REDUCTIVE GROUP SCHEMES

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Abstract. — We develop the relative theory of reductive group schemes, using dynamic techniques and algebraic spaces to streamline the original development in SGA3.

Résumé. — Nous développons la théorie relative des schémas en groupes réductifs, à l'aide da techniques dynamiques et des espaces algébriques afin de simplifier le développement original dans SGA3.

To the memory of Robert Steinberg

Introduction

These notes present the theory of reductive group schemes, simplifying the original proofs via tools developed after 1963 (see "What's new?" at the end of this Introduction). We assume familiarity with the structure theory over an algebraically closed field k (as developed in [**Bo91**], [**Hum87**], [**Spr**]), but a review is given in § 1 to fix terminology, set everything in the framework of k-schemes (rather than classical varieties), and provide a convenient reference for the scheme-theoretic developments.

We give complete proofs of the main results in the theory (conjugacy theorems, scheme of maximal tori, construction of root groups and root datum, structure of open cell, parameterization of parabolics, schemes of Borel and parabolic subgroups, Existence and Isomorphism theorems, existence of automorphism scheme), apart from some calculations with low-rank root systems. We do *not* assume the Existence and Isomorphism Theorems over a general algebraically closed field k because the scheme-theoretic approach proves these over any non-empty base scheme granting only the Existence Theorem over $k = \mathbf{C}$ (which we prove in Appendix D).

Although the structure theory of connected reductive groups is developed over a general field in [**Bo91**] and [**Spr**], we take on faith only the case of an algebraically closed ground field (as in [**SGA3**] via its reliance on [**BIBLE**]). Some results are proved here in less generality than in [**SGA3**], since our aim is to reach the structure theory of reductive group schemes as quickly as possible. In a small number of places we refer to [**SGA3**] for omitted proofs, and the interested reader can readily check that this does not create circular arguments in our development of the general theory.

Background. — We assume familiarity with smooth and étale morphisms (e.g., functorial criteria for each; cf. [**BLR**, 2.1–2.2]), faithfully flat descent (see [**BLR**, 6.1–6.2]), and the functorial approach to group schemes (e.g., scheme-theoretic kernels, intersections of closed subgroup schemes, and quotient morphisms via sheaves for a Grothendieck topology). We also use Cartier duality for finite flat commutative group schemes (of finite presentation).

In the arguments after § 1 we use multiplicative type group schemes [**SGA3**, IX, X]. This material is covered in Appendix B, building on Oesterlé's lectures [**Oes**]. We use a more restricted notion of "multiplicative type" than in [**SGA3**] and [**Oes**]. In these notes, by definition *multiplicative type groups are required to be of finite type* over the base, and are required to split *fppf-locally* on the base: they are fppf-local forms of diagonalizable groups of finite type. The fpqc topology is used instead of the fppf topology in the definition in [**SGA3**, IX, 1.1] and [**Oes**] (allowing fpqc groups that are not of finite type). Our restriction to the fppf topology is harmless, as we explain in Appendix B (due to the "finite type" requirement that we impose). We often use that multiplicative type groups are necessarily split étale-locally on the base. It is inconvenient to include this in the *definition* of "multiplicative type", so a proof of étale-local splitting is given in Proposition B.3.4 (as a mild variant of the proof of an analogous result in [**SGA3**, IX, 4.5]). The equivalence with fpqc-local triviality (under a finite-type hypothesis on the group) is proved in Corollary B.4.2(1) but is never used.

We require the notion of Lie algebra for a group scheme (not necessarily smooth). A reference that covers what we need (and beyond) is [**CGP**, A.7]. One of the properties we use for the Lie functor is that it commutes with fiber products of groups (and so is left exact), as is verified by considering points valued in the dual numbers over the base.

For a reader unfamiliar with algebraic spaces, in a few places it will be necessary to accept that algebraic spaces are a useful mechanism to equip certain set-valued functors with enough "geometric structure" that it makes sense to carry over concepts for schemes (e.g., properness, flatness, quasi-finiteness, etc.) to such functors; an excellent reference for this is [**Knut**]. All algebraic spaces we use will almost immediately be proved to be schemes, so our use of algebraic spaces will be similar to the use of distributions to provide function solutions to elliptic partial differential equations (i.e., they appear in the middle of a construction, the end result of which is an object of a more familiar type that we prefer to use).

Theory over a field. — In Example 5.3.9 we relate the approach in [**SGA3**] to other constructions in the literature (e.g., [**St67**]) for the group of field-valued points of a split semisimple group. Many arithmetic applications (as well as applications over $\mathbf{R}$) require a structure theory for connected reductive groups over general fields k without assuming the existence of a split (geometrically) maximal k-torus. Such a structure theory is due to Borel and Tits (see [**BoTi**] and [**Bo91**, § 20–24]), and is not discussed here. In [**SGA3**, XXVI, 6.16–6.18, § 7] and [**G**, § 4–§ 5] the Borel–Tits theory is partially generalized to reductive groups over connected semi-local non-empty schemes; see [**PS**] for recent work in this direction.

There is a result at the foundation of the Borel–Tits structure theory for which we do provide a proof: Grothendieck's theorem [**SGA3**, XIV, 1.1] (later proved in more elementary terms by Borel and Springer) that any smooth affine group G over a field k admits a k-torus $\mathrm{T} \subset \mathrm{G}$ such that $\mathrm{T}_{\overline{k}}$ is maximal in $\mathrm{G}_{\overline{k}}$. This existence result implies that for *every* k-torus T' in G not contained in a strictly larger k-torus and for every extension field K/k, T'_{K} is not contained in a strictly larger K-torus of G_{K} (see Remark A.1.2 for a proof). In particular, the concept of "maximal k-torus" in a smooth affine group over k is insensitive to ground field extension. (We only use this in the trivial case $k = \overline{k}$.)

Since Grothendieck's existence theorem for maximal tori over arbitrary fields is not needed in the development of the theory of reductive groups over general schemes, we have relegated our discussion of his theorem to Appendix A (which provides a scheme-theoretic version of the Borel–Springer proof of Grothendieck's theorem). What matters for our purposes is the existence of a "geometrically maximal" torus defined over a finite *separable* extension of the ground field. We present Grothendieck's construction of such a torus, using the existence and smoothness properties of a "scheme of maximal tori". The proof of Grothendieck's finer result that such a torus exists over the ground field uses a detailed study of Lie algebras. It can also be deduced from the deeper result that the scheme of maximal tori is a rational variety [**SGA3**, XIV, 6.1] (coupled with special arguments for finite ground fields).

What's new?— We take advantage of three post-1963 developments to streamline or simplify some of the original proofs:

(i) (Artin approximation) In the study of lifting problems, infinitesimal methods allow one to build liftings over the *completion* of a local noetherian ring (when starting with an algebro-geometric object over the residue field).

The *Artin approximation theorem* ([**Ar69a**], [**BLR**, 3.6/16]), whose statement we recall in Theorem 3.1.7, provides a method to use such lifts over a "formal" neighborhood of a point to construct lifts over an étale neighborhood of a point. This solves global problems over an étale cover, a dramatic improvement on having solutions only in formal neighborhoods of points. We will use this reasoning to study liftings of tori in smooth affine groups (e.g., see the proof of Theorem 3.2.6).

(ii) (algebraic spaces) A representability result of Murre [**Mur**, § 3, Thm. 2, Cor. 2] is used in [**SGA3**] to build certain quotients by flat equivalence relations (see [**SGA3**, XVI, § 2]). This underlies Grothendieck's quotient constructions in [**SGA3**, XVI, 2.4]. Murre's result is a precursor to Artin's criteria for a functor on schemes to be an algebraic space. The work of Artin ([**Ar69b**], [**Ar74**]) and Knutson [**Knut**] on algebraic spaces provides an ideal framework for a geometric theory of quotients by flat equivalence relations in algebraic geometry (and includes a sufficient criterion for an algebraic space to be a scheme), so we use algebraic spaces in place of Murre's result; e.g., see the proof of Theorem 2.3.1.

(iii) (dynamic method) There is a "dynamic" approach to describing parabolic subgroups as well as their unipotent radicals and Levi factors in connected reductive groups over an algebraically closed field. This method involves the limiting behavior along orbits under the conjugation action of a 1-parameter subgroup; it is a standard tool in the classical setting (see [**Bo91**, 13.8(1),(2)], [**Spr**, 8.4.5]) and also arises in Mumford's GIT.

The relative version of the dynamic viewpoint for group schemes over any ring was introduced and developed in [**CGP**, 2.1], where it was used to study pseudo-reductive groups over imperfect fields. In the present paper, this leads to simplifications in several places. For instance, arguments in [**SGA3**, XX] for constructing closed root groups in split reductive group schemes and classifying the split semisimple-rank 1 case over any scheme rest on elaborate computations. The relative dynamic method, which rests on elementary arguments, eliminates the need for most of the computations in [**SGA3**, XX] (see § 4.2). We review what we need from the dynamic method in § 4.1, referring to [**CGP**, 2.1] for proofs.

Acknowledgements. I am grateful to the referees and A. Auel, O. Gabber, S. Garibaldi, B. Gross, L. Illusie, J. Parson, P. Polo, G. Prasad, J-P. Serre, A. Venkatesh, and J-K. Yu for many helpful suggestions. Kęstutis Česnavičius and Jessica Fintzen provided a huge amount of expository and mathematical feedback on an earlier version, for which I am especially appreciative. This work was supported by NSF grants DMS-0917686 and DMS-1100784.

Contents

1. Review of the classical theory

We fix an algebraically closed field k and consider smooth affine group schemes over k (not necessarily connected). Such groups are called *linear algebraic groups* over k because they are precisely the smooth k-groups G of finite type for which there exists a k-homomorphism

$$j : \mathrm{G} \hookrightarrow \mathrm{GL(V)} \simeq \mathrm{GL}_n$$

that is a closed immersion [**Bo91**, 1.10]. Although we will be working in the classical setting, we use the viewpoint of schemes rather than classical varieties. For instance, we do not use any "universal domain" as in [**Bo91**], and a linear algebraic group over k is a scheme rather than a set of k-points equipped with extra structure. Likewise, the *kernel* of a k-homomorphism always means the scheme-theoretic kernel (e.g., $\ker(\mathrm{SL}_n \to \mathrm{PGL}_n)$ is identified with the k-group scheme $\mu_n = \ker(t^n : \mathbf{G}_{\mathrm{m}} \to \mathbf{G}_{\mathrm{m}})$ even if $\mathrm{char}(k)|n$), and an intersection of closed linear algebraic k-subgroups of a linear algebraic k-group always means scheme-theoretic intersection (which may be non-smooth when $\mathrm{char}(k) > 0$; e.g., the central $\mathbf{G}_{\mathrm{m}}$ in GL_n meets SL_n in precisely μ_n, even if $\mathrm{char}(k)|n$).

We do not require linear algebraic groups to be connected because there are interesting disconnected examples (such as orthogonal groups) and group-theoretic operations (centralizers, intersection of subgroups, kernels, etc.) can lead to disconnected (possibly non-smooth) groups. The identity component G^0 of any linear algebraic k-group is irreducible (as for any smooth connected non-empty k-scheme).

The basic structure theory of linear algebraic groups is developed in Borel's book [**Bo91**] and Springer's book [**Spr**], as well as Chevalley's book [**BIBLE**]. In this section we summarize some of the highlights of this theory, as a review of classical notions and results that are needed in the developments over a general base scheme in [**SGA3**], and it is assumed that the reader has prior experience with the classical case. We also discuss some aspects of representation theory over an algebraically closed field.

Before we get started, it will be helpful to check the consistency between the notions of G/H in the classical and scheme-theoretic theories, with H a linear algebraic subgroup of a linear algebraic group G. This goes as follows. Classically, a smooth quasi-projective quotient G/H is built so that $q : \mathrm{G} \to \mathrm{G/H}$ identifies $(\mathrm{G/H})(k)$ with $\mathrm{G}(k)/\mathrm{H}(k)$ and the map $\mathrm{Tan}_e(q)$ is the natural surjection $\mathfrak{g} \twoheadrightarrow \mathfrak{g}/\mathfrak{h}$ [**Bo91**, 6.7, 6.8]. In the scheme-theoretic approach the "quotient" morphism $\mathrm{G} \to \mathrm{Q}$ by the right H-action on G is uniquely characterized (if it exists!) by the condition that it that represents the fppf-sheafification of the functor $\mathrm{S} \rightsquigarrow \mathrm{G(S)/H(S)}$ on the category of k-schemes. We claim that the classical quotient construction satisfies this property, so it is the desired quotient in the sense of [**SGA3**].

To compare these concepts, first observe (via $G(k)$-translations) that q is surjective on tangent spaces at all $g \in G(k)$, so q is a smooth morphism ("submersion theorem" [**BLR**, 2.2/8(c)]). The agreement with the scheme-theoretic notion of quotient amounts to saying that q is the quotient of G modulo the flat equivalence relation $R := G \times H \hookrightarrow G \times G$ defined by $(g, h) \mapsto (g, gh)$. The map q is fppf, so by descent theory q is the quotient of G modulo the flat equivalence relation $R' = G \times_{G/H} G \to G \times G$. The relation R' is a closed subscheme that is smooth (since $\mathrm{pr}_1 : R' \to G$ is a base change the smooth q, and G is smooth), and the same clearly holds for R. We just have to show that $R = R'$ as subschemes of $G \times G$, so by smoothness it is equivalent to show $R(k) = R'(k)$ inside $G(k) \times G(k)$. Since $G(k)/H(k) \to (G/H)(k)$ is a bijection, the desired equality is clear.

1.1. Solvable groups and reductive groups. — By the closed orbit lemma [**Bo91**, 1.8], if $f : G' \to G$ is a k-homomorphism between linear algebraic k-groups then the image $f(G')$ is a smooth closed k-subgroup of G. In particular, since $G' \to f(G')$ is a surjective map between smooth k-schemes, by homogeneity using translation by k-points over $k = \overline{k}$ it follows that generic flatness propagates everywhere, so $G' \to f(G')$ is faithfully flat. Thus, descent theory gives that $G'/(\ker f) \simeq f(G')$; i.e., $f(G')$ represents the quotient sheaf $G'/(\ker f)$ for the fppf topology, where $\ker f$ is the scheme-theoretic kernel. If **P** is a property of scheme morphisms that is fppf local on the base (e.g., being finite, smooth, or an isomorphism) then the fppf map $G' \to G'/(\ker f)$ satisfies **P** if and only if $\ker f \to \operatorname{Spec} k$ does, since $G' \times_{G'/(\ker f)} G' \simeq \ker f \times G'$ via $(g'_1, g'_2) \mapsto (g'_1 {g'_2}^{-1}, g'_2)$. In particular, since $\ker f$ is finite as a k-scheme if and only if $(\ker f)(k)$ is finite (as $k = \overline{k}$), we get:

Proposition 1.1.1. — *If* $(\ker f)(k)$ *is finite then* f *is finite flat onto its smooth closed image, and if* $\ker f = 1$ *then* f *is a closed immersion. If* f *is surjective and* $\ker f = 1$ *then* f *is an isomorphism.*

Example 1.1.2. — The natural k-homomorphism $\pi : \mathrm{SL}_n \to \mathrm{PGL}_n$ between finite type k-schemes is surjective on k-points, hence surjective as a scheme map, and the scheme-theoretic kernel $\ker \pi = \mu_n$ is a finite k-scheme of degree n, so π is finite flat of degree n. If $n = p = \mathrm{char}(k) > 0$ then $\ker \pi$ is infinitesimal, π is purely inseparable, and π is bijective on k-points but is not an isomorphism.

It follows from Proposition 1.1.1 that if a linear algebraic group G' is a k-subgroup of another such group G in the sense that there is given a monic homomorphism $G' \to G$ then G' is a closed k-subgroup of G. In particular, if a linear representation $G \to \mathrm{GL}(V)$ is *faithful* in the sense that it has trivial (scheme-theoretic) kernel then it is necessarily a closed immersion.

Example 1.1.3. — The action of SL_n on the vector space $\mathrm{V} = \mathrm{Mat}_n$ of $n \times n$ matrices via conjugation defines a linear representation $\rho : \mathrm{SL}_n \to \mathrm{GL}(\mathrm{V})$ whose kernel is μ_n. In particular, if $n = \mathrm{char}(k) = p > 0$ then this representation is injective on k-points but it is not faithful in the scheme-theoretic sense (i.e., $\ker \rho \neq 1$).

Remark 1.1.4. — It is important that Proposition 1.1.1 has a variant without smoothness hypotheses: if $f : \mathrm{G}' \to \mathrm{G}$ is a k-homomorphism between affine k-group schemes of finite type and $\ker f = 1$ then f is a closed immersion. We do not use this in § 1, but it is used in the development of the relative theory over rings. See [**SGA3**, $\mathrm{VI_B}$, 1.4.2] for a proof.

The closed immersion property for monic homomorphisms between linear algebraic k-groups is a wonderful feature of the theory over fields; it is not true for smooth affine groups over rings (see Example 3.1.2) and criteria to ensure it over rings can lie quite deep (see Theorem 5.3.5).

We now recall notions related to Jordan decomposition for a linear algebraic group G (see [**Bo91**, 4.4] for proofs). Fix a faithful linear representation $j : \mathrm{G} \hookrightarrow \mathrm{GL}(\mathrm{V})$. For any $g \in \mathrm{G}(k)$, we say that g is *semisimple* if the linear endomorphism $j(g)$ of V is semisimple in the sense of linear algebra; i.e., $j(g)$ is diagonalizable. Likewise, we say that g is *unipotent* if $j(g)$ is unipotent as a linear endomorphism of V. These properties are independent of the choice of j and are preserved under any k-homomorphism $f : \mathrm{G} \to \mathrm{H}$ to another linear algebraic k-group; that is, if $g \in \mathrm{G}(k)$ is semisimple (resp. unipotent) then so is $f(g) \in \mathrm{H}(k)$.

In general for any $g \in \mathrm{G}(k)$ there are unique commuting elements $g_{\mathrm{ss}}, g_{\mathrm{u}} \in \mathrm{G}(k)$ such that g_{ss} is semisimple, g_{u} is unipotent, and $g = g_{\mathrm{ss}} g_{\mathrm{u}} = g_{\mathrm{u}} g_{\mathrm{ss}}$. We call these the *semisimple part* and *unipotent part* of g respectively, and refer to these product expressions for g as its *Jordan decomposition*. The existence of this Jordan decomposition demonstrates an advantage of linear algebraic groups over Lie algebras: the former is defined and studied in a characteristic-free way (over algebraically closed fields), whereas the development of the latter is entirely different in characteristic 0 and in positive characteristic (see [**Sel**, V.7.2] and [**Hum98**]).

The formation of the Jordan decomposition is functorial in the sense that if $f : \mathrm{G} \to \mathrm{H}$ is a k-homomorphism to another linear algebraic k-group then

$$f(g_{\mathrm{ss}}) = f(g)_{\mathrm{ss}}, \;\; f(g_{\mathrm{u}}) = f(g)_{\mathrm{u}}.$$

In particular, if $\mathrm{H} = \mathrm{GL}(\mathrm{W})$ for a finite-dimensional vector space W then $f(g)$ and $f(g_{\mathrm{ss}})$ have the same characteristic polynomial. This is a very useful fact.

Definition 1.1.5. — A linear algebraic group G is *solvable* if $\mathrm{G}(k)$ is solvable, and is *unipotent* if $g = g_{\mathrm{u}}$ for all $g \in \mathrm{G}(k)$.

Remark 1.1.6. — A more elegant definition of unipotence, avoiding the crutch of GL_n-embeddings, is given in [**SGA3**, XVII, 1.3]; its equivalence with the above definition is given in [**SGA3**, XVII, 2.1].

By the Lie–Kolchin theorem [**Bo91**, 10.5], if G is solvable and connected then every linear representation $\mathrm{G} \to \mathrm{GL}_\mathrm{N}$ can be conjugated to have image inside the upper triangular subgroup B_N. Also, *without* a connectedness hypothesis, G is unipotent precisely when it occurs as a closed subgroup of the strictly upper-triangular subgroup $\mathrm{U}_n \subset \mathrm{GL}_n$ for some n [**Bo91**, 4.8]. In particular, every unipotent linear algebraic group is solvable.

Example 1.1.7. — The group $\mathbf{G}_\mathrm{a}$ is unipotent, due to the faithful representation $x \mapsto \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$.

The strictly upper triangular subgroup U_n of GL_n admits a composition series whose successive quotients are $\mathbf{G}_\mathrm{a}$, so every unipotent linear algebraic group U admits a composition series whose successive quotients are $\mathbf{G}_\mathrm{a}$ or a finite étale subgroup of $\mathbf{G}_\mathrm{a}$ (as $k = \overline{k}$, so we can use the smooth underlying reduced schemes of intersections of U with the composition series for some U_n). When U is connected, we may arrange the composition series so that the successive quotients are all equal to $\mathbf{G}_\mathrm{a}$ [**Bo91**, 10.6(2), 10.9].

Functoriality of the Jordan decomposition implies that every homomorphic image or smooth closed subgroup of a unipotent linear algebraic group is unipotent, and likewise that if a linear algebraic group G contains a unipotent normal linear algebraic subgroup U such that G/U is also unipotent then G is unipotent. Since $\mathbf{G}_\mathrm{a}$ has no nontrivial finite subgroups when $\mathrm{char}(k) = 0$ and its finite étale subgroups are p-groups when $\mathrm{char}(k) = p > 0$, the unipotence of the finite étale component group U/U^0 of a unipotent linear algebraic group U implies that unipotent linear algebraic k-groups are connected when $\mathrm{char}(k) = 0$ and have component group that is a p-group when $\mathrm{char}(k) = p > 0$.

A key result at the start of the theory of linear algebraic groups is [**Bo91**, 11.1, 11.2]:

Theorem 1.1.8. — *Let* G *be a linear algebraic group over* k*. The maximal connected solvable linear algebraic* k*-subgroups of* G *are all* $\mathrm{G}(k)$*-conjugate to each other, and these are precisely the connected solvable linear algebraic subgroups* B *such that the quasi-projective quotient scheme* G/B *is projective.*

A *Borel subgroup* of G is a maximal connected solvable linear algebraic subgroup of G. A *parabolic subgroup* of G is a linear algebraic subgroup $\mathrm{P} \subset \mathrm{G}$ such that G/P is projective. All Borel subgroups are parabolic, by Theorem 1.1.8, so any linear algebraic subgroup containing a Borel subgroup is also parabolic. In fact, the parabolic subgroups are precisely the subgroups that contain Borel subgroups [**Bo91**, 11.2, Cor.]. Note that the definition of parabolicity does not require connectedness. The following result is fundamental (see [**Bo91**, 11.16] for a proof):

***Theorem 1.1.9* (Chevalley).** — *The parabolic subgroups* P *of any connected linear algebraic group* G *are connected, and* $\mathrm{N}_{\mathrm{G}(k)}(\mathrm{P}) = \mathrm{P}(k)$.

In Corollary 5.2.8 this will be improved to a scheme-theoretic equality $\mathrm{N}_{\mathrm{G}}(\mathrm{P}) = \mathrm{P}$ when G is connected reductive.

Example 1.1.10. — For $\mathrm{G} = \mathrm{SL}_n$, the subgroup B of upper triangular matrices is clearly connected and solvable, and under the natural transitive G-action on the complete variety F of full flags in k^n the smooth group B is the stabilizer scheme of the standard flag F_0. Thus, the orbit map $\mathrm{G} \to \mathrm{F}$ through F_0 is faithfully flat (even smooth) and induces a scheme isomorphism $\mathrm{G}/\mathrm{B} \simeq \mathrm{F}$, so B is also parabolic, hence a Borel subgroup.

Some obvious parabolic subgroups of G containing B are labelled by ordered partitions $\vec{a} = (a_1, \ldots, a_r)$ of n into non-empty parts (i.e., all $a_i > 0$, $\sum a_i = n$): we associate to $\vec{a}$ the subgroup $\mathrm{P}_{\vec{a}}$ consisting of points of SL_n that preserve each of the subspaces V_j spanned by the first $b_j := a_1 + \cdots + a_j$ standard basis vectors. (Note that $\vec{a} \mapsto \{b_1, \ldots, b_{r-1}\}$ is a bijection from the set of such $\vec{a}$ onto the set of subsets of $\{1, \ldots, n-1\}$.) These 2^{n-1} parabolic subgroups are the only parabolic subgroups containing B, so they represent (without repetition; see Corollary 1.4.9) the conjugacy classes of parabolic subgroups of G. (In Example 1.4.8 this example is addressed more fully.)

Let G be a linear algebraic group over k. If $\mathrm{U}, \mathrm{U}' \subset \mathrm{G}$ are connected *normal* unipotent linear algebraic subgroups of G then the normal closed subgroup $\mathrm{U} \cdot \mathrm{U}'$ that they generate is unipotent (as it is a quotient of $\mathrm{U} \ltimes \mathrm{U}'$). Hence, by dimension considerations there exists a unique maximal connected unipotent normal linear algebraic subgroup $\mathscr{R}_u(\mathrm{G}) \subset \mathrm{G}$, called the *unipotent radical* of G. In a similar manner, there is a maximal connected solvable normal linear algebraic subgroup $\mathscr{R}(\mathrm{G})$, called the *radical* of G.

If H is a normal linear algebraic subgroup of G then $\mathscr{R}_u(\mathrm{H})$ is normal in G, so $\mathscr{R}_u(\mathrm{H}) = (\mathrm{H} \bigcap \mathscr{R}_u(\mathrm{G}))^0_{\mathrm{red}}$; the same holds for radicals. (Note that the formation of the underlying reduced scheme is a local operation for the Zariski topology, so the formation of identity component and underlying reduced scheme of a finite type k-group scheme commute. In particular, there is no ambiguity in notation such as $\mathscr{G}^0_{\mathrm{red}}$ for such a group scheme $\mathscr{G}$.) These notions also behave well with respect to quotients: if $\pi : \mathrm{G} \twoheadrightarrow \mathrm{G}'$ is a surjective homomorphism between linear algebraic groups then $\mathscr{R}_u(\mathrm{G}') = \pi(\mathscr{R}_u(\mathrm{G}))$ and likewise for radicals. However, the proof of this result for quotient maps is nontrivial when $\mathrm{char}(k) > 0$ (if $\ker \pi$ is not smooth); see [**Bo91**, 14.11] for an argument that works regardless of the smoothness properties of π.

Definition 1.1.11. — A *reductive* k-group is a linear algebraic k-group G such that $\mathscr{R}_u(\mathrm{G}) = 1$ (i.e., G contains no nontrivial unipotent normal connected linear algebraic k-subgroup). A *semisimple* k-group is a linear algebraic k-group G such that $\mathscr{R}(\mathrm{G}) = 1$.

It is immediate from the good behavior of radicals and unipotent radicals with respect to normal subgroups and quotients that semisimplicity and reductivity are inherited by normal linear algebraic subgroups and images of homomorphisms.

Example 1.1.12. — A basic example of a connected reductive group is the open unit group GL(V) in the affine space of linear endomorphisms of V. This follows from the Lie–Kolchin theorem; it recovers $\mathbf{G}_\mathrm{m}$ when $\dim \mathrm{V} = 1$. The radical $\mathscr{R}(\mathrm{GL}(\mathrm{V}))$ coincides with the "scalar" $\mathbf{G}_\mathrm{m}$ (see Example 1.1.16).

Examples of connected semisimple groups are: the smooth irreducible hypersurface SL(V) in GL(V) (which recovers SL_2 when $\dim \mathrm{V} = 2$), its quotient PGL(V), and symplectic groups $\mathrm{Sp}(\mathrm{V}, \psi)$ for non-degenerate alternating forms ψ on a nonzero finite-dimensional vector space V. (The connectedness of symplectic groups can be proved via induction on dim V via a fibration argument, and the smoothness can be proved by the infinitesimal criterion.)

Special orthogonal groups $\mathrm{SO}(\mathrm{V}, q)$ associated to non-degenerate quadratic spaces (V, q) of dimension $\geqslant 3$ are also connected semisimple, but special care is needed to give a characteristic-free development of such groups that works well in characteristic 2 without parity restrictions on dim V; see Exercise 1.6.10 and Definition C.2.10. Smoothness and connectedness of special orthogonal groups are proved in Appendix C (see § C.2–§ C.3).

For proofs of reductivity of these groups, see Exercises 1.6.11(i) and 1.6.16.

Remark 1.1.13. — There is no universal convention as to whether or not reductive groups should be required to be connected. If G is connected and reductive then the centralizer in G of any torus is connected and reductive (see Theorem 1.1.19(3)), so for arguments with torus centralizers there is no harm in requiring connectedness. A more subtle case is the centralizer $\mathrm{Z}_\mathrm{G}(g)$ for connected reductive G and semisimple $g \in \mathrm{G}(k)$. (The k-group $\mathrm{Z}_\mathrm{G}(g)$ is smooth even when defined to represent a "centralizer functor", due to [**Bo91**, 9.2, Cor.] and Exercise 1.6.9(i); a vast generalization is provided by Lemma 2.2.4, recovering $\mathrm{Z}_\mathrm{G}(g)$ by taking Y there to be the Zariski closure of $g^{\mathbf{Z}}$.) Such centralizers appear in orbital integrals in the trace formula for automorphic forms. In general $\mathrm{Z}_\mathrm{G}(g)^0$ is reductive (see Theorem 1.1.19(3)) but $\mathrm{Z}_\mathrm{G}(g)$ may be disconnected (e.g., $\mathrm{G} = \mathrm{PGL}_2$ and $g = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ with $\mathrm{char}(k) \neq 2$; for SL_2 the analogous centralizer is connected).

In [**Bo91**] and [**Spr**], as well as in these notes, reductive groups over a field are not assumed to be connected. This is a contrast with [**SGA3**, XIX, 2.7], and it may seem to present a slightly inconsistency with the general definition over schemes, but no real confusion should arise. For reductive groups over schemes, the usefulness of a fibral connectedness condition is more compelling than in the theory over a field (see Example 3.1.2).

Remark 1.1.14. — For linear algebraic groups over an algebraically closed field of characteristic 0, the property of reductivity for the identity component is equivalent to the semisimplicity of all linear representations. This fails in positive characteristic; see Exercise 1.6.11.

In the context of connected solvable linear algebraic groups, the opposite extreme from the unipotent groups is the following class of groups:

Definition 1.1.15. — A *torus* over the algebraically closed field k is a k-group T that is isomorphic to a power $(\mathbf{G}_{\mathrm{m}})^r$ for some $r \geqslant 0$.

In view of the general structure of connected solvable linear algebraic groups (as $\mathrm{T} \ltimes \mathrm{U}$ for a torus T and unipotent radical U [**Bo91**, 10.6(4)]), we see that the solvable connected reductive groups are precisely the tori.

Example 1.1.16. — Let G be connected reductive. The solvable connected $\mathscr{R}(\mathrm{G})$ is a torus (as it must be reductive). Since the automorphism scheme $\mathrm{Aut}_{\mathrm{T}/k}$ of a k-torus T is a constant group (see the proof of Theorem 1.1.19), a normal torus T' in any *connected* linear algebraic group G' must be central (as the k-homomorphism $\mathrm{G}' \to \mathrm{Aut}_{\mathrm{T}'/k}$ giving the conjugation action has to be trivial). Hence, $\mathrm{Z} := \mathscr{R}(\mathrm{G})$ is a central torus such that G/Z is semisimple. In other words, every connected reductive group is a central extension of a connected semisimple group by a torus. For example, if $\mathrm{G} = \mathrm{GL}(\mathrm{V})$ then $\mathrm{G}/\mathrm{Z} = \mathrm{PGL}(\mathrm{V})$.

Deeper structure theory of reductive groups [**Bo91**, 14.2] ensures that the derived group $\mathscr{D}(\mathrm{G})$ is semisimple and that the commutative reductive quotient $\mathrm{G}/\mathscr{D}(\mathrm{G})$ (which must be a torus) is an isogenous quotient of Z. In other words, G is also canonically an extension of a torus by a connected semisimple group. For $\mathrm{G} = \mathrm{GL}(\mathrm{V})$ this is the exact sequence

$$1 \to \mathrm{SL}(\mathrm{V}) \to \mathrm{GL}(\mathrm{V}) \stackrel{\det}{\to} \mathbf{G}_{\mathrm{m}} \to 1,$$

and the isogeny of tori $\mathrm{Z} \to \mathrm{G}/\mathscr{D}(\mathrm{G})$ in this case is identified with the endomorphism $t \mapsto t^n$ of $\mathbf{G}_{\mathrm{m}}$ for $n = \dim \mathrm{V}$.

The natural homomorphism $\mathrm{Z} \times \mathscr{D}(\mathrm{G}) \to \mathrm{G}$ is an isogeny for any connected reductive k-group G. Because of this, for many (but not all!) problems in the classical theory of reductive groups one can reduce to a separate treatment of tori and semisimple groups. Note also that the semisimple $\mathscr{D}(\mathrm{G})$ must in fact be equal to its own derived group. Indeed, the quotient $\mathscr{D}(\mathrm{G})/\mathscr{D}^2(\mathrm{G})$ is connected, semisimple (inherited from $\mathscr{D}(\mathrm{G})$), and commutative, hence trivial.

A fundamental result (whose proof rests on the theory of Borel subgroups) is that a connected linear algebraic group G over $k = \overline{k}$ is a torus when all k-points are semisimple [**Bo91**, 11.5(1)]. If instead G contains no nontrivial tori then it must be unipotent [**Bo91**, 11.5(2)]. This is very useful: it implies that a general connected linear algebraic k-group G either admits a strictly upper triangular faithful representation or it contains a nontrivial k-torus. (This dichotomy is also true without assuming $k = \overline{k}$, but requires Grothendieck's deep result on the existence of geometrically maximal k-tori, proved in Appendix A.) Thus, if $\dim \mathrm{G} > 1$ then G contains either $\mathbf{G}_{\mathrm{a}}$ or $\mathbf{G}_{\mathrm{m}}$ as a proper k-subgroup (since $k = \overline{k}$).

A very effective way to work with tori is by means of some associated lattices. To be precise, since $\mathrm{End}(\mathbf{G}_{\mathrm{m}}) = \mathbf{Z}$ via the endomorphisms $t \mapsto t^n$, for any torus T the commutative groups

$$\mathrm{X}(\mathrm{T}) = \mathrm{Hom}_{k\text{-gp}}(\mathrm{T}, \mathbf{G}_{\mathrm{m}}), \ \ \mathrm{X}_*(\mathrm{T}) = \mathrm{Hom}_{k\text{-gp}}(\mathbf{G}_{\mathrm{m}}, \mathrm{T})$$

are finite free $\mathbf{Z}$-modules of rank $\dim \mathrm{T}$ that are respectively contravariant and covariant in T. (Elements of $\mathrm{X}_*(\mathrm{T})$ are called *cocharacters* of T.) For historical reasons via the theory of compact Lie groups, it is a standard convention to use additive notation when discussing elements of the character and cocharacter lattices of T. For example, if $a, b : \mathrm{T} \rightrightarrows \mathbf{G}_{\mathrm{m}}$ are two characters then $a + b$ denotes $t \mapsto a(t)b(t)$ and $-a$ denotes $t \mapsto 1/a(t)$ (and 0 denotes the trivial character). For this reason, $a(t)$ is often denoted as t^a.

Evaluation defines a perfect duality of lattices

$$\langle \cdot, \cdot \rangle : \mathrm{X}(\mathrm{T}) \times \mathrm{X}_*(\mathrm{T}) \to \mathrm{End}(\mathbf{G}_{\mathrm{m}}) = \mathbf{Z}$$

via $\langle a, \lambda \rangle = a \circ \lambda$. (Here, by "perfect" we mean that the $\mathbf{Z}$-bilinear form identifies each lattice with the $\mathbf{Z}$-dual of the other. In terms of matrices relative to a $\mathbf{Z}$-basis, it means that the matrix of the bilinear form has determinant in $\mathbf{Z}^\times = \{\pm 1\}$.) In terms of the cocharacter group $\mathrm{X}_*(\mathrm{T})$ we have

$$\mathrm{X}_*(\mathrm{T}) \otimes_{\mathbf{Z}} k^\times \simeq \mathrm{T}(k)$$

via $\lambda \otimes c \mapsto \lambda(c)$. In fact the algebraic group T (and not just its group of k-points) can be reconstructed from its cocharacter group, by considering the functor $\mathrm{X}_*(\mathrm{T}) \otimes_{\mathbf{Z}} \mathbf{G}_{\mathrm{m}}$ that assigns to any k-algebra A the group $\mathrm{X}_*(\mathrm{T}) \otimes_{\mathbf{Z}} \mathrm{A}^\times = \mathrm{Hom}(\mathrm{X}(\mathrm{T}), \mathrm{A}^\times)$. More specifically, the contravariant functors

$$\mathrm{T} \rightsquigarrow \mathrm{X}(\mathrm{T}), \ \ \mathrm{M} \rightsquigarrow \mathrm{M}^\vee \otimes_{\mathbf{Z}} \mathbf{G}_{\mathrm{m}}$$

(where $\mathrm{M}^\vee := \mathrm{Hom}(\mathrm{M}, \mathbf{Z})$) are inverse anti-equivalences between the categories of tori and finite free $\mathbf{Z}$-modules.

A fundamental fact about tori is that their linear representations are completely reducible in *any* characteristic. This can be expressed in the following canonical form.

Proposition 1.1.17. — *Let* T *be a torus,* $\mathrm{M} = \mathrm{X}(\mathrm{T})$ *its character group, and* V *a finite-dimensional linear representation of* T *over* k*. For each* $a \in \mathrm{M}$*, let* V_a *be the space of* $v \in \mathrm{V}$ *such that* $t.v = a(t)v$ *for all* $t \in \mathrm{T}(k)$*. The natural* T*-equivariant map* $\bigoplus_{a \in \mathrm{M}} \mathrm{V}_a \to \mathrm{V}$ *is an isomorphism.*

In this way, the category of linear representations of T *is equivalent to the category of* M*-graded* k*-vector spaces.*

We call V_a the *a-weight space* for T acting on V. Note that these vanish for all but finitely many a; the *weights* for the T-action on V are the a such that $\mathrm{V}_a \neq 0$. For the dual representation space V^*, we have $(\mathrm{V}^*)_a = (\mathrm{V}_{-a})^*$ for all $a \in \mathrm{M}$. Hence, if V is a self-dual representation of T then its set of weights is stable under negation.

Example 1.1.18. — For $\mathrm{T} = \mathbf{G}_{\mathrm{m}}$ we have $\mathrm{X}(\mathrm{T}) = \mathrm{End}(\mathbf{G}_{\mathrm{m}}) = \mathbf{Z}$, so a k-homomorphism $\mathbf{G}_{\mathrm{m}} \to \mathrm{GL}(\mathrm{V})$ is the "same" as a k-linear $\mathbf{Z}$-grading $\mathrm{V} = \bigoplus_{n \in \mathbf{Z}} \mathrm{V}(n)$, with $t \in \mathbf{G}_{\mathrm{m}} = \mathrm{T}$ acting on $\mathrm{V}(n)$ via t^n-scaling. (There is an analogous result for linear $\mathbf{G}_{\mathrm{m}}^r$-actions using $\mathbf{Z}^r$-gradings, as well as for linear μ_n-actions using $\mathbf{Z}/n\mathbf{Z}$-gradings. A common generalization for linear representations of any diagonalizable k-group $\mathrm{D}_k(\mathrm{M})$ is expressed in terms of M-gradings, but we will not need it here. See [**CGP**, A.8.1–A.8.9].)

Choose $v \in \mathrm{V}$, and write $v = \sum_{n \in \mathbf{Z}} v_n$ with $v_n \in \mathrm{V}(n)$, so $t.v = \sum_{n \in \mathbf{Z}} t^n v_n$. Viewing V as an affine space over k, clearly the orbit map $\mathbf{G}_{\mathrm{m}} \to \mathrm{V}$ defined by $t \mapsto t.v$ extends to a k-scheme morphism $\mathbf{A}^1 \to \mathrm{V}$ if and only if $v \in \mathrm{V}_{\geqslant 0} := \bigoplus_{n \geqslant 0} \mathrm{V}(n)$. For any such v, we define $\lim_{t \to 0} t.v$ to be the image of $0 \in \mathbf{A}^1(k) = k$ under the extension $\mathbf{A}^1 \to \mathrm{V}$ of the orbit map. This limiting value is v_0, so the space of $v \in \mathrm{V}$ for which $\lim_{t \to 0} t.v$ exists and vanishes is $\mathrm{V}_{>0} := \bigoplus_{n > 0} \mathrm{V}(n)$.

By dimension considerations, any torus in a linear algebraic k-group is contained in a maximal such torus. Here are some important properties of tori in linear algebraic groups:

Theorem 1.1.19. — *Let* G *be a linear algebraic group over* k*.*

1. *For any torus* T' *in* G*, the Zariski-closed centralizer* $\mathrm{Z}_{\mathrm{G}}(\mathrm{T}')$ *has finite index in the Zariski-closed normalizer* $\mathrm{N}_{\mathrm{G}}(\mathrm{T}')$*, and if* G *is connected then* $\mathrm{Z}_{\mathrm{G}}(\mathrm{T}')$ *is connected.*
2. *All maximal tori* T *in* G *are* $\mathrm{G}(k)$*-conjugate.*
3. *Assume* G *is connected reductive. The centralizer* $\mathrm{Z}_{\mathrm{G}}(\mathrm{T}')$ *is connected reductive for any torus* T' *in* G*, and if* T *is a maximal torus in* G *then* $\mathrm{Z}_{\mathrm{G}}(\mathrm{T}) = \mathrm{T}$*. If* $g \in \mathrm{G}(k)$ *is semisimple then* $\mathrm{Z}_{\mathrm{G}}(g)^0$ *is reductive.*

In this theorem we take $\mathrm{N}_{\mathrm{G}}(\mathrm{T}')$ and $\mathrm{Z}_{\mathrm{G}}(\mathrm{T}')$ to have the classical meaning, as smooth closed subgroup schemes corresponding to $\mathrm{N}_{\mathrm{G}(k)}(\mathrm{T}')$ and $\mathrm{Z}_{\mathrm{G}(k)}(\mathrm{T}')$

respectively. (This coincides with the functorial viewpoint on normalizers and centralizers that is addressed in Proposition 2.1.2, Definition 2.2.1, Lemma 2.2.4, and Exercise 2.4.4.)

Proof. — Since the endomorphism functor $\mathrm{R} \rightsquigarrow \mathrm{End}_{\mathrm{R}\text{-gp}}(\mathbf{G}_{\mathrm{m}})$ on k-algebras is represented by the constant k-group $\mathbf{Z}$ [**Oes**, I, § 5.2], the endomorphism functor of $\mathbf{G}_{\mathrm{m}}^r$ is represented by the constant k-group $\mathrm{Mat}_r(\mathbf{Z})$. Thus, the automorphism functor of $\mathrm{T}' \simeq \mathbf{G}_{\mathrm{m}}^r$ is represented by the locally finite type constant k-group $\mathrm{GL}_r(\mathbf{Z})$. In particular, all quasi-compact k-subschemes of this automorphism scheme are finite and closed. Since $\mathrm{N}_{\mathrm{G}}(\mathrm{T}')$ is finite type, its image in the automorphism scheme must therefore be finite. But the kernel of the action of $\mathrm{N}_{\mathrm{G}}(\mathrm{T}')$ on T' has underlying space $\mathrm{Z}_{\mathrm{G}}(\mathrm{T}')$ (in fact this kernel is smooth, but we do not need it here), so the finite-index claim in (1) is proved. The connectedness in (1) is [**Bo91**, 11.12]. The conjugacy of maximal tori is [**Bo91**, 11.3(1)]. The assertions in (3) concerning torus centralizers are part of [**Bo91**, 13.17, Cor. 2], and the reductivity of $\mathrm{Z}_{\mathrm{G}}(g)^0$ for semisimple $g \in \mathrm{G}(k)$ is [**Bo91**, 13.19] □

Remark 1.1.20. — For $\mathrm{G} = \mathrm{GL}_n$, every element can be conjugated into an upper triangular form and every semisimple element can be diagonalized. In other words, $\mathrm{G}(k)$ is covered by the subgroups $\mathrm{B}(k)$ as B varies through the Borel subgroups, and the subset of semisimple elements in $\mathrm{G}(k)$ is covered by the subgroups $\mathrm{T}(k)$ as T varies through the maximal tori. These properties are valid for every connected linear algebraic group G. Indeed, for solvable G the result can be deduced from the structure of connected solvable groups (see [**Bo91**, 10.6(5)]), and so by the general conjugacy of Borel subgroups and maximal tori it suffices to show that all elements of $\mathrm{G}(k)$ lie in a Borel subgroup. See [**Bo91**, 11.10] for this result.

In our later work with reductive groups over schemes we will define scheme-theoretic notions of centralizer and normalizer by closed subgroup schemes of a linear algebraic group, and in the case of a torus T in a linear algebraic group G over k we will show that the scheme-theoretic notions of $\mathrm{Z}_{\mathrm{G}}(\mathrm{T})$ and $\mathrm{N}_{\mathrm{G}}(\mathrm{T})$ are smooth (and so coincide with their classical counterparts); see Proposition 2.1.2 and Lemma 2.2.4. The quotient $\mathrm{N}_{\mathrm{G}}(\mathrm{T})/\mathrm{Z}_{\mathrm{G}}(\mathrm{T})$ will then be a finite étale k-group, denoted as $\mathrm{W}_{\mathrm{G}}(\mathrm{T})$, and it is the constant group associated to its group of k-points, which is $\mathrm{N}_{\mathrm{G}(k)}(\mathrm{T})/\mathrm{Z}_{\mathrm{G}(k)}(\mathrm{T})$. This latter group is the *Weyl group* associated to (G, T) in the classical theory, and it agrees with the group of k-points of the étale quotient construction that will be used in the relative scheme-theoretic theory. The Weyl group $\mathrm{W}_{\mathrm{G}}(\mathrm{T})$ is especially important when T is *maximal* in a connected reductive k-group G, in which case it is $\mathrm{N}_{\mathrm{G}}(\mathrm{T})/\mathrm{T}$.

For a linear algebraic group G, the centralizers $\mathrm{Z}_{\mathrm{G}}(\mathrm{T})$ of maximal tori T are called the *Cartan subgroups*. These are visibly $\mathrm{G}(k)$-conjugate to each other;

their common dimension is called the *nilpotent rank* of G, and the common dimension of the maximal tori is called the *reductive rank* of G. In the special case of connected reductive G the Cartan subgroups are precisely the maximal tori, and their common dimension is then simply called the *rank* of G.

1.2. Roots and coroots. — Let G be a connected reductive k-group, T a maximal torus in G, and $\mathfrak{g}$ and $\mathfrak{t}$ the respective Lie algebras. In contrast with characteristic zero, the adjoint action of G on $\mathfrak{g}$ can sometimes fail to be semisimple when $\mathrm{char}(k) > 0$. For instance, if $\mathrm{char}(k) = p > 0$ and $\mathrm{G} = \mathrm{SL}_p$ then $\mathfrak{g} = \mathfrak{sl}_p = \mathfrak{gl}_p^{\mathrm{Tr}=0}$ (the kernel of the trace map) and this contains the diagonal scalar subspace of $\mathfrak{gl}_p$ on which the adjoint action of G is trivial. This line admits no G-equivariant complement (see Exercise 1.6.4).

However, the T-action on $\mathfrak{g}$ is completely reducible, as for any linear representation of a torus. When $\mathfrak{g}$ is equipped with this extra structure then it becomes a useful invariant of G in any characteristic. To see this, consider the weight space decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus (\oplus_{a\in\Phi}\mathfrak{g}_a) \tag{1.2.1}$$

for a finite subset $\Phi = \Phi(\mathrm{G},\mathrm{T}) \subset \mathrm{X}(\mathrm{T}) - \{0\}$. The following properties hold: the subspace $\mathfrak{g}_0 = \mathfrak{g}^{\mathrm{T}}$ coincides with $\mathfrak{t}$, for each $a \in \Phi$ the weight space $\mathfrak{g}_a$ is 1-dimensional, and Φ is stable under negation in X(T) [**Bo91**, 13.18(1),(4a),(4b)]. Beware that even though the set of T-weights on $\mathfrak{g}$ is stable under negation, just like self-dual representations of G, when $\mathrm{char}(k) > 0$ the G-representation space $\mathfrak{g}$ can fail to be self-dual; see Exercise 1.6.4.

Letting $r = \dim \mathrm{T}$ denote the rank of G, the 1-dimensionality of the weight spaces $\mathfrak{g}_a$ for the nontrivial T-weights on $\mathfrak{g}$ implies that the characteristic polynomial of the T-action on $\mathfrak{g}$ is

$$\det(x\mathrm{I} - \mathrm{Ad}_{\mathrm{G}}(t)|\mathfrak{g}) = (x-1)^r \prod_{a\in\Phi}(x - a(t))$$

for $t \in \mathrm{T}(k)$. For this reason, the elements of Φ are called the *roots* of (G, T); the corresponding 1-dimensional weight spaces $\mathfrak{g}_a$ are called *root spaces* in $\mathfrak{g}$.

Example 1.2.1. — Take $\mathrm{G} = \mathrm{GL}(\mathrm{V}) = \mathrm{GL}_n$ with $n > 0$ and let $\mathrm{T} = \mathbf{G}_{\mathrm{m}}^n$ be the torus consisting of diagonal matrices

$$t = \mathrm{diag}(c_1, \ldots, c_n).$$

It is easy to check that $\mathrm{T}(k) = \mathrm{Z}_{\mathrm{G}(k)}(\mathrm{T})$, so T is a maximal torus. We have

$$\mathrm{X}(\mathrm{T}) = \bigoplus \mathbf{Z}e_i, \ \ \mathrm{X}_*(\mathrm{T}) = \bigoplus \mathbf{Z}e_i^\vee$$

where $e_i(\mathrm{diag}(c_1, \ldots, c_n)) = c_i$ and $e_i^\vee(c) = \mathrm{diag}(1, \ldots, c, \ldots, 1)$ (with c as the ith diagonal entry, all others being 1). The normalizer $\mathrm{N}_{\mathrm{G}(k)}(\mathrm{T})$ is the group of invertible monomial matrices (i.e., one nonzero entry in each row and in each column) and $N_{G(k)}(T)/T(k) = \mathfrak{S}_n$ is represented by the group of $n \times n$ permutation matrices (relative to the standard basis).

The Lie algebra $\mathfrak{g}$ is $\mathrm{End}(V) = \mathrm{Mat}_n(k)$, and the Lie subalgebra $\mathfrak{t}$ is the subspace of diagonal matrices. The roots are the characters

$$a_{ij}(\mathrm{diag}(c_1, \ldots, c_n)) = c_i/c_j$$

for $1 \leqslant i \neq j \leqslant n$; in other words, $a_{ij} = e_i - e_j$. The corresponding root space $\mathfrak{g}_{a_{ij}} \subset \mathrm{Mat}_n(k)$ consists of matrices with vanishing entries away from the ij-entry.

Example 1.2.2. — For $G = SL_2$, the diagonal torus $T = \{\mathrm{diag}(c, 1/c)\}$ is maximal and is usually identified with $\mathbf{G}_m$ via $\lambda : c \mapsto \mathrm{diag}(c, 1/c)$. The Lie algebra $\mathfrak{g}$ is the space $\mathfrak{sl}_2 = \mathfrak{gl}_2^{\mathrm{Tr}=0}$ of traceless 2×2 matrices over k, in which a basis is given by

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The line kH coincides with $\mathfrak{t}$, and the lines kE and kF are the root spaces. Explicitly,

$$\mathrm{Ad}_G(\lambda(c))(E) = \lambda(c)E\lambda(c)^{-1} = c^2E, \ \mathrm{Ad}_G(\lambda(c))(F) = \lambda(c)F\lambda(c)^{-1} = c^{-2}F,$$

so the roots in $X(T) = X(\mathbf{G}_m) = \mathbf{Z}$ are ± 2 with corresponding root spaces $\mathfrak{g}_2 = kE$ and $\mathfrak{g}_{-2} = kF$. Note that H is central in $\mathfrak{sl}_2$ when $\mathrm{char}(k) = 2$ (and the central μ_2 in SL_2 has Lie algebra $\mathrm{Lie}(\mu_2) = \mathfrak{t} = kH$ when $\mathrm{char}(k) = 2$).

Although our survey of the classical theory in § 1 largely uses the classical convention to treat normalizers and centralizers as reduced schemes, we now briefly digress to discuss a special case in the classical theory for which schematic centralizers are smooth. (In § 2 we will take up the relative scheme-theoretic notions of centralizer and normalizer.)

For a smooth affine k-group G and $g \in G(k)$, define the *schematic centralizer* $Z_G(g)$ to be the scheme-theoretic fiber over e for the morphism $G \to G$ defined by $x \mapsto gxg^{-1}x^{-1}$. (For any k-algebra R, $Z_G(g)(R)$ consists of those $x \in G(R)$ that commute with $g_R \in G(R)$.) For any smooth closed k-subgroup H, consider the closed subgroup scheme

$$Z_G(H) := \bigcap_{h \in H(k)} Z_G(h)$$

in G, where $Z_G(h)$ is taken in the scheme-theoretic sense as just defined.

Proposition 1.2.3. — *The k-subgroup scheme $Z_G(H)$ represents the functorial centralizer of H in G: for any k-algebra R, $Z_G(H)(R)$ coincides with the set of $g \in G(R)$ such that g-conjugation on G_R restricts to the identity on H_R. Moreover, $\mathrm{Lie}(Z_G(H)) = \mathfrak{g}^H$ inside $\mathfrak{g}$.*

Proof. — We first prove that $Z_G(H)$ represents the functorial centralizer. We have to show for any k-algebra R and $g \in G(R)$, the R-morphism $H_R \to G_R$ defined by $x \mapsto xgx^{-1}g^{-1}$ is the constant map $x \mapsto 1$ if and only if $g \in Z_G(H)(R) = \bigcap_{h \in H(k)} Z_G(h)(R)$, which is to say if and only if $hgh^{-1}g^{-1} = 1$ in $G(R)$ for all $h \in H(k)$.

More generally, for any reduced affine k-scheme X of finite type and (possibly non-reduced) k-algebra R, we claim that an R-morphism $f : X_R \to Y$ to an affine R-scheme Y is uniquely determined by the collection of values $f(x) \in Y(R)$ for all $x \in X(k)$. It suffices to show that an R-morphism $X_R \to Y \times_{\operatorname{Spec} R} Y$ factors through the diagonal if it does so at all points in $X(R)$ arising from $X(k)$. Since the diagonal is cut out by an ideal in the coordinate ring of $Y \times Y$, it suffices to show that if an element $a \in R \otimes_k k[X]$ vanishes in R after specialization at all k-points of the coordinate ring $k[X]$ then $a = 0$. In other words, it suffices to show that the map of R-algebras

$$h_R : R \otimes_k k[X] \to \prod_{x \in X(k)} R$$

defined by $a \mapsto (a(x))$ is injective for any k-algebra R. (This says that $H(k)$ is "relatively schematically dense" in H over k in the sense of [**EGA**, IV_3, 11.10.8]; see [**EGA**, IV_3, 11.9.13].)

Since $k[X]$ is a reduced k-algebra of finite type, by the Nullstellensatz the map h_k is injective. The map h_R is the composition of $R \otimes_k h_k$ and the natural map

$$R \otimes_k \prod_{x \in X(k)} k \to \prod_{x \in X(k)} R$$

defined by $r \otimes (a_x) \mapsto (ra_x)$. Hence, it suffices to prove a general fact in linear algebra: if W is a (possibly infinite-dimensional) vector space over a field k and if $\{V_i\}$ is a collection of k-vector spaces then the natural map $W \otimes_k \prod_i V_i \to \prod_i (W \otimes_k V_i)$ is injective. Any element in the kernel is a finite sum of elementary tensors, so we easily reduce to the case when W is finite-dimensional. The case $W = 0$ is obvious, and otherwise by choosing a k-basis of W we reduce to the trivial case $W = k$. This completes the proof that $Z_G(H)$ represents the functorial centralizer.

To prove $\mathrm{Lie}(Z_G(H)) = \mathfrak{g}^H$, we give an argument using just the functor of points of $Z_G(H)$ and not the smoothness of H. Since $Z_G(H)$ represents the functorial centralizer, $\mathrm{Lie}(Z_G(H))$ is the subset of elements in $\mathfrak{g} \subset G(k[\epsilon])$ on which $H_{k[\epsilon]}$-conjugation is trivial. Thus, for $v \in \mathfrak{g}$ we have to show that $\mathrm{Ad}_G(h)(v) = v$ in $\mathfrak{g}_R \subset G(R[\epsilon])$ for all k-algebras R and $h \in H(R)$ if and only if $H_{k[\epsilon]}$-conjugation on $G_{k[\epsilon]}$ leaves v fixed. Using the universal point of $H_{k[\epsilon]}$ (namely, its identity automorphism), for the latter condition it suffices to check triviality on v_R under conjugation against $H(R[\epsilon])$ for all

k-algebras R (such as $\mathrm{R} = k[\mathrm{H}]$). For any $h \in \mathrm{H}(\mathrm{R}[\epsilon])$, the specialization $h_0 \in \mathrm{H}(\mathrm{R})$ at $\epsilon = 0$ can be promoted to an $\mathrm{R}[\epsilon]$-point (still denoted h_0) via $\mathrm{R} \to \mathrm{R}[\epsilon]$, so $h = h_0 h'$ for $h' \in \ker(\mathrm{H}(\mathrm{R}[\epsilon]) \to \mathrm{H}(\mathrm{R})) = \mathrm{Lie}(\mathrm{H}_\mathrm{R})$. But the commutative addition on $\mathrm{Lie}(\mathrm{G}_\mathrm{R}) = \mathfrak{g}_\mathrm{R}$ is induced by the group law on $\mathrm{G}(\mathrm{R}[\epsilon])$, so $v_\mathrm{R} h v_\mathrm{R}^{-1} h^{-1} = v_\mathrm{R} h_0 v_\mathrm{R}^{-1} h_0^{-1} = v - \mathrm{Ad}_\mathrm{G}(h_0)(v)$. Since every point in $\mathrm{H}(\mathrm{R})$ arises in the form h_0, the desired equivalence is proved. □

Corollary 1.2.4. — *Let* G *be a smooth affine* k*-group and* T *a* k*-torus in* G*. The schematic centralizer* $\mathrm{Z}_\mathrm{G}(\mathrm{T})$ *is* k*-smooth with Lie algebra* $\mathfrak{g}^\mathrm{T}$*, and if* G *is connected reductive and* T *is maximal in* G *then* $\mathrm{Z}_\mathrm{G}(\mathrm{T}) = \mathrm{T}$ *(i.e., for any* k*-algebra* R*, if* $g \in \mathrm{G}(\mathrm{R})$ *centralizes* T_R *then* $g \in \mathrm{T}(\mathrm{R})$*). In particular, for connected reductive* G *and maximal tori* T *in* G*, every central closed subgroup scheme of* G *lies in* T.

The smoothness is proved in another way in Lemma 2.2.4 (via the infinitesimal criterion), avoiding recourse to the classical theory.

Proof. — First we consider the case of connected reductive case with maximal T. In such cases, $\mathrm{Lie}(\mathrm{Z}_\mathrm{G}(\mathrm{T})) = \mathfrak{g}^\mathrm{T} = \mathfrak{t}$. Thus, the inclusion of group schemes $\mathrm{T} \hookrightarrow \mathrm{Z}_\mathrm{G}(\mathrm{T})$ that is an equality on k-points (by the classical theory) is also an equality on Lie algebras (again, by the classical theory), so $\dim \mathrm{Z}_\mathrm{G}(\mathrm{T}) = \dim \mathrm{T} = \dim \mathfrak{t} = \dim \mathrm{Lie}(\mathrm{Z}_\mathrm{G}(\mathrm{T}))$. Hence, $\mathrm{Z}_\mathrm{G}(\mathrm{T})$ is k-smooth by the tangential criterion. The equality $\mathrm{T}(k) = \mathrm{Z}_{\mathrm{G}(k)}(\mathrm{T}) = \mathrm{Z}_\mathrm{G}(\mathrm{T})(k)$ of k-points therefore implies an equality $\mathrm{T} = \mathrm{Z}_\mathrm{G}(\mathrm{T})$ as k-schemes.

In general, by the same argument, the smoothness of $\mathrm{Z}_\mathrm{G}(\mathrm{T})$ amounts to showing that $\mathrm{Lie}(\mathrm{Z}_\mathrm{G}(\mathrm{T})_{\mathrm{red}}) = \mathfrak{g}^\mathrm{T}$. This is a special case of [**Bo91**, Cor. 9.2] (setting H, L there equal to G, T respectively, and working throughout with reduced k-schemes). □

For $a \in \Phi$, there is a unique subgroup $\mathrm{U}_a \subset \mathrm{G}$ normalized by T such that $\mathrm{U}_a \simeq \mathbf{G}_\mathrm{a}$ and $\mathrm{Lie}(\mathrm{U}_a) = \mathfrak{g}_a$ [**Bo91**, 13.18(4d)]. This is the *root group* associated to a. Explicitly, by Exercise 1.6.2(iv) and the T-equivariant identification $\mathrm{Lie}(\mathrm{U}_a) \simeq \mathfrak{g}_a$, the T-action on $\mathrm{U}_a \simeq \mathbf{G}_\mathrm{a}$ is $t.x = a(t)x$ (so $\mathrm{T} \cap \mathrm{U}_a \subset \mathrm{U}_a^\mathrm{T} = 1$ as k-schemes).

Example 1.2.5. — For $\mathrm{G} = \mathrm{SL}_2$ and $\mathrm{T} = \mathrm{D}$ the diagonal torus, the root groups are the strictly upper and lower triangular unipotent subgroups $\mathrm{U}^\pm$. The same holds for PGL_2 and its diagonal torus $\overline{\mathrm{D}}$, using the strictly upper and lower triangular unipotent subgroups $\overline{\mathrm{U}}^\pm$.

The following lemma will turn out to be a generalization of the classical fact that SL_2 is generated by the root groups $\mathrm{U}^\pm$.

Lemma 1.2.6. — *Let* $\mathrm{T}_a = (\ker a)^0_{\mathrm{red}}$ *be the unique codimension-*1 *torus in* T *killed by* $a \in \Phi$. *The root groups* U_a *and* U_{-a} *generate* $\mathrm{G}_a := \mathscr{D}(\mathrm{Z}_{\mathrm{G}}(\mathrm{T}_a))$, *and* G_a *is a closed subgroup admitting* PGL_2 *as an isogenous quotient.*

Proof. — Fix an isomorphism $u_{\pm a} : \mathbf{G}_{\mathrm{a}} \simeq \mathrm{U}_{\pm a}$, so $tu_{\pm a}(x)t^{-1} = u_{\pm a}(a(t)^{\pm 1}x)$ for $t \in \mathrm{T}$, by consideration of the Lie algebra and Exercise 1.6.2(iv). Hence, T_a must centralize $\mathrm{U}_{\pm a}$, so these root groups lie in $\mathrm{Z}_{\mathrm{G}}(\mathrm{T}_a)$. The group $\mathrm{Z}_{\mathrm{G}}(\mathrm{T}_a)$ is a connected reductive subgroup of G in which the maximal torus T contains the central subtorus T_a of codimension 1. The Lie algebra $\mathrm{Lie}(\mathrm{Z}_{\mathrm{G}}(\mathrm{T}_a))$ is equal to $\mathfrak{g}^{\mathrm{T}_a}$ [**Bo91**, 9.4], and this in turn is equal to $\mathfrak{t} \oplus \mathfrak{g}_a \oplus \mathfrak{g}_{-a}$ [**Bo91**, 13.18(4a)]. In particular, T is noncentral in $\mathrm{Z}_{\mathrm{G}}(\mathrm{T}_a)$ since its adjoint action on $\mathrm{Lie}(\mathrm{Z}_{\mathrm{G}}(\mathrm{T}_a))$ is nontrivial, so T_a is the maximal central torus in $\mathrm{Z}_{\mathrm{G}}(\mathrm{T}_a)$ (and hence by Example 1.1.16 it coincides with $\mathscr{R}(\mathrm{Z}_{\mathrm{G}}(\mathrm{T}_a))$). We conclude that the quotient $\mathrm{Z}_{\mathrm{G}}(\mathrm{T}_a)/\mathrm{T}_a$ is semisimple with the 1-dimensional T/T_a as a maximal torus. Equivalently, the isogenous G_a is semisimple with a 1-dimensional maximal torus $\mathrm{T}'_a = (\mathrm{T} \bigcap \mathrm{G}_a)^0_{\mathrm{red}}$ (an isogeny-complement to T_a in T); since T is maximal in $\mathrm{Z}_{\mathrm{G}}(\mathrm{T}_a)$, the maximality of T'_a in G_a is a special case of Exercise 1.6.12.

Since the only nontrivial T-weights on $\mathrm{Lie}(\mathrm{Z}_{\mathrm{G}}(\mathrm{T}_a))$ are $\pm a$, the semisimple G_a must have Lie algebra $\mathfrak{t}'_a \oplus \mathfrak{g}_a \oplus \mathfrak{g}_{-a}$, with $\mathfrak{t}'_a := \mathrm{Lie}(\mathrm{T}'_a)$. Thus, G_a is 3-dimensional. By [**Bo91**, 13.13(5)], there exists an isogeny $\mathrm{G}_a \to \mathrm{PGL}_2$, and by conjugacy of maximal tori it can be arranged to carry T'_a onto the diagonal torus $\overline{\mathrm{D}}$. Thus, this isogeny carries $\mathrm{U}_{\pm a}$ onto the root groups $\overline{\mathrm{U}}^{\pm}$ for $\overline{\mathrm{D}}$. But the pair of subgroups $\overline{\mathrm{U}}^{\pm}$ visibly generates PGL_2 (since the subgroups $\mathrm{U}^{\pm}$ generate SL_2), so we conclude that indeed $\mathrm{U}_{\pm a}$ generate G_a. □

We wish to introduce coroots: to each root $a \in \Phi$ we will attach a canonical nontrivial cocharacter $a^\vee : \mathbf{G}_{\mathrm{m}} \to \mathrm{T}$ that generalizes the cocharacter $c \mapsto \mathrm{diag}(c, 1/c)$ in SL_2 attached to the root $\mathrm{diag}(c, 1/c) \mapsto c^2$ for D. The definition of coroots rests on the classification of semisimple k-groups of rank 1. This classification is the assertion that any such group G is isomorphic to either SL_2 or PGL_2. In other words, there exists an isogeny $\mathrm{SL}_2 \to \mathrm{G}$ whose kernel is contained in the central μ_2. When combined with Lemma 1.2.6, this provides interesting homomorphisms from SL_2 into nontrivial connected semisimple groups. Here is the statement in the form that we will need.

Theorem 1.2.7. — *For each* $a \in \Phi(\mathrm{G}, \mathrm{T})$, *there exists a homomorphism* $\varphi_a : \mathrm{SL}_2 \to \mathrm{G}$ *carrying the diagonal torus* D *into* T *and the strictly upper triangular and strictly lower triangular unipotent subgroups* $\mathrm{U}^{\pm}$ *isomorphically onto the respective root groups* $\mathrm{U}_{\pm a}$.

Such a homomorphism φ_a *is an isogeny onto* G_a *with* $\ker \varphi_a \subset \mu_2$, *it is unique up to* $\mathrm{T}(k)$*-conjugation on* G, *and it carries the standard Weyl element* $w = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ *to an element* $n_a \in \mathrm{N}_{\mathrm{G}(k)}(\mathrm{T}) - \mathrm{T}(k)$.

Proof. — Let $\mathrm{T}'_a = (\mathrm{T} \bigcap \mathrm{G}_a)^0_{\mathrm{red}}$, a maximal torus of G_a that is an isogeny complement to T_a in T. Assume there is a homomorphism φ_a with the desired properties on D and $\mathrm{U}^{\pm}$. By Lemma 1.2.6 we know $\mathrm{U}_{\pm a} \subset \mathrm{G}_a$, so the classical fact that $\mathrm{U}^{\pm}(k)$ generate $\mathrm{SL}_2(k)$ implies that φ_a must land inside G_a. Likewise, $\varphi_a(\mathrm{D})$ must be contained in $(\mathrm{T} \bigcap \mathrm{G}_a)^0_{\mathrm{red}} = \mathrm{T}'_a$. Thus, we can replace (G, T) with $(\mathrm{G}_a, \mathrm{T}'_a)$.

Now G is semisimple with a 1-dimensional maximal torus T and $\Phi(\mathrm{G}, \mathrm{T}) = \{\pm a\}$. In particular, $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{g}_a \oplus \mathfrak{g}_{-a}$ is 3-dimensional, so $\dim \mathrm{G} = 3$. The key point is to show that G is isomorphic to either SL_2 or PGL_2. This is not proved in [**Bo91**] (and correspondingly, coroots are not discussed in [**Bo91**]), so we provide a proof below.

Step 1. We first show that G admits an isogeny onto PGL_2 with scheme-theoretic kernel Z that is isomorphic to 1 or μ_2 and is scheme-theoretically central in G (i.e., Z(R) is central in G(R) for every k-algebra R). Note that the case $\mathrm{char}(k) = 2$ is "non-classical" since μ_2 is non-reduced for such k, but our arguments will be characteristic-free.

Since $\mathrm{T} \cap \mathrm{U}_a = 1$ as k-schemes inside G, the map $\mathrm{B} := \mathrm{T} \ltimes \mathrm{U}_a \to \mathrm{G}$ is a closed k-subgroup (see Proposition 1.1.1). By dimension considerations, B is a Borel subgroup of G containing T, and another is $\mathrm{B}' := \mathrm{T} \ltimes \mathrm{U}_{-a}$. The multiplication map

$$\mu : \mathrm{U}_{-a} \times \mathrm{B} = \mathrm{U}_{-a} \times \mathrm{T} \times \mathrm{U}_a \to \mathrm{G} \tag{1.2.2}$$

is étale at the identity due to the tangential criterion, so (by left $\mathrm{U}_{-a}(k)$-translation and right $\mathrm{B}(k)$-translation) it is étale everywhere. Since the closed k-subgroup scheme $\mathrm{U}_{-a} \bigcap \mathrm{B}$ is étale (due to transversality: $\mathfrak{u}_{-a} \bigcap \mathfrak{b} = 0$) and normalized by T (see the T-equivariant description of root groups at the start of the proof of Lemma 1.2.6), yet the k-group $\mathrm{U}_{-a} \simeq \mathbf{G}_{\mathrm{a}}$ clearly contains no nontrivial finite étale subgroups normalized by T, it follows that the étale map μ is injective on k-points. Thus, μ is étale and radiciel, hence (by [**EGA**, IV_4, 17.9.1]) an isomorphism onto its open image Ω; i.e., μ is an open immersion. We conclude that $\mathrm{B} \bigcap \mathrm{B}' = \mathrm{T}$ scheme-theoretically.

Since G is semisimple of rank 1, $\mathrm{G}/\mathrm{B} \simeq \mathbf{P}^1$ [**Bo91**, 13.13(4)]. The left translation action of G on G/B then defines a k-homomorphism to the automorphism scheme

$$f : \mathrm{G} \to \mathrm{Aut}_{\mathbf{P}^1_k/k} = \mathrm{PGL}_2$$

(see Exercise 1.6.3(iv),(v)) whose scheme-theoretic kernel K is a normal subgroup scheme that is contained in B. We shall now prove that $\mathrm{K} = \ker a$ and that this is a finite central subgroup scheme of G (so f is an isogeny with central kernel). By normality K is contained in the $\mathrm{G}(k)$-conjugate B' of B, so $\mathrm{K} \subset \mathrm{B} \bigcap \mathrm{B}' = \mathrm{T}$ as closed subschemes of G. The left translation action by

T on G preserves the open subscheme $\Omega = \mathrm{U}_{-a} \times \mathrm{T} \times \mathrm{U}_a$ via the formula

$$t.(u_- t' u_+) = (t u_- t^{-1})(t t') u_+,$$

so the left T-action on G/B preserves the open subscheme $\Omega/\mathrm{B} = \mathrm{U}_{-a} \simeq \mathbf{G}_\mathrm{a}$ on which it acts via scaling through $-a : \mathrm{T} \to \mathbf{G}_\mathrm{m}$. Hence, K must be contained in $\ker a$. But the group scheme $\ker a$ visibly centralizes the dense open subscheme Ω in the smooth group G, so it centralizes G. Since $\ker a \subset \mathrm{T} \subset \mathrm{B}$, we obtain the reverse inclusion $\ker a \subset \mathrm{K}$.

To summarize, we have built a short exact sequence of group schemes

$$1 \to \mathrm{Z} \to \mathrm{G} \xrightarrow{f} \mathrm{PGL}_2 \to 1 \tag{1.2.3}$$

with $\mathrm{Z} = \ker a$ a finite subgroup scheme of $\mathrm{T} = \mathbf{G}_\mathrm{m}$ that is central in G. In particular, $\mathrm{Z} \simeq \mu_n$ for some $n \geqslant 1$. The Weyl group $\mathrm{W}_\mathrm{G}(\mathrm{T}) := \mathrm{N}_{\mathrm{G}(k)}(\mathrm{T})/\mathrm{T}(k)$ has order 2 [**Bo91**, 13.13(2)], so conjugation by a representative $n \in \mathrm{N}_{\mathrm{G}(k)}(\mathrm{T})$ of the nontrivial element in $\mathrm{W}_\mathrm{G}(\mathrm{T})$ acts on $\mathrm{T} = \mathbf{G}_\mathrm{m}$ by its only nontrivial automorphism, namely inversion. But such conjugation must be trivial on the *central* subgroup scheme Z in G, so inversion on μ_n is trivial. This forces $n|2$. If $n = 1$ then f is an isomorphism, and if $n = 2$ then the conjugation action by $\mathrm{T} = \mathbf{G}_\mathrm{m}$ on $\mathrm{U}_{\pm a} = \mathbf{G}_\mathrm{a}$ must be scaling by the only two characters of $\mathbf{G}_\mathrm{m}$ with kernel μ_2, namely $t \mapsto t^{\pm 2}$.

Step 2. Now we relate G to SL_2 in case $\mathrm{Z} = \mu_2$, and in general we adjust f so that it relates T and $\mathrm{U}_{\pm a}$ to $\overline{\mathrm{D}}$ and $\overline{\mathrm{U}}^{\pm}$ respectively. The isogeny f must carry T onto a maximal torus of PGL_2, so by composing f with a suitable conjugation we can arrange that $f(\mathrm{T})$ is the diagonal torus $\overline{\mathrm{D}}$. It then follows that $\mathrm{T} = f^{-1}(\overline{\mathrm{D}})$ scheme-theoretically because $\mathrm{Z} \subset \mathrm{T}$.

Since the k-subgroup $\mathrm{Z} = \ker a$ has trivial scheme-theoretic intersection with the root groups $\mathrm{U}_{\pm a}$ (as even $\mathrm{U}_{\pm a} \bigcap \mathrm{T} = 1$), these root groups are carried *isomorphically* by f onto their images in PGL_2. By the unique characterization of root groups [**Bo91**, 13.18(4d)], it follows that (as unordered pairs) $\{f(\mathrm{U}_a), f(\mathrm{U}_{-a})\}$ is the set of root groups $\{\overline{\mathrm{U}}^+, \overline{\mathrm{U}}^-\}$ for $\overline{\mathrm{D}}$, so by composing f with conjugation by the standard Weyl element of $(\mathrm{PGL}_2, \overline{\mathrm{D}})$ if necessary we can arrange that $f(\mathrm{U}_a) = \overline{\mathrm{U}}^+$ and $f(\mathrm{U}_{-a}) = \overline{\mathrm{U}}^-$.

Suppose $\mathrm{Z} = \mu_2$. There is a unique isomorphism $\mathrm{T} \simeq \mathbf{G}_\mathrm{m}$ carrying the degree-2 isogeny $a : \mathrm{T} \to \mathbf{G}_\mathrm{m}$ over to the map $t^2 : \mathbf{G}_\mathrm{m} \to \mathbf{G}_\mathrm{m}$. Combining this with the isomorphisms $\mathrm{U}_{\pm a} \simeq \overline{\mathrm{U}}^{\pm} = \mathbf{G}_\mathrm{a}$ arising from f and the natural isomorphisms $\mathrm{U}^{\pm} \simeq \overline{\mathrm{U}}^{\pm}$, we identify the open subscheme $\Omega = \mathrm{U}_{-a} \times \mathrm{T} \times \mathrm{U}_a$ in G with the standard open subscheme

$$\mathbf{G}_\mathrm{a} \times \mathbf{G}_\mathrm{m} \times \mathbf{G}_\mathrm{a} \simeq \mathrm{U}^- \times \mathrm{D} \times \mathrm{U}^+ \subset \mathrm{SL}_2 \tag{1.2.4}$$

where the isomorphism (1.2.4) is defined by

$$(x', c, x) \mapsto \begin{pmatrix} 1 & 0 \\ x' & 1 \end{pmatrix} \begin{pmatrix} c & 0 \\ 0 & 1/c \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.$$

Continuing to assume $\mathrm{Z} = \mu_2$, let $\mathrm{V} \subset \Omega \times \Omega$ be the dense open locus of points (ω, ω') such that $\omega\omega' \in \Omega$ inside G. We claim that the open immersion $j : \Omega \simeq \mathrm{U}^- \times \mathrm{D} \times \mathrm{U}^+ \hookrightarrow \mathrm{SL}_2$ is a "birational homomorphism" in the sense that $j(\omega\omega') = j(\omega)j(\omega')$ for $(\omega, \omega') \in \mathrm{V}$. Since the composition of j with the canonical isogeny $q : \mathrm{SL}_2 \to \mathrm{PGL}_2$ is a homomorphism (namely, f), the map $\mathrm{V} \to \mathrm{SL}_2$ defined by $j(\omega\omega')j(\omega')^{-1}j(\omega)^{-1}$ factors through $\ker q = \mu_2$ and so is identically 1. Hence, indeed j is a birational homomorphism, so it extends uniquely to an isomorphism of k-groups (Exercise 1.6.6)!

Allowing either possibility for Z, $f : \mathrm{G} \to \mathrm{PGL}_2$ is either an isomorphism carrying T onto $\overline{\mathrm{D}}$ and carrying $\mathrm{U}_{\pm a}$ onto $\overline{\mathrm{U}}^{\pm}$ or else it factors through $q : \mathrm{SL}_2 \to \mathrm{PGL}_2$ via an isomorphism $\mathrm{G} \simeq \mathrm{SL}_2$ carrying T onto D and carrying $\mathrm{U}_{\pm a}$ onto $\mathrm{U}^{\pm}$. Either way, there is a unique homomorphism $\varphi : \mathrm{SL}_2 \to \mathrm{G}$ factoring q through f, and φ satisfies the desired properties to be φ_a except possibly uniqueness up to $\mathrm{T}(k)$-conjugation (e.g., $\varphi(\mathrm{D}) = \mathrm{T}$ since $\mathrm{T} = f^{-1}(\overline{\mathrm{D}})$).

Step 3. Finally, we prove the uniqueness of φ_a up to $\mathrm{T}(k)$-conjugation. First suppose f is an isomorphism. By using composition with f, it suffices to show that the only homomorphisms $\pi : \mathrm{SL}_2 \to \mathrm{PGL}_2$ carrying D into $\overline{\mathrm{D}}$ and $\mathrm{U}^{\pm}$ isomorphically onto $\overline{\mathrm{U}}^{\pm}$ respectively are $\overline{\mathrm{D}}(k)$-conjugates of q. Since the roots for PGL_2 have trivial kernel whereas the roots for SL_2 have kernel equal to the central $\mathrm{D}[2] = \mu_2$, it follows from the isomorphism condition on root groups that any such π must kill $\mathrm{D}[2]$ and so factors through q. In other words, to prove the uniqueness of π up to $\overline{\mathrm{D}}(k)$-conjugation it suffices to treat the analogous assertion for endomorphisms $\overline{\pi} : \mathrm{PGL}_2 \to \mathrm{PGL}_2$ that satisfy $\overline{\pi}(\overline{\mathrm{D}}) = \overline{\mathrm{D}}$ and $\overline{\pi} : \overline{\mathrm{U}}^{\pm} \simeq \overline{\mathrm{U}}^{\pm}$.

Since $\mathrm{Aut}(\mathbf{G}_{\mathrm{a}}) = k^{\times}$ and $\overline{\pi}$ carries $\overline{\mathrm{U}}^{+} = \mathbf{G}_{\mathrm{a}}$ isomorphically onto $\overline{\mathrm{U}}^{+}$, by composing with a $\overline{\mathrm{D}}(k)$-conjugation (which makes $\mathrm{diag}(t, 1) \in \overline{\mathrm{D}}(k)$ act on $\overline{\mathrm{U}}^{+} = \mathbf{G}_{\mathrm{a}}$ via $t.x = tx$) we may arrange that $\overline{\pi}$ is the *identity* map on $\overline{\mathrm{U}}^{+}$. By hypothesis $\overline{\pi}$ carries $\overline{\mathrm{D}}$ into $\overline{\mathrm{D}}$, so the faithfulness of the $\overline{\mathrm{D}}$-action on $\overline{\mathrm{U}}^{+} = \mathbf{G}_{\mathrm{a}}$ implies that $\overline{\pi}$ restricts to the identity on $\overline{\mathrm{D}}$. We will prove that $\overline{\pi}$ restricts to the identity map on the dense open $\overline{\Omega} := \overline{\mathrm{U}}^{-} \times \overline{\mathrm{D}} \times \overline{\mathrm{U}}^{+}$, thereby forcing $\overline{\pi}$ to be the identity map.

The restriction of $\overline{\pi}$ to the dense open direct product subscheme $\overline{\Omega} \subset \mathrm{PGL}_2$ is visibly an automorphism of $\overline{\Omega}$, so $\ker \overline{\pi}$ is étale. But an étale closed normal subgroup of a connected linear algebraic group is central, yet PGL_2 has no nontrivial central finite étale subgroup (as $\overline{\mathrm{D}}$ is its own centralizer on k-points and acts faithfully under conjugation on $\overline{\mathrm{U}}^{+}$), so $\ker \overline{\pi} = 1$. Thus, $\overline{\pi}$ is an isomorphism. In particular, $\overline{\pi}^{-1}(\overline{\Omega}) = \overline{\Omega}$.

Let $\overline{u}_{\pm} : \mathbf{G}_{\mathrm{a}} \simeq \overline{\mathrm{U}}^{\pm}$ be the parameterizations $x \mapsto \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$ and $x \mapsto \left(\begin{smallmatrix} 1 & 0 \\ x & 1 \end{smallmatrix}\right)$, so the calculations

$$\begin{aligned} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} &= \begin{pmatrix} 1+xy & x \\ y & 1 \end{pmatrix}, \\ \begin{pmatrix} 1 & 0 \\ x' & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & y' \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} a & ay' \\ ax' & ax'y' + a^{-1} \end{pmatrix} \end{aligned}$$

imply that the product $\overline{u}_{+}(x)\overline{u}_{-}(y)$ lies in $\overline{\Omega} = \overline{\mathrm{U}}^{-} \times \overline{\mathrm{D}} \times \overline{\mathrm{U}}^{+}$ if and only if $1 + xy \in \mathbf{G}_{\mathrm{m}}$. The restriction $\overline{\pi} : \overline{\mathrm{U}}^{-} \simeq \overline{\mathrm{U}}^{-}$ corresponds to an automorphism of $\mathbf{G}_{\mathrm{a}}$, which is to say $\overline{\pi}(\overline{u}_{-}(y)) = \overline{u}_{-}(cy)$ for some $c \in k^{\times}$. Thus, $\overline{\pi}(\overline{u}_{+}(x)\overline{u}_{-}(y)) = \overline{u}_{+}(x)\overline{u}_{-}(cy)$, so the equality $\overline{\pi}^{-1}(\overline{\Omega}) = \overline{\Omega}$ implies that $1 + xy \in \mathbf{G}_{\mathrm{m}}$ if and only if $1 + x \cdot cy \in \mathbf{G}_{\mathrm{m}}$. It follows that $c = 1$, so $\overline{\pi}$ is the identity on $\overline{\mathrm{U}}^{-}$ and we are done when f is an isomorphism.

Suppose instead that f is not an isomorphism, so (as we have seen above) $(\mathrm{G}, \mathrm{T}) \simeq (\mathrm{SL}_2, \mathrm{D})$ carrying $\mathrm{U}_{\pm a}$ to $\mathrm{U}^{\pm}$ respectively. It therefore suffices to show that the only endomorphisms φ of SL_2 carrying each of $\mathrm{D}, \mathrm{U}^{+}, \mathrm{U}^{-}$ into themselves are conjugation by elements of $\mathrm{D}(k)$. By the uniqueness established above, $q \circ \varphi : \mathrm{SL}_2 \to \mathrm{PGL}_2$ is the composition of q with conjugation against some $\overline{d} \in \overline{\mathrm{D}}(k)$. Hence, if $d \in \mathrm{D}(k)$ lifts $\overline{d}$ then $g \mapsto \varphi(g)(dgd^{-1})^{-1}$ is a scheme morphism from the smooth connected SL_2 into μ_2 and thus is the trivial map $g \mapsto 1$. That is, necessarily $\varphi(g) = dgd^{-1}$ for all g, as desired. □

Using any φ_a as in Theorem 1.2.7, the cocharacter

$$a^{\vee} : \mathbf{G}_{\mathrm{m}} \to \mathrm{D} \to \mathrm{T}$$

defined by $a^{\vee}(c) = \varphi_a(\mathrm{diag}(c, 1/c))$ is unaffected by $\mathrm{T}(k)$-conjugation on G, so it is intrinsic.

Definition 1.2.8. — The *coroot* associated to $(\mathrm{G}, \mathrm{T}, a)$ is the cocharacter $a^{\vee} \in \mathrm{X}_*(\mathrm{T}) - \{0\}$. The finite subset of coroots in $\mathrm{X}_*(\mathrm{T})$ is denoted $\Phi^{\vee}$.

Concretely, $a^{\vee}$ is a parameterization (with kernel 1 or μ_2) of the 1-dimensional torus $(\mathrm{T} \bigcap \mathscr{D}(\mathrm{Z}_{\mathrm{G}}(\mathrm{T}_a)))^0_{\mathrm{red}}$ that is an isogeny complement to T_a in T. The composition of any φ_a with transpose-inverse on SL_2 satisfies the requirements to be φ_{-a}. Since transpose-inverse acts by inversion of the diagonal torus of SL_2, we conclude that $(-a)^{\vee} = -a^{\vee}$.

Example 1.2.9. — Suppose $\mathrm{G} = \mathrm{SL}_2$ and T is the diagonal torus D. The roots for (G, T) are $\mathrm{diag}(c, 1/c) \mapsto c^{\pm 2}$. Let a be the root $\mathrm{diag}(c, 1/c) \mapsto c^2$, so $\mathfrak{g}_a$ is the subspace of upper triangular nilpotent matrices in $\mathfrak{sl}_2$ and $\mathfrak{g}_{-a}$ is the subspace of lower triangular nilpotent matrices in $\mathfrak{sl}_2$. By Example 1.2.5, the corresponding root groups are $\mathrm{U}_a = \mathrm{U}^{+}$ and $\mathrm{U}_{-a} = \mathrm{U}^{-}$, so φ_a can be taken to be the identity map. In particular, $a^\vee(c) = \mathrm{diag}(c, 1/c)$. Note that $\langle a, a^\vee\rangle = 2$ in $\mathrm{End}(\mathbf{G}_\mathrm{m}) = \mathbf{Z}$ (i.e., $a(a^\vee(c)) = c^2$): this follows from the calculation

$$\begin{pmatrix} c & 0 \\ 0 & 1/c \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c & 0 \\ 0 & 1/c \end{pmatrix}^{-1} = \begin{pmatrix} 1 & c^2 x \\ 0 & 1 \end{pmatrix},$$

which implies that the adjoint action of $a^\vee(c)$ on $\mathfrak{g}_a = \mathrm{Lie}(\mathrm{U}_a)$ is scaling by c^2. Observe also that in $\mathfrak{t}$ we have

$$\mathrm{Lie}(a^\vee)(t\partial_t) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \mathrm{H}_a = [\mathrm{E}_a, \mathrm{F}_a]$$

(where t is the standard coordinate on $\mathbf{G}_\mathrm{m} \simeq \mathrm{D}$ via the isomorphism $c \mapsto \mathrm{diag}(c, 1/c)$).

