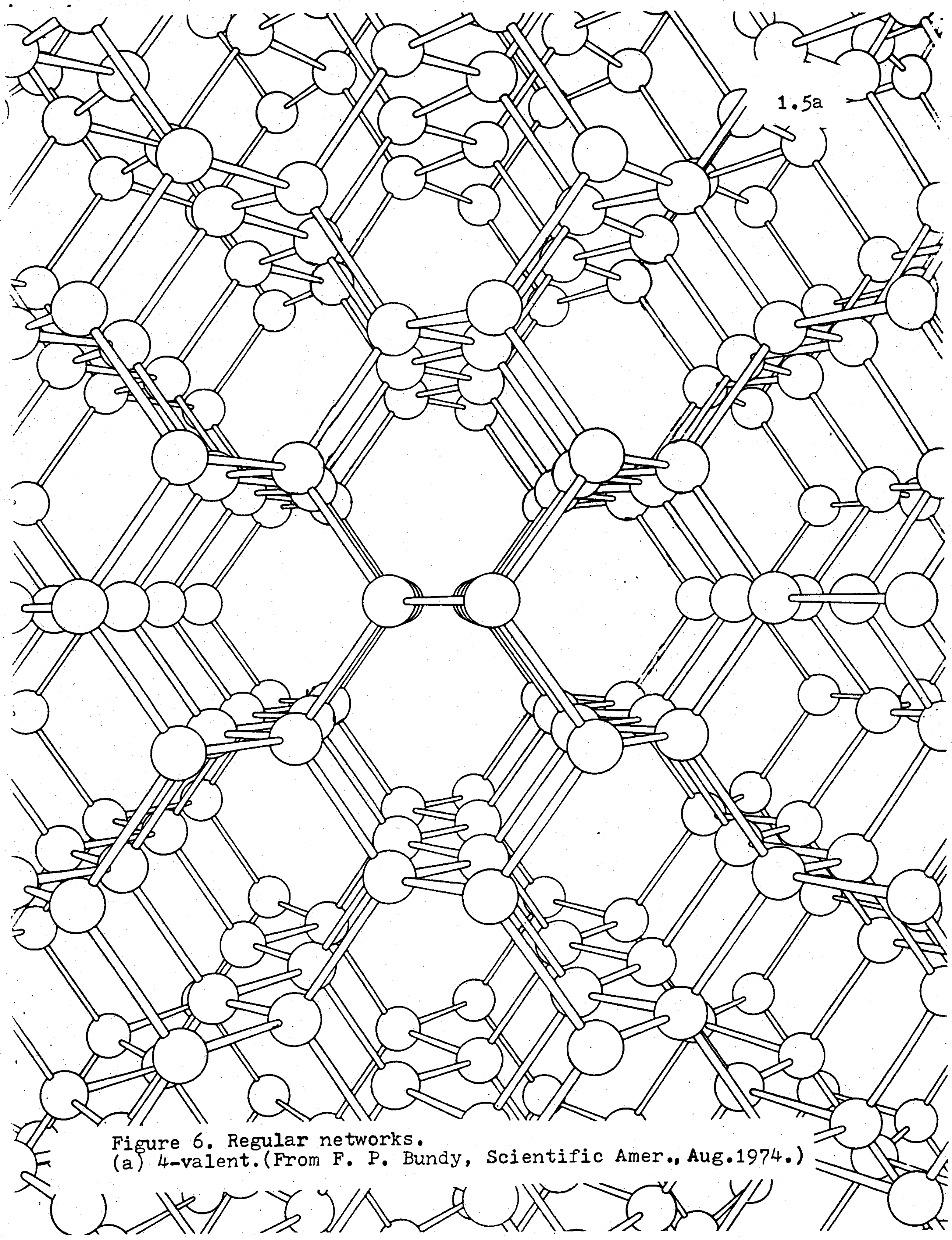


Figure 6. Regular networks.
(a) 4-valent.(From F. P. Bundy, Scientific Amer., Aug.1974.)

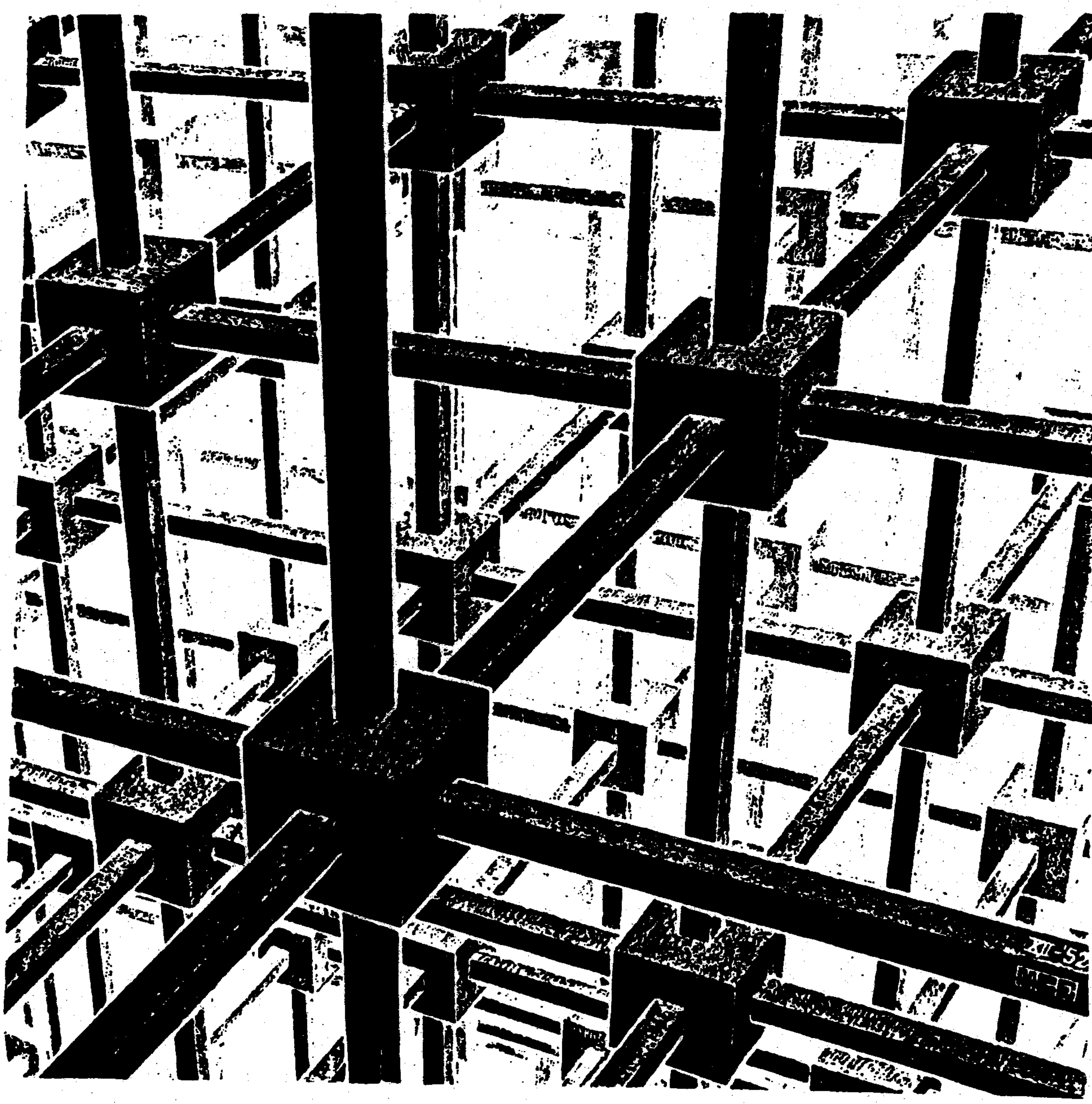


Figure 6(b). 6-valent (from 'The world of M. C. Escher', Abrams, New York 1971).

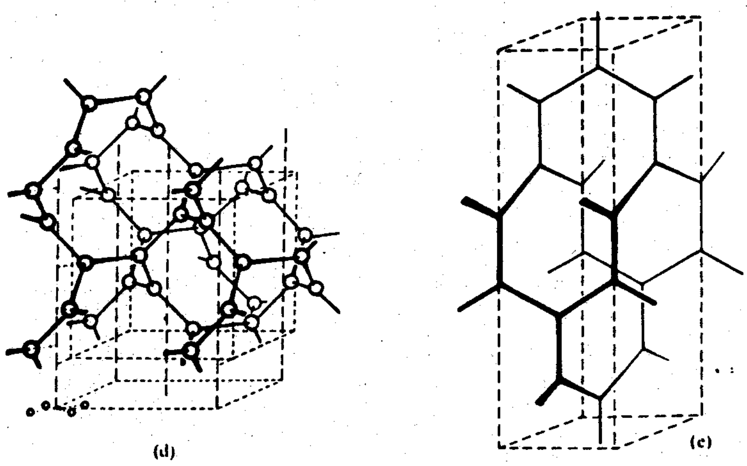


Figure 6(c). 3-valent.(From 'Models in Structural Inorganic Chemistry' by A. F. Wells, Oxford 1970.)

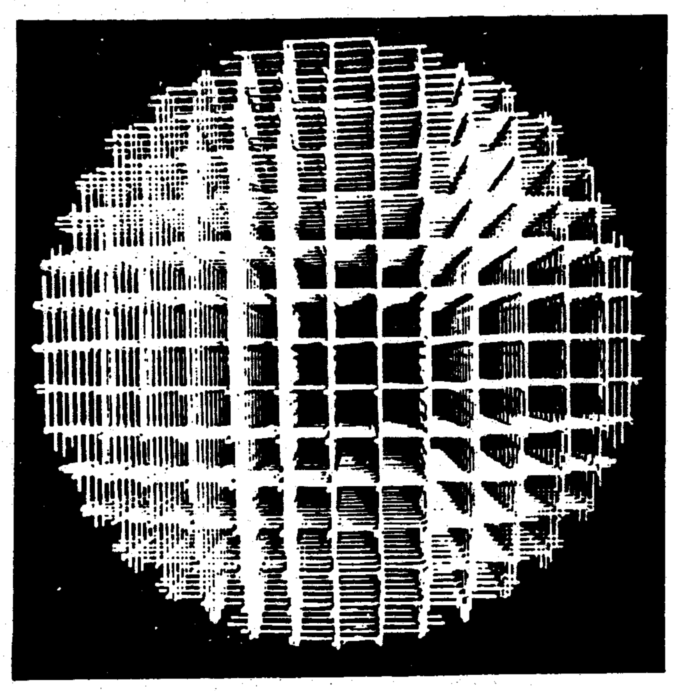
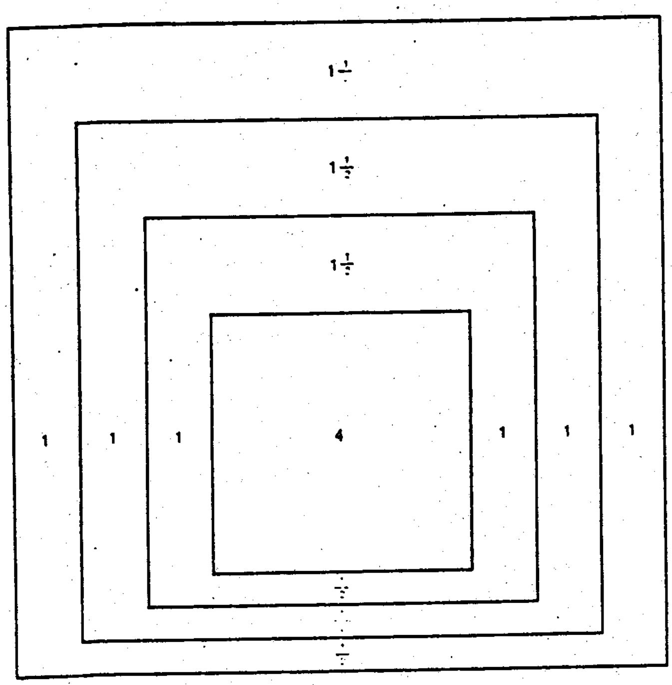
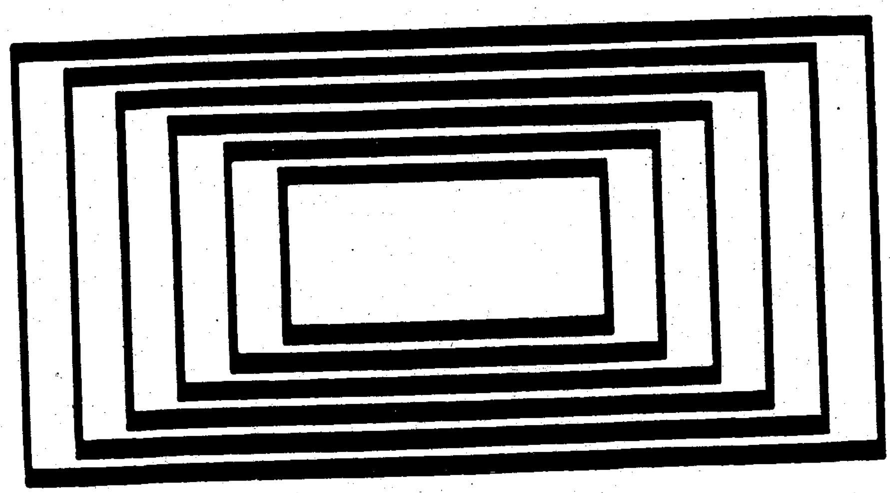


Figure 6(d). 6-valent.(Sculpture 'Sphere-Web' by F. Morellet, 1962.)



(a)



(b)

Figure 6. Some patterns in the work of J. Albers.
(a) Relative sizes of differently colored regions in several paintings.
(b) Part of a white and black painting.