For $\mathrm{G} = \mathrm{PGL}_2$ and $\mathrm{T} = \overline{\mathrm{D}}$, we have an isomorphism $\mathbf{G}_\mathrm{m} \simeq \mathrm{T}$ via $c \mapsto \mathrm{diag}(c, 1) \bmod \mathbf{G}_\mathrm{m}$. The inverse isomorphism $\overline{a} : \mathrm{T} \simeq \mathbf{G}_\mathrm{m}$ is a root whose root space consists of the upper triangular nilpotent matrices in $\mathfrak{pgl}_2 = \mathfrak{gl}_2/\mathfrak{gl}_1$, and similarly for the root $-\overline{a}$ using lower triangular nilpotent matrices. Thus, $\varphi_{\overline{a}} : \mathrm{SL}_2 \to \mathrm{G}$ can be taken to be the canonical projection. In particular, $\overline{a}^\vee : \mathbf{G}_\mathrm{m} \to \mathrm{T}$ is $c \mapsto \mathrm{diag}(c, 1/c) \bmod \mathbf{G}_\mathrm{m} = \mathrm{diag}(c^2, 1) \bmod \mathbf{G}_\mathrm{m}$, so $\langle \overline{a}, \overline{a}^\vee\rangle = 2$.

Remark 1.2.10. — It follows from Example 1.2.9 (for both SL_2 and PGL_2) that in general $\langle a, a^\vee\rangle = 2$ for any connected reductive k-group G and maximal torus $\mathrm{T} \subset \mathrm{G}$.

Example 1.2.11. — Let $\mathrm{G} = \mathrm{GL}(\mathrm{V}) = \mathrm{GL}_n$, and take T to be the diagonal torus D_n. For the root $a = e_i - e_j$ ($1 \leqslant i \neq j \leqslant n$), the associated codimension-1 subtorus T_a is defined by the equality $\{c_i = c_j\}$ between ith and jth diagonal entries. The centralizer $\mathrm{Z}_\mathrm{G}(\mathrm{T}_a)$ consists of elements of GL_n whose off-diagonal entries vanish away from the ij and ji positions, and the subgroup $\mathrm{G}_a := \mathscr{D}(\mathrm{Z}_\mathrm{G}(\mathrm{T}_a))$ is $\mathrm{SL}(ke_i \oplus ke_j) = \mathrm{SL}_2$. In particular, $a^\vee = e_i^\vee - e_j^\vee$. This makes explicit that $\langle a, a^\vee\rangle = 2$ for all $a \in \Phi(\mathrm{GL}_n, \mathrm{D}_n)$.

1.3. Root datum, root system, and classification theorem. — Let G be a connected reductive k-group, and let $\mathrm{T} \subset \mathrm{G}$ be a maximal torus. In the dual lattices $\mathrm{X} = \mathrm{X}(\mathrm{T})$ and $\mathrm{X}^\vee = \mathrm{X}_*(\mathrm{T})$ we have defined finite subsets $\Phi = \Phi(\mathrm{G}, \mathrm{T}) \subset \mathrm{X} - \{0\}$ and $\Phi^\vee = \Phi(\mathrm{G}, \mathrm{T})^\vee \subset \mathrm{X}^\vee - \{0\}$ and a bijection $a \mapsto a^\vee$ between Φ and $\Phi^\vee$ such that (i) $\langle a, a^\vee\rangle = 2$ for all a, (ii) for each $a \in \Phi$, $\Phi \bigcap \mathbf{Q}a = \{\pm a\}$ inside $\mathrm{X}_\mathbf{Q}$.

There are additional properties satisfied by this combinatorial data. To formulate them, we introduce some reflections. (If V is a nonzero finite-dimensional vector space over a field of characteristic 0, a *reflection* $r : \mathrm{V} \to \mathrm{V}$ is an automorphism with order 2 such that -1 occurs as an eigenvalue with multiplicity one, or equivalently such that $\mathrm{V}^{r=1}$ is a hyperplane.) For each

$a \in \Phi$, define the linear endomorphisms $s_a : \mathrm{X} \to \mathrm{X}$ and $s_{a^\vee} : \mathrm{X}^\vee \to \mathrm{X}^\vee$ by

$$s_a(x) = x - \langle x, a^\vee \rangle a, \quad s_{a^\vee}(\lambda) = \lambda - \langle a, \lambda \rangle a^\vee. \tag{1.3.1}$$

Since $\langle a, a^\vee \rangle = 2$, it is easy to check that $s_a(a) = -a$ and $s_{a^\vee}(a^\vee) = -a^\vee$. Clearly s_a fixes pointwise the hyperplane $\ker a^\vee \subset \mathrm{X}_{\mathbf{Q}}$ complementary to $\mathbf{Q}a$, and similarly for $s_{a^\vee}$ and $\ker a \subset \mathrm{X}^\vee_{\mathbf{Q}}$. Moreover, $s_a^2 = 1$ since

$$s_a^2(x) = x - \langle x, a^\vee \rangle a - \langle x - \langle x, a^\vee \rangle a, a^\vee \rangle a = x - 2\langle x, a^\vee \rangle a + \langle a, a^\vee \rangle \langle x, a^\vee \rangle a = x,$$

and similarly $s_{a^\vee}^2 = 1$, so on $\mathrm{X}_{\mathbf{Q}}$ and $\mathrm{X}^\vee_{\mathbf{Q}}$ the automorphisms s_a and $s_{a^\vee}$ are reflections in the lines spanned by a and $a^\vee$ respectively. It is also easy to check that $s_{a^\vee}$ is dual to s_a; this amounts to the identity

$$\langle x - \langle x, a^\vee \rangle a, \lambda \rangle = \langle x, \lambda - \langle a, \lambda \rangle a^\vee \rangle.$$

(Some introductory accounts of the theory of root systems impose a Euclidean structure on $\mathrm{X}_{\mathbf{Q}}$ or $\mathrm{X}_{\mathbf{R}}$ at the outset, such as in [**Hum72**, III], but this is not necessary. To that end, note that we have not imposed any positive-definite quadratic form on $\mathrm{X}_{\mathbf{Q}}$ or $\mathrm{X}_{\mathbf{R}}$.)

Example 1.3.1. — Let $\mathrm{G} = \mathrm{GL}_n$ and let T be the diagonal torus. For $1 \leqslant i \neq j \leqslant n$ and $1 \leqslant h \leqslant n$, we have $s_{e_i - e_j}(e_h) = e_h$ when $h \neq i, j$ and the reflection $s_{e_i - e_j}$ swaps e_i and e_j. In particular, $s_a(\Phi) = \Phi$ for all $a \in \Phi$.

For $n = 2$ this amounts to the fact that the standard Weyl element $w = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ acts on the diagonal torus of SL_2 via inversion.

The importance of the reflections s_a and $s_{a^\vee}$ in general is:

Proposition 1.3.2. — *For all $a \in \Phi$, $s_a(\Phi) = \Phi$ and $s_{a^\vee}(\Phi^\vee) = \Phi^\vee$.*

Proof. — Visibly $\mathrm{T}_a \cap \mathrm{T}'_a$ is finite (as T'_a is 1-dimensional and $a|_{\mathrm{T}'_a} \neq 1$), so $\mathrm{T}'_a \times \mathrm{T}_a \to \mathrm{T}$ is an isogeny. This identifies $\mathrm{X}(\mathrm{T})_{\mathbf{Q}}$ with $\mathrm{X}(\mathrm{T}'_a)_{\mathbf{Q}} \oplus \mathrm{X}(\mathrm{T}_a)_{\mathbf{Q}}$, carrying $\mathrm{X}(\mathrm{T}_a)_{\mathbf{Q}}$ onto the hyperplane spanned over $\mathbf{Q}$ by the characters that kill $\mathrm{T}'_a = a^\vee(\mathbf{G}_{\mathrm{m}})$ and carrying $\mathrm{X}(\mathrm{T}'_a)_{\mathbf{Q}}$ onto the line spanned over $\mathbf{Q}$ by the characters that kill T_a (i.e., it is the $\mathbf{Q}$-span of a).

Let $n_a \in \mathrm{G}_a = \mathscr{D}(\mathrm{Z}_{\mathrm{G}}(\mathrm{T}_a))$ be a representative in $\mathrm{N}_{\mathrm{G}_a(k)}(\mathrm{T}'_a)$ for the nontrivial element in the group $\mathrm{W}_{\mathrm{G}_a}(\mathrm{T}'_a)$ of order 2. The conjugation action by n_a on the almost direct product $\mathrm{T} = \mathrm{T}'_a \cdot \mathrm{T}_a$ is trivial on T_a and inversion on $\mathrm{T}'_a = a^\vee(\mathbf{G}_{\mathrm{m}})$, so n_a acts on $\mathrm{X}(\mathrm{T})$ as an involution whose effect on $\mathrm{X}(\mathrm{T})_{\mathbf{Q}}$ negates the line spanned by a and fixes pointwise the hyperplane $\mathrm{X}(\mathrm{T}_a)_{\mathbf{Q}}$.

We conclude that n_a acts on $\mathrm{X}(\mathrm{T})$ as a reflection r_a. This reflection visibly preserves Φ (as does the effect of any element of $\mathrm{N}_{\mathrm{G}(k)}(\mathrm{T})$), and we claim that $r_a = s_a$ (so $s_a(\Phi) = \Phi$). The reflections r_a and s_a negate the same line $\mathbf{Q}a$, so it suffices to show that they restrict to the identity on a common hyperplane, namely $\mathrm{X}(\mathrm{T}_a)_{\mathbf{Q}}$. But this hyperplane is the annihilator of $\mathbf{Q}a^\vee \subset \mathrm{X}_*(\mathrm{T})_{\mathbf{Q}}$, so the definition of s_a makes it clear that s_a is the identity on $\mathrm{X}(\mathrm{T}_a)_{\mathbf{Q}}$.

The equality $r_a = s_a$ implies that the dual reflection $s_a^\vee = s_{a^\vee}$ on $X_*(T)$ is also induced by n_a-conjugation on T. Thus, $s_{a^\vee}$ preserves $\Phi^\vee$. $\square$

We have shown that $R(G, T) := (X(T), \Phi(G, T), X_*(T), \Phi(G, T)^\vee)$ satisfies the requirements in the following definition introduced in [**SGA3**, XXI]:

Definition 1.3.3 **(Demazure)**. — A *root datum* is a 4-tuple $(X, \Phi, X^\vee, \Phi^\vee)$ consisting of a pair of finite free $\mathbf{Z}$-modules X and $X^\vee$ equipped with

— a perfect duality $\langle\cdot,\cdot\rangle : X \times X^\vee \to \mathbf{Z}$,

— finite subsets $\Phi \subset X - \{0\}$ and $\Phi^\vee \subset X^\vee - \{0\}$ stable under negation for which there exists a bijection $a \mapsto a^\vee$ from Φ to $\Phi^\vee$ such that $\langle a, a^\vee\rangle = 2$ and the resulting reflections $s_a : X \simeq X$ and $s_{a^\vee} : X^\vee \simeq X^\vee$ as in (1.3.1) satisfy $s_a(\Phi) = \Phi$ and $s_{a^\vee}(\Phi^\vee) = \Phi^\vee$. If moreover $\mathbf{Q}a \bigcap \Phi = \{\pm a\}$ inside $X_{\mathbf{Q}}$ for all $a \in \Phi$ then the root datum is *reduced*.

Remark 1.3.4. — The bijection $a \mapsto a^\vee$ in the definition of a root datum is uniquely determined. For a proof, see [**CGP**, Lemma 3.2.4].

Remark 1.3.5. — It is immediate from the axioms that if $(X, \Phi, X^\vee, \Phi^\vee)$ is a root datum then so is $(X^\vee, \Phi^\vee, X, \Phi)$. This is called the *dual* root datum.

In the study of connected semisimple k-groups "up to central isogeny" (see Exercise 1.6.13), it is convenient to work with a coarser notion than a root datum, in which we relax the $\mathbf{Z}$-structure to a $\mathbf{Q}$-structure and remove the explicit mention of the coroots. This leads to the notion of a *root system* (which historically arose much earlier than the notion of a root datum, in the classification of semisimple Lie algebras over $\mathbf{C}$, and is extensively studied in [**Bou2**, Ch. VI]): this is a pair (V, Φ) consisting of a finite-dimensional $\mathbf{Q}$-vector space V and a finite spanning set $\Phi \subset V - \{0\}$ such that for each $a \in \Phi$ there exists a reflection $s_a : v \mapsto v - \lambda(v)a$ with $\lambda \in V^*$ such that $s_a(\Phi) = \Phi$, $s_a(a) = -a$ (equivalently, $\lambda(a) = 2$), and $\lambda(\Phi) \subset \mathbf{Z}$. Such a reflection is *unique* (even without the integrality condition on λ) because if s' is another then by inspecting the effects on $\mathbf{Q}a$ and $V/\mathbf{Q}a$ we see that $s' \circ s^{-1}$ is a unipotent automorphism of V yet it preserves the *finite* spanning set Φ and hence has finite order, forcing $s' \circ s^{-1} = 1$ (as $\mathrm{char}(\mathbf{Q}) = 0$).

Example 1.3.6. — If $(X, \Phi, X^\vee, \Phi^\vee)$ is a root datum then the $\mathbf{Q}$-span V of Φ in $X_{\mathbf{Q}}$ equipped with the subset Φ is a root system.

The difference between root systems and root data is analogous to the difference between connected semisimple k-groups considered up to central isogeny and connected reductive k-groups considered up to isomorphism: the possible failure of Φ to span $X_{\mathbf{Q}}$ is analogous to the possibility that a connected reductive group may have a nontrivial central torus (i.e., fail to be semisimple), and the use of $\mathbf{Q}$-structures rather than $\mathbf{Z}$-structures amounts to considering groups up to central isogeny.

More explicitly, if (G, T) is a connected reductive k-group equipped with a maximal torus T, then the saturation in $\mathrm{X}(\mathrm{T})$ of the $\mathbf{Z}$-span $\mathbf{Z}\Phi$ of Φ is $\mathrm{X}(\mathrm{T}/\mathrm{Z})$ where Z is the maximal central torus of G (since this saturation is $\mathrm{X}(\mathrm{T}/\mathscr{T})$ for the largest torus $\mathscr{T}$ killed by Φ, and a torus T' in G is central if and only if $a(\mathrm{T}') = 1$ for all $a \in \Phi$ [**Bo91**, 14.2(1)]). Thus, Φ spans $\mathrm{X}(\mathrm{T})_{\mathbf{Q}}$ if and only if G has no nontrivial central torus, which is to say that G is semisimple.

Remark 1.3.7. — If a root datum is not reduced then $\mathbf{Q}a \bigcap \Phi$ equals $\{\pm a\}$, $\{\pm a, \pm 2a\}$ (the *multipliable* case), or $\{\pm a, \pm a/2\}$ (the *divisible* case). Indeed, this is a property of the underlying root system, so it holds by [**Bou2**, VI, § 1.3, Prop. 8(i)]. The study of connected reductive groups over fields $k \neq k_s$ gives rise to non-reduced root data; e.g., this occurs for $k = \mathbf{R}$ in the study of non-compact connected semisimple Lie groups, as well as in the study of special unitary groups over general k admitting a separable quadratic extension.

In general, the quotient $\mathrm{X}(\mathrm{T})/\mathbf{Z}\Phi$ is the Cartier dual $\mathrm{Hom}(\mathrm{Z}_{\mathrm{G}}, \mathbf{G}_{\mathrm{m}})$ of the scheme-theoretic center Z_{G} of G. This asserts that the inclusion $\mathrm{Z}_{\mathrm{G}} \subset \bigcap_{a\in\Phi} \ker a$ is an equality, and holds because G is generated by T and the root groups U_a (as even holds for Lie algebras). As a special case, $\mathbf{Z}\Phi$ has finite index in $\mathrm{X}(\mathrm{T})$ if and only if Z_{G} is finite; i.e., there is no nontrivial central torus (equivalently, G is semisimple). Here are some illustrations of the relations between the root datum and the (scheme-theoretic) center.

Example 1.3.8. — The set Φ spans $\mathrm{X}(\mathrm{T})$ over $\mathbf{Z}$ if and only if the scheme-theoretic center is trivial (the "adjoint semisimple" case). For example, $(\mathrm{PGL}_2, \overline{\mathrm{D}})$ has roots $\pm\overline{a}$ that are isomorphisms $\overline{\mathrm{D}} \simeq \mathbf{G}_{\mathrm{m}}$, so these each span $\mathrm{X}(\overline{\mathrm{D}}) = \mathbf{Z}$, whereas $(\mathrm{SL}_2, \mathrm{D})$ has roots $\pm a : \mathrm{D} \to \mathbf{G}_{\mathrm{m}}$ with kernel $\mathrm{D}[2] = \mu_2$, so these each span the unique index-2 subgroup of $\mathrm{X}(\mathrm{D}) = \mathbf{Z}$. This encodes the fact that SL_2 has scheme-theoretic center μ_2 whereas PGL_2 has trivial scheme-theoretic center.

Example 1.3.9. — The set $\Phi(\mathrm{G}, \mathrm{T})$ is empty if and only if G is a torus (i.e., $\mathrm{G} = \mathrm{T}$), or equivalently G is solvable. This is immediate from the weight space decomposition (1.2.1) since $\mathfrak{g}_0 = \mathfrak{t}$.

Example 1.3.10. — Suppose G is semisimple. In this case we have the containment of lattices

$$\mathrm{Q} := \mathbf{Z}\Phi \subset \mathrm{X}(\mathrm{T}) \subset (\mathbf{Z}\Phi^\vee)^* =: \mathrm{P}, \tag{1.3.2}$$

where the dual lattice $(\mathbf{Z}\Phi^\vee)^*$ in $\mathrm{X}(\mathrm{T})_{\mathbf{Q}}$ is $\mathbf{Z}$-dual to the lattice $\mathbf{Z}\Phi^\vee \subset \mathrm{X}(\mathrm{T})^\vee = \mathrm{X}(\mathrm{T})^*$. Thus, the two "extreme" cases are $\mathrm{X}(\mathrm{T}) = \mathbf{Z}\Phi$ and $\mathrm{X}(\mathrm{T}) = (\mathbf{Z}\Phi^\vee)^*$ (i.e., $\mathbf{Z}\Phi^\vee = \mathrm{X}_*(\mathrm{T})$). In the language of root systems, these cases respectively correspond to the cases when (i) the base for a positive system of roots $\Phi(\mathrm{G},\mathrm{T})$ (see Definition 1.4.1) is a basis of the character group of T, and (ii) the base for a positive system of coroots is a basis of the cocharacter group of T. The first of these two extremes is the case of adjoint G (such as PGL_n) and the second is the case of simply connected G (such as SL_n); see Exercise 1.6.13(ii).

The above examples (and the theory over **C**; see Proposition D.4.1) inspire:

Definition 1.3.11. — A reduced root datum $\mathrm{R} = (\mathrm{X}, \Phi, \mathrm{X}^\vee, \Phi^\vee)$ is *semisimple* if Φ spans $\mathrm{X}_{\mathbf{Q}}$ over $\mathbf{Q}$. In the semisimple case, it is *adjoint* if $\mathbf{Z}\Phi = \mathrm{X}$ and is *simply connected* if $\mathbf{Z}\Phi^\vee = \mathrm{X}^\vee$.

If $(\mathbf{Q}\Phi^\vee)^\perp$ denotes the annihilator in $\mathrm{X}_{\mathbf{Q}}$ of the subspace $\mathbf{Q}\Phi^\vee \subset \mathrm{X}^\vee_{\mathbf{Q}}$ then the natural map $(\mathbf{Q}\Phi) \oplus (\mathbf{Q}\Phi^\vee)^\perp \to \mathrm{X}_{\mathbf{Q}}$ is an isomorphism [**Spr**, Exer. 7.4.2] (mirroring the isogeny decomposition of a connected reductive group into the almost direct product of a torus and a connected semisimple group as in Example 1.1.16). Thus, the subgroup

$$\mathrm{W}(\mathrm{R}) = \langle s_a \mid a \in \Phi \rangle \subset \mathrm{Aut}(\mathrm{X})$$

is trivial on $(\mathbf{Q}\Phi^\vee)^\perp$ and acts on $\mathbf{Q}\Phi$ through permutations of a finite spanning set. It follows that $\mathrm{W}(\mathrm{R})$ is finite; it is called the *Weyl group* of the root datum, and is naturally isomorphic to the Weyl group of the associated root system:

Example 1.3.12. — If (G,T) is a connected reductive k-group equipped with a maximal torus T then since $\mathrm{Z}_{\mathrm{G}(k)}(\mathrm{T}) = \mathrm{T}(k)$, the action of $\mathrm{N}_{\mathrm{G}(k)}(\mathrm{T})$ on T identifies $\mathrm{W}_{\mathrm{G}}(\mathrm{T}) = \mathrm{N}_{\mathrm{G}(k)}(\mathrm{T})/\mathrm{T}(k)$ with a finite subgroup of $\mathrm{Aut}(\mathrm{X}(\mathrm{T}))$. This subgroup is the Weyl group of the root datum associated to (G,T) [**Bo91**, 14.8]. Here is the idea of the proof. In the proof of Proposition 1.3.2 we showed each reflection s_a in the Weyl group of the root datum $\mathrm{R}(\mathrm{G},\mathrm{T})$ lies in $\mathrm{W}_{\mathrm{G}}(\mathrm{T})$, so $\mathrm{W}(\mathrm{R}(\mathrm{G},\mathrm{T})) \subset \mathrm{W}_{\mathrm{G}}(\mathrm{T})$. The reverse containment is proved in § 1.4 by relating Borel subgroups to positive systems of roots in the associated root system (see Definition 1.4.1 and Proposition 1.4.4) and using that the Weyl group of a root system acts simply transitively on the set of positive systems of roots ([**Bou2**, VI, § 1.5, Thm. 2(i); § 1.6, Thm. 3]).

Definition 1.3.13. — A root system (V, Φ) is *non-empty* if $\Phi \neq \varnothing$ (equivalently, $\mathrm{V} \neq 0$). The *direct product* of root systems (V_1, Φ_1) and (V_2, Φ_2) is $(\mathrm{V}_1, \Phi_1) \times (\mathrm{V}_2, \Phi_2) = (\mathrm{V}_1 \oplus \mathrm{V}_2, \Phi_1 \coprod \Phi_2)$. A root system (V, Φ) is *irreducible* if it is non-empty and not a direct product of two non-empty root systems.

Remark 1.3.14. — Every non-empty root system is uniquely a direct product of irreducible root systems [**Bou2**, VI, § 1.2, Prop. 6] and there is a classification of irreducible root systems ([**Bou2**, VI, § 4.2, Prop. 1, Thm. 3] in the reduced case, and [**Bou2**, VI, § 1.4, Prop. 13] and [**Bou2**, VI, § 4.14] in the non-reduced case). This is very useful in the study of root data.

For each irreducible root system Φ, there is a positive-definite $\mathbf{Q}$-valued quadratic form Q invariant under the action of the Weyl group [**Bou2**, VI, § 1.1, Prop. 3]. This quadratic form is unique up to scaling whether we take V to be finite-dimensional over $\mathbf{Q}$ (as we have done above) or over $\mathbf{R}$ (as in most literature) [**Bou2**, VI, § 1.2, Prop. 7]. In particular, it is intrinsic to compare ratios $\mathrm{Q}(a)/\mathrm{Q}(b)$ for $a, b \in \Phi$ when (V, Φ) is irreducible. After extending scalars to $\mathbf{R}$, this equips irreducible root systems with a canonical inner product (up to scaling), and thereby puts the study of root systems into the framework of Euclidean geometry.

For example, suppose $\mathrm{R} = (\mathrm{X}, \Phi, \mathrm{X}^\vee, \Phi^\vee)$ is a root datum such that Φ spans $\mathrm{V} := \mathrm{X}_{\mathbf{Q}}$ and (V, Φ) is an irreducible root system. Choose a W(R)-invariant inner product $(\cdot|\cdot)$ on $\mathrm{V}_{\mathbf{R}}$. We use this inner product to identify $\mathrm{X}_{\mathbf{R}}$ with its linear dual $\mathrm{X}_{\mathbf{R}}^\vee$. In this way the coroot $a^\vee$ is identified with the element $2a/(a|a) \in \mathrm{X}_{\mathbf{R}}$ [**Bou2**, VI, § 1.1, Lemma 2]. This is the most convenient way for drawing pictures of low-rank irreducible root data. Also, it is intrinsic to compare *ratios* of root lengths, and in cases with distinct root lengths there are exactly two, with $(a|a)/(b|b) \in \{2, 3\}$ when $(a|a) > (b|b)$ [**Bou2**, VI, § 1.4, Prop. 12].

There is an evident notion of *isomorphism* between root data (required to respect the given perfect duality between X and $\mathrm{X}^\vee$). If $f : (\mathrm{G}, \mathrm{T}) \simeq (\mathrm{G}', \mathrm{T}')$ is an isomorphism, then it is clear that the induced isomorphisms $\mathrm{X}(\mathrm{T}') \simeq \mathrm{X}(\mathrm{T})$ and $\mathrm{X}_*(\mathrm{T}) \simeq \mathrm{X}_*(\mathrm{T}')$ respect the dualities and the subsets of roots and coroots, so we get an isomorphism of root data $\mathrm{R}(\mathrm{G}', \mathrm{T}') \simeq \mathrm{R}(\mathrm{G}, \mathrm{T})$. If we change f via composition with a $\mathrm{T}'(k)$-conjugation or $\mathrm{T}(k)$-conjugation then the isomorphism between the root data is unchanged. Now we can finally state a fundamental result in the classical theory over $k = \overline{k}$:

Theorem 1.3.15 **(Existence and Isomorphism Theorems)**
The reduced root datum $\mathrm{R}(\mathrm{G}, \mathrm{T})$ *associated to a connected reductive* k*-group* G *and maximal torus* $\mathrm{T} \subset \mathrm{G}$ *determines* (G, T) *uniquely up to isomorphism. More precisely, for any two pairs* (G, T) *and* $(\mathrm{G}', \mathrm{T}')$*, every isomorphism* $\mathrm{R}(\mathrm{G}', \mathrm{T}') \simeq \mathrm{R}(\mathrm{G}, \mathrm{T})$ *arises from an isomorphism* $(\mathrm{G}, \mathrm{T}) \simeq (\mathrm{G}', \mathrm{T}')$ *that is unique up to the conjugation actions of* $\mathrm{T}'(k)$ *and* $\mathrm{T}(k)$*, and every reduced root datum is isomorphic to* $\mathrm{R}(\mathrm{G}, \mathrm{T})$ *for some pair* (G, T) *over* k*.*

A remarkable aspect of this theorem is that the root datum has nothing to do with k or $\mathrm{char}(k)$. There is a finer version of the theorem that also classifies isogenies in terms of a notion of "isogeny" between root data; this encodes characteristic-dependent concepts (such as the Frobenius isogeny in characteristic $p > 0$), but we postpone the statement and proof of this Isogeny

Theorem until we discuss the Existence and Isomorphism Theorems over a general non-empty base scheme in § 6. The proof of these theorems over a general base scheme will *not* require the classical version over a general algebraically closed field. It only requires the Existence Theorem for connected semisimple groups over a single algebraically closed field of characteristic 0; see Appendix D.

Example 1.3.16. — For any (G, T), consider the root datum R' dual to $\mathrm{R}(\mathrm{G}, \mathrm{T})$ in the sense of Remark 1.3.5. By the Existence Theorem, there exists another pair $(\mathrm{G}', \mathrm{T}')$ over k unique up to isomorphism for which $\mathrm{R}(\mathrm{G}', \mathrm{T}') \simeq \mathrm{R}'$. Since the isomorphism in $(\mathrm{G}', \mathrm{T}')$ is only ambiguous up to a $\mathrm{T}'(k)$-conjugation, and such conjugation has *no* effect on T', it is reasonable to consider the representation theory of $(\mathrm{G}', \mathrm{T}')$ (incorporating T'-weight space information) as a structure that is intrinsically associated to (G, T). This is a version of *Langlands duality.*

A basic example is $\mathrm{G} = \mathrm{GL}_n$, in which case $\mathrm{G}' = \mathrm{GL}_n$. Slightly more interesting is the case $\mathrm{G} = \mathrm{SL}_n$, for which $\mathrm{G}' = \mathrm{PGL}_n$. In general, Langlands duality in the semisimple case swaps adjoint and simply connected groups (see Example 1.3.10).

Remark 1.3.17. — Operations with root systems have analogues for connected semisimple groups. For example, the decomposition of a non-empty root system into its irreducible components corresponds to the fact that every nontrivial connected semisimple k-group G has only finitely many minimal (necessarily semisimple) nontrivial normal smooth connected subgroups G_i, each G_i is simple (i.e., has no nontrivial proper connected normal linear algebraic subgroup), and these pairwise commute and define a central isogeny $\prod \mathrm{G}_i \to \mathrm{G}$ via multiplication. See [**Bo91**, 14.10] for proofs based on the structure of automorphisms of semisimple groups, and see Proposition 5.1.17ff. for a simple proof based on the structure of the "open cell" (using (1.4.2)).

1.4. Positive systems of roots and parabolic subgroups. — Let G be a connected reductive k-group and T a maximal torus. There are only finitely many parabolic subgroups P of G containing T, and in particular only finitely many Borel subgroups B of G containing T. These can be described in terms of the following combinatorial notion applied to the root system $\Phi(\mathrm{G}, \mathrm{T})$:

Definition 1.4.1. — Let (V, Φ) be a non-empty root system. A *positive system of roots* is a subset $\Phi^+ \subset \Phi$ such that $\Phi^+ = \Phi_{\lambda > 0} := \{a \in \Phi \mid \lambda(a) > 0\}$ for some $\lambda \in \mathrm{V}^*$ that is non-vanishing on Φ; i.e., Φ^+ is the part of Φ lying in an open half-space of $\mathrm{V}_{\mathbf{R}}$ whose boundary hyperplane is disjoint from Φ.

For any positive system of roots Φ^+ in Φ, the subset $\Delta \subset \Phi^+$ of elements of Φ^+ that cannot be expressed as a sum of two elements of Φ^+ turns out to be a basis of V and every element of Φ has the form $\sum_{a\in\Delta} m_a a$ with integers m_a that are either all $\geqslant 0$ or all $\leqslant 0$ [**Bou2**, VI, § 1.6, Thm. 3]. The elements of Δ are called the *simple positive roots* relative to Φ^+, and any such Δ is called a *base* for the root system Φ.

Note that if we fix an enumeration $\{a_i\}$ of such a Δ and equip $\mathrm{V} = \bigoplus \mathbf{Q} a_i$ with the lexicographical ordering then Φ^+ consists of the elements of Φ that are positive relative to this ordering. Conversely, for any ordered vector space structure on V the set of positive elements of Φ turns out to be a positive system of roots [**Bou2**, VI, § 1.7, Cor. 2] (this is useful in the study of "highest weight vectors" in representation theory; see § 1.5).

Example 1.4.2. — For $\mathrm{G} = \mathrm{GL}_n$ $(n \geqslant 2)$ and $\mathrm{T} = \mathbf{G}_{\mathrm{m}}^n$ the diagonal torus, identify $\mathrm{X}(\mathrm{T})_{\mathbf{Q}} = \mathrm{X}(\mathbf{G}_{\mathrm{m}})^n_{\mathbf{Q}}$ with $\mathbf{Q}^n$ via the canonical identification $\mathrm{X}(\mathbf{G}_{\mathrm{m}}) = \mathbf{Z}$. In other words, to each $\chi : \mathbf{G}_{\mathrm{m}}^n \to \mathbf{G}_{\mathrm{m}}$ associate the n-tuple $(c_1, \ldots, c_n) \in \mathbf{Z}^n$ for which $\chi(t_1, \ldots, t_n) = \prod t_i^{c_i}$. In this way, $\Phi(\mathrm{G}, \mathrm{T})$ is a root system in the hyperplane $\mathrm{V} = \{\vec{c} \in \mathbf{Q}^n \mid \sum c_j = 0\}$. Equip $\mathbf{Q}^n$ with the lexicographical ordering, and V with the induced ordering. The root system $\Phi(\mathrm{G}, \mathrm{T})$ consists of the differences $e_i - e_j$ for $1 \leqslant i \neq j \leqslant n$, and the positive ones for this ordering are $e_i - e_j$ for $i < j$. The corresponding base Δ consists of the roots $e_i - e_{i+1}$ for $1 \leqslant i \leqslant n-1$.

This positive system of roots is exactly the set of roots that occur as nontrivial T-weights on the Lie algebra of the Borel subgroup B of upper triangular matrices in G. See Proposition 1.4.4 for the general result of which this is a special case.

Every element of Φ is a simple positive root for some choice of Φ^+ (equivalently, for any Δ the $\mathrm{W}(\Phi)$-orbits of the elements of Δ cover Φ) [**Bou2**, VI, § 1.5, Prop. 15], and if we fix a choice of Φ^+ and let Δ be the corresponding base then $\mathrm{W}(\Phi)$ is generated by the reflections s_a for $a \in \Delta$. In fact, $\mathrm{W}(\Phi)$ has a presentation as a reflection group generated by the reflections in the simple positive roots [**Bou2**, VI, § 1.5, Thm. 2(vii), Rem. 3, (11)]:

$$\mathrm{W}(\Phi) = \langle \{s_a\}_{a\in\Delta} \mid (s_a s_b)^{m_{ab}} = 1 \text{ for } a, b \in \Delta \rangle \tag{1.4.1}$$

where $m_{aa} = 1$ for all $a \in \Delta$, $m_{ab} = 2$ (equivalently, $s_a s_b = s_b s_a$) when a and b are in distinct irreducible components of Φ or are orthogonal in the same component, and otherwise $m_{ab} = 3$ for non-orthogonal $a, b \in \Delta$ with the same length and $m_{ab} = 2\langle a, b^\vee\rangle\langle b, a^\vee\rangle \in \{4, 6\}$ for non-orthogonal $a, b \in \Delta$ with distinct lengths.

Remark 1.4.3. — An important invariant of a reduced root system (V, Φ) is its *Dynkin diagram* $\mathrm{Dyn}(\Phi)$, a graph with extra structure on certain edges. This intervenes in the classification of root systems, which is an ingredient in the proof of the Existence and Isomorphism Theorems (at least in low rank), and it can be defined in terms of Euclidean geometry or combinatorics with a root system (see [**Bou2**, VI, § 4.2]). We will give both definitions. In all cases, the vertices of the graph are the elements of a base Δ for a positive system of roots (and the simply transitive action of $\mathrm{W}(\Phi)$ on the set of all Δ's identifies vertices with certain $\mathrm{W}(\Phi)$-orbits in Φ, eliminating the dependence on the choice of Δ).

For the Euclidean definition we shall define $\mathrm{Dyn}(\Phi) = \coprod \mathrm{Dyn}(\Phi_i)$ for the irreducible components (V_i, Φ_i) of (V, Φ), so suppose Φ is irreducible. Recall from Remark 1.3.14 that there exists a $\mathrm{W}(\Phi)$-invariant inner product $(\cdot|\cdot)$ on $\mathrm{V}_{\mathbf{R}}$ (even a $\mathbf{Q}$-valued $\mathrm{W}(\Phi)$-invariant positive-definite quadratic form on V), it is unique up to scaling, and as we vary through $a, b \in \Phi$ with $(a|a) \geqslant (b|b)$, the ratio $(a|a)/(b|b)$ is $\mathbf{Z}$-valued and takes on at most two possible values. The diagram $\mathrm{Dyn}(\Phi)$ has as its vertices the elements of Δ, and there exists an edge linking vertices a and b precisely when $(a|b) \neq 0$. If $(a|a) > (b|b)$ then this edge is assigned multiplicity $(a|a)/(b|b)$ and a direction pointing from a to b. The graph is always connected (for irreducible Φ).

In the combinatorial definition we do not need to pass to the irreducible components: $\mathrm{Dyn}(\Phi)$ is a graph whose vertices are the elements of Δ, and an edge joins a and b precisely when $a + b \in \Phi$. (Note that $a - b \notin \Phi$ since $a, b \in \Delta$.) For such a and b, the multiplicity and direction of the edge joining a and b are defined in terms of the sets

$$\mathrm{I}_{a,b} = \{j \in \mathbf{Z} \,|\, a + jb \in \Phi\}, \;\; \mathrm{I}_{b,a} = \{j \in \mathbf{Z} \,|\, b + ja \in \Phi\}$$

as follows. Since $a, b \in \Delta$, we have $\mathrm{I}_{a,b} = \mathbf{Z} \bigcap [0, \dots, -\langle a, b^\vee \rangle]$ and $\mathrm{I}_{b,a} = \mathbf{Z} \bigcap [0, \dots, -\langle b, a^\vee \rangle]$ [**Bou2**, VI, § 1.3, Prop. 9]. An inspection of cases [**Bou2**, VI, § 1.3, Rem.] shows that either both sets coincide with $\{0, 1\}$, in which case a and b are joined by a single undirected edge, or one of them is $\{0, 1\}$ and the other is either $\{0, 1, 2\}$ or $\{0, 1, 2, 3\}$. In these latter cases if we arrange the labels so that $\mathrm{I}_{a,b}$ contains 2 or 3 then a and b are joined by an edge with multiplicity $-\langle a, b^\vee \rangle$ pointing from a to b.

Returning to the setting of a connected reductive k-group G and a maximal torus T, if B is a Borel subgroup containing T then the subalgebra $\mathfrak{b} = \mathrm{Lie}(\mathrm{B}) \subset \mathfrak{g}$ contains $\mathfrak{t}$ and so has a weight space decomposition $\mathfrak{b} = \mathfrak{t} \bigoplus (\bigoplus_{a \in \Phi(\mathrm{B},\mathrm{T})} \mathfrak{g}_a)$ for a subset $\Phi(\mathrm{B}, \mathrm{T}) \subset \Phi(\mathrm{G}, \mathrm{T})$. Such subsets are positive systems of roots in $\Phi(\mathrm{G}, \mathrm{T})$ [**Bo91**, 14.1]. If $\mathrm{U} = \mathscr{R}_u(\mathrm{B})$ then for any enumeration $\{a_i\}$ of $\Phi(\mathrm{B}, \mathrm{T})$ the multiplication map

$$\prod \mathrm{U}_{a_i} \to \mathrm{U} \tag{1.4.2}$$

is an isomorphism of schemes (see [**Bo91**, 14.4] or [**CGP**, 3.3.6, 3.3.7, 3.3.11]); we say U is "directly spanned in any order" by the root groups contained in U.

Since $\mathrm{W_G(T)}$ acts simply transitively on the set of all $\mathrm{B} \supset \mathrm{T}$ [**Bo91**, 13.10(2)], and $\mathrm{W}(\Phi)$ acts simply transitively on the set of positive systems of roots in Φ [**Bou2**, VI, § 1.5, Thm. 2(i)], the inclusion $\mathrm{W}(\Phi) \subset \mathrm{W_G(T)}$ is forced to be an equality and we obtain:

Proposition 1.4.4. — *The map* $\mathrm{B} \mapsto \Phi(\mathrm{B},\mathrm{T})$ *is a bijection from the set of* $\mathrm{B} \supset \mathrm{T}$ *onto the set of positive systems of roots in* $\Phi(\mathrm{G},\mathrm{T})$.

The generalization of Proposition 1.4.4 to the case of parabolic subgroups containing T requires another class of distinguished subsets of a root system:

Definition 1.4.5. — Let (V,Φ) be a root system. A *parabolic subset* of Φ is a subset $\Psi \subset \Phi$ of the form $\Psi = \Phi_{\lambda \geqslant 0} := \{a \in \Phi \mid \lambda(a) \geqslant 0\}$ for some $\lambda \in \mathrm{V}^*$; i.e., Ψ is the part of Φ lying in a closed half-space of V (or $\mathrm{V}_{\mathbf{R}}$).

The reason for this terminology is that it will turn out that for any parabolic subgroup $\mathrm{P} \subset \mathrm{G}$ containing T, the set $\Phi(\mathrm{P},\mathrm{T})$ of nontrivial T-weights occurring in $\mathfrak{p} = \mathrm{Lie(P)}$ is a parabolic subset of $\Phi(\mathrm{G},\mathrm{T})$. Before we can state the precise bijective correspondence in this direction, it will be convenient to discuss some alternative formulations of the definition of parabolicity for a subset of a root system (V,Φ).

It is clear from the definition that any parabolic subset Ψ of Φ satisfies the following two properties: (i) $\Phi = \Psi \bigcup -\Psi$, and (ii) Ψ is a *closed* set in Φ (i.e., if $a, b \in \Psi$ and $a + b \in \Phi$ then $a + b \in \Psi$). Conversely, any subset of Ψ satisfying (i) and (ii) is parabolic. The equivalence is proved in [**CGP**, 2.2.8] (where parabolicity is defined using (i) and (ii), as in [**Bou2**, Ch. IV]), and a more explicit description of parabolic sets is provided there: they are precisely the subsets $\Phi^+ \bigcup [\mathrm{I}]$, where Φ^+ is a positive system of roots, I is a subset of the corresponding base Δ, and [I] denotes the set of roots that are $\mathbf{Z}$-linear combinations of elements of I. In particular, every parabolic set contains a positive system of roots.

Remark 1.4.6. — Let $\Psi \subset \Phi := \Phi(\mathrm{G},\mathrm{T})$ be a closed set of roots that is contained in a positive system of roots Φ^+ for Φ. For the unique Borel subgroup $\mathrm{B} \subset \mathrm{G}$ containing T that satisfies $\Phi(\mathrm{B},\mathrm{T}) = \Phi^+$, the smooth connected subgroup $\mathrm{U}_\Psi \subset \mathrm{G}$ generated by the root groups $\{\mathrm{U}_a\}_{a \in \Psi}$ is contained in $\mathscr{R}_u(\mathrm{B})$ and hence is unipotent. But much more is true: the group U_Ψ is *directly spanned in any order* by the groups $\{\mathrm{U}_a\}_{a\in\Psi}$ in the same sense as for the case $\Psi = \Phi^+$ considered in (1.4.2). That is, for *any* enumeration $\{a_1, \dots, a_m\}$ of Ψ, the multiplication map $\prod \mathrm{U}_{a_i} \to \mathrm{U}_\Psi$ between pointed schemes is an isomorphism. A proof using the structure theory of reductive groups is given in [**Bo91**, 14.5, Prop. (2)], and a proof via general dynamical principles is given in [**CGP**, 3.3.11, 3.3.13(1)].

A given parabolic set $\Psi \subset \Phi$ can contain more than one positive system of roots, just as a parabolic subgroup P can contain more than one Borel subgroup containing a fixed maximal torus T in P. Nonetheless, for *any* positive system of roots Φ^+ (with corresponding base Δ) contained in Ψ, we have $\Psi = \Phi^+ \bigcup [\mathrm{I}]$ for a unique $\mathrm{I} \subset \Delta$ [**Bou2**, VI, § 1.7, Lemma 3].

Proposition 1.4.7. — *The map* $\mathrm{P} \mapsto \Phi(\mathrm{P}, \mathrm{T})$ *is a bijective correspondence between the set of parabolic subgroups of* G *containing* T *and the set of parabolic subsets of* $\Phi = \Phi(\mathrm{G}, \mathrm{T})$*, and the following are equivalent:* $\mathrm{P} \subset \mathrm{P}'$*,* $\Phi(\mathrm{P}, \mathrm{T}) \subset \Phi(\mathrm{P}', \mathrm{T})$*, and* $\mathrm{Lie}(\mathrm{P}) \subset \mathrm{Lie}(\mathrm{P}')$ *inside* $\mathfrak{g}$.

Proof. — Since Lie(P) is spanned by Lie(T) and the weight spaces $\mathfrak{g}_c = \mathrm{Lie}(\mathrm{U}_c)$ for $c \in \Phi(\mathrm{P}, \mathrm{T})$, to prove the equivalence of containment assertions it suffices to prove that if $c \in \Phi(\mathrm{P}, \mathrm{T})$ then $\mathrm{U}_c \subset \mathrm{P}$ (as then P is generated by T and such U_c by Lie algebra considerations since P is connected). We reduce to the rank-1 case as follows.

Any P contains a Borel subgroup B of G that contains T (due to the conjugacy of maximal tori in P). For any subtorus $\mathrm{S} \subset \mathrm{T}$, the group $(\mathrm{P} \bigcap \mathrm{Z}_{\mathrm{G}}(\mathrm{S}))_{\mathrm{red}} = \mathrm{Z}_{\mathrm{P}}(\mathrm{S})$ contains the subgroup $(\mathrm{B} \bigcap \mathrm{Z}_{\mathrm{G}}(\mathrm{S}))_{\mathrm{red}} = \mathrm{Z}_{\mathrm{B}}(\mathrm{S})$ that is a Borel subgroup of $\mathrm{Z}_{\mathrm{G}}(\mathrm{S})$ [**Bo91**, 11.15], so $(\mathrm{P} \bigcap \mathrm{Z}_{\mathrm{G}}(\mathrm{S}))_{\mathrm{red}}$ is a parabolic subgroup of $\mathrm{Z}_{\mathrm{G}}(\mathrm{S})$ containing T. The bijective correspondence between the sets of parabolic subgroups of a connected reductive group H and of its derived group $\mathscr{D}(\mathrm{H})$ is defined by $\mathrm{P}' \mapsto (\mathrm{P}' \bigcap \mathscr{D}(\mathrm{H}))_{\mathrm{red}}$, so $(\mathrm{P} \bigcap \mathscr{D}(\mathrm{Z}_{\mathrm{G}}(\mathrm{S})))_{\mathrm{red}}$ is a parabolic subgroup of $\mathscr{D}(\mathrm{Z}_{\mathrm{G}}(\mathrm{S}))$ containing the maximal torus $(\mathrm{T} \bigcap \mathscr{D}(\mathrm{Z}_{\mathrm{G}}(\mathrm{S})))_{\mathrm{red}}$.

Taking S to be the codimension-1 subtorus $\mathrm{T}_c := (\ker c)^0_{\mathrm{red}} \subset \mathrm{T}$, $\mathrm{P}_c := (\mathrm{P} \bigcap \mathscr{D}(\mathrm{Z}_{\mathrm{G}}(\mathrm{T}_c)))_{\mathrm{red}}$ is a parabolic subgroup of the 3-dimensional connected semisimple group $\mathscr{D}(\mathrm{Z}_{\mathrm{G}}(\mathrm{T}_c)) = \langle \mathrm{U}_c, \mathrm{U}_{-c} \rangle$ containing the maximal torus $\mathrm{T} \bigcap \mathscr{D}(\mathrm{Z}_{\mathrm{G}}(\mathrm{T}_c)) = c^\vee(\mathbf{G}_{\mathrm{m}})$. Since the root groups of $\mathscr{D}(\mathrm{Z}_{\mathrm{G}}(\mathrm{T}_c))$ relative to $c^\vee(\mathbf{G}_{\mathrm{m}})$ are $\mathrm{U}_{\pm c}$, we can replace $(\mathrm{G}, \mathrm{T}, \mathrm{P})$ with $(\mathscr{D}(\mathrm{Z}_{\mathrm{G}}(\mathrm{T}_c)), c^\vee(\mathbf{G}_{\mathrm{m}}), \mathrm{P}_c)$ to reduce to the case when G is semisimple of rank 1. We can then choose an isomorphism from G onto either SL_2 or PGL_2 such that T is carried to the diagonal torus and U_c is carried to the upper triangular unipotent subgroup. There are three parabolic subgroups containing the diagonal torus: the entire group and the upper and lower triangular Borel subgroups. The condition $c \in \Phi(\mathrm{P}, \mathrm{T})$ rules out the lower triangular Borel subgroup, and inspection of the two remaining possibilities shows that $\mathrm{U}_c \subset \mathrm{P}$.

It remains to show each parabolic subset Ψ of Φ is $\Phi(\mathrm{P}, \mathrm{T})$ for some parabolic subgroup P containing T. Since $\mathrm{W}_{\mathrm{G}}(\mathrm{T})$ acts transitively on the set of positive systems of roots in Φ, we may restrict attention to Ψ that contain $\Phi^+ := \Phi(\mathrm{B}, \mathrm{T})$ for a fixed Borel subgroup B of G containing T. Thus, $\Psi = \Phi^+ \cup [\mathrm{I}]$ for a unique subset I of the base Δ of Φ^+. Since T-conjugation on $\mathrm{U}_c \simeq \mathbf{G}_{\mathrm{a}}$ is scaling through c, for the subtorus $\mathrm{T}_{\mathrm{I}} = (\cap_{a \in \mathrm{I}} \ker a)^0_{\mathrm{red}}$ we see that $\mathrm{Z}_{\mathrm{G}}(\mathrm{T}_{\mathrm{I}})$ contains as its T-root groups exactly U_c for $c \in \Phi$ that kill T_{I}. Let U_{I} be the smooth connected subgroup of $\mathscr{R}_u(\mathrm{B})$ directly spanned in any order by the root groups U_b for b in the closed subset $(\Phi - [\mathrm{I}]) \cap \Phi^+ \subset \Phi^+$ (Remark 1.4.6).

The connected reductive group $\mathrm{Z_G(T_I)}$ normalizes $\mathrm{U_I}$. Indeed, its root system with respect to T is [I] (as Δ is a basis of the character group of the adjoint torus $\mathrm{T/Z_G}$), so it is generated by T and the root groups U_c for $c \in [\mathrm{I}]$. Clearly T normalizes $\mathrm{U_I}$, and if $c \in -\Phi^+ \bigcap [\mathrm{I}]$ then U_c even centralizes $\mathrm{U_I}$ since $c+b$ cannot be a root for any $b \in \Phi^+ \cap (\Phi - [\mathrm{I}])$. If instead $c \in [\mathrm{I}] \cap \Phi^+$ then for any $b \in (\Phi - [\mathrm{I}]) \cap \Phi^+$ the roots of the form $ib + jc$ with $i, j \geqslant 1$ obviously lie in $(\Phi - [\mathrm{I}]) \cap \Phi^+$, so points of U_c conjugate U_b into $\mathrm{U_I}$ for all such b. That is, U_c normalizes $\mathrm{U_I}$ for all such c, so $\mathrm{Z_G(T_I)}$ normalizes $\mathrm{U_I}$ as claimed. Moreover, the schematic intersection $\mathrm{Z_G(T_I)} \cap \mathrm{U_I}$ is trivial. Indeed, T-weight space considerations show that the Lie algebra of this intersection vanishes, so the intersection is étale, yet it is also connected since $\mathrm{Z_{T_I \ltimes U_I}(T_I)} = \mathrm{T_I} \times (\mathrm{Z_G(T_I)} \cap \mathrm{U_I})$ and torus centralizers in smooth connected affine groups are connected. Thus, $\mathrm{P_I} := \mathrm{Z_G(T_I)} \ltimes \mathrm{U_I}$ makes sense as a subgroup of G.

The smooth subgroup $\mathrm{P_I}$ clearly contains B (so it is a parabolic subgroup containing T) and satisfies $\Phi(\mathrm{P_I}, \mathrm{T}) = \Phi^+ \cup [\mathrm{I}] = \Psi$. (This recovers the parameterization of "standard" parabolic subgroups of G in [**Bo91**, 14.18] via a different proof. Note that $\mathscr{R}_u(\mathrm{P_I}) = \mathrm{U_I}$.) □

Example 1.4.8. — Let $\mathrm{G} = \mathrm{SL}_n$ and consider the upper triangular B and diagonal T with character group $\mathbf{Z}^n/\mathrm{diag}(\mathbf{Z})$. Then $\Delta = \{e_{i+1} - e_i\}_{1 \leqslant i \leqslant n-1} = \{1, \dots, n-1\}$, and for $\mathrm{I} \subset \{1, \dots, n-1\}$ the parabolic $\mathrm{P_I}$ corresponds via Example 1.1.10 to the ordered partition $\vec{a} = (a_1, \dots, a_r)$ of n into non-empty parts for which the associated subset $\{b_j = a_1 + \cdots + a_j\}_{1 \leqslant j < r}$ in $\{1, \dots, n-1\}$ is the *complement* of I; e.g., $\mathrm{P}_\varnothing = \mathrm{B}$.

Corollary 1.4.9. — *The number of* $\mathrm{G}(k)$*-conjugacy classes of parabolic subgroups of* G *is* $2^{r_{\mathrm{ss}}}$*, where* $r_{\mathrm{ss}} = \mathrm{rank}(\mathscr{D}(\mathrm{G}))$ *is the dimension of the maximal tori of* $\mathscr{D}(\mathrm{G})$. (*Equivalently,* $r_{\mathrm{ss}} = \#\Delta$ *for a base* Δ *of a positive system of roots in* $\Phi(\mathrm{G}, \mathrm{T})$ *for maximal tori* T *of* G.)

An explicit description of conjugacy class representatives is in Exercise 4.4.8.

Proof. — We just have to show that for a fixed Borel subgroup $\mathrm{B} \subset \mathrm{G}$ and a maximal torus $\mathrm{T} \subset \mathrm{B}$, parabolic subgroups $\mathrm{P_I}$ and $\mathrm{P_J}$ associated to subsets I and J of the base Δ of $\Phi^+ = \Phi(\mathrm{B}, \mathrm{T})$ are $\mathrm{G}(k)$-conjugate if and only if $\mathrm{I} = \mathrm{J}$. More conceptually, if P and P′ are parabolic subgroups of G containing B and if $\mathrm{P}' = g\mathrm{P}g^{-1}$ for some $g \in \mathrm{G}(k)$ then we claim that $\mathrm{P}' = \mathrm{P}$. The Borel subgroups B and $g\mathrm{B}g^{-1}$ in P′ are $\mathrm{P}'(k)$-conjugate, so by replacing g with a suitable left $\mathrm{P}'(k)$-translate we can arrange that $g\mathrm{B}g^{-1} = \mathrm{B}$. But then $g \in \mathrm{N}_{\mathrm{G}(k)}(\mathrm{B}) = \mathrm{B}(k) \subset \mathrm{P}(k)$, so $\mathrm{P}' = \mathrm{P}$. □

Remark 1.4.10. — Note the remarkable consequence of Proposition 1.4.7 that in arbitrary characteristic, a parabolic subgroup P is uniquely determined by its Lie algebra $\mathfrak{p}$ as a subalgebra of $\mathfrak{g}$ *given* that P is a parabolic subgroup containing a specific maximal torus T. In fact, P is determined by $\mathfrak{p}$ without reference to T: under the adjoint representation of G on $\mathfrak{g}$, $\mathrm{P}(k)$ is the stabilizer in $\mathrm{G}(k)$ of the subspace $\mathfrak{p}$. This goes beyond the classical result $\mathrm{P}(k) = \mathrm{N}_{\mathrm{G}(k)}(\mathrm{P})$ in Theorem 1.1.9, but there are some subtleties because the scheme $\mathrm{N}_{\mathrm{G}}(\mathfrak{p})$ can be non-reduced in characteristic 2; see Exercise 2.4.8 for further discussion.

We call $\Phi(\mathrm{B}, \mathrm{T})$ the positive system of roots *associated to* B (and T). For each B, the unique Borel subgroup $\mathrm{B}' \supset \mathrm{T}$ for which $\Phi(\mathrm{B}', \mathrm{T}) = -\Phi(\mathrm{B}, \mathrm{T})$ is the *opposite Borel subgroup* (relative to T), and its unipotent radical U' is directly spanned in any order by the negative root groups U_{-a} for a in the positive system of roots $\Phi(\mathrm{B}, \mathrm{T})$. (Explicitly, $\mathrm{B}' = \mathrm{U}' \rtimes \mathrm{T}$.) For example, if $\mathrm{G} = \mathrm{GL}_n$, T is the diagonal torus, and B is the Borel subgroup of upper triangular matrices then B' is the Borel subgroup of lower triangular matrices.

Proposition 1.4.11. — *The natural multiplication map*

$$\mathrm{U}' \times \mathrm{T} \times \mathrm{U} = \mathrm{U}' \times \mathrm{B} \to \mathrm{G} \tag{1.4.3}$$

is an open immersion.

Proof. — This map is visibly étale at the origin, corresponding to the decomposition $\mathfrak{g} = \mathfrak{g}_{<0} \oplus \mathfrak{t} \oplus \mathfrak{g}_{>0}$ relative to an ordering on $\mathbf{Q}\Phi$ that makes $\Phi(\mathrm{B}, \mathrm{T})$ positive. Thus, it is everywhere étale, due to $\mathrm{U}'(k)$-translation on the left and $\mathrm{B}(k)$-translation on the right. Now we can argue exactly as in the analysis of (1.2.2) provided that $\mathrm{U}'(k)$ has no nontrivial finite subgroup Γ normalized by $\mathrm{T}(k)$. Since T is connected, such a subgroup must be centralized by T, yet $\mathrm{Z}_{\mathrm{G}(k)}(\mathrm{T}) = \mathrm{T}(k)$ and $\mathrm{T}(k) \cap \mathrm{U}'(k) = 1$. □

The image Ω of (1.4.3) is the *open cell* for $(\mathrm{G}, \mathrm{T}, \mathrm{B})$, though this terminology is also often used for the "inverse" $\mathrm{U} \times \mathrm{T} \times \mathrm{U}' \subset \mathrm{G}$ involving multiplication in the opposite order. It is also commonly called the *big cell*.

The open cell naturally arises in the description of double cosets for B in G. To make this precise, first note that by the orbit lemma (applied to $\mathrm{B} \times \mathrm{B}$ acting on G through the commuting operations of left and right multiplication by B), each double coset $\mathrm{B}g\mathrm{B}$ in G (for $g \in \mathrm{G}(k)$) is locally closed. Recall that $\mathrm{W}(\Phi)$ acts simply transitively on the set of positive systems of roots. In particular, there is a unique $w_0 \in \mathrm{W}(\Phi)$ such that $w_0(\Phi^+) = -\Phi^+$. This w_0 is the unique longest element relative to the reflections in the simple positive roots for $\Phi^+ = \Phi(\mathrm{B}, \mathrm{T})$, by [**Bou2**, VI, §1.6, Cor. 3 to Prop. 17]. For any representative $n_0 \in \mathrm{N}_{\mathrm{G}(k)}(\mathrm{T})$ of w_0 we have $\mathrm{U}' = n_0^{-1}\mathrm{U}n_0$, so $\mathrm{B}n_0\mathrm{B} = n_0\Omega$. This is the only open double coset since distinct double cosets are disjoint.

***Theorem 1.4.12* (Bruhat decomposition).** — *Let* G *be a connected reductive group,* T *a maximal torus, and* B *a Borel subgroup. For each* $w \in \mathrm{W} := \mathrm{W}_{\mathrm{G}}(\mathrm{T})$*, let* $n_w \in \mathrm{N}_{\mathrm{G}}(\mathrm{T})(k)$ *be a representative. Each locally closed double coset* $\mathrm{B}n_w\mathrm{B} \subset \mathrm{G}$ *only depends on* w*, and set-theoretically*

$$\mathrm{G} = \coprod_{w \in \mathrm{W}} \mathrm{B}n_w\mathrm{B}.$$

If $\Phi^+ := \Phi(\mathrm{B}, \mathrm{T})$ *then* $\mathrm{B}n_w\mathrm{B} = \mathrm{U}_w \times n_w\mathrm{B}$ *as* k*-schemes via multiplication, where* U_w *is a unipotent smooth connected subgroup directly spanned in any order by the positive root groups* U_a *for* a *in the closed set* Φ^+_w *of* $a \in \Phi^+$ *such that* $w^{-1}(a) \in -\Phi^+$.

Proof. — Since $\mathrm{T} \subset \mathrm{B}$ and n_w is well-defined modulo $\mathrm{T}(k)$, it is clear that $\mathrm{B}n_w\mathrm{B}$ only depends on w. The asserted decomposition of the underlying set of the scheme G as a disjoint union of locally closed sets amounts to an equality on k-points: $\mathrm{G}(k) = \coprod_{w \in \mathrm{W}} \mathrm{B}(k)n_w\mathrm{B}(k)$. This equality is proved in [**Bo91**, § 14.12] (and it is a special case of a general decomposition for any abstract group equipped with a "BN-pair" [**Bou2**, IV, § 2.1, Def. 1], which in turn was inspired by axiomatizing arguments for reductive groups; see [**Bo91**, 14.15]). To establish the desired description of each $\mathrm{B}n_w\mathrm{B}$ as a product scheme, since $\mathrm{B} = \mathrm{U} \rtimes \mathrm{T}$ and the k-group U is directly spanned in any order by the U_a's for $a \in \Phi^+$, we see via the equalities $n_w^{-1}\mathrm{T}n_w = \mathrm{T}$ and $n_w^{-1}\mathrm{U}_a n_w = \mathrm{U}_{w^{-1}(a)}$ that $\mathrm{B}n_w\mathrm{B} = \mathrm{U}_w n_w \mathrm{B}$ where U_w is directly spanned (in any order) by the root groups U_a for $a \in \Phi^+_w$ (Remark 1.4.6 applied to $\Psi = \Phi^+_w$).

The multiplication map $\mathrm{U}_w \times n_w\mathrm{B} \to \mathrm{G}$ is a locally closed immersion because applying left translation by n_w^{-1} yields the map $(n_w^{-1}\mathrm{U}_w n_w) \times \mathrm{B} \to \mathrm{G}$ (via multiplication) that is the restriction to a closed subscheme of the open immersion $\mathrm{U}^- \times \mathrm{B} \to \mathrm{G}$. □

Corollary 1.4.13. — *For a Borel subgroup* B *of a connected reductive* k*-group* G*, the stabilizer of* $\mathfrak{b}$ *under the adjoint action of* $\mathrm{G}(k)$ *on* $\mathfrak{g}$ *is* $\mathrm{B}(k)$.

Proof. — Choose a maximal torus T in B and let $\Phi^+ = \Phi(\mathrm{B}, \mathrm{T})$ be the positive system of roots associated to B. Choose $g \in \mathrm{G}(k)$ whose adjoint action preserves $\mathfrak{b}$. By the Bruhat decompostion, $g = bnb'$ for some $b, b' \in \mathrm{B}(k)$ and $n \in \mathrm{N}_{\mathrm{G}(k)}(\mathrm{T})$. It is harmless to replace g with $b^{-1}gb'^{-1}$, so g normalizes T. Then g represents some $w \in \mathrm{W}_{\mathrm{G}}(\mathrm{T})$, and the adjoint action of g carries $\mathfrak{b} = \mathfrak{t} \oplus (\oplus_{a \in \Phi^+} \mathfrak{g}_a)$ to $\mathfrak{t} \oplus (\oplus_{a' \in w(\Phi^+)} \mathfrak{g}_{a'})$. Hence, the preservation of $\mathfrak{b}$ forces $w(\Phi^+) = \Phi^+$, so $w = 1$ and hence $g \in \mathrm{T}(k) \subset \mathrm{B}(k)$. □

In translation arguments with split reductive group schemes over a general scheme S, the following covering result on geometric fibers will be very useful.

Corollary 1.4.14. — *For any* $(\mathrm{G},\mathrm{T},\mathrm{B})$*, with associated open cell* $\Omega = \mathrm{U}'\times\mathrm{B}$ *and Weyl group* $\mathrm{W} = \mathrm{W}_{\mathrm{G}}(\mathrm{T})$*,* $\mathrm{G} = \bigcup_{w\in\mathrm{W}} n_w\Omega$ *for any set of representatives* $\{n_w\}_{w\in\mathrm{W}} \subset \mathrm{N}_{\mathrm{G}(k)}(\mathrm{T})$ *of* W.

Proof. — For each $w \in \mathrm{W}$, $n_w^{-1}\mathrm{U}_w n_w\mathrm{B} \subset \mathrm{U}' \times \mathrm{B} = \Omega$. Thus, $\mathrm{B}n_w\mathrm{B} = \mathrm{U}_w n_w\mathrm{B} \subset n_w\Omega$. □

1.5. Based root datum, pinnings, and representation theory. — In this section, we will explore consequences of the Existence and Isomorphism Theorems over k. Some concepts introduced (such as a "based root datum" and a "pinning" of a triple $(\mathrm{G},\mathrm{T},\mathrm{B})$) will be used in the proof of the Existence and Isomorphism Theorems over general base schemes in § 6.

One often considers triples $(\mathrm{G},\mathrm{T},\mathrm{B})$ rather than pairs (G,T) (for a connected reductive k-group G), and likewise for root data $\mathrm{R} = (\mathrm{X},\Phi,\mathrm{X}^\vee,\Phi^\vee)$ it is often convenient to choose a positive system of roots $\Phi^+ \subset \Phi$. By Proposition 1.4.4, the choice of Φ^+ corresponds to a choice of $\mathrm{B} \supset \mathrm{T}$ when R arises from a connected reductive k-group G equipped with a maximal k-torus T. Another way to express the choice of Φ^+ is to specify its base Δ. The set $\Delta^\vee$ of coroots $a^\vee$ for $a \in \Delta$ is a base for $\Phi^\vee$ (Exercise 1.6.17), and we call the 6-tuple $(\mathrm{X},\Phi,\Delta,\mathrm{X}^\vee,\Phi^\vee,\Delta^\vee)$ (or the triple $(\mathrm{R},\Delta,\Delta^\vee)$, or simply (R,Δ)) a *based root datum*.

There are evident notions of *isomorphism* and *duality* for based root data. For a based root datum $(\mathrm{R},\Delta,\Delta^\vee)$ with associated root system Φ, $\mathrm{W}(\Phi)$ is normal in $\mathrm{Aut}(\mathrm{R})$ because $\theta s_a\theta^{-1} = s_{\theta(a)}$ for each $\theta \in \mathrm{Aut}(\mathrm{R})$, as is seen from (1.3.1). Thus, if Θ is the automorphism group of the based root datum then

$$\mathrm{Aut}(\mathrm{R}) = \mathrm{W}(\Phi) \rtimes \Theta \tag{1.5.1}$$

because the action of $\mathrm{W}(\Phi)$ on X is simply transitive on the set of choices of $(\Delta,\Delta^\vee)$. If Φ spans $\mathrm{X}_{\mathbf{Q}}$ (or equivalently, Δ is a $\mathbf{Q}$-basis of $\mathrm{X}_{\mathbf{Q}}$) and the root system $(\mathrm{X}_{\mathbf{Q}},\Phi)$ is reduced then Θ is naturally identified with a subgroup of the automorphism group of the Dynkin diagram $\mathrm{Dyn}(\Phi)$ of Φ. The interest in Θ is due to the fact (discussed below; see (1.5.2)) that it computes the outer automorphism group of a connected reductive k-group, so for the connected semisimple case (corresponding to the condition that Φ spans $\mathrm{X}_{\mathbf{Q}}$) it is useful to know sufficient criteria (in terms of terminology in Definition 1.3.11) for Θ to exhaust the automorphism group of the Dynkin diagram:

Proposition 1.5.1. — *Assume* Φ *spans* $\mathrm{X}_{\mathbf{Q}}$ *and that* $(\mathrm{X}_{\mathbf{Q}},\Phi)$ *is reduced. The inclusion* $\Theta \subset \mathrm{Aut}(\mathrm{Dyn}(\Phi))$ *is an equality if the root datum is adjoint or simply connected, or if* $(\mathrm{X}_{\mathbf{Q}},\Phi)$ *is irreducible and* $(\mathbf{Z}\Phi^\vee)^*/\mathbf{Z}\Phi$ *is cyclic.*

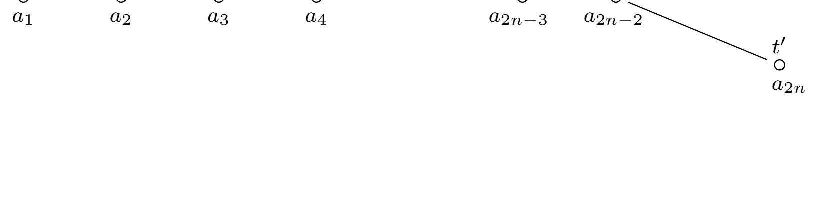
Proof. — In the adjoint cases of Example 1.3.10, the root datum is a direct product of root data whose associated root system is irreducible and adjoint;

we have likewise in the simply connected case. Thus, we can assume $(\mathrm{X}_{\mathbf{Q}}, \Phi)$ is irreducible. Hence, there exists a $\mathrm{W}(\Phi)$-invariant inner product on $\mathrm{X}_{\mathbf{R}}$ that is unique up to scaling, and it is uniquely determined by imposing the condition that the shortest root length is 1. The resulting inner products among the roots are encoded in the Dynkin diagram, as is the resulting identification of the coroots with elements of $\mathrm{X}_{\mathbf{R}}$ (see Remark 1.3.14 and Remark 1.4.3).

Identify the set of vertices of the diagram with Δ. The diagram encodes the Cartan matrix of integers $\langle a, b^\vee\rangle$ for $a, b \in \Delta$, so any automorphism of the diagram uniquely extends to an automorphism of $\mathrm{X}_{\mathbf{R}}$ preserving Φ and hence $\Phi^\vee$ [**Bou2**, VI, § 1.5, Cor.]. This automorphism of $\mathrm{X}_{\mathbf{R}}$ carries X onto itself (or equivalently (by duality) carries $\mathrm{X}^\vee$ onto itself). Indeed, the adjoint and simply connected cases are obvious, and if $\Pi := (\mathbf{Z}\Phi^\vee)^*/\mathbf{Z}\Phi$ is cyclic then every automorphism of Π carries each subgroup of Π onto itself (with X uniquely determined by the corresponding subgroup in Π). $\square$

Here is a visualization of Proposition 1.5.1 when $\mathrm{R} = \mathrm{R}(\mathrm{G}, \mathrm{T})$ for a connected semisimple group G. In Exercise 1.6.13(ii) we define the *simply connected central cover* $\widetilde{\mathrm{G}} \to \mathrm{G}$; this is a central isogeny onto G from a connected semisimple group that is initial among such central isogenies onto G. (If $\mathrm{char}(k) = 0$ then $\widetilde{\mathrm{G}}$ is simply connected as a variety, in the sense that it has trivial étale fundamental group; this ultimately rests on comparison with the theory of compact Lie groups for $k = \mathbf{C}$. In contrast, if $\mathrm{char}(k) > 0$ then $\widetilde{\mathrm{G}}$ never has trivial étale fundamental group if $\mathrm{G} \neq 1$.) Every automorphism of a connected semisimple group G lifts (uniquely) to $\widetilde{\mathrm{G}}$, but an automorphism $\widetilde{f}$ of $\widetilde{\mathrm{G}}$ descends to G if and only if its restriction to $\mathrm{Z}_{\widetilde{\mathrm{G}}}$ preserves the central subgroup $\mu = \ker(\widetilde{\mathrm{G}} \to \mathrm{G})$. If $\widetilde{f}$ is inner or $\mathrm{Z}_{\widetilde{\mathrm{G}}}$ is cyclic then preservation of μ is automatic (because any automorphism of a finite cyclic group preserves every subgroup). In general, Θ is the group of outer automorphisms of $\widetilde{\mathrm{G}}$ that arise from lifting automorphisms of G (see (1.5.2) below).

Example 1.5.2. — The only cases with *irreducible* Φ when non-cyclicity occurs in Proposition 1.5.1 are type D_{2n} $(n \geqslant 2)$, with $(\mathbf{Z}\Phi^\vee)^*/(\mathbf{Z}\Phi) = (\mathbf{Z}/2\mathbf{Z})^2$. Let $\widetilde{\mathrm{G}} = \mathrm{Spin}_{4n}$ (type D_{2n}). Choose a maximal torus $\mathrm{T} \subset \widetilde{\mathrm{G}}$ and label $\mathrm{Dyn}(\Phi(\widetilde{\mathrm{G}}, \mathrm{T}))$ as follows:



where $(t, t') \in \mu_2 \times \mu_2 \simeq Z_{\widetilde{G}}$. We have $T = \prod_j a_j^\vee(\mathbf{G}_m)$, and the center $Z_{\widetilde{G}}$ is $\mu_2 \times \mu_2$ embedded in $T[2]$ by diagonally mapping each μ_2 into the 2-torsion of the indicated coroot groups.

There are three intermediate groups G strictly between $\widetilde{G}$ and $\widetilde{G}/Z_{\widetilde{G}}$ (one for each copy μ of μ_2 in $\mu_2 \times \mu_2 \subset Z_{\widetilde{G}}$), and $\mathrm{Aut}(\mathrm{Dyn}(\Phi))$ has order 2 when $n \geqslant 3$ and is $\mathfrak{S}_3 = \mathrm{GL}_2(\mathbf{F}_2)$ when $n = 2$. Thus, if $n = 2$ then each $\mu \simeq \mu_2 \subset Z_{\widetilde{G}}$ has $\mathrm{Aut}(\mathrm{Dyn}(\Phi))$-stabilizer of order 2, and if $n \geqslant 3$ then only one such μ is preserved under the order-2 group $\mathrm{Aut}(\mathrm{Dyn}(\Phi))$; we claim that this corresponds to the non-adjoint quotient $G = \mathrm{SO}_{4n}$.

More generally, noting that Spin_{2n} has center μ_4 for odd $n \geqslant 3$ (e.g., $\mathrm{Spin}_6 = \mathrm{SL}_4$), it suffices to show that for all $n \geqslant 2$, the group SO_{2n} has Θ of order 2, rather than of order 1. (In contrast, SO_{2n+1} for $n \geqslant 1$ has $\Theta = 1$ since $\mathrm{Aut}(\mathrm{Dyn}(\mathrm{B}_n)) = 1$ for $n \geqslant 1$; see Example C.6.2 for $n = 1$.) In view of (1.5.2) below, this says exactly that the outer automorphism of SO_{2n} has order 2 (rather than order 1). Where does a non-inner automorphism come from? Conjugation by O_{2n}! Indeed, if $g \in \mathrm{O}_{2n}(k) - \mathrm{SO}_{2n}(k)$ and g-conjugation on SO_{2n} is inner then by replacing g with a suitable left $\mathrm{SO}_{2n}(k)$-translate we would get that g centralizes SO_{2n}. But the "diagonal" maximal torus T in SO_{2n} has O_{2n}-centralizer equal to T (by explicit computation), so this is impossible.

The Isomorphism Theorem and (1.5.1) lead to the determination of the automorphism group of a connected reductive k-group G, as follows. Inside $\mathrm{Aut}(G)$ there is the normal subgroup $G(k)/Z_{G(k)}$ of inner automorphisms; for $g \in G(k)$ let c_g denote the inner automorphism $x \mapsto gxg^{-1}$. To describe the quotient group $\mathrm{Out}(G)$ of outer automorphisms, fix a choice of (B, T) as usual and consider an automorphism φ of G. By composing φ with a suitable inner automorphism, we can arrange that $\varphi(B) = B$ and $\varphi(T) = T$. The only $g \in G(k)$ such that $\varphi \circ c_g$ preserves B and T are $g \in N_{G(k)}(B) \bigcap N_{G(k)}(T) = B(k) \bigcap N_{G(k)}(T) = T(k)$, and this is precisely the ambiguity that arises by passing from φ to the induced automorphism of the root datum $R(G, T)$. The automorphism φ induces an automorphism $\overline{\varphi}$ of the based root datum associated to (G, T, B), and by (1.5.1) the element $\overline{\varphi} \in \Theta$ determines the outer automorphism class of φ. Moreover, every element of Θ arises in this way, due to the Isomorphism Theorem. We thereby obtain a short exact sequence of abstract groups

$$1 \to G(k)/Z_{G(k)} \to \mathrm{Aut}(G) \to \Theta \to 1. \tag{1.5.2}$$

Corollary 1.5.3. — *The natural map* $\mathrm{Aut}(G, T) \to \mathrm{Aut}(R(G, T))$ *is surjective with kernel* $T(k)/Z_{G(k)}$.

Proof. — Since $\mathrm{Aut}(G, T) \cap (G(k)/Z_{G(k)}) = N_{G(k)}(T)/Z_{G(k)}$ inside $\mathrm{Aut}(G)$, the quotient group $\mathrm{Aut}(G, T)/(T(k)/Z_{G(k)})$ contains $W_G(T)$ as a subgroup that is carried isomorphically onto the subgroup $\mathrm{W}(\Phi) \subset \mathrm{Aut}(\mathrm{R}(\mathrm{G},\mathrm{T}))$. The quotient of $\mathrm{Aut}(\mathrm{G},\mathrm{T})/(\mathrm{T}(k)/\mathrm{Z}_{\mathrm{G}(k)})$ modulo its normal subgroup $\mathrm{W}_{\mathrm{G}}(\mathrm{T})$ is clearly $\mathrm{Out}(\mathrm{G})$, and likewise $\mathrm{Aut}(\mathrm{R}(\mathrm{G},\mathrm{T}))/\mathrm{W}(\Phi) = \Theta$. This is compatible with the isomorphism $\Theta \simeq \mathrm{Out}(\mathrm{G})$ defined via (1.5.2). □

Remarkably, the short exact sequence (1.5.2) splits as a semi-direct product. To formulate this, we need to more structure beyond the triple $(\mathrm{G},\mathrm{T},\mathrm{B})$:

Definition 1.5.4. — A *pinning* of $(\mathrm{G},\mathrm{T},\mathrm{B})$ is the specification of an isomorphism $p_a : \mathbf{G}_{\mathrm{a}} \simeq \mathrm{U}_a$ for each $a \in \Delta$; equivalently, it is the choice of a nonzero $\mathrm{X}_a \in \mathfrak{g}_a$ for each $a \in \Delta$ (via $\mathrm{Lie}(p_a)(\partial_x) = \mathrm{X}_a$, with x the standard coordinate on $\mathbf{G}_{\mathrm{a}}$). The data $(\mathrm{G},\mathrm{T},\mathrm{B},\{\mathrm{X}_a\}_{a\in\Delta})$ is a *pinned* connected reductive group.

(In [**Bou3**, IX, § 4.10, Def. 3] the analogous notion for a connected compact Lie group is called a *framing*. Kottwitz, Langlands, and Shelstad use the terminology *splitting*.) There is an evident notion of *isomorphism* for pinned connected reductive groups. Pinnings remove $\mathrm{T}(k)$-conjugation ambiguity in the Isomorphism Theorem:

Proposition 1.5.5. — *For* $(\mathrm{G},\mathrm{T},\mathrm{B},\{\mathrm{X}_a\}_{a\in\Delta})$ *and* $(\mathrm{G}',\mathrm{T}',\mathrm{B}',\{\mathrm{X}'_{a'}\}_{a'\in\Delta'})$, *the natural map*

$$\begin{array}{c}\mathrm{Isom}((\mathrm{G},\mathrm{T},\mathrm{B},\{\mathrm{X}_a\}_{a\in\Delta}),(\mathrm{G}',\mathrm{T}',\mathrm{B}',\{\mathrm{X}'_{a'}\}_{a'\in\Delta'}))\\ \big\downarrow \\ \mathrm{Isom}((\mathrm{R}(\mathrm{G},\mathrm{T}),\Delta,\Delta^\vee),(\mathrm{R}(\mathrm{G}',\mathrm{T}'),\Delta',\Delta'^\vee))\end{array}$$

is bijective. In particular, if f is an automorphism of $(\mathrm{G},\mathrm{T},\mathrm{B})$ *that is the identity on* T *and on the simple positive root groups then f is the identity on* G*, and a choice of pinning on* $(\mathrm{G},\mathrm{T},\mathrm{B})$ *defines a homomorphic section to the quotient map* $\mathrm{Aut}(\mathrm{G}) \to \mathrm{Out}(\mathrm{G}) \simeq \Theta$.

Proof. — Since the elements of Δ are linearly independent in $\mathrm{X}(\mathrm{T})$, the map $\mathrm{T} \to \mathbf{G}_{\mathrm{m}}^{\Delta}$ defined by $t \mapsto (a(t))_{a\in\Delta}$ is surjective. (Indeed, otherwise the cokernel would map onto $\mathbf{G}_{\mathrm{m}}$ via a quotient map $\mathbf{G}_{\mathrm{m}}^{\Delta} \to \mathbf{G}_{\mathrm{m}}$ corresponding to a nonzero map $\Delta \to \mathbf{Z}$ and thereby give a nontrivial dependence relation on $\Delta \subset \mathrm{X}(\mathrm{T})$.) Thus, the group of inner automorphisms by $\mathrm{T}(k)$ acts transitively on the set of possible pinnings. Hence, the Isomorphism Theorem guarantees that if the based root data are isomorphic then there exists an isomorphism between the pinned data. We may therefore pass to the case of automorphisms.

Since the $\mathrm{T}(k)$-action is trivial on the root datum, it follows from the Isomorphism Theorem that any automorphism of the based root datum arises from an automorphism of $(\mathrm{G},\mathrm{T},\mathrm{B},\{\mathrm{X}_a\}_{a\in\Delta})$; i.e., surjectivity is proved. It remains to prove injectivity, so it suffices to show that if f is an automorphism of $(\mathrm{G},\mathrm{T},\mathrm{B})$ that is the identity on T (encoding being the identity on $\mathrm{X}(\mathrm{T})$)

and on the simple positive root groups (encoding preservation of the pinning) then f is the identity on G.

Corollary 1.5.3 implies that f must be conjugation against some $t \in \mathrm{T}(k)$, but the condition on the simple positive root groups implies that $a(t) = 1$ for all $a \in \Delta$, so $a(t) = 1$ for all $a \in \Phi$. It follows that t centralizes every root group, and hence an open cell, so t is central in G. That is, the conjugation action f by t is the identity map. □

We now discuss representation theory, which will only play a role in our proof of the Existence Theorem over **C** via analytic methods in Appendix D (and in a few exercises in § 1.6 that are not used anywhere else). The proof of the Existence Theorem over **Z** in § 6 rests on the Existence Theorem over **C**.

Fix a triple $(\mathrm{G}, \mathrm{T}, \mathrm{B})$, with G a connected reductive group over an algebraically closed field k of any characteristic. Let Δ be the base of simple positive roots of $\Phi^+ = \Phi(\mathrm{B}, \mathrm{T})$ in $\Phi = \Phi(\mathrm{G}, \mathrm{T})$, and $\mathrm{W} = \mathrm{W}_{\mathrm{G}}(\mathrm{T}) = \mathrm{W}(\Phi)$. By [**Bou2**, VI, § 1.5, Thm. 2(ii),(vi)], $\mathrm{X}(\mathrm{T})$ is covered by W-translates of the closed Weyl chamber

$$\mathrm{C} = \{\lambda \in \mathrm{X}(\mathrm{T}) \mid \langle \lambda, a^\vee \rangle \geqslant 0 \text{ for all } a \in \Delta\}. \tag{1.5.3}$$

For an irreducible representation $\mathrm{G} \to \mathrm{GL}(\mathrm{V})$, consider the W-stable finite set $\Omega_{\mathrm{V}} \subset \mathrm{X}(\mathrm{T})$ of T-weights on V.

***Theorem 1.5.6* (Theorem of the highest weight: group version)**
There is a unique weight $\lambda_{\mathrm{V}} \in \Omega_{\mathrm{V}} \bigcap \mathrm{C}$ that is highest in the sense that all weights in Ω_{V} have the form

$$\lambda_{\mathrm{V}} - \sum_{a \in \Delta} n_a a$$

with integers $n_a \geqslant 0$, and the λ_{V}-weight space is 1-dimensional and B*-stable. For each $\lambda \in \mathrm{X}(\mathrm{T})$ there exists a unique irreducible representation V_λ of* G *with highest weight λ.*

Proof. — See [**Hum87**, 31.2–31.4]. The relationship between constructions in characteristic 0 and positive characteristic is in [**Jan**, II, 2.2, 2.4, 2.6]. □

Theorem 1.5.6 is not used later, but to prove the Existence Theorem over **C** by analytic methods we need a link between the notions of "simply connected" in the sense of topology and in the sense of root data. This rests on a variant of Theorem 1.5.6 for Lie algebras that is used in the proof of Proposition D.4.1. To state this variant, let $\mathfrak{g}$ be a semisimple Lie algebra over an algebraically closed field k of characteristic 0, and let $\mathfrak{t}$ be a Cartan subalgebra. Fix a positive system of roots Φ^+ in $\Phi(\mathfrak{g}, \mathfrak{t}) \subset \mathfrak{t}^* - \{0\}$, and for each $a \in \Phi$ define an "infinitesimal coroot" $h_a \in \mathfrak{t}$ using $\mathfrak{sl}_2$ similarly to how coroots are defined in reductive k-groups using SL_2. (See [**Ser01**, VI, 3.1(iii)] for another formulation.) A linear form $\lambda \in \mathfrak{t}^*$ is *integral* if $\lambda(h_a) \in \mathbf{Z}$ for all $a \in \Phi$, and *dominant integral* if $\lambda(h_a) \in \mathbf{Z}_{\geqslant 0}$ for all $a \in \Phi^+$ (equivalently, for all simple positive a).

Theorem 1.5.7 **(Theorem of the highest weight: Lie algebra version)** *For each dominant integral $\lambda \in \mathfrak{t}^*$, there exists an irreducible finite-dimensional $\mathfrak{g}$-representation V_λ whose highest weight for $\mathfrak{t}$ is λ in the sense that its $\mathfrak{t}$-weights have the form $\lambda - \sum_{a\in\Delta} n_a a$ with integers $n_a \geqslant 0$. It is unique up to isomorphism and has a 1-dimensional λ-weight space.*

Proof. — See [**Hum72**, 20.2, 21.1–21.2] (and see [**Hum98**] for a version in positive characteristic). □

Existence in Theorem 1.5.7 is the key construction in the analytic proof of the Existence Theorem for connected reductive groups over $\mathbf{C}$. The character of V_λ in Theorem 1.5.6 is determined by its restriction to the regular semisimple locus of $\mathrm{G}(k)$, a dense open locus introduced in Exercise 1.6.9. Since the character is conjugation-invariant, it is even determined by its restriction to the dense open set of elements of $\mathrm{T}(k)$ that are *regular* in $\mathrm{G}(k)$ (see Exercise 1.6.9(ii)). This restriction is given by:

Theorem 1.5.8 **(Weyl character formula)**. — *Let* G *be a nontrivial connected semisimple group over an algebraically closed field k of characteristic* 0*,* T *a maximal torus ,and* B *a Borel subgroup containing* T*. Assume that* G *is simply connected. Let $\mathrm{W} = \mathrm{W}_{\mathrm{G}}(\mathrm{T})$, and let $\varepsilon = \varepsilon_{\mathrm{B}} : \mathrm{W} \to \{1, -1\}$ be the unique quadratic character carrying each simple reflection to* -1 (*see* (1.4.1)).

For each λ in the Weyl chamber of X(T) *corresponding to* B (*see* (1.5.3)) *and for each regular semisimple $t \in \mathrm{T}(k)$, the character of t on V_λ is given by*

$$\mathrm{Tr}(t|\mathrm{V}_\lambda) = \frac{\sum_{w\in\mathrm{W}} \varepsilon(w) t^{w(\lambda+\rho)}}{\sum_{w\in\mathrm{W}} \varepsilon(w) t^{w\rho}},$$

where $t^a := a(t)$ for $a \in \mathrm{X}(\mathrm{T})$ and $\rho := (1/2)\sum_{a\in\Phi^+} a \in \mathrm{X}(\mathrm{T})$.

Proof. — Using the weight space decomposition $\mathrm{V}_\lambda = \bigoplus_{\mu\in\mathrm{X}(\mathrm{T})} \mathrm{V}_\lambda(\mu)$, the character $\chi_\lambda(t) = \mathrm{Tr}(t|\mathrm{V}_\lambda)$ is equal to $\sum_\mu \dim \mathrm{V}_\lambda(\mu) t^\mu$. Hence, the assertion of the formula is the identity

$$\left(\sum_{\mu\in\mathrm{X}(\mathrm{T})} \dim \mathrm{V}_\lambda(\mu) t^\mu\right)\left(\sum_{w\in\mathrm{W}} \varepsilon(w) t^{w\rho}\right) = \sum_{w\in\mathrm{W}} \varepsilon(w) t^{w(\lambda+\rho)}$$

in the group ring $\mathbf{Z}[\mathrm{X}(\mathrm{T})]$. The T-weight spaces for a finite-dimensional G-representation V are the same as the $\mathfrak{t}$-weight spaces for V viewed as a $\mathfrak{g}$-module since G is connected and $\mathrm{char}(k) = 0$. Thus, by embedding $\mathrm{X}(\mathrm{T}) = \mathrm{Hom}(\mathrm{T}, \mathbf{G}_{\mathrm{m}})$ into $\mathfrak{t}^*$ in the natural way, it is equivalent to prove the analogue of the Weyl character formula for V_λ viewed as a $\mathfrak{g}$-module (by identifying $\Phi(G, T)$ with $\Phi(\mathfrak{g}, \mathfrak{t})$). For a proof of the Lie algebra version, see [**Bou3**, VIII, 9.1, Thm. 1]. (The group version can be reduced to the case $k = \mathbf{C}$, in which case it can be proved using integration on a compact form of the group; see [**FH**, 26.2].) $\square$

Remark 1.5.9. — The regularity condition on t is needed in the Weyl character formula to ensure that the denominator is nonzero; see Example 1.5.10. Also, the integrality of ρ can be made explicit: it is the sum of the elements in $X(T)$ dual to the $\mathbf{Z}$-basis $\Delta^\vee$ of $X_*(T)$ (a $\mathbf{Z}$-basis due to the simply connected condition on G; see Exercise 1.6.13(ii)). This is the "dual" of the situation in Exercise 4.4.8(iii) (which treats adjoint semisimple groups).