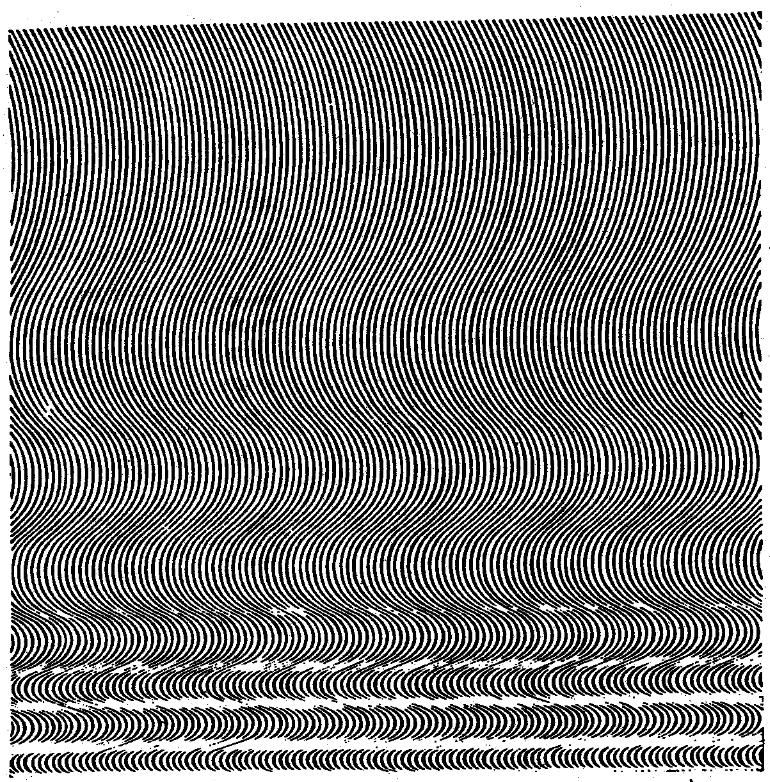


Figure 8. A painting by B. Riley (1963).

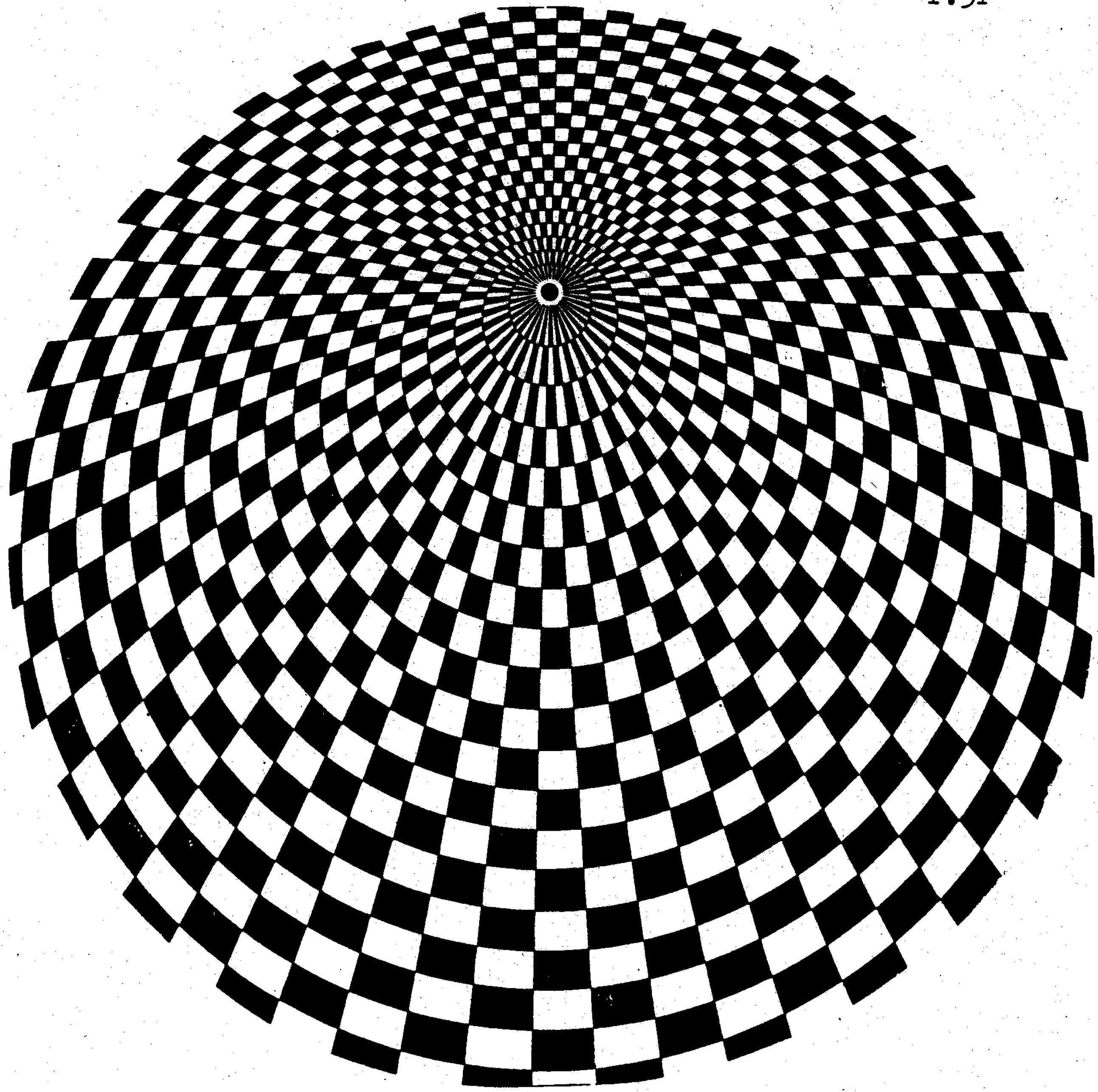


Figure 9. A design by S. Horemis (1970).

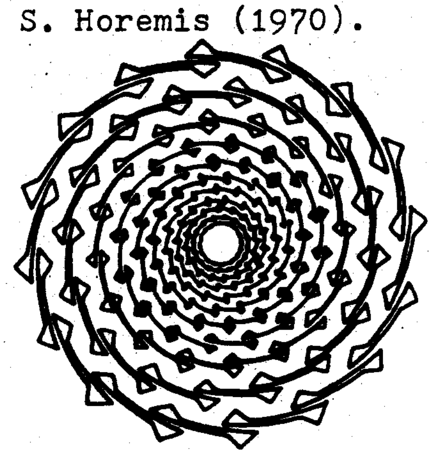


Figure 10. Design from an advertisment in the Scientific Amer.,April1969.

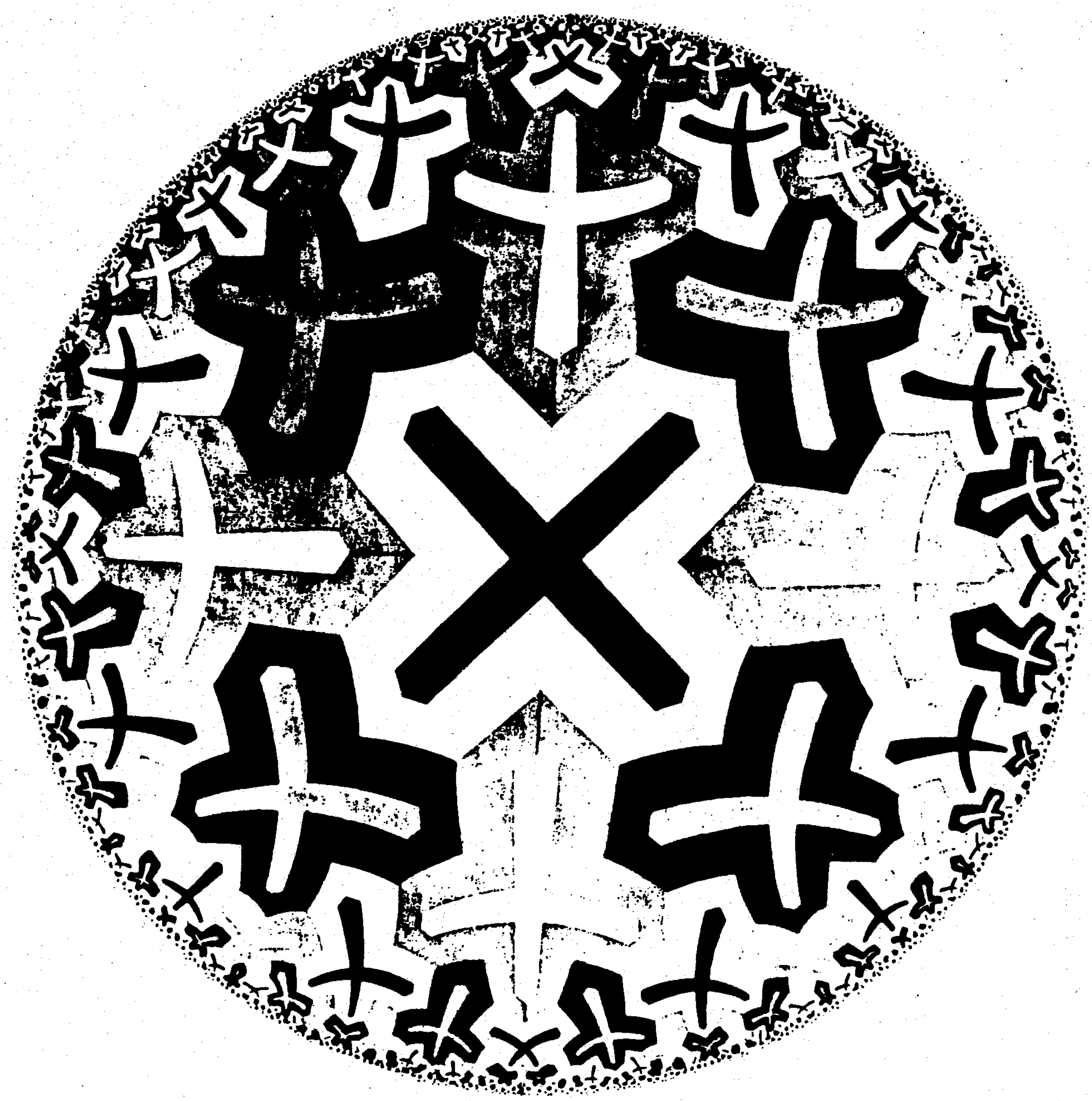


Figure 11. A woodcut by M. C. Echer.

(6) R. M. Robinson in his beautiful paper "Undecidability and nonperiodicity for tilings of the plane" (Inventiones Math. 12(1971), 177-209). It may be shown to have the remarkable property that for each finite part of it there exists a "critical size" such that any square in the plane that has that size contains a congruent copy of the chosen part. Possibly it would be worthwhile to investigate in general the consequences of this property, or of other traits found in different patterns.

One last (for today) group of problems is rooted in phenomena that are of interest to architects, engineers, some modern sculptors, and geometers: The rigidity or mobility of variously hinged systems of polygons, rods (= segments), cables, etc. Cauchy's theorem on the rigidity of polyhedra that have as faces rigid polygons hinged along common edges is probably the deepest known result. Another known fact is that every planar algebraic curve may be traced by a suitable
(7) planar linkage (that is, system of hinged rods); probably the most famous linkage is Peaucellier's, which draws a (segment of a) straight line, thus solving a problem which resisted Cayley, Sylvester and Chebišev - to mention just a few names. However, the related question of Hilbert whether every algebraic surface in 3-space can be traced by a suitable linkage, is still open. The engineering literature contains much that is of interest in this context; some of it is true, some possibly true but unproved, some false. For example, it is true (and follows at once from Cauchy's theorem) that if the edges of a simplicial convex polyhedron are replaced by rods hinged at their
(8) endpoints, the resulting structure will be rigid. The rigidity of the system formed by the edges of any convex polyhedron to which face-diagonals have been added in such a manner that each face is triangulated

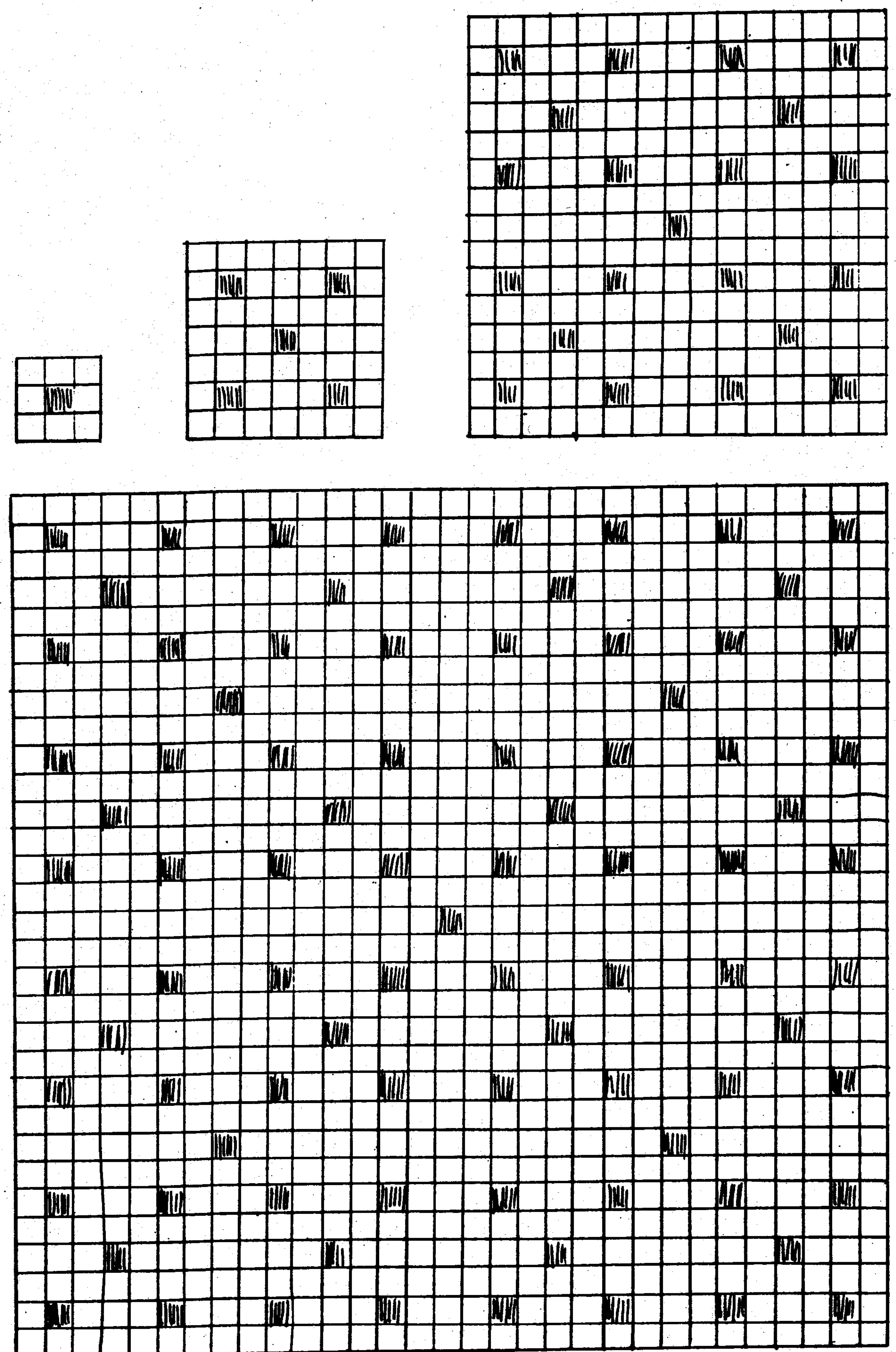


Figure 12. Stages in the formation of a non-periodic pattern underlying Robinson's contruction.

is frequently asserted. It is probably true, but I am not aware of any proof. The often repeated assertion that a non-planar rigid system with v vertices must have at least $3v-6$ edges (rods) is false. What happens if some rods are replaced by flexible but
9 inextensible cables has attracted the curiosity of architects (Buckminster Fuller, D. G. Emmerich), artists (Kenneth Snelson) and geometers (G.C. Shephard and myself) - but so far with very few results.

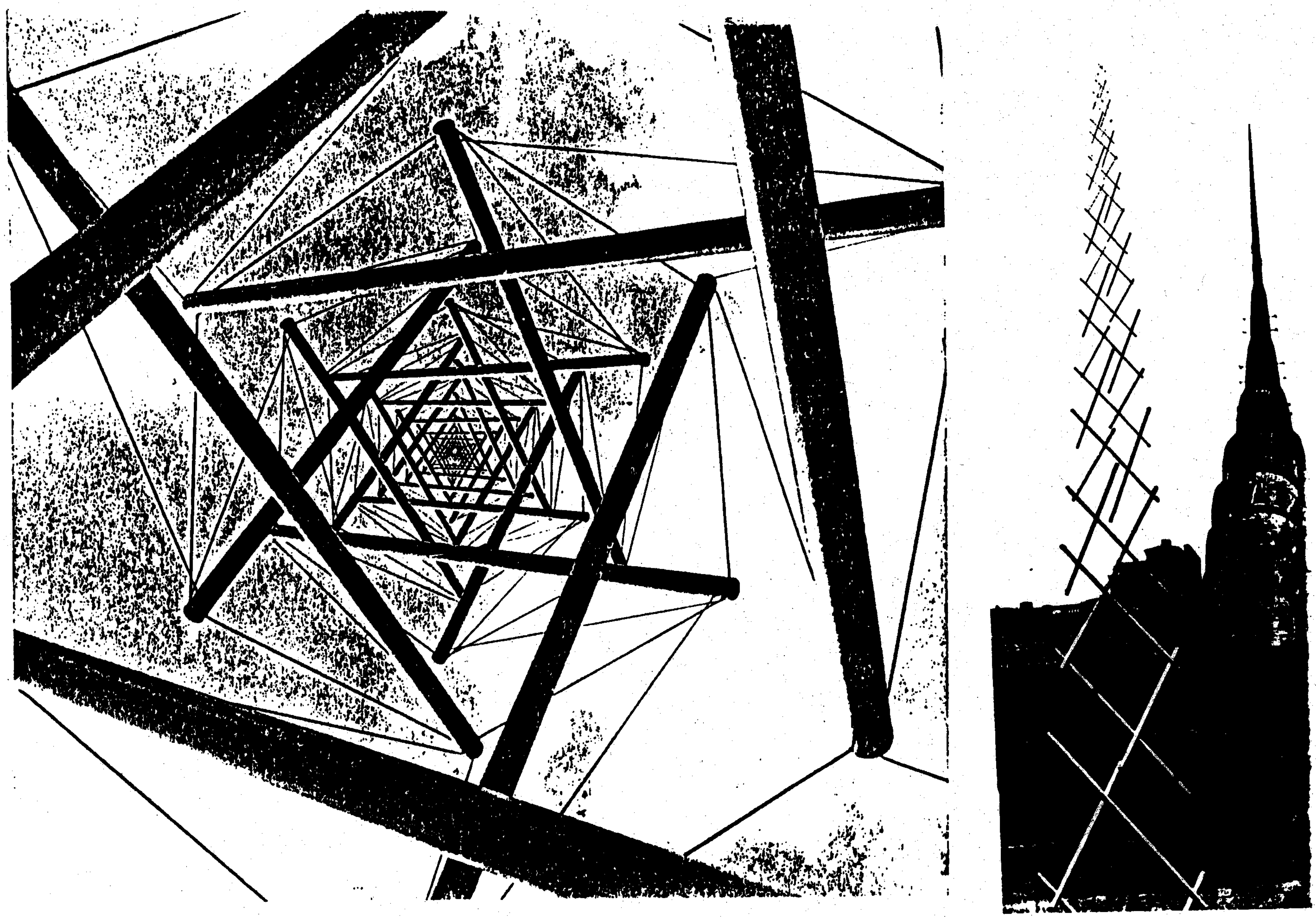


Figure 13. Two views of K. Snelson's 'Needle Tower', 1968.

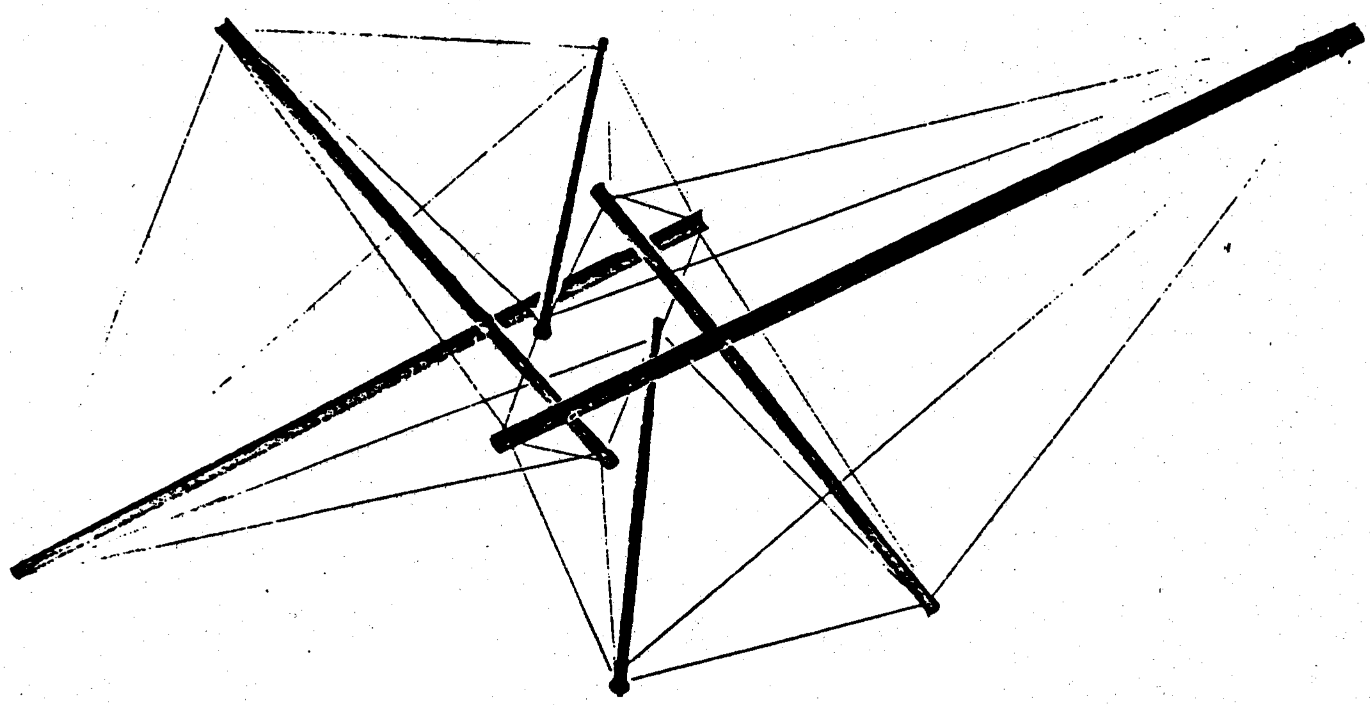


Figure 14. 'Northwood III' by K. Snelson, 1970.

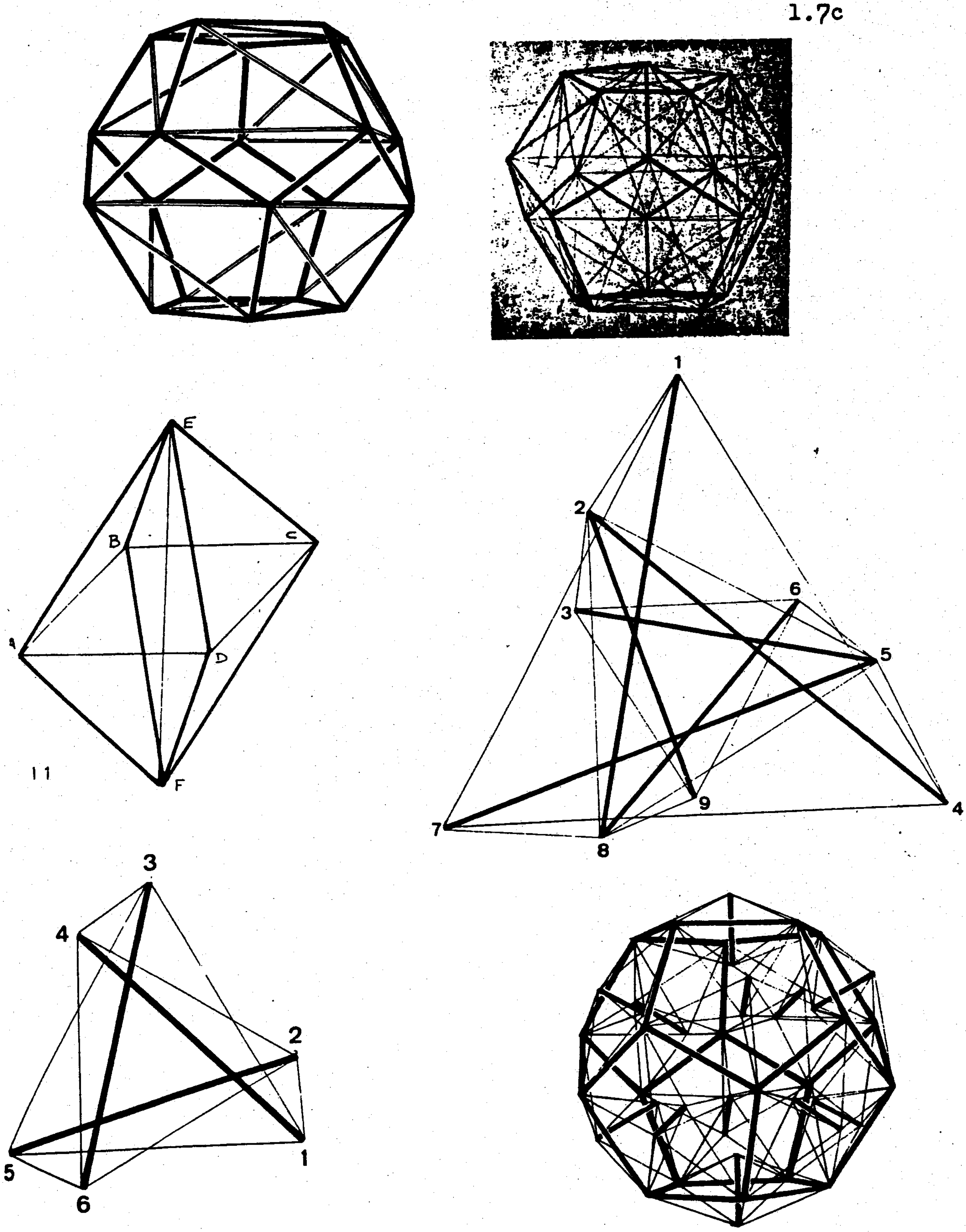
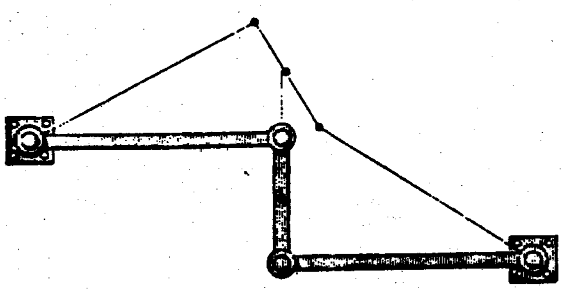
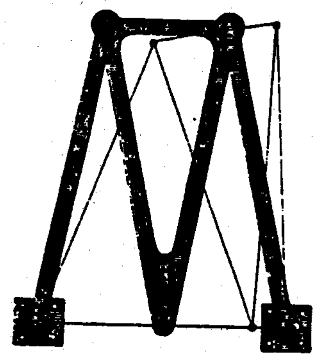


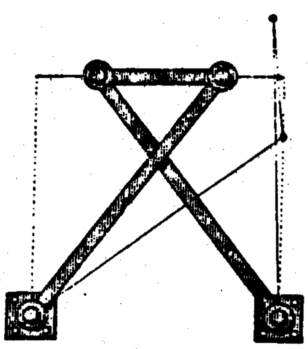
Figure 15. 'Structures autotendantes' from 'Exercices de Geometrie Constructive' by D. G. Emmerich, Paris 1967.



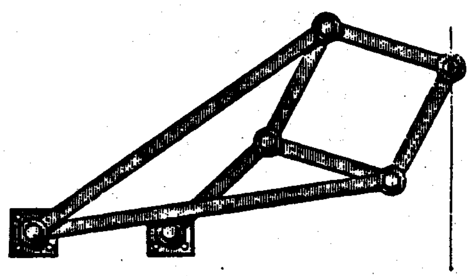
(a)



(b)



(c)



(d)

Figure 16. Various well-known simple linkages.
(a) Watt, (b) Roberts, (c) Čebyšev, (d) Peaucellier.

Branko Grünbaum:

University of Washington
Seattle, Fall 1975

Lectures on Lost Mathematics

2. Rigidity and stiffness.

A convenient starting place for this topic is the rather famous result known (since 1813) as Cauchy's rigidity theorem. In order to formulate it properly we need a few definitions.

Let P and P^* be convex polyhedra in the Euclidean 3-space E^3; we shall say that φ is an isomorphism between P and P^* provided φ is a one-to-one map from $V(P)$, the set of vertices of P, to $V(P^*)$, with the property that a subset $V \subset V(P)$ is the set of all vertices of a face of P is and only if $\varphi(V)$ is the set of all vertices of a face of P^*. (More intuitively, φ induces a one-to-one incidence-preserving map among the faces of the polyhedra.)