Example 1.5.10. — Consider $G = SL_2$ over an algebraically closed field k of characteristic 0, and let T be the diagonal torus. Identify $\mathbf{G}_m$ with T via $c \mapsto \mathrm{diag}(c, 1/c)$, so $X(T) = \mathbf{Z}$. Take B to be the upper triangular Borel subgroup, so $\Phi(B, T) = \{2\}$. The Weyl chamber C is $\mathbf{Z}_{\geqslant 0}$ and $\rho = 2/2 = 1$.

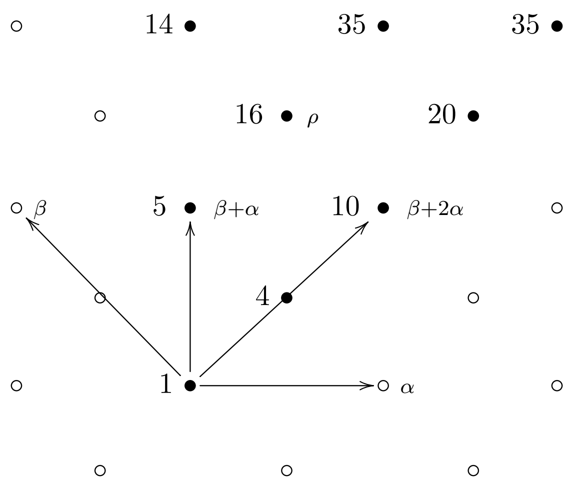
The irreducible representation of G with highest weight $n \geqslant 0$ is the symmetric power $V_n = \mathrm{Sym}^n(k^2)$. A regular semisimple element $t \in T(k)$ is any $t \in k^\times$ such that $t^2 \neq 1$. Clearly

$$\mathrm{Tr}(t|V_n) = t^n + t^{n-2} + \cdots + t^{-n} = \frac{t^{n+1} - t^{-(n+1)}}{t - t^{-1}},$$

and the right side is exactly the Weyl character formula in this case (likewise illustrating the need for the regularity condition on t in order that the denominator be nonzero).

If $\mathrm{char}(k) = p > 0$ then these symmetric powers can be non-semisimple; e.g., $\mathrm{Sym}^p(k^2)$ is not semisimple (Exercise 1.6.11(iii)) but it has a unique irreducible quotient with highest weight p. In general, the dimensions of "highest weight" representations are not known when $\mathrm{char}(k) > 0$, nor is there a formula for the character at regular semisimple points of $T(k)$.

Example 1.5.11. — Consider $G = Sp_4$ over an algebraically closed field of characteristic 0. The Dynkin diagram is $\underset{\alpha}{\circ} \Longleftarrow \underset{\beta}{\circ}$. The weight lattice $X(T)$ is represented by dots in the diagram below, and the arrows represent positive roots. Note that $X(T) = \mathbf{Z}\alpha + \mathbf{Z}(\beta/2)$ contains $\mathbf{Z}\alpha + \mathbf{Z}\beta$ as a subgroup of index 2 (so the half-sum $\rho = 2\alpha + (3/2)\beta$ of the positive roots lies in $X(T)$ as it should). The weights in the Weyl chamber (1.5.3) are denoted by black dots and labelled with the dimension of the corresponding irreducible representation.



We can describe some irreducible representations of small dimension in more familiar terms:

— The 4-dimensional representation V_4 is the standard representation of $G = Sp_4$.

— The 5-dimensional representation V_5 is $\wedge^2 V_4/L$, where L is the line in $\wedge^2 V_4$ fixed by G (i.e., the line spanned by the symplectic form). This can be regarded as the standard representation of $SO_5 \simeq Sp_4/\mu_2$ (see Example C.6.5).

— The 10-dimensional representation is $\mathrm{Sym}^2(V_4)$.

— The 14-dimensional representation is $\mathrm{Sym}^2(V_5)/L'$, where L' is the line $\mathrm{Sym}^2(V_5)^{SO_5}$ corresponding to the quadratic form.

— The 20-dimensional representation is $\mathrm{Sym}^3(V_4)$.

1.6. Exercises. — The first seven exercises below require no specific background in the classical theory beyond basic definitions that we assume are familiar to the reader. However, some of those exercises are referenced in § 1 to clarify features of the classical theory.

Exercise 1.6.1. — Let $\mathrm{G} \subset \mathrm{GL}_3$ be the non-reductive connected solvable subgroup

$$\begin{pmatrix} t & x & z \\ 0 & 1 & y \\ 0 & 0 & t^{-1} \end{pmatrix} \subset \mathrm{GL}_3$$

in which a maximal torus is given by $\mathrm{T} = \mathbf{G}_{\mathrm{m}}$ via $t \mapsto \mathrm{diag}(t, 1, 1/t)$. Consider the weight space decomposition for $\mathfrak{g}$ under the adjoint action of T.

Show that the nontrivial character $\chi(t) = t$ occurs with multiplicity 2, and that its inverse $\chi(t)^{-1} = 1/t$ occurs with multiplicity 0. (This contrasts with two important features of the reductive case, namely that root spaces are 1-dimensional and that the set of roots is stable under negation in the character lattice.)

Exercise 1.6.2. — (i) Prove that the only automorphisms of $\mathbf{G}_{\mathrm{a}}$ over a field F are the usual $\mathrm{F}^{\times}$-scalings, and conclude that over a reduced ring R the only automorphisms of $\mathbf{G}_{\mathrm{a}}$ over R are the usual $\mathrm{R}^{\times}$-scalings. (Hint: reduce to the case of noetherian R.)

(ii) Prove $\mathbf{G}_{\mathrm{m}}$ represents the automorphism functor of $\mathbf{G}_{\mathrm{a}}$ on the category of $\mathbf{Q}$-algebras. (Hint: reduce to the noetherian case, then induct on the nilpotence order of the nilradical.)

(iii) Let k be a field with $\mathrm{char}(k) = p > 0$. Prove that $\mathbf{G}_{\mathrm{m}}$ does not represent the automorphism functor of $\mathbf{G}_{\mathrm{a}}$ on k-algebras by giving an example of an automorphism of $\mathbf{G}_{\mathrm{a}}$ over the dual numbers $k[\epsilon]$ that does *not* arise from the usual $\mathbf{G}_{\mathrm{m}}$-action on $\mathbf{G}_{\mathrm{a}}$.

(iv) Despite (iii), prove any action by a torus T on $\mathbf{G}_{\mathrm{a}}$ over a field k of any characteristic is given by $t.x = \chi(t)x$ for a homomorphism of k-groups $\chi : \mathrm{T} \to \mathbf{G}_{\mathrm{m}}$. (Hint: use (i) to get a homomorphism $\chi : \mathrm{T}(k_s) \to k_s^{\times}$, and work over the function field $\mathrm{K} = k(\mathrm{T})$ of T to prove χ is "algebraic" over k_s and defined over k. Algebraicity is the main point.)

Exercise 1.6.3. — Define $\mathbf{Z}[x_{ij}]_{(\det)}$ to be the degree-0 part of $\mathbf{Z}[x_{ij}][1/\det]$ (i.e., the ring of fractions $f/\det^e$ with f homogenous and $\deg f = e\deg(\det) = en$). Define $\mathrm{PGL}_n = \mathrm{Spec}(\mathbf{Z}[x_{ij}]_{(\det)}) = \{\det \neq 0\} \subset \mathbf{P}_{\mathbf{Z}}^{n^2-1}$.

(i) Construct an injective map $\mathrm{GL}_n(\mathrm{R})/\mathrm{R}^{\times} \to \mathrm{PGL}_n(\mathrm{R})$ natural in rings R.

(ii) Prove PGL_n is the Zariski-sheafification of $\mathrm{S} \rightsquigarrow \mathrm{GL}_n(\mathrm{S})/\mathbf{G}_{\mathrm{m}}(\mathrm{S})$ on the category of schemes, and that it has a unique $\mathbf{Z}$-group structure making $\mathrm{GL}_n \to \mathrm{PGL}_n$ a homomorphism.

(iii) Prove $\mathrm{GL}_n(\mathrm{R})/\mathrm{R}^\times = \mathrm{PGL}_n(\mathrm{R})$ for local R, and construct a *counterexample* with $n = 2$ for any Dedekind domain R with $\mathrm{Pic}(\mathrm{R})[2] \neq 1$.

(iv) For local R show $\mathrm{Pic}(\mathbf{P}^{\mathrm{N}}_{\mathrm{R}}) = \mathbf{Z}$ (generated by $\mathscr{O}(1)$) by using deformation from the residue field and the theorem on formal functions (after reducing to the case of noetherian R). Deduce that the evident action of PGL_n on $\mathbf{P}^{n-1}_{\mathbf{Z}}$ via (ii) makes PGL_n represent the automorphism functor $\mathrm{S} \rightsquigarrow \mathrm{Aut}_{\mathrm{S}}(\mathbf{P}^{n-1}_{\mathrm{S}})$ on the category of schemes. (For example, the $\mathbf{Z}$-group PGL_2 represents the automorphism functor of $\mathbf{P}^1_{\mathbf{Z}}$.) By using intersection theory on $\mathbf{P}^1 \times \mathbf{P}^1$ over a field, adapt the deformation arguments to prove that $(\mathrm{PGL}_2 \times \mathrm{PGL}_2) \rtimes (\mathbf{Z}/2\mathbf{Z})$ represents the automorphism functor of $\mathbf{P}^1 \times \mathbf{P}^1$ in the evident manner. (Hint: prove that $\mathrm{Pic}_{(\mathbf{P}^1 \times \mathbf{P}^1)/\mathbf{Z}}$ is the constant $\mathbf{Z}$-group associated to $\mathbf{Z} \oplus \mathbf{Z}$.)

(v) Give a pre-Grothendieck proof (i.e., no functors, non-reduced schemes, or cohomology) that if a linear algebraic group G over a field k acts on $\mathbf{P}^{n-1}_k$ then the resulting homomorphism $\mathrm{G}(k_s) \to \mathrm{Aut}_{k_s}(\mathbf{P}^{n-1}_{k_s}) = \mathrm{PGL}_n(k_s)$ arises from a k-homomorphism $\mathrm{G} \to \mathrm{PGL}_n$.

Exercise 1.6.4. — Construct natural Lie algebra isomorphisms over $\mathbf{Z}$ between $\mathfrak{pgl}_n := \mathrm{Lie}(\mathrm{PGL}_n)$ and $\mathfrak{gl}_n/\mathfrak{gl}_1$, as well as between $\mathfrak{sl}_n := \mathrm{Lie}(\mathrm{SL}_n)$ with $\mathfrak{gl}_n^{\mathrm{Tr}=0}$ (kernel of the trace).

(i) Construct a GL_n-equivariant duality between $\mathfrak{sl}_n$ and $\mathfrak{pgl}_n$ over $\mathbf{Z}$.

(ii) Over any field k of characteristic $p > 0$, prove that $\mathfrak{sl}_p$ and $\mathfrak{pgl}_p$ are not isomorphic as representation spaces for the diagonal torus of SL_p. In particular, neither is self-dual as an SL_p-representation space.

(iii) In the setup of (ii), prove that the central line $\mathrm{Lie}(\mu_p) \subset \mathfrak{sl}_p$ consisting of scalar diagonal matrices does not admit an SL_p-equivariant complement.

(iv) Note that the conclusion of (ii) holds with $\mathfrak{sl}_p$ and $\mathfrak{pgl}_p$ respectively replaced by $\mathfrak{sl}_{p^e}$ and $\mathfrak{pgl}_{p^e}$. Verify that for $0 < f < e$, the central line in $\mathrm{Lie}(\mathrm{SL}_{p^e}/\mu_{p^f})$ does have an SL_{p^e}-equivariant complement, namely the image of $\mathfrak{sl}_{p^e}$.

Exercise 1.6.5. — Let X be a connected scheme of finite type over a field k, and assume that $\mathrm{X}(k) \neq \varnothing$. Prove that X is geometrically connected over k, which is to say that X_{K} is connected for any field extension K/k, or equivalently that $\mathrm{X}_{\overline{k}}$ is connected. (Hint: prove connectedness of X_{k_s} by considering the fiber over $x_0 \in \mathrm{X}(k)$ for the open and closed projection map $\mathrm{X}_{\mathrm{K}} \to \mathrm{X}$ with finite Galois extensions K/k.) Deduce that a connected k-group scheme of finite type is geometrically connected over k; this fact is often used without comment when working with fibers of finitely presented group schemes in relative situations.

Exercise 1.6.6. — Let G and G′ be smooth connected groups over a field k, and $f : \Omega \to \mathrm{G}'$ a k-morphism defined on a dense open subset $\Omega \subset \mathrm{G}$. Assume f is a "rational homomorphism" in the sense that for a dense open subset

$V \subset \Omega \times \Omega$ for which $\omega\omega' \in \Omega$ for all $(\omega, \omega') \in V$, the morphism $V \to G$ given by $(\omega, \omega') \mapsto f(\omega\omega')f(\omega')^{-1}f(\omega)^{-1}$ is identically 1. Use pre-Grothendieck arguments (i.e., no descent theory or scheme-theoretic methods) to prove that f uniquely extends to a k-homomorphism $\widetilde{f} : G \to G'$ and that if f is birational then $\widetilde{f}$ is an isomorphism.

Exercise 1.6.7. — Let a smooth finite type k-group G act linearly on a finite-dimensional k-vector space V. Let $\underline{V}$ be the affine space over k whose A-points are $V_A := V \otimes_k A$ for any k-algebra A. Define $\underline{V}^G(A)$ to be the set of $v \in V_A$ on which the A-group G_A acts trivially (i.e., $g.v = v$ in V_R for all A-algebras R and $g \in G(R)$).

(i) Prove $\underline{V}^G$ is represented by the closed subscheme associated to a k-subspace of V (denoted V^G). Hint: use Galois descent to reduce to the case $k = k_s$ and then prove $V^{G(k)}$ works.

(ii) For an extension field K/k, prove $(V_K)^{G_K} = (V^G)_K$ inside V_K.

Exercise 1.6.8. — Let G be a connected reductive group over an algebraically closed field k of characteristic $p > 0$, and let $T \subset G$ be a maximal k-torus. For any affine k-scheme X of finite type, let $X^{(p^n)}$ denote the scalar extension by the p^n-power endomorphism of k, and define $F_{X/k,n} : X \to X^{(p^n)}$ to be the natural k-morphism induced by the absolute p^n-Frobenius $F_X^n : X \to X$ over the p^n-Frobenius $F_k^n : \operatorname{Spec} k \to \operatorname{Spec} k$. This is the *$n$-fold relative Frobenius morphism for* X *over* k, also denoted $F_{X/k}$ when $n = 1$.

(i) Using the isomorphism $(\mathbf{A}_k^d)^{(p^n)} \simeq \mathbf{A}_k^d$ via the natural $\mathbf{F}_p$-descent of $\mathbf{A}_k^d$, compute $F_{\mathbf{A}_k^d,n}$ as an explicit k-endomorphism of $\mathbf{A}_k^d$. Do the same for $\mathbf{P}_k^d$.

(ii) Prove that $F_{X/k,n}$ is functorial in X, compatible with direct products in X (over k), and compatible with extension of the ground field. Deduce that if X is a k-group and $X^{(p^n)}$ is made into a k-group via scalar extension then $F_{X/k,n}$ is a k-homomorphism.

(iii) If X is smooth of pure dimension $d > 0$ then prove that $F_{X/k,n}$ is finite flat of degree p^{dn}. In particular, $F_{G/k,n} : G \to G^{(p^n)}$ is an isogeny of degree p^{dn} carrying T onto $T^{(p^n)}$.

(iv) Compute $F_{G/k,n}$ for GL(V) and SO(q), and prove $\mathrm{Lie}(F_{G/k}) = 0$ for any G. In general, compute the effect on root data arising from $F_{G/k,n} : (G, T) \to (G^{(p^n)}, T^{(p^n)})$.

Exercise 1.6.9. — Let G be a connected linear algebraic group over an algebraically closed field k. By Remark 1.1.20, the set of semisimple elements of $G(k)$ is the union $\bigcup_T T(k)$ as T varies through the maximal tori of G.

(i) Prove that for semisimple $g \in G(k)$, $\mathrm{Lie}(Z_G(g)) = \mathfrak{g}^{g=1}$ and $Z_G(g)^0$ contains some Cartan subgroup $Z_G(T)$ of G (these are the maximal tori when

G is reductive). Deduce that $\dim \mathrm{Z}_{\mathrm{G}}(g)$ coincides with the common dimension of the Cartan subgroups if and only if $\mathrm{Z}_{\mathrm{G}}(g)^0$ is a Cartan subgroup.

(ii) Prove that there exist semisimple $g \in \mathrm{G}(k)$ such that $\mathrm{Z}_{\mathrm{G}}(g)^0$ is a Cartan subgroup. (Hint: For a maximal torus T, consider the *finitely many* nontrivial T-weights that occur on $\mathfrak{g}$.) An element $g \in \mathrm{G}(k)$ is *regular* when $\mathrm{Z}_{\mathrm{G}}(g_{\mathrm{ss}})^0$ is a Cartan subgroup. Using that $g \in \mathrm{Z}_{\mathrm{G}}(g_{\mathrm{ss}})^0$ [**Bo91**, 11.12], deduce that g is regular if and only if it belongs to a unique Cartan subgroup. (For $\mathrm{G} = \mathrm{GL}(\mathrm{V})$, these are precisely the $g \in \mathrm{G}(k)$ whose characteristic polynomial has non-zero discriminant since g and g_{ss} have the same characteristic polynomial, so this is a Zariski-dense open locus in $\mathrm{GL}(\mathrm{V})$.)

(iii) By considering the multiplicity of $x-1$ as a factor of the characteristic polynomial for the adjoint action of G on $\mathfrak{g}$, prove that the regular locus of $\mathrm{G}(k)$ is a (non-empty) Zariski-open subset. (Hint: for any $g \in \mathrm{G}(k)$, $\mathrm{Ad}_{\mathrm{G}}(g)$ and $\mathrm{Ad}_{\mathrm{G}}(g_{\mathrm{ss}})$ have the same characteristic polynomial on $\mathfrak{g}$.)

(iv) Steinberg proved that all regular elements in $\mathrm{G}(k)$ are semisimple when G is reductive. Without using this fact, prove that for reductive G the non-empty locus of regular semisimple elements in $\mathrm{G}(k)$ is Zariski-open. (This is false if G is nontrivial and unipotent!) In other words, within the dense Zariski-open locus of regular elements, prove that the non-empty semisimple locus is open. (Hint: consider $\dim \mathrm{Z}_{\mathrm{G}}(g)$ rather than $\dim \mathrm{Z}_{\mathrm{G}}(g_{\mathrm{ss}})$, and apply semicontinuity of fiber dimension to a "universal centralizer scheme" over G.)

Exercise 1.6.10. — Let k be a field. To define special orthogonal groups over rings in a uniform way, we need a characteristic-free definition of non-degeneracy for a quadratic form $q : \mathrm{V} \to k$ on a finite-dimensional k-vector space V of dimension $d \geqslant 2$. We say q is *non-degenerate* when $q \neq 0$ and $(q = 0)$ is smooth in $\mathbf{P}(\mathrm{V}^*) := \mathrm{Proj}(\mathrm{Sym}(\mathrm{V}^*)) \simeq \mathbf{P}_k^{d-1}$.

(i) Let $\mathrm{B}_q : \mathrm{V} \times \mathrm{V} \to k$ be the symmetric bilinear form $(v, v') \mapsto q(v+v') - q(v) - q(v')$, and define $\mathrm{V}^{\perp} = \{v \in \mathrm{V} \,|\, \mathrm{B}_q(v, \cdot) = 0\}$; we call $\delta_q := \dim \mathrm{V}^{\perp}$ the *defect* of q. Prove that B_q uniquely factors through a non-degenerate symmetric bilinear form on $\mathrm{V}/\mathrm{V}^{\perp}$, and that B_q is non-degenerate precisely when the defect is 0. Also show that if $\mathrm{char}(k) = 2$ then B_q is alternating, and deduce that $\delta_q \equiv \dim \mathrm{V} \bmod 2$ for such k (so $\delta_q \geqslant 1$ if $\dim \mathrm{V}$ is odd).

(ii) Prove that if $\delta_q = 0$ then $q_{\overline{k}}$ admits one of the following "standard forms": $\sum_{i=1}^n x_{2i-1}x_{2i}$ if $\dim \mathrm{V} = 2n$ $(n \geqslant 1)$, and $x_0^2 + \sum_{i=1}^n x_{2i-1}x_{2i}$ if $\dim \mathrm{V} = 2n+1$ $(n \geqslant 1)$. Do the same if $\mathrm{char}(k) = 2$ and $\delta_q = 1$. (Distinguish whether or not $q|_{\mathrm{V}^{\perp}} \neq 0$.) How about the converse?

(iii) If $\mathrm{char}(k) \neq 2$, prove q is non-degenerate if and only if $\delta_q = 0$. For $\mathrm{char}(k) = 2$, prove q is non-degenerate if and only if $\delta_q \leqslant 1$ with $q|_{\mathrm{V}^{\perp}} \neq 0$ when $\delta_q = 1$. Hint: use (ii) to simplify calculations. (In [**EKM**, § 7] (V, q) is called *regular* if q has no nontrivial zeros on $\mathrm{V}^{\perp}$. This is equivalent to non-degeneracy if $\mathrm{char}(k) \neq 2$ or if $\mathrm{char}(k) = 2$ with $\dim \mathrm{V}^{\perp} \leqslant 1$. In general regularity is preserved by separable extension on k and is equivalent to the zero scheme $(q = 0) \subset \mathbf{P}(\mathrm{V}^*)$ being regular at k-points.)

Exercise 1.6.11. — Let G be a smooth affine group over a field $k = \overline{k}$ (allow $\mathrm{G} \neq \mathrm{G}^0$).

(i) If all finite-dimensional linear representations of G are completely reducible, or if there is even a single faithful semi-simple representation of G, then prove that G^0 is reductive. (Hint: use Lie–Kolchin and the behavior of semisimplicity under restriction to a normal subgroup.) Deduce that GL(V) is reductive and SL(V) is semisimple.

(ii) Conversely, assume G^0 is reductive and $\mathrm{char}(k) = 0$. Prove that finite-dimensional linear representations of G are completely reducible. (Hint: prove $\mathrm{Lie}(\mathscr{D}(\mathrm{G}))$ is semisimple.)

(iii) Let V be the standard 2-dimensional representation of SL_2 over k, and assume $\mathrm{char}(k) = p > 0$. In $\mathrm{Sym}^p(\mathrm{V})$, prove that the line of pth powers has no SL_2-equivariant complement (so $\mathrm{Sym}^p(\mathrm{V})$ is not semisimple as a representation, and hence $\mathrm{V}^{\otimes p}$ is not semisimple).

Exercise 1.6.12. — Let G be a linear algebraic group over an algebraically closed field k, N a normal linear algebraic subgroup (e.g., $\mathrm{N} = \mathscr{D}(\mathrm{G})$), and T a maximal torus in G. Prove that $(\mathrm{T} \bigcap \mathrm{N})^0_{\mathrm{red}}$ is a maximal torus in N. Hint: argue in reverse, starting with a maximal torus in N and extending it to a maximal torus in G, which in turn is conjugate to T. (See Example 2.2.6 and Exercise 5.5.1 for smoothness and connectedness properties of $\mathrm{T} \bigcap \mathrm{N}$.)

Exercise 1.6.13. — Let G be a connected semisimple group over an algebraically closed field k. The *central isogeny class* of G is the equivalence class of G generated by the relation on connected semisimple k-groups G' and G'' that there exists a central isogeny $\mathrm{G}' \to \mathrm{G}''$ or $\mathrm{G}'' \to \mathrm{G}'$ (i.e., an isogeny whose scheme-theoretic kernel is central). Composites of central isogenies between connected reductive groups are central (due to Corollary 3.3.5 applied over k, where the proof simplifies); this is false for general smooth connected affine groups in positive characteristic (Exercise 3.4.4(ii)).

(i) Prove that any central isogeny $f : \mathrm{G}' \to \mathrm{G}''$ can be arranged via composition with a suitable conjugation to satisfy $f(\mathrm{T}') = \mathrm{T}''$ and $f(\mathrm{B}') = \mathrm{B}''$ for any choices of Borel subgroups and maximal tori that they contain, and use the open cells to show that such an f must induce isomorphisms between root groups (false for Frobenius isogenies!). Deduce via Corollary 1.2.4, the Isomorphism Theorem, and (1.3.2) that if $\mathrm{G}' \simeq \mathrm{G}''$ abstractly then f must be an isomorphism and that if $\mathrm{G}' \not\simeq \mathrm{G}''$ then up to conjugacy f is the only central isogeny between G' and G'' (in either direction!). In particular, a central isogeny class is *partially ordered* (when members are considered up to abstract k-isomorphism).

(ii) Using the Existence and Isomorphism Theorems over k, prove the equivalence of the following conditions on G: the only central isogenies $\mathrm{G}' \to \mathrm{G}$ from connected semisimple groups are isomorphisms, G dominates all other members of its central isogeny class, and the simple positive coroots of (G, T) (relative to a choice of Φ^+) are a $\mathbf{Z}$-basis of the cocharacter group of T. Under these conditions, we say G is *simply connected* (e.g., Sp_{2n} and SL_n).

Likewise prove the equivalence of: Ad_{G} is a closed immersion, G is dominated by all members of its central isogeny class, and the simple positive roots of (G, T) (relative to a choice of Φ^+) are a $\mathbf{Z}$-basis of the character group of T. Under these conditions we say G is *adjoint* (e.g., PGL_n and SO_{2n+1}).

(iii) Let $\mathrm{T} \subset \mathrm{G}$ be a maximal torus. For each $a \in \Phi(\mathrm{G}, \mathrm{T})$, let $\mathrm{T}_a = (\ker a)^0_{\mathrm{red}}$ be the unique codimension-1 torus in T killed by a. By the structure theory underlying the *definition* of coroots, $\mathrm{G}_a := \mathscr{D}(\mathrm{Z}_{\mathrm{G}}(\mathrm{T}_a))$ is either SL_2 or PGL_2 with maximal torus $a^\vee(\mathbf{G}_{\mathrm{m}})$ having root groups U_a and U_{-a}.

If G is simply connected, prove that $\mathrm{G}_a = \mathrm{SL}_2$ for all $a \in \Phi(\mathrm{G}, \mathrm{T})$. Show that the converse is false by proving that $\mathrm{G}_a = \mathrm{SL}_2$ for all $a \in \Phi(\mathrm{G}, \mathrm{T})$ when $\mathrm{G} = \mathrm{PGL}_n$ with $n \geqslant 3$.

Exercise 1.6.14. — Let U be a nonzero finite-dimensional vector space over a field k, and U^* its dual. Define $\mathrm{W} = \mathrm{U} \bigoplus \mathrm{U}^*$.

(i) Define $\psi_0 : \mathrm{W} \times \mathrm{W} \to k$ by $\psi_0((v, f), (v', f')) = f'(v) - f(v')$. Show that (W, ψ_0) is a non-degenerate symplectic space. Let $\mathrm{GL}(\mathrm{U})$ act on W by $g.(v, f) = (g.v, f \circ g^{-1})$. Show that this defines a monic homomorphism (hence closed immersion) of k-groups $\mathrm{GL}(\mathrm{U}) \to \mathrm{GL}(\mathrm{W})$ and that the image lies in $\mathrm{Sp}(\mathrm{W}, \psi_0)$.

(ii) Define a quadratic form q_0 on W by $q_0(v, f) = f(v)$. Show that q_0 is non-degenerate and that the image of $\mathrm{GL}(\mathrm{U}) \to \mathrm{GL}(\mathrm{W})$ also lies in $\mathrm{SO}(\mathrm{W}, q_0)$.

Exercise 1.6.15. — Let $\mathrm{U} = k^2$ for an algebraically closed field k and define W, ψ_0, q_0 as in the previous exercise. Thus, we have inclusions $\mathrm{SL}_2 \subset \mathrm{GL}_2 \subset \mathrm{Sp}(\mathrm{W}, \psi_0)$ and $\mathrm{SL}_2 \subset \mathrm{SO}(\mathrm{W}, q_0)$. In this exercise, you can take for granted that the groups $\mathrm{SO}(\mathrm{V}, q)$ and $\mathrm{Sp}(\mathrm{V}, \psi)$ are connected reductive groups. For such groups, we shall interpret the maps φ_a from Theorem 1.2.7.

(i) Let (V, q) be a non-degenerate quadratic space of dimension $2n$ over k with $n \geqslant 2$. Consider the diagonal maximal torus T in $\mathrm{G} = \mathrm{SO}(\mathrm{V}, q)$ as in Proposition C.3.10 relative to a basis of V putting q into the "standard form" as in Exercise 1.6.10(ii). Compute $\Phi(\mathrm{G}, \mathrm{T})$. For each $a \in \Phi(\mathrm{G}, \mathrm{T})$, show that $\varphi_a : \mathrm{SL}_2 \to \mathrm{G}$ arises from the following construction: take an embedding $i : (\mathrm{W}, q_0) \hookrightarrow (\mathrm{V}, q)$ of quadratic spaces, show $\mathrm{W} \oplus \mathrm{W}^\perp = \mathrm{V}$, and use this to define an embedding $\mathrm{SO}(\mathrm{W}, q_0) \hookrightarrow \mathrm{SO}(\mathrm{V}, q)$ in an obvious manner (yielding $\mathrm{SL}_2 \hookrightarrow \mathrm{SO}(\mathrm{W}, q_0) \hookrightarrow \mathrm{SO}(\mathrm{V}, q)$).

(ii) Let (V, q) be a non-degenerate quadratic space of dimension $2n+1$ over k with $n \geqslant 1$. Consider the diagonal maximal torus T in $\mathrm{G} = \mathrm{SO}(\mathrm{V}, q)$ as in Proposition C.3.10 relative to a basis of V putting q in "standard form" as in Exercise 1.6.10(ii). Compute $\Phi(\mathrm{G},\mathrm{T})$. For each $a \in \Phi(\mathrm{G},\mathrm{T})$, show that $\varphi_a : \mathrm{SL}_2 \to \mathrm{G}$ arises from one of the following two constructions: (a) take an embedding $i : (\mathrm{W}, q_0) \hookrightarrow (\mathrm{V}, q)$ of quadratic spaces and form $\mathrm{SL}_2 \hookrightarrow \mathrm{SO}(\mathrm{W}, q_0) \hookrightarrow \mathrm{SO}(\mathrm{V}, q)$ as in (i); (b) take a 3-dimensional subspace V_3 of V such that (V_3, q) is non-degenerate, so $\mathrm{SO}(\mathrm{V}_3, q) \simeq \mathrm{PGL}_2$ (Example C.6.2), and show $\mathrm{V}_3 \oplus \mathrm{V}_3^{\perp} = \mathrm{V}$, yielding a homomorphism $\mathrm{SL}_2 \to \mathrm{SO}(\mathrm{V}_3, q) \hookrightarrow \mathrm{SO}(\mathrm{V}, q)$. Case (a) is for the long roots, and (b) is for the short roots.

(iii) Let (V, ψ) be a non-degenerate symplectic space of dimension $2n$ over k. Construct a maximal torus T of $\mathrm{G} = \mathrm{Sp}(\mathrm{V}, \psi)$ and compute $\Phi(\mathrm{G},\mathrm{T})$. For each $a \in \Phi(\mathrm{G},\mathrm{T})$, show that $\varphi_a : \mathrm{SL}_2 \to \mathrm{G}$ arises from one of the following two constructions: (a) take an embedding $i : (\mathrm{W}, \psi_0) \hookrightarrow (\mathrm{V}, \psi)$ and form $\mathrm{SL}_2 \hookrightarrow \mathrm{Sp}(\mathrm{W}, \psi_0) \hookrightarrow \mathrm{Sp}(\mathrm{V}, \psi)$; (b) take a 2-dimensional subspace V_2 of V such that (V_2, ψ) is non-degenerate, so we have $\mathrm{SL}_2 = \mathrm{Sp}(\mathrm{V}_2, \psi) \hookrightarrow \mathrm{Sp}(\mathrm{V}, \psi)$. (Determine which of (a) or (b) corresponds to long roots and short roots.)

Exercise 1.6.16. — (i) Let G be a connected linear algebraic group over an algebraically closed field k, and let T be a torus of G. Let $\mathfrak{g} = \bigoplus_{a \in \mathrm{X}(\mathrm{T})} \mathfrak{g}_a$ be the weight space decomposition of $\mathfrak{g} = \mathrm{Lie}\,\mathrm{G}$ for the action of T. Let $\Phi = \{a \in \mathrm{X}(\mathrm{T}) - \{0\} \mid \mathfrak{g}_a \neq 0\}$. Assume:

— $\mathfrak{g}_0 = \mathrm{Lie}\,\mathrm{T}$ (so T is maximal as a torus of G) and $\dim \mathfrak{g}_a \leqslant 1$ for all $a \in \Phi$;

— $\Phi = -\Phi$ and if $a, b \in \Phi$ are $\mathbf{Z}$-linearly dependent then $b = \pm a$;

— for all $a \in \Phi$ there exists a homomorphism $\varphi_a : \mathrm{SL}_2 \to \mathrm{G}$ with finite kernel such that the image commutes with the codimension-1 torus $\mathrm{T}_a = (\ker a)^0_{\mathrm{red}} \subset \mathrm{T}$.

Show that G is a reductive group. (Hint: prove $\mathrm{Lie}(\mathscr{R}_u(\mathrm{G}_{\overline{k}})) = 0$. Also see Lemma 3.1.10.)

(ii) Prove SO_n and Sp_{2n} are reductive (grant each is connected and smooth).

Exercise 1.6.17. — Let Δ be the base of a positive system of roots Φ^+ in a reduced root system (V, Φ). Consider the reduced dual root system $(\mathrm{V}^*, \Phi^\vee)$ arising from the coroots (where $a^\vee$ is the linear form that computes the unique reflection $s_a : \mathrm{V} \simeq \mathrm{V}$ preserving Φ and negating a; i.e., $s_a(v) = v - a^\vee(v)a$).

(i) Assume (V, Φ) is irreducible and fix a $\mathrm{W}(\Phi)$-invariant positive-definite quadratic form $\mathrm{Q} : \mathrm{V} \to \mathbf{Q}$ on V (unique up to scaling, by Remark 1.3.14). Prove that under the resulting isomorphism $\mathrm{V}^* \simeq \mathrm{V}$, $a^\vee$ is identified with $2a/\mathrm{Q}(a)$.

(ii) Let $\Delta^\vee$ be the set of coroots $a^\vee$ for $a \in \Delta$. Using (i), prove that $\Delta^\vee$ is a basis of V^* and that every element of $\Phi^\vee$ is a linear combination of $\Delta^\vee$ with all coefficients in $\mathbf{Q}_{\geqslant 0}$ or in $\mathbf{Q}_{\leqslant 0}$. Use the equivalent characterizations of bases of root systems in [**SGA3**, XXI, 3.1.5] to deduce that $\Delta^\vee$ is the base of a positive system of roots in $\Phi^\vee$.

(iii) Assume (V, Φ) is irreducible and choose a nonzero $\lambda \in \mathrm{V}^*$ such that $\Phi_{\lambda>0} = \Phi^+$ (so $\langle \lambda, a\rangle \neq 0$ for all $a \in \Phi$). For Q as in (i) and $v \in \mathrm{V}$ satisfying $\mathrm{B}_{\mathrm{Q}}(v, \cdot) = \lambda$, prove that v is not orthogonal to any coroot and that $\Delta^\vee$ is a base for the positive system of roots $\Phi^\vee_{v>0}$.

(iv) Using (iii), prove that if (V, Φ) is irreducible then the Dynkin diagrams for Φ and $\Phi^\vee$ coincide in the simply laced case (i.e., one root length) and otherwise are related by swapping the direction of the unique multiple edge. (Hence, by the classification of irreducible reduced root systems, the diagrams coincide except that B_n and C_n are swapped for $n \geqslant 3$.)

2. Normalizers, centralizers, and quotients

To motivate the need for a theory of reductive groups over rings, consider a connected reductive group G over a number field K with ring of integers R, Choose a faithful representation $G \hookrightarrow GL_{n,K}$ over K. The schematic closure of G in the R-group $GL_{n,R}$ is a flat closed R-subgroup $\mathscr{G}$ of $GL_{n,R}$. By direct limit considerations (and Exercise 1.6.5), if $a \in R$ is nonzero and sufficiently divisible (i.e., $\operatorname{Spec} R[1/a]$ is a sufficiently small neighborhood of the generic point in $\operatorname{Spec} R$) then $\mathscr{G}_{R[1/a]}$ is smooth over $\operatorname{Spec} R[1/a]$ with (geometrically) connected fibers (see [**EGA**, IV_3, 9.7.8]). Is $\mathscr{G}_{\overline{s}}$ reductive for all $s \in \operatorname{Spec} R[1/a]$, perhaps after making a more divisible in R, and if so then is the isomorphism type (of the associated root datum) independent of s? In such cases we wish to say that $\mathscr{G}_{R[1/a]}$ is a reductive group scheme over $R[1/a]$.

Although G may not have a Borel K-subgroup, G_{K_v} has a Borel K_v-subgroup for all but finitely many v. This assertion only involves the theory of reductive groups over fields (of characteristic 0), but its proof uses the notion of reductive group over localized rings of integers. The theory of reductive groups over discrete valuation rings links the theories over $\mathbf{F}_p$ and $\mathbf{Q}$, as well as relates finite groups of Lie type to complex semisimple Lie groups.

We also seek a conceptual understanding of the smooth affine $\mathbf{Z}$-groups constructed by Chevalley in [**Chev61**] (with $\mathbf{Q}$-split semisimple generic fiber). The Bruhat–Tits structure theory for p-adic groups, the study of integral models of Shimura varieties, and Galois deformation theory valued in reductive groups provide further motivation for the notion of reductive group over rings.

The main result that we are aiming for is this: over *any* non-empty scheme S, there are analogues of the Existence and Isomorphism Theorems for connected reductive groups over algebraically closed fields. More specifically, we will show that the category of "split" reductive S-groups (equipped with a suitable notion of isomorphism as the morphisms) is the same as the category of root data (equipped with isomorphisms as the morphisms); the latter has nothing to do with S!

In the classical theory, normalizers and centralizers of tori are ubiquitous tools for creating interesting subquotients of a non-solvable smooth connected affine group. In the relative theory these constructions remain essential, but we need to use torsion-levels in a torus because those are *finite* flat over the base (unlike the torus). Such finiteness makes the torsion-levels very useful in proving representability results for functorial normalizers and centralizers of tori. The method of passing to torsion-levels in tori has a useful variant when considering normalizers and centralizers of smooth closed subgroups with connected fibers: passing to infinitesimal neighborhoods of the identity (which are often finite flat closed subschemes, but generally not subgroups).

The construction of quotients by normalizers of subgroups will also use the consideration of "finite flat approximations" to subgroups.

Our use of torsion levels in tori forces us to consider general (possibly non-smooth) groups of multiplicative type. See Appendix B for the basics of the theory of group schemes of multiplicative type, including definitions and notation (and precise references within [**Oes**] for proofs of some fundamental results). The reader should learn the material in Appendix B before continuing on to the discussion that follows. (For example, by Lemma B.1.3, any monic homomorphism $\mathrm{H} \to \mathrm{G}$ from a multiplicative type S-group to an S-affine S-group of finite presentation is a closed immersion.)

2.1. Transporter schemes and Hom schemes. — Let G be an S-group, and $\mathrm{Y}, \mathrm{Y}' \rightrightarrows \mathrm{G}$ monic S-morphisms from S-schemes. The transporter functor on S-schemes is defined to be

$$\underline{\mathrm{Transp}}_{\mathrm{G}}(\mathrm{Y}, \mathrm{Y}') : \mathrm{S}' \rightsquigarrow \{g \in \mathrm{G}(\mathrm{S}') \,|\, g(\mathrm{Y}_{\mathrm{S}'})g^{-1} \subset \mathrm{Y}'_{\mathrm{S}'}\}.$$

In the special case $\mathrm{Y}' = \mathrm{Y}$ with Y of finite presentation over S, this coincides with the normalizer functor

$$\underline{\mathrm{N}}_{\mathrm{G}}(\mathrm{Y}) : \mathrm{S}' \rightsquigarrow \{g \in \mathrm{G}(\mathrm{S}') \,|\, g\mathrm{Y}_{\mathrm{S}'}g^{-1} = \mathrm{Y}_{\mathrm{S}'}\}$$

because the monic endomorphism of $\mathrm{Y}_{\mathrm{S}'}$ given by g-conjugation is an automorphism, as for any finitely presented scheme over any base [**EGA**, IV_4, 17.9.6]. When these functors are representable, we denote representing objects as $\mathrm{Transp}_{\mathrm{G}}(\mathrm{Y}, \mathrm{Y}')$ and $\mathrm{N}_{\mathrm{G}}(\mathrm{Y})$ respectively.

Definition 2.1.1. — A finitely presented S-subgroup G' in G is *normal* if $\underline{\mathrm{N}}_{\mathrm{G}}(\mathrm{G}') = \mathrm{G}$, or equivalently for all S-schemes S' the conjugation on $\mathrm{G}_{\mathrm{S}'}$ by any $g \in \mathrm{G}(\mathrm{S}')$ carries $\mathrm{G}'_{\mathrm{S}'}$ into (and hence isomorphically onto) itself.

Another formulation of this definition is that $\mathrm{G}'(\mathrm{S}')$ is a normal subgroup of $\mathrm{G}(\mathrm{S}')$ for every S-scheme S'. Normality of G' in G *cannot* be checked on geometric points even when G' is smooth and $\mathrm{S} = \mathrm{Spec}\, k$ for an algebraically closed field k; see Example 2.2.3.

Proposition 2.1.2. — *Let* G *be a finitely presented* S*-affine group scheme. Let* $\mathrm{Y}, \mathrm{Y}' \subset \mathrm{G}$ *be finitely presented closed subschemes. Assume* Y *is either a multiplicative type subgroup or is finite flat over* S*. The scheme* $\mathrm{Transp}_{\mathrm{G}}(\mathrm{Y}, \mathrm{Y}')$ *exists as a finitely presented closed subscheme of* G*. In particular, the normalizer* $\mathrm{N}_{\mathrm{G}}(\mathrm{Y})$ *exists as a finitely presented closed subgroup of* G*.*

If G *is smooth and* Y *and* Y' *are both multiplicative type subgroups of* G *then these subschemes are smooth.*

In [**SGA3**, XI, 2.4bis], the smoothness aspect of this proposition is proved without affineness hypotheses on G. The proof in the affine case is much simpler, and this case suffices for our needs. In Proposition 2.1.6 we will adapt the method of proof to apply to smooth closed subschemes $Y, Y' \subset G$ with geometrically connected fibers, but the *smoothness* of $\mathrm{Transp}_G(Y, Y')$ may then fail, even if G is smooth. Smoothness of $\mathrm{Transp}_G(H, H')$ for any H and H' of multiplicative type (when G is smooth) is remarkably useful.

Proof. — By "standard" direct limit arguments, we may and do assume that S is noetherian. To be precise, the problems are Zariski-local on the base, so we can assume $S = \mathrm{Spec}(A)$ is affine. Writing $A = \varinjlim A_i$ for the directed system of finitely generated **Z**-subalgebras $A_i \subset A$, the finite presentation hypotheses imply that for sufficiently large i_0 there is a finite type affine A_{i_0}-group scheme G_{i_0} descending G and finite type closed A_{i_0}-subgroup schemes Y'_{i_0} and Y_{i_0} of G_{i_0} that descend $Y', Y \subset G$. By increasing i_0 if necessary, it can be arranged that the property of being finite flat over the base descends, and likewise for smoothness by [**EGA**, IV_4, 17.8.7].

To descend the multiplicative type property, for a closed A-subgroup $H \subset G$ of multiplicative type and $i \geqslant i_0$ such that there is a closed A_i-subgroup $H_i \subset G_i := G_{i_0} \otimes_{A_{i_0}} A_i$ descending H, pick an fppf cover $\mathrm{Spec}\, A' \to \mathrm{Spec}\, A$ such that $H_{A'} \simeq D_{A'}(M)$ for some finitely generated abelian group M. By increasing i we can arrange that $\mathrm{Spec}\, A' \to \mathrm{Spec}\, A$ descends to an fppf cover $\mathrm{Spec}\, A'_i \to \mathrm{Spec}\, A_i$. Then the A'_i-groups $H_i \otimes_{A_i} A'_i$ and $D_{A'_i}(M)$ become isomorphic over A'. However, A' is the direct limit of the rings $A'_j := A'_i \otimes_{A_i} A_j$ over $j \geqslant i$, so by increasing i some more we can arrange that $H_i \otimes_{A_i} A'_i \simeq D_{A'_i}(M)$. Since $\mathrm{Spec}\, A'_i$ is an fppf cover of $\mathrm{Spec}\, A_i$, it follows that H_i is multiplicative type. Now it suffices to treat the case of noetherian S, so all closed subschemes of G are finitely presented over S and hence we do not need to keep track of the "finitely presented" property.

Granting the representability by a closed subscheme, let's address the S-smoothness when G is smooth and Y and Y' respectively coincide with multiplicative type subgroups H and H' of G. We shall verify the functorial criterion for smoothness. The condition is that for an affine scheme S' over S (which we may and do take to be noetherian, or even artinian) and a closed subscheme S'_0 defined by a square-zero quasi-coherent ideal on S', any $g_0 \in G(S'_0)$ conjugating $H_{S'_0}$ into $H'_{S'_0}$ admits a lift $g \in G(S')$ conjugating $H_{S'}$ into $H'_{S'}$. We may rename S' as S, and define $X_0 := X_{S_0}$ for S-schemes X. By S-smoothness of G we can lift g_0 to some $g \in G(S)$ but perhaps gHg^{-1} is not contained in H'. Nonetheless, these two subgroups of multiplicative type in G satisfy $(gHg^{-1})_0 = g_0 H_0 g_0^{-1} \subset H'_0$, so by renaming gHg^{-1} as H we may assume that $H_0 \subset H'_0$ and $g_0 = 1$.

Since H' is of multiplicative type, the multiplicative type S_0-subgroup H_0 in H'_0 uniquely lifts to a multiplicative type S-subgroup $\widetilde{H}$ of H' (as the uniqueness allows us to work étale-locally on S, so it suffices to consider the easy case that H' is S-split and H_0 is S_0-split). The multiplicative type S-subgroups H and $\widetilde{\mathrm{H}}$ in G have the same reduction in G_0, so by [**Oes**, IV, § 1] these subgroups are abstractly isomorphic in a manner lifting their residual identification inside G_0. Hence, if we choose such an isomorphism $\mathrm{H} \simeq \widetilde{\mathrm{H}}$ then by Corollary B.2.6 (applied to $\mathrm{H} \hookrightarrow \mathrm{G}$ and $\mathrm{H} \simeq \widetilde{\mathrm{H}} \hookrightarrow \mathrm{G}$) and the vanishing of higher Hochschild cohomology in [**Oes**, III, 3.3] there exists $g' \in \mathrm{G}(\mathrm{S})$ satisfying $g'_0 = 1$ and $g'\mathrm{H}g'^{-1} = \widetilde{\mathrm{H}} \subset \mathrm{H}'$. This completes the proof of smoothness of the transporter scheme (granting its existence).

It remains to prove that the transporter functor is represented by a closed subscheme of G (necessarily of finite presentation since S is noetherian) when Y is either finite flat or a closed subgroup of multiplicative type. We first reduce the second case to the first, so suppose $\mathrm{Y} = \mathrm{H}$ is a closed subgroup of multiplicative type. By the relative schematic density of $\{\mathrm{H}[n]\}_{n\geqslant 1}$ in H (in the sense of [**EGA**, IV_3, 11.10.8–11.10.10]), for any S-scheme S' the only closed subscheme $\mathrm{Z} \subset \mathrm{H}_{\mathrm{S}'}$ containing every $\mathrm{H}[n]_{\mathrm{S}'}$ is $\mathrm{Z} = \mathrm{H}_{\mathrm{S}'}$. Thus,

$$\underline{\mathrm{Transp}}_{\mathrm{G}}(\mathrm{H}, \mathrm{Y}') = \bigcap_{n>0} \underline{\mathrm{Transp}}_{\mathrm{G}}(\mathrm{H}[n], \mathrm{Y}')$$

as subfunctors of G, so it suffices to treat each of the pairs $(\mathrm{H}[n], \mathrm{Y}')$ separately. Since an arbitrary intersection of closed subschemes is a closed subscheme, now we may and do assume that Y is finite and flat over S (and S is noetherian).

To build $\mathrm{Transp}_{\mathrm{G}}(\mathrm{Y}, \mathrm{Y}')$ for finite flat Y, we will use Hom-schemes. Hence, we now prove the representability of Hom-functors in a special case.

Lemma 2.1.3. — *Let* S *be a scheme,* $\mathrm{X} \to \mathrm{S}$ *a finite flat and finitely presented map, and* $\mathrm{Y} \to \mathrm{S}$ *an affine morphism of finite presentation. The functor on* S*-schemes defined by*

$$\mathrm{S}' \rightsquigarrow \mathrm{Hom}_{\mathrm{S}'}(\mathrm{X}_{\mathrm{S}'}, \mathrm{Y}_{\mathrm{S}'})$$

is represented by an S*-affine* S*-scheme of finite presentation.*

Likewise, if G *and* G' *are finitely presented* S*-groups with* G *finite flat and* G' *affine over* S *then the functor* $\mathrm{S}' \rightsquigarrow \mathrm{Hom}_{\mathrm{S}'\text{-gp}}(\mathrm{G}_{\mathrm{S}'}, \mathrm{G}'_{\mathrm{S}'})$ *classifying group scheme homomorphisms is represented by an* S*-affine* S*-scheme of finite presentation.*

The representing schemes are denoted $\underline{\mathrm{Hom}}(\mathrm{X}, \mathrm{Y})$ and $\underline{\mathrm{Hom}}_{\mathrm{S}\text{-gp}}(\mathrm{G}, \mathrm{G}')$ respectively (since the notation without underlining has a categorical meaning).

Proof. — We may and do assume that S is affine, and then noetherian, say $\mathrm{S} = \mathrm{Spec}(\mathrm{A})$, with $\mathrm{X} = \mathrm{Spec}(\mathrm{B})$ for B that is finite free as an A-module (admitting an A-basis containing 1) and $\mathrm{Y} = \mathrm{Spec}(\mathrm{C})$ for C a finitely generated

A-algebra. Once the case of the functor of scheme morphisms is settled, the refinement for group schemes amounts to the formation of several fiber products (see Exercise 2.4.5). Thus, we just focus on the assertion for scheme morphisms with X and Y, avoiding any involvement of group schemes.

The basic idea of the construction of $\underline{\mathrm{Hom}}(\mathrm{X}, \mathrm{Y})$ is similar to the construction of Weil restriction of scalars $\mathrm{R}_{\mathrm{S}'/\mathrm{S}}(\mathrm{X}')$ for a finite flat map $\mathrm{S}' \to \mathrm{S}$ and affine finite type S'-scheme X' for affine noetherian S (see [**BLR**, 7.6]). The reason for the similarity is that $\underline{\mathrm{Hom}}(\mathrm{X}, \mathrm{Y}) = \mathrm{R}_{\mathrm{X}/\mathrm{S}}(\mathrm{X} \times_{\mathrm{S}} \mathrm{Y})$, since for any S-scheme S' the set $\mathrm{Hom}_{\mathrm{S}}(\mathrm{S}', \underline{\mathrm{Hom}}(\mathrm{X}, \mathrm{Y}))$ is identified with

$$\mathrm{Hom}_{\mathrm{S}}(\mathrm{S}' \times_{\mathrm{S}} \mathrm{X}, \mathrm{Y}) = \mathrm{Hom}_{\mathrm{X}}(\mathrm{S}' \times_{\mathrm{S}} \mathrm{X}, \mathrm{X} \times_{\mathrm{S}} \mathrm{Y}) = \mathrm{Hom}_{\mathrm{S}}(\mathrm{S}', \mathrm{R}_{\mathrm{X}/\mathrm{S}}(\mathrm{X} \times_{\mathrm{S}} \mathrm{Y})).$$

Let $\{e_1, \ldots, e_r\}$ be an A-basis of B with $e_1 = 1$, and let

$$\mathrm{A}[t_1, \ldots, t_n]/(f_1, \ldots, f_m) \simeq \mathrm{C}$$

be a presentation of C. For any A-algebra A' and the associated scalar extensions $\mathrm{B}' = \mathrm{A}' \otimes_{\mathrm{A}} \mathrm{B}$ and $\mathrm{C}' = \mathrm{A}' \otimes_{\mathrm{A}} \mathrm{C}$ over A', we identify $\mathrm{Hom}_{\mathrm{A}'\text{-alg}}(\mathrm{C}', \mathrm{B}')$ with the set of ordered n-tuples $b' = (b'_1, \ldots, b'_n) \in \mathrm{B}'^n$ such that $f_j(b') = 0$ in B' for all j. By expressing the A-algebra structure on the A-module $\mathrm{B} = \bigoplus \mathrm{A} e_\alpha$ in terms of "structure constants" in A, the specification of b' amounts to the specification of an ordered nr-tuple in A', and the relations $f_j(b') = 0$ amount to a "universal" system of polynomial conditions over A on this ordered nr-tuple. These polynomial conditions define the desired representing object as a closed subscheme of an affine space over A. □

Remark 2.1.4. — In [**SGA3**, XI, 4.1, 4.2], a fundamental result is proved: for any smooth S-affine S-group G and multiplicative type group H, the functor $\underline{\mathrm{Hom}}_{\mathrm{S}\text{-gp}}(\mathrm{H}, \mathrm{G})$ classifying S'-group homomorphisms $\mathrm{H}_{\mathrm{S}'} \to \mathrm{G}_{\mathrm{S}'}$ over S-schemes S' is represented by a smooth separated S-scheme and the functor $\underline{\mathrm{Mult}}_{\mathrm{G}/\mathrm{S}}$ classifying subgroups of G of multiplicative type is likewise represented by a smooth separated S-scheme. (Beware that each of these representing schemes is generally just *locally* of finite presentation over S; they are typically *not* quasi-compact over S.)

The construction of schemes representing $\underline{\mathrm{Hom}}_{\mathrm{S}\text{-gp}}(\mathrm{H}, \mathrm{G})$ and $\underline{\mathrm{Mult}}_{\mathrm{G}/\mathrm{S}}$ rests on deep representability criteria for functors in [**SGA3**, XI, § 3] that are specially designed for such applications. These moduli schemes in turn underlie Grothendieck's construction of quasi-affine quotients $\mathrm{G}/\mathrm{N}_{\mathrm{G}}(\mathrm{H})$ and $\mathrm{G}/\mathrm{Z}_{\mathrm{G}}(\mathrm{H})$ for multiplicative type subgroups H in G (see Remark 2.3.2 for a review of the notion of quasi-affine morphism). The quasi-affineness property of these quotients [**SGA3**, XI, 5.11] is crucial for applications with descent theory (as it ensures effectivity of descent).

Later we will use a variant on Grothendieck's method, requiring only the more elementary case of Hom-schemes as in Lemma 2.1.3 and establishing the existence and quasi-affineness of the quotient schemes $G/N_G(H)$ and $G/Z_G(H)$ for smooth G via an alternative procedure. The reason that we can succeed in this way is that we will appeal to general theorems from the theory of algebraic spaces to understand representability and geometric properties of quotient sheaves.

Returning to the proof of Proposition 2.1.2, by Lemma 2.1.3 the Hom-functors $\underline{\mathrm{Hom}}(Y, G)$ and $\underline{\mathrm{Hom}}(Y, Y')$ classifying scheme homomorphisms (over varying S-schemes) are represented by S-affine S-schemes of finite type. There is a natural G-action on $\underline{\mathrm{Hom}}(Y, G)$ via G-conjugation on G, so the S-point $j \in \underline{\mathrm{Hom}}(Y, G)(S) = \mathrm{Hom}(Y, G)$ corresponding to the given inclusion yields a G-orbit map

$$G \to \underline{\mathrm{Hom}}(Y, G) \tag{2.1.1}$$

over S defined by $g \mapsto (y \mapsto gj(y)g^{-1})$. Consider the pullback of the natural monomorphism

$$\underline{\mathrm{Hom}}(Y, Y') \to \underline{\mathrm{Hom}}(Y, G) \tag{2.1.2}$$

(defined by composition with the inclusion $Y' \hookrightarrow G$) under the map (2.1.1). This pullback recovers the subfunctor $\underline{\mathrm{Transp}}_G(Y, Y')$; i.e., we have cartesian diagram of functors

$$\begin{array}{ccc} \underline{\mathrm{Transp}}_G(Y, Y') & \longrightarrow & \underline{\mathrm{Hom}}_S(Y, Y') \\ \downarrow & & \downarrow \\ G & \longrightarrow & \underline{\mathrm{Hom}}_S(Y, G) \end{array}$$

This establishes the representability of the transporter functor, and to prove that it is a closed subscheme of G it suffices to prove that (2.1.2) is a closed immersion. Exactly as in the construction of these Hom-schemes in the proof of Lemma 2.1.3, the condition on an S′-scheme morphism $Y_{S'} \to G_{S'}$ (for an S-scheme S′) that it factors through the closed subscheme $Y'_{S'}$ is represented by an additional system of universal Zariski-closed conditions arising from generators of the ideal of Y′ in G (Zariski-locally over S). This establishes the required closed immersion property. □

See Exercise 2.4.4 for further discussion of constructions of transporters and normalizers. As an application of Proposition 2.1.2, we have a mild refinement of [**SGA3**, XI, 5.4bis]:

Corollary 2.1.5. — *Let* G *be a smooth* S*-affine* S*-group, and* H *and* H′ *a pair of subgroups of multiplicative type. For any* $s \in S$*, if there exists an extension field* $K/k(s)$ *such that* $(H_s)_K$ *is* $G_s(K)$*-conjugate to* $(H'_s)_K$ *then there exists an étale neighborhood* $U \to S$ *of* s *such that* H_U *is* $G(U)$*-conjugate to* H'_U.

Proof. — The hypothesis is that the smooth map $\mathrm{Transp}_{\mathrm{G}}(\mathrm{H}, \mathrm{H}') \to \mathrm{S}$ hits a K-point over s, so its open image V contains s. Any surjective smooth map of schemes admits sections étale-locally on the base. Applying this to $\mathrm{Transp}_{\mathrm{G}}(\mathrm{H}, \mathrm{H}') \twoheadrightarrow \mathrm{V}$ provides an étale neighborhood (U, u) of (S, s) such that $\mathrm{Transp}_{\mathrm{G}}(\mathrm{H}, \mathrm{H}')(\mathrm{U})$ is non-empty. This U does the job. Indeed, for $g \in \mathrm{G}(\mathrm{U})$ that conjugates H_{U} into H'_{U}, the inclusion $g\mathrm{H}_{\mathrm{U}}g^{-1} \subset \mathrm{H}'_{\mathrm{U}}$ is an equality on u-fibers since H_s and H'_s are abstractly isomorphic, and any containment between multiplicative type groups that is an equality on geometric fibers at one point is an equality over an open neighborhood (as we see by passing to an fppf or étale covering that splits both groups). □

In our later study of parabolic subgroups P of reductive group schemes G, it will be important to establish that P is self-normalizing, which is to say that P represents $\underline{\mathrm{N}}_{\mathrm{G}}(\mathrm{P})$ (a scheme-theoretic improvement on the result $\mathrm{N}_{\mathrm{G}(k)}(\mathrm{P}) = \mathrm{P}(k)$ in the classical theory over an algebraically closed field k). The proof of this property will rest on knowing a priori that $\underline{\mathrm{N}}_{\mathrm{G}}(\mathrm{P})$ is represented by *some* finitely presented closed subscheme of G, so we need a variant on Proposition 2.1.2 that is applicable to smooth closed subschemes Y of G such that all fibers Y_s are geometrically connected (e.g., $\mathrm{Y} = \mathrm{P}$):

Proposition 2.1.6. — *Let* G *be a smooth* S*-affine* S*-group and* $\mathrm{Y}, \mathrm{Y}' \subset \mathrm{G}$ *finitely presented closed subschemes such that* Y *is smooth with non-empty and geometrically connected fibers over* S*. The transporter functor* $\underline{\mathrm{Transp}}_{\mathrm{G}}(\mathrm{Y}, \mathrm{Y}')$ *is represented by a finitely presented closed subscheme of* G*. In particular, the normalizer functor*

$$\underline{\mathrm{N}}_{\mathrm{G}}(\mathrm{Y}) : \mathrm{S}' \rightsquigarrow \{g \in \mathrm{G}(\mathrm{S}') \mid g\mathrm{Y}_{\mathrm{S}'}g^{-1} = \mathrm{Y}_{\mathrm{S}'}\}$$

is represented by a finitely presented closed subgroup of G*.*

The main idea in the proof of this result is to reduce to the finite flat case treated in Proposition 2.1.2. The role of relative schematic density of the torsion-level subgroups in the proof of Proposition 2.1.2 will be replaced by an alternative notion of "relative density" applicable to the collection of infinitesimal neighborhoods of a section $y : \mathrm{S} \to \mathrm{Y}$ (which exists étale-locally on the base). This is easiest to understand in the classical setting. Before explaining this case, we introduce some convenient terminology:

Definition 2.1.7. — For a scheme S, a collection $\{\mathrm{S}_\alpha\}$ of closed subschemes of S is *weakly schematically dense* if the only closed subscheme $\mathrm{Z} \subset \mathrm{S}$ containing every S_α is $\mathrm{Z} = \mathrm{S}$.

This notion is *not* Zariski-local on S, as the following example illustrates.

Example 2.1.8. — Let Y be a smooth connected non-empty scheme over an algebraically closed field k (so Y is irreducible and reduced). Pick $y \in \mathrm{Y}(k)$, and let Y_n denote the nth infinitesimal neighborhood of y; i.e., Y_n is the infinitesimal closed subscheme defined by the vanishing of $\mathscr{I}_y^{n+1}$, where $\mathscr{I}_y$ is the ideal of y in $\mathscr{O}_Y$. We claim that the collection $\{Y_n\}$ is weakly schematically dense in Y. (This collection is supported at a single point, so it is rarely schematically dense in the sense of [**EGA**, IV_3, 11.10.2].) Let Z be a closed subscheme of Y containing every Y_n. If $\mathscr{J}$ is the ideal of Z in Y then the stalk $\mathscr{J}_y$ in the local ring $\mathscr{O}_{Y,y}$ vanishes in the completion $\mathscr{O}_{Y,y}^{\wedge}$, so $\mathscr{J}_y = 0$. But Y is integral, so $\mathscr{J} = 0$ as desired.

Example 2.1.9. — If $H \to S$ be a group scheme of multiplicative type then $\{H[n]\}_{n\geqslant 1}$ is weakly schematically dense in H. Indeed, we may pass to an étale cover of S so that $S = \mathrm{Spec}(A)$ is affine and $H = \mathbf{G}_m^N \times \prod \mu_{d_i}$ for some $\{d_1, \ldots, d_r\}$. Thus, it suffices to observe the elementary fact that if an element

$$b \in A[T_1^{\pm 1}, \ldots, T_N^{\pm 1}, X_1, \ldots, X_r]/(X_1^{d_1} - 1, \ldots, X_r^{d_r} - 1)$$

vanishes modulo $(T_j^n - 1)_j$ for all n divisible by the d_i's then $b = 0$.

Now we turn to the proof of Proposition 2.1.6.

Proof. — Since étale descent is effective for closed subschemes, and the functors in question are sheaves for the étale topology, the problem is étale-local on S. Thus, we may assume there exists $y \in Y(S)$. For each $n \geqslant 0$, let Y_n denote the nth infinitesimal neighborhood of y in Y; this is the closed subscheme defined by the $(n+1)$th-power of the ideal of the closed immersion $y : S \to Y$. The proof of Proposition 2.1.2 will be adapted by using these infinitesimal neighborhoods in the role of the n-torsion subgroups in that earlier proof.

By direct limit arguments we may assume that S is noetherian (see [**EGA**, IV_3, 8.3.3, 9.7.9; IV_4, 17.8.7]). We claim that each Y_n is finite flat over S, so its formation commutes with any base change, and that the collection $\{Y_n\}$ is weakly schematically dense in Y (in the sense of Definition 2.1.7) and remains so after any base change. Once this is established, the resulting equality

$$\underline{\mathrm{Transp}}_G(Y, Y') = \bigcap_{n\geqslant 0} \underline{\mathrm{Transp}}_G(Y_n, Y')$$

as subfunctors of G and the representability of $\underline{\mathrm{Transp}}_G(Y_n, Y')$ will complete the proof.

To prove that Y_n is flat over S, we can reduce to the case $S = \mathrm{Spec}\, A$ for an artin local ring A (by the local flatness criterion; see [**Mat**, 22.3(1),(5)]). Now y has a single physical point y_0, and by A-smoothness of Y near y the completion of $\mathscr{O}_{Y,y_0}$ along y is identified with $A[\![t_1, \ldots, t_d]\!]$ carrying the ideal of y over to the ideal $(t_1, \ldots, t_d)$. (This is seen by using an étale map $f : (Y, y) \to (\mathbf{A}_S^d, 0)$.) Thus, $Y_n \simeq A[\![t_1, \ldots, t_d]\!]/(t_1, \ldots, t_d)^n$, which is visibly S-flat.

Returning to a general noetherian S, Y_n is S-finite since $(Y_n)_{\rm red} = S_{\rm red}$ and Y_n is finite type over S. It remains to show $\{Y_n\}$ is weakly schematically dense in Y and remains so after any base change. By localizing we may assume S is local with closed point s_0. Let U be an affine open neighborhood of $y(s_0)$ in Y, so $y \in U(S)$ inside Y(S). Since $Y \to S$ is smooth surjective with geometrically connected fibers, $U \to S$ is fiberwise dense and hence $U \to Y$ is relatively schematically dense over S [**EGA**, IV_3, 11.10.10]. Thus, a closed subscheme of Y containing U coincides with Y, and likewise after base change on S, so we may replace Y with U to reduce to the case that Y is affine.

It now suffices to show that if A is a noetherian local ring and B is a smooth A-algebra such that $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective with geometrically connected fibers and there is an A-algebra section $s : B \to A$ then for any $\{t_1, \ldots, t_n\}$ in $J = \ker(s)$ lifting an A-basis of J/J^2 and any local homomorphism $A \to A'$, the natural map $h : B \otimes_A A' \to A'[\![T_1, \ldots, T_n]\!]$ to the $J \otimes_A A'$-adic completion is injective. Writing A' as a direct limit of noetherian local A-subalgebras A'_i with local inclusion $A'_i \hookrightarrow A'$, we reduce to the case that A' is noetherian and so we may assume $A' = A$. We may also assume A is complete, so h is identified with the natural map from B to its completion at the closed point of the section. Hence, any $b \in \ker(h)$ vanishes in the local ring of B at that closed point, so b vanishes on an open neighborhood of the section in Spec(B). The schematic density argument as above then implies that $b = 0$. □

Example 2.1.10. — In contrast with the smoothness of normalizers in Proposition 2.1.2 (and centralizers in § 2.2) for multiplicative type subgroups in smooth affine groups, the normalizers in Proposition 2.1.6 can fail to be flat when G is smooth and Y is a smooth subgroup with connected fibers. In particular, normality on a fiber does not imply normality on nearby fibers, even when working with smooth groups. More specifically, a family of non-normal smooth closed subgroups can degenerate to a normal subgroup.

We give an example over $S = \mathbf{A}^1$ (with coordinate t) using the S-group $G = \mathbf{G}_a \times SL_2$. Consider the closed S-subgroup $G' = \mathbf{G}_a$ defined by the closed immersion

$$j : (u, t) \mapsto (u, \left(\begin{smallmatrix} 1 & tu \\ 0 & 1 \end{smallmatrix}\right), t)$$

over the t-line S. The fiber map over $t = 0$ is the inclusion of $\mathbf{G}_a$ into the first factor of $\mathbf{G}_a \times SL_2$, but for $t \neq 0$ the fiber map j_t has non-normal image. Hence, the normalizer $N_G(G')$ has fiber G_0 over $t = 0$ but its fiber over any $t \neq 0$ has strictly smaller dimension. It follows that $N_G(G')$ is not S-flat.

2.2. Centralizer schemes. — In the classical theory of linear algebraic groups over an algebraically closed field k, the centralizer of a smooth closed subscheme Y of a smooth affine group G is defined by brute force: it is the reduced (hence smooth) Zariski-closed subgroup structure on $\bigcap_{y\in \mathrm{Y}(k)} \mathrm{Z}_{\mathrm{G}}(y)$. This definition makes it unclear what the Lie algebra is when Y is a subgroup of G. The scheme-theoretic approach defines the centralizer in a more functorial way that makes it easy to identify the Lie algebra, but shifts the burden of work to proving that (in favorable circumstances) this subgroup is actually *smooth*.

Definition 2.2.1. — Let $\mathrm{G} \to \mathrm{S}$ be a group scheme and Y a closed subscheme. The *functorial centralizer* $\underline{\mathrm{Z}}_{\mathrm{G}}(\mathrm{Y})$ on S-schemes assigns to any S-scheme S' the set of $g \in \mathrm{G}(\mathrm{S}')$ such that g centralizes $\mathrm{Y}_{\mathrm{S}'}$ inside $\mathrm{G}_{\mathrm{S}'}$.

A closed subgroup of G representing $\underline{\mathrm{Z}}_{\mathrm{G}}(\mathrm{Y})$, if one exists, is denoted $\mathrm{Z}_{\mathrm{G}}(\mathrm{Y})$ and is called the *centralizer* of Y in G. In the special case $\mathrm{Y} = \mathrm{G}$, such a subgroup scheme is called the *center* of G (if it exists!) and is denoted Z_{G}.

The general existence of centralizers is delicate, but in special cases there are affirmative results. One favorable situation is when $\mathrm{S} = \mathrm{Spec}(k)$ for a field k, in which case $\mathrm{Z}_{\mathrm{G}}(\mathrm{Y})$ exists for any Y; see Exercise 2.4.4. The case of smooth Y with connected fibers is [**SGA3**, XI, 6.11] (subject to some mild hypotheses on Y and G), but for later purposes over a general base S we must allow any Y of multiplicative type (so Y may not be S-smooth and may have disconnected fibers). Such cases are handled in Lemma 2.2.4 below, where we also show that the centralizer of Y is smooth when G is smooth (even if Y is not smooth!).

Definition 2.2.2. — A subgroup scheme G' in G is *central* when G-conjugation on G is trivial on G' (equivalently, $\underline{\mathrm{Z}}_{\mathrm{G}}(\mathrm{G}') = \mathrm{G}$).

Another way to express the centrality condition is that for every S-scheme S' and $g' \in \mathrm{G}'(\mathrm{S}')$, the g'-conjugation action on $\mathrm{G}_{\mathrm{S}'}$ is trivial.

Example 2.2.3. — If G is a *smooth* finite type group over an algebraically closed field k, a closed subgroup scheme G' is central if G' is centralized by all elements of $\mathrm{G}(k)$; see Exercise 2.4.4(iv). This is false when G is not assumed to be smooth, even when G' is smooth. For example, in characteristic $p > 0$ the usual semi-direct product $\mathrm{G} = \mathbf{G}_{\mathrm{m}} \ltimes \alpha_p$ and its smooth closed subgroup $\mathrm{G}' = \mathbf{G}_{\mathrm{m}}$ have the same geometric points, but G' is not normal in G.

As another example of a central subgroup scheme, for any scheme S the diagonal μ_n in the S-group SL_n is central for any $n > 1$. See Example 3.3.7 for a stronger centrality property of μ_n in SL_n.

Lemma 2.2.4. — *Let* $\mathrm{G} \to \mathrm{S}$ *be a finitely presented* S-*affine* S-*group, and* Y *a finitely presented closed subscheme of* G. *Assume that* Y *is a subgroup of multiplicative type or that* $\mathrm{Y} \to \mathrm{S}$ *either is smooth with each fiber non-empty and connected* (*hence geometrically connected*) *or is finite flat. Then* $\mathrm{Z}_{\mathrm{G}}(\mathrm{Y})$

exists as a finitely presented closed subgroup; it is smooth when G *is smooth and* Y *is a subgroup of multiplicative type, in which case* $\mathrm{Lie}(\mathrm{Z_G}(\mathrm{Y})) = \mathrm{Lie}(\mathrm{G})^{\mathrm{Y}}$ *and this Lie algebra represents the functor of* Y*-invariants under* $\mathrm{Ad_G}$*.*

For G smooth and Y a subgroup of multiplicative type, this lemma is part of [**SGA3**, XI, 5.3] (aside from the description of $\mathrm{Lie}(\mathrm{Z_G}(\mathrm{Y}))$ in such cases).

Proof. — We may assume S is noetherian, and for the representability assertions it suffices to restrict to functors on the category of noetherian S-schemes (since for affine $\mathrm{S} = \mathrm{Spec}(\mathrm{A})$, the functor $\underline{\mathrm{Z}}_{\mathrm{G}}(\mathrm{Y})$ on A-algebras commutes with the formation of direct limits).

First we treat the existence when Y is finite flat over S. In this case Lemma 2.1.3 provides an S-scheme $\underline{\mathrm{Hom}}(\mathrm{Y}, \mathrm{G})$ that is affine of finite type over S. The given inclusion of Y into G corresponds to an S-point j of $\underline{\mathrm{Hom}}(\mathrm{Y}, \mathrm{G})$, and $j : \mathrm{S} \to \underline{\mathrm{Hom}}(\mathrm{Y}, \mathrm{G})$ is a closed immersion since it is a section to a separated map. The pullback of the morphism j under the orbit map $\mathrm{G} \to \underline{\mathrm{Hom}}(\mathrm{Y}, \mathrm{G})$ through j (via the G-action on $\underline{\mathrm{Hom}}(\mathrm{Y}, \mathrm{G})$ through conjugation) is a closed subscheme of G representing $\underline{\mathrm{Z}}_{\mathrm{G}}(\mathrm{Y})$.

Suppose Y is a subgroup H of multiplicative type. Since $\{\mathrm{H}[n]\}$ is weakly schematically dense in H in the sense of Definition 2.1.7 and remains so after any base change on S (Example 2.1.9), $\underline{\mathrm{Z}}_{\mathrm{G}}(\mathrm{H}) = \bigcap_{n>0} \underline{\mathrm{Z}}_{\mathrm{G}}(\mathrm{H}[n])$ as subfunctors of G since the condition of equality for two maps $\mathrm{H} \rightrightarrows \mathrm{G}$ can be expressed using the closed relative diagonal of G over S. Each $\mathrm{H}[n]$ is finite flat, and $\bigcap_{n>0} \mathrm{Z_G}(\mathrm{H}[n])$ represents $\underline{\mathrm{Z}}_{\mathrm{G}}(\mathrm{H})$. Likewise, if Y is a smooth closed subscheme with geometrically connected non-empty fibers then by working étale-locally on S we can assume that $\mathrm{Y} \to \mathrm{S}$ admits a section y. Then we can argue exactly as in the multiplicative type case by using the finite flat infinitesimal neighborhoods Y_n of y in the role of the torsion-levels $\mathrm{H}[n]$ (due to the weak schematic density of $\{\mathrm{Y}_n\}$ in Y that persists after any base change, as established in the proof of Proposition 2.1.6).

Finally, for smooth G and general H of multiplicative type we prove that $\mathrm{Z_G}(\mathrm{H})$ is smooth and compute its Lie algebra inside $\mathfrak{g} = \mathrm{Lie}(\mathrm{G})$. By the functorial smoothness criterion and the generality of the base scheme, to prove smoothness it suffices to show that if S is affine and S_0 is a closed subscheme of S defined by a square-zero quasi-coherent ideal then any $g_0 \in \mathrm{G}_0(\mathrm{S}_0)$ centralizing H_0 lifts to some $g \in \mathrm{G}(\mathrm{S})$ that centralizes H. By smoothness of G we can pick some $g \in \mathrm{G}(\mathrm{S})$ lifting g_0, so $g\mathrm{H}g^{-1}$ and H are multiplicative type subgroups of G with the same reduction in G_0. As in the proof of Proposition 2.1.2, by Corollary B.2.6 and [**Oes**, III, 3.3] there exists $g' \in \mathrm{G}(\mathrm{S})$ lifting $1 \in \mathrm{G}_0(\mathrm{S}_0)$ such that g' conjugates $g\mathrm{H}g^{-1}$ to H, so $g'g$ normalizes H. But over S_0 this conjugation endomorphism of H_0 is conjugation by $(g'g)_0 = g_0$, which is the trivial action on H_0. Since H is multiplicative type and S_0 is defined by a nilpotent ideal on S, it follows (by Corollary B.2.7) that $g'g$ must centralize H, so $g'g \in Z_G(H)(S)$ and this lifts $g_0 \in Z_G(H)(S_0)$.

Since $Z_G(H)$ is S-smooth, $\mathrm{Lie}(Z_G(H))$ is a subbundle of $\mathfrak{g}$ whose formation commutes with base change. Likewise, by fppf descent from the case of split $H = D_S(M) := \mathrm{Spec}(\mathscr{O}_S[M])$ (for which a linear representation of H on a vector bundle corresponds to an M-grading, by [**Oes**, III, 1.5] or [**CGP**, Lemma A.8.8]), there is a subbundle $\mathfrak{g}^H$ of $\mathfrak{g}$ representing the functor of H-invariants under Ad_G and its formation also commutes with any base change on S. There is an evident inclusion $\mathrm{Lie}(Z_G(H)) \subset \mathfrak{g}^H$ as subbundles of $\mathfrak{g}$, so to prove it is an equality it suffices to check on geometric fibers over S. Thus, we may assume $S = \mathrm{Spec}\, k$ for an algebraically closed field k, in which case the equality $\mathrm{Lie}(Z_G(H)) = \mathfrak{g}^H$ is shown in the proof of Proposition 1.2.3 (where that part of the proof was specifically written to only use the existence of $Z_G(H)$ as a closed subgroup of G and not any smoothness hypothesis on H). □

Remark 2.2.5. — An important special case of Lemma 2.2.4 is $Y = G$, for which this lemma asserts the existence of the scheme-theoretic center Z_G of a smooth S-affine S-group G such that all $G_{\overline{s}}$ are connected. (See [**SGA3**, XI, 6.11] for the removal of the fibral connectedness condition, taking $G = H$ there.) In general Z_G can fail to be flat (see the end of [**SGA3**, XVI, § 3] for an example), in which case it is not very useful.

See Exercise 2.4.6(ii) for examples of Lemma 2.2.4 with Z_G of multiplicative type, and Exercises 2.4.4 and 2.4.10 for generalizations of Lemma 2.2.4 over fields. In Theorem 3.3.4 we will show Z_G is of multiplicative type (hence flat) for any reductive group scheme $G \to S$.

Example 2.2.6. — Here are two applications of the smoothness aspect of Lemma 2.2.4 in the context of smooth affine groups over a field k. Let G be such a group, and T a torus in G (not necessarily maximal). The lemma ensures that the centralizer $Z_G(T)$ is always smooth (and it is connected when G is connected, by the classical theory over $\overline{k}$). This is the scheme-theoretic proof of a fact in the classical theory [**Bo91**, 9.2, Cor.]: the reduced structure on the closed subset of $G_{\overline{k}}$ corresponding to the centralizer of $T_{\overline{k}}$ in $G(\overline{k})$ descends to a smooth closed k-subgroup of G (namely $Z_G(T)$). As a special case, if G is connected reductive and T is a (geometrically) maximal torus then $Z_G(T) = T$ because such an equality between smooth closed subgroups can be checked on $\overline{k}$-points, where it follows from the classical theory.

For another application, if G' is a smooth closed k-subgroup of G that is normalized by T (e.g., a normal k-subgroup of G) then the scheme-theoretic intersection $Z_G(T) \bigcap G'$ represents the functorial centralizer for the T-action on G'. Although T may not be a k-subgroup of G' inside G, so Lemma 2.2.4 does not literally apply to T acting on G', there is a standard trick with semi-direct products that enables us to apply the lemma anyway to prove $Z_G(T) \bigcap G'$ is smooth: form the semi-direct product $G' \rtimes T$ in which T embeds along the second factor. Lemma 2.2.4 can be applied to this semi-direct product. Thus, $Z_{G' \rtimes T}(T)$ is smooth, and we know it is connected when G' is connected. But clearly

$$Z_{G' \rtimes T}(T) = (Z_G(T) \bigcap G') \rtimes T$$

as k-schemes, and the left side is smooth (and connected when G' is connected). Thus, the direct factor scheme $Z_G(T) \bigcap G'$ is smooth (and connected when G' is). For example, if $N \subset G$ is any smooth closed normal subgroup then $N \bigcap Z_G(T)$ is smooth (and connected when N is connected).

Remark 2.2.7. — Our construction of centralizers and normalizers relied on passage to Y that are finite flat over the base. For an alternative approach in the presence of enough "module-freeness" for some coordinate rings (as algebras over a base ring), see [**SGA3**, VIII, § 6], [**CGP**, A.8.10(1)], and Exercise 2.4.4.

2.3. Some quotient constructions. — As was noted in Remark 2.1.4, for the construction of quotients in the relative setting we will bypass some of Grothendieck's techniques in [**SGA3**] in favor of the theory of algebraic spaces. This is illustrated in the following results, the first of which is a variant on [**SGA3**, XI, 5.3bis].

Theorem 2.3.1. — *Let* $G \to S$ *be a smooth* S*-affine group scheme, and* H *a subgroup of multiplicative type. The quotients* $G/Z_G(H)$ *and* $G/N_G(H)$ *exist as smooth quasi-affine* S*-schemes. Moreover, the quotient* $W_G(H) := N_G(H)/Z_G(H)$ *exists as a separated and étale* S*-group of finite presentation.*

In particular, if H *is normal in* G (*i.e.,* $N_G(H) = G$) *and the fibers* G_s *are connected* (*so* $W_G(H)_s = W_{G_s}(H_s) = 1$ *for all* $s \in S$ *and hence* $W_G(H) = 1$) *then* H *is central in* G.

The final centrality assertion admits a more elementary proof; see the self-contained Lemma 3.3.1(1). Before we prove Theorem 2.3.1, we briefly digress to make some remarks.

Remark 2.3.2. — As in [**EGA**, II, 5.1.1], a map of schemes $f : X \to Y$ is *quasi-affine* if, over the constituents of some affine open cover of Y, it factors as a quasi-compact open immersion into an affine scheme (so f is quasi-compact and separated). In [**EGA**, II, 5.1.2, 5.1.6] there are several equivalent versions of this definition. A more "practical" description is provided in [**EGA**, II, 5.1.9] when $f : X \to Y$ is finite type and Y is noetherian (or more generally, when Y is quasi-compact and quasi-separated): such an f is quasi-affine if and only if f factors as a quasi-compact open immersion followed by an affine map $Y' \to Y$ *of finite type.*

Since fppf descent is always effective for schemes that are quasi-affine over the base, the quasi-affineness in Theorem 2.3.1 is useful; e.g., it will underlie our later construction of the "scheme of maximal tori" in a reductive group scheme (even when there is no torus over the given base scheme that is maximal on all geometric fibers).

Remark 2.3.3. — We will build the quotients in Theorem 2.3.1 via an orbit argument, as is also done when $\mathrm{S} = \mathrm{Spec}(k)$ for a field k. However, unlike the case over a field, in the relative setting we do *not* have a plentiful supply of linear representations. In fact, we do not even know if every smooth affine group over the dual numbers $k[\epsilon]$ is a closed subgroup scheme of some GL_n! (See [**SGA3**, $\mathrm{VI_B}$, 13.2, 13.5] and [**SGA3**, XI, 4.3] for further discussion in this direction.) Thus, the construction of quotients G/H modulo flat and finitely presented closed subgroups H is rather subtle when the base is not a field.

Generally such quotients G/H are algebraic spaces. In some cases we will prove that G/H is a scheme by using the following modification of the classical orbit argument. We will identify the quotient sheaf G/H as a G-equivariant subfunctor of a scheme X on which G acts, but we have no general analogue in the relative setting of the result in the classical case that G-orbits are always smooth and locally closed when G is smooth. To show that the algebraic space G/H is a scheme, we will use a general result of Knutson which only requires that the subfunctor inclusion $j : \mathrm{G/H} \hookrightarrow \mathrm{X}$ into a scheme is quasi-finite and separated (with X noetherian). In the situations that arise in the proof of Theorem 2.3.1 one can prove (see Remark 2.3.5) that the morphism j is étale (hence an open immersion, so *a fortiori* G/H is locally closed in X), but this fact is not used in our proof of Theorem 2.3.1.

Now we turn to the proof of Theorem 2.3.1.

Proof. — We may and do assume that S is noetherian. The relative schematic density of $\{\mathrm{H}[n]\}$ in H implies that $\mathrm{Z_G(H)} = \bigcap_{n>0} \mathrm{Z_G(H}[n])$ and $\mathrm{N_G(H)} = \bigcap_{n>0} \mathrm{N_G(H}[n])$ as closed subschemes of G. But the noetherian condition on G implies that any descending chain of closed subschemes of G stabilizes, so $\mathrm{Z_G(H)} = \mathrm{Z_G(H}[n])$ and $\mathrm{N_G(H)} = \mathrm{N_G(H}[n])$ for sufficiently large n. Hence, it suffices to treat each $\mathrm{H}[n]$ in place of H, so we may assume that H is S-finite. Now Lemma 2.1.3 provides the scheme $\underline{\mathrm{Hom}}_{\text{S-gp}}(\mathrm{H},\mathrm{G})$ that is S-affine of finite type. By Proposition 2.1.2 and Lemma 2.2.4, $\mathrm{N_G(H)}$ and $\mathrm{Z_G(H)}$ are S-smooth.

Consider the natural G-action on this Hom-scheme via composition with the conjugation action of G on itself, and the S-point corresponding to the given inclusion $j : \mathrm{H} \to \mathrm{G}$. The orbit map $\mathrm{G} \to \underline{\mathrm{Hom}}_{\text{S-gp}}(\mathrm{H},\mathrm{G})$ through j is right-invariant by the stabilizer scheme $\mathrm{Z_G(H)}$ of the S-point j, so the quotient sheaf $\mathrm{G/Z_G(H)}$ for the fppf (or étale) topology is naturally a subfunctor of the Hom-scheme. By a general theorem of Artin [**Ar74**, Cor. 6.3], for any finite type S-scheme X and equivalence relation R on X that is represented by a closed subscheme in $X \times_S X$ and for which both projections $R \rightrightarrows X$ are flat, the fppf quotient sheaf X/R is a separated algebraic space of finite type over S. Thus, $G/Z_G(H)$ is such an algebraic space. The monomorphism $G/Z_G(H) \to \underline{\mathrm{Hom}}_{S\text{-gp}}(H, G)$ over S must be separated and finite type with finite fibers, so $G/Z_G(H)$ is separated and quasi-finite over a scheme.

By a result of Knutson [**Knut**, II, 6.15], an algebraic space that is quasi-finite and separated over a noetherian scheme is a scheme. (See [**LMB**, Thm. A.2] for a generalization without noetherian hypotheses.) Hence, $G/Z_G(H)$ is a scheme that is moreover quasi-finite and separated over the scheme $\underline{\mathrm{Hom}}_{S\text{-gp}}(H, G)$. By Zariski's Main Theorem [**EGA**, IV_3, 8.12.6], any quasi-finite and separated map between noetherian schemes is quasi-affine, so $G/Z_G(H)$ is quasi-affine over the S-affine $\underline{\mathrm{Hom}}_{S\text{-gp}}(H, G)$ and hence it is quasi-affine over S. The quotient $G/Z_G(H)$ is S-smooth, since $G \to G/Z_G(H)$ is an fppf cover by the smooth G. (In Remark 2.3.5 we show the orbit map $G/Z_G(H) \to \underline{\mathrm{Hom}}_{S\text{-gp}}(H, G)$ is an open immersion.)

By the same reasoning, the quotient sheaves $G/N_G(H)$ and $W_G(H) = N_G(H)/Z_G(H)$ are smooth and separated algebraic spaces of finite type over S. The map $W_G(H) \to G/Z_G(H)$ is a closed immersion, via fppf descent of the closed immersion property for its pullback $N_G(H) \to G$ along the smooth covering $G \to G/Z_G(H)$, so $W_G(H)$ is a scheme as well. To prove it is S-étale we may pass to geometric fibers over S. With $S = \mathrm{Spec}(k)$ for an algebraically closed field k, the automorphism functor of H is represented by a disjoint union $\mathrm{Aut}_{H/k}$ of rational points (opposite to the automorphism functor of the constant dual of H) and $N_G(H)/Z_G(H)$ is a finite type k-group equipped with a monic homomorphism to the étale k-group $\mathrm{Aut}_{H/k}$. This forces the smooth $N_G(H)/Z_G(H)$ to be *finite*, hence étale.

Exhibiting $G/N_G(H)$ as quasi-finite and separated over a quasi-affine S-scheme requires a new idea. We shall exhibit it as a subfunctor of a scheme that is finite type and quasi-affine over S. For this, we replace the scheme $\underline{\mathrm{Hom}}_{S\text{-gp}}(H, G)$ classifying *homomorphisms* of H into G with the scheme classifying *closed subgroup schemes* of G that are "twists" of H:

Lemma 2.3.4. — *Let* $G \to S$ *be a smooth* S*-affine group scheme, and* $H \subset G$ *a subgroup of multiplicative type with finite fibers. There is a quasi-affine* S*-scheme* $\mathrm{Twist}_{H/G}$ *of finite presentation that represents the functor* $\underline{\mathrm{Twist}}_{H/G}$ *assigning to any* S*-scheme* S' *the set of multiplicative type subgroups* $H' \subset G_{S'}$ *such that* $H'_{\overline{s}'} \simeq H_{\overline{s}'}$ *for all geometric points* $\overline{s}'$ *of* S'.

This lemma is a special case of the deeper result [**SGA3**, XI, 4.1] that the functor classifying *all* multiplicative type subgroups of G is represented by a smooth and separated S-scheme (without requiring that G contains any such nontrivial subgroups over S). Whereas Lemma 2.3.4 will be deduced from Proposition 2.1.3, which is a special case of [**SGA3**, XI, 4.2], in [**SGA3**] the logic goes the other way: [**SGA3**, XI, 4.2] is deduced from [**SGA3**, XI, 4.1].

Proof. — We may and do assume S is noetherian. Since étale descent is effective for schemes that are quasi-affine over the base, to construct the quasi-affine $\mathrm{Twist}_{\mathrm{H/G}} \to \mathrm{S}$ representing $\underline{\mathrm{Twist}}_{\mathrm{H/G}}$ we may work étale-locally on S so that H has *constant* Cartier dual M. Thus, $\underline{\mathrm{Aut}}_{\text{S-gp}}(\mathrm{H}) = \Gamma_{\mathrm{S}}$ for the ordinary finite group $\Gamma = \mathrm{Aut}(\mathrm{M}^{\vee})$, where $\mathrm{M}^{\vee} := \mathrm{Hom}(\mathrm{M}, \mathbf{Q}/\mathbf{Z})$.

Inside $\underline{\mathrm{Hom}}_{\text{S-gp}}(\mathrm{H},\mathrm{G})$, the monicity condition on S′-homomorphisms $\mathrm{H}_{\mathrm{S}'} \to \mathrm{G}_{\mathrm{S}'}$ is represented by an open subscheme V. Indeed, if B is noetherian and $f : \mathrm{K} \to \mathrm{B}$ is a finite group scheme with $\mathrm{K}_b = 1$ for some $b \in \mathrm{B}$ then $\mathrm{K}|_{\mathrm{U}} = 1$ for some open $\mathrm{U} \subset \mathrm{B}$ around b (by applying Nakayama's Lemma to the stalks of the ideal sheaf $\ker(e^* : f_*(\mathscr{O}_{\mathrm{K}}) \to \mathscr{O}_{\mathrm{B}})$ of the identity section over B). Applying this to the kernel of the universal homomorphism over $\mathrm{B} := \underline{\mathrm{Hom}}_{\text{S-gp}}(\mathrm{H},\mathrm{G})$ gives the open V. The natural right action of Γ_{S} on the finite type S-affine scheme $\underline{\mathrm{Hom}}_{\text{S-gp}}(\mathrm{H},\mathrm{G})$ via $\gamma.f = f \circ \gamma$ leaves V stable and is free on V, so by [**SGA3**, V, Thm. 4.1(iv)] there exists a finite étale quotient map

$$\mathrm{V} \to \mathrm{Q} := \mathrm{V}/\Gamma_{\mathrm{S}}.$$

This quotient is constructed via Γ-invariants over open affines in S; it is quasi-affine over S by [**EGA**, II, 6.6.1, 5.1.6(c′)] (applied to the finite étale cover V that is quasi-affine over S).

It remains to show $\mathrm{V}/\Gamma_{\mathrm{S}}$ represents $\underline{\mathrm{Twist}}_{\mathrm{H/G}}$. For the evident Γ_{S}-invariant map $\mathrm{V} \to \underline{\mathrm{Twist}}_{\mathrm{H/G}}$, the induced map $\mathrm{V}/\Gamma_{\mathrm{S}} \to \underline{\mathrm{Twist}}_{\mathrm{H/G}}$ is a monomorphism because if $j, j' : \mathrm{H}_{\mathrm{S}'} \rightrightarrows \mathrm{G}_{\mathrm{S}'}$ are S′-subgroup inclusions whose images agree as closed subschemes then $j' = j \circ \gamma$ for some $\gamma \in \mathrm{Aut}_{\mathrm{S}'}(\mathrm{H}_{\mathrm{S}'}) = \Gamma_{\mathrm{S}}(\mathrm{S}')$.

It remains to prove that the map $\mathrm{V} \to \underline{\mathrm{Twist}}_{\mathrm{H/G}}$ between sheaves for the étale topology is a surjection (thereby forcing the inclusion $\mathrm{V}/\Gamma_{\mathrm{S}} \hookrightarrow \underline{\mathrm{Twist}}_{\mathrm{H/G}}$ to be an equality). Pick an S-scheme S′ and an S′-subgroup H′ of $\mathrm{G}_{\mathrm{S}'}$ such that $\mathrm{H}'_{\overline{s}'} \simeq \mathrm{H}_{\overline{s}'}$ for all geometric points $\overline{s}'$ of S′. In particular, H′ has finite fibers. It suffices to find an étale cover S″ of S′ over which the pullbacks of H′ and H become isomorphic as group schemes. Passing to an étale cover brings us to the trivial case that H′ and H each have constant Cartier dual. □

To finish the proof of Theorem 2.3.1, note that the conjugation action of G on itself defines a left action of G on $\mathrm{Twist}_{\mathrm{H/G}}$, and $\mathrm{N}_{\mathrm{G}}(\mathrm{H})$ is the stabilizer of the S-point of $\mathrm{Twist}_{\mathrm{H/G}}$ corresponding to the given copy of H in G. Thus, the separated and finite type algebraic space $\mathrm{G}/\mathrm{N}_{\mathrm{G}}(\mathrm{H})$ over S is a subfunctor of $\mathrm{Twist}_{\mathrm{H/G}}$, so applying Knutson's schematic criterion [**Knut**, II, 6.15] proves that $\mathrm{G}/\mathrm{N}_{\mathrm{G}}(\mathrm{H})$ is a scheme that is separated and quasi-finite over the S-scheme $\mathrm{Twist}_{\mathrm{H/G}}$ that we know is quasi-affine over S. Applying Zariski's

Main Theorem, the monomorphism $G/N_G(H) \hookrightarrow \mathrm{Twist}_{H/G}$ must be quasi-affine (in Remark 2.3.5 we show it is an open immersion). Hence, $G/N_G(H)$ is quasi-affine over S since $\mathrm{Twist}_{H/G}$ is. $\square$

Remark 2.3.5. — A common difficulty with algebraic spaces is the intervention of monomorphisms $j : X \to Y$ such that it is not obvious if j is a (locally closed) immersion. In the proof of Theorem 2.3.1 we encountered two such maps, namely the orbit maps $j : G/Z_G(H) \to \underline{\mathrm{Hom}}_{\text{S-gp}}(H, G)$ and $j' : G/N_G(H) \to \mathrm{Twist}_{H/G}$ for H a *finite* S-group of multiplicative type (with noetherian S). Ignorance of the immersion property for these maps is irrelevant for our purposes. We shall now prove that these monomorphisms are open immersions.

Let $f : X \to Y$ be a map of finite type between noetherian schemes. It is an open immersion if and only if it is an étale monomorphism [**EGA**, IV_4, 17.9.1], or equivalently a smooth monomorphism. We apply this criterion for open immersions to the orbit maps

$$j : G/Z_G(H) \to \underline{\mathrm{Hom}}_{\text{S-gp}}(H, G), \;\; j' : G/N_G(H) \to \mathrm{Twist}_{H/G}$$

with G and H as in Theorem 2.3.1 and H finite over S (a noetherian scheme). These maps are monomorphisms of finite type, and (by [**EGA**, IV_4, 17.14.2, 17.7.1(ii)]) to verify the functorial criterion for étaleness we may assume $S = \mathrm{Spec}\, R$ for an artin local ring $(R, \mathfrak{m})$ with algebraically closed residue field k. Let $R_0 = R/J$ for an ideal $J \subset R$ satisfying $J^2 = 0$. Consider an R-homomorphism $f : H \to G$ (resp. an R-subgroup $H' \subset G$ as in Lemma 2.3.4 with $S' = S$). For any R-algebra A and $g \in G(A)$, let c_g denote conjugation on G_A by g. Assuming that f_0 arises from $(G/Z_G(H))(R_0)$ (resp. H' arises from $(G/N_G(H))(R_0)$), we seek to prove that f arises from $(G/Z_G(H))(R)$ (resp. H' arises from $(G/N_G(H))(R)$).

The quotient maps $G \to G/Z_G(H)$ and $G \to G/N_G(H)$ are smooth since $Z_G(H)$ and $N_G(H)$ are smooth. Any R_0-point lifts through a smooth surjection since the residue field is algebraically closed, so $(G/Z_G(H))(R_0) = G(R_0)/Z_G(H)(R_0)$ and similarly for $G/N_G(H)$.

Consider the case of $G/Z_G(H)$, so $f : H \to G$ is an R-homomorphism that lifts $c_{g_0}|_{H_0}$ for some $g_0 \in G(R_0)$. By R-smoothness of G, g_0 lifts to some $g \in G(R)$, so $c_g|_H : H \to G$ and f are R-homomorphisms with the same reduction. By Corollary B.3.5, we can change the choice of g lifting g_0 if necessary so that $f = c_g|_H$.

The case of $G/N_G(H)$ goes similarly. Indeed, by hypothesis $H'_0 = c_{g'_0}(H_0)$ for some $g'_0 \in G(R_0)$, and we choose $g' \in G(R)$ lifting g'_0, so $c_{g'}(H)$ and H' are multiplicative type subgroups of G that lift H'_0. By the deformation theory of multiplicative type subgroups of smooth affine groups (Corollary B.2.6 and

[**Oes**, III, 3.3]) we can change the choice of g' lifting g'_0 if necessary so that $c_{g'}(\mathrm{H}) = \mathrm{H}'$.

Whereas Theorem 2.3.1 concerns quasi-affine quotients, we know from the classical theory that it is also necessary to consider quotients G/H that turn out to be projective. The criterion we will use to make such quotients as schemes, and not merely as algebraic spaces, is a self-normalizer hypothesis (see Corollary 5.2.8 for an important class of examples):

Theorem 2.3.6. — *Let* $\mathrm{G} \to \mathrm{S}$ *be a smooth* S*-affine group scheme with connected fibers, and* H *a smooth closed subgroup with connected fibers such that* $\mathrm{H} = \mathrm{N}_{\mathrm{G}}(\mathrm{H})$.

1. *The quotient sheaf* G/H *is represented by a smooth* S*-scheme that is quasi-projective Zariski-locally over* S*, and it coincides with the functor* $\underline{\mathrm{Twist}}_{\mathrm{H}/\mathrm{G}}$ *of closed subgroups of* G *that are conjugate to* H *étale-locally on the base.*
2. *Assume that the geometric fibers* $(\mathrm{G}/\mathrm{H})_{\overline{s}} = \mathrm{G}_{\overline{s}}/\mathrm{H}_{\overline{s}}$ *are projective. The morphism* $\mathrm{G}/\mathrm{H} \to \mathrm{S}$ *is proper and admits as a canonical* S*-ample line bundle* $\det(\mathrm{Lie}(\mathscr{H}))^*$ *where the* G/H*-subgroup* $\mathscr{H} \hookrightarrow \mathrm{G} \times (\mathrm{G}/\mathrm{H})$ *is the universal étale-local conjugate of* H *in* G*. In particular,* $\mathrm{G}/\mathrm{H} \to \mathrm{S}$ *is projective Zariski-locally on the base.*

The following preliminary remarks should clarify aspects of Theorem 2.3.6 before we undertake the proof. By Proposition 2.1.6 (and Exercise 1.6.5) the given smoothness and connectedness hypotheses on H imply that the normalizer $\mathrm{N}_{\mathrm{G}}(\mathrm{H})$ does exist a priori. If we do not assume that $\mathrm{N}_{\mathrm{G}}(\mathrm{H}) = \mathrm{H}$ then typically $\mathrm{N}_{\mathrm{G}}(\mathrm{H})$ may not be S-flat (so there would not be a useful notion of quotient $\mathrm{G}/\mathrm{N}_{\mathrm{G}}(\mathrm{H})$); see Example 2.1.10. By Corollary 2.1.5, if H is of multiplicative type then the criterion defining the functor $\underline{\mathrm{Twist}}_{\mathrm{H}/\mathrm{G}}$ in part (1) can be expressed on geometric fibers as in Lemma 2.3.4.

The notion of "S-ample" that we use in part (2) means "ample on fibers"; by [**EGA**, IV_3, 9.6.4], for proper and finitely presented S-schemes this implies the usual notion of ampleness over affine opens in S. Part (2) of Theorem 2.3.6 is [**SGA3**, XXII, 5.8.2], apart from the explicit S-ample line bundle G/H (which is borrowed from the proof of [**SGA3**, XVI, 2.4]).

The existence of a canonical S-ample line bundle on G/H in part (2) will be crucial in our later construction of the "scheme of Borel subgroups" of a reductive group scheme G. The reason is that in general G does not admit a Borel subgroup over S, so it is necessary to pass to an étale cover $\mathrm{S}' \to \mathrm{S}$ in order that there exists a Borel subgroup $\mathrm{B}' \subset \mathrm{G}_{\mathrm{S}'}$. We will then be faced with a descent problem for $\mathrm{G}_{\mathrm{S}'}/\mathrm{B}'$ relative to $\mathrm{S}' \to \mathrm{S}$. The *canonical* S'-ample line bundle on $\mathrm{G}_{\mathrm{S}'}/\mathrm{B}'$ (arising from the Lie algebra of the universal Borel subgroup in the $G_{S'}/B'$-group $G_{S'} \times (G_{S'}/B'))$ will ensure the effectivity of the descent. The fibral projectivity hypothesis in part (2) is a familiar condition in the classical theory.

Remark 2.3.7. — In the classical theory over an algebraically closed field k, it is well-known that the quotient G/P modulo a parabolic subgroup has a canonical ample line bundle: the anti-canonical bundle $\det(\Omega^1_{(G/P)/k})^*$. This is the line bundle in Theorem 2.3.6(2) when $S = \operatorname{Spec} k$, up to a twist against the k-line $\det(\mathrm{Lie}(G))^*$. To explain this link in the relative setting, view $f : X = G/H \to S$ as the moduli scheme classifying closed subgroups of G that are conjugate to H étale-locally on the base. We claim that the S-ample line bundle $\det(\mathrm{Lie}(\mathscr{H}))^*$ arising from the universal closed X-subgroup $j : \mathscr{H} \hookrightarrow G \times_S X$ is canonically isomorphic to $(\det_X(\Omega^1_{X/S}))^* \otimes f^*(\det_S \mathrm{Lie}(G))^*$.

Let $\mathscr{I}_j$ be the ideal defining j, so we obtain (see [**EGA**, IV_4, 17.2.5]) an exact sequence of vector bundles on $\mathscr{H}$:

$$0 \to \mathscr{I}_j/\mathscr{I}_j^2 \to j^*(\Omega^1_{(G \times_S X)/X}) \to \Omega^1_{\mathscr{H}/X} \to 0.$$

Pulling back along the identity section $e : X \to \mathscr{H}$ yields an exact sequence

$$0 \to e^*(\mathscr{I}_j/\mathscr{I}_j^2) \to f^*(\mathrm{Lie}(G)^*) \to \mathrm{Lie}(\mathscr{H})^* \to 0$$

of vector bundles on X. But the left term is identified with $\Omega^1_{X/S}$ due to the cartesian square

$$\begin{tikzcd} \mathscr{H} \arrow[r, "j"] \arrow[d, "\pi"'] & G \times_S X \arrow[d, "a"] \\ X \arrow[r, "\Delta_{X/S}"'] & X \times_S X \end{tikzcd}$$

in which $a(g, x) := (gx, x)$ and π satisfies $\pi \circ e = \mathrm{id}_X$. (Indeed, $\mathscr{I}_j/\mathscr{I}_j^2 \simeq \pi^*(\mathscr{I}_\Delta/\mathscr{I}_\Delta^2) = \pi^*(\Omega^1_{X/S})$, so applying e^* gives the identification.) Thus,

$$\det \Omega^1_{X/S} \otimes \det(\mathrm{Lie}(\mathscr{H}))^* \simeq f^*(\det_S \mathrm{Lie}(G))^*,$$

yielding the asserted description of $\det(\mathrm{Lie}(\mathscr{H}))^*$.

Proof of Theorem 2.3.6. — Since $N_G(H) = H$ and H is smooth (so $G \to G/H$ admits sections étale-locally on G/H), it follows by effective descent for closed subschemes that the quotient sheaf G/H coincides with the functor of smooth closed subgroups of G that are conjugate to H étale-locally on the base. To prove the representability of this functor and its properties as asserted in (1) and (2), we may and do assume that S is noetherian.

For $n \geqslant 0$, let H_n denote the nth infinitesimal neighborhood of the identity in H (i.e., the closed subscheme defined by the $(n+1)$th power of the ideal of the identity section). The construction of $N_G(H)$ in the proof of Proposition 2.1.6

gives a description of $N_G(H)$ as an infinite descending intersection, namely the intersection of the normalizers of the closed subschemes $\{H_n\}_{n\geqslant 1}$ that are finite flat over S (where we define the "normalizer" of H_n in the evident manner; this makes sense even though H_n is usually not a subgroup scheme of H, and it exists by Proposition 2.1.2). The noetherian property of G implies that the intersection stabilizes for large n. In other words, for sufficiently large n we have an equality

$$N_G(H) = N_G(H_n)$$

inside G. Fix such an $n > 0$. By hypothesis $H = N_G(H)$, so $N_G(H_n) = H$.

Since G and H are smooth, with H closed in G, we can write the finite flat S-schemes G_n and H_n in the form $G_n = \mathrm{Spec}_S(A_n)$ and $H_n = \mathrm{Spec}_S(B_n)$ for coherent $\mathscr{O}_S$-algebras A_n and B_n that are locally free over $\mathscr{O}_S$, with B_n a quotient of A_n. The degrees of H_n and G_n over S are given by some universal formulas in terms of n and the relative dimensions of H and G; let N denote the degree of H_n over S. Consider the Grassmannian $\mathrm{Gr}_N(G_n)$ that classifies quotient vector bundles of A_n with rank N. The conjugation action of G on itself induces an action of G on G_n, and hence an action of G on $\mathrm{Gr}_N(G_n)$. Under this action, the S-point ξ of $\mathrm{Gr}_N(G_n)$ corresponding to $H_n \subset G_n$ has functorial stabilizer $N_G(H_n) = N_G(H) = H$. We conclude that the orbit map $G \to \mathrm{Gr}_N(G_n)$ through ξ identifies the quotient sheaf G/H for the étale (or equivalently, fppf) topology with a subfunctor of the projective S-scheme $\mathrm{Gr}_N(G_n)$. Now we can run through the same argument with algebraic spaces (and Zariski's Main Theorem) as in the proof of Theorem 2.3.1 to conclude that G/H is a smooth S-scheme that is quasi-affine over $\mathrm{Gr}_N(G_n)$. This completes the proof of (1).

Finally, assume every geometric fiber $G_{\overline{s}}/H_{\overline{s}} = (G/H)_{\overline{s}}$ is projective. The properness of $G/H \to S$ in such cases is a consequence of the following general fact: if $f : X \to S$ is a separated flat surjective map of finite type (with S noetherian) and if the fibers X_s are proper and geometrically connected then f is proper. To prove this fact (a special case of [**EGA**, IV_3, 15.7.10]), by direct limit considerations we may pass to local rings on S; i.e., we can assume S is local. Then by fpqc descent for the properness property of morphisms (which we only need in the quasi-projective case, for which it reduces to the topological property of closedness for a locally closed immersion), we may assume $S = \mathrm{Spec}(A)$ for a *complete* local noetherian ring A.

By a deep result of Grothendieck on algebraization for formal A-schemes (see [**EGA**, III_1, 5.5.1]), properness of the special fiber X_0 provides an S-proper open and closed subscheme $Z \subset X$ with $Z_0 = X_0$. (The existence of Z is a simple consequence of the theorem on formal functions if we assume X is open in a proper A-scheme $\overline{X}$, as is automatic when X is quasi-projective. To see this, note that the open subscheme $X_0 \subset \overline{X}_0$ is closed by properness of

X_0, so it is the zero scheme of an idempotent e_0 on $\overline{\mathrm{X}}_0$. Idempotents uniquely lift through infinitesimal thickenings, so by the theorem on formal functions we can lift e_0 uniquely to an idempotent e on $\overline{\mathrm{X}}$. Then the S-proper open and closed subscheme $\overline{\mathrm{Z}}$ of $\overline{\mathrm{X}}$ defined by the vanishing of e meets X in an open subscheme Z of $\overline{\mathrm{Z}}$ that is open and closed in X and has special fiber $\mathrm{Z}_0 = \overline{\mathrm{Z}}_0 \cap \mathrm{X}_0 = \overline{\mathrm{Z}}_0$, so the closed complement of Z in the S-proper $\overline{\mathrm{Z}}$ is empty because its special fiber is empty. Such an A-proper $\overline{\mathrm{X}}$ actually exists even when X is not quasi-projective, due to the Nagata compactification theorem, but that lies much deeper than Grothendieck's construction of Z in general. Anyway, we only need the case of quasi-projective X, namely G/H above.) Since f is an open and closed map (as it is flat and proper) and $\mathrm{Z}_0 = \mathrm{X}_0 \neq \varnothing$, so $f(\mathrm{Z}) = \mathrm{S}$ because the local S is connected, we conclude that Z_s is non-empty for all $s \in \mathrm{S}$. But each X_s is connected by hypothesis, and Z_s is open and closed in X_s, so $\mathrm{Z}_s = \mathrm{X}_s$ for all s. Hence, $\mathrm{Z} = \mathrm{X}$, so X is S-proper.

We conclude from the properness of G/H that the monomorphism i : $\mathrm{G/H} \to \mathrm{Gr}_{\mathrm{N}}(\mathrm{G}_n)$ constructed above over S is proper, and hence by [**EGA**, IV_3, 8.11.5] i is a *closed immersion* (compare with Remark 2.3.5). Let $\mathscr{O}_{\mathrm{G}_n}$ denote the structure sheaf on G_n, viewed as a vector bundle on S. The canonical S-ample line bundle $\mathscr{N}$ on the Grassmannian equips G/H with a line bundle $\mathscr{L} = i^*(\mathscr{N})$ that is S-ample (since i is a closed immersion). Although $\mathscr{L}$ depends on the choice of Grassmannian $\mathrm{Gr}_{\mathrm{N}}(\mathrm{G}_n)$ (i.e., depends on the choice of n), so it is not canonically attached to (G, H), nonetheless $\mathscr{L}$ serves a useful purpose: we will prove that it is a non-negative power of $\det(\mathrm{Lie}(\mathscr{H}))^*$, so this dual determinant bundle is also S-ample, as desired.

By construction of the Plücker embedding of the Grassmannian, $\mathscr{N}$ is the determinant of the universal rank-N quotient bundle $\mathscr{Q}$ of $\mathscr{O}_{\mathrm{G}_n}$ over $\mathrm{Gr}_{\mathrm{N}}(\mathrm{G}_n)$. The nth infinitesimal neighborhood $\mathscr{H}_n$ of $\mathscr{H}$ along its identity section is finite locally free over G/H of rank N, so the structure sheaf $\mathscr{O}_{\mathscr{H}_n}$ may be viewed as a rank-N vector bundle quotient of $\mathscr{O}_{\mathrm{G}_n \times (\mathrm{G/H})}$ over G/H. By the definition of i, we have $i^*(\mathscr{Q}) = \mathscr{O}_{\mathscr{H}_n}$ as quotients of $\mathscr{O}_{\mathrm{G}_n \times (\mathrm{G/H})}$, so $\mathscr{L} = \det_{\mathrm{G/H}}(\mathscr{O}_{\mathscr{H}_n})$.

There is an evident filtration of $\mathscr{O}_{\mathscr{H}_n}$ by powers of the augmentation ideal of $\mathscr{H}$, and this filtration has successive quotients $\mathrm{Sym}^i_{\mathrm{G/H}}(\mathrm{Lie}(\mathscr{H})^*)$ for $0 \leqslant i \leqslant n$. For any vector bundle $\mathscr{E}$ of rank r, naturally $\det(\mathrm{Sym}^i(\mathscr{E})) \simeq (\det \mathscr{E})^{m(i,r)}$ for an exponent $m(i, r) \geqslant 0$ depending only on i and r. Hence, the S-ample line bundle $\mathscr{L} = \det \mathscr{O}_{\mathscr{H}_n}$ is a non-negative power of $\det(\mathrm{Lie}(\mathscr{H}))^*$. (The power depends on n and on the relative dimensions of H and G over S.) $\square$

2.4. Exercises. —

Exercise 2.4.1. — Let M be a finitely generated abelian group, and k a field. Prove that every closed k-subgroup scheme H of $\mathrm{D}_k(\mathrm{M})$ has the form $\mathrm{D}_k(\mathrm{M}/\mathrm{N})$ for a subgroup $\mathrm{N} \subset \mathrm{M}$. (Hint: reduce to the case $k = \overline{k}$, so $\mathrm{H}_{\mathrm{red}}$ is a *smooth* subgroup. By considering $\mathrm{H}^0_{\mathrm{red}}$ and a decomposition of M into a product of a finite free **Z**-module and a finite abelian group, first treat the case when M is free and H is smooth and connected. Then pass to $\mathrm{D}_k(\mathrm{M})/\mathrm{H}^0_{\mathrm{red}}$ in general to reduce to the case of finite H, for which Cartier duality can be used.)

Exercise 2.4.2. — Let $\mathrm{H} \to \mathrm{S}$ be a group of multiplicative type, and let $j : \mathrm{K} \hookrightarrow \mathrm{H}$ be a finitely presented quasi-finite closed subgroup. This exercise proves K is *finite* over S (a special case of [**SGA3**, IX, 6.4]).

(i) Reduce to the case of noetherian S. Using that proper monomorphisms are closed immersions, reduce to $\mathrm{S} = \mathrm{Spec}\,\mathrm{R}$ for a discrete valuation ring R (hint: valuative criterion). Further reduce to the case when $\mathrm{H} = \mathrm{D}_{\mathrm{R}}(\mathrm{M})$ for a finitely generated abelian group M.

(ii) With $\mathrm{S} = \mathrm{Spec}\,\mathrm{R}$ as in (i), use Exercise 2.4.1 to show that the schematic closure in H of the generic fiber of K is $\mathrm{D}_{\mathrm{R}}(\mathrm{M}/\mathrm{M}')$ for $\mathrm{M}' \subset \mathrm{M}$ of finite index.

(iii) Pass to quotients by $\mathrm{D}_{\mathrm{R}}(\mathrm{M}/\mathrm{M}')$ to reduce to the case $\mathrm{M}' = 0$, and conclude via Zariski's Main Theorem.

Exercise 2.4.3. — Let $f : \mathrm{H} \to \mathrm{G}$ be a homomorphism from a multiplicative type group H to an S-affine group G of finite presentation over a scheme S. Prove as follows that $\mathrm{K} := \ker f$ is multiplicative type (so f factors through the multiplicative type fppf quotient H/K that is an S-subgroup of G); this is part of [**SGA3**, IX, 6.8].

(i) Reduce to split $\mathrm{H} = \mathrm{D}_{\mathrm{S}}(\mathrm{M})$ and $\mathrm{S} = \mathrm{Spec}\,\mathrm{A}$ for local noetherian A.

(ii) Prove that K is closed in H, so by Exercise 2.4.1 the special fiber K_0 equals $\mathrm{D}_k(\mathrm{M}/\mathrm{M}')$ for a subgroup $\mathrm{M}' \subset \mathrm{M}$ (with k the residue field of A). Prove that the map $\mathrm{D}_{\mathrm{A}}(\mathrm{M}/\mathrm{M}') \to \mathrm{G}$ vanishes (hint: use Corollary B.3.5), so $\mathrm{D}_{\mathrm{A}}(\mathrm{M}/\mathrm{M}') \subset \mathrm{K}$.

(iii) Replace H with $\mathrm{H}/\mathrm{D}_{\mathrm{A}}(\mathrm{M}/\mathrm{M}') = \mathrm{D}_{\mathrm{A}}(\mathrm{M}')$ so that $\mathrm{K}_0 = 1$. By considering the special fiber of each *finite* (perhaps non-flat?) S-group $\mathrm{K}[n]$ with $n \geqslant 1$, prove $\mathrm{K}_s = 1$ for all $s \in \mathrm{S}$. Use Lemma B.3.1 to show that $e : \mathrm{S} \to \mathrm{K}$ is an isomorphism.

(iv) Relax the affineness hypothesis on G to separatedness.

Exercise 2.4.4. — Let X, Y, Z be schemes over a ring k, and $\alpha : \mathrm{X} \times \mathrm{Y} \to \mathrm{Z}$ a k-morphism. For a closed subscheme $\iota : \mathrm{Z}' \hookrightarrow \mathrm{Z}$ and k-algebra R, let $\underline{\mathrm{Transp}}_{\mathrm{X}}(\mathrm{Y}, \mathrm{Z}')(\mathrm{R})$ be the set of $x \in \mathrm{X}(\mathrm{R})$ such that $\alpha(x, \cdot)$ carries Y_{R} into Z'_{R}.

(i) If Y is affine and $k[\mathrm{Y}]$ is k-free, prove $\underline{\mathrm{Transp}}_{\mathrm{X}}(\mathrm{Y}, \mathrm{Z}')$ is represented by a closed subscheme $\mathrm{Transp}_{\mathrm{X}}(\mathrm{Y}, \mathrm{Z}')$ of X. (Hint: Reduce to affine X. If $\{e_i\}$ is a

k-basis of $k[\mathrm{Y}]$ and $\{f_j\}$ generates the ideal in $k[\mathrm{X} \times \mathrm{Y}] = k[\mathrm{X}] \otimes_k k[\mathrm{Y}]$ of the pullback of ι, with $f_j = \sum h_{ij} \otimes e_i$, consider the zero scheme of the h_{ij} in X.) Remove the affineness hypothesis on Y if k is artinian and Y is k-flat (and see [**SGA3**, VIII, § 6] for further generalizations).

(ii) Let Y be a closed subscheme of a separated k-group G. Using (i) with $\Delta_{\mathrm{G}/k}$ as ι, construct a closed subscheme $\mathrm{Z_G(Y)} \subset \mathrm{G}$ if k is a field or if Y is affine and $k[\mathrm{Y}]$ is k-free. Discuss the case Y = G. For a closed subscheme $\mathrm{Y}' \subset \mathrm{G}$, construct $\mathrm{Transp_G(Y, Y')}$. How about $\mathrm{N_G(Y)}$?

(iii) Compute equations for $\mathrm{N_G(G')} \subset \mathrm{G}$ over $k = \mathbf{Z}[t]$ in Example 2.1.10.

(iv) Consider a finite type group G over a field $k = k_s$ and a closed subscheme $\mathrm{Y} \subset \mathrm{G}$. If Y is smooth then prove $\mathrm{Z_G(Y)} = \bigcap_{y \in \mathrm{Y}(k)} \mathrm{Z_G}(y)$, and if G is smooth then prove Y is normalized (resp. centralized) by G if it is normalized (resp. centralized) by $\mathrm{G}(k)$.

Exercise 2.4.5. — Let $\pi : \mathrm{G} \to \mathrm{S}$ a finite group scheme, with $\pi_*(\mathscr{O}_\mathrm{G})$ locally free over $\mathscr{O}_\mathrm{S}$. Let G′ be an S-affine S-group of finite presentation. Recall that the functor $\underline{\mathrm{Hom}}(\mathrm{X}, \mathrm{Y})$ is represented by an S-affine S-scheme of finite presentation under the hypotheses of the first part of Lemma 2.1.3. Use several fiber products to represent $\underline{\mathrm{Hom}}_{\text{S-gp}}(\mathrm{G}, \mathrm{G}')$ by an S-affine S-scheme of finite presentation, thereby proving the second part of Lemma 2.1.3.

Exercise 2.4.6. — (i) Let G be SL_n or PGL_n over a ring k and T the diagonal torus, or let $\mathrm{G} = \mathrm{Sp}_{2n}$ and T the torus of points $\left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right)$ for diagonal $t \in \mathrm{GL}_n$. In all cases prove $\mathrm{Z_G(T)} = \mathrm{T}$ (so T is a maximal torus on all geometric fibers) by using the Lie algebra.

(ii) Using (i), prove $\mathrm{Z}_{\mathrm{SL}_n} = \mu_n$, $\mathrm{Z}_{\mathrm{PGL}_n} = 1$, and $\mathrm{Z}_{\mathrm{Sp}_{2n}} = \mu_2$ as schemes.

Exercise 2.4.7. — Consider a field k and a k-group H acting on a separated k-scheme Y. For a k-scheme S, let $\mathrm{Y^H(S)}$ be the set of $y \in \mathrm{Y(S)}$ invariant by the $\mathrm{H_S}$-action on $\mathrm{Y_S}$.

(i) Adapt Exercise 2.4.4(ii) to prove that $\mathrm{Y^H}$ is represented by a closed subscheme of H. In case H is smooth and $k = k_s$, prove $\mathrm{Y^H} = \bigcap_{h \in \mathrm{H}(k)} \mathrm{Y}^h$ where $\mathrm{Y}^h := \alpha_h^{-1}(\Delta_{\mathrm{Y}/k})$ for the map $\alpha_h : \mathrm{Y} \to \mathrm{Y} \times \mathrm{Y}$ defined by $y \mapsto (y, h.y)$. Relate this to Exercise 1.6.7.

(ii) For Y of finite type over k and $y \in \mathrm{Y^H}(k)$, prove $\mathrm{Tan}_y(\mathrm{Y^H}) = \mathrm{Tan}_y(\mathrm{Y})^\mathrm{H}$ for a suitable H-action on $\mathrm{Tan}_y(\mathrm{Y})$.

(iii) Assume H is a closed subgroup of a k-group G of finite type. Let $\mathfrak{g} := \mathrm{Lie(G)}$ and $\mathfrak{h} := \mathrm{Lie(H)}$. Prove $\mathrm{Tan}_e(\mathrm{Z_G(H)}) = \mathfrak{g}^\mathrm{H}$ (schematic invariants under the adjoint action). Also prove $\mathrm{Tan}_e(\mathrm{N_G(H)}) = \bigcap_{h \in \mathrm{H}(k)} (\mathrm{Ad_G}(h) - 1)^{-1}(\mathfrak{h})$ when $k = k_s$ and H is smooth.

Exercise 2.4.8. — Let G be a smooth connected affine group over a field k. For a smooth connected k-subgroup $\mathrm{H} \subset \mathrm{G}$, the proof of Theorem 2.3.6

constructs $n > 0$ so that the nth-order infinitesimal neighborhood H_n of 1 in H satisfies $\mathrm{N_G}(\mathrm{H}_n) = \mathrm{N_G}(\mathrm{H})$.

(i) For any subspace $\mathrm{V} \subset \mathfrak{g}$, show that the $\mathrm{Ad_G}$-stabilizer $\mathrm{N_G}(\mathrm{V})$ of V in G has Lie algebra $\mathfrak{n}_{\mathfrak{g}}(\mathrm{V})$ equal to the normalizer of V in $\mathfrak{g}$.

(ii) Assume G is reductive and let $\mathrm{P} \subset \mathrm{G}$ be parabolic with Lie algebra $\mathfrak{p} \subset \mathfrak{g}$. Show the inclusion $\mathrm{P} \subset \mathrm{N_G}(\mathfrak{p})$ identifies P with $\mathrm{N_G}(\mathfrak{p})_{\mathrm{red}}$. (Hint: Reduce to $k = \overline{k}$ and choose a maximal torus $\mathrm{T} \subset \mathrm{P}$. Let $\mathrm{Q} := \mathrm{N_G}(\mathfrak{p})_{\mathrm{red}}$, so Q is parabolic since $\mathrm{P} \subset \mathrm{Q}$. For a Borel $\mathrm{B} \subset \mathrm{P}$ containing T and the associated basis Δ of $\Phi(\mathrm{G}, \mathrm{T})$, $\mathrm{P} = \mathrm{P_I}$ and $\mathrm{Q} = \mathrm{P_J}$ for subsets $\mathrm{I} \subset \mathrm{J}$ of Δ. If there exists $a \in \mathrm{J} - \mathrm{I}$ then by identifying $\mathrm{G}_a = \mathscr{D}(\mathrm{Z_G}(\mathrm{T}_a))$ with SL_2 or PGL_2 show for $u \in \mathrm{U}_{-a}(k) - \{1\}$ that $\mathrm{Ad_G}(u)(\mathfrak{g}_a)$ has nonzero component in the $-a$-weight space.) If $\mathrm{char}(k) \neq 2$ (so $\mathfrak{sl}_2 = \mathfrak{pgl}_2$!) then show $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{p}) = \mathfrak{p}$ and deduce $\mathrm{N_G}(\mathfrak{p}) = \mathrm{P}$ as schemes.

(iii) Assume $\mathrm{char}(k) = 2$ and $\mathrm{G} = \mathrm{SL}_2$. Let B be the upper triangular Borel subgroup. Show $\mathrm{N_G}(\mathrm{B})$ has Lie algebra $\mathfrak{sl}_2$, and for the k-algebra $\mathrm{R} = k[\epsilon]$ of dual numbers and $g = \left(\begin{smallmatrix} 1 & 0 \\ \epsilon & 1 \end{smallmatrix}\right) \in \mathrm{G}(\mathrm{R})$ show $g\mathrm{B_R}g^{-1}$ and $\mathrm{B_R}$ are distinct Borel R-subgroups of SL_2 with the same Lie algebra over R. Also show that all maximal k-tori T in SL_2 have the *same* Lie algebra in $\mathfrak{sl}_2$ (hint: $\mathrm{Lie}(\mathrm{Z}_{\mathrm{SL}_2}) = \mathrm{Lie}(\mathrm{T})$), so membership of T in B *cannot* be detected on Lie algebras. Show that if $n = 2$ then $\mathrm{N}_{\mathrm{SL}_2}(\mathrm{D}_n) = \mathrm{N}_{\mathrm{SL}_2}(\mathrm{D})$ for the diagonal torus D.

(iv) Let G be semisimple of *adjoint type* over $k = \overline{k}$. Consider $(\mathrm{P}, \mathrm{T}, \mathrm{B}, \Delta)$ as in (ii), so Δ is a $\mathbf{Z}$-basis of $\mathrm{X}(\mathrm{T})$ and hence $\{\mathrm{Lie}(a)\}_{a \in \Delta}$ is a basis of the dual space $\mathfrak{t}^*$ via the canonical identification $\mathrm{Lie}(\mathbf{G}_{\mathrm{m}}) = k$. Using that $[x_a, v] = -\mathrm{Lie}(a)(v)x_a$ for $x_a \in \mathfrak{g}_a$ and $v \in \mathfrak{t}$, show the T-stable $\mathfrak{g}$-transporter of $\mathfrak{t}$ into $\mathfrak{p}$ is exactly $\mathfrak{p}$. Adapt the argument for (ii) to prove $\mathrm{N_G}(\mathfrak{p}) = \mathrm{P}$ as schemes, without restriction on $\mathrm{char}(k)$. (See [**SGA3**, XXII, 5.1.7(a), 5.3.2].)

Exercise 2.4.9. — Let $\Gamma = \mathrm{Gal}(k_s/k)$ for a field k. For a k-group M of multiplicative type, the *character group* $\mathrm{X}(\mathrm{M}) = \mathrm{Hom}_{k_s}(\mathrm{M}_{k_s}, \mathbf{G}_{\mathrm{m}})$ is a discrete Γ-module in an evident manner.

(i) If k'/k is a finite subextension of k_s, prove the Weil restriction $\mathrm{R}_{k'/k}(\mathrm{M}')$ is of multiplicative type over k when M' is of multiplicative type over k'. (For $\mathrm{M}' = \mathbf{G}_{\mathrm{m}}$ this is "${k'}^{\times}$ viewed as a k-group".) By functorial considerations, prove $\mathrm{X}(\mathrm{R}_{k'/k}(\mathrm{M}')) = \mathrm{Ind}_{\Gamma'}^{\Gamma}(\mathrm{X}(\mathrm{M}'))$ with $\Gamma' = \mathrm{Gal}(k_s/k')$. For every k-torus T, construct a surjective k-homomorphism $\prod_i \mathrm{R}_{k'_i/k}(\mathbf{G}_m) \twoheadrightarrow \mathrm{T}$ for finite separable extensions k'_i/k. Conclude that k-tori are *unirational* over k.

(ii) For a local field and k-torus T, prove T is k-anisotropic if and only if $\mathrm{T}(k)$ is compact.

(iii) For a finite extension field k'/k, define a *norm* map $\mathrm{N}_{k'/k} : \mathrm{R}_{k'/k}(\mathbf{G}_{\mathrm{m}}) \to \mathbf{G}_m$. Prove its kernel is a torus when k'/k is separable. What if k'/k is not separable?

Exercise 2.4.10. — Let X be a smooth separated scheme locally of finite type over a field k, and T a k-group of multiplicative type with a left action on X. This exercise is devoted to proving that X^T (as in Exercise 2.4.7) is *smooth*, generalizing Lemma 2.2.4 over fields.

(i) Reduce to the case $k = \overline{k}$. Fix a finite local k-algebra R with residue field k, and an ideal J in R with $J\mathfrak{m}_R = 0$. Choose $\overline{x} \in X^T(R/J)$, and for R-algebras A let E(A) be the fiber of $X(A) \twoheadrightarrow X(A/JA)$ over $\overline{x}_{A/JA}$. Let $x_0 = \overline{x} \bmod \mathfrak{m}_R \in X^T(k)$ and $A_0 = A/\mathfrak{m}_R A$. Prove $E(A) \neq \varnothing$ and make it a torsor over the A_0-module $F(A) := JA \otimes_k \mathrm{Tan}_{x_0}(X) = JA \otimes_{A_0} (A_0 \otimes_k \mathrm{Tan}_{x_0}(X))$ naturally in A (action $v.x$ denoted as $v + x$).

(ii) Define an A_0-linear $T(A_0)$-action on F(A) (hence a T_R-action on F), and prove that E(A) is T(A)-stable in X(A) with $t.(v + x) = t_0.v + t.x$ for $x \in E(A)$, $t \in T(A)$, $v \in F(A)$, and $t_0 = t \bmod \mathfrak{m}_R A$.

(iii) Choose $\xi \in E(R)$ and define a map of functors $h : T_R \to F$ by $t.\xi = h(t) + \xi$ for points t of T_R; check it is a 1-cocycle, and is a 1-coboundary if and only if $E^{T_R}(R) \neq \varnothing$. For $V_0 = J \otimes_k \mathrm{Tan}_{x_0}(X)$ use h to define a 1-cocycle $h_0 : T \to \underline{V}_0$, and prove $t.(v, c) := (t.v + ch_0(t), c)$ is a k-linear representation of T on $V_0 \bigoplus k$. Use a T-equivariant k-linear splitting (!) to prove h_0 (and then h) is a 1-coboundary; deduce X^T is smooth.

Exercise 2.4.11. — Let S be a scheme, $\mathscr{G}$ an fppf S-affine S-group, and $H^1(S, \mathscr{G})$ the set of isomorphism classes of right $\mathscr{G}$-torsors over S for the fppf topology. For any homomorphism $\mathscr{G} \to \mathscr{G}'$ between such groups, define $H^1(S, \mathscr{G}) \to H^1(S, \mathscr{G}')$ via pushout of torsors: $E \mapsto E \times^{\mathscr{G}} \mathscr{G}'$ (the quotient of $E \times \mathscr{G}'$ by the anti-diagonal $\mathscr{G}$-action $(e, g').g = (e.g, g^{-1}g')$).

(i) Prove that any right $\mathscr{G}$-torsor E is necessarily S-affine (and fppf), and that the quotient $E \times^{\mathscr{G}} \mathscr{G}'$ exists as a scheme. Also use the affineness to prove that $H^1(S, \mathscr{G})$ can be computed (functorially in $\mathscr{G}$!) by a non-commutative version of the usual Čech-type procedure generalizing non-abelian degree-1 Galois cohomology.

(ii) For any fppf S-affine S-group G and fppf closed S-subgroup H such that the fppf quotient sheaf G/H is represented by a scheme (see Theorem 2.3.6 for a sufficient criterion), identify $G(S)\backslash(G/H)(S)$ with the kernel of the map $H^1(S, H) \to H^1(S, G)$.

Exercise 2.4.12. — For $n \geqslant 1$, let $X_n = \mathrm{Spec}\, A_n$ be the $\mathbf{C}$-scheme obtained by gluing 2^n affine lines in a loop, with 0 on the ith line glued to 1 on the $(i+1)$th line ($i \in \mathbf{Z}/2^n\mathbf{Z}$).

(i) Prove that $\pi_1(X_n) = \widehat{\mathbf{Z}}$ and the finite étale covers of X_n split Zariski-locally on X_n.

(ii) Define $X_{n+1} \to X_n$ by collapsing odd-indexed lines to points and sending the $2j$th line in X_{n+1} to the jth line in X_n. Prove $X_\infty := \varprojlim X_n$ is reducible and its local rings are discrete valuation rings or fields. (This is a slight variant of an example in the Stacks Project.)

(iii) Construct a nontrivial $\mathbf{Z}$-torsor $\mathrm{E}_1 \to \mathrm{X}_1$ that is split Zariski-locally on X_1 but not by any finite étale cover of X_1. Prove $\mathrm{E}_\infty := \mathrm{E}_1 \times_{\mathrm{X}_1} \mathrm{X}_\infty \to \mathrm{X}_\infty$ is a non-split $\mathbf{Z}$-torsor that splits Zariski-locally on X_∞, and construct a rank-2 torus $\mathrm{T} \to \mathrm{X}_\infty$ that splits Zariski-locally on X_∞ but is not isotrivial. Thus, "irreducible" cannot be relaxed to "connected" in Corollary B.3.6. (The preceding construction was suggested by Gabber.)

3. Basic generalities on reductive group schemes

3.1. Reductivity and semisimplicity. — In [**SGA3**, XIX, 2.7] a connectedness condition is imposed in the relative theory of reductive groups:

Definition 3.1.1. — Let S be a scheme. An S-*torus* is an S-group $\mathrm{T} \to \mathrm{S}$ of multiplicative type with smooth connected fibers. A *reductive* S-*group* is a smooth S-affine group scheme $\mathrm{G} \to \mathrm{S}$ such that the geometric fibers $\mathrm{G}_{\overline{s}}$ are connected reductive groups. A *semisimple* S-*group* is a reductive S-group whose geometric fibers are semisimple.

In this definition, it suffices to check reductivity (resp. semisimplicity) for a single geometric point over each $s \in \mathrm{S}$ because for any linear algebraic group H over an algebraically closed field k and any algebraically closed extension K/k the inclusions $\mathscr{R}_u(\mathrm{H})_{\mathrm{K}} \subset \mathscr{R}_u(\mathrm{H}_{\mathrm{K}})$ and $\mathscr{R}(\mathrm{H})_{\mathrm{K}} \subset \mathscr{R}(\mathrm{H}_{\mathrm{K}})$ are equalities (see Exercise 3.4.1). By Proposition B.3.4, any S-torus becomes a power of $\mathbf{G}_{\mathrm{m}}$ étale-locally on S (also see Corollary B.4.2(1) and [**Oes**, II, § 1.3]).

In the theory of linear algebraic groups G over an algebraically closed field k, reductive groups are often permitted to be disconnected. One reason is that if $g \in \mathrm{G}(k)$ is semisimple then $\mathrm{Z}_{\mathrm{G}}(g)$ may be disconnected (as happens already for PGL_2) but $\mathrm{Z}_{\mathrm{G}}(g)^0$ is always reductive. Also, the Galois cohomological classification of connected semisimple groups G over a field k leads to the consideration of the automorphism scheme $\mathrm{Aut}_{\mathrm{G}/k}$, and this is a smooth affine k-group whose identity component is semisimple but is usually disconnected when the Dynkin diagram has nontrivial automorphisms (e.g., if $n > 2$ then $\mathrm{Aut}_{\mathrm{SL}_n/k} = \mathrm{PGL}_n \rtimes \mathbf{Z}/2\mathbf{Z}$ with component group generated by transpose-inverse). In § 7.1 we will discuss the existence and structure of automorphism schemes of reductive group schemes.

In the relative theory over a scheme that is not a single point, the disconnectedness of fibers presents new phenomena not seen in the classical case. For example, if $\mathrm{G} \to \mathrm{S}$ is a smooth S-affine group scheme then the orders of the geometric fibral component groups $\pi_0(\mathrm{G}_{\overline{s}})$ can vary with $\overline{s}$, so these component groups can fail to arise as the fibers of a *finite* étale S-group (see Example 3.1.4).

Requiring connectedness of fibers is not unreasonable. By Exercise 1.6.5, a connected group scheme of finite type over a field is geometrically connected (as for any connected finite type scheme X over a field k when $\mathrm{X}(k)$ is non-empty), so for a group scheme of finite type the property of having connected fibers is preserved by any base change. Also, the identity component varies well in smooth families of groups: for any smooth group scheme $\mathrm{G} \to \mathrm{S}$ of finite presentation there exists a unique open subgroup scheme $\mathrm{G}^0 \subset \mathrm{G}$ such that $(\mathrm{G}^0)_s$ is the identity component of G_s for all $s \in \mathrm{S}$ [**EGA**, IV_3, 15.6.5]. The formation of G^0 commutes with any base change on S since each G^0_s is geometrically connected, so by reduction to the case of noetherian S we see that G^0 is finitely presented over S. Imposing connectedness of fibers amounts to passing to G^0 in place of G. Beware that passage to G^0 can exhibit some peculiar behavior relative to the theory over a field:

Example 3.1.2. — For smooth S-affine S-groups G, the open subgroup G^0 may not be closed. An interesting example is given in [**SGA3**, XIX, § 5] over $\operatorname{Spec} k[t]$ for any field k of characteristic 0. To describe this example, let $\mathfrak{g}$ be the Lie algebra over $k[t]$ whose underlying $k[t]$-module is free with basis $\{X, Y, H\}$ satisfying the bracket relations

$$[H, X] = X, \ \ [H, Y] = -Y, \ \ [X, Y] = 2tH.$$

Over $\{t \neq 0\}$ this becomes isomorphic to $\mathfrak{sl}_2$ over the degree-2 finite étale cover given by $\sqrt{t}$, using the $\mathfrak{sl}_2$-triple $(X/\sqrt{t}, Y/\sqrt{t}, 2H)$, but the fiber at $t = 0$ is solvable. Explicit computations (see [**SGA3**, XIX, 5.2–5.10]) show that the group scheme G of automorphisms of $\mathfrak{g}$ that lie in $SL(\mathfrak{g})$ is smooth, and that $G|_{t\neq 0}$ is an étale form of PGL_2 but the fiber G_0 at $t = 0$ is solvable with two geometric components. Consequently, G^0 cannot be closed since it is a dense open subscheme of G that is distinct from G.

In this example the inclusion morphism $G^0 \to G$ is affine (see [**SGA3**, XIX, 5.13]), so G^0 is S-affine. There are pairs (G, S) with smooth S-affine G such that G^0 is not S-affine (so it is not closed in G). See [**Ra**, VII, § 3, (iii)] for such an example over $S = \mathbf{A}^2_k$ with k of characteristic 0.

It turns out that with a reductivity hypothesis on fibral identity components, the problems in Example 3.1.2 do not arise. This is made precise by the following result that we will never use and which rests on many later developments in the theory:

Proposition 3.1.3. — *Let* $G \to S$ *be a smooth separated group scheme of finite presentation such that* $G^0_{\overline{s}}$ *is reductive for all* $s \in S$*. Then* G^0 *is a reductive* S*-group that is open and closed in* G*, and* G/G^0 *exists as a separated étale* S*-group of finite presentation.*

Proof. — The open subgroup $G^0 \to S$ is smooth and finitely presented with connected reductive fibers. Incredibly, in the definition of a reductive group scheme we can replace “affine” with “finitely presented” [**SGA3**, XVI, 5.2(i)], so G^0 is a reductive S-group (in particular, G^0 is S-affine). Moreover, in Theorem 5.3.5 we will show that any monic homomorphism from a reductive group scheme to a finitely presented and separated group scheme is always a closed immersion. Hence, for such G we see that G^0 is both open *and* closed in G, so the quotient $E := G/G^0$ that is initially a finitely presented and étale algebraic space over S is also *separated* over S. Thus, after a reduction to the case of noetherian S, we may apply Knutson's criterion (as in the proof of Theorem 2.3.1) to conclude that E is a scheme. □

Example 3.1.4. — The conclusion in Proposition 3.1.3 is "best possible", in the sense that the relative component group G/G^0, which is always quasi-finite, separated, and étale over S, may not be finite over S. For example, let S be a connected $\mathbf{Z}[1/2]$-scheme and $\mathrm{O}(q)$ the orthogonal group of a nondegenerate quadratic space (V, q) over S, so $\mathrm{O}(q)^0 = \mathrm{SO}(q)$ and $\mathrm{O}(q)/\mathrm{SO}(q) = (\mathbf{Z}/2\mathbf{Z})_{\mathrm{S}}$. Let $\mathrm{U} \subset \mathrm{S}$ be a finitely presented non-empty open subscheme with $\mathrm{U} \neq \mathrm{S}$, so U is not closed in S. The open subgroup $\mathrm{E} \subset (\mathbf{Z}/2\mathbf{Z})_{\mathrm{S}}$ obtained by removing the closed non-identity locus over $\mathrm{S} - \mathrm{U}$ is not S-finite, so its open preimage $\mathrm{G} \subset \mathrm{O}(q)$ satisfies the hypotheses in Proposition 3.1.3 but $\mathrm{G}/\mathrm{G}^0 = \mathrm{E}$ is not S-finite.

In the classical theory of connected reductive groups, it is a fundamental fact that torus centralizers are again connected reductive [**Bo91**, 13.17, Cor. 2]. In the relative case this remains valid: if G is a reductive S-group scheme and T is an S-torus in G then the closed subgroup $\mathrm{Z}_{\mathrm{G}}(\mathrm{T})$ is S-smooth (see Lemma 2.2.4) and its geometric fibers are connected reductive by the classical theory, so $\mathrm{Z}_{\mathrm{G}}(\mathrm{T})$ is reductive over S. This can be pushed a bit further, as explained in the following Remark that we will never use.

Remark 3.1.5. — Let H be a multiplicative type subgroup of a reductive S-group G. Even over an algebraically closed field, H need not lie in a maximal torus of G. Nonetheless, $\mathrm{Z}_{\mathrm{G}}(\mathrm{H})$ is smooth (by Lemma 2.2.4) and the geometric fibers $\mathrm{Z}_{\mathrm{G}}(\mathrm{H})_{\overline{s}}$ have reductive identity component (by [**CGP**, A.8.12], which applies to a wider class of fibers H_s; its proof rests on a hard affineness theorem of Borel and Richardson for coset spaces modulo connected reductive groups ([**Bo85**], [**Ri**])). The fibers of $\mathrm{Z}_{\mathrm{G}}(\mathrm{H})$ can be disconnected. By Proposition 3.1.3, the open subgroup $\mathrm{Z}_{\mathrm{G}}(\mathrm{H})^0$ is reductive (hence affine) over S and closed in $\mathrm{Z}_{\mathrm{G}}(\mathrm{H})$.

To affirm that the notion of reductive group scheme is reasonable, we want to prove that reductivity of a fiber is inherited by nearby fibers for any smooth affine group scheme with connected fibers. This requires an improvement on the lifting of tori over adic noetherian rings in Corollary B.3.5, replacing completions with étale neighborhoods:

Proposition 3.1.6. — *Let* $\mathrm{G} \to \mathrm{S}$ *be a smooth* S*-affine group scheme, and* H_0 *a multiplicative type subgroup of the fiber* G_s *over some* $s \in \mathrm{S}$*. There exists an étale neighborhood* (S', s') *of* (S, s) *with* $k(s') = k(s)$ *and a multiplicative type subgroup* $\mathrm{H}' \subset \mathrm{G}_{\mathrm{S}'}$ *such that* $\mathrm{H}'_{s'} = \mathrm{H}_0$ *inside* $(\mathrm{G}_{\mathrm{S}'})_{s'} = \mathrm{G}_s$*.*

This result is [**SGA3**, XI, 5.8(a)]. Note that by Lemma B.1.3, the monomorphism $\mathrm{H}' \to \mathrm{G}_{\mathrm{S}'}$ must be a closed immersion.

Proof. — We may assume S is noetherian and affine, and even finite type over $\mathbf{Z}$ (by expressing a noetherian ring as a direct limit of its $\mathbf{Z}$-subalgebras of finite type; see [**EGA**, IV$_4$, 17.8.7] for the descent of smoothness through such direct limits). Let A denote the completion $\mathscr{O}^{\wedge}_{\mathrm{S},s}$. By Corollary B.3.5, there exists a multiplicative type A-subgroup $\widehat{\mathrm{H}}$ in G_{A} lifting H_0 in the special fiber G_s. Let $\{\mathrm{A}_\alpha\}$ be the directed system of finite type $\mathscr{O}_{\mathrm{S},s}$-subalgebras of A, so $\mathrm{A} = \varinjlim \mathrm{A}_\alpha$. By the argument at the start of the proof of Proposition 2.1.2, we can choose α large enough so that $\widehat{\mathrm{H}}$ descends to an A_α-group H_α of multiplicative type. Since $\mathscr{O}_{\mathrm{S},s}$ is essentially of finite type over $\mathbf{Z}$, and A_α is finite type over $\mathscr{O}_{\mathrm{S},s}$, we can apply the powerful Artin approximation theorem:

***Theorem 3.1.7* (Artin).** — *Let* R *be a local ring that is essentially of finite type over* $\mathbf{Z}$*, and* B *a finite type* R*-algebra equipped with a map* $f : \mathrm{B} \to \widehat{\mathrm{R}}$ *over* R*. Pick* $\mathrm{N} \geqslant 0$*.*

The map f *admits an* N*th-order "étale" approximation over* R *in the sense that there exists a residually trivial local-étale extension* $\mathrm{R} \to \mathrm{R}'$ *and an* R*-algebra map* $\varphi : \mathrm{B} \to \mathrm{R}'$ *such that the induced map to the completion*

$$\widehat{\varphi} : \mathrm{B} \to \widehat{\mathrm{R}'} = \widehat{\mathrm{R}}$$

agrees with f *modulo* $\mathfrak{m}_{\widehat{\mathrm{R}}}^{\mathrm{N}+1}$*.*

This theorem says that any solution in $\widehat{\mathrm{R}}$ to a finite system of polynomial equations over R is well-approximated by a solution in the henselization of R (equivalently, a solution in some residually trivial local-étale extension of R). The Artin approximation theorem actually allows any excellent Dedekind domain in place of $\mathbf{Z}$ in Theorem 3.1.7; see [**BLR**, 3.6/16] for the proof in that generality.

We apply Theorem 3.1.7 to $\mathrm{R} = \mathscr{O}_{\mathrm{S},s}$, $\mathrm{B} = \mathrm{A}_\alpha$, $\mathrm{N} = 0$, and the inclusion $\mathrm{B} \to \widehat{\mathrm{R}}$ to obtain a residually trivial local-étale extension $\mathrm{R} \to \mathrm{R}'$ and an R-algebra map $\mathrm{A}_\alpha \to \mathrm{R}'$ that agrees residually with the reduction of the given inclusion $\mathrm{A}_\alpha \hookrightarrow \widehat{\mathrm{R}}$. Thus, the R'-group $\mathrm{H}_\alpha \otimes_{\mathrm{A}_\alpha} \mathrm{R}'$ of multiplicative type in $\mathrm{G}_{\mathrm{R}'}$ has special fiber H_0 in G_s! In other words, we have found a multiplicative type subgroup lifting H_0 over a local-étale neighborhood of (S, s). Spreading this out over an étale neighborhood of (S, s) then does the job. $\square$

Corollary 3.1.8. — *Let* $\mathrm{G} \to \mathrm{S}$ *be an fppf* S*-affine group scheme with connected fibers, and assume that* G_s *is a torus for some* $s \in \mathrm{S}$*. Then* G_{U} *is a torus for some open neighborhood* U *of* s *in* S*.*

This result is [**SGA3**, X, 4.9].

Proof. — First we assume that G is smooth, and then we reduce the general case to the smooth case. Since the property of being a torus is étale-local on the base, we may work in an étale neighborhood of (S, s). Hence, by the smoothness of G we can use Proposition 3.1.6 to arrange that G contains a multiplicative type subgroup H such that $\mathrm{H}_s = \mathrm{G}_s$, and that G has constant fiber dimension. Passing to a further étale neighborhood makes H split, say $\mathrm{H} = \mathrm{D_S}(\mathrm{M})$ for a finitely generated abelian group M. Since H_s is a torus, M is free. Hence, H is a torus. But the inclusion $\mathrm{H} \hookrightarrow \mathrm{G}$ between smooth S-affine groups with connected fibers is an equality on s-fibers, so by smoothness and constancy of fiber dimensions over S it follows that $\mathrm{H}_{s'} = \mathrm{G}_{s'}$ for all $s' \in \mathrm{S}$. That is, $\mathrm{H} \to \mathrm{G}$ induces an isomorphism on fibers, so it is an isomorphism by the fibral isomorphism criterion (Lemma B.3.1).

In general (with G only assumed to be fppf rather than smooth over S), we just need to prove that G is automatically smooth over an open neighborhood of s. We may reduce to the case when S is local noetherian, and the smoothness of the s-fiber and the fppf hypothesis on G implies that $\mathrm{G} \to \mathrm{S}$ is smooth at all points of G_s, and so on an open neighborhood of G_s in G. This open neighborhood has open image in S, so by shrinking around $s \in \mathrm{S}$ we can arrange that this open image is equal to S. In particular, each fiber group scheme $\mathrm{G}_{s'}$ has a non-empty smooth locus, so the fibers are smooth (due to homogeneity considerations on geometric fibers). □

Proposition 3.1.9. — *Let* $\mathrm{G} \to \mathrm{S}$ *be a smooth* S*-affine group scheme and suppose* $\mathrm{G}^0_{\overline{s}}$ *is reductive for some* $s \in \mathrm{S}$.

1. *There is an open* U *around* s *in* S *such that* $\mathrm{G}^0_{\overline{u}}$ *is reductive for all* $u \in \mathrm{U}$. *The same holds for semisimplicity.*
2. *If* $\mathrm{T} \subset \mathrm{G}$ *is a torus such that* $\mathrm{T}_{\overline{s}}$ *is maximal in* $\mathrm{G}^0_{\overline{s}}$ *for some* $s \in \mathrm{S}$ *then there exists an open* V *around* s *in* S *such that* $\mathrm{Z_G(T)}^0_{\mathrm{V}} = \mathrm{T_V}$ *and* $\mathrm{T}_{\overline{v}}$ *is a maximal torus in* $\mathrm{G}^0_{\overline{v}}$ *for all* $v \in \mathrm{V}$.

This result is essentially [**SGA3**, XIX, 2.6] (where it is assumed that each G_s is connected). Note that in part (2) we *assume* T exists. The existence of such a T étale-locally around s follows from Proposition 3.1.6 if we admit Theorem A.1.1, but we will prove such étale-local existence for reductive G in Corollary 3.2.7 without using Theorem A.1.1; also see Exercise 3.4.8. (See [**SGA3**, XIV, 3.20] for a generalization using the Zariski topology, building on Theorem A.1.1.)

Proof. — First we prove (2). The centralizer $\mathrm{Z_G(T)}$ in G is a smooth closed S-subgroup by Lemma 2.2.4, and obviously $\mathrm{T} \subset \mathrm{Z_G(T)}^0$. By working Zariski-locally around s, we may assume that the smooth S-groups $\mathrm{T} \to \mathrm{S}$ and $\mathrm{Z_G(T)} \to \mathrm{S}$ have constant fiber dimension. These fiber dimensions agree at s, so they agree on all fibers. For any $\xi \in \mathrm{S}$, the closed subgroup $\mathrm{Z}_{\mathrm{G}_\xi}(\mathrm{T}_\xi) =$

$Z_G(T)_\xi$ in G_ξ is smooth and contains T_ξ. But the dimensions agree, so $T_\xi = (Z_G(T)_\xi)^0 = (Z_G(T)^0)_\xi$ for all $\xi \in S$. We conclude that the S-map $T \hookrightarrow Z_G(T)^0$ between smooth S-schemes is an isomorphism on fibers over S, so it is an isomorphism (Lemma B.3.1).

Although G^0 might not be S-affine, for any torus $T' \subset G$ it is clear that $G^0 \cap Z_G(T')$ represents $\underline{Z}_{G^0}(T')$, so we denote it as $Z_{G^0}(T')$. In the classical theory it is shown that the centralizer of *any* torus in a smooth connected affine group over an algebraically closed field is connected, so $Z_{G^0}(T) = Z_G(T)^0 = T$. Also, for any geometric point $\overline{\xi}$ over $\xi \in S$ we have $T_{\overline{\xi}} = Z_{G^0}(T)_{\overline{\xi}} = Z_{G^0_{\overline{\xi}}}(T_{\overline{\xi}})$, so $T_{\overline{\xi}}$ must be maximal as a torus in $G^0_{\overline{\xi}}$. This proves (2).

We next turn to (1), and we may assume $G^0_s \neq 1$. Fix an algebraic geometric point $\overline{s} = \mathrm{Spec}(\overline{k(s)})$ over s. Any maximal torus in $G^0_{\overline{s}}$ descends to a split torus in G^0_K for some finite extension $K/k(s)$ contained in $\overline{k(s)}$, and it is harmless to work fppf-locally around s. Thus, we can pass to an fppf neighborhood of (S, s) to increase $k(s)$ to coincide with such a K. Now G^0_s contains a split torus T_s such that $T_{\overline{s}}$ is maximal in $G^0_{\overline{s}}$. By Proposition 3.1.6, we may make a further étale base change on S around s to get to the case that T_s lifts to a torus T in G (hence in G^0), and that T is even split. By (2), after some further Zariski-localization around s we may assume $Z_{G^0}(T) = T$ and that

$$T \simeq D_S(M) := \underline{\mathrm{Hom}}_{S\text{-gp}}(M_S, \mathbf{G}_m)$$

for a finite free $\mathbf{Z}$-module M.

The T-action on the vector bundle $\mathfrak{g} = \mathrm{Lie}(G) = \mathrm{Lie}(G^0)$ over S decomposes it into a direct sum of quasi-coherent weight spaces indexed by elements of M, and the formation of these weight spaces commutes with base change on S. These weight spaces are vector bundles (being direct summands of $\mathfrak{g}$), so the weight space decomposition on the s-fiber encodes the weight spaces on the nearby fibers (by Nakayama's Lemma at s). Thus, by shrinking around s we can arrange that all weight spaces have constant rank, so the only characters $m \in M$ for which the weight space $\mathfrak{g}_m$ is nonzero are $m = 0$ and the elements of the root system $\Phi = \Phi(G^0_s, T_s)$ for the connected reductive geometric fiber $G^0_{\overline{s}} \neq 1$.

For each $\xi \in S$, the weight space $\mathfrak{g}_0$ for the trivial weight has ξ-fiber

$$\mathrm{Lie}(Z_{G^0_\xi}(T_\xi)) = \mathrm{Lie}(Z_{G^0}(T)_\xi) = \mathrm{Lie}(T_\xi) = \mathrm{Lie}(T)_\xi,$$

but the subbundle $\mathfrak{t} := \mathrm{Lie}(T)$ in $\mathfrak{g}$ is clearly contained in $\mathfrak{g}_0$, so the inclusion $\mathfrak{t} \subset \mathfrak{g}_0$ between subbundles must be an equality over S for rank reasons. In other words, the weight space decomposition over S is

$$\mathfrak{g} = \mathfrak{t} \bigoplus (\bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha)$$

where each $\mathfrak{g}_\alpha$ is a line bundle on S.

For each $\alpha \in \Phi$, let $\mathrm{T}_\alpha \subset \mathrm{T} = \mathrm{D_S(M)}$ be the unique relative codimension-1 subtorus contained in $\ker \alpha$; explicitly, $\mathrm{T}_\alpha = \mathrm{D_S(M/L)}$ where $\mathrm{L} \subset \mathrm{M}$ is the saturation of $\mathbf{Z}\alpha$ in M. Since Φ is a reduced root system (by the classical theory, applied to $(\mathrm{G}^0_{\overline{s}}, \mathrm{T}_{\overline{s}})$), elements of Φ apart from $\pm\alpha$ are linearly independent from α and so cannot vanish on any fiber of T_α. By the following lemma, which is a variant of Exercise 1.6.16(ii), a geometric fiber $\mathrm{G}^0_{\overline{\xi}}$ is reductive provided that each fiber $\mathrm{Z}_{\mathrm{G}^0}(\mathrm{T}_\alpha)_{\overline{\xi}} = \mathrm{Z}_{\mathrm{G}^0_{\overline{\xi}}}((\mathrm{T}_{\overline{\xi}})_\alpha)$ is reductive.

Lemma 3.1.10. — *Let* G *be a (not necessarily reductive) smooth connected affine group over an algebraically closed field k, and* T *a maximal torus in* G *such that* $\mathrm{Z_G(T)} = \mathrm{T}$. *For each nonzero* T*-weight α on $\mathfrak{g}$, let* T_α *be the codimension*-1 *subtorus* $(\ker\alpha)^0_{\mathrm{red}}$.

The group G *is reductive if and only if the smooth connected subgroup* $\mathrm{Z_G}(\mathrm{T}_\alpha)$ *is reductive for each α.*

Proof. — The implication "$\Rightarrow$" is part of the classical theory of reductive groups (cf. Theorem 1.1.19(3)). For the converse, let $\mathrm{U} = \mathscr{R}_u(\mathrm{G})$, so U is connected by definition. Thus, $\mathfrak{u} := \mathrm{Lie(U)}$ is a T-stable subspace of $\mathfrak{g}$, so it has a weight space decomposition. Each intersection $\mathrm{Z_G}(\mathrm{T}_\alpha) \bigcap \mathrm{U}$ is smooth and connected by Example 2.2.6, yet is also visibly unipotent and normal in $\mathrm{Z_G}(\mathrm{T}_\alpha)$ since U is unipotent and normal in G. The reductivity of $\mathrm{Z_G}(\mathrm{T}_\alpha)$ then forces $\mathrm{Z_G}(\mathrm{T}_\alpha) \bigcap \mathrm{U} = 1$. The formation of Lie algebras of closed subgroup schemes is compatible with the formation of intersections (as one sees by consideration of dual numbers), so we conclude that $\mathfrak{u}$ has trivial intersection with $\mathrm{Lie}(\mathrm{Z_G}(\mathrm{T}_\alpha))$. But the functorial definition of $\mathrm{Z_G}(\mathrm{T}_\alpha)$ implies that $\mathrm{Lie}(\mathrm{Z_G}(\mathrm{T}_\alpha)) = \mathfrak{g}^{\mathrm{T}_\alpha}$ (see Proposition 1.2.3), so this contains the entire weight space for α on $\mathfrak{g}$. In particular, $\mathfrak{u}$ has vanishing intersection with each such weight space.

We conclude that $\mathfrak{u}$ supports no nontrivial T-weights, so $\mathfrak{u} \subset \mathfrak{g}^{\mathrm{T}} = \mathrm{Lie}(\mathrm{Z_G(T)})$. Thus,

$$\mathrm{Lie}(\mathrm{U} \bigcap \mathrm{Z_G(T)}) = \mathfrak{u} \bigcap \mathrm{Lie}(\mathrm{Z_G(T)}) = \mathfrak{u}.$$

But $\mathrm{U} \bigcap \mathrm{Z_G(T)}$ is smooth and connected (by Example 2.2.6), yet we have just seen that the containment $\mathrm{U} \bigcap \mathrm{Z_G(T)} \subset \mathrm{U}$ induces an equality on Lie algebras, so it must be an equality. In other words, necessarily $\mathrm{U} \subset \mathrm{Z_G(T)}$. We assumed $\mathrm{Z_G(T)} = \mathrm{T}$, so the smooth connected unipotent U must be trivial. □

Returning to the relative setting, by smoothness of $\mathrm{Z}_{\mathrm{G}^0}(\mathrm{T}_\alpha)$ the inclusion $\mathrm{Lie}(\mathrm{Z}_{\mathrm{G}^0}(\mathrm{T}_\alpha)) \subset \mathfrak{g}^{\mathrm{T}_\alpha} = \mathfrak{t} \bigoplus \mathfrak{g}_\alpha \bigoplus \mathfrak{g}_{-\alpha}$ as subbundles of $\mathfrak{g}$ is an equality since this holds on geometric fibers. To verify that G^0 has reductive fibers at all points of S near s, it suffices to treat each $\mathrm{Z_G}(\mathrm{T}_\alpha)$ separately in place of G (by Lemma 3.1.10), so we have reduced to the case $\Phi = \{\alpha, -\alpha\}$.

Clearly $\mathrm{G}^0 \cap \mathrm{N}_{\mathrm{G}}(\mathrm{T})$ represents $\underline{\mathrm{N}}_{\mathrm{G}^0}(\mathrm{T})$, so we denote this as $\mathrm{N}_{\mathrm{G}^0}(\mathrm{T})$. Likewise, $\mathrm{W}_{\mathrm{G}^0}(\mathrm{T}) := \mathrm{N}_{\mathrm{G}^0}(\mathrm{T})/\mathrm{T}$ is an open and finitely presented S-subgroup of the S-group $\mathrm{W}_{\mathrm{G}}(\mathrm{T}) = \mathrm{N}_{\mathrm{G}}(\mathrm{T})/\mathrm{T}$ that is separated, étale, and quasi-finite (Theorem 2.3.1), so $\mathrm{W}_{\mathrm{G}^0}(\mathrm{T})$ inherits these properties. The fiber $\mathrm{W}_{\mathrm{G}^0}(\mathrm{T})_{\overline{s}}$ has order 2 and nontrivial element that acts by inversion on $(\mathrm{T}/\mathrm{T}_\alpha)_{\overline{s}}$ (and so swaps α and $-\alpha$). Étaleness of $\mathrm{W}_{\mathrm{G}^0}(\mathrm{T})$ implies that by passing to an étale neighborhood of (S, s) we can arrange that $\mathrm{W}_{\mathrm{G}^0}(\mathrm{T}) \to \mathrm{S}$ admits a section w that is the nontrivial point in the s-fiber. Thus, the w-action on the rank-1 torus $\mathrm{T}/\mathrm{T}_\alpha$ is inversion over a neighborhood of s since it is inversion on the s-fiber.

By localizing more around s to lift w to a section of $\mathrm{N}_{\mathrm{G}^0}(\mathrm{T})$, we obtain that for *all* geometric points $\overline{\xi}$ of S the fiber $\mathrm{N}_{\mathrm{G}^0}(\mathrm{T})_{\overline{\xi}}$ contains an element that does not centralize $\mathrm{T}_{\overline{\xi}}$. But in any smooth connected *solvable* group over an algebraically closed field, the normalizer of a maximal torus is equal to its centralizer (as is immediately verified by considering the description of any such group as a semi-direct product of a maximal torus against the unipotent radical). It follows that all fibers $\mathrm{G}^0_{\overline{\xi}}$ are *non-solvable.*

We have arranged that each fiber $\mathrm{G}^0_{\overline{\xi}}$ contains the central torus $(\mathrm{T}_\alpha)_{\overline{\xi}}$ with codimension 3 and is not solvable. Thus, the quotient $(\mathrm{G}/\mathrm{T}_\alpha)^0_{\overline{\xi}}$ is a non-solvable 3-dimensional smooth connected affine group. This leaves no room for a nontrivial unipotent radical (as the quotient by such a radical would be a smooth connected group of dimension at most 2, forcing solvability). Hence, every $\mathrm{G}^0_{\overline{\xi}}$ is reductive.

It remains to check that when $\mathrm{G}^0 \to \mathrm{S}$ has reductive geometric fibers, semisimplicity of a geometric fiber $\mathrm{G}^0_{\overline{s}}$ implies semisimplicity of geometric fibers at points near s. In the presence of reductivity, semisimplicity of a connected linear algebraic group H over an algebraically closed field can be read off from the root system Ψ: it is equivalent to the condition that the $\dim \mathrm{H} = \#\Psi + \mathrm{rank}(\Psi)$. The preceding arguments (using a weight space decomposition of $\mathfrak{g}$ fppf-locally near s) show that these invariants are inherited by geometric fibers at points near s, so we are done. $\square$

In later arguments involving reduction to the noetherian case, we need to ensure that reductivity (and semisimplicity) hypotheses can be descended:

Corollary 3.1.11. — *Let $\{\mathrm{A}_i\}$ be a directed system of rings with direct limit* A*, and G_{i_0} a smooth affine A_{i_0}-group for some i_0. For all $i \geqslant i_0$ define $\mathrm{G}_i = \mathrm{G}_{i_0} \otimes_{\mathrm{A}_{i_0}} \mathrm{A}_i$ and $\mathrm{G} = \mathrm{G}_{i_0} \otimes_{\mathrm{A}_{i_0}} \mathrm{A}$. The fibers of $\mathrm{G}^0 \to \mathrm{Spec}\,\mathrm{A}$ are reductive if and only if the fibers of $\mathrm{G}^0_i \to \mathrm{Spec}\,\mathrm{A}_i$ are reductive for all sufficiently large $i \geqslant i_0$, and* G *is a reductive* A*-group if and only if G_i is a reductive A_i-group for all sufficiently large $i \geqslant i_0$. The same holds for semisimplicity.*

Proof. — For $i \geqslant i_0$, let $\mathrm{U}_i \subset \operatorname{Spec} \mathrm{A}_i$ be the locus of points at which the geometric fiber of G_i^0 is reductive (resp. semisimple), and define $\mathrm{U} \subset \operatorname{Spec} \mathrm{A}$ similarly for G^0. By Proposition 3.1.9(1), these loci are open subsets. Our first problem is to prove that $\mathrm{U} = \operatorname{Spec} \mathrm{A}$ if and only if $\mathrm{U}_i = \operatorname{Spec} \mathrm{A}_i$ for all sufficiently large i.

Under the transition maps $\operatorname{Spec} \mathrm{A}_{i'} \to \operatorname{Spec} \mathrm{A}_i$ (resp. the maps $\operatorname{Spec} \mathrm{A} \to \operatorname{Spec} \mathrm{A}_i$), the preimage of U_i is $\mathrm{U}_{i'}$ (resp. U), so the same holds for the respective closed complements Z_i of U_i in $\operatorname{Spec} \mathrm{A}_i$ and Z of U in Spec A. We wish to show that Z is empty if and only if Z_i is empty for all sufficiently large i. Letting $\mathrm{J}_i \subset \mathrm{A}_i$ and $\mathrm{J} \subset \mathrm{A}$ be the respective radical ideals of these closed sets, we have $\mathrm{A}/\mathrm{J} = \varinjlim \mathrm{A}_i/\mathrm{J}_i$. Hence, $\mathrm{A}/\mathrm{J} = 0$ if and only if $\mathrm{A}_i/\mathrm{J}_i = 0$ for all sufficiently large i (by considering the equation $1 = 0$).

It remains to show that if $\mathrm{G}^0 = \mathrm{G}$ then $\mathrm{G}_i^0 = \mathrm{G}$ for all sufficiently large i. By [**EGA**, IV_3, 9.7.7(ii)], the subset $\mathrm{Y}_i \subset \operatorname{Spec} \mathrm{A}_i$ of points at which G_i has a geometrically connected fiber is constructible, and if $i' \geqslant i$ then the preimage of Y_i in $\operatorname{Spec} \mathrm{A}_{i'}$ is $\mathrm{Y}_{i'}$. The common preimage of all Y_i in Spec A is the entire space, so by [**EGA**, IV_3, 8.3.4] we have $\mathrm{Y}_i = \operatorname{Spec} \mathrm{A}_i$ for large i. □

In the proof of Proposition 3.1.9 we showed that when each $\mathrm{G}^0_{\overline{s}}$ is reductive and there is a split torus $\mathrm{T} \subset \mathrm{G}$ that is maximal on geometric fibers then for each $s_0 \in \mathrm{S}$ and varying $s \in \mathrm{S}$ near s_0 the root systems $\Phi(\mathrm{G}^0_{\overline{s}}, \mathrm{T}_{\overline{s}})$ may be identified with $\Phi(\mathrm{G}^0_{\overline{s}_0}, \mathrm{T}_{\overline{s}_0})$. This has the following interesting consequence:

Proposition 3.1.12. — *Let* $\mathrm{G} \to \mathrm{S}$ *be a smooth* S*-affine group such that each* G_s^0 *is reductive. The locus of* $s \in \mathrm{S}$ *such that* G_s *is connected is closed in* S*. In particular, if* S *is irreducible and the generic fiber is connected then all fibers are connected.*

Proposition 3.1.12 follows immediately from the claim in Proposition 3.1.3 that G/G^0 is a finitely presented *separated* étale S-group (ensuring that the locus $\{s \in \mathrm{S} \mid \#(\mathrm{G}/\mathrm{G}^0)_{\overline{s}} \geqslant n\}$ is open in S). The proof of such separatedness rests on Theorem 5.3.5 (which is proved much later), so we avoid Proposition 3.1.3 in our proof of Proposition 3.1.12.

Proof. — We may assume S is affine, and then by Corollary 3.1.11 we can assume S is noetherian. Since G_s is connected if and only if it is geometrically connected over $k(s)$, the set Y of points $s \in \mathrm{S}$ such that G_s is connected is constructible [**EGA**, IV_3, 9.7.7(ii)]. For any map $\mathrm{S}' \to \mathrm{S}$, the preimage of Y in S' is the locus of connected fibers for $\mathrm{G}_{\mathrm{S}'} \to \mathrm{S}'$, so by the specialization criterion for closedness of a constructible set we may assume $\mathrm{S} = \operatorname{Spec} \mathrm{R}$ for a discrete valuation ring R and that the generic fiber is *connected.* We seek to prove that the special fiber is connected.

Without loss of generality, R is complete with an algebraically closed residue field k, so G_k^0 contains a maximal torus T_0 and by Proposition 3.1.6 there is an R-torus $T \subset G$ lifting $T_0 \subset G_0$. The completeness of R ensures that T is split. Clearly $T \subset Z_G(T)$, so T is closed in $Z_G(T)$. Consider the special fibers T_0 and $Z_G(T)_k = Z_{G_k}(T_0)$. The reductive identity component G_k^0 meets $Z_{G_k}(T_0)$ in $Z_{G_k^0}(T_0) = T_0$ (equality due to the maximality of T_0).

Let K be the fraction field of R. The closed subgroups T and $Z_G(T)$ in G are both R-smooth, and we have shown that their special fibers have the same identity component, so their relative dimensions agree. Hence, on K-fibers the inclusion $T_K \subset Z_G(T)_K = Z_{G_K}(T_K)$ must be an equality, as $Z_{G_K}(T_K)$ is smooth and connected (since G_K is connected). In other words, the complement $Z_G(T) - T$ is the union of the non-identity components of the special fiber of $Z_G(T)$, so it is closed in $Z_G(T)$. Hence, T is also open in $Z_G(T)$. But $Z_G(T)$ is R-flat with irreducible generic fiber, so the total space of $Z_G(T)$ is connected (even irreducible). Thus, the open and closed subset T in $Z_G(T)$ must be the entire space, so the closed immersion $T \hookrightarrow Z_G(T)$ must be an isomorphism due to R-smoothness.

By the observation immediately preceding the present proposition, the split reductive pairs (G_K, T_K) and (G_k^0, T_k) have isomorphic root systems. Let Φ denote the common isomorphism class of $\Phi(G_K, T_K)$ and $\Phi(G_k^0, T_k)$. The normalizer $N_G(T)$ is a smooth closed subgroup of G, so the quotient

$$W_G(T) := N_G(T)/Z_G(T) = N_G(T)/T$$

is a separated étale R-group of finite type (Theorem 2.3.1). The generic fiber $W_G(T)_K = W_{G_K}(T_K)$ is the finite constant group over K associated to the ordinary finite group $W(\Phi)$. Likewise, $W_G(T)_k = W_{G_k}(T_k)$ is finite étale over k and contains as an open subgroup $W_{G_k^0}(T_k) \simeq W(\Phi)$. But the fiber degree for a separated étale map of finite type can only "drop" under specialization (by Zariski's Main Theorem [**EGA**, IV_3, 8.12.6]), so we conclude that $W_G(T)_k = W_{G_k^0}(T_k)$. In other words, $N_G(T) \subset G^0$.

For any $g \in G_k$, the conjugate gT_kg^{-1} is a maximal torus of G_k^0. Hence, there exists $g' \in G_k^0(k)$ such that $gT_kg^{-1} = g'T_kg'^{-1}$, so $g^{-1}g' \in N_G(T)_k \subset G_k^0$. This forces $g \in G_k^0$. □

See [**PY06**, Thm. 1.2] for a generalization of Proposition 3.1.12, and note that the locus of connected fibers can fail to be open (Example 3.1.4).

3.2. Maximal tori. — There are several ways to define the notion of "maximal torus" in a smooth affine group scheme. Different approaches do lead to the same concept for reductive groups, at least Zariski-locally on the base. For our purposes, the following definition (taken from [**SGA3**, XII, 1.3]) is suitable for developing the general theory.

Definition 3.2.1. — A *maximal torus* in a smooth S-affine group scheme $G \to S$ is a torus $T \subset G$ such that for each geometric point $\overline{s}$ of S the fiber $T_{\overline{s}}$ is not contained in any strictly larger torus in $G_{\overline{s}}$.

There are two sources of ambiguity in this definition. One is rather minor, namely to check that for any $s \in S$ it suffices to consider a single geometric point over s. This is a special case of the general principle that properties of finitely presented structures in algebraic geometry are insensitive to scalar extension from one algebraically closed ground field to a bigger such field; this principle pervades the classical approach to linear algebraic groups ("independence of the universal domain"). In the case of interest, it comes out as follows.

Proposition 3.2.2. — *If* K/k *is an extension of algebraically closed fields and* $T \subset G$ *is a torus in a smooth affine* k*-group* G *then* T *is not contained in a strictly larger torus of* G *if and only if* T_K *is not contained in a strictly larger torus of* G_K*.*

An immediate consequence of Proposition 3.2.2 is that if $T \subset G$ is a torus in a smooth relatively affine group scheme over a scheme S and if $S' \to S$ is surjective then T satisfies Definition 3.2.1 in G over S if and only if $T_{S'}$ does in $G_{S'}$ over S'.