A map φ from a set A to a set B is an isometry of A and B provided the distance between any two points of A is the same as the distance between their images in B. It is well known (although usually not stated explicitly) and it follows easily from the theorems on congruence of triangles, that if φ is an isometry between the vertex sets $V(A)$ and $V(B)$ of convex polygons A and B in the plane, then φ may be extended to an isometry between the polygons A and B themselves. Cauchy's theorem is, essentially, an extension of this statement to convex polyhedra.

Theorem 1. (Cauchy [1813]) If φ is an isomorphism between convex polyhedra P and $P^* = \varphi(P)$, and if, for each face F of P, φ induces an isometry between $V(F)$ and $V(\varphi(F))$ (and hence also between F and $\varphi(F)$), then P and P^* are isometric (=congruent).

Less formally, Cauchy's theorem may be (and often is) stated as follows:

Theorem 1*. Two convex polyhedra for which corresponding faces are equal (=congruent) and equally arranged, have equal corresponding dihedral angles, and are themselves congruent.

(Note that we use the word "congruent" synonymously with "isometric"; congruent sets can be made to coincide either by rigid motion alone - in which case they are sometimes said to be "directly congruent" - or by reflection and rigid motion.)

Still more intuitively:

Theorem 1**. A convex polyhedron put together from rigid polygons in a specified manner is itself rigid.

Note that this formulation is suggested by - and provides an explanation for - the experimental fact that models of polyhedra made from cardboard polygons scotch-taped along common edges, are quite rigid when completed, although many fail to be rigid if even a single face is removed. An example of this failure of rigidity may be observed with the square antiprism from which one of the squares has been removed.

(Historical notes and references: Cauchy's theorem seems to have been suspected or suggested already by Legendre [1794]. Cauchy's [1813] proof consists of an elementary-geometric part (see below, p. 2.13) and a topological part, and is very ingenious and elegant; it was highly regarded and praised, and reproduced in the more advanced texts on elementary geometry published during the XIXth century. However, technical defects in the topological part were observed by Hadamard [1907] (see also Steinitz [1916]); various repairs were given by

Lebesgue [1909] and others (see the different editions of Hadamard's [1910] "Cours de geométrie"). But these efforts helped only partially because - as pointed out by E. Steinitz (see Steinitz-Rademacher [1934, Chapter 16]) the elementary-geometric part of the proof was also incomplete. The first complete - though rather long - proof of Cauchy's theorem was given in Steinitz-Rademacher [1934]; it was essentially reproduced in Lyusternik [1956], and the English translations of that booklet are the only books in English that contain any proof
(10) of Cauchy's theorem. A different (and simpler) complete proof is given
(11) in Aleksandrov [1950]; related to it is the proof by Stoker [1968].
Relatively simple direct proofs of the elementary-geometric lemma have been given also by Egloff [1956] and Schoenberg-Zaremba [1967].)

The rather delicate nature of Cauchy's theorem is probably best
(12) underlined by examples showing that it is not possible to weaken most of its assumptions. This also means that the more informal versions of the theorem have to be interpreted with care.

In order to present these examples in a reasonable manner we should remark that the above definitions of isomorphism and isometry do not really depend on the convexity of the polyhedra in question, and remain valid for "geometric polyhedral complexes" -- that is, families of convex polygons (and segments) with the property that the intersection of any two polygons is either an edge of both, or a vertex of both, or empty.

Example 1. There exist pairs of non-congruent convex polyhedra that satisfy all the conditions of Cauchy's theorem except that the isometries of the corresponding faces are not in all cases induced by the isomorphism φ . An example is indicated in Figure 1.

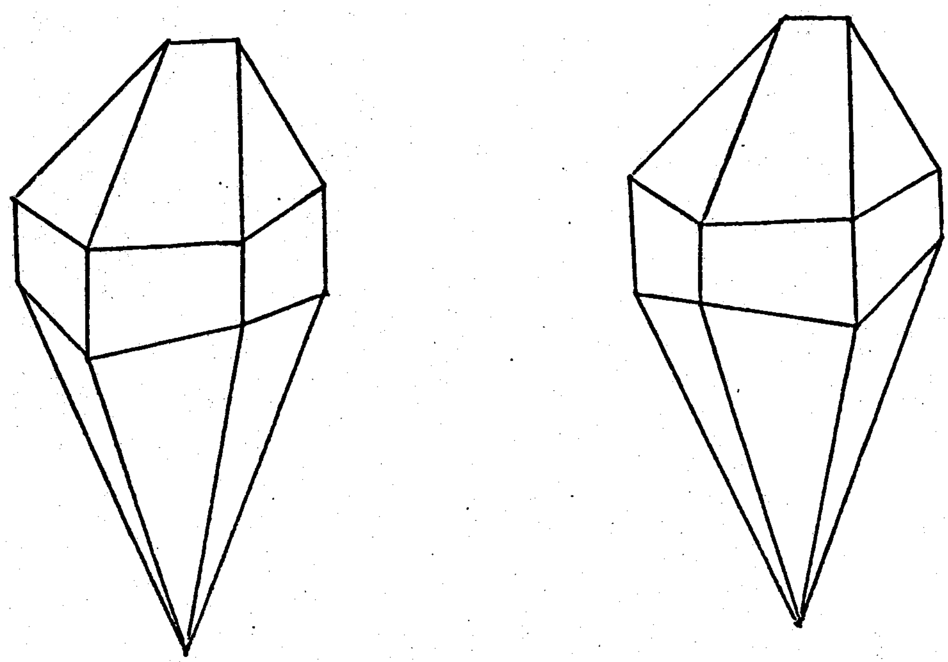


Figure 1.

Figure 2.

Example 2. If the convexity is not assumed for both P and P* Theorem 1 may fail; this may easily be seen by the example in Figure 2, that goes back at least to Legendre [1794].

It is tempting to suggest that the phenomenon exhibited in Example 2 is possible because in the two incongruent polyhedra with congruent faces certain pairs of edges that correspond to each other have different convexity or concavity character with respect to the whole polyhedron. However, following Wunderlich [1965] we have:

Example 3. There exist pairs of incongruent octahedra with corresponding faces congruent and corresponding edges of same convexity character.

From a cardboard model of one of those polyhedra the other is easily obtained by applying a slight twist. The simplest way to construct such "jumping polyhedra" is to start from a regular 3-sided right prism, and rotate the upper basis slightly (in its plane) with respect to the lower basis; the sides of the prism should be replaced by pairs of triangles, so as to obtain one concave edge in each side. In Figure 3 (adapted from Wunderlich [1965]) we show the view from above of such a pair of "jumping octahedra", together with their common "net".

An even more radical failure of the theorem occurs if the polyhedra in question are allowed to have selfintersections. (In other
13 words, if one considers polyhedral immersions, and not only embeddings, of the 2-sphere in E^3.) Following a query by Stephanos [1894], Bricard [1895] found selfintersecting octahedra for which there exists a whole one-parameter family of polyhedra with congruent corresponding faces, no two of which are congruent. These polyhedra are best described by their usual designation "movable octahedra". Later, Bricard [1897]

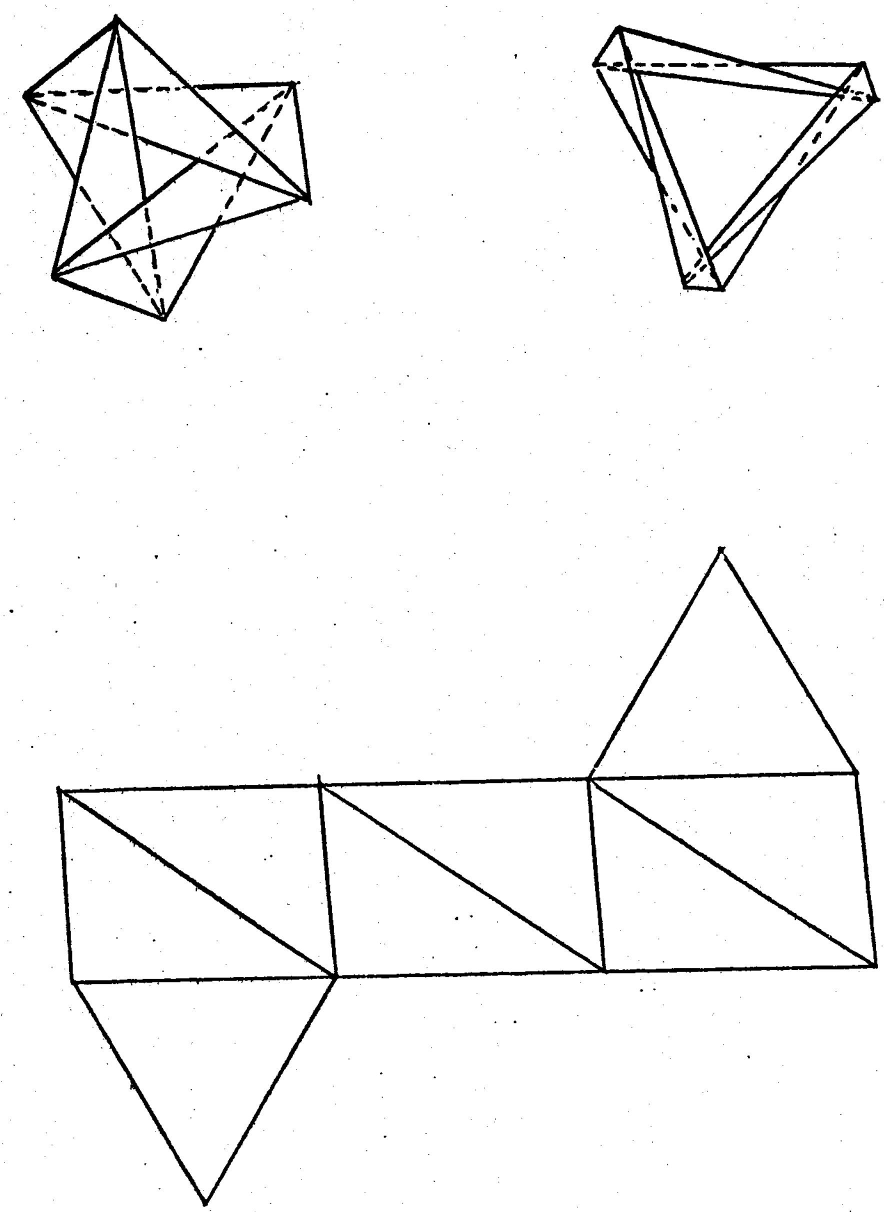


Figure 3. View from above (with top base removed), and net of a pair of "jumping octahedra".

gave a detailed analysis of all such possibilities, which showed that there are three distinct types of "movable octahedra" and completely determined their construction. Other papers on the same topic - extending Bricard's result, explaining it, or duplicating it (or parts of it) - are Bennett [1912], Wunderlich [1965], Lauwerier [1966], Bottema [1967], Lebesgue [1967], Dunitz-Waser [1972], Connelly [1975]. The topic has relations to the theory of linkages, some of which we shall mention later. It also is of interest in connection with the organic chemists' concerns about non-rigid ring-structures of molecules (see Dunitz-Waser [1972] and the references given there). Another direction of related investigations deals with the rigidity or mobility of "equilateral polygons"; see Grünbaum [1975] for results and references.

The simplest of Bricard's "movable octahedra" may be described as follows:

<u>Example 4</u>. Let ±a, ±b, ±c be the 6 points at unit distance from the origin on the coordinate axes. Let the triangles of an octahedron be as follows: a,-a,c; a,b,c; b,-b,c; -a,-b,c; a,-a,-c; a,b,-c; b,-b,-c; -a,-b,-c. If each of the triangles is rigid, but if the triangles are freely hinged along common edges, then the (selfintersecting) octahedron will be movable. (See Figure 4.)

Note that in this example the selfintersection occurs all along the segment c,-c ; in order to have physical movability it is simplest to delete two of the triangles, for example c,-b,c and a,-a,-c.

In order to describe the different types of failure in the attempted extensions of Cauchy's theorem exhibited by the above examples, and also to prepare for analogous situations in other contexts, we

(14) shall make the following definitions.

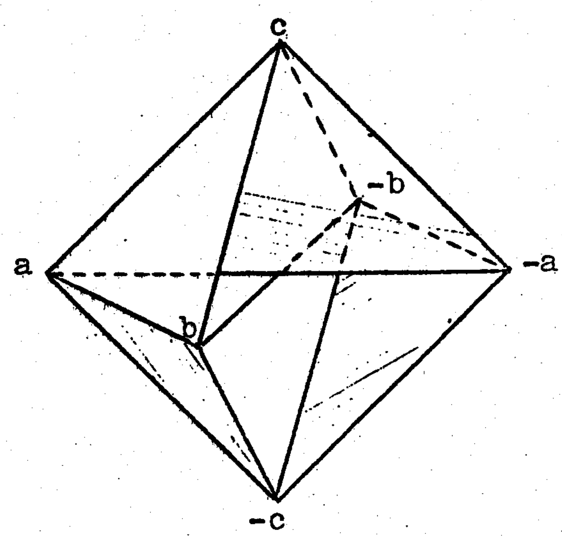


Figure 4.

We call a polyhedron P *rigid* (in a certain class of polyhedra) if it has the following property: Whenever P^* is a polyhedron (of that class) that is isomorphic to P and is such that the isomorphism induces isometries of the corresponding faces, then the isomorphism induces an isometry of P and P^*.

We call a polyhedron P *stiff* if for any family of polyhedra $P(t)$, with $0 \leq t \leq 1$, that depend continuously on t, the following holds: If $P(0) = P$ and if for each t $P(t)$ is isomorphic to $P(0)$ by an isomorphism that induces isometries of the corresponding faces, then $P(t)$ is isometric to P for all t.

Put simply, P is stiff if, assuming its faces are rigid, it may not be gradually deformed; P is rigid in a certain class if it may not be deformed even in a discontinuous manner (within the class in question). If P is not stiff we shall say it is *movable*.

(Other notions related to rigidity or stiffness have been
(15) considered in the literature, but we shall not dwell on them here. The most popular of these is the so-called "infinitesimal rigidity", which is motivated by mechanical as well as differential-geometric considerations; see, for example, Dehn [1916], Gluck [1975]. It should also be pointed out that the terminology varies from author to author, requiring care when comparing different texts.)

In the terminology we have just introduced, Cauchy's theorem asserts the rigidity of convex polyhedra in the class of convex polyhedra; Example 4 (and the example of the square antiprism from which one square was removed) deal with movable polyhedra, while those in Examples 2 and 3 are stiff but not rigid.

One of the most attractive open problems is this area is the following conjecture, a variant of which was made by Euler in 1766 (see Gluck [1975]):

(16) Conjecture 1. Every geometric polyhedral complex which has a closed 2-manifold embedded in E^3 as its underlying point set, is stiff.

It should be noted that various apparent counterexamples to Conjecture 1 -- Chinese lanterns, bellows, accordions, etc. -- depart from polyhedrality, rigidity, or isometry in the intermediate stages.

Conjecture 1 is also of a rather delicately balanced nature. As shown by Bricard's movable octahedra it fails if immersions are allowed; it also fails if the embedded manifold has a boundary, and it is invalid if "almost-manifolds" are allowed in which edges may belong to more than two faces. For rather spectacular "rings of tetrahedra" that demonstrate the last assertion see Ball-Coxeter [1974, pp.154, 215], Wheeler [1974]; particularly intriguing specimens, skillfuly decorated by suitable drawings of M. C. Escher, were demonstrated at the Annual Meeting of the American Mathematical Society in
(17) January 1975 by Prof. D. W. Schattschneider. The failure in the case of manifolds with boundary is very attractively demonstrated by the so-called flexagons (see, for example, Gardner [1959, pp. 1 - 14], [1961, pp.24-31], where also references to the literature may be found). As mildly supporting evidence for the conjecture we may mention the
(18) recent result of Gluck [1975] that "almost all" (in some sense) polyhedral spheres embedded in E^3 are stiff.

*   *   *

A somewhat different type of rigidity problems is of interest in mechanical engineering and architecture. Abstracting from questions arising in the design of buildings, bridges, towers, etc., we are led to the notion of *frameworks* (or *rod-structures*). Those are special geometric polyhedral complexes which are formed by edges and vertices only, without any 2-dimensional polygons. Thus we may think of a framework as consisting of rigid *rods* (*edges*), connected at their endpoints (*joints*, *vertices*), and we ask the obviously practical questions about the rigidity or stiffness of such structures.

A special class of frameworks are the *polyhedral skeleta*; a polyhedral skeleton is a framework the edges of which coincide with the edges of a convex polyhedron. Recalling that a polyhedron is called *simplicial* if all its faces are triangles, we have:

Theorem 2. Polyhedral skeleta of simplicial convex polyhedra are rigid (hence stiff).

For a proof we need only to observe that the lengths of the sides of a triangle determine the triangle (up to isometries), and to invoke Cauchy's rigidity theorem.

As a complement of Theorem 2 we have (compare Fuller [1975, p.319], where a vague statement of the same character is dogmatically affirmed):

(19) *Conjecture 2.* The polyhedral skeleton of a nonsimplicial convex polyhedron is never stiff in the class of frameworks.

Caution is needed in connection with Conjecture 2 since, as is easily seen, there exist nonsimplicial convex polyhedra such that their polyhedral skeletons are even rigid in the *class* of *polyhedral skeleta*; for example, each regular pyramid has that property. However, there also exist convex polyhedra for which the polyhedral skeleton is not rigid in the class of polyhedral skeleta; examples to that effect, that satisfy very stringent additional conditions, were given by Danzer [1967]. The polyhedral skeleton of a cube shows that even movability in the class of polyhedral skeleta is possible. The problem of characterizing polyhedra with skeleta rigid in the class of polyhedral skeleta was posed by Jessen [1967] and again by Shephard [1968], but at present there is not even a conjecture as to its solution. Similarly open is the question obtained by asking for stiffness instead of rigidity.

A result often quoted in the engineering literature (see, for example, Parkes [1974, p. 48] for an equivalent formulation (see also Section 3 of these notes)) asserts that a stiff framework with v vertices has at least $e = 3v-6$ rods. While not all the ramifications of that question are clear at present, the assertion is certainly false in the generality in which it is interpreted at least by some authors (for example, Cox [1936]; see below, page 2.14). Since the graph of a convex polyhedron P with v vertices has $3v-6$ edges if and only if P is simplicial, even for frameworks that are polyhedral skeleta the validity of the assertion depends on that of Conjecture 2. In the slightly more general case of <u>convex</u> frameworks (see definition below) the assertion is certainly false unless some additional restrictions are imposed. It seems that the basis of the above assertion is the confusion of necessary conditions with sufficient ones, and of convex frameworks with non-convex ones.

Engineering experience strongly suggests the following conjecture, which appears to be accepted as a fact in all relevant literature. It should be stressed, however, that the aims, the terminology, and the methodology of the engineering literature are so different from those in mathematics that a direct comparison or quotation are well-nigh impossible.

For convenience in the sequel, we shall call a framework <u>convex</u> provided all its rods (edges) are contained in the boundary of its convex hull - that is, in the boundary of the convex hull of its joints (vertices). Clearly polyhedral skeleta are convex frameworks, and so are the frameworks of Figures 5, 6, 7.

(21) <u>Conjecture 3</u>. If the edges of a convex framework triangulate the boundary of its convex hull then the framework is stiff.

(A similar question could be posed regarding rigidity; however, there the situation is complicated by the possibility of "folding",- compare Figure 8.)

There seems to be no way in which an affirmative answer to Conjecture 3 can be derived from Cauchy's rigidity theorem, or even from the stronger results of Aleksandrov [1950] on the determination of convex polyhedra through the "inner metric". The reason for the inapplicability of these results is the possibility of failure of stiffness by such a deformation of the framework that all the intermediate stages fail to be convex.

* * *

Before proceeding with the discussion of frameworks, we shall extend the scope of the considerations by introducing the more general
(22) <u>tensed frameworks</u>. The mechanical interpretation of a tensed framework is by a collection of rods and inextensible (but flexible) <u>cables</u>. In purely mathematical terms, a <u>tensed framework</u> is a (finite) set of points (its <u>vertices</u>, or <u>joints</u>)for certain pairs of which their distance apart is prescribed (those are the <u>edges</u>, or <u>rods</u>) while for other pairs only an upper bound on the mutual distance is given (<u>cables</u> of the tensed framework). A <u>member</u> of a tensed framework is either a rod or a cable. Clearly, frameworks and tensed frameworks are objects

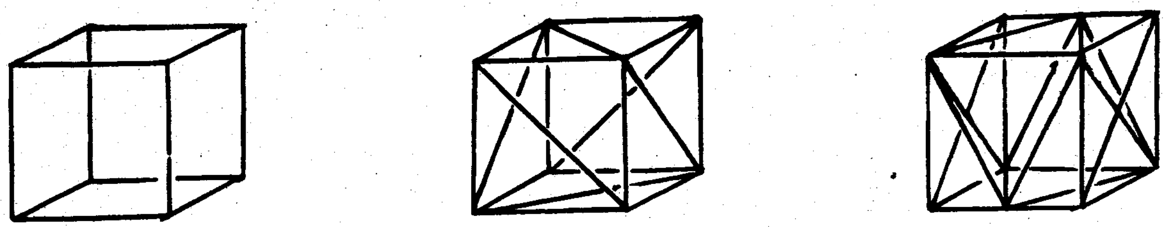


Figure 5.

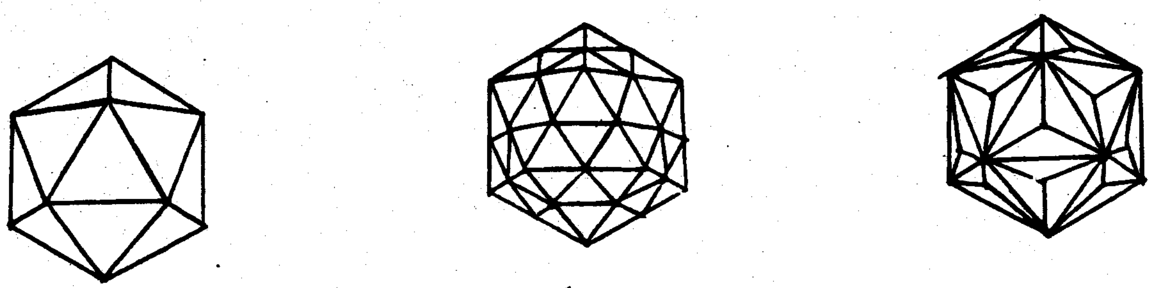


Figure 6.
(Front view only.)

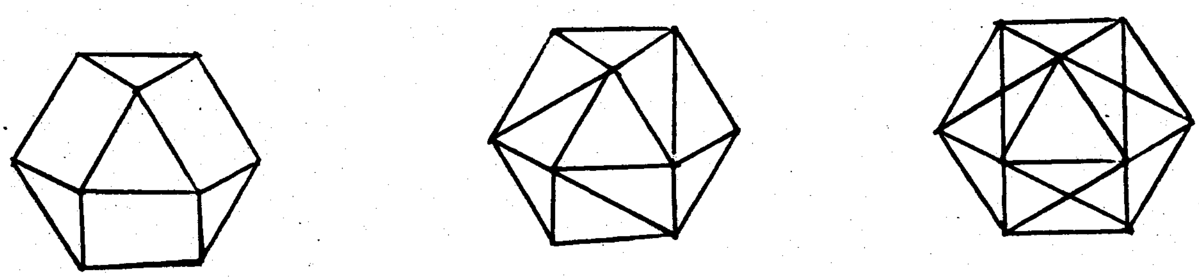


Figure 7.
(Front view only.)

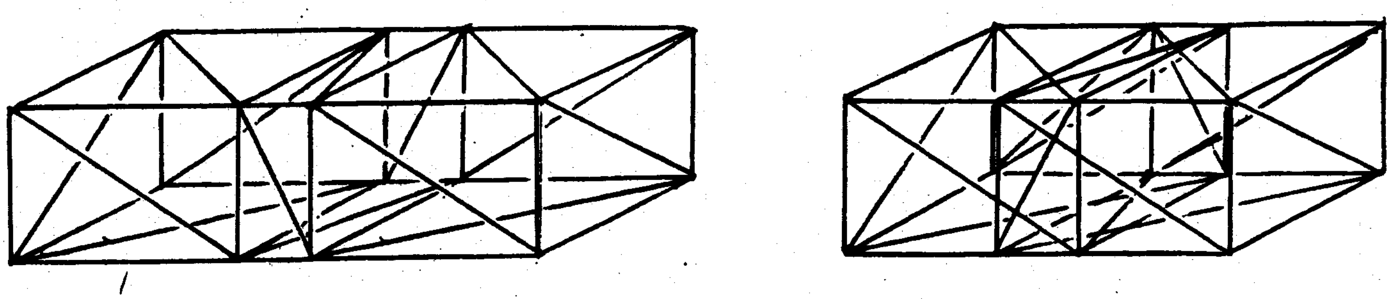


Figure 8.

that fit well into the domain of "metric geometry" (see Blumenthal [1953] for a detailed exposition of this branch of mathematics, and for references to the literature), although it seems that they have not been investigated from this point of view.

It is obvious from the definition of tensed frameworks that we on purpose disregard the problems that arise in the mechanical-physical realizations from the possibilities of interpenetration of cables with each other or with rods, and of knotting of cables or of "sideways" support of rods by other rods or cables. In this respect even tensed frameworks consisting exclusively of rods are more general than frameworks, since the definition of the latter excludes the "crossing" of rods.

In Figures 9 - 20 we illustrate some tensed frameworks; other examples were given in Figures 13, 14, and 15 of Section 1. In all these illustrations rods are indicated by heavy lines, cables by thin ones, and joints by small circles.

A tensed framework F is <u>convex</u> provided all segments between pairs of points that are the endpoints of a member of F belong to the boundary of the <u>convex hull</u> of F (that is, of the vertices of F). A <u>face</u> of a tensed framework F is the tensed framework formed by those members of F that have both endpoints in a fixed face of the convex hull of F .

The notions of <u>rigidity</u> and <u>stiffness</u> extend to tensed frameworks in the obvious way.

Questions of stiffness of tensed frameworks arise from and in the work of architectural designers like Fuller and Emmerich (see, for example, Fuller [1975], Emmerich [1967]) and the artist Snelson (see Snelson [1971]). Of those, only Emmerich appears to have been aware of the possibility of stiffening the polyhedral skeleton of a nonsimplicial

Figure 9.

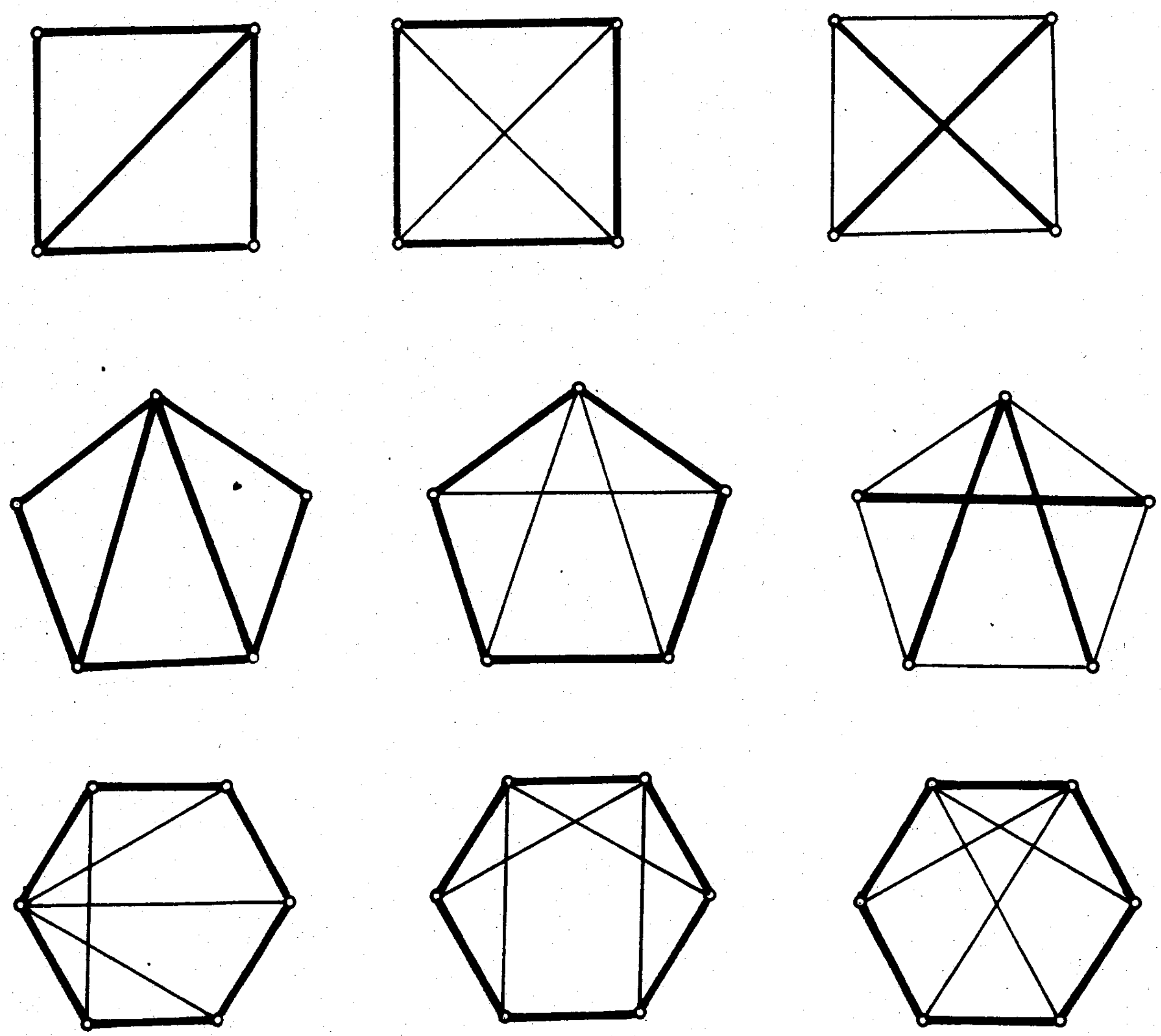


Figure 10. Examples of planarly-stiff planar tensed frameworks.

polyhedron through the addition of cables that turn it into a convex
(23) tensed framework (see the example at top right of Figure 15 in Section 1). Independently, the general conjecture that every polyhedral skeleton can be stiffened by inserting as cables sufficiently many (possibly all)
(24) the diagonals of its faces was made by Grünbaum-Shephard [1975].