Proof. — The implication "$\Leftarrow$" is obvious. To verify the converse, we shall use the general technique of "spreading out and specialization". Express K as the direct limit of its k-subalgebras A_i of finite type over k, so by limit considerations, any strict containment $T_K \subsetneq T'$ between K-tori in G_K descends to a strict containment $T_{A_i} \subsetneq T'_i$ between A_i-tori in G_{A_i} for some i. (This is the "spreading out" step, as $\mathrm{Spec}(A_i)$ is a finite type k-scheme, unlike Spec(K) in general. Note that an argument is required to verify that T'_i is a torus for large i; e.g., it follows from the limit argument used at the start of the proof of Proposition 2.1.2.)

Viewing $G_{A_i} = G \times \mathrm{Spec}(A_i)$ as a reduced k-scheme of finite type in which T_{A_i} and T'_i are reduced closed subschemes (due to smoothness over the reduced $\mathrm{Spec}(A_i)$), since $k = \overline{k}$ and T_{A_i} is a proper closed subset of T'_i it follows from the Nullstellensatz that some k-point of T'_i is not contained in T_{A_i}. This point lies over some $\xi \in \mathrm{Spec}(A_i)(k)$, so we specialize there: passing to ξ-fibers yields a strict containment $T \subsetneq (T'_i)_\xi$ between k-tori in G, a contradiction. □

A more serious ambiguity in the terminology of Definition 3.2.1 is to determine its relation with maximality relative to inclusions among tori in G over S. Any T that is maximal in the sense of Definition 3.2.1 is maximal among tori of G; i.e., any containment $T \subset T'$ among tori of G must be an equality.

Indeed, the equality on fibers is clear, so by Lemma B.3.1 equality holds in G (and likewise $\mathrm{T_U}$ is maximal in $\mathrm{G_U}$ for all open subschemes U of S).

Consider the converse: if a torus $\mathrm{T} \subset \mathrm{G}$ is not contained in a strictly larger torus over S then is T maximal in the sense of Definition 3.2.1 (i.e., is $\mathrm{T}_{\overline{s}}$ maximal in $\mathrm{G}_{\overline{s}}$ for all geometric points $\overline{s}$ of S)? There are Zariski-local obstructions: $\mathrm{T_U}$ may lie in a strictly larger $\mathrm{G_U}$-torus for some open $\mathrm{U} \subset \mathrm{S}$. This is due to the fact that the dimension of the maximal tori in each geometric fiber $\mathrm{G}_{\overline{s}}$ may not be locally constant in s. For instance, in the Example at the end of [**SGA3**, XVI, § 3] there is given an explicit smooth affine group over a discrete valuation ring such that the generic fiber is $\mathbf{G}_{\mathrm{m}}$ and the special fiber is $\mathbf{G}_{\mathrm{a}}$ (so $\mathrm{T} = 1$ is maximal relative to containment over the entire base, but not over the open generic point). Here is a more natural example.

Example 3.2.3. — Let $\mathrm{S} = \mathrm{Spec(V)}$ for a complete discrete valuation ring V with $\mathrm{K} := \mathrm{Frac(V)}$, and consider a separable quadratic extension $\mathrm{K'/K}$ that is not unramified. Let $\mathrm{V'}$ be the valuation ring of $\mathrm{K'}$. The Weil restriction $\mathrm{G} = \mathrm{R_{V'/V}}(\mathbf{G}_{\mathrm{m}})$ (which represents the functor $\mathrm{B} \rightsquigarrow \mathbf{G}_{\mathrm{m}}(\mathrm{B} \otimes_{\mathrm{V}} \mathrm{V'}) = (\mathrm{B} \otimes_{\mathrm{V}} \mathrm{V'})^{\times}$ on V-algebras) is a smooth affine V-group of relative dimension 2; it is even an open subscheme of the V-scheme $\mathrm{R_{V'/V}}(\mathbf{A}^1_{\mathrm{V'}}) = \mathbf{A}^2_{\mathrm{V}}$ (see [**CGP**, Prop. A.5.2]). In particular, the fibers of G are connected (as is also a consequence of general connectedness results for Weil restriction of *smooth* schemes [**CGP**, Prop. A.5.9]). The fibral connectedness for G can also be seen by inspection: $\mathrm{G_K} = \mathrm{R_{K'/K}}(\mathbf{G}_{\mathrm{m}})$ is a torus since $\mathrm{K'/K}$ is separable, and since $\mathrm{K'/K}$ is not unramified the geometric special fiber is $\mathrm{R}_{\overline{k}[\epsilon]/\overline{k}}(\mathbf{G}_{\mathrm{m}}) = \mathbf{G}_{\mathrm{m}} \times \mathbf{G}_{\mathrm{a}}$ (via $(t, x) \mapsto t(1 + x\epsilon)$).

Thus, the evident $\mathbf{G}_{\mathrm{m}}$ as a V-subgroup of G is a torus not contained in any strictly larger torus of G (due to consideration of the special fiber), but its geometric generic fiber is not maximal in that of G.

For our purposes, "maximal torus" will always be taken in the sense of Definition 3.2.1. See Remark A.1.2 for the equivalence with another possible definition (using actual fibers rather than geometric fibers). The case of most interest to us is maximal tori in reductive group schemes. In such groups, the maximality property is robust with respect to the Zariski topology:

Example 3.2.4. — Proposition 3.1.9(2) implies that if T is a torus in a reductive group scheme $\mathrm{G} \to \mathrm{S}$ and it is maximal on the $\overline{s}$-fiber for some $s \in \mathrm{S}$ then it is maximal in the sense of Definition 3.2.1 over a Zariski-open neighborhood of s.

To study maximal tori via "reduction to the noetherian case", we require:

Lemma 3.2.5. — *Let* $\{\mathrm{A}_i\}$ *be a directed system of rings with direct limit* A, *and let* G *be a smooth affine* A*-group equipped with an* A*-torus* $\mathrm{T} \subset \mathrm{G}$. *Pick*

i_0 so that G *descends to a smooth affine* A_{i_0}*-group* G_{i_0} *and* T *descends to an* A_{i_0}*-torus* T_{i_0} *in* G_{i_0}*. For* $i \geqslant i_0$*, define the pair* $(\mathrm{G}_i, \mathrm{T}_i)$ *over* A_i *by scalar extension of the pair* $(\mathrm{G}_{i_0}, \mathrm{T}_{i_0})$ *over* A_{i_0}*.*

If T *is maximal in* G *then* T_i *is maximal in* G_i *for sufficiently large* i*.*

The proof of this lemma uses standard direct limit arguments as well as constructions that are specific to group schemes.

Proof. — We will use centralizers for tori (as in Lemma 2.2.4) and elementary affineness results for quotients by central tori (see [**SGA3**, VIII, 5.1; IX, 2.3]). It is harmless to replace G_{i_0} with the finitely presented affine centralizer $\mathrm{Z}_{\mathrm{G}_{i_0}}(\mathrm{T}_{i_0})$ (and replace G_i with $\mathrm{Z}_{\mathrm{G}_{i_0}}(\mathrm{T}_{i_0})_{\mathrm{A}_i} = \mathrm{Z}_{\mathrm{G}_i}(\mathrm{T}_i)$ for all $i \geqslant i_0$, and replace G with their common base change $\mathrm{Z}_{\mathrm{G}}(\mathrm{T})$ over A), so we may assume that T_{i_0} is central in G_{i_0} over A_{i_0} (and similarly over A_i for $i \geqslant i_0$, as well as over A). Then by passing to the affine quotient $\mathrm{G}_{i_0}/\mathrm{T}_{i_0}$ over A_{i_0}, and similarly over every A_i and over A, we reduce to the case that all T_i and T are trivial.

Letting $\mathrm{S} = \mathrm{Spec}(\mathrm{A})$ and $\mathrm{S}_i = \mathrm{Spec}(\mathrm{A}_i)$ for all i, we are brought to the case that the smooth affine geometric fibers $\mathrm{G}_{\overline{s}}$ all have no nontrivial tori (equivalently, $\mathrm{G}_{\overline{s}}^0$ is unipotent for all $s \in \mathrm{S}$). We seek to prove the same property holds for all geometric fibers of $\mathrm{G}_i \to \mathrm{S}_i$ when i is sufficiently large. We may and do assume that S is non-empty, and hence likewise all S_i are non-empty. Since $\mathrm{G}_{i_0} \to \mathrm{S}_{i_0}$ is finitely presented, the number of connected components for its geometric fibers is bounded [**EGA**, IV_3, 9.7.9]. The same bound is valid for the geometric fibers of every $\mathrm{G}_i \to \mathrm{S}_i$ for $i \geqslant i_0$, as well as for the geometric fibers of $\mathrm{G} \to \mathrm{S}$. Thus, if we choose a prime number p larger than such a bound then the unipotence of all $\mathrm{G}_{\overline{s}}^0$ implies that for all $s \in \mathrm{S}$ there are no nontrivial group homomorphisms $\mu_p \to \mathrm{G}_{\overline{s}}$. In other words, for the affine and finitely presented S-scheme $\mathrm{Y} := \underline{\mathrm{Hom}}_{\mathrm{S}\text{-gp}}(\mu_p, \mathrm{G})$ and the canonical section $\sigma : \mathrm{S} \to \mathrm{Y}$ over S corresponding to the *trivial* S-map $\mu_p \to \mathrm{G}$, the map σ is surjective.

Let $\mathrm{Y}_i = \underline{\mathrm{Hom}}_{\mathrm{S}_i\text{-gp}}(\mu_p, \mathrm{G}_i)$, so $\mathrm{Y}_i = \mathrm{Y}_{i_0} \otimes_{\mathrm{A}_{i_0}} \mathrm{A}_i$ compatibly with change in $i \geqslant i_0$ and the limit of these affine schemes is Y. Since the canonical sections $\sigma_i : \mathrm{S}_i \to \mathrm{Y}_i$ defined by the trivial S_i-maps $\mu_p \to \mathrm{G}_i$ yield the surjective map σ in the limit, for sufficiently large $i \geqslant i_0$ the map σ_i is surjective [**EGA**, IV_3, 8.10.5(vi)]. Fix such a large i, so the geometric fibers of the smooth S_i-group $\mathrm{G}_i \to \mathrm{S}_i$ do not contain μ_p as a subgroup scheme over S_i (since S_i is non-empty). It follows that the identity components of these fibers cannot contain a nontrivial torus. Hence, these identity components are unipotent, as desired. □

The following fundamental result concerning the "scheme of maximal tori" is the engine that makes the relative theory of reductive groups work (and it is proved in [**SGA3**, XII, 1.10] under weaker hypotheses than we impose):

Theorem 3.2.6. — *Let* $\mathrm{G} \to \mathrm{S}$ *be a smooth* S*-affine group scheme such that in the identity component* $\mathrm{G}^0_{\overline{s}}$ *of each geometric fiber the maximal tori are their own centralizers. Then the functor on* S*-schemes*

$$\underline{\mathrm{Tor}}_{\mathrm{G}/\mathrm{S}} : \mathrm{S}' \rightsquigarrow \{\textit{maximal tori in } \mathrm{G}_{\mathrm{S}'}\}$$

is represented by a smooth quasi-affine S*-scheme* $\mathrm{Tor}_{\mathrm{G}/\mathrm{S}}$*, and* $\mathrm{Tor}_{\mathrm{G}/\mathrm{S}} \to \mathrm{S}$ *is surjective.*

If T *is a maximal torus of* G *then the map* $\mathrm{G}/\mathrm{N}_{\mathrm{G}}(\mathrm{T}) \to \mathrm{Tor}_{\mathrm{G}/\mathrm{S}}$ *defined by* G*-conjugation against* T *is an isomorphism. In particular, any two maximal tori of* G *are conjugate étale-locally on* S.

The main case of interest for which we can verify the hypothesis "Cartan subgroups of (identity components of) geometric fibers are tori" is the case of reductive group schemes, but another interesting case that will arise later is parabolic subgroups of reductive group schemes (to be defined and studied in § 5.2). In [**SGA3**, XII, 5.4] it is shown that $\mathrm{Tor}_{\mathrm{G}/\mathrm{S}}$ is actually S-affine in Theorem 3.2.6. The proof of this finer property uses a hard representability theorem [**SGA3**, XI, 4.1] that we are avoiding. We do not need such affineness, so we say nothing further about it. See Exercise 3.4.8 for a generalization of Theorem 3.2.6 over fields.

Proof. — By effectivity of fppf descent for schemes that are quasi-affine over the base, it suffices to work fppf-locally on S (since the functor in question is clearly an fppf sheaf of sets). Choose $s \in \mathrm{S}$. As in the proof of Proposition 3.1.9(1), we may work fppf-locally around s to arrange that there exists a torus $\mathrm{T} \subset \mathrm{G}$ such that $\mathrm{T}_{\overline{s}}$ is maximal in $\mathrm{G}^0_{\overline{s}}$. The inclusion $\mathrm{T} \subset \mathrm{Z}_{\mathrm{G}}(\mathrm{T})^0$ between smooth S-groups is an isomorphism on geometric fibers at $\overline{s}$, so by connectedness and fiber dimension considerations it follows that the same holds on geometric fibers over points near s. Hence, by passing to a Zariski-open neighborhood of s in S we may arrange that $\mathrm{T} = \mathrm{Z}_{\mathrm{G}}(\mathrm{T})^0$, so T is maximal in G (in the sense of Definition 3.2.1). Now we will just use that $\mathrm{G} \to \mathrm{S}$ is a smooth S-affine group.

By Theorem 2.3.1, the quotient $\mathrm{G}/\mathrm{N}_{\mathrm{G}}(\mathrm{T})$ exists as a smooth quasi-affine S-scheme, and clearly $\mathrm{G}/\mathrm{N}_{\mathrm{G}}(\mathrm{T}) \to \mathrm{S}$ is surjective. There is an evident G-action on $\underline{\mathrm{Tor}}_{\mathrm{G}/\mathrm{S}}$ via conjugation, and the point $\mathrm{T} \in \underline{\mathrm{Tor}}_{\mathrm{G}/\mathrm{S}}(\mathrm{S})$ has stabilizer subfunctor in G represented by the transporter scheme $\mathrm{Transp}_{\mathrm{G}}(\mathrm{T},\mathrm{T}) = \mathrm{N}_{\mathrm{G}}(\mathrm{T})$. Thus, $\mathrm{G}/\mathrm{N}_{\mathrm{G}}(\mathrm{T})$ is a subfunctor of $\underline{\mathrm{Tor}}_{\mathrm{G}/\mathrm{S}}$, and we claim that it equals $\underline{\mathrm{Tor}}_{\mathrm{G}/\mathrm{S}}$. It suffices to prove that the map $\mathrm{G} \to \underline{\mathrm{Tor}}_{\mathrm{G}/\mathrm{S}}$ between fppf sheaves of sets is surjective, which is to say that for any S-scheme S′ and $\mathrm{T}' \in \underline{\mathrm{Tor}}_{\mathrm{G}/\mathrm{S}}(\mathrm{S}')$ there exists an fppf covering $\mathrm{S}'' \to \mathrm{S}'$ and a $g \in \mathrm{G}(\mathrm{S}'')$ such that $g\mathrm{T}'_{\mathrm{S}''}g^{-1} = \mathrm{T}_{\mathrm{S}''}$. Put another way, we are claiming that the maximal tori $\mathrm{T}_{\mathrm{S}'}$ and T′ in $\mathrm{G}_{\mathrm{S}'}$ are conjugate fppf-locally on S′. Upon renaming S′ as S, our task is to prove that any two maximal tori T and T' in G are conjugate fppf-locally on S.

Direct limit arguments (along with Lemma 3.2.5) allow us to arrange that S is noetherian and affine, and even finite type over $\mathbf{Z}$ (which will be relevant when we apply Artin approximation later in the argument). Pick $s \in \mathrm{S}$, so $\mathrm{T}_{\overline{s}}$ and $\mathrm{T}'_{\overline{s}}$ are $\mathrm{G}(\overline{s})$-conjugate for an algebraic geometric point $\overline{s}$ over s. Express $k(\overline{s})$ as a direct limit of subextensions of finite degree over $k(s)$, so there exists a finite extension $\mathrm{K}/k(s)$ such that T_{K} and T'_{K} are G(K)-conjugate and split. Passing to an fppf neighborhood of (S, s) then brings us to the case that T_s and T'_s are $\mathrm{G}(s)$-conjugate and split. Let A denote the completion $\mathscr{O}^{\wedge}_{\mathrm{S},s}$.

By the same style of Artin approximation argument as used in the proof of Proposition 3.1.6 (this is where we need that S is finite type over $\mathbf{Z}$), if T_{A} is G(A)-conjugate to T'_{A} then we can build an étale neighborhood U of (S, s) such that T_{U} is G(U)-conjugate to T'_{U}, so we would be done. Thus, we may and do replace S with Spec(A) to arrange that (i) S is the spectrum of a complete local noetherian ring $(\mathrm{A}, \mathfrak{m})$ with residue field denoted k, and (ii) the special fibers T_0 and T'_0 are $\mathrm{G}(k)$-conjugate and split. By lifting a conjugating element from $\mathrm{G}(k)$ into G(A) (as we may do since the local noetherian A is complete and G is smooth and affine), we can arrange that $\mathrm{T}_0 = \mathrm{T}'_0$ inside the special fiber G_k.

It follows from the completeness of A that T and T' are split, hence abstractly isomorphic. The splitting isomorphisms $\mathrm{T} \simeq \mathbf{G}_{\mathrm{m}}^r$ and $\mathrm{T}' \simeq \mathbf{G}_{\mathrm{m}}^r$ can be chosen to coincide on the special fibers (i.e., they induced the same splitting of the torus $\mathrm{T}_0 = \mathrm{T}'_0$ in G_0), so the inclusions of T and T' into G may be viewed as a pair of inclusions from a common A-torus $\mathbf{G}_{\mathrm{m}}^r$ into G that agree residually. Thus, by Corollary B.3.5, these inclusions are G(A)-conjugate. □

In earlier arguments we have built maximal tori in reductive group schemes by working fppf-locally on the base. Now we show that it always suffices to work étale-locally:

Corollary 3.2.7. — *Any reductive group scheme* $\mathrm{G} \to \mathrm{S}$ *admits a maximal torus étale-locally on* S. *In particular, any connected reductive group over a field k admits a geometrically maximal torus defined over a finite separable extension of k.*

Proof. — The structural morphism $\mathrm{Tor}_{\mathrm{G/S}} \to \mathrm{S}$ from the scheme of maximal tori is a smooth surjection, so it admits sections étale-locally on S. But to give a section over an S-scheme U is to give an element of the set $\mathrm{Tor}_{\mathrm{G/S}}(\mathrm{U})$, which is to say a maximal torus of G_{U}. □

The following result is a special case of part of [**SGA3**, XII, 5.4].

Proposition 3.2.8. — *Let* T *be a maximal torus in a reductive group scheme* $G \to S$*. The Weyl group* $W_G(T) := N_G(T)/Z_G(T)$ *is finite étale over* S*.*

Recall from Theorem 2.3.1 that a priori (without using reductivity) the quotient sheaf $W_G(T)$ is represented by an S-scheme that is separated, étale, and finitely presented.

Proof. — By Proposition 3.1.9 (and Corollary 3.1.11) we use direct limit arguments to reduce to the case that S is noetherian, so we can apply to $W_G(T) \to S$ a general finiteness criterion for quasi-compact separated étale maps: it suffices that the number of points in the geometric fibers is Zariski-locally constant on the base. This finiteness criterion follows from étale localization on the base and the local structure theorem for such morphisms over a henselian local base in [**EGA**, IV_4, 18.5.11(c)]. (See Exercise 3.4.2 for a generalization.) In other words, we claim that the order of the fibral Weyl group $W_G(T)_{\overline{s}} = W_{G_{\overline{s}}}(T_{\overline{s}})$ is locally constant in s.

Since the Weyl group of a connected reductive group over an algebraically closed field coincides with the Weyl group of the associated root system, it suffices to check that the isomorphism class of the root system $\Phi(G_{\overline{s}}, T_{\overline{s}})$ is locally constant in s. By passing to an étale neighborhood of a chosen point of S we can arrange that

$$T = D_S(M) = \underline{\mathrm{Hom}}_{S\text{-gp}}(M_S, \mathbf{G}_m)$$

for a finite free $\mathbf{Z}$-module M (so the abelian sheaf $\underline{\mathrm{Hom}}_{S\text{-gp}}(T, \mathbf{G}_m)$ is thereby identified with M_S). In the proof of Proposition 3.1.9(1) we saw that the vector bundle $\mathfrak{g}$ with its T-action decomposes into a direct sum of "root subbundles" and that this decomposition identifies the root system $\Phi(G_{\overline{\xi}}, T_{\overline{\xi}}) \subset X(T_{\overline{\xi}}) = M = X(T_{\overline{s}})$ with $\Phi(G_{\overline{s}}, T_{\overline{s}})$ for all ξ near s. □

3.3. Scheme-theoretic and reductive center. — The final topic in this section is the functorial center of a reductive group scheme. This material is developed in [**SGA3**, XII, § 4], and we will navigate our way towards a single result for reductive group schemes (which in turn is a special case of [**SGA3**, XII, 4.11]). We begin with two lemmas.

Lemma 3.3.1. — *Let* G *be a smooth* S*-affine* S*-group with connected fibers, and* H *a subgroup of multiplicative type.*

1. *If* H *is normal in* G *then it is central in* G*.*
2. *Assume* H *is normal in* G*, and let* $\overline{G} = G/H$ *be the associated smooth* S*-affine quotient* [**SGA3**, VIII, 5.1; IX, 2.3]*. For every central multiplicative type subgroup* $\overline{H}'$ *in* $\overline{G}$*, the preimage* H' *of* $\overline{H}'$ *in* G *is a central multiplicative type subgroup of* G*.*

Part (2) of this lemma serves a role in our treatment akin to the role of [**SGA3**, XII, 4.7] in the general development of [**SGA3**, XII].

Proof. — The normality of H in G implies that of each $\mathrm{H}[n]$ in G, and the weak schematic density of $\{\mathrm{H}[n]\}$ in H after any base change (Example 2.1.9) implies that H is central in G if all $\mathrm{H}[n]$ are central in G. Hence, to prove (1) we replace H with $\mathrm{H}[n]$ for arbitrary $n > 0$ so that H is S-finite. Then the automorphism functor of H is identified (up to inversion) with that of its finite étale Cartier dual, so this functor is represented by a finite étale S-scheme (via effective descent for *finite* étale schemes over the base, applied to the automorphism functor of the constant Cartier dual over an étale cover of S).

The conjugation action of G on H is classified by an S-group map from G to the finite étale automorphism scheme E of H. The identity section of $\mathrm{E} \to \mathrm{S}$ is an open and closed immersion (as for any section to a finite étale map), so the kernel of the action map $\mathrm{G} \to \underline{\mathrm{Aut}}_{\text{S-gp}}(\mathrm{H}) = \mathrm{E}$ is an S-subgroup scheme of G that is both open and closed. Thus, (1) is reduced to the obvious fact that for any map of schemes $\mathrm{X} \to \mathrm{S}$ with connected fibers, the only open and closed subscheme of X that maps onto S is X.

To prove (2), first note that H' is clearly finitely presented and flat (and affine) over S, and it is normal in G since $\overline{\mathrm{H}}'$ is central in $\overline{\mathrm{G}}$. In particular, by (1) it suffices to prove that H' is multiplicative type. We shall check directly that H' is central in G and hence is commutative, so then Corollary B.4.2(2) will ensure that H' is of multiplicative type. Since H is central in G and the quotient $\mathrm{H}'/\mathrm{H} = \overline{\mathrm{H}}'$ is central in $\mathrm{G}/\mathrm{H} = \overline{\mathrm{G}}$, the G-action on H' via conjugation is classified by a homomorphism from G to the Hom-functor

$$\underline{\mathrm{Hom}}_{\text{S-gp}}(\overline{\mathrm{H}}', \mathrm{H})$$

via $g \mapsto (\overline{h}' \mapsto gh'g^{-1})$ where h' is an fppf-local lift of $\overline{h}'$ to H', the choice of which does not matter. This map to the Hom-functor vanishes if and only if H' is central in G, so by the weak schematic density of $\{\overline{\mathrm{H}}'[n]\}$ in $\overline{\mathrm{H}}'$ after any base change it suffices to replace $\overline{\mathrm{H}}'$ with $\overline{\mathrm{H}}'[n]$ (for an arbitrary $n > 0$) and to replace H' with the preimage H'_n of $\overline{\mathrm{H}}'[n]$ in H'. But then we have the equality of functors

$$\underline{\mathrm{Hom}}_{\text{S-gp}}(\overline{\mathrm{H}}', \mathrm{H}) = \underline{\mathrm{Hom}}_{\text{S-gp}}(\overline{\mathrm{H}}', \mathrm{H}[n]),$$

and via Cartier duality the right side is identified with a Hom-functor between finite étale S-groups. In particular, this is represented by a finite étale S-group, so as in the proof of (1) any S-homomorphism from G to this Hom-scheme vanishes. Thus, H' is central in G. $\square$

The following lemma identifies an interesting property for multiplicative type subgroups that are central on geometric fibers.

Lemma 3.3.2. — *Let* G *be a smooth* S*-affine* S*-group with connected fibers, and* $\mathrm{H} \subset \mathrm{G}$ *an* S*-subgroup of multiplicative type. For all geometric points* $\overline{s}$ *of* S*, assume* $\mathrm{H}_{\overline{s}}$ *is central in* $\mathrm{G}_{\overline{s}}$ *and contains all central multiplicative type subgroups of* $\mathrm{G}_{\overline{s}}$*. Then* H *is central in* G *and for any* S*-scheme* S'*, every central multiplicative type subgroup of* $\mathrm{G}_{\mathrm{S}'}$ *is contained in* $\mathrm{H}_{\mathrm{S}'}$*.*

Proof. — By Lemma 2.2.4, the functorial centralizer of H in G is represented by a smooth closed S-subgroup $\mathrm{Z}_{\mathrm{G}}(\mathrm{H})$. The inclusion $\mathrm{Z}_{\mathrm{G}}(\mathrm{H}) \hookrightarrow \mathrm{G}$ between smooth S-affine S-groups is an isomorphism on s-fibers for all $s \in \mathrm{S}$ and hence (by Lemma B.3.1) is an isomorphism. Thus, H is central in G. Since each G_s is connected, by dimension and smoothness considerations the same method shows that the centrality of $\mathrm{H}_{\overline{s}}$ in $\mathrm{G}_{\overline{s}}$ for a single geometric point $\overline{s}$ over s implies that H_{U} is central in G_{U} for some open neighborhood U of s in S.

The fppf quotient sheaf G/H has a natural S-group structure due to centrality of H in G, and by [**SGA3**, VIII, 5.1; IX, 2.3] it is represented by an S-affine S-scheme of finite presentation. By Exercise 2.4.3 and Lemma 3.3.1(2) we may rename G/H as G so that each $\mathrm{G}_{\overline{s}}$ contains no nontrivial central closed subgroup of multiplicative type. This property persists after any base change $\mathrm{S}' \to \mathrm{S}$, so upon renaming S' as S it remains to show that a central multiplicative type subgroup $\mathrm{H}' \subset \mathrm{G}$ must be trivial. Every geometric fiber $\mathrm{H}'_{\overline{s}}$ is trivial, so consideration of the character group of the multiplicative type H' forces H' to be the trivial S-group. □

In view of the preceding lemma, the following definition (taken from [**SGA3**, XII, 4.1]) is reasonable as well as checkable in practice.

Definition 3.3.3. — Let $\mathrm{G} \to \mathrm{S}$ be a smooth S-affine group scheme with connected fibers. A *reductive center* of G is a central multiplicative type subgroup $\mathrm{H} \subset \mathrm{G}$ that satisfies the conditions in Lemma 3.3.2.

It is clear that if a reductive center exists then it is unique; we then call it *the* reductive center. Finally, we arrive at a special case of [**SGA3**, XII, 4.11]:

Theorem 3.3.4. — *Any reductive group scheme* $\mathrm{G} \to \mathrm{S}$ *admits a reductive center* Z*, and* Z *coincides with the scheme-theoretic center* Z_{G} *of* G*. In particular,* Z_{G} *is* S*-flat. Moreover,* Z *represents the kernel of the action map*

$$u : \mathrm{G} \to \underline{\mathrm{Aut}}_{\mathrm{S}}(\mathrm{Tor}_{\mathrm{G}/\mathrm{S}}).$$

Proof. — Since the formation of the reductive center commutes with base change if it exists, by effective fppf descent for schemes affine over the base we may work fppf-locally on S. Hence, we may and do assume that G contains a split maximal torus T. This identifies $\mathrm{G}/\mathrm{N}_{\mathrm{G}}(\mathrm{T})$ with $\mathrm{Tor}_{\mathrm{G}/\mathrm{S}}$ via $g \mapsto g\mathrm{T}g^{-1}$, so the natural G-action on $\mathrm{Tor}_{\mathrm{G}/\mathrm{S}}$ via conjugation goes over to the left translation action of G on $\mathrm{G}/\mathrm{N}_{\mathrm{G}}(\mathrm{T})$. We will first construct a reductive center Z, and then prove that it is the center and represents the functorial kernel $\ker u$.

We may and do assume S is affine and $\mathrm{T} = \mathrm{D}_{\mathrm{S}}(\mathrm{M}) \simeq \mathbf{G}_{\mathrm{m}}^r$ for $\mathrm{M} \simeq \mathbf{Z}^r$ with some $r \geqslant 0$. Consider the $\mathscr{O}_{\mathrm{S}}$-linear M-graded decomposition $\bigoplus_{m \in \mathrm{M}} \mathfrak{g}_m$ of $\mathfrak{g} = \mathrm{Lie}(\mathrm{G})$ into weight spaces under the T-action, with $t.v = m(t)v$ for $v \in \mathfrak{g}_m$ (see [**Oes**, III, 1.5] or [**CGP**, Lemma A.8.8]). These weight spaces are vector bundles since they are direct summands of $\mathfrak{g}$, and the formation of this decomposition commutes with base change on S. By working Zariski-locally on S, we may arrange that the weight spaces $\mathfrak{g}_{m_0}$ that are nonzero have constant rank (necessarily rank 1 for $m_0 \neq 0$, by the reductivity hypothesis). Let $\Phi \subset \mathrm{M}$ be the finite set of nontrivial weights that arise. Define

$$\mathrm{H} = \bigcap_{\alpha \in \Phi} \ker(\alpha : \mathrm{T} \twoheadrightarrow \mathbf{G}_{\mathrm{m}}).$$

This intersection of finitely many multiplicative type S-subgroups of T is a multiplicative type S-subgroup of T (corresponding to the quotient of M by the **Z**-submodule spanned by Φ), and by construction the H-action on G via conjugation induces the trivial action on $\mathfrak{g}$.

The centralizer $\mathrm{Z}_{\mathrm{G}}(\mathrm{H})$ is a smooth closed S-subgroup of G (Lemma 2.2.4) and its Lie algebra is $\mathfrak{g}^{\mathrm{H}}$ (Proposition 1.2.3). By design $\mathfrak{g}^{\mathrm{H}} = \mathfrak{g}$, so $\mathrm{Z}_{\mathrm{G}}(\mathrm{H}) \hookrightarrow \mathrm{G}$ is an isomorphism between s-fibers for all $s \in \mathrm{S}$ and hence is an equality by Lemma B.3.1. This shows that H is a central multiplicative type S-subgroup of G. But for any S-scheme S′ and $g' \in \mathrm{G}(\mathrm{S}')$ that centralizes $\mathrm{G}_{\mathrm{S}'}$, clearly $g' \in \mathrm{Z}_{\mathrm{G}(\mathrm{S}')}(\mathrm{T}_{\mathrm{S}'}) = (\mathrm{Z}_{\mathrm{G}}(\mathrm{T}))(\mathrm{S}') = \mathrm{T}(\mathrm{S}')$ (Proposition 3.1.9(2)) and the adjoint action of $\mathrm{T}_{\mathrm{S}'}$ on $\mathfrak{g}_{\mathrm{S}'} = \mathrm{Lie}(\mathrm{G}_{\mathrm{S}'})$ makes the point $g' \in \mathrm{T}(\mathrm{S}')$ act trivially, so g' is an S′-point of $\bigcap_{\alpha \in \Phi} \ker(\alpha_{\mathrm{S}'}) = \mathrm{H}_{\mathrm{S}'}$. This shows that H is a reductive center of G, and that it represents the functorial center of G. The centrality also forces $\mathrm{H} \subset \ker u$, so it remains to prove (after making a base change by any $\mathrm{S}' \to \mathrm{S}$ and renaming S′ as S) that any $g \in \mathrm{G}(\mathrm{S})$ with trivial conjugation action on $\mathscr{T} := \mathrm{Tor}_{\mathrm{G}/\mathrm{S}}$ necessarily lies in H(S).

By the usual limit arguments (including Corollary 3.1.11) we may reduce to the case when S is noetherian, and then artin local (by the Krull intersection theorem relative to the ideal of H in $\mathscr{O}_{\mathrm{G}}$), and finally artin local with algebraically closed residue field (via faithfully flat base change [**EGA**, 0_{III}, 10.3.1]). Writing $\mathrm{S} = \mathrm{Spec}(\mathrm{A})$ for an artin local ring A, the set of A-points of the A-smooth $\mathscr{T}$ is relatively schematically dense [**EGA**, IV_3, 11.9.13, 11.10.9]; i.e., for any A-algebra A′, an A′-morphism from $\mathscr{T}_{\mathrm{A}'}$ to a separated A′-scheme is uniquely determined by its effect on the points of $\mathscr{T}(\mathrm{A})$ (viewed as A′-points of $\mathscr{T}_{\mathrm{A}'}$). In other words, $\ker u$ is *represented* by the intersection of the closed subschemes $\mathrm{N}_{\mathrm{G}}(\mathrm{T}_\sigma)$ where T_σ is the maximal A-torus in G corresponding to the varying point $\sigma \in \mathscr{T}(\mathrm{A})$.

We have $\mathrm{H} \subset \ker u$ as closed subschemes of G, and seek to prove this is an equality. By Lemma B.3.1, this reduces to the consideration of the special fiber (as the artin local S has only one point). Now we are considering a connected reductive group G over an algebraically closed field k, so every central closed k-subgroup scheme is of multiplicative type since any maximal torus in G is its own schematic centralizer (Corollary 1.2.4). The above description of $\ker u$ becomes $\ker u = \bigcap_{\mathrm{T}'} \mathrm{N}_{\mathrm{G}}(\mathrm{T}')$ with T' varying through the maximal k-tori of G. Since $\mathrm{G}/\mathrm{Z}_{\mathrm{G}}$ is perfect, a normal closed subgroup scheme of G is central if and only if its identity component is central (see [**CGP**, Lemma 5.3.2] for a self-contained proof of this elementary fact). Thus, since $(\ker u)^0 \subset \bigcap_{\mathrm{T}'} \mathrm{T}'$, to show $\ker u = \mathrm{H}$ it suffices to prove that $\bigcap_{\mathrm{T}'} \mathrm{T}'$ is central in G.

Consider the smooth closed subgroup N of G generated by the maximal tori. This is normal, so G/N is a reductive group that contains no nontrivial tori (since quotient maps between linear algebraic k-groups carry maximal tori onto maximal tori). Hence, G/N is unipotent and therefore trivial. In other words, G is generated by its maximal tori, so there exists a finite set $\{\mathrm{T}'_i\}$ of maximal tori of G such that the multiplication map of k-schemes $q : \prod \mathrm{T}'_i \to \mathrm{G}$ is dominant. A dominant map between k-varieties is generically flat, so there are dense open $\mathrm{V} \subset \mathrm{G}$ and $\mathrm{V}' \subset q^{-1}(\mathrm{V})$ such that $\mathrm{V}' \to \mathrm{V}$ is faithfully flat. Any (functorial) point of $\bigcap_{\mathrm{T}'} \mathrm{T}' = \ker u$ centralizes all T'_i and hence centralizes $q(\mathrm{V}') = \mathrm{V}$, so $\ker u$ centralizes G (as any S-endomorphism of G_{S} for a k-scheme S is determined by its restriction to V_{S}, by [**EGA**, IV_3, 11.9.13, 11.10.1(d)]). $\square$

Corollary 3.3.5. — *Let* G *be a reductive* S*-group scheme, and* Z *a multiplicative type subgroup scheme of the center* Z_{G}*. The reductive quotient* G/Z *has center* $\mathrm{Z}_{\mathrm{G}}/\mathrm{Z}$*; in particular,* $\mathrm{G}/\mathrm{Z}_{\mathrm{G}}$ *has trivial center.*

Moreover, $\mathrm{T} \mapsto \mathrm{T}/\mathrm{Z}$ *defines a bijective correspondence between the set of maximal tori of* G *and the set of maximal tori of* G/Z*, with inverse given by scheme-theoretic preimage under the quotient map* $\mathrm{G} \to \mathrm{G}/\mathrm{Z}$*.*

Any such T contains the central Z in G since $\mathrm{T} = \mathrm{Z}_{\mathrm{G}}(\mathrm{T})$, so T/Z makes sense. This corollary is a special case of [**SGA3**, XII, 4.7(b),(c)].

Proof. — By Lemma 3.3.1, the set of central subgroups of multiplicative type in G/Z is in bijection with the set of central subgroups of multiplicative type in G containing Z, which is to say the multiplicative type subgroups of Z_{G} containing Z (as Z_{G} is the reductive center of G). Those subgroups correspond to multiplicative type subgroups of G/Z contained in $\mathrm{Z}_{\mathrm{G}}/\mathrm{Z}$. Thus, $\mathrm{Z}_{\mathrm{G}}/\mathrm{Z}$ is the reductive center of G/Z, so it is the center of G/Z.

To establish the bijective correspondence for maximal tori, we first note that the classical theory ensures that T/Z is maximal in G/Z when T is maximal in G. Hence, the proposed correspondence makes sense. It is also clear that T is the preimage of T/Z under the quotient map $\mathrm{G} \to \mathrm{G}/\mathrm{Z}$, so the only problem is to prove that every maximal torus $\overline{\mathrm{T}}$ in $\overline{\mathrm{G}} := \mathrm{G}/\mathrm{Z}$ has the form T/Z for some (necessarily unique) maximal torus T in G. Consider the preimage $\widetilde{\mathrm{T}}$ of $\overline{\mathrm{T}}$ in G. We seek to prove that $\widetilde{\mathrm{T}}$ is a maximal torus in G.

Since $\overline{\mathrm{G}}$ is the quotient of G modulo a central subgroup scheme, the conjugation action of G on itself factors through a left action of $\overline{\mathrm{G}}$ on G. As such, we get a left action of the torus $\overline{\mathrm{T}}$ on G, so the functorial centralizer $\mathrm{G}^{\overline{\mathrm{T}}}$ for this action is represented by a smooth closed subgroup of G with connected geometric fibers (by arguing as in Example 2.2.6: we form the $\overline{\mathrm{T}}$-centralizer in the reductive semi-direct product $\mathrm{G} \rtimes \overline{\mathrm{T}}$). Consideration of geometric fibers shows that $\mathrm{G}^{\overline{\mathrm{T}}}$ is reductive over S. But $\mathrm{Z} \subset \mathrm{G}^{\overline{\mathrm{T}}}$, so we get a natural monomorphism

$$j : \mathrm{G}^{\overline{\mathrm{T}}}/\mathrm{Z} \to \mathrm{Z}_{\overline{\mathrm{G}}}(\overline{\mathrm{T}}) = \overline{\mathrm{T}}$$

between smooth S-affine S-groups. To show j is an isomorphism, we pass to geometric fibers (by Lemma B.3.1) so the monic j is a closed immersion (Proposition 1.1.1). Then j is surjective since surjections between smooth connected affine groups over a field induce surjections between centralizers for a torus action [**Bo91**, 11.14, Cor. 2]. Thus, j is an isomorphism.

The isomorphism $\mathrm{G}^{\overline{\mathrm{T}}}/\mathrm{Z} = \overline{\mathrm{T}}$ implies that $\mathrm{G}^{\overline{\mathrm{T}}}$ is the preimage $\widetilde{\mathrm{T}}$ of $\overline{\mathrm{T}}$ in G. Apply Lemma 3.3.1(2) to the quotient map $\mathrm{G}^{\overline{\mathrm{T}}} \to \mathrm{G}^{\overline{\mathrm{T}}}/\mathrm{Z} = \overline{\mathrm{T}}$ to conclude that the reductive $\mathrm{G}^{\overline{\mathrm{T}}}$ is central in itself (i.e., commutative), so $\mathrm{G}^{\overline{\mathrm{T}}}$ is a torus by Corollary 3.2.7. It suffices to prove that it is a maximal torus in G (in the sense of Definition 3.2.1). Passing to geometric fibers over S, now $\mathrm{S} = \mathrm{Spec}(k)$ for an algebraically closed field k. Consider a torus T' of G containing $\mathrm{G}^{\overline{\mathrm{T}}}$. The image T'/Z in $\overline{\mathrm{G}}$ is a torus (as it is multiplicative type and smooth), yet this image contains the torus $\mathrm{G}^{\overline{\mathrm{T}}}/\mathrm{Z} = \overline{\mathrm{T}}$ that is maximal by hypothesis, so $\mathrm{T}'/\mathrm{Z} = \mathrm{G}^{\overline{\mathrm{T}}}/\mathrm{Z}$ and hence $\mathrm{T}' = \mathrm{G}^{\overline{\mathrm{T}}}$. Thus, $\mathrm{G}^{\overline{\mathrm{T}}}$ is a maximal torus in G. □

Corollary 3.3.6. — *Let* $\mathrm{G} \to \mathrm{S}$ *be a reductive group scheme, and* T *an* S-*torus in* G *that is maximal* (*in the sense of Definition* 3.2.1).

1. *The center* Z_G *is the kernel of the adjoint action* $\mathrm{T} \to \mathrm{GL}(\mathfrak{g})$.
2. *If* $\mathrm{S} = \mathrm{Spec}\, k$ *for an algebraically closed field* k *then* Z_G *coincides with the scheme-theoretic intersection of all maximal tori* T' *in* G.

This corollary is a special case of [**SGA3**, XII, 4.7(d), 4.10].

Proof. — To prove (1) we may work étale-locally on S so that T is split. In that case the equality in (1) was shown in the course of proving Theorem 3.3.4. For (2), since $\mathrm{Z}_\mathrm{G}(\mathrm{T}') = \mathrm{T}'$ for all T' (as subschemes of G), clearly every T' contains Z_G. It remains to show that the intersection of all T' is central in G (as a subgroup scheme), and we showed this in the proof of Theorem 3.3.4. □

Example 3.3.7. — Corollary 3.3.6(1) provides a way to compute $\mathrm{Z_G}$, since it is often easy to find a torus T such that $\mathrm{Z_G(T) = T}$ (equivalently $\mathfrak{g}^{\mathrm{T}} = \mathfrak{t}$, due to smoothness and fibral connectedness of torus centralizers in G).

We illustrate with $\mathrm{G} = \mathrm{SL}_n$ over any scheme S. The diagonal torus T is maximal, since the case of geometric fibers is well-known. The action of T on the Lie algebra $\mathfrak{sl}_n$ over any base S is the given by the habitual formulas, from which we see that the kernel of the action is $\mu_n \subset \mathrm{T}$, so $\mu_n = \mathrm{Z}_{\mathrm{SL}_n}$.

Proposition 3.3.8. — *Let* $\mathrm{G} \to \mathrm{S}$ *be a reductive group scheme. The center* $\mathrm{Z_G}$ *equals the kernel of the adjoint representation* $\mathrm{Ad_G} : \mathrm{G} \to \mathrm{GL}(\mathfrak{g})$.

Proof. — The schematic center $\mathrm{Z_G}$ is fppf over S (Theorem 3.3.4) and $\ker \mathrm{Ad_G}$ is finitely presented over S, so by Lemma B.3.1 the inclusion $\mathrm{Z_G} \hookrightarrow \ker \mathrm{Ad_G}$ is an isomorphism if it is so on (geometric) fibers over S. Thus, we may and do assume $\mathrm{S} = \mathrm{Spec}(k)$ for an algebraically closed field k. As we discussed in the proof of Theorem 3.3.4, a normal closed k-subgroup of G is central if its identity component is central. It therefore suffices to show that $(\ker \mathrm{Ad_G})^0$ is central in G. Thus, by Corollary 3.3.6(2), it suffices to show that $(\ker \mathrm{Ad_G})^0$ is contained in each maximal torus T of G, or equivalently is contained in the schematic centralizer $\mathrm{Z_G(T) = T}$ for each such T.

Normality of $\ker \mathrm{Ad_G}$ in the smooth affine k-group G implies the normality of its identity component in G. Hence, there is a T-action on this identity component via conjugation, and we just need to show that this action is trivial. Since T is of multiplicative type, its action on a connected k-group scheme H of finite type is trivial if and only its induced action on Lie(H) is trivial, by [**CGP**, Cor. A.8.11] (whose proof simplifies significantly for the action by a torus). Thus, we only need to verify that the adjoint action of T on $\mathrm{Lie}(\ker \mathrm{Ad_G})$ is trivial. But $\mathrm{Lie}(\ker \mathrm{Ad_G}) = \ker(\mathrm{ad}_{\mathfrak{g}})$ [**CGP**, Prop. A.7.5], so it suffices to prove that $\ker(\mathrm{ad}_{\mathfrak{g}}) \subset \mathrm{Lie(T)}$, or equivalently (via T-weight space considerations) that $\ker(\mathrm{ad}_{\mathfrak{g}})$ does not contain any root line $\mathfrak{g}_a$ for $a \in \Phi(\mathrm{G,T})$.

For any root a, consider the rank-1 semisimple subgroup $\mathrm{G}_a = \mathscr{D}(\mathrm{Z_G}(\mathrm{T}_a))$ with maximal torus $a^\vee(\mathbf{G}_{\mathrm{m}})$ whose root groups are $\mathrm{U}_{\pm a}$. By functoriality of the adjoint representation (applied to the inclusion $\mathrm{G}_a \hookrightarrow \mathrm{G}$), if $\mathfrak{g}_a \subset \ker(\mathrm{ad}_{\mathfrak{g}})$ then the analogue holds for $(\mathrm{G}_a, a^\vee(\mathbf{G}_{\mathrm{m}}), a)$ in place of $(\mathrm{G, T}, a)$. Thus, to get a contradiction we may replace G with G_a, so it suffices to treat the groups SL_2 and PGL_2, taking T to be the diagonal torus and a to correspond to the upper triangular unipotent subgroup U^+. Choose nonzero $v^\pm \in \mathfrak{u}^\pm$ and a nonzero $t \in \mathfrak{t}$. In the SL_2-case $[v^+, v^-] \neq 0$ and in the PGL_2-case $[v^+, t] \neq 0$. □

Passing to the Lie algebra, we conclude from Proposition 3.3.8 that

$$\mathrm{Lie(Z_G)} = \ker(\mathrm{Lie(Ad_G)}) = \ker(\mathrm{ad}_{\mathfrak{g}}).$$

We say G is *adjoint* if $Z_G = 1$; this can be checked on geometric fibers since Z_G is multiplicative type, and is equivalent to each G_s being adjoint semisimple.

Definition 3.3.9. — A homomorphism $f : G' \to G$ between smooth S-affine S-groups is an *isogeny* if it is a finite flat surjection, and is a *central isogeny* if also $\ker f$ is central in G'.

Over any field of characteristic > 0, the Frobenius isogeny of a nontrivial connected semisimple group is non-central. In the classical setting there exist examples of isogenies between connected semisimple groups such that the kernel is *commutative* and non-central, though these only exist in characteristic 2 (see Remark C.3.6 and [**PY06**, Lemma 2.2]).

If $f : G' \to G$ is a homomorphism between smooth S-affine S-groups and if f_s is an isogeny for all $s \in S$ then is f finite flat? Such an f is certainly surjective, and also flat due to the fibral flatness criterion [**EGA**, IV_3, 11.3.10]. Hence, $\ker f$ is a quasi-finite flat closed normal S-subgroup of G', and $G = G'/\ker f$ in the sense of fppf sheaves. By fppf descent, f is finite if and only if $\ker f$ is S-finite (cf. Exercise 3.4.6(iii)). But is $\ker f$ actually S-finite? And if moreover $\ker f_s$ is central in G_s for all $s \in S$ then is $\ker f$ central in G'? We shall give affirmative answers in the reductive case. The case of central isogenies will be treated now; the general case lies a bit deeper (see Proposition 6.1.10).

Proposition 3.3.10. — *A surjective homomorphism $f : G' \to G$ between reductive groups over a scheme* S *is a central isogeny if and only if $f_{\overline{s}}$ has finite central kernel for all $s \in S$.*

Proof. — The implication "⇒" is obvious. For the converse, the flatness of f follows from the fibral flatness criterion [**EGA**, IV_3, 11.3.10], so it remains to show that $K := \ker f$ is S-finite and central in G'. We may assume that S is noetherian, then local (by direct limit considerations), and finally complete (by faithfully flat descent).

First we show K is central. This asserts that the G'-action on K by conjugation is trivial, an identity that is sufficient to check on artin local points over S. Thus, we may assume $S = \operatorname{Spec} A$ for an artin local ring A, so the quasi-finite flat K is finite flat. The special fiber $K_0 \subset Z_{G'_0}$ uniquely lifts to a finite multiplicative type subgroup $\widetilde{K} \subset Z_{G'}$ since A is artin local and $Z_{G'}$ is of multiplicative type, and the map $\widetilde{K} \to G$ induced by f is trivial on the special fiber, so it is trivial by Corollary B.2.7. This implies $\widetilde{K} \subset \ker f = K$ inside G'. But this inclusion between finite flat A-groups induces an equality on special fibers (by construction of $\widetilde{K}$), so it is an equality. Hence, K is central in G'.

Over a general S, K is closed in the multiplicative type $Z_{G'}$. By Exercise 2.4.2, all finitely presented quasi-finite closed subgroups of a multiplicative type group are finite. Thus, K is S-finite. □

3.4. Exercises. —

Exercise 3.4.1. — Let K/k be an extension of algebraically closed fields, and G a connected reductive k-group. This exercise proves by contradiction that the smooth connected affine K-group G_{K} is reductive; the same method also handles semisimplicity.

(i) Assume G_{K} is not reductive. Show that G_{K} contains a nontrivial normal K-subgroup U admitting a finite composition series whose successive quotients are isomorphic to $\mathbf{G}_{\mathrm{a}}$.

(ii) In the setup of (i), by expressing K as a direct limit of its finitely generated k-subalgebras show that there is a finitely generated k-subalgebra $\mathrm{A} \subset \mathrm{K}$ such that G_{A} has a smooth affine normal closed A-subgroup $\mathscr{U} \subset \mathrm{G}_{\mathrm{A}}$ admitting an increasing finite sequence of smooth closed A-subgroups $1 = \mathscr{U}_0 \subset \cdots \subset \mathscr{U}_n = \mathscr{U}$ such that $n > 0$ and $\mathscr{U}_i$ is identified with the kernel of an fppf A-homomorphism $\mathscr{U}_{i+1} \twoheadrightarrow \mathbf{G}_{\mathrm{a}}$ for $0 \leqslant i < n$.

(iii) By specializing at a k-point of $\mathrm{Spec}(\mathrm{A})$, deduce that G is not reductive.

Exercise 3.4.2. — This exercise proves a very useful lemma of Deligne and Rapoport [**DR**, II, 1.19] that is a generalization of the finiteness criterion used in the proof of Proposition 3.2.8. Let $f : \mathrm{X} \to \mathrm{Y}$ be a quasi-finite flat and separated map between noetherian schemes, and assume its fiber degree is constant. We seek to prove that f is finite.

(i) Using that a proper quasi-finite map is finite, reduce to the case $\mathrm{Y} = \mathrm{Spec}\,\mathrm{R}$ for a discrete valuation ring R (hint: use the valuative criterion for properness).

(ii) By Zariski's Main Theorem, the quasi-finite separated X over Y admits an open immersion $j : \mathrm{X} \hookrightarrow \overline{\mathrm{X}}$ into a finite Y-scheme $\overline{\mathrm{X}}$. With $\mathrm{Y} = \mathrm{Spec}\,\mathrm{R}$ as in (i), arrange that $\overline{\mathrm{X}}$ is also R-flat and has the same generic fiber as X.

(iii) Using constancy of fiber degree, deduce that j is an isomorphism and conclude.

(iv) If f is étale, express the result in terms of specialization for constructible étale sheaves.

(v) Remove the noetherian hypotheses without requiring f to be of finite presentation.

Exercise 3.4.3. — This exercise directly proves the fibral isomorphism criterion (Lemma B.3.1) when Y, Y′, and S are noetherian. (The case of general S reduces to this case by standard limit arguments.)

(i) Reduce to the case of separated h by using that $\Delta_h : \mathrm{Y} \to \mathrm{Y} \times_{\mathrm{Y}'} \mathrm{Y}$ satisfies the given hypotheses and is separated.

(ii) Now taking h to be separated, use the result from Exercise 3.4.2 to reduce to proving h is flat, and then reduce to the case when S is artin local.

(iii) For artin local $\mathrm{S} = \mathrm{Spec}\,\mathrm{A}$, prove h is a closed immersion, and use S-flatness of Y to deduce that the ideal defining it in Y' vanishes modulo $\mathfrak{m}_{\mathrm{A}}$, so $\mathrm{Y} = \mathrm{Y}'$.

Exercise 3.4.4. — (i) Prove that Corollary 3.3.5 is valid when G is replaced with a parabolic subgroup of a connected reductive group.

(ii) Over any field k, show that the Heisenberg group $\mathrm{U} \subset \mathrm{GL}_3$ (the standard upper triangular unipotent subgroup) is a central extension of $\mathrm{U}'' \simeq \mathbf{G}_{\mathrm{a}} \times \mathbf{G}_{\mathrm{a}}$ by $\mathrm{U}' \simeq \mathbf{G}_{\mathrm{a}}$, with U' the scheme-theoretic center of U. Assuming $\mathrm{char}(k) = p > 0$, show that the Frobenius kernel $\ker \mathrm{F}_{\mathrm{U}/k}$ (see Exercise 1.6.8) is likewise a central extension of $\ker \mathrm{F}_{\mathrm{U}''/k} \simeq \alpha_p \times \alpha_p$ by $\ker \mathrm{F}_{\mathrm{U}'/k} \simeq \alpha_p$, and deduce that $\mathrm{U} \to \overline{\mathrm{U}} := \mathrm{U}/(\ker \mathrm{F}_{\mathrm{U}'/k})$ and $\overline{\mathrm{U}} \to \mathrm{U}/(\ker \mathrm{F}_{\mathrm{U}/k}) = \mathrm{U}^{(p)}$ are central isogenies between smooth connected affine k-groups such that the composite isogeny is not a central isogeny.

(iii) Using any nontrivial smooth connected unipotent group over a field, show both parts of Corollary 3.3.6 fail when "reductive" is relaxed to "smooth connected affine".

Exercise 3.4.5. — Let A be a finite-dimensional associative algebra over a field k. Consider the ring functor $\underline{\mathrm{A}} : \mathrm{R} \rightsquigarrow \mathrm{A} \otimes_k \mathrm{R}$ and the group functor $\underline{\mathrm{A}}^{\times} : \mathrm{R} \rightsquigarrow (\mathrm{A} \otimes_k \mathrm{R})^{\times}$ on k-algebras.

(i) Prove that $\underline{\mathrm{A}}$ is represented by an affine space over k. Using the k-scheme map $\mathrm{N}_{\mathrm{A}/k} : \underline{\mathrm{A}} \to \mathbf{A}^1_k$ defined functorially by $u \mapsto \det(m_u)$, where $m_u : \mathrm{A} \otimes_k \mathrm{R} \to \mathrm{A} \otimes_k \mathrm{R}$ is left multiplication by $u \in \underline{\mathrm{A}}(\mathrm{R})$, prove that $\underline{\mathrm{A}}^{\times}$ is represented by the open *affine* subscheme $\mathrm{N}_{\mathrm{A}/k}^{-1}(\mathbf{G}_{\mathrm{m}})$. (This is often called "$\mathrm{A}^{\times}$ viewed as a k-group", a phrase that is, strictly speaking, meaningless, since $\mathrm{A}^{\times}$ does not encode the k-algebra A.)

(ii) For $\mathrm{A} = \mathrm{Mat}_n(k)$ prove $\underline{\mathrm{A}}^{\times} = \mathrm{GL}_n$, and for $k = \mathbf{Q}$ and $\mathrm{A} = \mathbf{Q}(\sqrt{d})$ identify it with an explicit $\mathbf{Q}$-subgroup of GL_2 (depending on d). Prove $\underline{\mathrm{A}}^{\times}$ is connected reductive in general.

(iii) For $\mathrm{A} = \mathrm{Mat}_n(k)$, show that $\mathrm{N}_{\mathrm{A}/k} : \underline{\mathrm{A}}^{\times} \to \mathbf{G}_{\mathrm{m}}$ is $\det^n$.

Exercise 3.4.6. — A diagram $1 \to \mathrm{G}' \xrightarrow{j} \mathrm{G} \xrightarrow{\pi} \mathrm{G}'' \to 1$ of fppf groups over a ring k is *exact* if π is faithfully flat and $\mathrm{G}' = \ker \pi$.

(i) For any such diagram, prove $\mathrm{G}'' = \mathrm{G}/\mathrm{G}'$ via π as fppf sheaves on the category of k-schemes. Prove a diagram of k-groups of multiplicative type $1 \to \mathrm{H}' \to \mathrm{H} \to \mathrm{H}'' \to$ is exact if and only if the associated diagram of $\mathbf{G}_{\mathrm{m}}$-dual étale sheaves is exact.

(ii) Prove that G'' is smooth when G is smooth, even if G' is not smooth.

(iii) If G' is finite then prove that π is *finite flat* with fibral degree locally constant on $\mathrm{Spec}\,k$, and that $\pi_n : \mathrm{SL}_n \to \mathrm{PGL}_n$ has degree n. Compute $\mathrm{Lie}(\pi_n)$; when is it surjective?

(iv) Assume k is a field. Prove that the left exact sequence of Lie algebras arising from a short exact sequence of finite type k-groups as above is short exact if G' is smooth. Give a counterexample to short exactness on Lie algebras with smooth G and non-smooth G'.

Exercise 3.4.7. — Let k be a ring, and let $1 \to \mathrm{H}' \to \mathrm{E} \to \mathrm{H}'' \to 1$ be a short exact sequence of fppf group sheaves with H'' and H' group schemes of multiplicative type.

(i) Using descent theory and the affineness of H', prove that E is affine and fppf over k.

(ii) Assume E is commutative. Prove that if k is a field then E is of multiplicative type. What if k is an arbitrary ring? (See Corollary B.4.2(2).)

(iii) By considering the E-conjugation action on H' and étaleness of the automorphism functor of H', prove that if H'' has connected fibers (e.g., a torus) then H' is central in E and in fact E is commutative (hint: once centrality is proved, show the commutator of E factors through a bi-additive pairing $\mathrm{H}'' \times \mathrm{H}'' \to \mathrm{H}'$). What can we then conclude via (ii)?

Exercise 3.4.8. — Let G be a smooth affine group over a field k. Using fppf descent and the existence of a (geometrically!) maximal torus over some finite extension, generalize Theorem 3.2.6 to apply to G without restriction on the Cartan subgroups of $\mathrm{G}_{\overline{k}}$. Deduce that G admits such a torus over a finite *separable* extension k'/k.

4. Roots, coroots, and semisimple-rank 1

Let $G \to S$ be a reductive group scheme. By Corollary 3.2.7, at the cost of passing to an étale cover on S we may arrange that G contains a maximal torus T. By deeper work with Cartan subalgebras of $\mathfrak{g}$ one can even make a maximal torus *Zariski-locally* on the original S [**SGA3**, XIV, 3.20]; we do not use this result (but see Exercise 7.3.7(i)).

Suppose for a moment that T is split, so there is an isomorphism

$$T \simeq D_S(M) := \underline{\mathrm{Hom}}_{S\text{-gp}}(M_S, \mathbf{G}_m)$$

for a finite free $\mathbf{Z}$-module M. In the evident manner, we get a map of groups $M \to \mathrm{Hom}_{S\text{-gp}}(T, \mathbf{G}_m)$. The T-action on $\mathfrak{g} = \mathrm{Lie}(G)$ then corresponds to an $\mathscr{O}_S$-linear M-grading $\bigoplus_{m \in M} \mathfrak{g}_m$ of the vector bundle $\mathfrak{g}$, where $t \in T$ acts on the subbundle $\mathfrak{g}_m$ via multiplication by the unit $m(t)$ (see [**Oes**, III, 1.5] or [**CGP**, Lemma A.8.8]). The formation of each subbundle $\mathfrak{g}_m$ in $\mathfrak{g}$ commutes with any base change on S.

By passing to geometric fibers and using the classical theory of root spaces for connected reductive groups, we see that the locally constant rank of each vector bundle $\mathfrak{g}_m$ takes values in $\{0, 1\}$ when $m \neq 0$ and that $\mathfrak{g}_0 = \mathrm{Lie}(T)$ (since $\mathrm{Lie}(T) \subset \mathfrak{g}_0$ as subbundles of $\mathfrak{g}$, with equality on geometric fibers over S). We will use these observations to develop a general theory of root spaces and root groups, leading (in § 5) to both a complete classification in the split case when the geometric fibers have semisimple-rank 1 as well as a deeper understanding of the Weyl group scheme $W_G(T)$ in the split case.

4.1. Roots and the dynamic method. — When T is split as above, any character $T \to \mathbf{G}_m$ arises from an element of M Zariski-locally on S since (i) $T \simeq \mathbf{G}_m^r$ for some $r \geqslant 0$, and (ii) any endomorphism of $\mathbf{G}_m$ over S is given, Zariski-locally on S, by $t \mapsto t^n$ for some $n \in \mathbf{Z}$. This leads to the construction of some subbundles of $\mathfrak{g}$:

Definition 4.1.1. — Let S be a non-empty scheme. Assume that G admits a split maximal torus T over S and fix an isomorphism $T \simeq D_S(M)$ for a finite free $\mathbf{Z}$-module M. A *root* for (G, T) is a nonzero element $a \in M$ such that $\mathfrak{g}_a$ is a line bundle. We call such $\mathfrak{g}_a$ a *root space* for (G, T, M).

We may view roots as fiberwise nontrivial characters $T \to \mathbf{G}_m$ corresponding to *constant* sections of the étale sheaf $\underline{\mathrm{Hom}}_{S\text{-gp}}(T, \mathbf{G}_m) = M_S$ that are fiberwise nonzero and induce roots in the classical sense on geometric fibers. In Exercise 4.4.1 it is shown that any root for $(G_{\overline{s}}, T_{\overline{s}})$ arises from a root for (G_U, T_U) in the above sense for some Zariski-open neighborhood U of s in S.

We will later study root systems arising from reductive group schemes equipped with a split maximal torus, but for now we study a single root $a : T \to \mathbf{G}_m$. Since a is fiberwise nontrivial, its kernel $\ker a$ is S-flat by the fibral flatness criterion [**EGA**, IV_3, 11.3.10] and hence is an S-group of multiplicative type (by Corollary B.3.3).

Example 4.1.2. — In the relative theory, we encounter a new phenomenon that is never seen in the theory over a field: the root spaces $\mathfrak{g}_a$ are line bundles on S but they may be *nontrivial* as such, even in the presence of a split maximal torus. The most concrete version is seen over a Dedekind domain A with nontrivial class group: if J is a non-principal integral ideal of A then for the rank-2 vector bundle $\mathrm{M} = \mathrm{A} \oplus \mathrm{J}$ the A-group GL(M) has generic fiber GL_2 and its points valued in any A-algebra R consist of matrices in $\left(\begin{smallmatrix} \mathrm{R} & \mathrm{J}^{-1}\otimes_{\mathrm{A}}\mathrm{R} \\ \mathrm{J}\otimes_{\mathrm{A}}\mathrm{R} & \mathrm{R} \end{smallmatrix}\right)$ with unit determinant. This is a form of GL_2 over A whose root spaces relative to the split diagonal torus are $\mathrm{J}^{\pm 1}$ inside the standard root spaces for $\mathfrak{gl}_2$ over the fraction field of A.

More generally, consider a nontrivial line bundle $\mathscr{L}$ over a scheme S (such as $\mathscr{L} = \mathscr{O}(1)$ on $\mathrm{S} = \mathbf{P}^1_k$ for a field k). Let $\mathrm{G} = \mathrm{GL}(\mathscr{E})$ for $\mathscr{E} = \mathscr{O} \bigoplus \mathscr{L}$. This is a Zariski-twisted form of GL_2 over S, and it contains the split maximal torus $\mathrm{T} = \mathbf{G}_{\mathrm{m}}^2 = \mathrm{D}_{\mathrm{S}}(\mathbf{Z}^2)$ acting as ordinary unit scaling on both $\mathscr{O}$ and $\mathscr{L}$. In this case the Lie algebra of G is

$$\mathfrak{g} = \underline{\mathrm{End}}(\mathscr{E}) = \mathscr{E} \otimes \mathscr{E}^* = \mathscr{O}^{\oplus 2} \oplus \mathscr{L}^{-1} \oplus \mathscr{L},$$

where $\mathscr{O}^{\oplus 2} = \mathrm{Lie}(\mathrm{T})$ and the root spaces $\mathfrak{g}_{\pm a}$ are the subbundles $\mathscr{L}^{\mp 1}$ (with roots $\pm a : \mathrm{T} \to \mathbf{G}_{\mathrm{m}}$ corresponding to $(c_1, c_2) \mapsto (c_1/c_2)^{\pm 1}$). Since $\mathscr{L}$ a nontrivial line bundle, the root spaces are nontrival line bundles.

As in the classical theory, we describe characters of T using additive notation, so we write $-a$ rather than $1/a$, and $a+b$ rather than ab (and 0 denotes the trivial element of $\mathrm{Hom}_{\text{S-gp}}(\mathrm{T}, \mathbf{G}_{\mathrm{m}})$).

Lemma 4.1.3. — *Let* $\mathrm{T} = \mathrm{D}_{\mathrm{S}}(\mathrm{M})$ *be a split maximal torus of a reductive group scheme* G *over a non-empty scheme* S. *A fiberwise nontrivial* $a : \mathrm{T} \to \mathbf{G}_{\mathrm{m}}$ *is a root of* (G, T) *if and only if* $-a$ *is, in which case the common kernel* $\ker a = \ker(-a)$ *contains a unique subtorus* $\mathrm{T}_a = \mathrm{T}_{-a}$ *of relative codimension* 1 *in* T.

In such cases, the reductive centralizer $\mathrm{G}_a := \mathrm{Z}_{\mathrm{G}}(\mathrm{T}_a)$ *has geometric fibers with semisimple-rank* 1 *and Lie algebra* $\mathfrak{t} \bigoplus \mathfrak{g}_a \bigoplus \mathfrak{g}_{-a}$ *inside* $\mathfrak{g}$.

This is essentially [**SGA3**, XIX, 3.5]. Beware that our notation now deviates from the classical case, with G_a denoting $\mathrm{Z}_{\mathrm{G}}(\mathrm{T}_a)$ whereas in the classical theory it denotes $\mathscr{D}(\mathrm{Z}_{\mathrm{G}}(\mathrm{T}_a))$. There is no serious risk of confusion because in the relative setting for smooth affine group schemes there is no concept of "derived group" in the same generality as over fields. (This problem is overcome for reductive group schemes in Theorem 5.3.1ff.)

Proof. — An element $a \in \mathrm{M}$ is a root if and only if its negative $-a \in \mathrm{M}$ is a root, by the theory on geometric fibers. Suppose these are both roots (over S). To prove the rest, consider the relative codimension-1 torus T_a in T corresponding to the maximal torsion-free quotient of $\mathrm{M}/(\mathbf{Z}a)$, so $\mathrm{T}_a \subset \ker a$. Its uniqueness as a relative codimension-1 torus killed by a is clear on (geometric) fibers by the classical theory, and so uniqueness holds over S by the duality between tori and étale sheaves of finite free $\mathbf{Z}$-modules on S.

The centralizer $\mathrm{Z_G}(\mathrm{T}_a)$ has Lie algebra $\mathfrak{g}^{\mathrm{T}_a}$ that is a subbundle of $\mathfrak{g}$ whose formation commutes with any base change. This subbundle contains the subbundle $\mathfrak{t} \bigoplus \mathfrak{g}_a \bigoplus \mathfrak{g}_{-a}$ of $\mathfrak{g}$, so to prove that the containment is an equality we may pass to geometric fibers and use the classical theory. The semisimple-rank 1 property of the geometric fibers of G_a over S is likewise classical (and is evident from the description of the Lie algebra). $\square$

Now that we have built root spaces $\mathfrak{g}_a$, the next step is to build root groups. First we review the classical perspective on root groups so we can see why it cannot be used when working over a base scheme S.

Over an algebraically closed field k, if $a \in \Phi(\mathrm{G}, \mathrm{T})$ and $\mathrm{T}_a = (\ker a)^0_{\mathrm{red}}$ is the unique codimension-1 torus in T killed by a, then the centralizer scheme $\mathrm{Z_G}(\mathrm{T}_a)$ is a connected reductive subgroup of G with Lie algebra $\mathfrak{g}^{\mathrm{T}_a}$ containing the nonzero a-weight space $\mathfrak{g}_a$. Thus, the codimension-1 torus T_a in T is the maximal central torus in $\mathrm{Z_G}(\mathrm{T}_a)$, so $\mathrm{Lie}(\mathrm{Z_G}(\mathrm{T}_a)) = \mathfrak{g}^{\mathrm{T}_a}$ has as its T-roots precisely the nonzero $\mathbf{Q}$-multiples of a in $\Phi(\mathrm{G}, \mathrm{T}) \subset \mathrm{X}(\mathrm{T})_{\mathbf{Q}}$ since a is a nontrivial character of $\mathrm{T}/\mathrm{T}_a \simeq \mathbf{G}_{\mathrm{m}}$. In particular, $\mathscr{D}(\mathrm{Z_G}(\mathrm{T}_a))$ is a semisimple group having as a maximal torus the 1-dimensional isogeny complement $\mathrm{T}' := (\mathrm{T} \bigcap \mathscr{D}(\mathrm{Z_G}(\mathrm{T}_a)))^0_{\mathrm{red}}$ to T_a in T. By the semisimple rank-1 classification, the group $\mathscr{D}(\mathrm{Z_G}(\mathrm{T}_a))$ is isomorphic to either SL_2 or PGL_2, and the isomorphism can be chosen to carry T' over to the diagonal torus. In particular, the roots for (G, T) that are $\mathbf{Q}$-multiples of a are precisely $\pm a$, and (by composing with conjugation on SL_2 or PGL_2 if necessary) we obtain a central isogeny $q_a : \mathrm{SL}_2 \to \mathscr{D}(\mathrm{Z_G}(\mathrm{T}_a))$ carrying the diagonal torus D onto T' and carrying the standard upper triangular unipotent subgroup U^+ isomorphically onto a subgroup $\mathrm{U}_a \subset \mathscr{D}(\mathrm{Z_G}(\mathrm{T}_a))$ that is k-isomorphic to $\mathbf{G}_{\mathrm{a}}$ and normalized by T with $\mathrm{Lie}(\mathrm{U}_a) = \mathfrak{g}_a$. This q_a is unique up to $\mathrm{D}(k)$-conjugation.

A direct inspection of SL_2 and PGL_2 then shows that U_a is uniquely determined by these properties relative to $(\mathrm{G}, \mathrm{T}, a)$, and it is called the *root group* for a. The composition of q_a with the standard parameterization $t \mapsto \mathrm{diag}(t, 1/t)$ yields a unique cocharacter $a^\vee : \mathbf{G}_{\mathrm{m}} \to \mathrm{T}'$ satisfying $\langle a, a^\vee \rangle = 2$, and this is called the *coroot* for $(\mathrm{G}, \mathrm{T}, a)$. The definitions of root groups and coroots in the classical theory rest on notions of "derived group" and "unipotent subgroup" that are not (yet) available over a general base. Also, in the classical theory the definition of a coroot rests on the semisimple-rank 1 classification.

To introduce root groups and coroots attached to roots of a reductive group scheme over a general scheme S, we require an entirely different construction technique. This will *not* give a new approach in the classical case because in our proofs over S we will appeal to the known theory of root groups and coroots on geometric fibers. The current absence of a "derived group" for reductive S-groups (which will only become available in Theorem 5.3.1ff.) also makes the statement of the split semisimple-rank 1 classification over S more complicated than in the classical case, as we do not yet have a way to "split off" the maximal central torus (up to isogeny) in the relative setting.

In the relative theory, root groups will be instances of a general group construction that we now explain. For any finite-rank vector bundle $\mathscr{E}$ on S, let $\mathbf{W}(\mathscr{E}) \to \mathrm{S}$ denote the associated additive S-group whose set of S′-points is the additive group of global sections of $\mathscr{E}_{\mathrm{S}'}$ for any S-scheme S′. Explicitly, double duality for $\mathscr{E}$ provides a canonical isomorphism

$$\mathbf{W}(\mathscr{E}) = \mathrm{Spec}_{\mathrm{S}}(\mathrm{Sym}(\mathscr{E}^*))$$

as S-groups, and $\mathrm{Lie}(\mathbf{W}(\mathscr{E})) \simeq \mathscr{E}$ as vector bundles on S (with trivial Lie bracket) respecting base change on S. The relative approach to root groups is:

Theorem 4.1.4. — *Let* $\mathrm{G} \to \mathrm{S}$ *be a reductive group scheme over a non-empty scheme* S, $\mathrm{T} \simeq \mathrm{D}_{\mathrm{S}}(\mathrm{M})$ *a split maximal torus, and* $a \in \mathrm{M}$ *a root. Let* T *act on* $\mathbf{W}(\mathfrak{g}_a)$ *via* $t.v = a(t)v$ *using the vector bundle structure on* $\mathfrak{g}_a$*. There is a unique* S*-group homomorphism*

$$\exp_a : \mathbf{W}(\mathfrak{g}_a) \to \mathrm{G}$$

inducing the canonical inclusion $\mathfrak{g}_a \hookrightarrow \mathfrak{g}$ *on Lie algebras and intertwining the* T*-action on* G *via conjugation and the* T*-action on* $\mathbf{W}(\mathfrak{g}_a)$ *via a-scaling.*

The map $\exp_a$ *is also a closed immersion factoring through* $\mathrm{Z}_{\mathrm{G}}(\mathrm{T}_a)$*, its formation commutes with base change on* S*, and the multiplication map*

$$\mathbf{W}(\mathfrak{g}_{-a}) \times \mathrm{T} \times \mathbf{W}(\mathfrak{g}_a) \to \mathrm{Z}_{\mathrm{G}}(\mathrm{T}_a) \tag{4.1.1}$$

defined by $(\mathrm{X}', t, \mathrm{X}) \mapsto \exp_{-a}(\mathrm{X}')t\exp_a(\mathrm{X})$ *is an isomorphism onto an open subscheme* $\Omega_a \subset \mathrm{Z}_{\mathrm{G}}(\mathrm{T}_a)$*. Moreover, the semi-direct product of* T *against each* $\mathbf{W}(\mathfrak{g}_{\pm a})$ *is a closed* S*-subgroup of* G.

The closed subgroup $\exp_a(\mathbf{W}(\mathfrak{g}_a)) \subset \mathrm{G}$ will be called the *a-root group* for $(\mathrm{G}, \mathrm{T}, \mathrm{M})$. The proof of Theorem 4.1.4 (to be given in § 4.2) uses a "dynamic method" entirely different from the approach given in [**SGA3**, XX, 1.5–1.14] (which rests on Hochschild cohomology, deformation theory, and fpqc descent). To be precise, the method in [**SGA3**, XX, § 1] gives slightly less; the properties that the maps $\exp_{\pm a}$ are closed immersions (rather than mere monomorphisms) and that the semi-direct products $\mathrm{T} \ltimes \mathbf{W}(\mathfrak{g}_{\pm a})$ are closed S-subgroups of G are not obtained until [**SGA3**, XX, 5.9]. The dynamic approach will yield these properties immediately and provide a more streamlined route through the semisimple-rank 1 classification over S. Thus, we now digress and introduce the dynamic method.

As motivation for what will follow, we first explain how to construct the standard upper triangular Borel subgroup and its unipotent radical in SL_2 over an algebraically closed field by means of a $\mathbf{G}_{\mathrm{m}}$-action on SL_2 without ever needing to say "solvable subgroup" or "unipotent subgroup" as in the classical theory. Rather generally, if $\mathbf{G}_{\mathrm{m}} \times \mathrm{G} \to \mathrm{G}$ is an action of $\mathbf{G}_{\mathrm{m}}$ on a separated group scheme G over a base scheme S, for any $g \in \mathrm{G(S)}$ we say $\lim_{t\to 0} t.g$ *exists* if the orbit map $\mathbf{G}_{\mathrm{m}} \to \mathrm{G}$ defined by $t \mapsto t.g$ extends to an S-scheme morphism $\mathbf{A}^1_{\mathrm{S}} \to \mathrm{G}$; such an extension is unique if it exists since G is separated and $k[x] \subset k[x, 1/x]$ for any ring k. In such cases, the image of 0 in G(S) is called $\lim_{t\to 0} t.g$. This limit concept has the following interesting application for SL_2:

Example 4.1.5. — Let $\lambda : \mathbf{G}_{\mathrm{m}} \to \mathrm{G} := \mathrm{SL}_2$ be $t \mapsto \left(\begin{smallmatrix} t & 0 \\ 0 & 1/t \end{smallmatrix}\right)$. Define a $\mathbf{G}_{\mathrm{m}}$-action on G as follows: for any k-algebra k', $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{G}(k')$, and $t \in \mathbf{G}_{\mathrm{m}}(k') = k'^{\times}$, let

$$t.g = \lambda(t) \begin{pmatrix} a & b \\ c & d \end{pmatrix} \lambda(t)^{-1} = \begin{pmatrix} a & t^2 b \\ t^{-2} c & d \end{pmatrix}.$$

Thus, $\lim_{t\to 0} t.g$ exists if and only if $c = 0$ in k', and this limit exists and equals 1 if and only if $c = 0$ and $a = d = 1$ in k'.

In other words, the upper triangular subgroup $\mathrm{B} \subset \mathrm{G}$ represents the functor of points g of G for which $\lim_{t\to 0} t.g$ exists, and the strictly upper triangular subgroup U represents the functor of points g of G for which $\lim_{t\to 0} t.g$ exists and equals 1. This gives a mechanism for constructing B and U entirely in terms of G and the $\mathbf{G}_{\mathrm{m}}$-action on it, without reference to notions such as solvability or unipotence that are well-suited to working over a field but are not available (at least not in a useful manner) when the base is anything more complicated than a field (such as a discrete valuation ring or non-reduced artin local ring).

Note that if we replace λ with its reciprocal 1-parameter subgroup $-\lambda : t \mapsto \lambda(1/t)$ then the analogous limiting process above recovers the lower triangular subgroup B' and its strictly lower triangular subgroup U'.

The subgroup constructions in the preceding example can be carried out more generally. In [**CGP**, Ex. 2.1.1] the case $\mathrm{G} = \mathrm{GL}_n$ is worked out over any ring k when using a diagonal 1-parameter subgroup

$$\lambda(t) = \mathrm{diag}(t^{e_1}, \ldots, t^{e_n})$$

for integers $e_1 \geqslant \cdots \geqslant e_n$. In such cases one gets various "parabolic" subgroups and their "unipotent radicals" (keep in mind that the base ring k may not be a field, hence the quotation marks), all depending on the e_j's.

A general setting for these limit considerations uses a separated S-scheme X equipped with a left action by the S-group $\mathbf{G}_{\mathrm{m}}$ (only the case of S-affine X will be used below). For an S-scheme S′ and $x \in \mathrm{X}(\mathrm{S}')$, it makes sense to ask if the orbit map $\mathbf{G}_{\mathrm{m}} \to \mathrm{X}_{\mathrm{S}'}$ over S′ defined by $t \mapsto t.x$ extends to an S′-map $\mathbf{A}^1_{\mathrm{S}'} \to \mathrm{X}_{\mathrm{S}'}$. If such an extension exists then it is unique (because X is separated and a closed subscheme of $\mathbf{A}^1_{\mathrm{S}'}$ that contains $\mathbf{G}_{\mathrm{m}}$ must be the entire affine line); we then say that "$\lim_{t\to 0} t.x$ exists" and denote the image of $0 \in \mathbf{A}^1(\mathrm{S}')$ in $\mathrm{X}(\mathrm{S}')$ as this limit. A fundamental source of such examples arises as follows.

Definition 4.1.6. — For a ring k and an affine k-group G, a *1-parameter subgroup* of G is a k-homomorphism $\lambda : \mathbf{G}_{\mathrm{m}} \to \mathrm{G}$. (We allow that $\ker \lambda \neq 1$, and even that $\lambda = 1$, though the latter option is not very useful.)

Any 1-parameter subgroup defines a $\mathbf{G}_{\mathrm{m}}$-action on G via $t.g = \lambda(t)g\lambda(t)^{-1}$. By using a "weight space" decomposition of the coordinate ring $k[\mathrm{G}]$, a variant of the above procedures in SL_2 and GL_n can be carried out for 1-parameter subgroups $\lambda : \mathbf{G}_{\mathrm{m}} \to \mathrm{G}$ of rather general affine groups G *without* requiring the crutch of a GL_n-embedding of G (which is *not* known to exist locally on the base in general, even for smooth affine groups over the dual numbers over a field; cf. [**SGA3**, XI, 4.3, 4.6]). It will also be extremely useful to consider an abstract action of $\mathbf{G}_{\mathrm{m}}$ on G, not only actions arising from conjugation against a 1-parameter subgroup, so in our formulation below of a vast generalization of Example 4.1.5 we treat abstract $\mathbf{G}_{\mathrm{m}}$-actions. The following "dynamic" result summarizes the main conclusions in [**CGP**, 2.1]. It is the key to our approach to root groups in reductive group schemes.

Theorem 4.1.7. — *Let* G *be a finitely presented affine group over a ring k, and consider an action $\lambda : \mathbf{G}_{\mathrm{m}} \times \mathrm{G} \to \mathrm{G}$ by the k-group $\mathbf{G}_{\mathrm{m}}$ on the k-group* G*. Consider the following subfunctors of* G *on the category of k-algebras:*

$$\begin{aligned} \underline{\mathrm{P}}_{\mathrm{G}}(\lambda)(k') &= \{g \in \mathrm{G}(k') \mid \lim_{t\to 0} \lambda(t,g) \text{ exists}\}, \\ \underline{\mathrm{U}}_{\mathrm{G}}(\lambda)(k') &= \{g \in \underline{\mathrm{P}}_{\mathrm{G}}(\lambda)(k') \mid \lim_{t\to 0} \lambda(t,g) = 1\}. \end{aligned}$$

Likewise, let $\underline{\mathrm{Z}}_{\mathrm{G}}(\lambda)$ be the subfunctor of points of G *that commute with the $\mathbf{G}_{\mathrm{m}}$-action λ.*

1. *These functors are unaffected by replacing λ with λ^n for $n > 0$, and they are represented by respective finitely presented closed subgroups $\mathrm{P}_{\mathrm{G}}(\lambda)$, $\mathrm{U}_{\mathrm{G}}(\lambda)$, and $\mathrm{Z}_{\mathrm{G}}(\lambda)$ of* G*, with $\mathrm{U}_{\mathrm{G}}(\lambda)$ normalized by $\mathrm{Z}_{\mathrm{G}}(\lambda)$.*

2. *The fibers of* $\mathrm{U}_{\mathrm{G}}(\lambda) \to \mathrm{S}$ *are connected, and so are the fibers of* $\mathrm{P}_{\mathrm{G}}(\lambda)$ *and* $\mathrm{Z}_{\mathrm{G}}(\lambda)$ *if* $\mathrm{G} \to \mathrm{S}$ *has connected fibers.*
3. *The multiplication map* $\mathrm{Z}_{\mathrm{G}}(\lambda) \ltimes \mathrm{U}_{\mathrm{G}}(\lambda) \to \mathrm{P}_{\mathrm{G}}(\lambda)$ *is an isomorphism.*
4. *Assume* G *is smooth. The subgroups* $\mathrm{P}_{\mathrm{G}}(\lambda)$*,* $\mathrm{U}_{\mathrm{G}}(\lambda)$*, and* $\mathrm{Z}_{\mathrm{G}}(\lambda)$ *are smooth and the multiplication map*

$$\mathrm{U}_{\mathrm{G}}(-\lambda) \times \mathrm{P}_{\mathrm{G}}(\lambda) \to \mathrm{G}$$

is an open immersion. Writing $\bigoplus_{n \in \mathbf{Z}} \mathfrak{g}_n$ *for the weight space decomposition of* $\mathfrak{g}$ *under the* $\mathbf{G}_{\mathrm{m}}$*-action (so* $\mathfrak{g}_n = \{v \in \mathfrak{g} \,|\, t.v = t^n v\}$*), the Lie algebras of these subgroups are*

$$\mathfrak{z}_{\mathrm{G}}(\lambda) = \mathfrak{g}_0 = \mathfrak{g}^{\mathbf{G}_{\mathrm{m}}}, \;\; \mathfrak{u}_{\mathrm{G}}(\lambda) = \mathfrak{g}_+ := \bigoplus_{n>0} \mathfrak{g}_n, \;\; \mathfrak{p}_{\mathrm{G}}(\lambda) = \mathfrak{z}_{\mathrm{G}}(\lambda) \oplus \mathfrak{u}_{\mathrm{G}}(\lambda).$$

Also, the fibers of $\mathrm{U}_{\mathrm{G}}(\lambda) \to \mathrm{S}$ *are unipotent.*

In terms of the theory of connected reductive groups over an algebraically closed field, part (3) is analogous to a Levi decomposition of a parabolic subgroup and part (4) is analogous to an open Bruhat cell. When we apply Theorem 4.1.7 to examples in which the action arises from conjugation against a 1-parameter subgroup $\lambda : \mathbf{G}_{\mathrm{m}} \to \mathrm{G}$, we shall denote the resulting closed subgroups as $\mathrm{P}_{\mathrm{G}}(\lambda)$, $\mathrm{U}_{\mathrm{G}}(\lambda)$, and $\mathrm{Z}_{\mathrm{G}}(\lambda)$.

Proof. — Using the semi-direct product group $\mathrm{G}' = \mathrm{G} \rtimes \mathbf{G}_{\mathrm{m}}$ defined by the given action, the evident 1-parameter subgroup $t \mapsto (1, t)$ reduces the general case to the special case that λ arises from the conjugation action against a 1-parameter subgroup. This reduction step is explained in [**CGP**, Rem. 2.1.11], so we now may and do assume the action is conjugation against a 1-parameter subgroup, also denoted $\lambda : \mathbf{G}_{\mathrm{m}} \to \mathrm{G}$. The $\mathbf{G}_{\mathrm{m}}$-action on G yields a $\mathbf{G}_{\mathrm{m}}$-action on the k-module $k[\mathrm{G}]$, which in turn corresponds to a k-linear $\mathbf{Z}$-grading $\oplus_{n \in \mathbf{Z}} k[\mathrm{G}]_n$ of $k[\mathrm{G}]$, where $f(t.g) = t^n f(g)$ for $f \in k[\mathrm{G}]_n$ (see [**CGP**, (2.1.2)]).

It is clear from the definitions that the three subfunctors of G under consideration are subgroup functors. Their invariance under passage to λ^n with $n > 0$ is elementary; see Exercise 4.4.2(i) (or [**CGP**, Rem. 2.1.7]). The existence of $\mathrm{P}_{\mathrm{G}}(\lambda)$ as a closed subscheme of G is a special case of [**CGP**, Lemma 2.1.4] (defining the closed subscheme by the ideal of $k[\mathrm{G}]$ generated by the negative weight spaces $k[\mathrm{G}]_n$ for $n < 0$ relative to the $\mathbf{G}_{\mathrm{m}}$-action on $k[\mathrm{G}]$ through λ; this ideal is typically larger than the k-linear span of the negative weight spaces). Existence of $\mathrm{Z}_{\mathrm{G}}(\lambda)$ and $\mathrm{U}_{\mathrm{G}}(\lambda)$ is given in [**CGP**, Lemma 2.1.5], where the finite presentation property is also established (by reduction to the case of noetherian k). Explicitly, $\mathrm{Z}_{\mathrm{G}}(\lambda) = \mathrm{P}_{\mathrm{G}}(\lambda) \bigcap \mathrm{P}_{\mathrm{G}}(-\lambda)$ and $\mathrm{U}_{\mathrm{G}}(\lambda)$ is the fiber over 1 for the limit morphism $\mathrm{P}_{\mathrm{G}}(\lambda) \to \mathrm{G}$ defined by $g \mapsto \lim_{t \to 0} t.g$. This settles (1). (The proof of the existence of $\mathrm{P}_{\mathrm{G}}(\lambda)$, $\mathrm{Z}_{\mathrm{G}}(\lambda)$, and $\mathrm{U}_{\mathrm{G}}(\lambda)$ as closed subgroup schemes does not require G to be finitely presented over k.)

Part (3) is [**CGP**, Prop. 2.1.8(2)], and part (4) apart from the Lie algebra and unipotence assertions is [**CGP**, Prop. 2.1.8(3)] (which is the hardest part of the proof). The description of the Lie algebras in (4) is [**CGP**, Prop. 2.1.8(1)] (proved by a functorial calculation with dual numbers). The fibral unipotence for $\mathrm{U_G}(\lambda)$ in (4) is part of [**CGP**, Lemma 2.1.5] (which does not require smoothness of G, once one has developed a good theory of unipotent group schemes over a field; see [**SGA3**, XVII, 1.3, 2.1]). Part (2) is [**CGP**, Prop. 2.1.8(4)]; the idea for proving $\mathrm{U_G}(\lambda)$ is fiberwise connected is that the limiting process provides paths $t \mapsto t.g$ linking all points g of $\mathrm{U_G}(\lambda)$ to 1, and for $\mathrm{P_G}(\lambda)$ and $\mathrm{Z_G}(\lambda)$ the fibral connectedness in the case of smooth G follows from part (4) whereas in the general case for (2) it requires further work that is specific to groups over fields (e.g., the existence of a GL_n-embedding). □

Example 4.1.8. — For $\mathrm{G} = \mathrm{SL}_n$ over any ring k and $\lambda(t) = \mathrm{diag}(t^{e_1}, \ldots, t^{e_n})$ for a strictly decreasing sequence of integers $e_1 > \cdots > e_n$, $\mathrm{P_G}(\lambda)$ is the standard upper triangular k-subgroup, $\mathrm{U_G}(\lambda)$ is its k-subgroup of strictly upper triangular matrices, and $\mathrm{Z_G}(\lambda)$ is the k-subgroup of diagonal elements. Passing to $-\lambda$ yields the lower triangular analogues, exactly as in Example 4.1.5. If the e_j's are pairwise distinct but not strictly monotone then $\mathrm{P_G}(\lambda)$ is the conjugate of the upper triangular subgroup by a suitable permutation matrix (corresponding to rearranging the e_j's to be strictly decreasing).

Example 4.1.9. — For any connected reductive group G over an algebraically closed field k and any maximal torus T in G, as λ varies through the cocharacters of T the resulting smooth connected subgroups $\mathrm{P_G}(\lambda)$ of G containing T are precisely the parabolic subgroups of G that contain T. We refer the reader to [**CGP**, Prop. 2.2.9] for a proof.

Note that $\mathrm{P_G}(\lambda) = \mathrm{Z_G}(\lambda) \ltimes \mathrm{U_G}(\lambda)$ with $\mathrm{U_G}(\lambda)$ a smooth connected unipotent normal subgroup and $\mathrm{Z_G}(\lambda)$ the centralizer of the torus $\lambda(\mathbf{G}_{\mathrm{m}})$ in G (so $\mathrm{Z_G}(\lambda)$ is connected reductive). Thus, $\mathrm{U_G}(\lambda)$ is the unipotent radical of $\mathrm{P_G}(\lambda)$ and $\mathrm{Z_G}(\lambda)$ is a Levi subgroup of $\mathrm{P_G}(\lambda)$.

It is immediate from the definitions that the formation of $\mathrm{P_G}(\lambda)$, $\mathrm{U_G}(\lambda)$, and $\mathrm{Z_G}(\lambda)$ commutes with any base change on k, and that Theorem 4.1.7 adapts to work over any base scheme S (not just affine schemes). Here are some easy but very useful "functorial" properties of these subgroups.

Proposition 4.1.10. — *Let* (G, λ) *be as in Theorem* 4.1.7.

1. *For any finitely presented closed subgroup* H *of* G *that is stable under the* $\mathbf{G}_{\mathrm{m}}$*-action, with the restricted action on* H *also denoted as* λ*,*

$$\mathrm{H} \bigcap \mathrm{P_G}(\lambda) = \mathrm{P_H}(\lambda), \ \ \mathrm{H} \bigcap \mathrm{U_G}(\lambda) = \mathrm{U_H}(\lambda), \ \ \mathrm{H} \bigcap \mathrm{Z_G}(\lambda) = \mathrm{Z_H}(\lambda).$$

In particular, if H *and* G *are smooth then* $\mathrm{H}\bigcap \mathrm{P_G}(\lambda)$ *is smooth and likewise with* $\mathrm{U_G}(\lambda)$ *and* $\mathrm{Z_G}(\lambda)$*.*

2. *Let* (G', λ') *be another such pair over* k*, and* $f : \mathrm{G} \to \mathrm{G}'$ *a* $\mathbf{G}_{\mathrm{m}}$*-equivariant map. Then* f *carries* $\mathrm{P_G}(\lambda)$ *into* $\mathrm{P_{G'}}(\lambda')$*,* $\mathrm{U_G}(\lambda)$ *into* $\mathrm{U_{G'}}(\lambda')$*, and* $\mathrm{Z_G}(\lambda)$ *into* $\mathrm{Z_{G'}}(\lambda')$*. When* f *is flat and surjective and* $\mathrm{G} \to \operatorname{Spec} k$ *has connected fibers then the maps*

$$\mathrm{P_G}(\lambda) \to \mathrm{P_{G'}}(\lambda'),\ \ \mathrm{U_G}(\lambda) \to \mathrm{U_{G'}}(\lambda'),\ \ \mathrm{Z_G}(\lambda) \to \mathrm{Z_{G'}}(\lambda') \tag{4.1.2}$$

are surjections that are flat between fibers over $\operatorname{Spec} k$*. If* f *is a flat surjection and* G *is* k*-smooth with connected fibers then the surjections in* (4.1.2) *are flat.*

Proof. — The assertion in (1) is immediate from the functorial definitions of these subgroups. The functoriality in (2) is likewise obvious, and for the surjectivity assertion it suffices to check on fibers over $\operatorname{Spec} k$. The surjectivity problem when k is a field is [**CGP**, Cor. 2.1.9], which also gives the flatness of the surjections in such cases. (Strictly speaking, [**CGP**, Cor. 2.1.9] considers the case of conjugation actions against 1-parameter subgroups. The general case reduces to this; see [**CGP**, Rem. 2.1.11].)

Suppose f is a flat surjection and G is k-smooth, so the finitely presented k-group G′ in (2) is smooth as well [**EGA**, IV_4, 17.7.7]. All of the subgroups of interest in (4.1.2) are smooth, by Theorem 4.1.7(4). Hence, in such cases the flatness of the induced surjective maps arising from f can be checked on fibers over $\operatorname{Spec} k$, for which we have already noted that the flatness holds. □

4.2. Root groups and coroots. — For a split maximal torus $\mathrm{T} = \mathrm{D_S}(\mathrm{M})$ in a reductive group scheme $\mathrm{G} \to \mathrm{S}$ over a non-empty scheme S, now we use dynamic constructions to build “root groups” $\mathrm{U}_{\pm a}$ in $\mathrm{G}_a = \mathrm{Z_G}(\mathrm{T}_a) = \mathrm{G}_{-a}$ for any root a of (G, T) (with $\mathrm{T}_a = \mathrm{T}_{-a}$ as in Lemma 4.1.3). Before we prove Theorem 4.1.4, we work out what it is saying for SL_2 over any ring.

Example 4.2.1. — We saw in Example 4.1.5 that for the standard 1-parameter subgroup $\lambda(t) = \mathrm{diag}(t, 1/t)$ in $\mathrm{G} = \mathrm{SL}_2$, the resulting subgroups $\mathrm{U}(\lambda)$ and $\mathrm{U}(-\lambda)$ are respectively the strictly upper and strictly lower triangular subgroups, corresponding to the roots $\pm a : \mathrm{T} \rightrightarrows \mathbf{G}_{\mathrm{m}}$ satisfying $a(\lambda(t)) = t^2$ and $(-a)(\lambda(t)) = t^{-2}$. The proof of Theorem 4.1.4 will show that these subgroups are respectively the images of $\exp_a$ and $\exp_{-a}$ from Theorem 4.1.4. Via the standard trivializations of $\mathfrak{g}_{\pm a}$ given by $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ respectively (to identify $\mathbf{W}(\mathfrak{g}_{\pm a})$ with $\mathbf{W}(\mathscr{O}_{\mathrm{S}}) = \mathbf{G}_{\mathrm{a}}$), we have $\exp_a(z) = \left(\begin{smallmatrix} 1 & z \\ 0 & 1 \end{smallmatrix}\right)$ and $\exp_{-a}(z) = \left(\begin{smallmatrix} 1 & 0 \\ z & 1 \end{smallmatrix}\right)$. Note that the “exponential” terminology is reasonable: the nilpotent matrices $n_+(z) = \left(\begin{smallmatrix} 0 & z \\ 0 & 0 \end{smallmatrix}\right)$ and $n_-(z) = \left(\begin{smallmatrix} 0 & 0 \\ z & 0 \end{smallmatrix}\right)$ satisfy $n_\pm(z)^2 = 0$, so we imagine that $e^{n_\pm(z)}$ should mean $1 + n_\pm(z) = \exp_{\pm a}(z)$.

Here is the dynamic proof of Theorem 4.1.4.

Proof. — By T-equivariance, since T_a acts trivially on $\mathbf{W}(\mathfrak{g}_a)$ it follows that if $\exp_a$ is to exist then it must factor through the reductive subgroup $\mathrm{Z}_{\mathrm{G}}(\mathrm{T}_a)$. We may therefore replace G with $\mathrm{Z}_{\mathrm{G}}(\mathrm{T}_a)$, so by Lemma 4.1.3 we are reduced to the case that G has geometric fibers of semisimple-rank 1 with T_a central in G and $\mathfrak{g} = \mathfrak{t} \bigoplus \mathfrak{g}_a \bigoplus \mathfrak{g}_{-a}$.

By the asserted uniqueness, compatibility with base change will be automatic and we may work étale-locally on S to prove the theorem. Choose a cocharacter $\lambda : \mathbf{G}_{\mathrm{m}} \to \mathrm{T}$ corresponding to an element of the dual lattice $\mathrm{M}^{\vee}$ such that the pairing $\langle a, \lambda \rangle \in \mathbf{Z}$ (corresponding to the element $a \circ \lambda \in \mathrm{End}_{\text{S-gp}}(\mathbf{G}_{\mathrm{m}}) = \mathbf{Z}_{\mathrm{S}}(\mathrm{S})$ that is a constant section) lies in $\mathbf{Z}_{>0}$. Note that it may be impossible to arrange that this pairing is 1, since a may be divisible by 2 in the character group of T (as happens for the long roots of Sp_{2n}).

The inclusion $\mathrm{T} \subset \mathrm{Z}_{\mathrm{G}}(\lambda)$ of smooth closed subgroups of G is an equality. Indeed, it suffices to prove equality on geometric fibers over S, both of which are connected, so it is enough to compare their Lie algebras inside $\mathfrak{g} = \mathfrak{t} \bigoplus \mathfrak{g}_a \bigoplus \mathfrak{g}_{-a}$. Since $\langle \pm a, \lambda \rangle \neq 0$, the description of $\mathrm{Lie}(\mathrm{Z}_{\mathrm{G}}(\lambda))$ in Theorem 4.1.7(4) implies that this Lie algebra must coincide with $\mathfrak{t}$, as desired.

Using the indicated T-action on $\mathbf{W}(\mathfrak{g}_a)$, composing with λ defines a $\mathbf{G}_{\mathrm{m}}$-action on $\mathbf{W}(\mathfrak{g}_a)$, namely a point c of $\mathbf{G}_{\mathrm{m}}$ acts via multiplication by $a(\lambda(c)) = c^{\langle a, \lambda \rangle}$. Since $\langle a, \lambda \rangle > 0$, it follows that for $\mathrm{H} := \mathbf{W}(\mathfrak{g}_a)$ we have $\mathrm{H} = \mathrm{U}_{\mathrm{H}}(\lambda)$. The T-equivariance requirement on $\exp_a$ implies that (if it exists) it must be $\mathbf{G}_{\mathrm{m}}$-equivariant, so it must carry H into $\mathrm{U}_{\mathrm{G}}(\lambda)$. By Theorem 4.1.7(4), $\mathrm{Lie}(\mathrm{U}_{\mathrm{G}}(\lambda))$ is the "positive" weight space for the $\mathbf{G}_{\mathrm{m}}$-action on $\mathfrak{g} = \mathfrak{t} \bigoplus \mathfrak{g}_a \bigoplus \mathfrak{g}_{-a}$. Since $\mathbf{G}_{\mathrm{m}}$ acts on $\mathfrak{t}$ trivially and on $\mathfrak{g}_{\pm a}$ via $t.v = t^{\langle \pm a, \lambda \rangle} v$ with $\langle a, \lambda \rangle > 0$, we conclude that $\mathrm{Lie}(\mathrm{U}_{\mathrm{G}}(\lambda)) = \mathfrak{g}_a$. Hence, if $\exp_a$ is to exist then it must factor through an S-homomorphism $\mathbf{W}(\mathfrak{g}_a) \to \mathrm{U}_{\mathrm{G}}(\lambda)$ that induces an *isomorphism* on Lie algebras. But $\mathrm{U}_{\mathrm{G}}(\lambda)$ is S-smooth with connected fibers, so if $\exp_a$ exists then it must be an étale homomorphism *onto* $\mathrm{U}_{\mathrm{G}}(\lambda)$.

We can do better: such an étale map $\mathbf{W}(\mathfrak{g}_a) \to \mathrm{U}_{\mathrm{G}}(\lambda)$ must be an isomorphism. Indeed, it suffices to check the isomorphism property on geometric fibers over S, and both sides have geometric fiber $\mathbf{G}_{\mathrm{a}}$ equipped with a $\mathbf{G}_{\mathrm{m}}$-action inducing the *same* scaling action $t.v = t^{\langle a, \lambda \rangle} v$ on the Lie algebra. But the only $\mathbf{G}_{\mathrm{m}}$-actions on $\mathbf{G}_{\mathrm{a}}$ over a field are $t.x = t^n x$ for $n \in \mathbf{Z}$, and the effect on the Lie algebra detects n by the same formula. In other words, the induced map between geometric fibers (if $\exp_a$ exists) must be an étale endomorphism of $\mathbf{G}_{\mathrm{a}}$ that is equivariant for the action $t.x = t^{\langle a, \lambda \rangle} x$, and it is easy to check that over a field F the only such endomorphisms are $x \mapsto cx$ for $c \in \mathrm{F}^{\times}$, which are visibly isomorphisms.

To summarize, if $\exp_a$ is to exist then it must be a $\mathbf{G}_{\mathrm{m}}$-equivariant isomorphism $\mathbf{W}(\mathfrak{g}_a) \simeq \mathrm{U}_{\mathrm{G}}(\lambda)$ (so in particular it must be a closed immersion into G).

Note conversely that any such $\mathbf{G}_{\mathrm{m}}$-equivariant isomorphism is T-equivariant as a map to G since T_a acts trivially on both sides and $\mathbf{G}_{\mathrm{m}} \times \mathrm{T}_a \to \mathrm{T}$ defined by $(c,t) \mapsto \lambda(c)t$ is an isogeny of tori (ensuring that T-equivariance is equivalent to the combination of T_a-equivariance and $\mathbf{G}_{\mathrm{m}}$-equivariance). Thus, for the existence and uniqueness of $\exp_a$ it is necessary and sufficient to prove the existence and uniqueness of a $\mathbf{G}_{\mathrm{m}}$-equivariant isomorphism of S-groups $\mathbf{W}(\mathfrak{g}_a) \simeq \mathrm{U}_{\mathrm{G}}(\lambda)$ that induces the identity map on Lie algebras. Once we have $\exp_{\pm a}$ in hand, the open immersion claim in the theorem is immediate from Theorem 4.1.7(4) since $\mathrm{T} = \mathrm{Z}_{\mathrm{G}}(\lambda)$ and necessarily $\exp_{\pm a}$ carries $\mathbf{W}(\mathfrak{g}_{\pm a})$ over to $\mathrm{U}_{\mathrm{G}}(\pm\lambda)$. This also gives that each $\mathrm{T} \ltimes \mathbf{W}(\mathfrak{g}_{\pm a})$ is a closed S-subgroup of G, as each is identified with $\mathrm{Z}_{\mathrm{G}}(\pm\lambda) \ltimes \mathrm{U}_{\mathrm{G}}(\pm\lambda) = \mathrm{P}_{\mathrm{G}}(\pm\lambda)$.

The uniqueness of $\exp_a$ amounts to the assertion that $\mathbf{W}(\mathfrak{g}_a)$ has no nontrivial automorphism that is $\mathbf{G}_{\mathrm{m}}$-equivariant and induces the identity on the Lie algebra. Working Zariski-locally so that $\mathfrak{g}_a$ admits a trivialization as a line bundle, this is the assertion that $\mathbf{G}_{\mathrm{a}}$ over S admits no nontrivial automorphism that is equivariant for the $\mathbf{G}_{\mathrm{m}}$-action $t.x = t^{\langle a,\lambda\rangle}x$ and induces the identity on $\mathrm{Lie}(\mathbf{G}_{\mathrm{a}})$. An endomorphism of $\mathbf{G}_{\mathrm{a}}$ over a ring k is precisely an additive polynomial, and equivariance for $t.x = t^n x$ with $n \neq 0$ says precisely that the polynomial is $x \mapsto cx$ for some $c \in k$, so the effect on the Lie algebra is multiplication by c. Thus, the identity condition on the Lie algebra forces $c = 1$, as desired.

It remains to prove the existence of $\exp_a$. The smooth S-group $\mathrm{U}_{\mathrm{G}}(\lambda)$ has fibers that are connected and unipotent (see Theorem 4.1.7(2),(4)) and of dimension 1 (as the Lie algebra is the line bundle $\mathfrak{g}_a$), so the geometric fibers of $\mathrm{U}_{\mathrm{G}}(\lambda)$ are $\mathbf{G}_{\mathrm{a}}$. Beware that it is not obvious that even the actual fibers of $\mathrm{U}_{\mathrm{G}}(\lambda)$ over S are isomorphic to $\mathbf{G}_{\mathrm{a}}$ (let alone that this holds Zariski-locally over S); the classification of forms of $\mathbf{G}_{\mathrm{a}}$ is rather subtle, even over (imperfect) fields, because the automorphism functor of $\mathbf{G}_{\mathrm{a}}$ is quite bad in positive characteristic (see [**Ru**]). The key to bypassing such difficulties is the $\mathbf{G}_{\mathrm{m}}$-action on $\mathrm{U}_{\mathrm{G}}(\lambda)$, as we now explain.

The uniqueness of $\exp_a$ allows us to work étale-locally to prove its existence, so we may assume that the smooth surjection $\mathrm{U}_{\mathrm{G}}(\lambda) \to \mathrm{S}$ admits a section σ disjoint from the identity section. In this case, we have:

Lemma 4.2.2. — *The* S*-group* $\mathrm{U}_{\mathrm{G}}(\lambda)$ *is isomorphic to* $\mathbf{G}_{\mathrm{a}}$*, via an isomorphism carrying its* $\mathbf{G}_{\mathrm{m}}$*-action over to* $t.x = t^{\langle a,\lambda\rangle}x$*.*

Granting this lemma, let us conclude the argument. Fix such an isomorphism of S-groups. This induces a canonical basis of $\mathrm{Lie}(\mathrm{U}_{\mathrm{G}}(\lambda)) = \mathfrak{g}_a$, which in turn identifies $\mathbf{W}(\mathfrak{g}_a)$ with $\mathbf{G}_{\mathrm{a}}$ carrying the $\mathbf{G}_{\mathrm{m}}$-action over to $t.x = t^{\langle a,\lambda\rangle}x$. Thus, visibly $\mathbf{W}(\mathfrak{g}_a)$ and $\mathrm{U}_{\mathrm{G}}(\lambda)$ are $\mathbf{G}_{\mathrm{m}}$-equivariantly isomorphic as S-groups (namely, isomorphic to $\mathbf{G}_{\mathrm{a}}$ with the indicated $\mathbf{G}_{\mathrm{m}}$-action). Pick one such isomorphism, so its effect on the Lie algebra is multiplication on $\mathfrak{g}_a$ by some global unit. Scaling on $\mathbf{W}(\mathfrak{g}_a)$ by the reciprocal of that unit then provides the desired $\exp_a$.

It remains to prove Lemma 4.2.2. By the functorial definition of $\mathrm{U}_{\mathrm{G}}(\lambda)$, the orbit map $\mathbf{G}_{\mathrm{m}} \to \mathrm{U}_{\mathrm{G}}(\lambda)$ defined by $t \mapsto t.\sigma$ extends to an S-scheme map

$$q : \mathbf{A}^1_{\mathrm{S}} \to \mathrm{U}_{\mathrm{G}}(\lambda)$$

that carries 1 to σ and is $\mathbf{G}_{\mathrm{m}}$-equivariant when using the *usual* $\mathbf{G}_{\mathrm{m}}$-scaling action on $\mathbf{A}^1_{\mathrm{S}}$ (as it suffices to check such equivariance on the open $\mathbf{G}_{\mathrm{m}}$ inside the affine S-line). Note also that $q(0) = 1$ by the definition of $\mathrm{U}_{\mathrm{G}}(\lambda)$. On fibers over a geometric point $\overline{s}$ of S, we may identify the group $\mathrm{U}_{\mathrm{G}}(\lambda)_{\overline{s}}$ with $\mathbf{G}_{\mathrm{a}}$ carrying the point $\sigma(\overline{s})$ over to 1, and the $\mathbf{G}_{\mathrm{m}}$-action on $\mathrm{U}_{\mathrm{G}}(\lambda)_{\overline{s}}$ goes over to scaling on $\mathbf{G}_{\mathrm{a}}$ by t^n for some $n \in \mathbf{Z}$. Inspecting $\mathrm{Lie}(\mathrm{U}_{\mathrm{G}}(\lambda)) = \mathfrak{g}_a$ shows $n = \langle a, \lambda \rangle$. Thus, $q_{\overline{s}}$ is identified with an endomorphism of the affine line over $k(\overline{s})$ that satisfies $q_{\overline{s}}(t) = t^{\langle a, \lambda \rangle}$ for $t \in \mathbf{G}_{\mathrm{m}}$ and hence for $t \in \mathbf{A}^1$.

Letting $n = \langle a, \lambda \rangle > 0$, we claim that μ_n acts trivially on $\mathrm{U}_{\mathrm{G}}(\lambda)$. Indeed, the centralizer subgroup scheme $\mathrm{U}_{\mathrm{G}}(\lambda)^{\mu_n}$ is a *smooth* closed subgroup of $\mathrm{U}_{\mathrm{G}}(\lambda)$ (as μ_n is multiplicative type), so this subgroup equals $\mathrm{U}_{\mathrm{G}}(\lambda)$ if and only if it does so on fibers over S. This reduces the μ_n-triviality claim on $\mathrm{U}_{\mathrm{G}}(\lambda)$ to the case of geometric fibers, where it is clear from our concrete description ($\mathrm{U}_{\mathrm{G}}(\lambda)_{\overline{s}} = \mathbf{G}_{\mathrm{a}}$ with $\mathbf{G}_{\mathrm{m}}$-action $t.x = t^n x$).

It follows from the $\mathbf{G}_{\mathrm{m}}$-equivariance of the map q that it is invariant under the natural μ_n-action on the affine line. The nth-power endomorphism of the affine S-line is a categorical quotient by the μ_n-action in the category of S-affine schemes, so q factors through an S-scheme map

$$\overline{q} : \mathbf{A}^1_{\mathrm{S}} \to \mathrm{U}_{\mathrm{G}}(\lambda)$$

carrying 0 to 1 that is $\mathbf{G}_{\mathrm{m}}$-equivariant when using the action $t.x = t^n x$ on the affine line (and the conjugation action on $\mathrm{U}_{\mathrm{G}}(\lambda)$ via λ). On geometric fibers over S, our earlier calculations with each $q_{\overline{s}}$ imply that each $\overline{q}_{\overline{s}}$ is identified with an *automorphism* of the affine line over $k(\overline{s})$ (as a scheme), so $\overline{q}$ is an isomorphism of S-schemes by Lemma B.3.1. Thus, it remains to prove that $\overline{q}$ is an S-homomorphism.

We have seen above that the $\mathbf{G}_{\mathrm{m}}$-action on $\mathrm{U}_{\mathrm{G}}(\lambda)$ makes μ_n act trivially, and the same holds for the $\mathbf{G}_{\mathrm{m}}$-action on the affine line over S that is the source for q. Thus, the domain of $\overline{q}$ inherits an action by the quotient $\mathbf{G}_{\mathrm{m}}/\mu_n \simeq \mathbf{G}_{\mathrm{m}}$ that makes the S-scheme isomorphism $\overline{q}$ identify $\mathrm{U}_{\mathrm{G}}(\lambda)$ with an S-group structure on $\mathbf{A}^1_{\mathrm{S}}$ that has 0 as the identity and is equivariant for the ordinary $\mathbf{G}_{\mathrm{m}}$-scaling. The S-homomorphism property for $\overline{q}$ is reduced to checking that addition is the only such group law on the affine S-line.

We may assume $\mathrm{S} = \operatorname{Spec} k$ for a ring k, so the abstract group law transferred from $\mathrm{U_G}(\lambda)$ via $\overline{q}$ is an $m \in k[x,y]$ satisfying $m(x,0) = x$, $m(0,y) = y$, and $m(tx,ty) = tm(x,y)$ for $t \in \mathbf{G}_{\mathrm{m}}$. It is clear by homogeneity considerations in t that the final condition forces $m = cx + c'y$ for some $c, c' \in k$, so the first two conditions imply $m = x + y$. □

Definition 4.2.3. — The image $\exp_a(\mathbf{W}(\mathfrak{g}_a)) \subset \mathrm{G}$ is denoted U_a and is called the *a-root group*.

Note that if the line bundle $\mathfrak{g}_a$ is trivialized (as may be done Zariski-locally on S) then U_a is identified with the additive group $\mathbf{G}_{\mathrm{a}}$ over S.

Remark 4.2.4. — The existence of the T-equivariant $\exp_a$ implies that to give an S-group isomorphism $p_a : \mathbf{G}_{\mathrm{a}} \simeq \mathrm{U}_a$ intertwining the T-action on U_a and the action $t.x = a(t)x$ on $\mathbf{G}_{\mathrm{a}}$ is precisely the same as to choose a global trivializing section X of the line bundle $\mathfrak{g}_a$ (via $p_a(z) = \exp_a(z\mathrm{X})$). This follows from the faithful flatness of $a : \mathrm{T} \to \mathbf{G}_{\mathrm{m}}$ and the easy fact that the only automorphisms of the S-group $\mathbf{G}_{\mathrm{a}}$ that are equivariant for the standard $\mathbf{G}_{\mathrm{m}}$-action are scaling by global units of S. Such isomorphisms p_a are called *parameterizations* of the root group U_a.

With root spaces $\mathfrak{g}_a$ and root groups U_a now constructed for roots a arising from triples $(\mathrm{G}, \mathrm{T}, \mathrm{M})$ over any non-empty scheme S, the remaining ingredient before we can discuss the split semisimple-rank 1 classification over S is the definition of coroots in the relative setting. As we have already noted, this requires a viewpoint rather different from the classical case over a field. Indeed, in the classical case coroots are defined using the classification of semisimple groups of rank 1 (over an algebraically closed field), whereas in the relative setting everything gets turned upside down: we need coroots even to state the split semisimple-rank 1 classification over a base scheme.

In the classical theory over an algebraically closed field, it makes sense to consider the closed subgroup $\langle \mathrm{U}_a, \mathrm{U}_{-a}\rangle$ of G generated by a pair of "opposite" root groups, and one shows that this is either SL_2 or PGL_2. But in the theory over rings it is unclear in what generality it makes sense to form a (smooth closed) subgroup "generated" by a pair of smooth closed subgroups of a smooth affine group. Likewise, we cannot use the alternative description $\mathscr{D}(\mathrm{Z_G}(\mathrm{T}_a))$ of $\langle \mathrm{U}_a, \mathrm{U}_{-a}\rangle$ since it is unclear in what generality the notion of "derived group" makes sense for smooth closed subgroups of a smooth affine group over a ring. The key to our success over S is to simultaneously characterize the coroot $a^\vee : \mathbf{G}_{\mathrm{m}} \to \mathrm{Z_G}(\mathrm{T}_a)$ and compatible trivializations of $\mathfrak{g}_a$ and $\mathfrak{g}_{-a}$ in intrinsic terms. For inspiration, once again we turn to the case of SL_2:

Example 4.2.5. — Let $\mathrm{G} = \mathrm{SL}_2$ over a non-empty scheme S and let D be the diagonal split maximal torus $\mathrm{D_S}(\mathbf{Z})$ equipped with the standard positive root $a : \mathrm{D} \simeq \mathbf{G}_{\mathrm{m}}$ given by $\lambda(t) \mapsto t^2$ where $\lambda : t \mapsto \mathrm{diag}(t, 1/t)$. This yields the standard "open cell"

$$\Omega_a = \mathrm{U}_{-a} \times \mathrm{D} \times \mathrm{U}_a = \mathrm{U}_{\mathrm{G}}(-\lambda) \times \mathrm{Z}_{\mathrm{G}}(\lambda) \times \mathrm{U}_{\mathrm{G}}(\lambda),$$

and likewise there is the other "open cell"

$$\Omega_{-a} = \mathrm{U}_a \times \mathrm{D} \times \mathrm{U}_{-a} \subset \mathrm{G}$$

(with the groups $\mathrm{U}_{\pm a}$ appearing in opposite order for the multiplication).

Consider the product $\mathrm{U}_a \times \mathrm{U}_{-a}$ in Ω_{-a}. Its points are

$$\exp_a(z)\exp_{-a}(z') = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ z' & 1 \end{pmatrix} = \begin{pmatrix} 1+zz' & z \\ z' & 1 \end{pmatrix}.$$

When does such a point lie in the other open cell $\Omega_a = \mathrm{U}_{-a} \times \mathrm{D} \times \mathrm{U}_a$? The points of Ω_a are those of the form

$$\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}\begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix}\begin{pmatrix} 1 & u' \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} c & cu' \\ cu & c^{-1}+cuu' \end{pmatrix},$$

which are precisely the points of SL_2 whose upper left entry is a unit. Hence, $\exp_a(z)\exp_{-a}(z')$ lies in Ω_a if and only if $1+zz'$ is a unit (with z, z' arbitrary points of $\mathbf{G}_{\mathrm{a}}$), in which case $c = 1+zz'$, $u = z'/(1+zz')$, and $u' = z/(1+zz')$.

Varying over points of SL_2 valued in all S-schemes, the duality between $\mathfrak{g}_a$ and $\mathfrak{g}_{-a}$ given by multiplication in $\mathscr{O}_{\mathrm{S}}$ characterizes when $\exp_a(z)\exp_{-a}(z')$ lies in Ω_a via the condition "$1+zz' \in \mathbf{G}_{\mathrm{m}}$". Likewise, since $\exp_a(z)\exp_{-a}(z')$ equals

$$\begin{pmatrix} 1 & 0 \\ z'/(1+zz') & 1 \end{pmatrix}\begin{pmatrix} 1+zz' & 0 \\ 0 & (1+zz')^{-1} \end{pmatrix}\begin{pmatrix} 1 & z/(1+zz') \\ 0 & 1 \end{pmatrix},$$

we see that the coroot $a^\vee(t) := \mathrm{diag}(t, 1/t)$ is recovered by noting that the D-component of $\exp_a(z)\exp_{-a}(z')$ in Ω_a is exactly $a^\vee(1+zz')$.

The calculations with SL_2 in Example 4.2.5 motivate:

Theorem 4.2.6. — *Let* G *be a reductive group over a scheme* S*, with fibers of semisimple-rank* 1*. Assume there exists a split maximal torus* $\mathrm{T} = \mathrm{D}_{\mathrm{S}}(\mathrm{M})$ *in* G *and a root* $a : \mathrm{T} \to \mathbf{G}_{\mathrm{m}}$ *arising from* M*. Let* $\mathrm{U}_{\pm a}$ *and* $\Omega_a := \mathrm{U}_{-a} \times \mathrm{T} \times \mathrm{U}_a$ *and* $\Omega_{-a} := \mathrm{U}_a \times \mathrm{T} \times \mathrm{U}_{-a}$ *be the associated root groups and "open cells" in* G *as in Theorem* 4.1.4*.*

There is a unique pair $(\beta_a, a^\vee)$ *consisting of an* $\mathscr{O}_{\mathrm{S}}$*-bilinear (hence* $\mathbf{G}_{\mathrm{m}}$*-equivariant) pairing of line bundles* $\beta_a : \mathfrak{g}_a \times \mathfrak{g}_{-a} \to \mathscr{O}_{\mathrm{S}}$ *(denoted* $(\mathrm{X}, \mathrm{Y}) \mapsto \mathrm{XY}$*) and an* S*-homomorphism* $a^\vee : \mathbf{G}_{\mathrm{m}} \to \mathrm{T}$ *such that the following conditions hold:*

1. *for any* S*-scheme* S′ *and points* $\exp_a(\mathrm{X}) \in \mathrm{U}_a(\mathrm{S}')$ *and* $\exp_{-a}(\mathrm{Y}) \in \mathrm{U}_{-a}(\mathrm{S}')$*, the* S′*-valued point*

$$\exp_a(\mathrm{X})\exp_{-a}(\mathrm{Y}) \in \Omega_{-a} \subset \mathrm{G}$$

lies in the "open cell" Ω_a *if and only if* $1+\mathrm{XY}$ *is a unit on* S'.

2. when this unit condition is satisfied,

(4.2.1)
$$\exp_a(\mathrm{X})\exp_{-a}(\mathrm{Y}) = \exp_{-a}\left(\frac{\mathrm{Y}}{1+\mathrm{XY}}\right) a^\vee(1+\mathrm{XY})\exp_a\left(\frac{\mathrm{X}}{1+\mathrm{XY}}\right) \in \Omega_a.$$

In particular, the formation of this bilinear pairing and $a^\vee$ *commute with base change on* S.

Moreover, the pairing $(\mathrm{X},\mathrm{Y}) \mapsto \mathrm{XY}$ *is a perfect duality, and* $a\circ a^\vee = 2$ (*i.e.,* $a(a^\vee(c)) = c^2$ *for* $c \in \mathbf{G}_\mathrm{m}$).

This result is [**SGA3**, XX, 2.1], whose proof there involves elaborate calculations. Our proof of Theorem 4.2.6 will be long, but it involves very few calculations and yields some auxiliary results that rapidly lead to a Zariski-local version of the classification of "split" reductive groups of semisimple-rank 1 (in Theorem 5.1.8). First, we make some observations.

An interesting consequence of the duality in Theorem 4.2.6 is that the line bundle $\mathfrak{g}_a$ is globally trivial if and only if $\mathfrak{g}_{-a}$ is, so likewise U_a admits a parameterization in the sense of Remark 4.2.4 if and only if U_{-a} does. When such parameterizations $p_{\pm a} : \mathbf{G}_\mathrm{a} \simeq \mathrm{U}_{\pm a}$ exist, we say (following [**SGA3**, XX, 2.6.1]) that they are *linked* if they correspond to dual bases for $\mathfrak{g}_a$ and $\mathfrak{g}_{-a}$; such dual bases are called *linked trivializations.* (An alternative convention, advocated by Demazure in more recent times, is to declare bases of $\mathfrak{g}_a$ and $\mathfrak{g}_{-a}$ to be linked when they are negative dual to each other. This has the advantage that the open subscheme $\Omega_a \bigcap \Omega_{-a}$ in Ω_a is defined by $\mathrm{XY} \neq 1$ rather than $\mathrm{XY} \neq -1$. It thereby eliminates signs in certain equations.)

Clearly for a given parameterization of U_a, there exists a unique one of U_{-a} to which it is linked. Note also that necessarily $(-a)^\vee = -a^\vee$. Indeed, this is a known fact in the classical theory, and in general it can be deduced from geometric fibers over S.

Proof of Theorem 4.2.6. — We may and do assume S is non-empty. To prove uniqueness, we first note that $\mathrm{Z_G} = \ker a$ by Corollary 3.3.6(1). Concretely, inside $\mathrm{T} = \mathrm{D_S}(\mathrm{M})$ we have $\mathrm{Z_G} = \ker a = \mathrm{D_S}(\mathrm{M}/\mathbf{Z}a)$. (Note that $\mathrm{M}/\mathbf{Z}a$ may not be torsion-free.)

The quotient $\mathrm{G}/\mathrm{Z_G}$ is a reductive group scheme in which $\mathrm{T}/\mathrm{Z_G} = \mathrm{D_S}(\mathbf{Z}a)$ is a maximal torus such that the induced character $\overline{a} : \mathrm{T}/\mathrm{Z_G} \to \mathbf{G}_\mathrm{m}$ is an isomorphism. In particular, since $\ker \overline{a} = 1$, it follows that $\mathrm{G}/\mathrm{Z_G}$ has trivial schematic center (Corollary 3.3.5). Moreover, the behavior of the open immersion (4.1.1) under $\mathrm{Z_G}$-scaling shows that (i) the natural maps $\mathbf{W}(\mathfrak{g}_{\pm a}) \rightrightarrows \mathrm{G}/\mathrm{Z_G}$ are isomorphisms onto closed subgroups $\mathrm{V}_{\pm\overline{a}}$ normalized by $\mathrm{T}/\mathrm{Z_G}$ and (ii) the adjoint action of $\mathrm{G}/\mathrm{Z_G}$ on its Lie algebra makes $\mathrm{T}/\mathrm{Z_G}$ have $\mathrm{Lie}(\mathrm{V}_{\pm\overline{a}})$ as a weight space for the character $\pm\overline{a}$. In other words, the quotient map $\mathrm{G} \to \mathrm{G}/\mathrm{Z}_{\mathrm{G}}$ induces an isomorphism $\mathrm{U}_{\pm a} \simeq \mathrm{U}_{\pm \overline{a}}$ and identifies the "open cell" $\Omega_{\pm \overline{a}}$ as $\Omega_{\pm a}/\mathrm{Z}_{\mathrm{G}}$.

We conclude that to construct the bilinear pairing between $\mathfrak{g}_a$ and $\mathfrak{g}_{-a}$ that characterizes the points of $\mathrm{U}_a \times \mathrm{U}_{-a}$ whose product in G lies in the open subscheme $\Omega_a = \mathrm{U}_{-a} \times \mathrm{T} \times \mathrm{U}_a$, it is harmless to pass to $\mathrm{G}/\mathrm{Z}_{\mathrm{G}}$. Likewise, for the proof of uniqueness it is harmless to pass to $\mathrm{G}/\mathrm{Z}_{\mathrm{G}}$ provided that we can settle uniqueness in general over an algebraically closed field, since any S-homomorphism $\mathbf{G}_{\mathrm{m}} \to \mathrm{T}$ is uniquely determined by its effect on geometric fibers over S. Thus, for the proof of uniqueness it suffices to treat two cases: $\mathrm{S} = \mathrm{Spec}(k)$ for an algebraically closed field k, and $\mathrm{Z}_{\mathrm{G}} = 1$ over a general S.

Consider the situation over an algebraically closed field k. In this case we know that $\mathrm{G}/\mathrm{Z}_{\mathrm{G}} = \mathrm{PGL}_2$ by the classical theory, and by conjugacy of maximal tori we can choose this identification to carry the maximal torus $\mathrm{T}/\mathrm{Z}_{\mathrm{G}}$ over to the diagonal torus $\overline{\mathrm{D}}$ parameterized by $\overline{\lambda} : t \mapsto \mathrm{diag}(t, 1)$. Applying conjugation by a representative of the nontrivial element of $\mathrm{W}_{\mathrm{PGL}_2}(\overline{\mathrm{D}})$ (such as the standard Weyl element) if necessary, we can also arrange that $\overline{a}$ goes over to the unique root for $\overline{\mathrm{D}}$ that satisfies $\overline{a}(\overline{\lambda}(t)) = t$. In this case, existence is settled by using the calculations in Example 4.2.5 and composing with the degree-2 central isogeny $\mathrm{SL}_2 \to \mathrm{PGL}_2$ (e.g., we take $\overline{a}^\vee$ to be the composition of $t \mapsto \mathrm{diag}(t, 1/t) \in \mathrm{SL}_2$ with the central isogeny to PGL_2, which is to say $\overline{a}^\vee(t) = \mathrm{diag}(t^2, 1)$). These calculations also imply uniqueness of the pairing of root spaces, since we can pass to $\mathrm{G}/\mathrm{Z}_{\mathrm{G}} = \mathrm{PGL}_2$ and observe that any possibility for the bilinear pairing must be a multiple of the standard one by some $c \in k$, yet the unit conditions on $1 + xy$ and $1 + cxy$ for varying $x, y \in k$ do not coincide unless $c = 1$ since k is an algebraically closed field.

The bilinear pairing between root spaces is uniquely determined (over the algebraically closed field k) by composing the formula (4.2.1) over k with projection to $\mathrm{G}/\mathrm{Z}_{\mathrm{G}}$, so any possibility for the coroot $a^\vee : \mathbf{G}_{\mathrm{m}} \to \mathrm{T}$ over k has composition with $\mathrm{T} \to \mathrm{T}/\mathrm{Z}_{\mathrm{G}} = \overline{\mathrm{D}} \subset \mathrm{PGL}_2$ given by $c \mapsto \mathrm{diag}(c, 1/c) = \mathrm{diag}(c^2, 1) \bmod \mathbf{G}_{\mathrm{m}}$. Thus, $a^\vee$ is unique up to multiplication against a cocharacter $\mu : \mathbf{G}_{\mathrm{m}} \to \mathrm{Z}_{\mathrm{G}}$. But any such cocharacter μ factors through the torus $(\mathrm{Z}_{\mathrm{G}})^0_{\mathrm{red}}$ that is the maximal central torus in G, and this has finite intersection with the connected semisimple $\mathscr{D}(\mathrm{G})$. Since (4.2.1) over k forces $a^\vee$ to be valued in $\langle \mathrm{U}_a, \mathrm{U}_{-a} \rangle = \mathscr{D}(\mathrm{G})$, it follows that μ is trivial, so $a^\vee$ is also unique. This completes the proof of existence and uniqueness over an algebraically closed field, and in such cases the additional properties (perfectness of the bilinear pairing, and the identity $a \circ a^\vee = 2$) are immediate from these calculations (since a factors through $\mathrm{T}/\mathrm{Z}_{\mathrm{G}}$).

Returning to the situation over a general (non-empty) base S, the results over an algebraically closed field imply uniqueness of the coroot in general, as well as perfectness of the bilinear pairing (if it exists) and the identity

$a \circ a^\vee = 2$ (as the latter concerns an endomorphism of $\mathbf{G}_\mathrm{m}$ and so can be checked on geometric fibers). We will next prove uniqueness in general, and then it will remain to address existence. As we have already noted, for the proof of uniqueness (of the bilinear pairing, as the case of the coroot is settled) we may and do assume $\mathrm{Z}_\mathrm{G} = 1$. In this case G has all geometric fibers isomorphic to PGL_2 by the classical theory, so the roots $\pm a : \mathrm{T} \to \mathbf{G}_\mathrm{m}$ are isomorphisms on geometric fibers over S and hence are isomorphisms over S. Thus, we can apply:

Proposition 4.2.7. — *Let* $\mathrm{G} \to \mathrm{S}$ *be a reductive group with trivial center and geometric fibers of semisimple-rank* 1*. If there exists a split maximal torus* $\mathrm{T} \subset \mathrm{G}$ *then Zariski-locally on* S *there exists a group isomorphism* $\mathrm{G} \simeq \mathrm{PGL}_2$*. This isomorphism may be chosen to carry* T *over to the diagonal torus.*

The Zariski-local nature of this result could be improved to a unique global isomorphism at the cost of using a relative notion of pinning and carrying out some preliminary arguments with the relative notion of Borel subgroup to prove that PGL_2 is its own automorphism functor. We postpone such considerations until we treat the general Existence and Isomorphism Theorems, as Zariski-local results will be entirely sufficient for our present purposes.

Proof. — The split property of T provides a weight space decomposition of the rank-3 vector bundle $\mathfrak{g}$, and by working Zariski-locally on S we may arrange that there exists a root $a : \mathrm{T} \simeq \mathbf{G}_\mathrm{m}$. Let $\lambda : \mathbf{G}_\mathrm{m} \simeq \mathrm{T}$ be the inverse of a, so $\mathrm{U}_\mathrm{G}(\pm\lambda) = \mathrm{U}_{\pm a}$. Let $\mathrm{B} = \mathrm{P}_\mathrm{G}(\lambda)$. By Exercise 4.4.2(i) and the PGL_2-variant of Example 4.1.5, on geometric fibers this is a Borel subgroup. Proposition 2.1.6 provides a closed normalizer subscheme $\mathrm{N}_\mathrm{G}(\mathrm{B}) \subset \mathrm{G}$. We have not shown this normalizer to be flat, but we claim more: the closed immersion $\mathrm{B} \hookrightarrow \mathrm{N}_\mathrm{G}(\mathrm{B})$ inside G is an equality. Since B is flat, by Lemma B.3.1 it suffices to prove equality on geometric fibers.

Now consider the situation over an algebraically closed field k. An elementary calculation with PGL_2 over k shows that B and $\mathrm{N}_\mathrm{G}(\mathrm{B})$ have the same k-points, so it suffices to show that $\mathrm{N}_\mathrm{G}(\mathrm{B})$ has the same Lie algebra as B inside of $\mathfrak{pgl}_2$. By dimension considerations, this is just a matter of ruling out the possibility that $\mathrm{Lie}(\mathrm{N}_\mathrm{G}(\mathrm{B})) = \mathfrak{pgl}_2$. But $\mathfrak{b}$ is an ideal in $\mathrm{Lie}(\mathrm{N}_\mathrm{G}(\mathrm{B}))$ and it is clearly not an ideal in $\mathfrak{pgl}_2$.

We conclude that $\mathrm{B} = \mathrm{N}_\mathrm{G}(\mathrm{B})$ as S-subgroups of G, so by Theorem 2.3.6 the quotient sheaf G/B is a smooth proper S-scheme admitting a canonical S-ample line bundle. The formation of G/B commutes with any base change, such as passage to geometric fibers over S, so these fibers are identified with the quotient scheme of PGL_2 modulo a Borel subgroup over an algebraically closed field, which is to say that the geometric fibers $(\mathrm{G}/\mathrm{B})_{\overline{s}}$ are isomorphic to $\mathbf{P}^1$. That is, $\mathrm{G}/\mathrm{B} \to \mathrm{S}$ is a smooth proper curve with connected geometric fibers of genus 0. Moreover, the identity section of $\mathrm{G} \to \mathrm{S}$ provides a section σ to $\mathrm{G}/\mathrm{B} \to \mathrm{S}$.

By standard arguments with cohomology and base change (applied to the direct image on S of the inverse of the ideal sheaf of σ on G/B, after reducing to the case of noetherian S), Zariski-locally on S there exists an isomorphism $\mathrm{G}/\mathrm{B} \simeq \mathbf{P}^1$ carrying the section σ over to ∞. Hence, we may assume $\mathrm{G}/\mathrm{B} \simeq \mathbf{P}^1_{\mathrm{S}}$ carrying 1 mod B to ∞. In particular, the automorphism functor $\underline{\mathrm{Aut}}_{(\mathrm{G}/\mathrm{B})/\mathrm{S}}$ of G/B on the category of S-schemes is represented by PGL_2 (Exercise 1.6.3(iv)) with the stabilizer of 1 mod B going over to the stabilizer of $\infty \in \mathbf{P}^1$, which is to say the standard upper triangular subgroup B_∞ of PGL_2.

The left translation action of G on G/B defines an S-homomorphism

$$\mathrm{G} \to \underline{\mathrm{Aut}}_{(\mathrm{G}/\mathrm{B})/\mathrm{S}} = \mathrm{PGL}_2.$$

On geometric fibers this is an isomorphism by the classical theory, so it is an isomorphism of S-groups (Lemma B.3.1). Points of B are carried into B_∞, and the resulting map $\mathrm{B} \to \mathrm{B}_\infty$ is an isomorphism since it is so on geometric fibers over S. Thus, the torus $\mathrm{T} \subset \mathrm{B}$ is carried over to a maximal torus of PGL_2 contained in B_∞.

It remains to prove that any maximal torus T of PGL_2 over S that is contained in B_∞ can be conjugated to the diagonal torus $\mathrm{D} = \mathbf{G}_{\mathrm{m}}$ Zariski-locally on S. Consider the smooth transporter scheme $\mathrm{Y} = \mathrm{Transp}_{\mathrm{B}_\infty}(\mathrm{T}, \mathrm{D})$ over S. All fibers Y_s are non-empty, and Y is stable under left multiplication by D in PGL_2. We claim that this makes Y a left D-torsor for the étale topology. Since the smooth surjection $\mathrm{Y} \to \mathrm{S}$ admits sections étale-locally on S, the torsor assertion is equivalent to the condition that the map

$$\mathrm{D} \times_{\mathrm{S}} \mathrm{Y} \to \mathrm{Y} \times_{\mathrm{S}} \mathrm{Y}$$

defined by $(d, y) \mapsto (d.y, y)$ is an isomorphism. By the smoothness of both sides it is sufficient to check the isomorphism property on fibers over geometric points $\overline{s}$ of S. But $\mathrm{T}_{\overline{s}}$ is $\mathrm{B}_\infty(\overline{s})$-conjugate to $\mathrm{D}_{\overline{s}}$ by the classical theory, so $\mathrm{Y}_{\overline{s}}$ is a torsor for the *smooth* normalizer scheme $\mathrm{N}_{\mathrm{B}_\infty}(\mathrm{D})_{\overline{s}}$ that is equal to $\mathrm{D}_{\overline{s}}$ (via computation on geometric points). Since $\mathrm{D} = \mathbf{G}_{\mathrm{m}}$, and every $\mathbf{G}_{\mathrm{m}}$-torsor for the étale topology is also a torsor for the Zariski topology (by descent theory for line bundles), it follows that $\mathrm{Y} \to \mathrm{S}$ admits sections Zariski-locally over S, so the desired B_∞-conjugation of T into D exists Zariski-locally on S. $\square$

By Proposition 4.2.7, for the proof of uniqueness in Theorem 4.2.6 we may assume $\mathrm{G} = \mathrm{PGL}_2$ with T the diagonal torus $\overline{\mathrm{D}}$ parameterized by $\lambda : \mathbf{G}_{\mathrm{m}} \simeq \overline{\mathrm{D}}$ via $\lambda(t) = \mathrm{diag}(t, 1)$. Any root $\overline{a} : \overline{\mathrm{D}} \to \mathbf{G}_{\mathrm{m}}$ must be inverse to one of $\pm\lambda$ Zariski-locally on S, as it suffices to check this on geometric fibers (where it follows from the classical theory). Thus, by working Zariski-locally on S and composing with conjugation by the standard Weyl element $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ if necessary, we can arrange that $\overline{a}$ is inverse to λ. It follows that $\langle \overline{a}, \lambda \rangle = 1 > 0$, so the root group $\mathrm{U}_{\overline{a}} = \mathrm{U}_{\mathrm{PGL}_2}(\lambda)$ is the strictly upper triangular subgroup of PGL_2 and the root group $\mathrm{U}_{-\overline{a}}$ is the strictly lower triangular subgroup. If we use the standard bases of the Lie algebras of these subgroups of PGL_2 then $\exp_{\pm\overline{a}}$ are the standard parameterizations of $\mathrm{U}_{\pm\overline{a}}$ (see Example 4.2.1, using the root $\mathrm{diag}(c, 1) \mapsto c$ and composition with the central isogeny $\mathrm{SL}_2 \to \mathrm{PGL}_2$). Thus, the calculations in SL_2 in Example 4.2.5 show that the standard duality between the root spaces (using their standard bases) and the 1-parameter subgroup $\overline{a}^\vee(c) = \mathrm{diag}(c^2, 1) = \mathrm{diag}(c, 1/c) \bmod \mathbf{G}_{\mathrm{m}}$ satisfy the requirements.

We have just proved existence for Theorem 4.2.6 over a general base S when $\mathrm{Z}_{\mathrm{G}} = 1$, and the argument gives uniqueness in such cases too. Indeed, any possibility for the bilinear pairing must be $(\mathrm{X}, \mathrm{Y}) \mapsto c\mathrm{XY}$ for some global unit c on S, and the equivalence of the unit conditions on $1 + \mathrm{XY}$ and $1 + c\mathrm{XY}$ on all S-schemes forces $c = 1 + \xi$ for some nilpotent ξ on S. Then the requirement (4.2.1) (applied to the modified pairing $(\mathrm{X}, \mathrm{Y}) \mapsto c\mathrm{XY}$) and the analogous established formula using the standard pairing and standard coroot force $\xi = 0$ (because $\mathrm{U}_{-a} \times \mathrm{T} \times \mathrm{U}_a \simeq \Omega_a$) and force any possibility for the coroot to agree with the standard coroot on any unit of the form $1 + xy$ with functions x and y on varying S-schemes. Any unit can be expressed in this form (take $y = 1$), so uniqueness is established in general when $\mathrm{Z}_{\mathrm{G}} = 1$. But we already noted above that uniqueness over a general base when $\mathrm{Z}_{\mathrm{G}} = 1$ implies uniqueness in general (without restriction on Z_{G}), since uniqueness is already known for the coroot (due to the case of an algebraically closed ground field, which has been completely settled).

Finally, it remains prove existence without assuming $\mathrm{Z}_{\mathrm{G}} = 1$. By the settled general uniqueness, we may work Zariski-locally on S for existence. Thus, we can arrange that $\mathrm{G}/\mathrm{Z}_{\mathrm{G}}$ and its split maximal torus $\mathrm{T}/\mathrm{Z}_{\mathrm{G}}$ admit an isomorphism

$$(\mathrm{G}/\mathrm{Z}_{\mathrm{G}}, \mathrm{T}/\mathrm{Z}_{\mathrm{G}}) \simeq (\mathrm{PGL}_2, \overline{\mathrm{D}})$$

by Proposition 4.2.7. Consider the pullback diagram

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{Z}_{\mathrm{G}} \arrow[r] \arrow[d, equal] & \widetilde{\mathrm{G}} \arrow[r] \arrow[d] & \mathrm{SL}_2 \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathrm{Z}_{\mathrm{G}} \arrow[r] & \mathrm{G} \arrow[r] & \mathrm{PGL}_2 \arrow[r] & 1
\end{tikzcd}$$

in which the top row is a central extension. By Proposition 4.3.1 below, the top row uniquely splits, so we get an isomorphism $\widetilde{\mathrm{G}} \simeq \mathrm{SL}_2 \times \mathrm{Z}_{\mathrm{G}}$ in which the preimage $\widetilde{\mathrm{T}}$ of T goes over to $\mathrm{D} \times \mathrm{Z}_{\mathrm{G}}$ (since the diagonal torus $\mathrm{D} \subset \mathrm{SL}_2$ is the full preimage of the diagonal torus $\overline{\mathrm{D}} \subset \mathrm{PGL}_2$). Thus, we get a cocharacter

$$a^\vee : \mathbf{G}_{\mathrm{m}} = \mathrm{D} \to \mathrm{T}$$

via the identification $\mu : \mathbf{G}_{\mathrm{m}} \simeq \mathrm{D}$ defined by $\mu : t \mapsto \mathrm{diag}(t, 1/t)$.

Composing $a^\vee$ with $\mathrm{T} \to \mathrm{T}/\mathrm{Z}_\mathrm{G} = \overline{\mathrm{D}}$ yields $\overline{\lambda} : t \mapsto \mathrm{diag}(t, 1/t) \bmod \mathbf{G}_\mathrm{m} = \mathrm{diag}(t^2, 1) = \lambda(t^2) \in \mathrm{PGL}_2$. Thus,

$$\langle a, a^\vee \rangle = \langle \overline{a}, a^\vee \bmod \mathrm{Z}_\mathrm{G} \rangle = \langle \overline{a}, \overline{\lambda} \rangle = \langle \overline{a}, 2\lambda \rangle = 2 > 0,$$

so the standard root groups $\mathrm{U}^\pm = \mathrm{U}_{\mathrm{SL}_2}(\pm\mu)$ in SL_2 map into $\mathrm{U}_\mathrm{G}(\pm\overline{\mu}) = \mathrm{U}_{\pm a}$ via $\widetilde{\mathrm{G}} \to \mathrm{G}$ and these maps $\mathrm{U}^\pm \to \mathrm{U}_{\pm a}$ are isomorphisms on geometric fibers for smoothness, dimension, and unipotence reasons because $\ker(\widetilde{\mathrm{G}} \to \mathrm{G}) = \mu_2$. Hence, the maps $\mathrm{U}^\pm \to \mathrm{U}_{\pm a}$ are isomorphisms. In this way, the induced map

$$\mathrm{U}^- \times (\mathrm{D} \times \mathrm{Z}_\mathrm{G}) \times \mathrm{U}^+ \to \mathrm{U}_{-a} \times \mathrm{T} \times \mathrm{U}_a = \Omega_a$$

is the quotient of translation by the central $\mathrm{D}[2] = \mu_2$, so it computes the full preimage of Ω_a in $\widetilde{\mathrm{G}}$; the same holds with the roles of $-a$ and a swapped. Thus, our explicit knowledge of the standard coroot and bilinear pairing of root spaces for $(\mathrm{SL}_2, \mathrm{D})$ (as in Example 4.1.5) imply that $a^\vee : \mathbf{G}_\mathrm{m} \to \mathrm{T}$ and the bilinear pairing constructed between $\mathfrak{g}_a$ and $\mathfrak{g}_{-a}$ satisfy the desired requirements that uniquely characterize the coroot and bilinear pairing of root spaces. This completes the proof of Theorem 4.2.6 conditional on the (unique) splitting of central extensions of SL_2, provided by Proposition 4.3.1 below. $\square$

4.3. Central extensions of SL_2. — This section is devoted to proving a general splitting result for central extensions of SL_2 (needed to complete the proof of Theorem 4.2.6) and recording an SL_2-variant of Proposition 4.2.7. We begin with the result on central extensions:

***Proposition 4.3.1* (Gabber).** — *Let* S *be a scheme, and* Z *a commutative separated* S*-group. Any fppf central extension of group sheaves*

$$1 \to \mathrm{Z} \to \mathrm{G}' \to \mathrm{SL}_2 \to 1$$

is uniquely split as a central extension. (*In particular,* G' *is an* S*-scheme.*)

We only need the case that G' is an S-affine S-group of finite presentation (possibly not smooth!) with Z of multiplicative type. I am grateful to Gabber for proving the result in the generality above (my original proof via deformation theory was only for Z of multiplicative type).

Proof. — In $\mathrm{G} := \mathrm{SL}_2$, define the usual parameterizations $h(t) = \mathrm{diag}(t, 1/t)$ of the diagonal torus D and $x(u) = \left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right)$ and $y(v) = \left(\begin{smallmatrix} 1 & 0 \\ v & 1 \end{smallmatrix}\right)$ for the strictly upper triangular subgroup U and strictly lower triangular subgroup V respectively. We first address the uniqueness of the splitting, which is to say (due to the centrality of Z in G') the vanishing of any S-homomorphism $\mathrm{G} = \mathrm{SL}_2 \to \mathrm{Z}$. Since $h(t)x(u)h(t)^{-1}x(u)^{-1} = x((t^2-1)u)$ and $h(t)y(v)h(t)^{-1}y(v)^{-1} = y((t^{-2}-1)v)$, and fppf-locally there is a unique t such that $t^{\pm 2} - 1$ are units, any homomorphism $f : \mathrm{G} \to \mathrm{Z}$ to an S-separated commutative target must kill the subgroups U and V. But for $g(t) := y(-1/t)x(t)y(-1/t)$ (with $t \in \mathbf{G}_\mathrm{m}$)

the standard formula $g(t)g(1)^{-1} = h(t)$ implies that f also kills D, so $\ker f$ contains the open cell $\Omega := \mathrm{U} \times \mathrm{D} \times \mathrm{V} \subset \mathrm{G}$. Thus, using relative schematic density [**EGA**, IV$_3$, 11.10.10] we reduce to the case over a field, where clearly $f = 1$. This proves the uniqueness.

To build a splitting of the given central extension, let $\mathrm{D}', \mathrm{U}', \mathrm{V}' \subset \mathrm{G}'$ respectively denote the preimages of $\mathrm{D}, \mathrm{U}, \mathrm{V} \subset \mathrm{G}$, so each is a central extension by Z of its image in G. A key point is to verify that D' is commutative. As for any central extension of one commutative group object by another, the commutator of D' factors through a bi-additive pairing

$$\mathrm{D} \times \mathrm{D} \to \mathrm{Z}$$

whose vanishing is equivalent to the commutativity of D' (see Exercise 4.4.3). Hence, it suffices to show there is no nontrivial bi-additive pairing $\mathbf{G}_{\mathrm{m}} \times \mathbf{G}_{\mathrm{m}} \to \mathrm{Z}$ into a separated commutative S-group. The collection of subgroups $\{\mu_n\}$ in $\mathbf{G}_{\mathrm{m}}$ is relatively schematically dense over S, so via S-separatedness of Z it suffices to prove any bi-additive $\mu_n \times \mathbf{G}_{\mathrm{m}} \to \mathrm{Z}$ vanishes. But $[n] : \mathbf{G}_{\mathrm{m}} \to \mathbf{G}_{\mathrm{m}}$ is an epimorphism of sheaves, so the vanishing is clear.

Next, we use commutativity of D' to prove commutativity of U' and V'. By symmetry, it suffices to treat U'. Note that D' normalizes U', and the G'-action on G' by conjugation factors through an action by the central quotient $\mathrm{G} = \mathrm{G}'/\mathrm{Z}$, so we get a natural action by D on U'. The bi-additive pairing

$$c : \mathrm{U} \times \mathrm{U} \to \mathrm{Z}$$

induced by the commutator on U' is clearly D-invariant in the sense that $c(h.u_1, h.u_2) = c(u_1, u_2)$ for all $h \in \mathrm{D} = \mathbf{G}_{\mathrm{m}}$ and $u_1, u_2 \in \mathrm{U} = \mathbf{G}_{\mathrm{a}}$. That is, for all $t \in \mathbf{G}_{\mathrm{m}}$ and $u_1, u_2 \in \mathbf{G}_{\mathrm{a}}$ we have $c(tu_1, tu_2) = c(u_1, u_2)$. Equivalently, $c(tu_1, u_2) = c(u_1, t^{-1}u_2)$. Consider fppf-local units t such that $t + 1$ is a unit and $t' := (t + 1)^{-1} - t^{-1} - 1$ is a unit. Bi-additivity of c gives

$$\begin{aligned} c(u_1, t'u_2) &= c(u_1, (t+1)^{-1}u_2)c(u_1, t^{-1}u_2)^{-1}c(u_1, u_2)^{-1} \\ &= c((t+1)u_1, u_2)c(tu_1, u_2)^{-1}c(u_1, u_2)^{-1} \\ &= c((t+1)u_1 - tu_1 - u_1, u_2) \\ &= 1 \end{aligned}$$

in Z. As an algebraic identity it is clear that such t' cover a relatively schematically dense open locus in $\mathbf{G}_{\mathrm{m}}$ (namely, the locus of $u \in \mathbf{G}_{\mathrm{m}}$ such that $u + 1 \in \mathbf{G}_{\mathrm{m}}$), so S-separatedness of Z then forces $c = 1$ as desired.

Lemma 4.3.2. — *The* D*-equivariant quotient maps* $\mathrm{U}' \to \mathrm{U}$ *and* $\mathrm{V}' \to \mathrm{V}$ *admit unique* D*-equivariant splittings.*

Proof. — By symmetry it suffices to treat U'. First we address the uniqueness, so then we may work fppf-locally on S to make the construction. Uniqueness amounts to the assertion that there are no nontrivial D-equivariant S-homomorphisms $f : \mathrm{U} \to \mathrm{Z}$. The torus $\mathrm{D} = \mathbf{G}_{\mathrm{m}}$ acts trivially on Z but acts on $\mathrm{U} = \mathbf{G}_{\mathrm{a}}$ via $t.x(u) = x(t^2u)$, so $\mathrm{U} - \{0\}$ is a single "D-orbit". Thus, relative schematic density of $\mathrm{U} - \{0\}$ in U and the S-separatedness of Z then give the vanishing of any such f.

Now working fppf-locally on S, we may arrange that there exists a unit t on S such that $t^2 - 1$ is also a unit, and also that the element $h(t) \in \mathrm{D}(\mathrm{S})$ admits a lift $h' \in \mathrm{D}'(\mathrm{S})$. The $h(t)$-action on U' is induced by h'-conjugation, and as an endomorphism of the commutative U' it induces the identity on the subgroup Z. Thus, the endomorphism $\varphi : u' \mapsto h'u'h'^{-1} - u' = h(t).u' - u'$ of the abstract commutative S-group U' kills Z and lies over the endomorphism $\overline{\varphi}$ of $\mathrm{U} = \mathbf{G}_{\mathrm{a}}$ given by $x(u) \mapsto x(t^2u)x(u)^{-1} = x((t^2-1)u)$. But $\overline{\varphi}$ is an automorphism, so φ factors through an S-homomorphism $\mathrm{U} = \mathrm{U}'/\mathrm{Z} \to \mathrm{U}'$ lifting an automorphism $\overline{\varphi}$ of U, and by construction it is D-equivariant (due to the commutativity of D'). Precomposing with the inverse of $\overline{\varphi}$ then provides the desired splitting. $\square$

Using the unique D-equivariant S-group isomorphisms $\mathrm{U}' \simeq \mathrm{U} \times \mathrm{Z}$ and $\mathrm{V}' \simeq \mathrm{V} \times \mathrm{Z}$ that split the central extensions, we obtain D-equivariant S-homomorphisms $x' : \mathbf{G}_{\mathrm{a}} \to \mathrm{U}'$ and $y' : \mathbf{G}_{\mathrm{a}} \to \mathrm{V}'$ lifting the respective standard parameterizations x of U and y of V. We'll use these to build an S-homomorphism $h' : \mathbf{G}_{\mathrm{m}} \to \mathrm{D}'$ lifting the parameterization $h : \mathbf{G}_{\mathrm{m}} \simeq \mathrm{D}$. Define $g'(t) = y'(-1/t)x'(t)y'(-1/t)$ for $t \in \mathbf{G}_{\mathrm{m}}$; this lifts the point $g(t) := y(-1/t)x(t)y(-1/t) = \left(\begin{smallmatrix} 0 & t \\ -1/t & 0 \end{smallmatrix}\right) \in \mathrm{SL}_2 = \mathrm{G}$, so $h'(t) := g'(t)g'(1)^{-1} \in \mathrm{G}'$ lifts $g(t)g(1)^{-1} = h(t)$ and hence is valued in D'.

Lemma 4.3.3. — *The map $h' : \mathbf{G}_{\mathrm{m}} \to \mathrm{D}'$ is a homomorphism lifting the parameterization $h : \mathbf{G}_{\mathrm{m}} \simeq \mathrm{D}$, and $h'(s)g'(t) = g'(st)$ for all $s, t \in \mathbf{G}_{\mathrm{m}}$.*

Proof. — Conjugation by $h'(s)$ on $g'(t) = y'(-1/t)x'(t)y'(-1/t)$ is the same as the action by $h(s)$, and so by the D-equivariance of the construction of x' and y' as respective lifts of x and y we have $h(s).y'(v) = y'(v/s^2)$ and $h(s).x'(u) = x'(s^2u)$ for any $u, v \in \mathbf{G}_{\mathrm{a}}$. Thus, $h(s).g'(t) = y'(-1/s^2t)x'(s^2t)y'(-1/s^2t) = g'(s^2t)$, or in other words

$$h'(s)g'(t)h'(s)^{-1} = g'(s^2t).$$

Multiplying this against the inverse of the case $t = 1$ gives $h'(s)h'(t)h'(s)^{-1} = g'(s^2t)g'(s^2)^{-1} = h'(s^2t)h'(s^2)^{-1}$. But D' is commutative, so we obtain $h'(t) = h'(s^2t)/h'(s^2)$ for any points $s, t \in \mathbf{G}_{\mathrm{m}}$. This establishes that h' is a homomorphism (visibly lifting h). By the definition of h', the identity

$h'(st) = h'(s)h'(t)$ says
$$g'(st)g'(1)^{-1} = h'(s)g'(t)g'(1)^{-1},$$
so $g'(st) = h'(s)g'(t)$. $\square$

By going back to the definition of g', the identity $h'(s)g'(t) = g'(st)$ says
$$h'(s)y'(-1/t)x'(t)y'(-1/t) = y'(-1/st)x'(st)y'(-1/st).$$
The D-equivariance of y' gives that $h'(s)y'(-1/t) = y'(-1/s^2t)h'(s)$, so
$$y'\left(-\frac{1}{s^2t}\right)h'(s)x'(t) = y'\left(-\frac{1}{st}\right)x'(st)y'\left(-\frac{1}{st}+\frac{1}{t}\right).$$
Multiplying by $y'(-1/st)^{-1}$ on the left, we arrive at the relation
$$y'\left(\frac{1}{s}\cdot\left(-\frac{1}{st}+\frac{1}{t}\right)\right)h'(s)x'(t) = x'(st)y'\left(-\frac{1}{st}+\frac{1}{t}\right)$$
in G' for any units s and t, or equivalently we have the following analogue of (4.2.1) (via the change of variables $u = st$ and $v = (s-1)/st$ making $1+uv = s$ and $u/(1+uv) = t$):
$$x'(u)y'(v) = y'\left(\frac{v}{1+uv}\right)h'(1+uv)x'\left(\frac{u}{1+uv}\right) \tag{4.3.1}$$
for $u, v \in \mathbf{G}_\mathrm{a}$ such that $1+uv$ and u are units.

We can establish (4.3.1) without the unit condition on u by working fppf-locally, as follows. For any points u, v of $\mathbf{G}_\mathrm{a}$ such that $1 + uv$ is a unit, fppf-locally we may write $u = u' + u''$ where $u', u'', 1+u''v \in \mathbf{G}_\mathrm{m}$ (as we see by treating separately the cases when v vanishes or does not vanish at a geometric point of interest), so by using that the commutation relation for h' against x' and y' coincides with that of h against x and y (due to the D-equivariance underlying the construction of x' and y') we get
$$\begin{aligned} x'(u)y'(v) &= x'(u')x'(u'')y'(v) \\ &\overset{(4.3.1)}{=} x'(u')y'\left(\frac{v}{1+u''v}\right)h'(1+u''v)x'\left(\frac{u''}{1+u''v}\right) \end{aligned}$$
when $1 + uv$ is a unit. Continuing to assume this unit condition, a further application of (4.3.1) transforms the 4-fold product into
$$y'\left(\frac{v}{1+uv}\right)h'\left(\frac{1+uv}{1+u''v}\right)x'\left(\frac{u'(1+u''v)}{1+uv}\right)h'(1+u''v)x'\left(\frac{u''}{1+u''v}\right).$$
Simplifying the three middle factors, this becomes
$$y'\left(\frac{v}{1+uv}\right)h'(1+uv)x'\left(\frac{u'}{(1+u''v)(1+uv)}\right)x'\left(\frac{u''}{1+u''v}\right),$$

which is just $y'(v/(1+uv))h'(1+uv)x'(u/(1+uv))$. Passing to inverses and negating u and v, we conclude that

$$y'(v)x'(u) = x'\left(\frac{u}{1+uv}\right)h'(1+uv)^{-1}y'\left(\frac{v}{1+uv}\right)$$

when $1+uv$ is a unit.

Recall that in SL_2, $x(u)y(v)$ lies in the open cell $\mathrm{U}^- \times \mathrm{D} \times \mathrm{U}^+$ precisely when $1+uv$ is a unit, and the preceding calculations show that x', h', y' satisfy the *same* commutation relations that govern the "S-birational" group law on the open cell $\Omega = \mathrm{U} \times \mathrm{D} \times \mathrm{V}$ of $\mathrm{G} = \mathrm{SL}_2$ (but beware that $\mathrm{U}' \times \mathrm{D}' \times \mathrm{V}' \to \mathrm{G}'$ is not an open immersion when $\mathrm{Z} \neq 1$, since Z lies in all three groups V', D', and U'). It follows that the S-morphism section $\sigma : \Omega \to \mathrm{G}'$ defined by $x(u)h(s)y(v) \mapsto x'(u)h'(s)y'(v)$ for $(u,s,v) \in \mathbf{G}_{\mathrm{a}} \times \mathbf{G}_{\mathrm{m}} \times \mathbf{G}_{\mathrm{a}}$ is "S-birationally multiplicative" in the sense that on the *fiberwise-dense* (hence relatively schematically dense [**EGA**, IV_3, 11.10.10]) open locus $m_{\mathrm{G}}^{-1}(\Omega) \bigcap (\Omega \times \Omega)$ in $\Omega \times \Omega$ consisting of points (ω_1, ω_2) whose product in $\mathrm{G} = \mathrm{SL}_2$ lies in Ω, we have $\sigma(\omega_1)\sigma(\omega_2) = \sigma(\omega_1\omega_2)$.

To show that σ extends (uniquely) to an S-homomorphism $\mathrm{G} \to \mathrm{G}'$ that is the desired splitting of the given central extension of G by Z, we use an alternative procedure that works when G' is just a group sheaf. Zariski-locally, every point of G either lies in the open cell Ω or its translate by the point $x(1) \in \mathrm{U}$ lies in Ω, so as a group sheaf G is generated by finite products of points of Ω. Hence, to construct the desired homomorphic section $\mathrm{G} \to \mathrm{G}'$ extending σ we just have to check that if $a_1, \dots, a_n$ and $b_1, \dots, b_m$ are points of Ω (valued in some S-scheme) such that $a_1 \cdots a_n = b_1 \cdots b_m$ in G then

$$\sigma(a_1) \cdots \sigma(a_n) \stackrel{?}{=} \sigma(b_1) \cdots \sigma(b_m). \tag{4.3.2}$$

To deduce this equality from the weaker "S-birational multiplicativity" already established for σ, observe that for such given a_i and b_j valued in some S-scheme S', fppf-locally on S' there exists a point ω of Ω such that $\omega a_1 \dots a_i$ and $\omega b_1 \cdots b_j$ lie in Ω for all $i, j \geqslant 0$. This holds because for any $x = (x_1, \dots, x_r) \in \mathrm{G}^r(\mathrm{S}')$ the map $\mathrm{X} := \bigcap_i \Omega_{\mathrm{S}'} x_i^{-1} \to \mathrm{S}'$ is fppf and hence admits sections fppf-locally on S'.

Letting $g = a_1 \cdots a_n = b_1 \cdots b_m$, we have

$$\omega g = (\omega a_1 \cdots a_{n-1})a_n, \;\; \omega g = (\omega b_1 \cdots b_{m-1})b_m.$$

Thus, the "S-birational multiplicativity" gives that

$$\sigma(\omega g) = \sigma(\omega a_1 \cdots a_{n-1})\sigma(a_n), \;\; \sigma(\omega g) = \sigma(\omega b_1 \cdots b_{m-1})\sigma(b_m).$$

Continuing in this way (using the conditions on ω), we get

$$\sigma(\omega a_1 \cdots a_{n-1}) = \sigma(\omega)\sigma(a_1) \cdots \sigma(a_{n-1})$$

and

$$\sigma(\omega b_1 \cdots b_{m-1}) = \sigma(\omega)\sigma(b_1)\cdots\sigma(b_{m-1}),$$

so (4.3.2) holds up to a harmless left multiplication by $\sigma(\omega)$ on both sides. $\square$

For later purposes, we now establish the SL_2-analogue of Proposition 4.2.7.

Proposition 4.3.4. — *Let* $\mathrm{G} \to \mathrm{S}$ *be a reductive group scheme whose geometric fibers have finite center of order* 2 *and semisimple-rank* 1*, and let* T *be a split maximal torus of* G*. Zariski-locally on* S *there is an* S*-group isomorphism* $\mathrm{G} \simeq \mathrm{SL}_2$*, and it can be chosen to identify* T *with the diagonal torus.*

Proof. — Since the center $\mathrm{Z_G}$ has fibers of order 2 and is of multiplicative type (see Theorem 3.3.4), its Cartier dual is finite étale of order 2. But $(\mathbf{Z}/2\mathbf{Z})_\mathrm{S}$ has no nontrivial étale S-forms since $(\mathbf{Z}/2\mathbf{Z})^\times = 1$. Hence, $\mathrm{Z_G} \simeq \mu_2$. Likewise, the geometric fibers $\mathrm{G}_{\overline{s}}$ must be SL_2 by the classical theory. The quotient $\mathrm{G}/\mathrm{Z_G}$ has trivial center (Corollary 3.3.5), and it has a split maximal torus $\mathrm{T}/\mathrm{Z_G}$, so by Proposition 4.2.7 there exists an isomorphism $\mathrm{G}/\mathrm{Z_G} \simeq \mathrm{PGL}_2$ Zariski-locally on S that moreover carries $\mathrm{T}/\mathrm{Z_G}$ over to the diagonal torus. In view of the bijective correspondence between maximal tori in G and $\mathrm{G}/\mathrm{Z_G}$, as well as in SL_2 and $\mathrm{SL}_2/\mu_2 = \mathrm{PGL}_2$ (apply Corollary 3.3.5), a lift of $\mathrm{G}/\mathrm{Z_G} \simeq \mathrm{PGL}_2$ to an isomorphism $\mathrm{G} \simeq \mathrm{SL}_2$ (if one exists, at least Zariski-locally on S) must carry T to the diagonal torus.

It remains to show that Zariski-locally on S, any smooth central extension G of PGL_2 by μ_2 over S with connected geometric fibers is isomorphic to SL_2 (as a central extension). Using central pushout along $\mu_2 \hookrightarrow \mathbf{G}_\mathrm{m}$ embeds G as a closed subgroup of a central extension G' of PGL_2 by $\mathbf{G}_\mathrm{m}$. Pulling back this latter extension by the μ_2-quotient map $\mathrm{SL}_2 \to \mathrm{PGL}_2$ yields a central extension

$$1 \to \mathbf{G}_\mathrm{m} \to \widetilde{\mathrm{G}}' \to \mathrm{SL}_2 \to 1$$

with $\widetilde{\mathrm{G}}'$ also a central μ_2-extension of G'. By Proposition 4.3.1 there exists a splitting $\widetilde{\mathrm{G}}' = \mathrm{SL}_2 \times \mathbf{G}_\mathrm{m}$, and Zariski-locally on S this can be arranged to carry any given maximal torus $\widetilde{\mathrm{T}}'$ of $\widetilde{\mathrm{G}}'$ over to $\mathrm{D} \times \mathbf{G}_\mathrm{m}$ (we just have to arrange that the corresponding isomorphism $\widetilde{\mathrm{G}}'/\mathrm{Z}_{\widetilde{\mathrm{G}}'} \simeq \mathrm{PGL}_2$ carries $\widetilde{\mathrm{T}}'/\mathrm{Z}_{\widetilde{\mathrm{G}}'}$ over to the diagonal torus, as can be done Zariski-locally on S by Proposition 4.2.7).

On geometric fibers over S, the central subgroup $\mu = \ker(\widetilde{\mathrm{G}}' \twoheadrightarrow \mathrm{G}') \simeq \mu_2$ in $\widetilde{\mathrm{G}}' = \mathrm{SL}_2 \times \mathbf{G}_\mathrm{m}$ must either be the center of the SL_2-factor or the diagonally embedded μ_2 in the product $\mathrm{SL}_2 \times \mathbf{G}_\mathrm{m}$. The first option cannot occur over any geometric point $\overline{s}$ of S: it would imply that $\mathrm{G}'_{\overline{s}}$ is a split extension $\mathrm{PGL}_2 \times \mathbf{G}_\mathrm{m}$, yet the smooth connected subgroup $\mathrm{G}_{\overline{s}} \subset \mathrm{G}'_{\overline{s}}$ is semisimple, so necessarily $\mathrm{G}_{\overline{s}} = \mathscr{D}(\mathrm{G}'_{\overline{s}}) = \mathrm{PGL}_2$, contradicting the hypothesis that the given central quotient map $\mathrm{G}_{\overline{s}} \to \mathrm{PGL}_2$ has a nontrivial kernel. Thus, the central $\mu \subset \mathrm{Z}_{\widetilde{\mathrm{G}}'} = \mu_2 \times \mu_2$ must be the diagonal S-subgroup since this holds on geometric fibers over S.

We conclude that G' is the pushout $\mathrm{SL}_2 \times^{\mu_2} \mathbf{G}_{\mathrm{m}} = \mathrm{GL}_2$ over S. Moreover, via the bijection between the sets of maximal tori in $\widetilde{\mathrm{G}}'$ and its central quotient $\mathrm{G}' = \widetilde{\mathrm{G}}'/\mu$ (Corollary 3.3.5), we see that Zariski-locally on S the isomorphism $\mathrm{G}' \simeq \mathrm{SL}_2 \times^{\mu_2} \mathbf{G}_{\mathrm{m}} = \mathrm{GL}_2$ can be arranged to carry any given maximal torus of G' over to $\mathrm{D} \times^{\mu_2} \mathbf{G}_{\mathrm{m}}$, which is the diagonal torus $\widetilde{\mathrm{D}}$ in GL_2. Apply this to the maximal torus $\mathrm{T}' := \mathrm{T} \times^{\mu_2} \mathbf{G}_{\mathrm{m}}$ inside $\mathrm{G} \times^{\mu_2} \mathbf{G}_{\mathrm{m}} = \mathrm{G}'$.

It suffices to prove that the subgroup $\mathrm{G} \subset \mathrm{G}'$ (which meets T' in T) coincides with the subgroup $\mathrm{SL}_2 \subset \mathrm{SL}_2 \times^{\mu_2} \mathbf{G}_{\mathrm{m}} = \mathrm{GL}_2$ (which meets $\mathrm{T} = \mathrm{D} \times^{\mu_2} \mathbf{G}_{\mathrm{m}} = \widetilde{\mathrm{D}}$ in D). Indeed, since the natural quotient map $\mathrm{G} \to \mathrm{G}/\mu_2 = \mathrm{G}'/\mathbf{G}_{\mathrm{m}} = \mathrm{SL}_2/\mu_2 = \mathrm{PGL}_2$ is the central quotient map $\mathrm{G} \to \mathrm{G}/\mathrm{Z}_{\mathrm{G}} \simeq \mathrm{PGL}_2$ arranged at the start, an equality $\mathrm{G} = \mathrm{SL}_2$ inside G' must respect the structures of both sides as central extensions of PGL_2 by μ_2, so we would be done.

To relate G and SL_2 inside G', we shall use root groups. More specifically, we may work Zariski-locally on S so as to acquire a pair of opposite roots $\pm a$ for (G, T) whose root spaces $\mathfrak{g}_{\pm a}$ are trivial line bundles over S. The torus T' is generated by T and the central $\mathbf{G}_{\mathrm{m}}$ in G', so the T-weight spaces on $\mathfrak{g}$ are also T'-weight spaces in $\mathfrak{g}'$. Letting $\pm a' : \mathrm{T}' \to \mathbf{G}_{\mathrm{m}}$ be the corresponding fiberwise nontrivial T'-weights, it is clear that $\mathrm{U}_{\pm a}$ equipped with $\exp_{\pm a}$ satisfies the properties uniquely characterizing $(\mathrm{U}_{\pm a'}, \mathfrak{g}'_{\pm a'})$. The identification

$$(\mathrm{G}', \mathrm{T}') = (\mathrm{GL}_2, \widetilde{\mathrm{D}})$$

must carry the root groups $\mathrm{U}_{\pm a'}$ for $(\mathrm{G}', \mathrm{T}')$ over to the standard root groups $\mathrm{U}^{\pm}$ on the right side.

We have shown that $\mathrm{U}_{\pm a'} = \mathrm{U}_{\pm a} \subset \mathrm{G}$ inside G', and obviously $\mathrm{U}^{\pm} \subset \mathrm{SL}_2$ inside $\mathrm{G}' = \mathrm{GL}_2$. For $x(u) = \left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{U}^+$ and $y(u) = \left(\begin{smallmatrix} 1 & 0 \\ u & 1 \end{smallmatrix}\right) \in \mathrm{U}^-$ we have

$$y(-1/t)x(t)y(-1/t)(y(-1)x(1)y(-1))^{-1} = \mathrm{diag}(t, 1/t)$$

for any $t \in \mathbf{G}_{\mathrm{m}}$, so the standard open cell $\Omega \subset \mathrm{SL}_2$ inside $\mathrm{G}' = \mathrm{GL}_2$ lies in the closed S-subgroup $\mathrm{G} \subset \mathrm{G}'$. By relative schematic density of Ω in SL_2 over S, it follows that $\mathrm{SL}_2 \subset \mathrm{G}$ as closed subgroups of G'. But G and SL_2 are each S-smooth with connected geometric fibers of dimension 3, so the inclusion $\mathrm{SL}_2 \subset \mathrm{G}$ is an equality on geometric fibers over S and hence is an equality as closed subgroups of G'. $\square$

4.4. Exercises. —

Exercise 4.4.1. — Let G be a reductive group scheme over a scheme S and let T be a split maximal torus of G, with a fixed isomorphism $\mathrm{T} \simeq \mathrm{D_S(M)}$ for a finite free **Z**-module M. For each $s \in \mathrm{S}$ and root $\alpha \in \Phi(\mathrm{G}_{\overline{s}}, \mathrm{T}_{\overline{s}})$ construct a Zariski-open neighborhood U of s in S and a root a of $(\mathrm{G_U}, \mathrm{T_U})$ in the sense of Definition 4.1.1 such that $a(\overline{s}) = \alpha$. Prove moreover that any two such a (for the same α) coincide on a Zariski-open neighborhood of s in S.

Exercise 4.4.2. — Let G be a finitely presented affine group over a ring k, and choose a 1-parameter subgroup $\lambda : \mathbf{G}_{\mathrm{m}} \to \mathrm{G}$ over k.

(i) Prove that $\mathrm{P_G}(\lambda^n) = \mathrm{P_G}(\lambda)$ for any $n > 0$, and likewise for $\mathrm{U_G}(\lambda^n)$ and $\mathrm{Z_G}(\lambda^n)$.

(ii) Suppose k/k_0 is a finite flat extension of noetherian rings, G_0 is the Weil restriction $\mathrm{R}_{k/k_0}(\mathrm{G})$ (an affine k_0-group of finite type), and $\lambda_0 : \mathbf{G}_{\mathrm{m}} \to \mathrm{G}_0$ is the k_0-morphism corresponding to the k-homomorphism $\lambda : \mathbf{G}_{\mathrm{m}} \to \mathrm{G}$ via the universal property of R_{k/k_0}. Prove that λ_0 is a k_0-homomorphism and $\mathrm{P}_{\mathrm{G}_0}(\lambda_0) = \mathrm{R}_{k/k_0}(\mathrm{P_G}(\lambda))$, and similarly for $\mathrm{U}_{\mathrm{G}_0}(\lambda_0)$ and $\mathrm{Z}_{\mathrm{G}_0}(\lambda_0)$.

Exercise 4.4.3. — Prove the following fact that was used in the proof of Proposition 4.3.1: for any central extension $1 \to \mathrm{Z} \to \mathrm{H}' \to \mathrm{H} \to 1$ of one commutative group sheaf by another (on any site), the commutator of H' factors through a bi-additive pairing

$$\mathrm{H} \times \mathrm{H} \to \mathrm{Z}$$

and the vanishing of this pairing is equivalent to the commutativity of H'. (This generalizes part of Exercise 3.4.7(iii).)