Making this conjecture more precise while at the same time generalizing Conjecture 3, we propose:

(25) Conjecture 4. A convex tensed framework F is stiff whenever each face of F is a planarly-stiff tensed framework.

Here a tensed framework F is called planarly-stiff provided F is contained in a plane and is stiff in the class of tensed frameworks contained in that plane. Examples of planar tensed frameworks that are planarly stiff, and of such that are not, are given in Figures 10 and 11.

(Planar frameworks that are planarly-infinitesimally rigid have been investigated by Laman [1970].)

As supporting evidence for Conjecture 4 we may mention - besides its esthetic appeal - the fact that its assertion is experimentally confirmed on rather large models in the two cases indicated in Figure 12 ; note that the dodecahedral example has 24 cables less than the model of Emmerich (Figure 15 of Section 1).

Planarly-stiff tensed frameworks are - it seems - not yet well understood either. To illustrate this contention, let a tensed rod-polygon be any planar tensed framework in which the rods form precisely the
(26) set of edges of a convex polygon. One affirmative result known about tensed rod-polygons may be formulated as follows (see illustrations in center column of Figure 10).

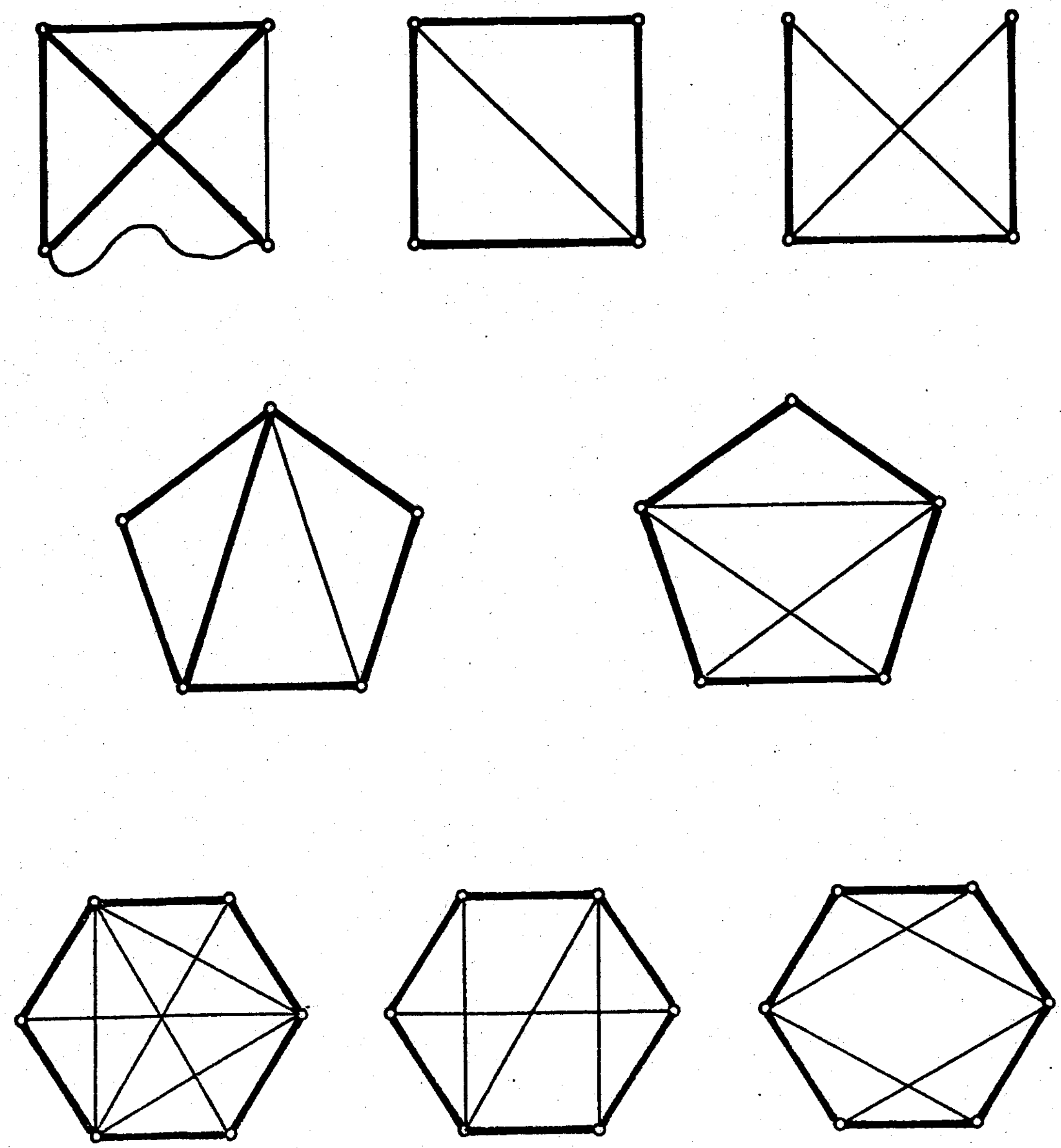


Figure 11. Examples of planar tensed frameworks that are not planarly-stiff.

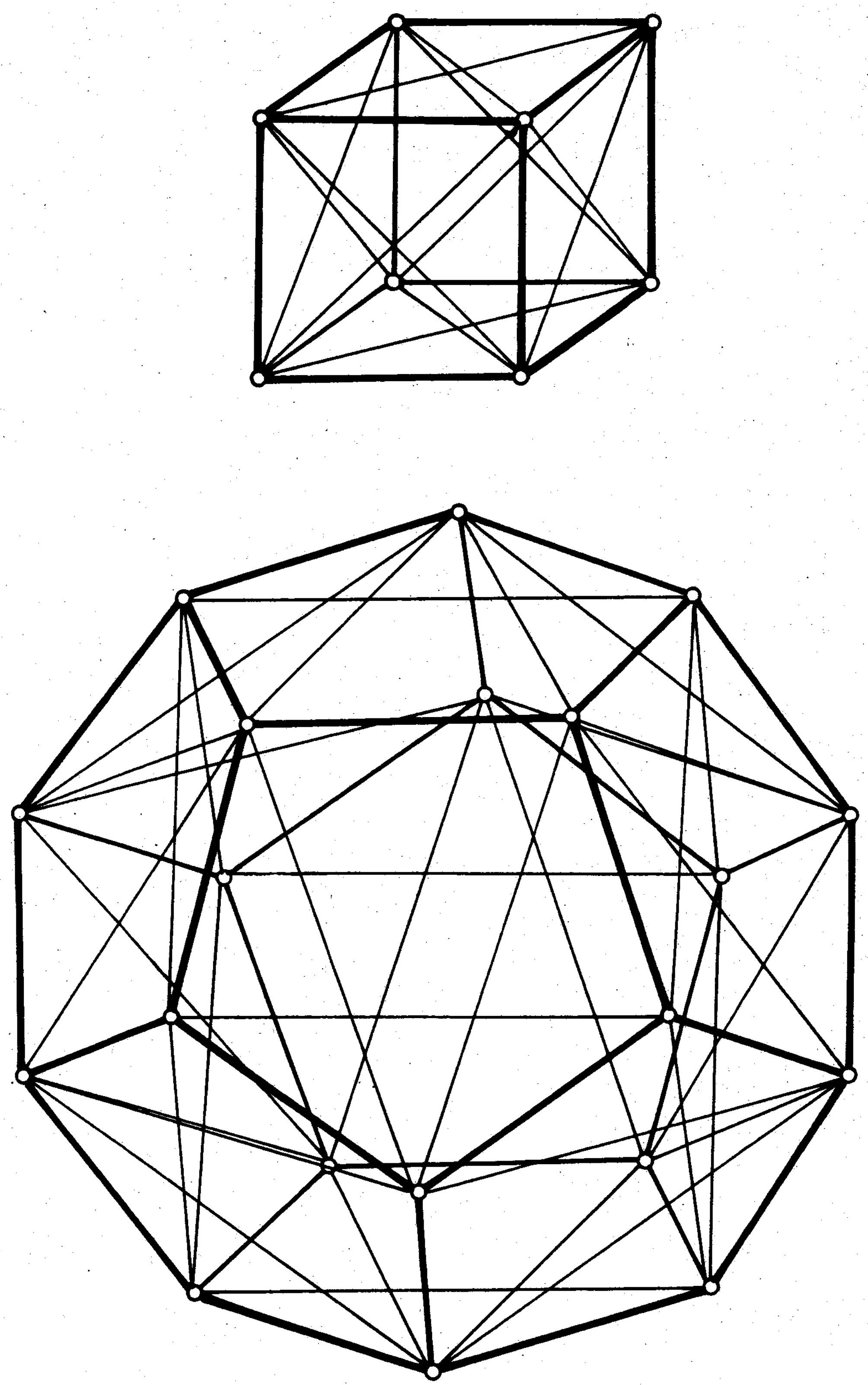


Figure 12. Two stiff convex tensed frameworks.

<u>Cauchy's lemma</u>. If a tensed rod-polygon P has cables of shortest possible length for n-2 consecutive diagonals spanning just two edges each, then P is planarly stiff.

For proof of the lemma we have only to observe that it is just the planar case of the elementary-geometric fact mentioned above, in connection with the proofs of Cauchy's rigidity theorem. Thus the books and papers mentioned on page 2.3 provide proofs of this lemma.

In this instance we see that if the (shortest possible) cables are present in a tensed rod-polygon P according to a certain pattern, then P is planarly stiff <u>regardless</u> of the metric proportions of P. (27) One of the open problems is whether that is typical; more precisely, if P and P* are tensed rod-polygons that are isomorphic (that is, have members of the same kind between corresponding vertices) and have shortest possible cables, are they simultaneously planarly-stiff ? In particular, we have no proof even for the following conjecture (see Figure 10):

(28) <u>Conjecture 5.</u> If the n-sided tensed rod-polygon P has n-2 shortest possible cables, n-3 of which share a vertex while the last cable connects the two neighbors of that vertex, then P is planarly-stiff.

The direct verification of Conjecture 5 is easy for n=4, 5, but no way has been found to escape the sharply increasing complexity needed in the proofs for larger n.

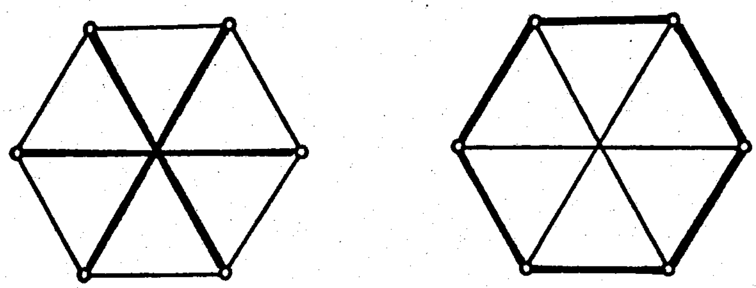
One other curious experimental observation that still lacks any explanation (or proof) is given by:

(29) Conjecture 6. Let P be a stiff tensed rod-polygon and let P^* be the tensed framework obtained by replacing all rods of P by shortest possible cables and the cables of P by rods. Then P^* is planarly-stiff.

The converse of Conjecture 6 is invalid, as may be verified on hand of the example shown in Figure 13.

* * *

Regarding frameworks and tensed frameworks that are not necessarily convex it shoud be pointed out, first, that very little of a mathematically interpretable and verifiable character seems to have been written. The condition $e \geq 3v-6$, alleged to be necessary for stiffness of frameworks (see above, page 2.9), is clearly violated by rigid frameworks like the one in Figure 14 which, if there are k edges on the diagonal, has $e = k+9$ edges and $v = k + 5$ vertices. Such examples are excluded by some authors as being of an "ill-conditioned" type ("...in which the members at a joint are nearly parallel." - Parkes [1974, p.24]). Examples with $e = \frac{3}{2}v + 1$ can be constructed in which this definition of "ill-conditioned framework" is not violated, although its probable intention is; a very simple such framework (actually a stiff tensed framework with $v = 8$ and with only 13 members, of which just 4 are rods) is shown in Figure 15, another (with $v = 6$ and $e = 11$) in Figure 16. More important is the observation that even "well-conditioned" frameworks with arbitrarily large v can be found so that $e = 2v$. An example of that nature (actually, again a stiff tensed framework, with $v = 8$, $e = 16$) is shown in Figure 17; it also contains only 4 rods. Examples with larger v (with $e = 2v$ and containing only $v/2$ rods) can easily be derived from the one in Figure 17.



Planarly-stiff Not planarly stiff

Figure 13.

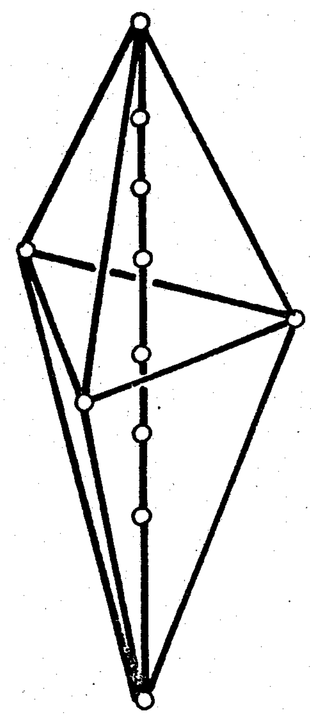


Figure 14.

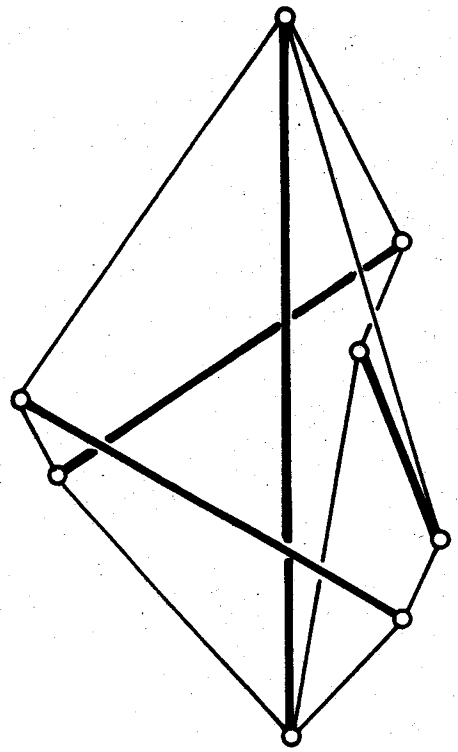


Figure 15.

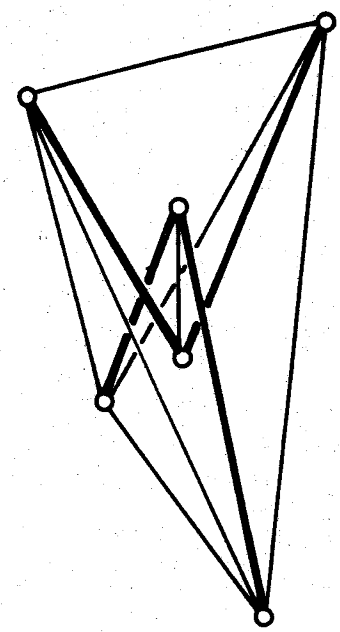


Figure 16.

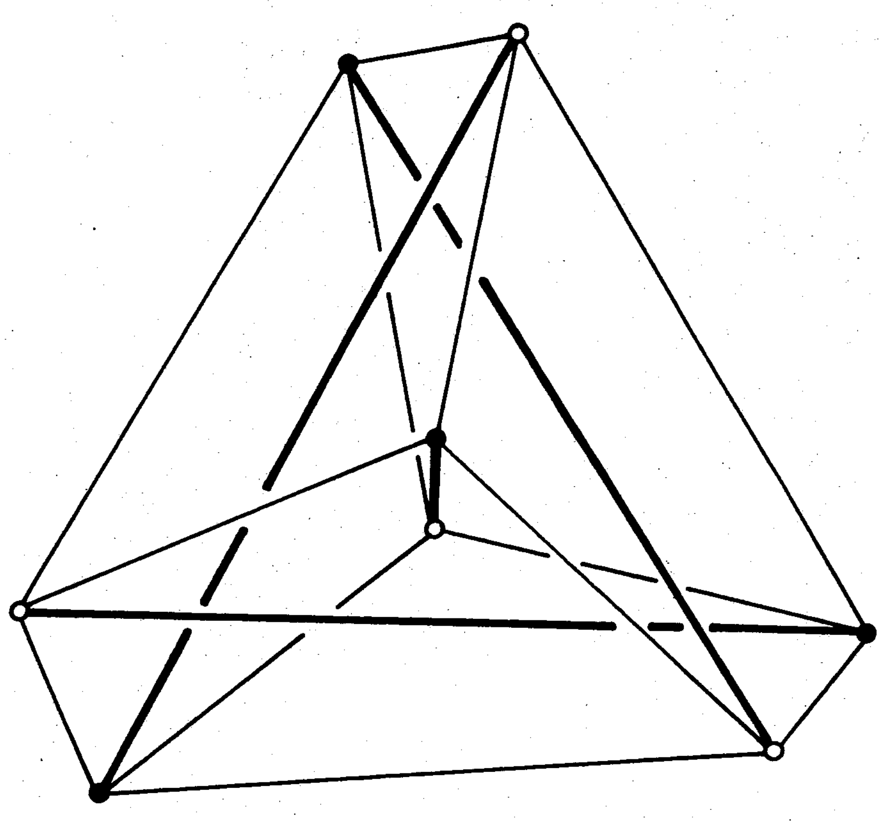


Figure 17.

Concerning the tensed framework represented in Figure 17 it may be observed that the graph determined by its vertices and members is the complete bipartite graph $K_{4,4}$. All its vertices are 4-valent, and its girth (length of shortest circuit) is also 4; this contrasts with the situation in the case of polyhedral skeleta, which must contain either triangles, or trivalent vertices.

* * *

Let a tensed framework be called <u>proper</u> provided it is stiff, its cables are shortest possible (that is, are representable by straight segments), its members are non-redundant (that is, the omission of any member renders the framework movable), each vertex meets at least one rod, and two members meet - if at all - only at a common endpoint. Rather obviously, proper tensed frameworks are the "nicest" and "best-behaved" type of tensed frameworks. One interesting question concerning them (that parallels similar questions about other types and realizations of graphs and complexes) is how to classify them. One possibility, patterned after the classification of planar polygons in Steinitz [1916] and Grünbaum [1975], is the following: We shall say that proper, tensed frameworks F_0 and F_1 have the same <u>form</u> provided there exists a family of proper tensed frameworks $F(t)$, $0 \leq t \leq 1$, such that each $F(t)$ is isomorphic to $F_0 = F(0)$, and $F(t)$ depends continuously on t. It seems rather obvious that proper tensed frameworks with r rods have only a finite number $f(r)$ of different forms; but I
30 have not seen a complete proof of that assertion. With just a little patience it may be shown that $f(3) = 1$, a representative of the only form being the proper tensed framework indicated in Figure 18. Already $f(4)$ is still undetermined; some forms are indicated in Figures 15, 16, 17, 19 and 20. It is known that $f(4) \geq 20$.

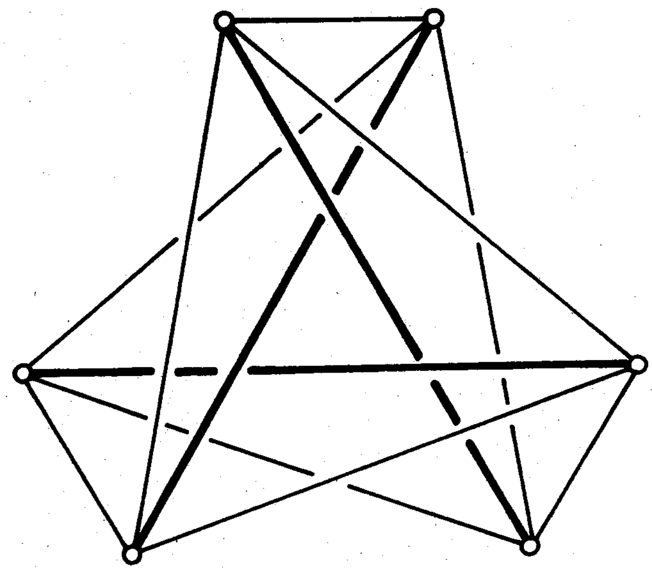


Figure 18.

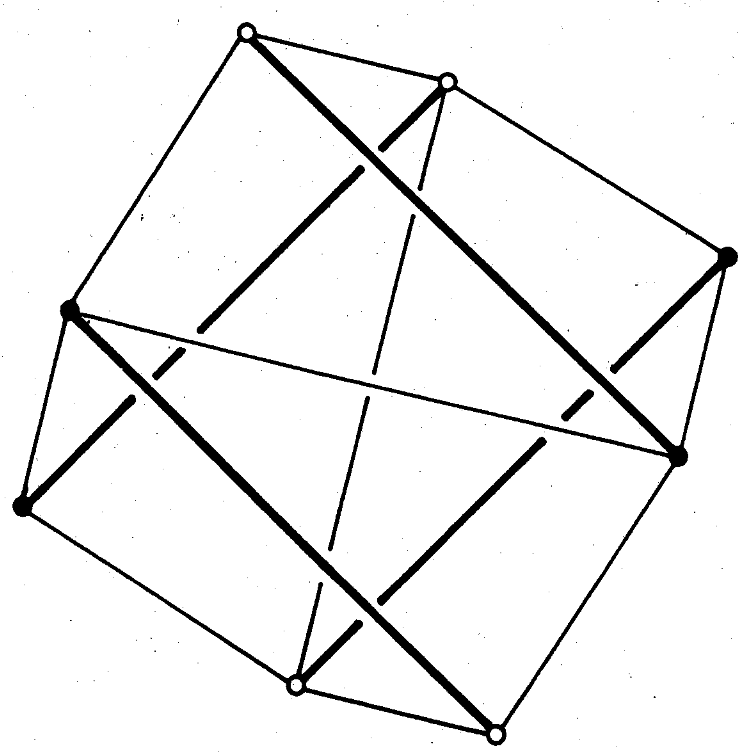


Figure 19.

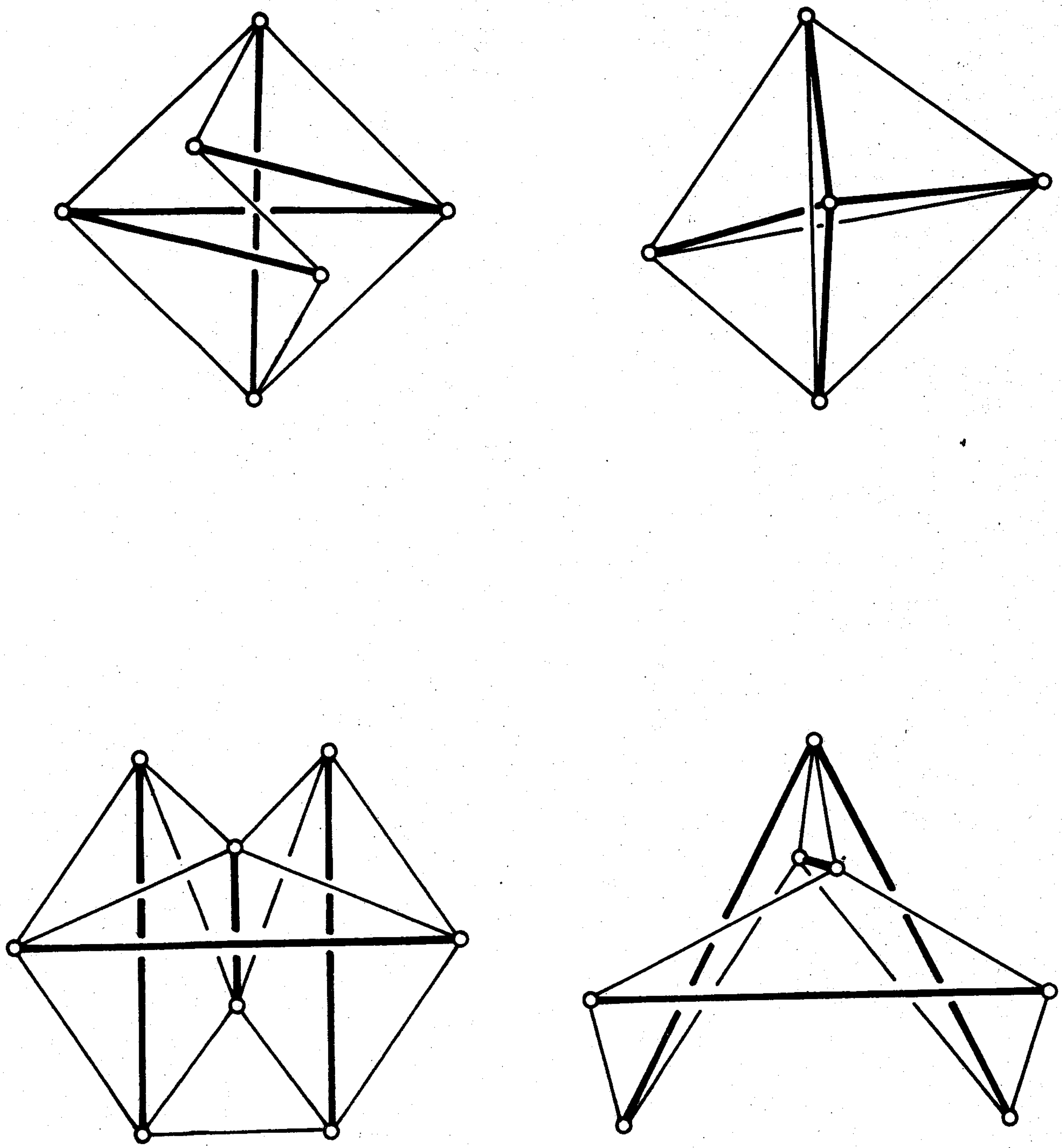


Figure 20.

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LECTURES ON "LOST MATHEMATICS"

3. Stiffness of frameworks.

As mentioned on page 2.9 and in other places in Section 2, the engineering literature contains many fallacious statements about stiffness and related topics. The following pages stem from an attempt to clarify and make precise and detailed this accusation. My interest in the topic derives as much from its intrinsic geometric contents as from the desire to find out how could such serious blunders have been committed by so many respected and otherwise reasonable investigators, and how could the detection of the errors have been so late in coming.

We shall investigate in detail the following assertion, parts of which are variously attributed to Mohr [1874], Levy [1874] and Föppl [1880]:

"Theorem". If a planar framework with v vertices (joints) and e edges (links, bars, rods) is planarly stiff, then $e \geq 2v - 3$. If a framework in 3-space, with v vertices and e edges, is stiff then $e \geq 3v - 6$.