Exercise 4.4.4. — Let A be a finite-dimensional associative algebra over a field k, and $\underline{\mathrm{A}}^{\times}$ the associated k-group of units as in Exercise 3.4.5. Prove $\mathrm{Tan}_e(\underline{\mathrm{A}}^{\times}) = \mathrm{A}$ naturally, and that the Lie algebra structure is $[a, a'] = aa' - a'a$. Using $\mathrm{A} = \mathrm{End(V)}$, recover $\mathfrak{gl}(\mathrm{V})$ without coordinates. Use this to compute the Lie algebras $\mathfrak{sl}(\mathrm{V})$, $\mathfrak{pgl}(\mathrm{V})$, $\mathfrak{sp}(\psi)$ (for a symplectic form ψ), $\mathfrak{gsp}(\psi)$, and $\mathfrak{so}(q)$ without coordinates.

Exercise 4.4.5. — Let K be a degree-2 finite étale algebra over a field k (i.e., a separable quadratic field extension or $k \times k$, the latter called the *split* case), and let σ be the unique nontrivial k-automorphism of K; note that $\mathrm{K}^{\sigma} = k$. A *σ-hermitian space* is a pair (V, h) consisting of a finite free K-module equipped with a perfect σ-semilinear form $h : \mathrm{V} \times \mathrm{V} \to \mathrm{K}$ (i.e., $h(cv, v') = ch(v, v')$, $h(v, cv') = \sigma(c)h(v, v')$, and $h(v', v) = \sigma(h(v, v'))$). Note that $v \mapsto h(v, v)$ is a quadratic form $q_h : \mathrm{V} \to k$ over k satisfying $q_h(cv) = \mathrm{N}_{\mathrm{K}/k}(c)q_h(v)$ for $c \in \mathrm{K}$ and $v \in \mathrm{V}$, and $\dim_k \mathrm{V}$ is even ($\mathrm{char}(k) = 2$ is allowed!). (One similarly defines the notion of a *σ-anti-hermitian space* by requiring $h(v', v) = -\sigma(h(v, v'))$.)

The *unitary group* $\mathrm{U}(h)$ over k is the subgroup of $\mathrm{R}_{\mathrm{K}/k}(\mathrm{GL}(\mathrm{V}))$ preserving h. Using $\mathrm{R}_{\mathrm{K}/k}(\mathrm{SL}(\mathrm{V}))$ gives the *special unitary group* $\mathrm{SU}(h)$. Example: $\mathrm{V} = \mathrm{F}$ finite étale over K with an involution σ' lifting σ, and $h(v, v') := \mathrm{Tr}_{\mathrm{F}/\mathrm{K}}(v\sigma'(v'))$; e.g., CM fields F and K, totally real k, and complex conjugations σ' and σ.

(i) If $\mathrm{K} = k \times k$, prove $\mathrm{V} \simeq \mathrm{V}_0 \times \mathrm{V}_0^*$ with $h((v, \ell), (v', \ell')) = (\ell'(v), \ell(v'))$ for a k-vector space V_0. Identify $\mathrm{U}(h)$ with $\mathrm{GL}(\mathrm{V}_0)$ carrying $\mathrm{SU}(h)$ to $\mathrm{SL}(\mathrm{V}_0)$. Compute q_h and prove that q_h is non-degenerate.

(ii) In the non-split case prove that $\mathrm{U}(h)_{\mathrm{K}} \simeq \mathrm{GL}_n$ carrying $\mathrm{SU}(h)$ to SL_n ($n = \dim_{\mathrm{K}} \mathrm{V}$). Prove $\mathrm{U}(h)$ is smooth and connected with derived group $\mathrm{SU}(h)$ and center $\mathbf{G}_{\mathrm{m}}$, and that q_h is non-degenerate. Compute $\mathfrak{su}(h)$.

(iii) Identify $\mathrm{U}(h)$ with a k-subgroup of $\mathrm{SO}(q_h)$. Discuss the split case, and the case $k = \mathbf{R}$.

Exercise 4.4.6. — Consider a k-torus $\mathrm{T} \subset \mathrm{GL}(\mathrm{V})$ containing $\mathrm{Z}_{\mathrm{GL}(\mathrm{V})} = \mathbf{G}_m$, with k infinite. Let $\mathrm{A}_{\mathrm{T}} \subset \mathrm{End}(\mathrm{V})$ be the commutative k-subalgebra generated by $\mathrm{T}(k)$.

(i) When $k = k_s$, prove A_{T} is a product of copies of k and that the inclusion $\mathrm{T}(k) \hookrightarrow \mathrm{A}_{\mathrm{T}}^{\times}$ is an equality.

(ii) Using Galois descent and the end of Exercise 2.4.9(i), prove $(\mathrm{A}_{\mathrm{T}})_{k_s} = \mathrm{A}_{\mathrm{T}_{k_s}}$, and deduce that $\mathrm{T}(k) = \mathrm{A}_{\mathrm{T}}^{\times}$. Construct a natural isomorphism $\mathrm{T} \simeq \mathrm{R}_{\mathrm{A}/k}(\mathbf{G}_{\mathrm{m}})$, and a bijection between the set of k-subtori in $\mathrm{GL}(\mathrm{V})$ containing $\mathrm{Z}_{\mathrm{GL}(\mathrm{V})}$ and the set of étale finite commutative k-subalgebras of $\mathrm{End}(\mathrm{V})$. Generalize to *finite* k, using Galois descent to reduce to the case just handled.

Exercise 4.4.7. — Let (V, q) be a non-degenerate quadratic space over a field k with $\dim \mathrm{V} \geqslant 2$.

(i) If $q(v) = 0$ for some $v \in \mathrm{V} - \{0\}$, prove that v lies in a hyperbolic plane H with $\mathrm{H} \bigoplus \mathrm{H}^{\perp} = \mathrm{V}$. (If $\mathrm{char}(k) = 2$ and $\dim \mathrm{V}$ is even, work over $\overline{k}$ to show $v \notin \mathrm{V}^{\perp}$.) Use this to construct a $\mathbf{G}_{\mathrm{m}}$ inside $\mathrm{SO}(q)$ over k.

(ii) If $\mathrm{SO}(q)$ contains a k-subgroup $\mathrm{S} \simeq \mathbf{G}_{\mathrm{m}}$, prove conversely that $q(v) = 0$ for some $v \in \mathrm{V} - \{0\}$. (Hint: prove that $\mathrm{V}^{\mathrm{S}} \neq \mathrm{V}$ and compute $q(tv)$ in two ways for $t \in \mathrm{S}$ and a nonzero v in a weight space for a nontrivial S-weight.)

Exercise 4.4.8. — Let G be a connected semisimple group over an algebraically closed field k, and let T be a maximal torus and B a Borel subgroup of G containing T. Let Δ be the set of simple positive roots relative to the positive system of roots $\Phi^+ = \Phi(\mathrm{B}, \mathrm{T})$.

(i) Using Corollary 3.3.6, prove that $\mathrm{Z}_{\mathrm{G}} = 1$ (equivalently, Ad_{G} is a closed immersion) if and only if Δ is a basis of $\mathrm{X}(\mathrm{T})$. (Do *not* assume the Existence and Isomorphism Theorems, as was done in Exercise 1.6.13(ii).)

(ii) Assume G is adjoint, and let $\{\omega_i^{\vee}\}$ denote the basis of $\mathrm{X}_*(\mathrm{T})$ dual to the basis $\{a_i\} = \Delta$ of $\mathrm{X}(\mathrm{T})$. For each subset $\mathrm{I} \subset \Delta$, let $\lambda_{\mathrm{I}} \in \mathrm{X}_*(\mathrm{T})$

be the cocharacter $\sum_{a_i \in \mathrm{I}} \omega_i^\vee$. Prove that the parabolic subgroup $\mathrm{P_G}(\lambda_\mathrm{I})$ coincides with the "standard" parabolic subgroup $\mathrm{P}_{\Delta - \mathrm{I}}$ containing B that arose in the proof of Proposition 1.4.7 (so $\mathrm{B} = \mathrm{P}_\varnothing = \mathrm{P_G}(\lambda_\Delta)$). This gives a "dynamic" description of the parabolic subgroups of G containing B. (Hint: By Proposition 1.4.7, it suffices to compare Lie algebras inside $\mathfrak{g}$.)

(iii) Prove that $\rho^\vee := \lambda_\Delta$ coincides with $(1/2)\sum_{a \in \Phi^+} a^\vee$. Equivalently (by consideration of the dual root datum and Exercise 1.6.17), for each $a_i \in \Delta$ prove $\langle \sum_{a \in \Phi^+} a, a_i^\vee \rangle = 2$. (Hint: Show that s_{a_i} permutes $\Phi^+ - \{a_i\}$ by using that the reflection $s_{a_i} : v \mapsto v - \langle v, a_i^\vee \rangle a_i$ preserves $\Phi = \Phi^+ \coprod -\Phi^+$. Apply the resulting "change of variables" $a \mapsto s_{a_i}(a)$ to show that $\langle \sum_{a \in \Phi^+ - \{a_i\}} a, a_i^\vee \rangle$ vanishes.)

5. Split reductive groups and parabolic subgroups

5.1. Split groups and the open cell. — In the theory of connected reductive groups G over a field k (not assumed to be algebraically closed), one says that G is *split* if it admits a maximal k-torus $\mathrm{T} \subset \mathrm{G}$ that is k-split. (Keep in mind that for us, "maximal" means "geometrically maximal", as in Definition 3.2.1. The equivalence with other possible notions of maximality over a field, which we never use, rests on Remark A.1.2 and Grothendieck's existence theorem for geometrically maximal tori over any field.) For such (G, T), the weight spaces $\mathfrak{g}_a$ in $\mathfrak{g}$ for the nontrivial weights a of T that occur on $\mathfrak{g}$ are all 1-dimensional (as may be inferred from the theory over $\overline{k}$). More specifically, each $\mathfrak{g}_a$ is free of rank 1 over k since k is a field. In the relative theory, such module-freeness for the root spaces $\mathfrak{g}_a$ must be imposed as a condition (following [**SGA3**, XXII, 1.13]):

Definition 5.1.1. — Let G be a reductive group over a non-empty scheme S. It is *split* if there exists a maximal torus T equipped with an isomorphism $\mathrm{T} \simeq \mathrm{D}_{\mathrm{S}}(\mathrm{M})$ for a finite free $\mathbf{Z}$-module M such that:

1. the nontrivial weights $a : \mathrm{T} \to \mathbf{G}_{\mathrm{m}}$ that occur on $\mathfrak{g}$ arise from elements of M (so in particular, such a are roots for (G, T) and are "constant sections" of M_{S}),
2. each root space $\mathfrak{g}_a$ is free of rank 1 over $\mathscr{O}_{\mathrm{S}}$,
3. each coroot $a^\vee : \mathbf{G}_{\mathrm{m}} \to \mathrm{T}$ arises from an element of the dual lattice $\mathrm{M}^\vee$ (i.e., $a^\vee$ as a global section of $\mathrm{M}_{\mathrm{S}}^\vee$ over S is a constant section).

The definition of $a^\vee$ is given by applying Theorem 4.2.6 to the reductive subgroup $\mathrm{Z}_{\mathrm{G}}(\mathrm{T}_a)$. Note that although conditions (1) and (3) are automatic when S is connected (as the global sections of M_{S} in general are the locally constant M-valued functions on S, and similarly for $\mathrm{M}_{\mathrm{S}}^\vee$), we do *not* assume S is connected. The reason is that when developing the theory of split reductive group schemes we want to work locally on S and use descent theory in some proofs, but localization on the base and (especially) descent theory often lead to disconnected base schemes. For this reason, we avoid the notation "$\Phi(\mathrm{G}, \mathrm{T})$" except when S is connected (e.g., $\mathrm{S} = \operatorname{Spec} k$ for a field or domain k).

Example 5.1.2. — Let $\mathrm{S} = \operatorname{Spec}(k_1 \times k_2) = \operatorname{Spec} k_1 \coprod \operatorname{Spec} k_2$ for fields k_1 and k_2, so an S-group G is precisely $\mathrm{G} = \mathrm{G}_1 \coprod \mathrm{G}_2$ where G_i is a k_i-group. In the case $\mathrm{G}_1 = \mathrm{PGL}_2 \times \mathbf{G}_{\mathrm{m}}$ and $\mathrm{G}_2 = \mathrm{GL}_2$ (with their split diagonal tori identified via $(\operatorname{diag}(t, 1), t') \mapsto \operatorname{diag}(tt', t')$), conditions (1) and (2) in Definition 5.1.1 are satisfied but condition (3) fails.

Lemma 5.1.3. — *Any reductive group scheme over a non-empty scheme becomes split étale-locally on the base.*

A basic example of this lemma is that a connected reductive group over a field k becomes split over a finite separable extension of k.

Proof. — There exists a maximal torus T étale-locally on the base (Corollary 3.2.7), and further étale localization provides an isomorphism $\mathrm{T} \simeq \mathrm{D_S(M)}$. Working Zariski-locally then makes the weight space decomposition for $\mathfrak{g}$ under the T-action have all nontrivial T-weights a arise from M, with $\mathfrak{g}_a$ having constant rank 1. A final Zariski-localization (which is necessary, by Example 5.1.2) makes each coroot arise from $\mathrm{M}^\vee$. □

There is a case when the split property is automatic in the presence of a maximal torus:

Example 5.1.4. — Let S be a (non-empty) connected normal noetherian scheme with trivial Picard group and trivial étale fundamental group. For example, S may be $\mathrm{Spec}\,\mathbf{Z}$ or $\mathbf{A}^n_k$ for an algebraically closed field k of characteristic 0. We claim that every reductive group scheme G over S admitting a maximal torus T over S is automatically split. Over $\mathbf{Z}$ this corresponds to the fact that the action of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ on $\mathrm{X}(\mathrm{T}_{\overline{\mathbf{Q}}})$ is unramified at all primes and hence is trivial (Minkowski). The general case proceeds along similar lines, as follows.

By Corollary B.3.6 and the hypotheses on S, tori over S correspond to continuous representations $\pi_1(\mathrm{S}, \overline{\eta})$ on discrete $\mathbf{Z}$-lattices. But $\pi_1(\mathrm{S}, \overline{\eta}) = 1$ by hypothesis, so T is split. Hence, we may choose an isomorphism $\mathrm{T} \simeq \mathrm{D_S(M)}$. The global sections of $\mathrm{M_S}$ are the elements of M since S is connected, and likewise the global sections of $\mathrm{M}_\mathrm{S}^\vee$ are the elements of $\mathrm{M}^\vee$. Finally, each $\mathfrak{g}_a$ for $a \neq 0$ has constant rank (0 or 1) since S is connected, and when nontrivial it is free of rank 1 since $\mathrm{Pic(S)} = 1$.

Remark 5.1.5. — The split semisimple groups over $\mathbf{Z}$ are called (semisimple) *Chevalley groups.* By Example 5.1.4, among all semisimple $\mathbf{Z}$-groups these are precisely the ones that admit a maximal $\mathbf{Z}$-torus (in the sense of Definition 3.2.1). A connected semisimple $\mathbf{Q}$-group admits a semisimple $\mathbf{Z}$-model if and only if it is split over $\mathbf{Q}_p$ for all p. (The implication "⇒" is a consequence of (i) the existence of a split $\mathbf{Z}$-form, (ii) the triviality of $\pi_1(\mathrm{Spec}\,\mathbf{Z})$, and (iii) the structure of the automorphism scheme of a semisimple group over a ring; see [**Con14**, Prop. 3.9ff.]. For the converse, we can first spread out to a semisimple group over some $\mathbf{Z}[1/\mathrm{N}]$, and then we can "glue" with split models over $\mathbf{Z}_p$ for each $p|\mathrm{N}$ to make a semisimple $\mathbf{Z}$-model; see the proof of [**Con14**, Lemma 4.3] and references therein for further discussion of this gluing process over a Dedekind base.) There are semisimple $\mathbf{Z}$-groups that are not split, or equivalently do not admit a maximal torus, such as special orthogonal groups of even unimodular lattices; e.g., the E_8 and Leech lattices.

The set of elements of M that occur in condition (1) of Definition 5.1.1 is denoted Φ (with the choice of isomorphism $\mathrm{T} \simeq \mathrm{D_S(M)}$ understood from context), and the set of corresponding coroots in $\mathrm{M}^\vee$ is denoted $\Phi^\vee$. We have

$$\mathfrak{g} = \mathfrak{t} \bigoplus (\bigoplus_{a \in \Phi} \mathfrak{g}_a).$$

The subsets $\Phi \subset \mathrm{M} - \{0\}$ and $\Phi^\vee \subset \mathrm{M}^\vee - \{0\}$ inherit combinatorial properties from the classical theory on geometric fibers [**SGA3**, XXII, 1.14, 3.4]:

Proposition 5.1.6. — *The* 4*-tuple* $(\mathrm{M}, \Phi, \mathrm{M}^\vee, \Phi^\vee)$ *is a reduced root datum. The Weyl group* $\mathrm{W_G(T)} = \mathrm{N_G(T)/T} \subset \underline{\mathrm{Aut}}(\mathrm{M}^\vee_\mathrm{S}) = \mathrm{Aut}(\mathrm{M}^\vee)_\mathrm{S}$ *is the constant subgroup* $\mathrm{W}(\Phi)_\mathrm{S}$.

Proof. — The required combinatorial conditions to be a root datum can be checked on a single geometric fiber (recall that $\mathrm{S} \neq \varnothing$), where it follows from the classical theory. Likewise, since $\mathrm{W_G(T)}$ is a finite étale S-subgroup of the (opposite group of the) automorphism scheme of $\mathrm{M_S}$ (as $\mathrm{W_G(T)}$ acts faithfully on $\mathrm{T} = \mathrm{D_S(M)}$), to compare it with the constant subgroup arising from $\mathrm{W}(\Phi)$ we may again pass to geometric fibers and appeal to the classical theory. $\square$

We sometimes call (G, T, M) a "split" group (or *split triple*), with the isomorphism $\mathrm{T} \simeq \mathrm{D_S(M)}$ and subset $\Phi \subset \mathrm{M}$ understood to be specified. This really comes in three parts: the pair (G, T), the root datum $(\mathrm{M}, \Phi, \mathrm{M}^\vee, \Phi^\vee)$, and the isomorphism $\mathrm{T} \simeq \mathrm{D_S(M)}$ that carries Φ over to roots for (G, T). Keep in mind that the axioms for a root datum uniquely determine the bijection $a \mapsto a^\vee$ between roots and coroots (Remark 1.3.4).

Example 5.1.7. — Consider a split triple (G, T, M). The center $\mathrm{Z_G}$ is $\mathrm{D_S(M/Q)}$, where $\mathrm{Q} \subset \mathrm{M}$ is the **Z**-span of the roots (the *root lattice*). Indeed, by Corollary 3.3.6, $\mathrm{Z_G}$ is the kernel of the adjoint action of $\mathrm{T} = \mathrm{D_S(M)}$ on $\mathfrak{g}$, and by definition the nontrivial weights for this action are the elements of $\Phi \subset \mathrm{M}$ viewed as characters on T. Hence, $\mathrm{D_S(M/Q)} \subset \mathrm{Z_G}$, and to prove equality we may pass to geometric fibers, where it is clear (since all multiplicative type groups over an algebraically closed field are split).

Since semisimplicity is equivalent to finiteness of the center, it follows that G is semisimple if and only if the elements of Φ span $\mathrm{M}_\mathbf{Q}$ over **Q**. Now suppose that G is semisimple. In such cases G has trivial center (i.e., it is *adjoint*) precisely when Φ spans M over **Z**. Since M lies inside the *weight lattice* P in $\mathrm{M}_\mathbf{Q}$ that is (by definition) dual to the coroot lattice (i.e., the **Z**-span of $\Phi^\vee$) in $\mathrm{M}^\vee_\mathbf{Q}$, the center $\mathrm{Z_G} = \mathrm{D_S(M/Q)}$ is a quotient of $\mathrm{D_S(P/Q)}$. In particular, if $\mathrm{M} = \mathrm{P}$ (i.e., if M is as big as possible) then the geometric fibers of G admit no nontrivial central isogenous cover: if $\mathrm{M} = \mathrm{P}$ then we say G is *simply connected*. The Existence Theorem implies that a split semisimple group scheme is simply connected if and only if it has no nontrivial central extension by a finite group scheme of multiplicative type; see Exercise 6.5.2.

To make the constancy of $\mathrm{W_G(T)}$ over S in Proposition 5.1.6 more concrete, note that for each root a there exists a natural map $\mathrm{W}_{\mathrm{Z_G(T}_a)}(\mathrm{T}) \to \mathrm{W_G(T)}$ that on geometric fibers computes the order-2 subgroup generated by the involution $s_a : t \mapsto t/a^\vee(a(t))$ of $\mathrm{T} = \mathrm{D_S(M)}$ (dual to the involution $m \mapsto m - a^\vee(m)a$ of M). Since endomorphisms of multiplicative type S-groups are uniquely determined by their effect on geometric fibers over S, we conclude that each subgroup $\mathrm{W}_{\mathrm{Z_G(T}_a)}(\mathrm{T})$ is identified with $(\mathbf{Z}/2\mathbf{Z})_\mathrm{S}$ having the unique everywhere-nontrivial section correspond to $m \mapsto m - a^\vee(m)a$. Hence, to construct elements $n_a \in \mathrm{N_G(T)(S)}$ representing the reflections s_a in the Weyl group of the root datum (as in the classical theory) the problem is reduced to the case of split reductive groups with semisimple-rank 1. In such cases we wish to show that $\mathrm{N_G(T)(S)} \to \mathrm{W_G(T)(S)}$ is surjective by exhibiting an explicit element $n_a \in \mathrm{N_G(T)(S)}$ representing s_a. This will be deduced (in Corollary 5.1.11) from the following classification of split semisimple-rank 1 groups Zariski-locally on the base.

Theorem 5.1.8. — *Let* $(\mathrm{G, T, M})$ *be a split reductive group with geometric fibers of semisimple-rank* 1 *over a non-empty scheme* S. *Up to forming a direct product against a split central torus, Zariski-locally on* S *the pair* $(\mathrm{G, T})$ *is isomorphic to exactly one of the following:*

— $(\mathrm{SL}_2, \mathrm{D})$ *with* D *the diagonal torus,*
— $(\mathrm{PGL}_2, \overline{\mathrm{D}})$ *with* $\overline{\mathrm{D}}$ *the diagonal torus,*
— $(\mathrm{GL}_2, \widetilde{\mathrm{D}})$ *with* $\widetilde{\mathrm{D}}$ *the diagonal torus.*

We will later refine this result by constructing a unique such isomorphism globally, subject to some additional conditions that can always be imposed in the split case (such as compatibility with linked trivializations of the root spaces $\mathfrak{g}_{\pm a}$). This provides an explicit isomorphism $\mathrm{W_G(T)(S)} \simeq \mathrm{W}(\Phi)$ as in [**SGA3**, XXII, 3.4].

Proof. — The three proposed cases are fiberwise non-isomorphic, so there are no repetitions in the list. The roots $\pm a$ provide a central torus T_a of relative codimension 1 in T, and the classical theory on geometric fibers implies that the center $\mathrm{Z_G/T}_a$ of $\mathrm{G/T}_a$ is finite. That is, each geometric fiber of $\mathrm{G/T}_a$ is either SL_2 or PGL_2. Note that T_a is the split torus corresponding to the quotient of M by the saturation of $\mathbf{Z}a \subset \mathrm{M}$. Since the root datum determines the structure of the center $\mathrm{Z_G} \subset \mathrm{T}$ on geometric fibers, it follows that $\mathrm{Z_G/T}_a$ must have constant fiber degree, either 1 or 2. We will treat the two possibilities separately.

First suppose that G/T_a has center of order 2. By Proposition 4.3.4, working Zariski-locally on S provides an isomorphism $G/T_a \simeq SL_2$ carrying T/T_a over to D. Thus, G is a central extension of SL_2 by the split torus T_a. Applying Proposition 4.3.1, there exists a unique splitting $G = SL_2 \times T_a$, and clearly T must then go over to $D \times T_a$.

Next, suppose that G/T_a has trivial center. By Proposition 4.2.7, working Zariski-locally on S provides an isomorphism $G/T_a \simeq PGL_2$ carrying T/T_a over to $\overline{D}$. Pulling back along the central isogeny $q : SL_2 \to PGL_2$ yields a central extension $\widetilde{G}$ of SL_2 by T_a that is also a central extension of G by μ_2:

$$\begin{tikzcd} 1 \arrow[r] & T_a \arrow[r] \arrow[d, equal] & \widetilde{G} \arrow[r] \arrow[d] & SL_2 \arrow[r] \arrow[d, "q"] & 1 \\ 1 \arrow[r] & T_a \arrow[r] & G \arrow[r] & PGL_2 \arrow[r] & 1 \end{tikzcd}$$

The top row forces $\widetilde{G}$ to be a reductive group scheme, and G is a central quotient of $\widetilde{G}$ by μ_2, so by Corollary 3.3.5 there is a unique maximal torus $\widetilde{T}$ of $\widetilde{G}$ satisfying $\widetilde{T}/\mu_2 = T$ inside $\widetilde{G}/\mu_2 = G$.

By Proposition 4.3.1, $\widetilde{G} = SL_2 \times T_a$ with $\widetilde{T}$ going over to $D \times T_a$. The central subgroup $\mu_2 \subset \widetilde{G} = SL_2 \times T_a$ has two possibilities on fibers: it is μ_2 in the SL_2-factor or it is a diagonally embedded μ_2 in $SL_2 \times T_a$ via some inclusion $\mu_2 \hookrightarrow T_a$. These cases (on fibers) are distinguished by whether or not the projection to T_a kills this central subgroup.

Since a homomorphism between multiplicative type S-groups is determined over a Zariski-open neighborhood of a point $s \in S$ by its effect on $\overline{s}$-fibers, we conclude that Zariski-locally on S either (i) $G = PGL_2 \times T_a$ with $T = \overline{D} \times T_a$, or (ii) $G = SL_2 \times^{\mu_2} T_a$ with $T = D \times^{\mu_2} T_a$ for some inclusion $\mu_2 \hookrightarrow T_a$. Case (i) corresponds to $Z_G = T_a$ being a torus, and the second case corresponds to $Z_G \simeq \mu_2 \times T_a$ not being a torus since the structure of Z_G is determined across all fibers by the root datum (so its fibral isomorphism class is "constant"). Thus, it remains to address the situation when (G, T) falls into case (ii) Zariski-locally on S. In such cases the inclusion $\mu_2 \hookrightarrow T_a$ corresponds Zariski-locally on S to an index-2 subgroup of the constant group dual to T_a, so we can Zariski-locally split off this μ_2 inside a $\mathbf{G}_m$-factor of T_a. This provides a description of G (Zariski-locally on S) as the direct product of a split torus against $SL_2 \times^{\mu_2} \mathbf{G}_m = GL_2$ equipped with the maximal torus $D \times^{\mu_2} \mathbf{G}_m = \widetilde{D}$. □

Corollary 5.1.9. — *Let* G *be a reductive group over a non-empty scheme* S, *and* T *a maximal torus for which there exists a root* a (*and hence a root* $-a$). *Let* $\mathbf{W}(\mathfrak{g}_a)^\times$ *denote the open complement of the identity section in* $\mathbf{W}(\mathfrak{g}_a)$. *For every section* X *of* $\mathbf{W}(\mathfrak{g}_a)^\times$, *let* X^{-1} *denote the dual section of* $\mathbf{W}(\mathfrak{g}_{-a})^\times$.

Define $w_a : \mathbf{W}(\mathfrak{g}_a)^\times \to \mathrm{G}$ *by*

$$w_a(\mathrm{X}) := \exp_a(\mathrm{X}) \exp_{-a}(-\mathrm{X}^{-1}) \exp_a(\mathrm{X}).$$

1. *The values of* w_a *lie in* $\mathrm{N}_{\mathrm{Z_G(T}_a)}(\mathrm{T})$ *and represent the unique everywhere nontrivial section of* $\mathrm{W}_{\mathrm{Z_G(T}_a)}(\mathrm{T}) = (\mathbf{Z}/2\mathbf{Z})_\mathrm{S}$*; i.e.,*

$$t \mapsto w_a(\mathrm{X}) t w_a(\mathrm{X})^{-1}$$

is the reflection $t \mapsto t/a^\vee(a(t))$ *in* $\mathrm{W}(\Phi(\mathrm{G}_{\overline{s}}, \mathrm{T}_{\overline{s}}))$ *associated to* $a_{\overline{s}}$ *for all geometric points* $\overline{s}$ *of* S.

2. *For any unit* c *on* S *and sections* X, X' *of* $\mathbf{W}(\mathfrak{g}_a)^\times$,

$$w_a(c\mathrm{X}) = a^\vee(c) w_a(\mathrm{X}) = w_a(\mathrm{X}) a^\vee(c)^{-1}, \quad w_a(\mathrm{X}) w_a(\mathrm{X}') = a^\vee(-\mathrm{X}\mathrm{X}'^{-1}).$$

3. *Conjugation by* $w_a(\mathrm{X})$ *on* $\mathrm{U}_a \subset \mathrm{G}$ *is valued in* U_{-a} *and given by*

$$w_a(\mathrm{X}) \exp_a(\mathrm{X}') w_a(\mathrm{X})^{-1} = \exp_{-a}(-(\mathrm{X}^{-1}\mathrm{X}')\mathrm{X}^{-1}).$$

In particular, $w_a(\mathrm{X}) \exp_a(\mathrm{X}) w_a(\mathrm{X})^{-1} = \exp_{-a}(-\mathrm{X}^{-1})$ *and the adjoint action of* $w_a(\mathrm{X})$ *on* $\mathfrak{g}$ *satisfies* $\mathrm{Ad_G}(w_a(\mathrm{X}))(\mathrm{X}') = -(\mathrm{X}^{-1}\mathrm{X}')\mathrm{X}^{-1}$.

4. *For any* X *we have* $w_{-a}(\mathrm{X}^{-1}) = w_a(\mathrm{X})^{-1} = w_a(-\mathrm{X})$, $w_a(\mathrm{X}) w_{-a}(\mathrm{Y}) = a^\vee(\mathrm{XY})$, *and* $w_a(\mathrm{X})^2 = a^\vee(-1) \in \mathrm{T}$.

Proof. — Since T_a centralizes $\mathrm{U}_{\pm a}$, it is clear that w_a takes its values in $\mathrm{Z_G}(\mathrm{T}_a)$. We may therefore replace G with its reductive closed subgroup $\mathrm{Z_G}(\mathrm{T}_a)$ (that contains T and $\mathrm{U}_{\pm a}$) to reduce to the case that G has all geometric fibers of semisimple-rank 1. The asserted identities are all fppf-local on the base, so by working étale-locally (or fppf-locally) we can assume that T is split. Thus, by Theorem 5.1.8 we get an explicit description of (G, T) up to forming a direct product against a split torus. Such an additional central torus factor has no effect on the root groups or the proposed relations, so we are reduced to the three special cases in Theorem 5.1.8. The third case in Theorem 5.1.8 reduces to the first case because in the central pushout $\mathrm{GL}_2 = \mathrm{SL}_2 \times^{\mu_2} \mathbf{G}_\mathrm{m}$ the subtorus $\mathbf{G}_\mathrm{m}$ is central and SL_2 contains the root groups for D.

Summarizing, we are reduced to checking the special cases SL_2 and PGL_2 equipped with their diagonal maximal torus. By composing with the conjugation by the standard Weyl element if necessary (which induces inversion on the diagonal torus), we may arrange that a is the root whose root group U_a is the strictly upper triangular subgroup. For the SL_2-case, Example 4.2.1 makes everything explicit. To be precise, in this case

$$w_a(\mathrm{X}) = \begin{pmatrix} 1 & \mathrm{X} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\mathrm{X}^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & \mathrm{X} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{X} \\ -\mathrm{X}^{-1} & 0 \end{pmatrix},$$

so for $t = \mathrm{diag}(c, 1/c)$ in part (1) we have $w_a(\mathrm{X})tw_a(\mathrm{X})^{-1} = \mathrm{diag}(1/c, c) = t^{-1}$ and the two formulas in part (2) simply assert

$$\begin{pmatrix} 0 & c\mathrm{X} \\ -c^{-1}\mathrm{X}^{-1} & 0 \end{pmatrix} = \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} \begin{pmatrix} 0 & \mathrm{X} \\ -\mathrm{X}^{-1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{X} \\ -\mathrm{X}^{-1} & 0 \end{pmatrix} \begin{pmatrix} c^{-1} & 0 \\ 0 & c \end{pmatrix},$$

$$\begin{pmatrix} 0 & \mathrm{X} \\ -\mathrm{X}^{-1} & 0 \end{pmatrix} \begin{pmatrix} 0 & \mathrm{X}' \\ -\mathrm{X}'^{-1} & 0 \end{pmatrix} = \begin{pmatrix} -\mathrm{X}\mathrm{X}'^{-1} & 0 \\ 0 & -\mathrm{X}'\mathrm{X}^{-1} \end{pmatrix}.$$

The identities in parts (3) and (4) are readily verified as well. (Replacing X' with $t\mathrm{X}'$ in the displayed formula in part (3) and differentiating at $t = 0$ yields the formula for $\mathrm{Ad}_{\mathrm{G}}(w_a(\mathrm{X}))(\mathrm{X}')$ via the Chain Rule and the definition of Ad_{G}.)

The formulas in Example 4.2.1 are inherited by the central quotient PGL_2 when using the diagonal torus and associated root $\overline{a} : \mathrm{diag}(c, 1) \mapsto c$ and coroot $\overline{a}^\vee : c \mapsto \mathrm{diag}(c^2, 1) = \mathrm{diag}(c, 1/c) \bmod \mathbf{G}_{\mathrm{m}}$, so now everything is reduced to straightforward calculations with the standard root groups, roots, and coroots for SL_2 and PGL_2 equipped with their diagonal tori. □

Remark 5.1.10. — In SL_2 we have

$$\begin{aligned} \begin{pmatrix} 0 & \mathrm{X} \\ -\mathrm{X}^{-1} & 0 \end{pmatrix} \begin{pmatrix} 1 & \mathrm{X} \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} 0 & \mathrm{X} \\ -\mathrm{X}^{-1} & -1 \end{pmatrix}, \\ \begin{pmatrix} 0 & \mathrm{X} \\ -\mathrm{X}^{-1} & -1 \end{pmatrix} \begin{pmatrix} -1 & -\mathrm{X} \\ \mathrm{X}^{-1} & 0 \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \end{aligned}$$

so it follows from the proof of Corollary 5.1.9 that

$$(w_a(\mathrm{X}) \exp_a(\mathrm{X}))^3 = 1$$

for any section X of $\mathbf{W}(\mathfrak{g}_a)^\times$; this encodes the identity $\left(\begin{smallmatrix} 0 & 1 \\ -1 & -1 \end{smallmatrix}\right)^3 = 1$ in SL_2. This relation is needed when constructing homomorphisms from reductive group schemes to other group schemes (e.g., isogenies or isomorphisms between reductive group schemes); see Theorem 6.2.4. Our approach to the group-theoretic relations among $w_{\pm a}$ and $\exp_{\pm a}$ involves reduction to calculations with SL_2 and PGL_2 (because we have already obtained a Zariski-local classification result). The approach in [**SGA3**, XX] rests on calculations via a more indirect method.

Here is the surjectivity of $\mathrm{N_G(T)(S)} \to \mathrm{W_G(T)(S)}$ in the split case:

Corollary 5.1.11. — *Let* $(\mathrm{G}, \mathrm{T}, \mathrm{M})$ *be a split reductive group over a non-empty scheme* S. *Fix linked trivializations* $\mathrm{X}_a \in \mathbf{W}(\mathfrak{g}_a)^\times(\mathrm{S})$ *for all* $a \in \Phi$ *(so* X_{-a} *is dual to* X_a*). The natural map* $\mathrm{N_G(T)(S)} \to \mathrm{W_G(T)(S)}$ *is surjective, with* $n_a := w_a(\mathrm{X}_a)$ *mapping to the reflection* s_a *in* (1.3.1) *and satisfying*

$$n_{-a} = a^\vee(-1)n_a,\ \ n_a^2 = a^\vee(-1),\ \ n_a \exp_a(\mathrm{X}_a)n_a^{-1} = \exp_{-a}(-\mathrm{X}_{-a}).$$

Proof. — Since $W_G(T)$ is the constant group associated to $W(\Phi)$, an element of $W_G(T)(S)$ is a locally constant function valued in $W(\Phi)$. Thus, by passing to the constituents of a covering of S by pairwise disjoint open sets, for the proof of surjectivity on S-points we can focus on constant functions. But $W(\Phi)$ is generated by the reflections s_a, so it remains to prove the assertions concerning n_a, which are special cases of (3) and (4) in Corollary 5.1.9. □

The infinitesimal version of relations among roots and root spaces [**SGA3**, XX, 2.10] will not be used in what follows, but we record it for completeness:

Corollary 5.1.12. — *Let* G*,* T*, and* a *be as in Corollary* 5.1.9*. Let* $\overline{\pm a} = \mathrm{Lie}(\pm a) : \mathfrak{t} \to \mathscr{O}_S$ *and define* $H_a = \mathrm{Lie}(a^\vee)(1) \in \mathfrak{t}$ *using the canonical basis of* $\mathrm{Lie}(\mathbf{G}_m)$*. Then*

$$\overline{-a} = -\overline{a}, \quad H_{-a} = -H_a, \quad \overline{a}(H_a) = 2,$$

and for all local sections t *of* T *and* $X, X' \in \mathfrak{g}_a$*,* $Y \in \mathfrak{g}_{-a}$*, and* $H \in \mathfrak{t}$*, we have:*

$$\mathrm{Ad}_G(t)(H) = H, \ \ \mathrm{Ad}_G(t)(X) = a(t)X, \ \ \mathrm{Ad}_G(t)(Y) = a(t)^{-1}Y, \tag{5.1.1}$$

$$\mathrm{Ad}_G(\exp_a(X))(H) = H - \overline{a}(H)X, \quad \mathrm{Ad}_G(\exp_a(X))(X') = X', \tag{5.1.2}$$

$$\mathrm{Ad}_G(\exp_a(X))(Y) = Y + (XY)H_a - (XY)X, \tag{5.1.3}$$

$$[H, X] = \overline{a}(H)X, \ \ [H, Y] = -\overline{a}(H)Y, \ \ [X, Y] = (XY)H_a. \tag{5.1.4}$$

Proof. — As in the proof of Corollary 5.1.9, we reduce to considering the groups SL_2 and PGL_2 equipped with their diagonal tori and standard linked root space trivializations, with a as the standard positive root (whose root group consists of the strictly upper triangular matrices). In these cases the assertions are straightforward (and classical) calculations. □

The following relative version of the "open cell" generalizes (4.1.1) by replacing $Z_G(T_a)$ with G (subject to the hypothesis that (G, T) is split).

Theorem 5.1.13. — *Let* (G, T, M) *be a split reductive group over a non-empty scheme* S*. Fix a positive system of roots* $\Phi^+ \subset \Phi$*.*

For any enumeration $\{a_i\}$ *of* Φ^+*, the multiplication map* $\prod_i U_{a_i} \to G$ *is an isomorphism onto a smooth closed subgroup* U_{Φ^+} *that is normalized by* T*, independent of the choice of enumeration, and has connected unipotent fibers. The multiplication map*

$$U_{-\Phi^+} \times T \times U_{\Phi^+} \to G$$

is an isomorphism onto an open subscheme Ω_{Φ^+} *and the semi-direct product* $T \ltimes U_{\Phi^+} \to G$ *is a closed* S*-subgroup.*

The existence of Ω_{Φ^+} is given in [**SGA3**, XXII, 4.1.2], but the construction of U_{Φ^+} as a *closed* subgroup in [**SGA3**, XXII, § 5] is completely different, resting on a detailed study of Lie algebras and smoothness properties of normalizers in G for certain subalgebras of $\mathfrak{g}$ (see [**SGA3**, XXII, 5.3.4, 5.6.5]). Our proof gives closedness results as a consequence of the construction of U_{Φ^+} via the dynamic method, which builds the desired "Borel subgroup" and its "unipotent radical" without any considerations with root groups.

Proof. — Pick $\lambda \in \mathrm{M}^\vee$ so that the open half-space $\{\lambda > 0\}$ in $\mathrm{M}_{\mathbf{R}}$ meets Φ in Φ^+. Interpreting λ as a cocharacter $\mathbf{G}_{\mathrm{m}} \to \mathrm{T}$, it makes sense to form the smooth closed S-subgroups $\mathrm{U}_{\mathrm{G}}(\lambda)$, $\mathrm{P}_{\mathrm{G}}(\lambda)$, and $\mathrm{Z}_{\mathrm{G}}(\lambda)$ with connected fibers as in Theorem 4.1.7. The tangent space to $\mathrm{Z}_{\mathrm{G}}(\lambda)$ coincides with $\mathfrak{t} = \mathfrak{g}_0$ since the only T-weight on $\mathfrak{g}$ whose pairing with λ vanishes is the trivial weight (as $\Phi = \Phi^+ \coprod -\Phi^+$). Thus, the inclusion $\mathrm{T} \subset \mathrm{Z}_{\mathrm{G}}(\lambda)$ between smooth closed S-subgroups with connected fibers induces an equality on Lie algebras (inside $\mathfrak{g}$) and hence is an equality inside G. That is, $\mathrm{Z}_{\mathrm{G}}(\lambda) = \mathrm{T}$, so $\mathrm{P}_{\mathrm{G}}(\lambda) = \mathrm{Z}_{\mathrm{G}}(\lambda) \ltimes \mathrm{U}_{\mathrm{G}}(\lambda) = \mathrm{T} \ltimes \mathrm{U}_{\mathrm{G}}(\lambda)$. The S-group $\mathrm{U}_{\mathrm{G}}(\lambda)$ has (connected) unipotent fibers, by Theorem 4.1.7(4), so $\mathrm{P}_{\mathrm{G}}(\lambda)$ has (connected) solvable fibers.

By Theorem 4.1.7, the multiplication map

$$\mathrm{U}_{\mathrm{G}}(-\lambda) \times \mathrm{T} \times \mathrm{U}_{\mathrm{G}}(\lambda) \to \mathrm{G}$$

is an open immersion. Note that $\mathrm{U}_{\mathrm{G}}(\lambda)$ has nothing to do with a choice of enumeration of Φ^+. Also, for $a \in \Phi$ the root group U_a is normalized by T and hence is normalized by the $\mathbf{G}_{\mathrm{m}}$-action through conjugation by λ, with $\lambda(t)$ acting on $\mathrm{Lie}(\mathrm{U}_a) = \mathfrak{g}_a$ via scaling by $t^{\langle a, \lambda\rangle}$. Thus, $\mathrm{U}_{\mathrm{U}_a}(\lambda)$ has Lie algebra $\mathfrak{g}_a$ if $\langle a, \lambda\rangle > 0$ (i.e., if $a \in \Phi^+$) and has vanishing Lie algebra otherwise (i.e., if $a \in -\Phi^+$). Since $\mathrm{U}_{\mathrm{U}_a}(\lambda)$ must be S-smooth with connected fibers (as U_a is!), this S-subgroup of U_a vanishes when $a \in -\Phi^+$ and coincides with U_a when $a \in \Phi^+$. In particular, $\mathrm{U}_a \subset \mathrm{U}_{\mathrm{G}}(\lambda)$ for all $a \in \Phi^+$.

It now suffices to show that for any enumeration $\{a_i\}$ of Φ^+, the multiplication mapping

$$\prod_i \mathrm{U}_{a_i} \to \mathrm{U}_{\mathrm{G}}(\lambda)$$

is an isomorphism of S-schemes. By smoothness of both sides, it suffices to check the isomorphism property on geometric fibers, so we may and do assume $\mathrm{S} = \operatorname{Spec} k$ for an algebraically closed field k. The k-group $\mathrm{P}_{\mathrm{G}}(\lambda) = \mathrm{T} \ltimes \mathrm{U}_{\mathrm{G}}(\lambda)$ is connected and solvable, so dimension considerations with its Lie algebra imply that it is a Borel subgroup with $\mathrm{U}_{\mathrm{G}}(\lambda)$ as its unipotent radical and that the subgroups U_{a_i} must be its root groups. In the classical theory it is proved that the unipotent radical of a Borel subgroup is directly spanned (in any order) by its root groups, though this also follows from general considerations using just the reducedness of the root system; see [**CGP**, Thm. 3.3.11] for such an alternative proof of direct spanning in the classical case (applying [**CGP**, Thm. 3.3.11] to the smooth connected unipotent $\mathrm{U}_{\mathrm{G}}(\lambda)$). □

As in the classical case, we say that U_{Φ^+} is *directly spanned* (in any order) by the U_{a_i}'s for $a_i \in \Phi^+$, and we call Ω_{Φ^+} the *open cell* (or *big cell*) associated to Φ^+. The link between the root system and the commutation relations among positive root groups carries over as in the classical theory:

Proposition 5.1.14. — *Let* $(\mathrm{G}, \mathrm{T}, \mathrm{M})$ *be a split reductive group over a non-empty scheme* S. *Pick roots* $a, b \in \Phi$ *such that* $b \neq \pm a$. *Choose trivializations of the root spaces* $\mathfrak{g}_c$ *for all roots* $c = ia + jb$ *with integers* $i, j > 0$. *Consider the associated parameterizations* $p_c : \mathbf{G}_{\mathrm{a}} \simeq \mathrm{U}_c$, *and fix an enumeration of this set of roots* c.

The root groups U_a *and* U_b *commute if there are no roots of the form* $ia+jb$ *with integers* $i, j > 0$, *and in general the commutation relation is given by*

$$(p_a(x), p_b(y)) := p_a(x)p_b(y)p_a(-x)p_b(-y) = \prod_{i,j>0} p_{ia+jb}(\mathrm{C}_{i,j,a,b}x^i y^j) \tag{5.1.5}$$

where the product is taken over all roots $ia+jb$ *with* $i, j > 0$ *and the coefficients* $\mathrm{C}_{i,j,a,b}$ *are global functions on* S.

As in the classical case, the "structure constants" $\mathrm{C}_{i,j,a,b}$ are mysterious at this stage of the theory; a detailed study of rank-2 cases will be required to clean them up. Note that these structure constants depend on the choice of ordering among the terms in the product on the right side of (5.1.5) (and on the choice of trivializations p_c of the root spaces $\mathfrak{g}_c$); also, this product involves at most 6 terms (as we see by inspecting the classification of reduced rank-2 root systems).

Proof. — The classical argument via T-equivariance will carry over, using more care due to the base scheme being rather general. Pick a positive system of roots Φ^+ containing a and b (as we can do since Φ is reduced and $b \neq \pm a$), and choose an enumeration $\{c_m\}$ of Φ^+ extending the choice of enumeration of the roots of the form $ia + jb$ with $i, j > 0$.

A priori the commutator $(p_a(x), p_b(y))$ lies in $\mathrm{U}_{\Phi^+} = \prod_m \mathrm{U}_{c_m}$, and we have to show that the only factors U_c which can have a nontrivial component are for the roots $c = ia+jb$ with $i, j > 0$, and that the factor in such a component has the form $p_{ia+jb}(\mathrm{C}_{i,j,a,b}x^i y^j)$.

Consider the expression

$$(p_a(x), p_b(y)) = \prod_{c\in\Phi^+} p_c(h_c(x, y))$$

where the product on the right side is taken in the order according to the chosen enumeration of Φ^+ and where $h_c : \mathrm{U}_a \times \mathrm{U}_b \to \mathrm{U}_c$ is a T-equivariant map of S-schemes. That is, $h_c(a(t)x, b(t)y) = c(t)h_c(x, y)$ for the S-map $h_c : \mathbf{G}_\mathrm{a} \times \mathbf{G}_\mathrm{a} \to \mathbf{G}_\mathrm{a}$ given by some 2-variable polynomial Zariski-locally over S. Writing $h_c = \sum_{i,j \geqslant 0} f_{i,j} x^i y^j$ for some Zariski-local functions $f_{i,j}$ on S, we have $f_{i,j} a(t)^i b(t)^j = c(t) f_{i,j}$ for all i, j. Setting $y = 0$ gives $f_{i,0} = 0$ for all i since $p_b(0) = 1$, and likewise $f_{0,j} = 0$ for all j. If $i, j > 0$ then $f_{i,j}$ is killed by the character $c - (ia + jb)$ on $\mathrm{T} = \mathrm{D_S(M)}$ that arises from an element of M. Such a character is either trivial or fiberwise nontrivial (and hence faithfully flat onto $\mathbf{G}_\mathrm{m}$), so $f_{i,j} = 0$ except possibly when $c = ia + jb$, in which case such i and j are uniquely determined by c (since the distinct positive roots a and b are linearly independent). In other words, each h_c that is not identically zero is a single monomial of some constant bi-degree (i, j) such that $ia + jb \in \Phi^+$ and $i, j > 0$. In particular, if there are no such roots $ia + jb$ with $i, j > 0$ then U_a commutes with U_b. □

It is useful to generalize the construction of U_{Φ^+} by constructing fiberwise unipotent smooth closed subgroups $\mathrm{U}_\Psi \subset \mathrm{G}$ directly spanned in any order by certain subsets $\Psi \subset \Phi$. To characterize the Ψ that we shall consider, we make a brief digression concerning general root systems.

Let (V, Φ) be a (possibly non-reduced) root system, with V a $\mathbf{Q}$-vector space. Recall that a subset $\Psi \subset \Phi$ is called *closed* if $a + b \in \Psi$ for any $a, b \in \Psi$ such that $a + b \in \Phi$. Examples of such Ψ are $\Phi_{\lambda > q} = \{a \in \Phi \,|\, \lambda(a) > q\}$ and $\Phi_{\lambda \geqslant q} = \{a \in \Phi \,|\, \lambda(a) \geqslant q\}$ for $\lambda \in \mathrm{V}^*$ and $q \in \mathbf{Q}$, as well as the sets of roots

$$[a, b] = \{ia + jb \in \Phi \,|\, i, j \geqslant 0\}, \;\; (a, b) = \{ia + jb \in \Phi \,|\, i, j \geqslant 1\},$$

$$[a, b) = \{ia + jb \,|\, i \geqslant 0, j \geqslant 1\}$$

for linearly independent $a, b \in \Phi$.

By [**CGP**, 2.2.7], the closed sets in Φ are precisely the subsets of the form $\Phi \bigcap \mathrm{A}$ for a subset $\mathrm{A} \subset \mathrm{V}$ that is a subsemigroup (i.e., $a + a' \in \mathrm{A}$ for all $a, a' \in \mathrm{A}$; we allow A to be empty). When $\Psi = \Phi_{\lambda > q}$ we can use $\mathrm{A} = \{v \in \mathrm{V} \,|\, \lambda(v) > q\}$, but when $\Psi = [a, b)$ for linearly independent $a, b \in \Phi$ there is no "obvious" choice for A. For any closed $\Psi \subset \Phi$ there is a unique minimal choice for A, namely the subsemigroup $\langle \Psi \rangle$ generated by Ψ (which is empty when Ψ is empty). We are interested in closed Ψ that lie in a positive system of roots. Such a positive system of roots is not uniquely determined by Ψ, but there is a simple characterization for when one exists:

Lemma 5.1.15. — *Let* (V, Φ) *be a root system. For* $\Psi \subset \Phi$*, the following are equivalent:*

1. Ψ *is closed and is contained in a positive system of roots;*
2. $\Psi = \Phi \bigcap \mathrm{A}$ *for a subsemigroup* $\mathrm{A} \subset \mathrm{V}$ *such that* $0 \notin \mathrm{A}$*;*
3. Ψ *is closed and* $\Psi \bigcap -\Psi$ *is empty.*

Proof. — Consider a closed set Ψ, so $\Psi = \Phi \bigcap \mathrm{A}$ for $\mathrm{A} = \langle\Psi\rangle$. The positive systems of roots in Φ are precisely the subsets $\Phi^+ = \Phi_{\lambda>0}$ with $\lambda \in \mathrm{V}^*$ that is nonzero on all roots. If Ψ is contained in some $\Phi^+ = \Phi_{\lambda>0}$ then $\langle\Psi\rangle$ lies in $\{v \in \mathrm{V} \,|\, \lambda(v) > 0\}$, so $0 \notin \langle\Psi\rangle$. Thus, (1) implies (2). The implication "(2) $\Rightarrow$ (3)" is trivial , and "(3) $\Rightarrow$ (1)" is precisely [**Bou2**, VI, §1.7, Prop. 22] (due to the characterization of positive systems of roots in Φ in terms of Weyl chambers for Φ in $\mathrm{V}_{\mathbf{R}}$, given by [**Bou2**, VI, §1.7, Cor. 1, Cor. 2 to Prop. 20]). □

Here is a generalization of U_{Φ^+} (inspired by [**SGA3**, XXII, 5.9.5]).

Proposition 5.1.16. — *Let* $(\mathrm{G}, \mathrm{T}, \mathrm{M})$ *be a split reductive group over a non-empty scheme* S*, and let* Ψ *be a closed set in* Φ *such that* $\Psi \bigcap -\Psi = \varnothing$*.*

1. *For any enumeration* $\{a_i\}$ *of* Ψ*, the multiplication map* $\prod \mathrm{U}_{a_i} \to \mathrm{G}$ *is an isomorphism onto a smooth closed subgroup* U_Ψ*. This subgroup is normalized by* T*, independent of the choice of enumeration, and has connected unipotent fibers.*
2. *Choose* $\lambda \in \mathrm{M}^\vee$ *that is non-vanishing on* Φ *such that the positive system of roots* $\Phi^+ := \Phi_{\lambda>0}$ *contains* Ψ*. For every integer* $n > 0$*, let* $\Psi_{\geqslant n} = \Psi \bigcap \Phi_{\lambda \geqslant n}$*. The subgroups* $\mathrm{U}_{\geqslant n} := \mathrm{U}_{\Psi_{\geqslant n}}$ *are normal in* U_Ψ *and the multiplication map*
$$\prod_{a \in \Psi, \lambda(a)=n} \mathrm{U}_a \to \mathrm{U}_{\geqslant n}/\mathrm{U}_{\geqslant n+1}$$
(*with the product taken in any order*) *is an* S*-group isomorphism. In particular,* $\mathrm{U}_{\geqslant n}/\mathrm{U}_{\geqslant n+1}$ *is a power of* $\mathbf{G}_{\mathrm{a}}$ *as an* S*-group.*

By Lemma 5.1.15, there always exists λ as in (2).

Proof. — Choose λ as in (2) and let $\Phi^+ = \Phi_{\lambda>0}$. Since U_{Φ^+} is directly spanned in any order by the root groups U_a for $a \in \Phi^+$, for any enumeration $\{a_i\}$ of Ψ the multiplication map $\prod \mathrm{U}_{a_i} \to \mathrm{G}$ is an isomorphism onto a smooth closed subscheme of U_{Φ^+}. If we can prove that this closed subscheme is an S-subgroup for one choice of enumeration then for *any* enumeration the multiplication map is an isomorphism onto the *same* closed S-subgroup (because a monic endomorphism of a finitely presented scheme is necessarily an isomorphism, by [**EGA**, IV_4, 17.9.6]). Thus, to prove (1) it suffices to consider a single enumeration. Also, once the existence of $\mathrm{U}_{\Psi_{\geqslant n}}$ is proved for all $n \geqslant 1$, it is immediate from (5.1.5) in Proposition 5.1.14 that $\mathrm{U}_{\Psi_{\geqslant n}}$ is normal in U_Ψ for all $n \geqslant 1$. It is also obvious that such subgroups are normalized by T.

For large m, $\Psi_{\geqslant m}$ is empty. By descending induction on m we shall prove (1) for $\Psi_{\geqslant m}$ and then (2) for $\Psi_{\geqslant m}$ when using our initial choice of λ. Since

$\Psi = \Psi_{\geqslant 1}$ and λ was arbitrary, this induction will prove (1) and (2) in general. The base of the induction (large m) is obvious, with $\mathrm{U}_{\Psi_{\geqslant m}} = 1$ for large m.

Now suppose the cases $m' \geqslant m+1$ are settled, and consider (1) and (2) for $\Psi_{\geqslant m}$. We know that to prove (1) for $\Psi_{\geqslant m}$ it suffices to consider one enumeration. We will use an enumeration that is adapted to λ. It is immediate from (5.1.5) that $\mathrm{U}_{\geqslant m+1}$ is normalized by U_a for all $a \in \Psi_{\geqslant m}$. Likewise, if $a, b \in \Psi_{\geqslant m}$ then for any S-scheme S′ and $u_a \in \mathrm{U}_a(\mathrm{S}')$ and $u_b \in \mathrm{U}_b(\mathrm{S}')$ we see that $u_a u_b u_a^{-1} u_b^{-1} \in \mathrm{U}_{\geqslant m+1}(\mathrm{S}')$ since $\lambda(ia + jb) \geqslant m+1$ for any $i, j \geqslant 1$. Letting $\Psi_m = \{a \in \Psi \mid \lambda(a) = m\}$, it follows that for any S-scheme S′ and $a, b \in \Psi_m$ the subgroups $\mathrm{U}_a(\mathrm{S}')$ and $\mathrm{U}_b(\mathrm{S}')$ in $\mathrm{G}(\mathrm{S}')$ commute modulo the subgroup $\mathrm{U}_{\geqslant m+1}(\mathrm{S}')$ that they normalize. Thus, for any enumeration $\{c_i\}$ of Ψ_m the monic multiplication map $\prod \mathrm{U}_{c_i}(\mathrm{S}') \times \mathrm{U}_{\geqslant m+1}(\mathrm{S}') \to \mathrm{G}(\mathrm{S}')$ has image that is a subgroup. This proves (1) for $\Psi_{\geqslant m}$, and (2) for $n = m$ is now obvious (by consideration of S′-valued points for any S-scheme S′). □

We end this section with applications of the open cell over a field. (See [**Bo91**, 14.10] for an alternative approach via the structure of automorphisms of connected semisimple groups).

Proposition 5.1.17. — *Let* G *be a split nontrivial connected semisimple group over a field k. The set $\{\mathrm{G}_i\}_{i \in \mathrm{I}}$ of minimal nontrivial normal smooth connected k-subgroups of* G *is finite, the* G_i*'s pairwise commute with each other, and the multiplication homomorphism $\prod \mathrm{G}_i \to \mathrm{G}$ is a central isogeny.*

Proof. — Let T be a split maximal k-torus in G, and $\Phi = \Phi(\mathrm{G}, \mathrm{T}) \neq \varnothing$. For each irreducible component Φ_i of Φ, let G_i be the smooth connected k-subgroup of G generated by the root groups U_a for $a \in \Phi_i$. For any $i' \neq i$ and roots $a' \in \Phi_{i'}$ and $a \in \Phi_i$, $a + a' \notin \Phi$ and we can put a and a' into a common positive system of roots. Hence, U_a and $\mathrm{U}_{a'}$ commute (Proposition 5.1.14), so G_i and $\mathrm{G}_{i'}$ commute. Since the root groups U_a and U_{-a} generate a subgroup containing $a^\vee(\mathbf{G}_\mathrm{m})$, and the coroots generate a finite-index subgroup of $\mathrm{X}_*(\mathrm{T})$ (as G is semisimple), the collection of all root groups generates a smooth closed subgroup containing all factors of the open cell in Theorem 5.1.13. Hence, the G_i's generate G, so each G_i is normal in G and the product map

$$\pi : \prod \mathrm{G}_i \to \mathrm{G}$$

is a surjective homomorphism. Normality of G_i in G implies that G_i inherits semisimplicity from G. For $i' \neq i$, the subgroups G_i and $\mathrm{G}_{i'}$ commute and are nontrivial and semisimple, so $\mathrm{G}_i \neq \mathrm{G}_{i'}$.

By induction, if $\{\mathrm{N}_j\}$ is a finite collection of pairwise commuting normal smooth closed k-subgroups of a smooth k-group H of finite type then the multiplication homomorphism $\prod \mathrm{N}_j \to \mathrm{H}$ has central kernel. (This can be generalized to the setting of group sheaves, as the interested reader can check.) Hence, $\ker \pi$ is central. But $\prod \mathrm{G}_i$ is semisimple, so π is a central isogeny.

It remains to show that these G_i are precisely the minimal nontrivial normal smooth connected k-subgroups of G. We may and do assume $k = \overline{k}$ since the formation of the G_i commutes with extension of the ground field.

Let N be a nontrivial normal smooth connected k-subgroup of G, so N inherits semisimplicity from G. Hence, N is non-commutative. Since G is generated by the pairwise commuting subgroups G_i, there must be some i such that the commutator subgroup $(\mathrm{N}, \mathrm{G}_i)$ is nontrivial. But $(\mathrm{N}, \mathrm{G}_i)$ is normal in G and is contained in G_i, so it suffices to show that each G_i is minimal as a nontrivial normal smooth connected k-subgroup of G. Now we can assume N is contained in some G_{i_0} and we seek to show that $\mathrm{N} = \mathrm{G}_{i_0}$. By normality of N in G and Exercise 5.5.1(i), $\mathrm{S} := \mathrm{T} \bigcap \mathrm{N}$ is a maximal torus in N (so $\mathrm{S} \neq 1$).

Since T is a split maximal torus of G and π is an isogeny, each $\mathrm{T}_i := \mathrm{T} \bigcap \mathrm{G}_i$ is a split maximal torus of G_i and π carries $\prod \mathrm{T}_i$ isogenously onto T. Clearly $\mathrm{S} = \mathrm{N} \bigcap \mathrm{T}_{i_0}$, and the connected reductive subgroup $\mathrm{N} \cdot \mathrm{T}_{i_0}$ in G has maximal torus T_{i_0} and derived group N (as N is semisimple), so T_{i_0} is the almost direct product of S and the maximal central torus Z of $\mathrm{N} \cdot \mathrm{T}_{i_0}$. Hence, the isomorphism $\mathrm{X}(\mathrm{T}_{i_0})_{\mathbf{Q}} \simeq \mathrm{X}(\mathrm{S})_{\mathbf{Q}} \oplus \mathrm{X}(\mathrm{Z})_{\mathbf{Q}}$ induces a bijection $\Phi(\mathrm{N} \cdot \mathrm{T}_{i_0}, \mathrm{T}_{i_0}) \simeq \Phi(\mathrm{N}, \mathrm{S}) \times \{0\}$. In this way $\Phi(\mathrm{N}, \mathrm{S})$ spans a nonzero subspace of $\mathrm{X}(\mathrm{T}_{i_0})_{\mathbf{Q}}$ stable under the action of $\mathrm{W}_{\mathrm{G}_{i_0}}(\mathrm{T}_{i_0}) = \mathrm{W}(\Phi_{i_0})$. The Weyl group of an irreducible root system (V, Ψ) acts irreducibly on V [**Bou2**, VI, § 1.2, Cor.], so if $\Phi(\mathrm{G}_{i_0}, \mathrm{T}_{i_0})$ is irreducible then $\Phi(\mathrm{N}, \mathrm{S})$ spans $\mathrm{X}(\mathrm{T}_{i_0})_{\mathbf{Q}} = \mathrm{X}(\mathrm{S})_{\mathbf{Q}} \oplus \mathrm{X}(\mathrm{Z})_{\mathbf{Q}}$, so $\mathrm{S} = \mathrm{T}_{i_0}$ for dimension reasons. This would force $\mathrm{T}_{i_0} \subset \mathrm{N}$, so the connected semisimple $\mathrm{G}_{i_0}/\mathrm{N}$ would have trivial maximal torus and thus $\mathrm{N} = \mathrm{G}_{i_0}$ as desired.

Finally, we show that each $(\mathrm{X}(\mathrm{T}_i)_{\mathbf{Q}}, \Phi(\mathrm{G}_i, \mathrm{T}_i))$ is irreducible by relating $\Phi(\mathrm{G}_i, \mathrm{T}_i)$ to the irreducible component Φ_i of Φ. The center of a connected reductive group lies in any maximal torus, so the direct product structure of open cells in Theorem 5.1.13 implies that a central isogeny $\mathrm{H}' \to \mathrm{H}$ between connected reductive k-groups induces (for compatible maximal tori of H' and H) a natural bijection between the collections of root groups as well as an isomorphism between the root systems and root groups for corresponding roots (see Exercise 1.6.9(i)). More specifically, the isomorphism $\mathrm{X}(\mathrm{T})_{\mathbf{Q}} \simeq \prod \mathrm{X}(\mathrm{T}_i)_{\mathbf{Q}}$ identifies $\Phi(\mathrm{G}, \mathrm{T})$ with $\coprod \Phi(\mathrm{G}_i, \mathrm{T}_i)$. But if $i' \neq i$ then $\mathrm{T}_{i'}$ centralizes G_i and hence centralizes all root groups of $(\mathrm{G}_i, \mathrm{T}_i)$, so each $a \in \Phi_i$ kills the image of $\mathrm{T}_{i'}$ in T. Thus, $\Phi_i \subset \Phi(\mathrm{G}_i, \mathrm{T}_i)$ inside $\mathrm{X}(\mathrm{T})_{\mathbf{Q}}$, so the definition of the Φ_i as the irreducible components of Φ forces $\Phi_i = \Phi(\mathrm{G}_i, \mathrm{T}_i)$ for all i. $\square$

A connected semisimple group H over a field k is *k-simple* if $\mathrm{H} \neq 1$ and H has no nontrivial normal smooth connected proper k-subgroup, and *absolutely simple* if H_{K} is K-simple for some (equivalently, any) algebraically closed extension K/k. For the groups G_i in Proposition 5.1.17, if T is a split maximal torus of G then $\mathrm{T}_i := \mathrm{T} \bigcap \mathrm{G}_i$ is a split maximal torus of G_i and the proof of Proposition 5.1.17 shows that the isomorphism $\mathrm{X}(\mathrm{T})_{\mathbf{Q}} \simeq \prod \mathrm{X}(\mathrm{T}_i)_{\mathbf{Q}}$ identifies the $\Phi(\mathrm{G}_i, \mathrm{T}_i)$ with the irreducible components of Φ. In particular:

Corollary 5.1.18. — *A nontrivial connected semisimple group* G *over a field is absolutely simple if and only if the root system for* G_{k_s} *is irreducible.*

It follows that the G_i in Proposition 5.1.17 are absolutely simple. Here is a generalization of Proposition 5.1.17 beyond the split case.

Theorem 5.1.19 **(Decomposition theorem for semisimple groups)**
Let G *be a nontrivial connected semisimple group over a field* k. *The set* $\{\mathrm{G}_i\}_{i\in\mathrm{I}}$ *of minimal nontrivial normal smooth connected* k*-subgroups of* G *is finite, each* G_i *is* k*-simple, the* G_i*'s pairwise commute, and the multiplication homomorphism*

$$\prod \mathrm{G}_i \to \mathrm{G}$$

is a central isogeny.

For each $\mathrm{J} \subset \mathrm{I}$ *the normal connected semisimple* k*-subgroup* $\mathrm{G}_\mathrm{J} \subset \mathrm{G}$ *generated by* $\{\mathrm{G}_i\}_{i\in\mathrm{J}}$ *has as its minimal nontrivial normal smooth connected* k*-subgroups precisely the* G_i *for* $i \in \mathrm{J}$, *and every normal smooth connected* k*-subgroup* $\mathrm{N} \subset \mathrm{G}$ *equals* G_J *for a unique* J. *In particular, for each* N *there exists a unique* N' *that commutes with* N *and makes the multiplication homomorphism* $\mathrm{N} \times \mathrm{N}' \to \mathrm{G}$ *a central isogeny.*

See Exercise 5.5.2 for the generalization to connected reductive k-groups G.

Proof. — We first treat the case $k = k_s$, and then will deduce the general case by Galois descent. Assuming $k = k_s$, G is split and we can apply Proposition 5.1.17. Letting the G_i be as in that result, we have proved their simplicity and that they pairwise commute and define a central isogeny $\prod \mathrm{G}_i \to \mathrm{G}$. Note also that each G_J is semisimple, due to normality in G.

For each non-empty J the natural map $\prod_{i\in\mathrm{J}} \mathrm{G}_i \to \mathrm{G}_\mathrm{J}$ is a central isogeny, so by root system considerations (applying Proposition 5.1.17 to the split G_J), the set of minimal nontrivial normal smooth connected subgroups of G_J is exhausted by the G_i's for $i \in \mathrm{J}$. To show that every N has the form G_J for some J, we can assume $\mathrm{N} \neq 1$. Thus, N contains some G_{i_0} and (by consideration of root systems) the minimal nontrivial normal smooth connected subgroups of $\overline{\mathrm{G}} = \mathrm{G}/\mathrm{G}_{i_0}$ are the images $\overline{\mathrm{G}}_i$ of the G_i for $i \neq i_0$. By induction on dimension, $\overline{\mathrm{N}} := \mathrm{N}/\mathrm{G}_{i_0}$ is equal to $\overline{\mathrm{G}}_{\mathrm{J}_0}$ for a subset $\mathrm{J}_0 \subset \mathrm{I} - \{i_0\}$, so $\mathrm{N} = \mathrm{G}_\mathrm{J}$ for $\mathrm{J} = \mathrm{J}_0 \bigcup \{i_0\}$.

Finally, we consider general k. Let $\Gamma = \mathrm{Gal}(k_s/k)$. Let $\{\mathrm{G}'_i\}_{i\in\mathrm{I}}$ be the set of minimal nontrivial normal smooth connected k_s-subgroups of $\mathrm{G}' = \mathrm{G}_{k_s}$, so Γ naturally permutes these subgroups and hence acts on the index set I. For each Γ-stable subset $\mathrm{J} \subset \mathrm{I}$, G'_J descends to a normal smooth connected

k-subgroup $\mathrm{G_J} \subset \mathrm{G}$, and by Galois descent these $\mathrm{G_J}$ are precisely the normal smooth connected k-subgroups of G. Hence, the minimal nontrivial ones are the groups $\mathrm{G_J}$ for J a Γ-orbit in I. Since the Γ-stable subsets of I are precisely the unions of Γ-orbits, we are done. □

The G_i in Theorem 5.1.19 are called the *k-simple factors* of G. The formation of the set of G_i's is sensitive to extension of the ground field:

Example 5.1.20. — Consider the Weil restriction $\mathrm{G} = \mathrm{R}_{k'/k}(\mathrm{G}')$ for a finite separable extension k'/k and an *absolutely simple* and semisimple k'-group G'. Since $k' \otimes_k k_s$ is a product of copies of k_s indexed by the set of k-embeddings $\sigma : k' \to k_s$, $\mathrm{G}_{k_s} = \prod_\sigma \mathrm{G}'_\sigma$ where $\mathrm{G}'_\sigma = k_s \otimes_{\sigma,k'} \mathrm{G}'$. In particular, G is connected semisimple and its simple factors over k_s are the G'_σ. But these are permuted transitively by $\mathrm{Gal}(k_s/k)$, so G is k-simple. If $\mathrm{G}'_{k'_s}$ has a root datum that is semisimple and simply connected (resp. adjoint) then so does G_{k_s}.

Theorem 5.1.19 shows that, up to central isogeny, to classify connected semisimple groups over a field k, the main case is the k-simple case. Remarkably, the k-simple case is always related to the absolutely simple case over a finite separable extension via the construction in Example 5.1.20 up to a simply connected hypothesis. We will address this more fully in Example 6.4.6, as an application of classification theorems in terms of root data.

5.2. Parabolic subgroups and conjugacy.

— In the classical theory one defines parabolic subgroups $\mathrm{P} \subset \mathrm{G}$ in terms of the structure of G/P and uses this to infer properties such as $\mathrm{P} = \mathrm{N_G(P)}$ (at least on geometric points) and the connectedness of such subgroups. In the version over a base scheme we will first prove that parabolic subgroups are their own schematic normalizers and *use* that fact to construct G/P as a scheme projective over the base.

Definition 5.2.1. — A *parabolic subgroup* of a reductive group scheme $\mathrm{G} \to \mathrm{S}$ is a smooth S-affine S-group P equipped with a monic homomorphism $\mathrm{P} \to \mathrm{G}$ such that $\mathrm{P}_{\overline{s}}$ is parabolic in $\mathrm{G}_{\overline{s}}$ (i.e., $\mathrm{G}_{\overline{s}}/\mathrm{P}_{\overline{s}}$ is proper) for all $s \in \mathrm{S}$.

Note that all fibers $\mathrm{P}_{\overline{s}}$ are connected, by the classical theory. We do not require $\mathrm{P} \to \mathrm{G}$ to be a closed immersion, but it will soon be proved that this condition does hold. Here is a natural class of examples arising from the dynamic method in § 4.1.

Example 5.2.2. — Let $\mathrm{T} \subset \mathrm{G}$ be a maximal S-torus, and $\lambda : \mathbf{G}_{\mathrm{m}} \to \mathrm{T}$ a cocharacter. The smooth closed S-subgroup $\mathrm{P_G}(\lambda)$ is parabolic. Indeed, its fiber at a geometric point $\overline{s}$ of S is $\mathrm{P}_{\mathrm{G}_{\overline{s}}}(\lambda_{\overline{s}})$, and the classical theory implies that such subgroups are always parabolic (see Example 4.1.9).

The dynamic description of parabolic subgroups over an algebraically closed field (Example 4.1.9) admits a relative formulation over any scheme:

Proposition 5.2.3. — *Let* $\mathrm{G} \to \mathrm{S}$ *be a reductive group scheme, and* Q *a parabolic subgroup of* G. *Then* $\mathrm{Q} \to \mathrm{G}$ *is a closed immersion, and étale-locally on* S *there exists a maximal torus* T *of* G *such that* $\mathrm{T} \subset \mathrm{Q}$. *If* G *admits a maximal torus* $\mathrm{T} \subset \mathrm{Q}$ *and* (G, T) *is split then Zariski-locally on* S *there exists* $\lambda : \mathbf{G}_{\mathrm{m}} \to \mathrm{T}$ *such that* $\mathrm{Q} = \mathrm{P}_{\mathrm{G}}(\lambda)$.

Our proof of the closedness of Q in G uses the dynamic method; the proof in [**SGA3**, XXII, 5.8.5] is rather different.

Proof. — On geometric fibers over S, since $\mathrm{Q}_{\overline{s}}$ is parabolic in $\mathrm{G}_{\overline{s}}$ we see that a maximal torus in $\mathrm{Q}_{\overline{s}}$ is its own scheme-theoretic centralizer in $\mathrm{Q}_{\overline{s}}$. Hence, we may apply Theorem 3.2.6 to Q, so by working étale-locally on S (as we may do for verifying that $\mathrm{Q} \to \mathrm{G}$ is a closed immersion) we obtain a maximal torus T of Q. Obviously T is also a maximal torus of G (as this is an assertion on geometric fibers that is trivial to verify).

By further étale localization on S, we may suppose that (G, T) arises from a split triple $(\mathrm{G}, \mathrm{T}, \mathrm{M})$ and that $\mathrm{S} \neq \varnothing$. For each $s \in \mathrm{S}$, the cocharacters of $\mathrm{T}_{\overline{s}}$ coincide with the cocharacters of T_s over $k(s)$ since T_s is split, so by Example 4.1.9 there exists a cocharacter $\lambda_s : \mathbf{G}_{\mathrm{m}} \to \mathrm{T}_s$ over $k(s)$ such that $\mathrm{Q}_{\overline{s}} = \mathrm{P}_{\mathrm{G}_{\overline{s}}}(\lambda_{\overline{s}}) = \mathrm{P}_{\mathrm{G}_s}(\lambda_s)_{\overline{s}}$. Hence, $\mathrm{Q}_s = \mathrm{P}_{\mathrm{G}_s}(\lambda_s)$. The split condition on T provides a Zariski-open neighborhood of s over which λ_s lifts to a cocharacter $\lambda : \mathbf{G}_{\mathrm{m}} \to \mathrm{T}$. Then we may work Zariski-locally around s in S to arrange that $\lambda \in \mathrm{M}^{\vee}$. Clearly $\mathrm{P}_{\mathrm{G}}(\lambda)$ is a *closed* parabolic subgroup of G that contains T and has s-fiber Q_s. We will prove that $\mathrm{Q} = \mathrm{P}_{\mathrm{G}}(\lambda)$ over a Zariski-open neighborhood of s in S.

Consider the Lie algebra $\mathfrak{q}$ of Q inside $\mathfrak{g}$. Although we do not yet know that Q is closed in G, nonetheless the inclusion $\mathfrak{q} \hookrightarrow \mathfrak{g}$ of $\mathscr{O}_{\mathrm{S}}$-modules is a subbundle because on geometric fibers over S the inclusion $\mathrm{Q}_{\overline{s}} \hookrightarrow \mathrm{G}_{\overline{s}}$ is a closed immersion. Since $\mathfrak{q}$ is stable under the adjoint action on $\mathfrak{g}$ by the split torus $\mathrm{T} \subset \mathrm{Q}$, by working Zariski-locally on S around s we can arrange that $\mathfrak{q}$ is a direct sum of $\mathfrak{t}$ and weight spaces $\mathfrak{g}_a$ for some roots $a \in \Phi \subset \mathrm{M}$. Since $\mathrm{Q}_{\overline{s}} = \mathrm{P}_{\mathrm{G}}(\lambda)_{\overline{s}}$, the a which arise in this way are precisely those that satisfy $\lambda(a) \geqslant 0$. Hence, the smooth closed subgroup $\mathrm{P}_{\mathrm{Q}}(\lambda) \subset \mathrm{Q}$ has full Lie algebra, forcing $\mathrm{P}_{\mathrm{Q}}(\lambda) = \mathrm{Q}$.

By the functoriality in Proposition 4.1.10(2) (applied to the $\mathbf{G}_{\mathrm{m}}$-equivariant homomorphism $\mathrm{Q} \to \mathrm{G}$), the map $\mathrm{Q} \to \mathrm{G}$ factors through $\mathrm{P}_{\mathrm{G}}(\lambda)$. The resulting map of smooth S-affine S-groups $j : \mathrm{Q} \to \mathrm{P}_{\mathrm{G}}(\lambda)$ is an isomorphism on Lie algebras inside $\mathfrak{g}$. On geometric fibers over S the map j induces a closed immersion between smooth connected affine groups, so the isomorphism property on Lie algebras forces j to be an isomorphism between geometric fibers over S. Hence, j is an isomorphism. In particular, Q is closed in G. □

Remark 5.2.4. — In the Borel–Tits structure theory for connected reductive groups over fields, the dynamic description $\mathrm{P}_{\mathrm{G}}(\lambda)$ of Q in terms of a cocharacter $\lambda : \mathbf{G}_{\mathrm{m}} \to \mathrm{T}$ over the base field is valid without a split hypothesis on (G, T) (see [**CGP**, Prop. 2.2.9]). Thus, if $\mathrm{S} = \operatorname{Spec} \mathrm{R}$ for a henselian local ring R then we can remove the split hypothesis on T in Proposition 5.2.3. Indeed, it suffices to show that any cocharacter $\lambda_0 : \mathbf{G}_{\mathrm{m}} \to \mathrm{T}_0$ over the residue field k lifts to a cocharacter $\mathbf{G}_{\mathrm{m}} \to \mathrm{T}$ over R. In terms of the étale sheaf E dual to T, this is precisely the surjectivity of $\mathrm{E}(\mathrm{R}) \to \mathrm{E}(k)$, which in turn is an immediate consequence of the henselian property of R. Removing the split hypothesis over more general rings (and hence removing it from the end of Proposition 5.2.3 over more general schemes S) is rather more delicate, and we will return to this near the end of § 5.4.

We now get many nice consequences, which we give in a series of corollaries.

Corollary 5.2.5. — *Let* $\mathrm{G} \to \mathrm{S}$ *be a reductive group scheme, and* $\mathrm{P} \subset \mathrm{G}$ *a parabolic subgroup. There is a unique smooth closed normal* S*-subgroup* $\mathscr{R}_u(\mathrm{P}) \subset \mathrm{P}$ *whose geometric fiber* $\mathscr{R}_u(\mathrm{P})_{\overline{s}}$ *coincides with the unipotent radical* $\mathscr{R}_u(\mathrm{P}_{\overline{s}})$ *for all* $s \in \mathrm{S}$.

The quotient $\mathrm{P}/\mathscr{R}_u(\mathrm{P})$ *is represented by a reductive group scheme, and any surjective homomorphism from* P *onto a reductive* S*-group uniquely factors through* $\mathrm{P}/\mathscr{R}_u(\mathrm{P})$.

We call $\mathscr{R}_u(\mathrm{P})$ the *unipotent radical* of P. The proof of Corollary 5.2.5 uses the dynamic method; an alternative is in [**SGA3**, XXII, 5.11.3, 5.11.4(ii)].

Proof. — In view of the uniqueness we may work étale-locally on S, so by Proposition 5.2.3 we may arrange that G contains a split maximal torus T and that $\mathrm{P} = \mathrm{P}_{\mathrm{G}}(\lambda) = \mathrm{Z}_{\mathrm{G}}(\lambda) \ltimes \mathrm{U}_{\mathrm{G}}(\lambda)$ for some cocharacter $\lambda : \mathbf{G}_{\mathrm{m}} \to \mathrm{T}$ (so $\mathrm{T} \subset \mathrm{Z}_{\mathrm{G}}(\lambda) \subset \mathrm{P}_{\mathrm{G}}(\lambda) = \mathrm{P}$). It is clear that $\mathrm{U}_{\mathrm{G}}(\lambda)$ satisfies the requirements to be $\mathscr{R}_u(\mathrm{P})$ except possibly for the uniqueness and the universal mapping property relative to homomorphisms from P onto reductive S-groups.

Suppose that $\mathrm{N} \subset \mathrm{P}$ is a smooth closed normal S-subgroup such that $\mathrm{N}_{\overline{s}} = \mathscr{R}_u(\mathrm{P}_{\overline{s}})$ for all $s \in \mathrm{S}$. Normality of N in P implies that N is normalized by T (as $\mathrm{T} \subset \mathrm{P}$), so it makes sense to form the smooth closed S-subgroup $\mathrm{U}_{\mathrm{N}}(\lambda)$ in N. The $\overline{s}$-fiber of $\mathrm{U}_{\mathrm{N}}(\lambda)$ has the same Lie algebra as N (since $\mathrm{N}_{\overline{s}} = \mathscr{R}_u(\mathrm{P}_{\overline{s}}) = \mathrm{U}_{\mathrm{G}_{\overline{s}}}(\lambda_{\overline{s}})$), so the closed immersion $\mathrm{U}_{\mathrm{N}}(\lambda) \hookrightarrow \mathrm{N}$ between smooth S-groups is an isomorphism on Lie algebras and hence an isomorphism on fibers (due to the *connectedness* of each $\mathrm{N}_{\overline{s}}$). It follows that $\mathrm{N} = \mathrm{U}_{\mathrm{N}}(\lambda) \subset \mathrm{U}_{\mathrm{G}}(\lambda)$. But by hypothesis this inclusion between smooth closed S-subgroups of G induces an equality on geometric fibers over S, so $\mathrm{N} = \mathrm{U}_{\mathrm{G}}(\lambda)$. This establishes the uniqueness of $\mathscr{R}_u(\mathrm{P})$.

Next, consider a surjective homomorphism $f : \mathrm{P} \twoheadrightarrow \overline{\mathrm{G}}$ onto a reductive S-group. We want f to kill $\mathscr{R}_u(\mathrm{P})$. Working étale-locally on S, by Proposition 5.2.3 we may arrange that $\mathrm{P} = \mathrm{P}_{\mathrm{G}}(\lambda)$, so $\mathscr{R}_u(\mathrm{P}) = \mathrm{U}_{\mathrm{G}}(\lambda)$. By Proposition 4.1.10(2), f carries $\mathrm{U}_{\mathrm{G}}(\lambda)$ onto $\mathrm{U}_{\overline{\mathrm{G}}}(f \circ \lambda)$ (so f makes $\mathrm{U}_{\mathrm{G}}(\lambda)$ an fppf cover of $\mathrm{U}_{\overline{\mathrm{G}}}(f \circ \lambda)$, by the fibral flatness criterion). But $\mathrm{U}_{\mathrm{G}}(\lambda)$ is normal in P, so since f and its restriction $\mathrm{U}_{\mathrm{G}}(\lambda) \to \mathrm{U}_{\overline{\mathrm{G}}}(f \circ \lambda)$ are fppf covers, it follows that $\mathrm{U}_{\overline{\mathrm{G}}}(f \circ \lambda)$ is normal in $\overline{\mathrm{G}}$. The S-smooth $\mathrm{U}_{\overline{\mathrm{G}}}(f \circ \lambda)$ has connected unipotent fibers, so normality in the reductive S-group $\overline{\mathrm{G}}$ forces $\mathrm{U}_{\overline{\mathrm{G}}}(f \circ \lambda)$ to have relative dimension 0 and therefore be trivial. This says exactly that f kills $\mathscr{R}_u(\mathrm{P})$. $\square$

Remark 5.2.6. — As an application of Corollary 5.2.5, we can construct many smooth closed subgroups of G directly spanned in any order by certain collections of root groups. This rests on the notion of *parabolicity* for subsets Ψ of a root system Φ; see [**Bou2**, VI, § 1.7, Def. 4]. These are the subsets

$$\Phi_{\lambda \geqslant 0} := \{a \in \Phi \mid \lambda(a) \geqslant 0\}$$

for linear forms λ on the $\mathbf{Q}$-span of Φ (see [**CGP**, Prop. 2.2.8] for a proof), and each contains a positive system of roots. (See Definition 1.4.5ff.)

Consider subsets $\Psi \subset \Phi$ whose complement is parabolic; i.e., $\Psi = \Phi_{\lambda<0}$ for some λ (or equivalently $\Psi = \Phi_{\lambda>0}$ for some λ). An interesting example of such a subset for reduced Φ is $\Psi = \Phi^+ - \{a\}$ for a positive system of roots Φ^+ and a root a in the base Δ of Φ^+. To see that this Ψ has the asserted form, we may assume $\#\Delta > 1$ (as otherwise Ψ is empty, a trivial case). Enumerating Δ as $\{a = a_1, \ldots, a_m\}$, we have $\Psi = \Phi_{\lambda>0}$ where $\lambda := \sum_j a_j^*$ for the basis $\{a_j^*\}$ dual to the basis $\{a_j\}$ of the $\mathbf{Q}$-span of Ψ.

For a split reductive group $(\mathrm{G}, \mathrm{T}, \mathrm{M})$ and the complement Ψ of a parabolic subset in the associated root system Φ, so $\Psi = \Phi_{\lambda>0}$ for some λ, we claim that $\mathrm{U}_{\mathrm{G}}(\lambda)$ coincides with the S-group U_Ψ from Proposition 5.1.16 (so it depends only on Ψ, not on the choice of λ). Since $\mathrm{Lie}(\mathrm{U}_{\mathrm{G}}(\lambda))$ is spanned by the weight spaces $\mathfrak{g}_a$ for such a, as is $\mathrm{Lie}(\mathrm{U}_\Psi)$, if there is an inclusion $\mathrm{U}_\Psi \subset \mathrm{U}_{\mathrm{G}}(\lambda)$ as smooth closed S-subgroups of G then it must be an equality (as we may check on geometric fibers, where smoothness and connectedness reduces the problem to the known equality of Lie algebras). Hence, it suffices to show that $\mathrm{U}_{\mathrm{G}}(\lambda)$ contains U_a for all $a \in \Phi_{\lambda>0}$. The explicit description of the T-conjugation action on U_a and the functorial definition of $\mathrm{U}_{\mathrm{G}}(\lambda)$ imply that $\mathrm{U}_a \subset \mathrm{U}_{\mathrm{G}}(\lambda)$ since $\langle \lambda, a \rangle > 0$.

The dependence of $\mathrm{U}_{\mathrm{G}}(\lambda)$ on Ψ rather than on λ can be proved in another way: this S-group is the unipotent radical of the parabolic subgroup $\mathrm{P}_{\mathrm{G}}(\lambda)$ containing T (in the sense of Corollary 5.2.5), and $\mathrm{P}_{\mathrm{G}}(\lambda)$ only depends on Ψ due to Corollary 5.2.7(2) below.