Since this "theorem" is the basis of several engineering methods of practical importance, it is extensively used and there is no hope of giving a complete list of references for it; the following are among the publications I have consulted that contain the above formulation (or part of it) or some closely related statement:

Henneberg [1892, p.576], [1903, pp.388, 413], Schur [1897, p.153] Lamb [1928, p.94], Föppl [1926, pp. 167, 236], Timoshenko-Young [1945, pp. 45, 189], Housner-Hudson [1949, p. 140], Parkes [1974, p.48].

Some of these books do not pretend to prove anything. As for those that do attempt to prove the "Theorem", it is in all cases some combination of vagueness of definitions coupled with the appeal to non-existent "facts" that lead to the meaningless "proofs".

Concerning the vagueness, the principal problem is in the definition of stiffness. Those authors that at all care to be precise (Schur [1897] gives no explanation at all, but presumably follows Henneberg [1892]) define a framework (consisting of vertices and edges) as stiff (some say "rigid") if it is "incapable of deformation without alteration of length of at least one of its bars" (Lamb [1928, pp. 93-94]), or by some equivalent wording (Föppl [1880], Henneberg [1892,p.576], [1903, p. 387]); this coincides with the definition given above (page 2.6) for polyhedra and used also for frameworks. It is understood in all those writings that in case of planar frameworks the stiffness is with respect to deformations in the same plane; we called this property "planarly-stiff" (see page 2.12). However, several of the authors later change their mind by asserting, when convenient, that this definition was meant to include also the prohibition of "infinitesimal deformations" (Henneberg [1892, p. 577], [1903, p. 386], Föppl [1926, pp. 175-176], Lamb [1928, pp. 96, 129]), although the latter is a much more stringent condition; we shall discuss it later. At any rate, most of the shortcomings in the "proofs" of the "theorem" are not dependent on the choice of meaning attributed to "stiffness".

In much of the literature a framework is called "just stiff" (or "simply stiff") if it ceases to be stiff whenever one of its edges is deleted, The pattern in one type of "proofs" given for the "Theorem" is best seen in the following quotation from Lamb [1928, p. 94]:

"There is a definite relation between the number of joints and bars in a plane frame which is just rigid. Let the number of joints be n. Suppose one bar, with its two joints, to be fixed; this will by hypothesis fix the frame. The positions, relative to this bar, of the remaining $n-2$ joints will involve $2(n-2)$ coordinates (Cartesian or other); and these must be completely determined by the equations which express that the remaining bars have given lengths. These equations must therefore be $2n-4$ in number, i.e. the total number of bars must be $2n-3$." [Emphasis supplied, B.G.]

The fallacy is committed in the underlined sentence; while in a system of linear equations it is true that the variables can be uniquely determined only if the number of equations exceeds, or equals to, the number of variables, - nothing of the sort holds for real solutions of quadratic equations of the type $(x_i-x_j)^2+(y_i-y_j)^2=a_{ij}^2$ involved in the statement that the bar with endpoints (x_i,y_i) and (x_j,y_j) has length a_{ij}. Indeed, it is obvious from the example of the planar framework indicated in Figure 1 that there exist stiff planar frameworks that have v vertices and only $e = v+2$ edges, as well as (Figure 2) stiff frameworks in 3-space that have v vertices and only $e = v+5$ edges.

The "proof" in Henneberg [1892, p.577] is very similar to Lamb's.

The second type of "proofs" of the "Theorem" uses results (from the kinematics of solids) on the instantaneous centers of rotation, applied to the assembly of rods obtained by omitting one of the rods from a just stiff framework. The results from kinematics are asserted to apply to this situation since "the framework obtained by omitting

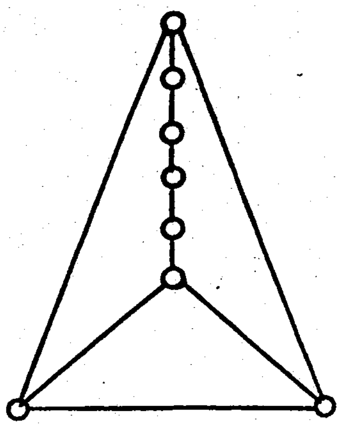

Figure 1.

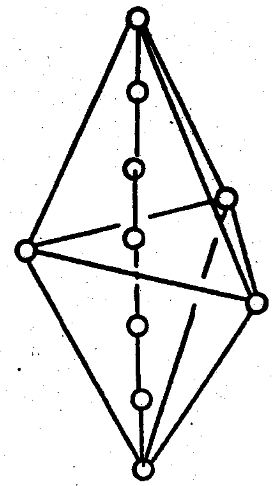

Figure 2.

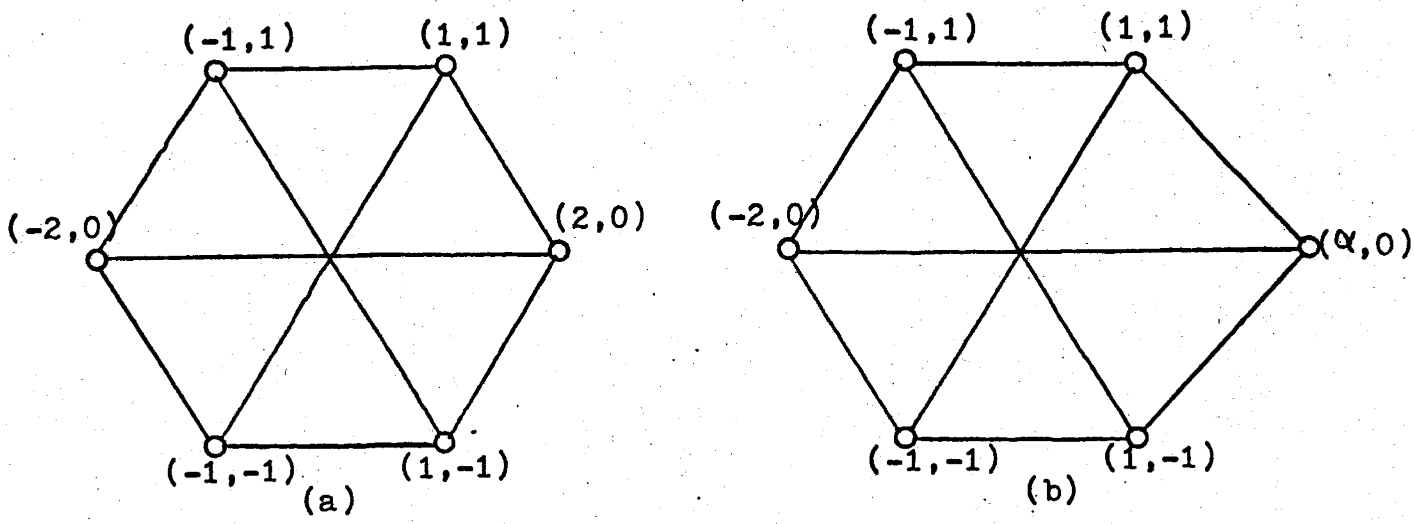

Figure 3.

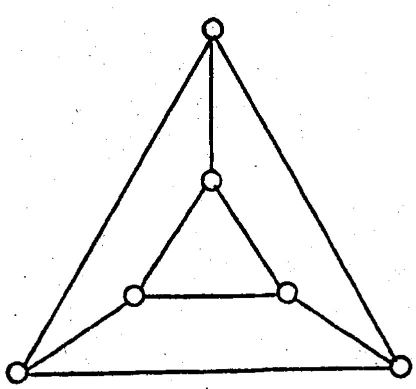

Figure 4.

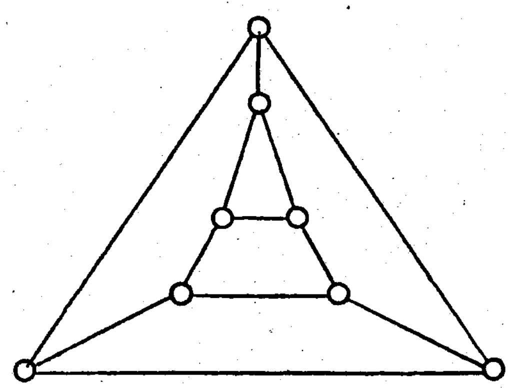

Figure 5.

a rod from a just stiff framework has precisely one degree of freedom, and its points are constrained to move along well-determined arcs" (free translation from Schur [1897, p. 149]). The fallacy here stems from the acceptance of the quoted assertion as fact - while actually, as shown by the examples in Figure 1 and 2, the omission of any one rod from a stiff framework may lead to frameworks with many degrees of freedom of motion.

Arguments similar to this "proof" appear also in Föppl [1926, p.198], Lamb [1928, p.128].

The third "proof" relies on a failure to distinguish sufficient conditions from necessary ones. As an examplary specimen of this fallacy we may mention the argumentation in Timoshenko-Young [1945, pp. 44 - 45]. They say:

"Beginning with a rigid triangle ABC ... and attaching to this the bars AD and BD, which are pinned together at D, we obtain the rigid frame ABCD. ... In the same way, the rigid truss ABCDE ... is obtained by adding to the rigid portion ABCD the two bars DE and CE, which are pinned together at E. Since the procedure above may be continued indefinitely, we conclude that <u>a rigid plane truss can always be formed by beginning with three bars pinned together at their ends in the form of a triangle and then adding to these two new bars for each new pin</u>."

Almost identical "reasoning" by jumps from examples to the totality appears in Föppl [1926, pp. 167-168].

* * * *

Since the above "Theorem" happens to be valid for "infinitesimally stiff" frameworks (see below), it could be argued that the authors of the above "proofs" were actually interested in infinitesimally stiff frameworks, and not in merely stiff ones; in other words, that their worst error consists of using sloppy and inaccurate definitions. While this excuse may indeed apply to a certain extent in some cases, most objections remain in effect. (It may be mentioned that Nielsen [1935] avoids these difficulties by formulating and proving the "Theorem" for "mechanically determined" frameworks; he does not mention "stiffness" or "infinitesimal stiffness" at all.) Among the objections to the usual treatment are the following ones:

1. None of the authors quoted gave anything resembling a precise and usable definition of infinitesimal stiffness. Since that notion is more complicated than that of stiffness, the shying away from formal definition is understandable, - but not excusable. Rigorous definitions that do not involve "infinitely small deformations" are possible (see, for example, Laman [1970]), and rather simple criteria may be given for the infinitesimal stiffness of frameworks.

2. Although many authors are trying to present infinitesimal stiffness as the only notion of rigidity that is natural and important from the engineering point of view, this attitude is very debatable on several grounds. On the one hand, the lack of infinitesimal stiffness is in many cases a singularity in the totality of possible realizations of a certain structure. For example, the six-vertex, nine-rods framework shown in Figure 3(a) is stiff but not infinitesimally stiff; however, there exist frameworks (Figure 3(b)) that are infinitesimally stiff and arbitrarily close to the framework in Figure 3(a); it is enough to choose $\alpha \neq 2$ to be sufficiently close to 2. Rather obviously, a classification in which a framework is "bad" if a certain length equals 2 but is "good" whenever that length differs from 2, can not have a serious claim

to engineering reality or importance. (It may be mentioned that the framework of Figure 3(a) is asserted to be "rigid" by Housner-Hudson [1949, p.150].)

On the other hand frameworks that are stiff but not infinitesimally stiff (such as those in Figures 4 and 5, and in Figure 15 to 20 of Section 2 with cables replaced by rods, as well as the structures of Snelson, Fuller, and Emmerich mentioned in Sections 1 and 2) may very well be practically usefull and have engineering importance.

3. The defects in the logic of the proofs mentioned above are not affected by the distinction between stiff and infinitesimally stiff frameworks. Hence, although with the substitution of "infinitesimally stiff" for "stiff" the planar case of the "Theorem" is true and has been proved by Laman [1970], and the 3-dimensional case is probably also true and provable by similar methods, the "proofs" mentioned above are still invalid.

4. The customary terminology, in which <u>infinitesimally</u> stiff frameworks (or surfaces) are designated "rigid" or "stiff" is (at least psychologically) misleading, and has actually caused or contributed to errors. For example, Timoshenko-Young [1945, pp. 80 - 81] speak of trusses that are "not completely rigid" (meaning stiff but not infinitesimally stiff ones) and even invite the student "to demonstrate the incomplete rigidity of this truss [our Figure 3(a) above] by direct experiment". As should be expected, in actual experiments the stiffness of this framework appers to be just about the same as that of the "completely rigid" ones in Figure 3(b) with $\alpha \neq 2$ but close to 2.

Another example of errors in thinking that are probably induced by the laxity in terminology is the following: Many authors seem to be convinced that the only frameworks that are stiff but not infinitesimally stiff are those that satisfy the equation $e = 3v - 6$ (or $e = 2v - 3$ in case of planarly-stiff planar frameworks) but are "critical" due to some collinearity, parallelism or coplanarity of "too many" edges. Explanations, examples, and comments in this spirit are given, among others, in Föppl [1926, Sections 56, 57], Lamb [1928, Sections 55, 56], Timoshenko-Young [1945, pp. 80-82]. This completely ignores the possibility of stiff (but not infinitesimally stiff) frameworks such as the one in Figure 5 (with $e = 12 < < 2v - 3 = 13$) or the other ones mentioned on page 3.6.

* * *

The infinitesimally stiff frameworks have the following property: If any system of forces in equilibrium is applied to the vertices of the framework, longitudinal tensions and/or compressions may be formed in the rods so that the forces at each vertex are in equilibrium. Conversely, any framework that is "statically determined" in this sense is also infinitesimally stiff. The intellectually untidy situation concerning the "Theorem" that was described above probably arose from a combination of two forces: The failure to keep apart "motion" from "infinitesimal deformation" provided the push -- away from the naive "stiffness". It combined with the pull exerted by the equivalence of the (obviously meaningful) concept of "statically determined" with "infinitesimally stiff". The resulting aggregate proved its utility by enabling the computation of forces and stresses, - so why worry about "minor" difficulties with the basic notions? Moreover, since infinitesimal stiffness allows (and even invites) the application of various powerful methods of analysis, and is related to analogous notions in the theory of surfaces, - it is not hard to understand that vagueness in the basics is a price gladly paid for a selfcontained body of interconnected facts and computational methods. The arrangement is completely safe from an engineering point of view -- its only "practical" drawback (that is, not counting the intellectual discomfort) being the exclusion of the stiff but not infinitesimally stiff frameworks from consideration. This situation appears to be slowly changing (see the Introduction of Laman [1970]), - but I have not found traces of the change in textbooks on structural engineering.

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