Corollary 5.2.7. — *Let* $\mathrm{G} \to \mathrm{S}$ *be a reductive group,* $s \in \mathrm{S}$ *a point, and* $\mathrm{P}, \mathrm{Q} \subset \mathrm{G}$ *parabolic subgroups.*

1. *If* $\mathrm{P}_{\overline{s}}$ *is conjugate to* $\mathrm{Q}_{\overline{s}}$ *for some* $s \in \mathrm{S}$ *then there exists an étale neighborhood* U *of* (S, s) *such that* P_{U} *is* $\mathrm{G}(\mathrm{U})$*-conjugate to* Q_{U}*. In particular, if* P *and* Q *are conjugate over all geometric points of* S *then they are conjugate étale-locally on* S.
2. *Assume* P *and* Q *contain a common maximal torus* T *of* G*. If* $\mathrm{Lie}(\mathrm{Q}_{\overline{s}}) \subset \mathrm{Lie}(\mathrm{P}_{\overline{s}})$ *inside* $\mathfrak{g}_{\overline{s}}$ *then* $\mathrm{Q}_{\mathrm{V}} \subset \mathrm{P}_{\mathrm{V}}$ *for some Zariski-open neighborhood* V *of* s *in* S*. In particular, if* $\mathrm{Lie}(\mathrm{P}_{\overline{s}}) = \mathrm{Lie}(\mathrm{Q}_{\overline{s}})$ *inside* $\mathfrak{g}_{\overline{s}}$ *then* P *and* Q *coincide over a Zariski-open neighborhood of* s *in* S.

This result is a special case of [**SGA3**, XXII, 5.3.7, 5.3.11].

Proof. — We first treat (2). Since étale maps are open, we may pass to an étale neighborhood of (S, s) to split T. By working Zariski-locally around s we can arrange that (G, T) arises from a split triple $(\mathrm{G}, \mathrm{T}, \mathrm{M})$. Further Zariski localization brings us to the case $\mathrm{P} = \mathrm{P}_{\mathrm{G}}(\lambda)$ and $\mathrm{Q} = \mathrm{P}_{\mathrm{G}}(\mu)$ for cocharacters $\lambda, \mu : \mathbf{G}_{\mathrm{m}} \rightrightarrows \mathrm{T}$ arising from $\mathrm{M}^{\vee}$ (Proposition 5.2.3). The containment $\mathrm{Lie}(\mathrm{Q}_{\overline{s}}) \subset \mathrm{Lie}(\mathrm{P}_{\overline{s}})$ implies that $\Phi_{\mu \geqslant 0} \subset \Phi_{\lambda \geqslant 0}$ inside Φ, as these are precisely the roots that appear in the respective Lie algebras of $\mathrm{Q}_{\overline{s}}$ and $\mathrm{P}_{\overline{s}}$ inside $\mathfrak{g}_{\overline{s}}$. But the containment $\Phi_{\mu \geqslant 0} \subset \Phi_{\lambda \geqslant 0}$ implies that the smooth closed subgroup $\mathrm{P}_{\mathrm{Q}}(\lambda) \subset \mathrm{Q}$ has full Lie algebra, so $\mathrm{Q} = \mathrm{P}_{\mathrm{Q}}(\lambda) \subset \mathrm{P}_{\mathrm{G}}(\lambda) = \mathrm{P}$ inside G.

Now consider (1). By working étale-locally around (S, s) we may assume P contains a split maximal torus T of G and that Q contains a split maximal torus T' of G. By Corollary 2.1.5 we may pass to a further étale neighborhood so that T and T' are $\mathrm{G}(\mathrm{S})$-conjugate, so applying such conjugacy brings us to the case that $\mathrm{T}' = \mathrm{T}$; i.e., P and Q contain a common split maximal torus T. Further Zariski localization provides a split triple $(\mathrm{G}, \mathrm{T}, \mathrm{M})$. By hypothesis there exists $g \in \mathrm{G}(\overline{s})$ such that $g\mathrm{P}_{\overline{s}}g^{-1} = \mathrm{Q}_{\overline{s}}$. Since $\mathrm{T}_{\overline{s}}$ and $g\mathrm{T}_{\overline{s}}g^{-1}$ are maximal tori in $\mathrm{Q}_{\overline{s}}$, there exists $q \in \mathrm{Q}(\overline{s})$ such that $qg\mathrm{T}_{\overline{s}}g^{-1}q^{-1} = \mathrm{T}_{\overline{s}}$, so in other words $qg \in \mathrm{N}_{\mathrm{G}}(\mathrm{T})(\overline{s})$ and its class $w_0 \in \mathrm{W}_{\mathrm{G}}(\mathrm{T})(\overline{s})$ carries $\mathrm{P}_{\overline{s}}$ to $\mathrm{Q}_{\overline{s}}$.

The S-group $\mathrm{W}_{\mathrm{G}}(\mathrm{T}) \to \mathrm{S}$ is the finite constant group $\mathrm{W}(\Phi)_{\mathrm{S}}$ (Proposition 5.1.6), so w_0 spreads over a Zariski-open neighborhood of (S, s). Further Zariski-localization then lifts the resulting point of $\mathrm{W}_{\mathrm{G}}(\mathrm{T})(\mathrm{S}) = (\mathrm{N}_{\mathrm{G}}(\mathrm{T})/\mathrm{T})(\mathrm{S})$ to $\mathrm{N}_{\mathrm{G}}(\mathrm{T})(\mathrm{S})$ since T is S-split (see Corollary 5.1.11). Passing to such a neighborhood yields some $n \in \mathrm{N}_{\mathrm{G}}(\mathrm{T})(\mathrm{S})$ such that $n\mathrm{P}n^{-1}$ and Q have the same s-fiber inside G_s. But these contain T, so by (2) there exists a Zariski-open neighborhood of s in S over which $n\mathrm{P}n^{-1}$ and Q coincide. □

Corollary 5.2.8. — *For any parabolic subgroup* P *in a reductive group scheme* G*, the normalizer functor* $\underline{\mathrm{N}}_{\mathrm{G}}(\mathrm{P})$ *is represented by* P.

The quotient sheaf G/P *for the étale topology on the category of* S*-schemes coincides with the functor of subgroups of* G *étale-locally conjugate to* P*, and it* *is represented by a smooth proper* S*-scheme equipped with a canonical* S*-ample line bundle. Explicitly, if* $\mathscr{P} \subset \mathrm{G} \times (\mathrm{G}/\mathrm{P})$ *is the "universal parabolic subgroup locally conjugate to* P*" then* $\det(\mathrm{Lie}(\mathscr{P}))^*$ *is* S*-ample on* G/P*.*

The self-normalizer property in the special case $\mathrm{G} = \mathrm{PGL}_2$ was handled in the proof of Proposition 4.2.7, using the smallness of $\dim \mathrm{PGL}_2$ to give a simple argument. The general case is proved in [**SGA3**, XXII, 5.8.5] by a method different from the one below.

Proof. — Since the S-smooth P is closed in G and has connected fibers, Proposition 2.1.6 ensures that $\underline{\mathrm{N}}_{\mathrm{G}}(\mathrm{P})$ is represented by a finitely presented closed subscheme $\mathrm{N}_{\mathrm{G}}(\mathrm{P})$ of G. Beware that whereas normalizers of multiplicative type subgroups in smooth affine groups are always smooth (Proposition 2.1.2), in the setting of Proposition 2.1.6 not even flatness of the normalizer is assured. Nonetheless, we do have an inclusion $\mathrm{P} \subset \mathrm{N}_{\mathrm{G}}(\mathrm{P})$ as finitely presented closed subschemes of G, with P flat (even smooth) over S. Hence, by Lemma B.3.1, this inclusion is an equality if it is so on geometric fibers over S. That is, we are reduced to the classical case $\mathrm{S} = \mathrm{Spec}\, k$ for an algebraically closed field k.

By the classical theory, $\mathrm{P}(k) = \mathrm{N}_{\mathrm{G}}(\mathrm{P})(k)$ inside $\mathrm{G}(k)$. Hence, the closed k-subgroup scheme $\mathrm{N}_{\mathrm{G}}(\mathrm{P})$ in G has the same dimension as P. We need to establish $\mathrm{P} = \mathrm{N}_{\mathrm{G}}(\mathrm{P})$ as schemes. It suffices to verify that $\mathrm{N}_{\mathrm{G}}(\mathrm{P})$ is smooth, or equivalently (since its dimension is $\dim \mathrm{P}$) that $\mathrm{Lie}(\mathrm{N}_{\mathrm{G}}(\mathrm{P})) = \mathrm{Lie}(\mathrm{P})$ inside $\mathfrak{g}$. Consider the explicit description $\mathrm{P} = \mathrm{P}_{\mathrm{G}}(\lambda)$ for some cocharacter $\lambda : \mathbf{G}_{\mathrm{m}} \to \mathrm{G}$ valued in a maximal torus T of G that necessarily lies in P (since $\mathrm{P}_{\mathrm{G}}(\lambda) \supset \mathrm{Z}_{\mathrm{G}}(\lambda) \supset \mathrm{Z}_{\mathrm{G}}(\mathrm{T}) = \mathrm{T}$). If the T-equivariant inclusion $\mathrm{Lie}(\mathrm{P}) \subset \mathrm{Lie}(\mathrm{N}_{\mathrm{G}}(\mathrm{P}))$ is not an equality then there is some $\mathrm{X} \in \mathrm{Lie}(\mathrm{N}_{\mathrm{G}}(\mathrm{P}))$ not in $\mathrm{Lie}(\mathrm{P}) = \mathfrak{g}_{\lambda \geqslant 0}$ that is a T-eigenvector for some weight $a : \mathrm{T} \to \mathbf{G}_{\mathrm{m}}$, so $\langle a, \lambda \rangle < 0$. In particular, $a \neq 1$, so there is $t \in \mathrm{T}(k)$ such that $a(t) \neq 1$. Since $\mathrm{X} \in \mathrm{Lie}(\mathrm{N}_{\mathrm{G}}(\mathrm{P})) = \ker(\mathrm{N}_{\mathrm{G}}(\mathrm{P})(k[\epsilon]) \to \mathrm{N}_{\mathrm{G}}(\mathrm{P})(k))$, $\mathrm{Ad}_{\mathrm{G}}(h)(\mathrm{X}) - \mathrm{X} \in \mathrm{Lie}(\mathrm{P})$ for all $h \in \mathrm{P}(k)$. Taking $h = t$, $\mathrm{Ad}_{\mathrm{G}}(t)(\mathrm{X}) - \mathrm{X} = (a(t) - 1)\mathrm{X}$ with $a(t) - 1 \in k^{\times}$. This is a contradiction, so there is no such X.

The established equality $\mathrm{P} = \mathrm{N}_{\mathrm{G}}(\mathrm{P})$ and Theorem 2.3.6 imply the assertions concerning the functorial meaning of G/P and its existence as a smooth proper S-scheme equipped with a canonical S-ample line bundle. □

Corollary 5.2.9. — *Let* $\mathrm{G} \to \mathrm{S}$ *be a reductive group scheme. The functor on* S*-schemes*

$$\underline{\mathrm{Par}}_{\mathrm{G}/\mathrm{S}} : \mathrm{S}' \rightsquigarrow \{\textit{parabolic subgroups of } \mathrm{G}_{\mathrm{S}'}\}$$

is represented by a smooth proper S*-scheme* $\mathrm{Par}_{\mathrm{G}/\mathrm{S}}$ *equipped with the canonical* S*-ample line bundle* $\det(\mathrm{Lie}(\mathscr{P}))^*$*, where* $\mathscr{P} \subset \mathrm{G} \times \mathrm{Par}_{\mathrm{G}/\mathrm{S}}$ *is the universal parabolic subgroup of* G*.*

The existence aspect of this corollary is part of [**SGA3**, XXVI, 3.3(ii)]. Over an algebraically closed field k the proof shows $\mathrm{Par}_{\mathrm{G}/k} = \coprod(\mathrm{G}/\mathrm{P}_i)$ where P_i varies through representatives of the finite set of conjugacy classes of parabolic subgroups of G (parameterized by the set of parabolic subsets of the root system Φ for G containing a fixed positive system of roots Φ^+, or equivalently by the set of subsets of the base Δ of Φ^+). In particular, in the classical case $\mathrm{Par}_{\mathrm{G}/k}$ is generally disconnected.

Proof. — Since we aim to construct $\mathrm{Par}_{\mathrm{G/S}}$ as a proper S-scheme equipped with a canonical S-ample line bundle arising from the universal parabolic subgroup, by effective descent in the presence of a relatively ample line bundle it suffices to work étale-locally on S. Thus, by Theorem 3.2.6 we can assume that G admits a split maximal torus T.

The isomorphism class of the fibral root system for (G, T) is Zariski-locally constant on S, so we can arrange that $\mathrm{S} \neq \varnothing$ and there exists a split 4-tuple $(\mathrm{G}, \mathrm{T}, \mathrm{M}, \Phi)$. Choose a positive system of roots $\Phi^+ \subset \Phi$ and let $\{\lambda_j\}_{j\in \mathrm{J}} \subset \mathrm{M}^\vee$ be a finite set of cocharacters such that $\Phi_{\lambda_j \geqslant 0}$ varies (without repetition) through the finitely many parabolic subsets of Φ containing Φ^+. By Corollary 5.2.7, parabolic subgroups of G are precisely the subgroups conjugate to some $\mathrm{P}_{\mathrm{G}}(\lambda_j)$ étale-locally on the base. Now we apply Corollary 5.2.8 to every $\mathrm{P}_{\mathrm{G}}(\lambda_j)$ to conclude that $\coprod_j(\mathrm{G}/\mathrm{P}_{\mathrm{G}}(\lambda_j))$ represents $\underline{\mathrm{Par}}_{\mathrm{G/S}}$. (Keep in mind that the functor of points of a disjoint union $\coprod_{i\in \mathrm{I}} \mathrm{X}_i$ of S-schemes X_i indexed by a set I assigns to every S-scheme S′ a disjoint union decomposition $\coprod \mathrm{S}'_i$ of S′ indexed by I and a point in $\mathrm{X}_i(\mathrm{S}'_i)$ for each $i \in \mathrm{I}$.)

Letting $\mathscr{P}$ denote the universal parabolic subgroup over $\mathrm{Par}_{\mathrm{G/S}}$, Theorem 2.3.6 ensures that the line bundle $\det(\mathrm{Lie}(\mathscr{P}))^*$ on $\mathrm{Par}_{\mathrm{G/S}}$ is S-ample. □

Definition 5.2.10. — A *Borel subgroup* of a reductive group scheme $\mathrm{G} \to \mathrm{S}$ is a parabolic subgroup $\mathrm{P} \subset \mathrm{G}$ such that $\mathrm{P}_{\overline{s}}$ is a Borel subgroup of $\mathrm{G}_{\overline{s}}$ for all $s \in \mathrm{S}$. A reductive group $\mathrm{G} \to \mathrm{S}$ is *quasi-split* over S if it admits a Borel subgroup scheme over S. (The relative notion of "quasi-split" is defined with additional requirements in [**SGA3**, XXIV, 3.9], especially involving the scheme of Dynkin diagrams, but for semi-local S the two notions coincide [**SGA3**, XXIV, 3.9.1]. For our purposes, the definition we have given will be sufficient.)

Theorem 5.2.11. — *Let* $\mathrm{G} \to \mathrm{S}$ *be a reductive group scheme.*

1. *Let* $\mathrm{P} \subset \mathrm{G}$ *be a parabolic subgroup. If* $\mathrm{P}_{\overline{s}}$ *is a Borel subgroup of* $\mathrm{G}_{\overline{s}}$ *then* P_{U} *is a Borel subgroup of* G_{U} *for some open* $\mathrm{U} \subset \mathrm{S}$ *around* s*, and the open locus of* $s \in \mathrm{S}$ *such that* $\mathrm{P}_{\overline{s}}$ *is a Borel subgroup is also closed.*
2. *Any two Borel subgroups of* G *are conjugate étale-locally on* S.

3. *The functor on* S*-schemes*
$$\underline{\mathrm{Bor}}_{\mathrm{G/S}} : \mathrm{S}' \rightsquigarrow \{\textit{Borel subgroups of } \mathrm{G}_{\mathrm{S}'}\}$$
is represented by a smooth proper S*-scheme* $\mathrm{Bor}_{\mathrm{G/S}}$ *equipped with the canonical* S*-ample line bundle* $\det(\mathrm{Lie}(\mathscr{B}))^*$*, where* $\mathscr{B} \subset \mathrm{G} \times \mathrm{Bor}_{\mathrm{G/S}}$ *is the universal Borel subgroup of* G*.*

The existence and properties of $\mathrm{Bor}_{\mathrm{G/S}}$ are part of [**SGA3**, XXII, 5.8.3(i)].

Proof. — Fiber dimension considerations for the smooth map $\mathrm{P} \to \mathrm{S}$ settle (1), since the isomorphism class of the fibral root datum for $\mathrm{G} \to \mathrm{S}$ is locally constant over S. Part (2) follows from Corollary 5.2.7. Finally, $\mathrm{Bor}_{\mathrm{G/S}}$ is the open and closed subscheme of $\mathrm{Par}_{\mathrm{G/S}}$ over which fibers of the universal parabolic subgroup are Borel subgroups (use part (1)). $\square$

Proposition 5.2.12. — *Let* $\mathrm{G} \to \mathrm{S}$ *be a reductive group scheme, and* $\mathrm{T} \subset \mathrm{G}$ *a maximal torus. If* $\mathrm{B} \subset \mathrm{G}$ *is a Borel subgroup containing* T *then there exists a unique Borel subgroup* $\mathrm{B}' \subset \mathrm{G}$ *satisfying* $\mathrm{B}' \bigcap \mathrm{B} = \mathrm{T}$*.*

Proof. — The uniqueness assertion allows us to work étale-locally on S, so we may assume that $\mathrm{S} \neq \varnothing$ and (G, T) arises from a split triple $(\mathrm{G}, \mathrm{T}, \mathrm{M})$ such that $\mathrm{B} = \mathrm{P}_{\mathrm{G}}(\lambda)$ for some $\lambda \in \mathrm{M}^\vee$. The inclusion $\mathrm{T} \subset \mathrm{Z}_{\mathrm{G}}(\lambda)$ is an equality (by checking on geometric fibers) and $\mathrm{B}' := \mathrm{P}_{\mathrm{G}}(-\lambda) = \mathrm{Z}_{\mathrm{G}}(-\lambda) \ltimes \mathrm{U}_{\mathrm{G}}(-\lambda) = \mathrm{T} \ltimes \mathrm{U}_{\mathrm{G}}(-\lambda)$ is a Borel subgroup of G containing T. By Theorem 4.1.7(4), $\mathrm{B}' \bigcap \mathrm{B} = \mathrm{Z}_{\mathrm{G}}(\lambda) = \mathrm{T}$. To establish uniqueness of B', note that if B'' is another such Borel subgroup then $\mathrm{B}'' \supset \mathrm{T}$, so Corollary 5.2.7(2) reduces uniqueness to the case of geometric fibers over S, which is Proposition 1.4.4. $\square$

In the split case, we get a Zariski-local conjugacy result for Borel subgroups:

Corollary 5.2.13. — *Let* $(\mathrm{G}, \mathrm{T}, \mathrm{M})$ *be a split reductive group over a non-empty scheme* S*, and let* B *be a Borel subgroup of* G*.*

1. *Every point* $s \in \mathrm{S}$ *admits an open neighborhood* U *such that some* $\mathrm{G}(\mathrm{U})$*-conjugate of* B_{U} *contains* T_{U}*.*
2. *Any two Borel subgroups of* G *that contain* T *are* $\mathrm{N}_{\mathrm{G}}(\mathrm{T})$*-conjugate Zariski-locally over* S*.*

Proof. — Let $\Phi \subset \mathrm{M} - \{0\}$ be the root system, and $\Phi^+ \subset \Phi$ a positive system of roots, so $\Phi_{\lambda \geqslant 0} = \Phi^+$ for some $\lambda \in \mathrm{M}^\vee$. Consideration of Lie algebras shows that the parabolic subgroup $\mathrm{B}' := \mathrm{P}_{\mathrm{G}}(\lambda)$ containing T is a Borel subgroup. Since B and B' are G-conjugate étale-locally on S, $\mathrm{Transp}_{\mathrm{G}}(\mathrm{B}, \mathrm{B}')$ provided by Proposition 2.1.6 is a torsor for $\mathrm{N}_{\mathrm{G}}(\mathrm{B}')$ in the étale topology, and $\mathrm{N}_{\mathrm{G}}(\mathrm{B}') = \mathrm{B}'$ by Corollary 5.2.8. Thus, to prove (1) it is enough to construct Zariski-local sections for any B'-torsor in the étale topology on S.

We will construct a composition series for the S-group B' consisting of smooth closed S-subgroups such that the successive quotients are $\mathbf{G}_{\mathrm{a}}$ or $\mathbf{G}_{\mathrm{m}}$. Since torsors for $\mathbf{G}_{\mathrm{a}}$ or $\mathbf{G}_{\mathrm{m}}$ in the étale topology are always Zariski-locally trivial, we will then get the desired Zariski-local sections since the composition series provides a succession of exact sequences in the étale topology. To build the composition series for B', we use its description as $\mathrm{P}_{\mathrm{G}}(\lambda) = \mathrm{Z}_{\mathrm{G}}(\lambda) \ltimes \mathrm{U}_{\mathrm{G}}(\lambda)$. The choice of λ implies that the inclusion $\mathrm{T} \subset \mathrm{Z}_{\mathrm{G}}(\lambda)$ is an equality, so since T is S-split we are reduced to considering $\mathrm{U}_{\mathrm{G}}(\lambda)$. The desired composition series for $\mathrm{U}_{\mathrm{G}}(\lambda) = \mathscr{R}_u(\mathrm{P}_{\mathrm{G}}(\lambda))$ (see Corollary 5.2.5) is given by the subgroups $\mathrm{U}_{\Phi_{\lambda \geqslant n}}$ as in Proposition 5.1.16(2).

Now we turn to the proof of (2). By Corollary 5.2.7, it suffices to show that for any $s \in \mathrm{S}$ there exists an open neighborhood U of s and $n \in \mathrm{N}_{\mathrm{G}}(\mathrm{T})(\mathrm{U})$ such that $n\mathrm{B}_{\mathrm{U}}n^{-1}$ and B'_{U} have the same s-fiber. Since T is split, by Corollary 5.1.11 the map $\mathrm{N}_{\mathrm{G}}(\mathrm{T}) \to \mathrm{W}_{\mathrm{G}}(\mathrm{T})$ is surjective for the Zariski topology and by Proposition 5.1.6 the finite S-group $\mathrm{W}_{\mathrm{G}}(\mathrm{T})$ is constant (so it has Zariski-local sections through any point of a fiber over S). Hence, we just need to recall the fact from the classical theory that the geometric fiber $\mathrm{W}_{\mathrm{G}}(\mathrm{T})_{\overline{s}} = \mathrm{W}_{\mathrm{G}_{\overline{s}}}(\mathrm{T}_{\overline{s}})$ acts transitively on the set of Borel subgroups of $\mathrm{G}_{\overline{s}}$ that contain $\mathrm{T}_{\overline{s}}$. □

Corollary 5.2.14. — *Let* G *be a reductive group over a henselian local ring* R *with finite residue field. Then* G *is quasi-split and it becomes split over a finite étale extension of* R.

See Definition 5.2.10 for the notion "quasi-split" for reductive group schemes.

Proof. — By [**Bo91**, 16.6] (or Exercise 6.5.6), the special fiber G_0 over the finite residue field k admits a Borel subgroup B_0. By Corollary 3.2.7, there exists a finite (separable) extension k'/k such that $(\mathrm{G}_0)_{k'}$ admits a split maximal torus T'_0. Since $\mathrm{Bor}_{\mathrm{G}/\mathrm{R}}$ is R-smooth and R is henselian local, any k-point in the special fiber lifts to an R-point. Hence, B_0 lifts to a Borel subgroup of G. The scheme $\mathrm{Tor}_{\mathrm{G}_0/k}$ has a k'-point corresponding to T'_0, and this may be viewed as a k'-point of $\mathrm{Tor}_{\mathrm{G}/\mathrm{R}}$. But $\mathrm{Tor}_{\mathrm{G}/\mathrm{R}}$ is a smooth scheme over the henselian local ring R, so any k'-point must lift to an R'-point, where $\mathrm{R} \to \mathrm{R}'$ is the local finite étale extension inducing the residual extension k'/k. Hence, $\mathrm{Tor}_{\mathrm{G}/\mathrm{R}}(\mathrm{R}') \neq \varnothing$, so $\mathrm{G}_{\mathrm{R}'}$ contains an R'-torus that lifts the k'-split T'_0 and hence is R'-split too (due to the henselian property of R'). □

Corollary 5.2.14 is very useful when R is the valuation ring of a non-archimedean local field. It says nothing about the quasi-split property for reductive groups given only over the fraction field of R, and conversely Steinberg's theorem that reductive groups over the maximal unramified extension of Frac(R) are quasi-split does not imply anything in the direction of the quasi-split property over R for G as in the corollary.

5.3. Applications to derived groups and closed immersions. — In the classical theory, there is a good notion of derived group for any smooth affine group. In the relative theory over a scheme, a new idea is needed to construct a satisfactory analogue (at least in the reductive case). The structure of the open cell (Theorem 5.1.13) and the construction of parabolic subgroups via cocharacters in the split case (Proposition 5.2.3) will enable us to build the "derived group" of any reductive group scheme (as in [**SGA3**, XXII, 6.1–6.2]):

Theorem 5.3.1. — *Let* $\mathrm{G} \to \mathrm{S}$ *be a reductive group scheme. There is a unique semisimple closed normal* S*-subgroup* $\mathscr{D}(\mathrm{G}) \subset \mathrm{G}$ *such that* $\mathrm{G}/\mathscr{D}(\mathrm{G})$ *is a torus. Moreover,* $\mathscr{D}(\mathrm{G})$ *represents the fppf-sheafification of the "commutator subfunctor"* $\mathrm{S}' \rightsquigarrow [\mathrm{G}(\mathrm{S}'), \mathrm{G}(\mathrm{S}')]$ *on the category of* S*-schemes.*

In particular, the quotient map $\mathrm{G} \to \mathrm{G}/\mathscr{D}(\mathrm{G})$ *is initial among all homomorphisms from* G *to an abelian sheaf, and the formation of* $\mathscr{D}(\mathrm{G})$ *commutes with any base change on* S*.*

Proof. — By the asserted uniqueness, we may use étale descent to arrange that S is non-empty and $\mathrm{T} = \mathrm{D}_{\mathrm{S}}(\mathrm{M})$ for a finite free $\mathbf{Z}$-module M such that $(\mathrm{G}, \mathrm{T}, \mathrm{M})$ is a split triple encoding a root datum $(\mathrm{M}, \Phi, \mathrm{M}^{\vee}, \Phi^{\vee})$.

Pick a positive system of roots Φ^{+} in Φ. Consider the resulting open cell

$$\Omega := \mathrm{U}_{-} \times \mathrm{T} \times \mathrm{U}_{+} \subset \mathrm{G}$$

where $\mathrm{U}_{+} = \mathrm{U}_{\Phi^{+}}$ and $\mathrm{U}_{-} = \mathrm{U}_{-\Phi^{+}}$. Let $\mathrm{T}' \subset \mathrm{T}$ be the split subtorus "generated" by the coroots; i.e., $\mathrm{T}' = \mathrm{D}_{\mathrm{S}}(\mathrm{M}/\mathrm{L})$ where $\mathrm{L} \subset \mathrm{M}$ is the saturated sublattice that is the annihilator of the coroot lattice $\mathbf{Z}\Phi^{\vee} \subset \mathrm{M}^{\vee}$. This is the minimal subtorus of T through which all coroots factor, and for a geometric point $\overline{s}$ of S the fiber $\mathrm{T}'_{\overline{s}}$ is the subtorus of $\mathrm{T}_{\overline{s}}$ generated by the coroots for $(\mathrm{G}_{\overline{s}}, \mathrm{T}_{\overline{s}})$. The idea is to show that $\Omega' := \mathrm{U}_{-} \times \mathrm{T}' \times \mathrm{U}_{+}$ is the open cell for a split closed semsimple S-subgroup of G that will be $\mathscr{D}(\mathrm{G})$.

Define the S-morphism

$$f : \Omega \to \mathrm{T}/\mathrm{T}' \tag{5.3.1}$$

by $f(u_{-}tu_{+}) = t \bmod \mathrm{T}'$. There is clearly at most one S-homomorphism $\overline{f} : \mathrm{G} \to \mathrm{T}/\mathrm{T}'$ that extends f, and we will prove that $\overline{f}$ exists and is smooth with $\mathscr{D}(\mathrm{G}) := \ker \overline{f}$ satisfying the desired properties. The key result to be shown is that the condition $u_{+}u_{-} \in \Omega'$ holds over a fiberwise-dense open subscheme $\mathrm{V} \subset \mathrm{U}_{+} \times \mathrm{U}_{-}$. Indeed, we can take such a V to be the preimage under multiplication $\mathrm{U}_{+} \times \mathrm{U}_{-} \to \mathrm{G}$ of the open $\mathrm{V}_0 \subset \mathrm{G}$ provided by:

Lemma 5.3.2. — *There exists an open subscheme* $\mathrm{V}_0 \subset \mathrm{G}$ *containing the identity section such that for all* $u_{\pm} \in \mathrm{U}_{\pm}$*, if* $u_{+}u_{-} \in \mathrm{V}_0$ *then* $u_{+}u_{-} \in \Omega'$*.*

This result is [**SGA3**, XXII, 6.16], whose proof rests on "abstract" rank-1 calculations in [**SGA3**, XX, 3.12]. We provide a version of the argument that instead rests on the explicit classification in Theorem 5.1.8.

Proof. — Let $w_0 \in \mathrm{W}(\Phi)$ denote the long Weyl element relative to Φ^+ (i.e., the product in any order of the reflections in the positive roots), so the w_0-action swaps Φ^+ and $-\Phi^+$. By Corollary 5.1.11 we may choose $n_0 \in \mathrm{N_G(T)(S)}$ that is a representative for w_0, so n_0-conjugation swaps U_+ and U_- and therefore $n_0\Omega' n_0^{-1} = \mathrm{U}_+ \times \mathrm{T}' \times \mathrm{U}_-$. Hence, it suffices to find an open $\mathrm{V}_{n_0} \subset \mathrm{G}$ containing the identity section such that $n_0\Omega' n_0^{-1} \bigcap \mathrm{V}_{n_0} \subset \Omega'$ inside G (as we may then take $\mathrm{V}_0 = \mathrm{V}_{n_0}$). We shall prove the analogous result for any $n \in \mathrm{N_G(T)(S)}$. Note that this problem is Zariski-local on S.

Using the map $\mathrm{N_G(T)(S)} \to \mathrm{W_G(T)(S)}$ and the identification of $\mathrm{W_G(T)}$ with $\mathrm{W}(\Phi)_\mathrm{S}$, by working Zariski-locally on S we may assume that the image of n in $\mathrm{W_G(T)(S)}$ is a constant section arising from some $w \in \mathrm{W}(\Phi)$. For each a in the set Δ of simple positive roots we pick $n_a \in \mathrm{N_G(T)(S)}$ as in Corollary 5.1.11 representing the simple reflection s_a (viewed as a constant section in $\mathrm{W_G(T)(S)}$). Thus, $n_a \in \mathrm{Z_G}(\mathrm{T}_a)$ for all $a \in \Delta$ and clearly $n = n_{a_1} \cdots n_{a_m} t$ for some finite sequence $\{a_j\}$ in Δ and some $t \in \mathrm{T(S)}$. To construct an open $\mathrm{V}_n \subset \mathrm{G}$ around the identity section such that $n\Omega' n^{-1} \bigcap \mathrm{V}_n \subset \Omega'$, we will argue by induction on the length m of the sequence $\{a_i\}$ of simple roots that appear in the expression for n. The case $m = 0$ is trivial by taking $\mathrm{V}_n = \mathrm{G}$ (T normalizes $\mathrm{U}_\pm$), so in general we can arrange $t = 1$. For $m = 1$ we will soon show that we can take $\mathrm{V}_{n_a} = \Omega$ for all $a \in \Delta$. Granting this for a moment, when $m > 1$ we write $n = n_{a_1} n'$ and by induction may assume that $\mathrm{V}_{n'}$ has been found. Then for the open subscheme $\mathrm{V}_n := n_{a_1} \mathrm{V}_{n'} n_{a_1}^{-1} \bigcap \Omega \subset \mathrm{G}$ we have

$$\begin{aligned} n\Omega' n^{-1} \cap \mathrm{V}_n &= n_{a_1}(n'\Omega' n'^{-1} \cap \mathrm{V}_{n'}) n_{a_1}^{-1} \cap \Omega \\ &\subset n_{a_1}\Omega' n_{a_1}^{-1} \cap \Omega \\ &\subset \Omega', \end{aligned}$$

where the final containment follows from our temporary hypothesis that we may take $\mathrm{V}_{n_a} = \Omega$ for all $a \in \Delta$.

It remains to prove that $n_a \Omega' n_a^{-1} \bigcap \Omega \subset \Omega'$ for all $a \in \Delta$. By applying Remark 5.2.6 with $\Psi = \Phi^+ - \{a\}$, the root groups U_b for $b \in \Phi^+ - \{a\}$ directly span *in any order* a smooth closed S-subgroup $\mathrm{U}_{\widehat{a}} \subset \mathrm{U}_{\Phi^+} = \mathrm{U}_+$ (clearly normalized by T). In particular, $\mathrm{U}_a \times \mathrm{U}_{\widehat{a}} = \mathrm{U}_+$ via multiplication. We similarly get $\mathrm{U}_{\widehat{-a}} \subset \mathrm{U}_-$ such that $\mathrm{U}_{\widehat{-a}} \times \mathrm{U}_{-a} = \mathrm{U}_-$ via multiplication, so every point ω' of Ω' valued in an S-scheme S' has the form

$$\omega' = g_- \exp_{-a}(\mathrm{X}_-) t' \exp_a(\mathrm{X}_+) g_+$$

where $g_\pm \in \mathrm{U}_{\widehat{\pm a}}(\mathrm{S}')$, $t' \in \mathrm{T}'(\mathrm{S}')$, and $\mathrm{X}_\pm \in \mathfrak{g}_{\pm a}(\mathrm{S}')$. The action on Φ by the reflection s_a preserves $\Phi^+ - \{a\}$ (as all elements of $\Phi^+ - \{a\}$ have a positive coefficient away from a somewhere in their Δ-expansion, and this property is not affected by applying s_a), so n_a-conjugation preserves each $\mathrm{U}_{\widehat{\pm a}}$. Thus, the property $n_a \omega' n_a^{-1} \in \Omega'$ is insensitive to replacing ω' with $\exp_{-a}(\mathrm{X}_-) t' \exp_a(\mathrm{X}_+)$ (as Ω' is stable under left multiplication by U_- and right multiplication by U_+). Likewise, by definition of Ω, the condition $n_a \omega' n_a^{-1} \in \Omega$ is insensitive to replacing ω' with $\exp_{-a}(\mathrm{X}_-) t' \exp_a(\mathrm{X}_+)$. In other words, we may pass to the case $g_\pm = 1$.

With $g_\pm = 1$, by Lemma 4.1.3 our problem now takes place within the split reductive group $\mathrm{Z_G}(\mathrm{T}_a)$ with semisimple-rank 1 and root system $\{\pm a\}$ relative to its split maximal torus T, so we can assume $\mathrm{G} = \mathrm{Z_G}(\mathrm{T}_a)$. To be precise, since $\mathrm{Z_G}(\mathrm{T}_a) \cap \Omega = \mathrm{U}_{-a} \times \mathrm{T} \times \mathrm{U}_a$, it suffices to show that for $\omega' \in (\mathrm{U}_{-a} \times \mathrm{T}' \times \mathrm{U}_a)(\mathrm{S}')$ such that $n_a \omega' n_a^{-1} \in (\mathrm{U}_{-a} \times \mathrm{T} \times \mathrm{U}_a)(\mathrm{S}')$, the product $n_a \omega' n_a^{-1}$ lies in $(\mathrm{U}_{-a} \times \mathrm{T}' \times \mathrm{U}_a)(\mathrm{S}')$. We may work fppf-locally on S', so the T'-component t' of ω' may be arranged to have the form $t'_a z$ for $t'_a \in a^\vee(\mathbf{G}_\mathrm{m})(\mathrm{S}')$ and $z \in (\mathrm{T}_a \cap \mathrm{T}')(\mathrm{S}')$. Since z is central in $\mathrm{Z_G}(\mathrm{T}_a)$, we can assume $z = 1$. Thus, t' lies in the S-torus $a^\vee(\mathbf{G}_\mathrm{m})$ that is the "T'" associated to $(\mathrm{Z_G}(\mathrm{T}_a), \mathrm{T}, \{\pm a\})$. Hence, we may assume $\mathrm{G} = \mathrm{Z_G}(\mathrm{T}_a)$.

If $\mathrm{G} = \mathrm{G}_1 \times \mathrm{T}_1$ for a split torus T_1 then our problem takes place inside G_1, so we may pass to G_1. Thus, after Zariski-localization on the base we may assume that (G, T) is one of the three explicit pairs in the split semisimple-rank 1 classification in Theorem 5.1.8 (up to direct product against a split torus), and conjugating by the standard Weyl element $w = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ if necessary allows us to arrange that a is the standard positive root (i.e., U_a is the subgroup of strictly upper triangular matrices in SL_2 or PGL_2). In cases (1) and (2) of Theorem 5.1.8 we have $\mathrm{T}' = \mathrm{D} = \mathrm{T}$, so $\Omega = \Omega'$ and there is nothing to do. Thus, we may assume (G, T) is as in case (3): $\mathrm{G} = \mathrm{SL}_2 \times^{\mu_2} \mathbf{G}_\mathrm{m} = \mathrm{GL}_2$ and T is the diagonal torus. The torus T' generated by $a^\vee$ is the diagonal torus D in the subgroup SL_2, and replacing n_a with a suitable left $\mathrm{T}(\mathrm{S}')$-multiple (as we may do) allows us to take $n_a = w$.

Now Ω' is the open subgroup of SL_2 given by the unit condition on the upper left entry (see Example 4.2.5), and $\Omega = \Omega' \times^{\mu_2} \mathbf{G}_\mathrm{m}$. Thus, $n\Omega' n^{-1} \bigcap \Omega \subset \mathrm{SL}_2 \bigcap \Omega = \Omega'$ inside G. □

For the choice of $\mathrm{V} \subset \mathrm{U}_+ \times \mathrm{U}_-$ built using Lemma 5.3.2, the relatively schematically dense open

$$\mathrm{U}_- \times \mathrm{T} \times \mathrm{V} \times \mathrm{T} \times \mathrm{U}_+ \subset \Omega \times \Omega$$

lies in $m_\mathrm{G}^{-1}(\Omega)$ and the map f in (5.3.1) satisfies $f((u_1^- t_1 u_1^+)(u_2^- t_2 u_2^+)) = t_1 t_2 \bmod \mathrm{T}'$. This "S-birational multiplicativity" for f implies via the self-contained and elementary [**SGA3**, XVIII, 2.3(i)] that f extends uniquely to an S-homomorphism $\overline{f}$. On fibers over a geometric point $\overline{s}$ of S we have a good theory of the semisimple derived group $\mathscr{D}(\mathrm{G}_{\overline{s}})$. In particular, $\mathrm{T}'_{\overline{s}}$ is a maximal torus of $\mathscr{D}(\mathrm{G}_{\overline{s}})$ and the identification of root systems $\Phi(\mathscr{D}(\mathrm{G}_{\overline{s}}), \mathrm{T}'_{\overline{s}}) = \Phi(\mathrm{G}_{\overline{s}}, \mathrm{T}_{\overline{s}})$ via the isogeny $\mathrm{T}'_{\overline{s}} \times (\mathrm{Z}_{\mathrm{G}_{\overline{s}}})^0_{\mathrm{red}} \to \mathrm{T}_{\overline{s}}$ identifies Φ^+ with a positive system of roots for $\Phi(\mathscr{D}(\mathrm{G}_{\overline{s}}), \mathrm{T}'_{\overline{s}})$. The corresponding open cell in $\mathscr{D}(\mathrm{G}_{\overline{s}})$ is clearly $\Omega'_{\overline{s}}$. Hence, $(\ker \overline{f}_{\overline{s}})^0 = \mathscr{D}(\mathrm{G}_{\overline{s}})$, so $f_{\overline{s}}$ factors as

$$\mathrm{G}_{\overline{s}} \to \mathrm{G}_{\overline{s}}/\mathscr{D}(\mathrm{G}_{\overline{s}}) \to \mathrm{T}_{\overline{s}}/\mathrm{T}'_{\overline{s}},$$

where the second map is an étale isogeny of tori. This isogeny is an isomorphism since the maximal torus $\mathrm{T}_{\overline{s}}$ maps onto the torus quotient $\mathrm{G}_{\overline{s}}/\mathscr{D}(\mathrm{G}_{\overline{s}})$ and $\mathrm{T}'_{\overline{s}} \subset \mathscr{D}(\mathrm{G}_{\overline{s}})$. Thus, $\overline{f}_{\overline{s}}$ is the quotient by the derived group of $\mathrm{G}_{\overline{s}}$, so $\overline{f}$ is a smooth surjection whose kernel $\ker \overline{f}$ has $\overline{s}$-fiber $\mathscr{D}(\mathrm{G}_{\overline{s}})$. Thus, $\ker \overline{f}$ is a semisimple S-group closed and normal in G; define $\mathscr{D}(\mathrm{G})$ to be this subgroup.

By construction, $\mathscr{D}(\mathrm{G})$ contains $\mathrm{U}_{\pm}$ and T' as subgroups, with T' a split maximal torus of $\mathscr{D}(\mathrm{G})$. There is a split triple $(\mathscr{D}(\mathrm{G}), \mathrm{T}', \mathrm{M}')$, where M' is the maximal torsion-free quotient of M that kills $(\mathbf{Z}\Phi^\vee)^\perp \subset \mathrm{M}$ and we let $\Phi' \subset \mathrm{M}'$ be the image of Φ. Let Φ'^+ be the positive system of roots corresponding to Φ^+ under the bijection $\Phi \to \Phi'$. The direct product subfunctor Ω' in Ω lies in $\mathscr{D}(\mathrm{G})$. It is obvious that Ω' must be the open cell of $\mathscr{D}(\mathrm{G})$ associated to Φ'^+. The commutativity of $\mathrm{G}/\mathscr{D}(\mathrm{G}) = \mathrm{T}/\mathrm{T}'$ implies that $\mathscr{D}(\mathrm{G})$ contains the commutator subsheaf of G. Provided that every semisimple S-group (such as $\mathscr{D}(\mathrm{G})$) coincides with its own commutator subsheaf (for the fppf topology), the asserted uniqueness of $\mathscr{D}(\mathrm{G})$ will be clear and so we will be done.

Finally, consider a semisimple S-group G. We seek to show that G is its own commutator subsheaf. We may work étale-locally on the base, so we can assume $\mathrm{S} \neq \varnothing$ and that G is part of a split triple $(\mathrm{G}, \mathrm{T}, \mathrm{M})$ over S. Let $\Phi \subset \mathrm{M} - \{0\}$ be the set of roots, and pick a positive system of roots in Φ. The associated open cell generates G for the fppf topology (as for any open neighborhood of the identity section of a smooth group scheme with connected fibers), and the map $\mathbf{G}_{\mathrm{m}}^r \to \mathrm{T}$ defined by the simple positive coroots is an isogeny. Thus, to prove that G is its own commutator subsheaf (for the fppf topology), the structure of the open cell as a direct product scheme reduces the problem to the case of semisimple groups with semisimple-rank 1 case, and more specifically to the case of SL_2 equipped with its diagonal torus (due to Theorem 5.1.8).

Once again passing to the standard open cell reduces us to some classical identities, as follows. For $x(u) = \left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right)$, $y(v) = \left(\begin{smallmatrix} 1 & 0 \\ v & 1 \end{smallmatrix}\right)$, and $h(t) = \mathrm{diag}(t, 1/t)$ we have $h(t)x(u)h(t)^{-1}x(u)^{-1} = x((t^2-1)u)$ and $h(t)y(v)h(t)^{-1}y(v)^{-1} = y((t^{-2}-1)v)$, so the commutator subsheaf contains both standard root groups. These in turn generate the diagonal torus via the identity

$$h(t) = y(-1/t)x(t)y(-1/t)(y(-1)x(1)y(-1))^{-1}, \tag{5.3.2}$$

so SL_2 is indeed its own commutator subsheaf. □

The group $\mathscr{D}(\mathrm{G})$ is called the *derived group* of G. Note that by uniqueness, $\mathscr{D}(\mathrm{G}) = \mathrm{G}$ if and only if G is semisimple, and by the universal property the formation of $\mathscr{D}(\mathrm{G})$ is functorial in G. It follows formally that any S-homomorphism from a semisimple S-group to G must factor through $\mathscr{D}(\mathrm{G})$.

Corollary 5.3.3. — *Let* $\mathrm{G} \to \mathrm{S}$ *be a reductive group scheme,* $\mathscr{D}(\mathrm{G})$ *its derived group. Let* $\mathrm{Z} \subset \mathrm{Z}_{\mathrm{G}}$ *be the maximal central torus of* G*, and* $\mathrm{T}' = \mathrm{G}/\mathscr{D}(\mathrm{G})$ *the maximal torus quotient. The natural* S*-homomorphisms* $f : \mathrm{Z} \times \mathscr{D}(\mathrm{G}) \to \mathrm{G}$ *and* $h : \mathrm{G} \to \mathrm{T}' \times (\mathrm{G}/\mathrm{Z})$ *are central isogenies.*

The result in this corollary is [**SGA3**, XXII, 6.2.4]. We refer the reader to Definition 3.3.9 for the relative notion of a central isogeny used here.

Proof. — The classical theory implies that f_s and h_s are central isogenies for all $s \in \mathrm{S}$, so we may apply Proposition 3.3.10 to conclude. (See Remark B.1.2 for the existence and compatibility with base change of a subtorus containing all others in any multiplicative type group, thereby providing Z inside Z_{G} compatibly with any base change.) □

Proposition 5.3.4. — *Let* $\mathrm{G} \to \mathrm{S}$ *be a reductive group scheme, and* Z *the maximal torus of* Z_{G}*. The map* $\mathrm{T} \mapsto \mathrm{T} \bigcap \mathscr{D}(\mathrm{G})$ *is a bijective correspondence between the set of maximal tori in* G *and the set of maximal tori in* $\mathscr{D}(\mathrm{G})$*. Conversely, if* T' *is a maximal torus of* $\mathscr{D}(\mathrm{G})$ *then* $\mathrm{T}' \times \mathrm{Z} \to \mathrm{G}$ *is an isogeny onto a maximal torus* T *of* G*, and this reverses the bijective correspondence.*

The same holds for parabolic subgroups, with the analogous procedures using intersection and product against Z*.*

Proof. — Consider the correspondence for maximal tori. By Example 2.2.6, since $\mathscr{D}(\mathrm{G})$ is normalized by T it follows that $\mathrm{T} \bigcap \mathscr{D}(\mathrm{G})$ is smooth with connected fibers, so it is a torus. The classical theory implies that this intersection is a maximal torus of $\mathscr{D}(\mathrm{G})$. Likewise, the classical theory shows that if we define $\mathrm{T}' := \mathrm{T} \bigcap \mathscr{D}(\mathrm{G})$ then the multiplication map $\mathrm{Z} \times \mathrm{T}' \to \mathrm{T}$ between tori is an isogeny. Conversely, for a maximal torus T' of $\mathscr{D}(\mathrm{G})$, we have to show that the multiplication map $\mathrm{T}' \times \mathrm{Z} \to \mathrm{G}$ is an isogeny onto a maximal torus of G. This map factors through $\mathrm{Z}_{\mathrm{G}}(\mathrm{T}')$, so it suffices to show that the reductive group $\mathrm{Z}_{\mathrm{G}}(\mathrm{T}')$ is a torus and $\mathrm{T}' \times \mathrm{Z} \to \mathrm{Z}_{\mathrm{G}}(\mathrm{T}')$ is an isogeny of tori. It suffices to check these assertions on geometric fibers over S, where they are well-known.

Now we turn to the consideration of parabolic subgroups. For any parabolic subgroup $\mathrm{P} \subset \mathrm{G}$, we have $\mathrm{Z}_{\mathrm{G}} \subset \mathrm{P}$. Indeed, it suffices to check this étale-locally on S, and by Proposition 5.2.3 we can perform such localization on S so that $\mathrm{P} = \mathrm{P}_{\mathrm{G}}(\lambda)$ for some $\lambda : \mathbf{G}_{\mathrm{m}} \to \mathrm{G}$. Thus, $\mathrm{Z}_{\mathrm{G}} \subset \mathrm{Z}_{\mathrm{G}}(\lambda) \subset \mathrm{P}_{\mathrm{G}}(\lambda) = \mathrm{P}$. It follows that $Z \subset P$, so the isogeny $Z \times \mathscr{D}(G) \to G$ implies that P is uniquely determined by $P' = P \bigcap \mathscr{D}(G)$, and P' is smooth by Proposition 4.1.10(1). For each $s \in S$ we have $\mathscr{D}(G)_{\overline{s}}/P'_{\overline{s}} \simeq G_{\overline{s}}/P_{\overline{s}}$, so $P'_{\overline{s}}$ is a parabolic subgroup of $\mathscr{D}(G)_{\overline{s}}$. Hence, P' is parabolic in $\mathscr{D}(G)$. The multiplication map $m : Z \times P' \to P$ is an isogeny on fibers, so m is a quasi-finite flat surjection, and $\ker m$ is visibly central. But $\ker m$ is closed in the S-finite kernel of $Z \times \mathscr{D}(G) \to G$, so it is S-finite and hence m is a central isogeny.

Finally, it remains to show that every parabolic subgroup Q of $\mathscr{D}(G)$ arises as $P \bigcap \mathscr{D}(G)$ for a parabolic subgroup P of G. In view of the uniqueness of such a P we may work étale-locally on S, so by Proposition 5.2.3 we may arrange that $Q = P_{\mathscr{D}(G)}(\mu)$ for some $\mu : \mathbf{G}_m \to \mathscr{D}(G)$. But then $P := P_G(\mu)$ is a parabolic subgroup of G (Example 5.2.2) and $P \bigcap \mathscr{D}(G) = P_{\mathscr{D}(G)}(\mu) = Q$. $\square$

By Proposition 3.3.8, if G is adjoint semisimple then the S-homomorphism $\mathrm{Ad}_G : G \to \mathrm{GL}(\mathfrak{g})$ has trivial kernel. In view of Remark B.1.4 it is not obvious if Ad_G is a closed immersion in the adjoint semisimple case over a general scheme. In fact, it is always a closed immersion for such G, because any monomorphism from a *reductive* group scheme to a separated group of finite presentation is a closed immersion. We will never use this result, but we provide a proof of it below (after some brief preparations).

For a split reductive group scheme (G, T, M) and parabolic subgroup $P = P_G(\lambda)$ with $\lambda \in M^\vee$, Proposition 5.1.16(2) provides a composition series $\{U_{\geqslant n}\}_{n \geqslant 1}$ for the unipotent radical $U = \mathscr{R}_u(P) = U_G(\lambda)$ with $U_{\geqslant n} = U_{\Phi_{\lambda \geqslant n}}$ a smooth closed subgroup directly spanned in any order by the root groups U_c for c satisfying $\lambda(c) \geqslant n$. Moreover, $U_{\geqslant n+1}$ is normal in $U_{\geqslant n}$ and the quotient $U_{\geqslant n}/U_{\geqslant n+1}$ is commutative and identified (as an S-group) with the direct product of the root groups $U_c \simeq \mathbf{G}_a$ with $\lambda(c) = n$. We shall use this general filtration of unipotent radicals of parabolic subgroups and the dynamic method of § 4.1 to prove:

Theorem 5.3.5. — *For a reductive group* $G \to S$, *any monic homomorphism* $f : G \to G'$ *to a separated* S-*group of finite presentation is a closed immersion. In particular, if* G *is an adjoint semisimple* S-*group then* $\mathrm{Ad}_G : G \to \mathrm{GL}(\mathfrak{g})$ *is a closed immersion.*

Theorem 5.3.5 is proved in another way (without the dynamic method) in [**SGA3**, XVI, 1.5(a)]. When G' is not S-affine, our proof uses a difficult theorem due to Raynaud. Note also that if G' is allowed to merely be locally of finite presentation (and separated) then the conclusion is false; counterexamples are provided by the Néron lft model of a split torus [**BLR**, 10.1/5]. Also, Example 3.1.2 gives counterexamples if reductivity of G is relaxed to "smooth affine with connected fibers" (and $G' \to S$ is smooth and affine).

Proof. — The application to Ad_{G} in the adjoint case is immediate from the rest via Proposition 3.3.8. In general, monicity means that the diagonal $\Delta_f : \mathrm{G} \to \mathrm{G} \times_{\mathrm{G}'} \mathrm{G}$ is an isomorphism, so by direct limit arguments we can reduce to the case when S is noetherian. Since f is a monomorphism, it is a closed immersion if and only if it is proper [**EGA**, IV_3, 8.11.5]. Thus, by the valuative criterion for properness, we are reduced to checking that if R is a discrete valuation ring with fraction field K and $\mathrm{Spec}\,\mathrm{R} \to \mathrm{S}$ is a morphism of schemes then $\mathrm{G}(\mathrm{R}) = \mathrm{G}(\mathrm{K}) \bigcap \mathrm{G}'(\mathrm{R})$ inside $\mathrm{G}'(\mathrm{K})$. Applying base change along $\mathrm{Spec}\,\mathrm{R} \to \mathrm{S}$ then reduces us to the case $\mathrm{S} = \mathrm{Spec}\,\mathrm{R}$.

Next, we reduce to the case when G' is affine and R-flat. (The reader who is only interested in the case of affine G' can ignore this step.) The map on generic fibers $\mathrm{G}_{\mathrm{K}} \to \mathrm{G}'_{\mathrm{K}}$ is a closed immersion, so the schematic closure $\overline{\mathrm{G}}$ of G in G' is an R-flat separated subgroup of G' through which G factors (since G is R-flat). But $\overline{\mathrm{G}}$ is a separated flat R-group of finite type with affine generic fiber, so it must be affine by a result of Raynaud (see [**SGA3**, VI_{B}, 12.10(iii), 12.10.1] or [**PY06**, Prop. 3.1]). Hence, we may replace G' with $\overline{\mathrm{G}}$ to reduce to the case that G' is affine and R-flat.

Over a regular base of dimension $\leqslant 1$, every flat affine group scheme of finite type is a closed subgroup of some GL_n. This is easy to prove by adapting arguments from the case when the base is a field (see Exercise 5.5.7); in fact, the result is true over any regular affine base of dimension $\leqslant 2$ [**SGA3**, VI_{B}, 13.2]. Thus, we may identify G' as a closed subgroup of some GL_n, so we can replace G' with GL_n. This reduces the problem to the case that G' is a reductive S-group, but we allow S to be an arbitrary scheme (to clarify the generality of the steps that follow).

By working étale-locally on S, we may assume G arises from a split triple $(\mathrm{G}, \mathrm{T}, \mathrm{M})$ (and that S is non-empty). Choose $\lambda \in \mathrm{M}^{\vee}$ not vanishing on any root, so $\mathrm{T} = \mathrm{Z}_{\mathrm{G}}(\lambda)$ and $\mathrm{B} := \mathrm{P}_{\mathrm{G}}(\lambda)$ is a Borel subgroup of G. Let $\mathrm{U}_+ = \mathrm{U}_{\mathrm{G}}(\lambda) = \mathscr{R}_u(\mathrm{B})$ and $\mathrm{U}_- = \mathrm{U}_{\mathrm{G}}(-\lambda) = \mathscr{R}_u(\mathrm{B}_-)$, where $\mathrm{B}_- = \mathrm{P}_{\mathrm{G}}(-\lambda)$ is the opposite Borel subgroup of G containing T (see Proposition 5.2.12). Let $\Omega \subset \mathrm{G}$ be the open cell $\mathrm{U}_- \times \mathrm{B}$.

For $\lambda' = f \circ \lambda$ we likewise get smooth closed subgroups $\mathrm{Z}_{\mathrm{G}'}(\lambda'), \mathrm{U}_{\mathrm{G}'}(\pm\lambda') \subset \mathrm{G}'$ such that the multiplication map

$$\mathrm{U}_{\mathrm{G}'}(-\lambda) \times \mathrm{Z}_{\mathrm{G}'}(\lambda') \times \mathrm{U}_{\mathrm{G}'}(\lambda') \to \mathrm{G}'$$

is an open immersion; we let $\Omega' \subset \mathrm{G}'$ denote this open subscheme. Since $\mathrm{Z}_{\mathrm{G}'}(\lambda')$ is reductive (as it is a torus centralizer in a reductive group scheme), by working étale-locally on S we can arrange that $\mathrm{Z}_{\mathrm{G}'}(\lambda')$ contains a split maximal torus T'. Clearly T' is maximal in G' (by the classical theory on geometric fibers), and λ' factors through T' (since $\mathrm{T}' = \mathrm{Z}_{\mathrm{G}'}(\mathrm{T}')$). Further localization on S brings us to the case that $(\mathrm{G}', \mathrm{T}')$ arises from a split triple $(\mathrm{G}', \mathrm{T}', \mathrm{M}')$. Hence, for $\Psi' = \Phi'_{\lambda'>0}$, by Remark 5.2.6 the group $\mathrm{U}_{\mathrm{G}'}(\pm\lambda')$

coincides with the subgroup $\mathrm{U}_{\pm\Psi'}$ from Proposition 5.1.16 that is directly spanned by the root groups $\mathrm{U}_{c'}$ for $c' \in \pm\Psi' = \Phi'_{\pm\lambda'>0}$.

It is harmless to work étale-locally on S and to compose the given monomorphism $\mathrm{G} \to \mathrm{G}'$ with conjugation by some element of $\mathrm{G}'(\mathrm{S})$. Thus, to reduce to the case that f carries T into T′ it suffices (by étale-local conjugacy of maximal tori in smooth relatively affine group schemes) to prove:

Lemma 5.3.6. — *For any homomorphism $f : \mathrm{T} \to \mathrm{H}$ from a torus into a smooth relatively affine group over a scheme* S*, étale-locally on* S *it factors through a maximal torus of* H.

Proof. — By replacing H with $\mathrm{T} \ltimes \mathrm{H}$ (via the action $t.h = f(t)hf(t)^{-1}$) it suffices to show that if f is the inclusion of T as a closed S-subgroup of H then étale-locally on S it is contained in a maximal torus of H. The centralizer $\mathrm{Z_H(T)}$ is smooth, and by the classical theory on geometric fibers we see that its maximal tori are also maximal in H. Hence, we may replace H with $\mathrm{Z_H(T)}$ to arrange that T is central. By passing to an étale cover of S we can assume that $\mathrm{Z_H(T)/T}$ admits a maximal torus. Corollary B.4.2(2) ensures that the preimage of this maximal torus in $\mathrm{Z_H(T)}$ is a torus, and by the classical theory on geometric fibers it is a maximal torus. □

Now we may and do assume $\mathrm{T} \subset \mathrm{T}'$. Proposition 4.1.10(2) gives that $\Omega \subset f^{-1}(\Omega')$, and the key point is that this containment is an *equality*. To verify this equality between open subschemes of G we may pass to geometric fibers over S, in which case the equality is [**CGP**, Prop. 2.1.8(3)] (which has nothing to do with smoothness or reductivity). Thus, the restriction of f over the open subscheme $\Omega' \subset \mathrm{G}'$ is the map $\Omega \to \Omega'$ that is the direct product of the maps

$$\mathrm{U_G}(-\lambda) \to \mathrm{U_{G'}}(-\lambda'),\ \ \mathrm{T} = \mathrm{Z_G}(\lambda) \to \mathrm{Z_{G'}}(\lambda'),\ \ \mathrm{U_G}(\lambda) \to \mathrm{U_{G'}}(\lambda').$$

We will prove that each of these three maps is a closed immersion, so f is a closed immersion when restricted over Ω'.

Since $\mathrm{T} \to \mathrm{T}'$ is a monic homomorphism between tori, it is a closed immersion. To prove that $\mathrm{U} := \mathrm{U_G}(\lambda) \to \mathrm{U_{G'}}(\lambda') =: \mathrm{U}'$ is a closed immersion, consider the filtrations $\{\mathrm{U}_{\geqslant n}\}_{n\geqslant 1}$ and $\{\mathrm{U}'_{\geqslant n}\}_{n\geqslant 1}$ on these as described immediately before Theorem 5.3.5. We claim that f carries $\mathrm{U}_{\geqslant n}$ into $\mathrm{U}'_{\geqslant n}$ for all n. More specifically, keeping in mind that we arranged $\mathrm{T} \subset \mathrm{T}'$ via f, we have:

Lemma 5.3.7. — *For $c \in \Phi^+$, $f|_{\mathrm{U}_c}$ factors through $\mathrm{U}_{\mathrm{Z}_{\mathrm{G}'}(\ker c)}(c^\vee) \subset \mathrm{G}'$, and this closed subgroup of* G′ *is directly spanned in any order by the root groups $\mathrm{U}'_{c'}$ for $c' \in \Phi'$ such that $c'|_{\mathrm{T}}$ is a positive integral multiple of c.*

The group $\mathrm{Z}_{\mathrm{G}'}(\ker c)$ is smooth by Lemma 2.2.4 since $\ker c$ is multiplicative type, but beware that its fibers over S may not be connected (since $\ker c$ may not be a torus).

Proof. — Since U_c is normalized by T with trivial action by $\ker c$, it is carried into the smooth closed subgroup $\mathrm{Z}_{\mathrm{G}'}(\ker c)$. It follows that U_c is carried into $\mathrm{U}_{\mathrm{Z}_{\mathrm{G}'}(\ker c)}(c^\vee)$ since $\langle c, c^\vee\rangle = 2 > 0$. Likewise, if $c' \in \Phi'$ satisfies $c'|_{\mathrm{T}} = nc$ with $n \geqslant 1$ then c' kills $\ker c$ and $\langle c', c^\vee\rangle = \langle nc, c^\vee\rangle = 2n > 0$, so $\mathrm{U}'_{c'}$ is contained in $\mathrm{U}_{\mathrm{Z}_{\mathrm{G}'}(\ker c)}(c^\vee)$. It remains to show that for any choice of enumeration $\{c'_i\}$ of the set of such c', the multiplication map of S-schemes

$$\prod \mathrm{U}'_{c'_i} \to \mathrm{U}_{\mathrm{Z}_{\mathrm{G}'}(\ker c)}(c^\vee) \tag{5.3.3}$$

is an isomorphism. Since this is a map between smooth S-schemes, we may pass to geometric fibers, so $\mathrm{S} = \operatorname{Spec} k$ for an algebraically closed field k.

Connectedness of $\mathrm{U}_{\mathrm{Z}_{\mathrm{G}'}(\ker c)}(c^\vee)$ [**CGP**, 2.1.8(4)] implies that it equals $\mathrm{U}_{\mathrm{Z}_{\mathrm{G}'}(\ker c)^0}(c^\vee)$. The group $\mathrm{Z}_{\mathrm{G}'}(\ker c)^0$ is smooth since $\ker c$ is of multiplicative type, and its Lie algebra is the trivial weight space $\mathrm{Lie}(\mathrm{G}')^{\ker c}$ for the linear action of the split multiplicative type group $\ker c$ on $\mathrm{Lie}(\mathrm{G}')$. Hence, the T'-weights on $\mathrm{Lie}(\mathrm{Z}_{\mathrm{G}'}(\ker c)^0)$ are the elements $c' \in \Phi'$ that are trivial on $\ker c$, which is to say $c'|_{\mathrm{T}}$ is an integral multiple of c. For such c', the condition $\langle c', c^\vee\rangle > 0$ says exactly that $c'|_{\mathrm{T}}$ is a positive integral multiple of c. It follows that (5.3.3) is an isomorphism on tangent spaces at the identity, so $\mathrm{U}_{\mathrm{Z}_{\mathrm{G}'}(\ker c)}(c^\vee)$ is generated by the $\mathrm{U}'_{c'_i}$. Thus, we just have to check that these root groups directly span (in any order) a unipotent smooth connected subgroup of G'. The subset $\{c'_i\} \subset \Phi'$ is closed and disjoint from its negative (since $\langle c'_i, c^\vee\rangle > 0$ for all i), so Proposition 5.1.16 provides this direct spanning result. (See [**Bo91**, 14.5(2)] and [**CGP**, 3.3.11, 3.3.13(1)] for related direct spanning results in the theory over a field.) $\square$

By Lemma 5.3.7, for every $n \geqslant 1$ we get homomorphisms $\mathrm{U}_{\geqslant n} \to \mathrm{U}'_{\geqslant n}$. Consider the resulting homomorphisms between vector groups

$$f_n : \mathrm{U}_{\geqslant n}/\mathrm{U}_{\geqslant n+1} \to \mathrm{U}'_{\geqslant n}/\mathrm{U}'_{\geqslant n+1}.$$

We claim that each f_n is a closed immersion. By construction, f_n is $\mathbf{G}_{\mathrm{m}}$-equivariant with source and target identified with a power of $\mathbf{G}_{\mathrm{a}}$ on which $\mathbf{G}_{\mathrm{m}}$ acts through the nth-power map. Thus, f_n is a linear map of vector bundles, so to check if it is a closed immersion it suffices to pass to geometric fibers and verify injectivity on Lie algebras. For this purpose we may now assume that $\mathrm{S} = \operatorname{Spec} k$ for an algebraically closed field k.

The Lie algebra of $\mathrm{U}_{\geqslant n}/\mathrm{U}_{\geqslant n+1}$ is the direct product of the root groups U_c for $c \in \Phi$ such that $\lambda(c) = n$, and similarly for $\mathrm{U}'_{\geqslant n}/\mathrm{U}'_{\geqslant n+1}$ using $\lambda' = f \circ \lambda$. If $\mathrm{Lie}(f_n)$ is not injective then by the equivariance of f_n with respect to the closed immersion of tori $\mathrm{T} \hookrightarrow \mathrm{T}'$ it follows that $\ker(\mathrm{Lie}(f_n))$ would have to contain some root space $\mathfrak{g}_c$ for $c \in \Phi$ satisfying $\lambda(c) = n$. The vanishing of $\mathrm{Lie}(f_n)$ on $\mathfrak{g}_c$ implies that $\mathrm{Lie}(f)$ carries $\mathfrak{g}_c$ into the span of the root spaces $\mathfrak{g}'_{c'}$ for $c' \in \Phi'$ such that $\lambda'(c') \geqslant n+1$. But the T-action on that span has as its weights precisely the T-restrictions of these roots c', so the containment of $\mathfrak{g}_c$ in the space forces $c'|_{\mathrm{T}} = c$ for at least one such c'. For that c' we have $n + 1 \leqslant \lambda'(c') = \lambda(c'|_{\mathrm{T}}) = \lambda(c) = n$, a contradiction.

Returning to the relative setting over S, since the maps $\mathrm{U}_{\geqslant n}/\mathrm{U}_{\geqslant n+1} \to \mathrm{U}'_{\geqslant n}/\mathrm{U}'_{\geqslant n+1}$ are all closed immersions, the map $\mathrm{U} \to \mathrm{U}'$ is a closed immersion by repeated applications of:

Lemma 5.3.8. — *In a commutative diagram of short exact sequences of flat, separated, finitely presented* S*-group schemes*

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{H}'_1 \arrow[r] \arrow[d, "j'"'] & \mathrm{H}_1 \arrow[r, "q_1"] \arrow[d, "j"] & \mathrm{H}''_1 \arrow[r] \arrow[d, "j''"] & 1 \\
1 \arrow[r] & \mathrm{H}'_2 \arrow[r] & \mathrm{H}_2 \arrow[r, "q_2"'] & \mathrm{H}''_2 \arrow[r] & 1
\end{tikzcd}$$

if the outer vertical maps are closed immersions then so is the middle one.

Proof. — As usual, we may reduce to the case that S is noetherian. Clearly j is a monomorphism, so it suffices to prove that it is proper. Using the valuative criterion for properness, it suffices to show that if R is a discrete valuation ring with fraction field K then the containment $\mathrm{H}_1(\mathrm{R}) \subset \mathrm{H}_1(\mathrm{K}) \bigcap \mathrm{H}_2(\mathrm{R})$ is an equality. Choose $h_1 \in \mathrm{H}_1(\mathrm{K}) \bigcap \mathrm{H}_2(\mathrm{R})$, so the image h''_1 of h_1 in $\mathrm{H}''_1(\mathrm{K})$ lies in $\mathrm{H}''_1(\mathrm{K}) \bigcap \mathrm{H}''_2(\mathrm{R}) = \mathrm{H}''_1(\mathrm{R})$. To extend h_1 to an R-point of H_1 it is harmless to replace R with a flat local extension by another discrete valuation ring. Since $\mathrm{H}_1 \to \mathrm{H}''_1$ is fppf, we can choose such an extension $\widetilde{\mathrm{R}}$ of R so that h''_1 viewed as an $\widetilde{\mathrm{R}}$-point of H''_1 lifts to $\mathrm{H}_1(\widetilde{\mathrm{R}})$. By renaming $\widetilde{\mathrm{R}}$ as R and multiplying h_1 by the inverse of an R-lift of h''_1, we reduce to the case that h''_1 is trivial, so h_1 arises from some $h'_1 \in \mathrm{H}'_1(\mathrm{K})$. The image $h_2 \in \mathrm{H}_2(\mathrm{K})$ of h_1 lies in $\mathrm{H}_2(\mathrm{R}) \bigcap \mathrm{H}'_2(\mathrm{K}) = \mathrm{H}'_2(\mathrm{R})$, so $h'_1 \in \mathrm{H}'_1(\mathrm{K}) \bigcap \mathrm{H}'_2(\mathrm{R}) = \mathrm{H}'_1(\mathrm{R})$. Hence, $h_1 \in \mathrm{H}_1(\mathrm{R})$. □

We have completed the proof that $\mathrm{U}_{\mathrm{G}}(\lambda) \to \mathrm{U}_{\mathrm{G}'}(\lambda')$ is a closed immersion. Likewise, $\mathrm{U}_{\mathrm{G}}(-\lambda) \to \mathrm{U}_{\mathrm{G}'}(-\lambda')$ is a closed immersion, so $\Omega = f^{-1}(\Omega') \to \Omega'$ is a closed immersion. That is, $f : \mathrm{G} \to \mathrm{G}'$ is a closed immersion when restricted over the open subscheme Ω' in the reductive group G'. Since (G, T) is split, $\mathrm{N}_{\mathrm{G}}(\mathrm{T})(\mathrm{S}) \to \mathrm{W}_{\mathrm{G}}(\mathrm{T})(\mathrm{S})$ is surjective (Corollary 5.1.11) and $\mathrm{W}_{\mathrm{G}}(\mathrm{T}) = \mathrm{W}(\Phi)_{\mathrm{S}}$ is a finite constant S-group (Proposition 5.1.6). Thus, by the Bruhat decomposition on geometric fibers (see Corollary 1.4.14), G is covered by $\mathrm{N}_{\mathrm{G}}(\mathrm{T})(\mathrm{S})$-translates of Ω. Hence, f is a closed immersion into the open union of translates of Ω' by the image in $\mathrm{G}'(\mathrm{S})$ of representatives in $\mathrm{N_G(T)(S)}$ for the finitely many elements of $\mathrm{W}(\Phi)$. We conclude that f is a (finitely presented) closed immersion into a (finitely presented) open subscheme, or in other words it is a quasi-compact immersion.

By the valuative criterion for properness, to prove f is a closed immersion we may assume (after limit arguments to reduce to the noetherian case) that $\mathrm{S} = \mathrm{Spec}\,\mathrm{R}$ for a discrete valuation ring R with fraction field K. The schematic closure $\overline{\mathrm{G}}$ of the locally closed G in G' is then a closed flat S-subgroup with generic fiber $\mathrm{G_K}$ and it contains G as an open subgroup. In particular, the special fiber $\overline{\mathrm{G}}_0$ of $\overline{\mathrm{G}}$ has reductive identity component, so $\overline{\mathrm{G}}$ is smooth. Thus, by Proposition 3.1.12, $\mathrm{G} = \overline{\mathrm{G}}$. Hence, G is closed in G' as desired. □

Example 5.3.9. — As an application (not to be used later) of the open cell and the closed immersion property for the adjoint representation of a semisimple group scheme of adjoint type in Theorem 5.3.5, consider such groups G over $\mathbf{Z}$ that are split; these are the (semisimple) Chevalley groups of adjoint type. Let T be a split maximal $\mathbf{Z}$-torus, and $\Phi = \Phi(\mathrm{G},\mathrm{T})$. Fix a positive system of roots Φ^+ in Φ, and let $\Delta \subset \Phi^+$ be the base of simple positive roots. For any field k, we claim $\mathrm{G}(k)$ is the subgroup of $\mathrm{Aut}(\mathfrak{g}_k)$ generated by the elements $\mathrm{Ad_G}(\exp_{\pm a}(\mathrm{X}))$ for $\mathrm{X} \in \mathfrak{g}_{\pm a}$ (with $a \in \Delta$) and the elements $\mathrm{Ad_G}(t)$ where $t \in \prod_{a\in\Delta} k^\times$ via the isomorphism $\mathrm{T} \simeq \prod_{a\in\Delta} \mathbf{G}_\mathrm{m}$ defined by $t \mapsto (a(t))$ (isomorphism due to the adjoint property; see Exercise 5.5.4).

The groups $\mathrm{U}_{\pm a}(k)$ generate the representative $w_a(\mathrm{X}_a) \in \mathrm{N_G(T)}(k)$ of $s_a \in \mathrm{W}(\Phi)$ using any $\mathrm{X}_a \in \mathfrak{g}_a - \{0\}$. These reflections s_a generate $\mathrm{W}(\Phi)$, so conjugation by the elements $w_c(\mathrm{X}_c)$ for $c \in \Delta$ carries the groups $\mathrm{U}_a(k)$ for $a \in \Delta$ to the groups $\mathrm{U}_b(k)$ for all roots b. This provides the factors $\mathrm{U}_{\pm\Phi^+}(k)$ in the k-points of the open cell Ω. The standard locally closed Bruhat cells over $\overline{k}$ are clearly defined over k (using representatives for $\mathrm{W}(\Phi)$ in $\mathrm{N_G(T)}(k)$, such as via the elements $w_a(\mathrm{X}_a)$), so the Bruhat decomposition over $\overline{k}$ implies that $\Omega(k)$ generates $\mathrm{G}(k)$, yielding the desired list of generators by applying the inclusion $\mathrm{Ad_G} : \mathrm{G}(k) \hookrightarrow \mathrm{Aut}(\mathfrak{g}_k)$.

We can go further via the split semisimple-rank 1 classification and Existence Theorem over $\mathbf{Z}$, as follows. The Existence Theorem provides a simply connected central cover $\widetilde{\mathrm{G}} \to \mathrm{G}$ over $\mathbf{Z}$ (see Exercise 6.5.2), so $\mathrm{G} = \widetilde{\mathrm{G}}/\mathrm{Z}_{\widetilde{\mathrm{G}}}$. We claim that inside $\mathrm{Aut}(\mathfrak{g}_k)$, the elements $\mathrm{Ad_G}(\exp_{\pm a}(\mathrm{X}_a))$ for $\mathrm{X}_a \in \mathfrak{g}_{\pm a}(k)$ (with $a \in \Delta$) generate the image of $\widetilde{\mathrm{G}}(k)/\mathrm{Z}_{\widetilde{\mathrm{G}}}(k)$ in $\mathrm{G}(k) = (\widetilde{\mathrm{G}}/\mathrm{Z}_{\widetilde{\mathrm{G}}})(k)$.

To prove this, first note that (as for any semisimple central extension of G by a finite group scheme of multiplicative type) the preimage $\widetilde{\mathrm{T}}$ of T in $\widetilde{\mathrm{G}}$ is a split maximal torus in $\widetilde{\mathrm{G}}$, and there exists a natural identification of root systems and isomorphisms between corresponding root groups for (G,T) and $(\widetilde{\mathrm{G}},\widetilde{\mathrm{T}})$ (Exercise 1.6.13(i) on geometric fibers). The simply connected property for $\widetilde{G}$ implies that the simple positive coroots are a basis of the cocharacter group of $\widetilde{T}$, so no coroot is divisible in the cocharacter lattice. Thus, each pair of opposite root groups of $\widetilde{G}_k$ generates an SL_2 and not a PGL_2. But in $SL_2(k)$ the diagonal points are generated by the k-points of the standard root groups (see (5.3.2) for a classical formula), so the subsets $U_{\pm a}(k)$ in $G(k)$ for $a \in \Delta$ generate the image of $\widetilde{G}(k)$ in $G(k)$. This establishes our claim.

By using a well-chosen $\mathbf{Z}$-basis X_b of $\mathfrak{g}_b$ for every $b \in \Phi$ (a "Chevalley system", as in the proof of the Existence Theorem; see Definition 6.3.2 and especially Remark 6.3.5), the Lie algebra $\mathfrak{g}$ over $\mathbf{Z}$ and elements $\exp_a(tX_a) \in \mathrm{Aut}(\mathfrak{g}_{\mathbf{Z}[t]})$ can be described explicitly in terms of the combinatorics of Φ. For varying fields k this yields a "universal" formula for $\mathfrak{g}_k$ and $\exp_a(tX_a)$ for all $t \in k$, recovering the definition of "Chevalley group of adjoint type" over k as given in [**St67**], especially for finite k and irreducible Φ; this coincides with $\widetilde{G}(k)/Z_{\widetilde{G}}(k)$. The simplicity of $\widetilde{G}(k)/Z_{\widetilde{G}}(k)$ for finite k and irreducible Φ away from the counterexamples $SL_2(\mathbf{F}_2)$, $SL_2(\mathbf{F}_3)/\mathbf{F}_3^\times$, $Sp_4(\mathbf{F}_2) \simeq \mathfrak{S}_6$, and $G_2(\mathbf{F}_2)$ can be established via the structure theory of split semisimple groups over general fields (using (B, N)-pairs). More sophisticated methods are required to prove that $\widetilde{G}(\mathbf{Z})$ is generated by its subgroups $U_a(\mathbf{Z})$ for $a \in \Phi$ (so the subgroup of $\mathrm{Aut}(\mathfrak{g})$ generated by $\exp_a(tX_a)$ for $t \in \mathbf{Z}$ and $a \in \Phi$ coincides with $\widetilde{G}(\mathbf{Z})/Z_{\widetilde{G}}(\mathbf{Z})$); the case of SL_2 is classical, and one can bootstrap from this case (see [**St67**, § 8, Cor. 3 to Thm. 18], [**Ma65**, Thm. 1, Cor. 2(ii)], [**Stein**, Thms. 2.2, 4.1], or [**CGP13**, 5.1, 5.2]).

5.4. Applications to Levi subgroups. — A further application of our study of parabolic subgroups in the relative case is an existence result for Levi subgroups over an affine base. Consider a finite-dimensional Lie algebra $\mathfrak{g}$ over a field k of characteristic 0. The *radical* $\mathfrak{r}$ of $\mathfrak{g}$ is the largest solvable ideal and $\mathfrak{g}/\mathfrak{r}$ is semisimple. A *Levi subalgebra* of $\mathfrak{g}$ is a subalgebra $\mathfrak{s}$ such that $\mathfrak{s} \to \mathfrak{g}/\mathfrak{r}$ is an isomorphism, or equivalently the natural map $\mathfrak{s} \ltimes \mathfrak{r} \to \mathfrak{g}$ is an isomorphism. (In particular, $\mathfrak{s}$ is semisimple.) By the theorem of Levi–Malcev [**Bou1**, I, § 6.8, Thm. 5], Levi subalgebras exist and any two are related through the action of a k-point in the unipotent radical of the linear algebraic k-group $\mathrm{Aut}_{\mathfrak{g}/k}$ (representing the automorphism functor of $\mathfrak{g}$ on the category of k-schemes).

Now consider a smooth affine group G over a general field k. (The case of most interest will be when G is a parabolic subgroup of a connected reductive k-group.) A *Levi k-subgroup* of G is a smooth closed k-subgroup $L \subset G$ such that $L_{\overline{k}} \to G_{\overline{k}}/\mathscr{R}_u(G_{\overline{k}})$ is an isomorphism; equivalently, $L_{\overline{k}} \ltimes \mathscr{R}_u(G_{\overline{k}}) \to G_{\overline{k}}$ is an isomorphism. Informally, L is a k-rational complement to the geometric unipotent radical. (Based on the analogy with Lie algebras, one might consider to define Levi subgroups as complements to the geometric radical. Experience with parabolic subgroups of connected reductive groups shows that complements to the geometric unipotent radical are more useful.)

If k is perfect then $\mathscr{R}_u(\mathrm{G}_{\overline{k}})$ descends to a k-subgroup $\mathscr{R}_u(\mathrm{G}) \subset \mathrm{G}$ and (when L exists!) $\mathrm{L} \ltimes \mathscr{R}_u(\mathrm{G}) \to \mathrm{G}$ is an isomorphism. If $\mathrm{char}(k) = p > 0$ then such an L can fail to exist, even if k is algebraically closed. A counterexample is "$\mathrm{SL}_n(\mathrm{W}_2(k))$ as a k-group" for any $n \geqslant 2$, where W_2 denotes the ring-functor of length-2 Witt vectors. (See [**CGP**, Prop. A.6.4] for a precise formulation and proof, with SL_n replaced by any Chevalley group. This rests on an analysis of root groups relative to suitable maximal tori to reduce the fact that the natural quotient map $\mathrm{W}_2 \twoheadrightarrow \mathbf{G}_{\mathrm{a}}$ has no additive section.)

***Proposition 5.4.1* (Mostow)**. — *If* $\mathrm{char}(k) = 0$ *then Levi k-subgroups of* G *exist and* $(\mathscr{R}_u(\mathrm{G}))(k)$*-conjugation is transitive on the set of such k-subgroups.*

Proof. — More generally, consider a (possibly disconnected) reductive k-group $\overline{\mathrm{G}}$, a unipotent k-group U, and an exact sequence of affine algebraic k-groups

$$1 \to \mathrm{U} \to \mathrm{G} \to \overline{\mathrm{G}} \to 1.$$

We claim that this splits over k as a semi-direct product, and that any two splittings are related through u-conjugation for some $u \in \mathrm{U}(k)$. Using a filtration of U by its (characteristic) derived series, we reduce to the case where U is commutative *provided* that we also show $\mathrm{H}^1(k, \mathrm{U}) = 0$ in the commutative case (so k-rational points conjugating one splitting into another can be lifted through stages of the derived series of U in general).

Since $\mathrm{char}(k) = 0$ and U is commutative, $\mathrm{U} \simeq \mathbf{G}_{\mathrm{a}}^n$ for some n (see Exercise 5.5.10). The endomorphism functor of $\mathbf{G}_{\mathrm{a}}$ on the category of k-algebras is represented by $\mathbf{G}_{\mathrm{a}}$ (i.e., the only additive polynomials over a k-algebra R are $r\mathrm{X}$ for $r \in \mathrm{R}$) since $\mathrm{char}(k) = 0$, so the endomorphism functor of $\mathbf{G}_{\mathrm{a}}^n$ is represented by Mat_n and hence the automorphism functor of $\mathbf{G}_{\mathrm{a}}^n$ is represented by GL_n. It follows that there is a *unique* linear structure on U lifting the one on its Lie algebra, so this structure is compatible with extension on k and equivariant for the natural action of $\overline{\mathrm{G}} = \mathrm{G}/\mathrm{U}$ on the commutative normal k-subgroup U of G. Thus, G is an extension of the possibly disconnected $\overline{\mathrm{G}}$ by a linear representation V of $\overline{\mathrm{G}}$, with $\overline{\mathrm{G}}^0$ a reductive group. The vanishing of $\mathrm{H}^1(k, \mathrm{V})$ is a consequence of additive Hilbert 90, and our task is to show

$$1 \to \mathrm{V} \to \mathrm{G} \to \overline{\mathrm{G}} \to 1$$

splits over k as a semi-direct product, with any two splittings related by v-conjugacy for some $v \in \mathrm{V}(k)$.

Observe that $q : \mathrm{G} \to \overline{\mathrm{G}}$ is a V-torsor for the étale topology on $\overline{\mathrm{G}}$. Before we show that q admits a k-homomorphic section, let's show that it admits a morphic section: the underlying V-torsor (ignoring the group structure of G) is trivial. More generally, for any k-scheme S (such as $\overline{\mathrm{G}}$) the set of V-torsors over S (up to isomorphism) is $\mathrm{H}^1(\mathrm{S}_{\acute{e}t}, \mathrm{V} \otimes_k \mathscr{O}_{\mathrm{S}})$, so it suffices to prove that this cohomology group vanishes when S is affine. By choosing a k-basis of V it suffices to treat the case $\mathrm{V} = k$. By descent theory for quasi-coherent sheaves, $\mathrm{H}^1(\mathrm{S}_{\acute{e}t}, \mathscr{O}_{\mathrm{S}})$ classifies the set of quasi-coherent extensions of $\mathscr{O}_{\mathrm{S}}$ by $\mathscr{O}_{\mathrm{S}}$. Writing $\mathrm{S} = \mathrm{Spec}\,\mathrm{A}$, this corresponds to the set of A-linear extensions of A by itself as an A-module, and any such extension is clearly split. Thus, q admits a section σ as a map of affine k-schemes.

We will modify σ to make it a homomorphism by studying Hochschild cohomology that imitates group cohomology via "algebraic cochains". (See [**Oes**, III, § 3] and § B.2 for a review of this cohomology theory.) Consider the Hochschild cohomology $\mathrm{H}^2(\overline{\mathrm{G}}, \mathrm{V})$. As in the classical setting of group cohomology, by Proposition B.2.5 the obstruction to modifying the choice of σ to make it a homomorphism is a canonically associated class in $\mathrm{H}^2(\overline{\mathrm{G}}, \mathrm{V})$, and if this class vanishes then the set of $\mathrm{V}(k)$-conjugacy classes of such splittings is a torsor for $\mathrm{H}^1(\overline{\mathrm{G}}, \mathrm{V})$. It therefore suffices to show that the higher Hochschild cohomology of $\overline{\mathrm{G}}$ with coefficients in a linear representation vanishes. The formation of such Hochschild cohomology commutes with extension of the ground field (Proposition B.2.2), so we can assume $k = \overline{k}$.

By Lemma B.2.3, for an affine algebraic group scheme H over a field k, its Hochschild cohomology (as a functor on the category of not necessarily finite-dimensional algebraic linear representations for the group) is the derived functor of the functor of H-invariants. Consider algebraic linear representations W of H (i.e., k-vector spaces W equipped with an R-linear action of H(R) on $\mathrm{R} \otimes_k \mathrm{W}$ functorially in all k-algebras R). By [**Wat**, § 3.1–§ 3.3] any such W is the direct limit of its finite-dimensional algebraic subrepresentations, and the formation of Hochschild cohomology commutes with direct limits, so if H has completely reducible finite-dimensional (algebraic) linear representation theory then the higher cohomology vanishes. Now it remains to solve Exercise 1.6.11(ii): if $k = \overline{k}$ with $\mathrm{char}(k) = 0$ then any linear algebraic group H over k with reductive identity component has completely reducible finite-dimensional algebraic linear representation theory.

In view of the natural isomorphism $\mathrm{Hom}_{\mathrm{H}}(\mathrm{W}, \mathrm{W}') = (\mathrm{W}' \otimes \mathrm{W}^*)^{\mathrm{H}}$ for finite-dimensional linear representations W and W′ of H, it suffices to prove that the functor of H-invariants is right-exact. It suffices to separately treat H^0 and the finite constant H/H^0. The case of finite constant groups is settled via averaging since $\mathrm{char}(k) = 0$, and the connected reductive case reduces separately to the cases of split tori and connected semisimple groups. The case of split tori is well-known (in any characteristic), by consideration of graded modules as reviewed just before Proposition B.2.5. For a connected semisimple k-group H and finite-dimensional algebraic linear representation W of H, we have naturally $W^H = W^{\mathfrak{h}}$ via the associated Lie algebra representation on W since $\mathrm{char}(k) = 0$ and H is connected. Thus, it suffices to show that $\mathfrak{h}$ is semisimple when H is connected semisimple.

Suppose to the contrary, so the radical $\mathfrak{r}$ of $\mathfrak{h}$ is nonzero. This subspace of $\mathfrak{h}$ is stable under the adjoint action of H, so we can consider the weight space decomposition of $\mathfrak{r}$ under the restriction of Ad_H to a maximal torus $T \subset H$. If $\mathfrak{r} \subset \mathfrak{h}^T = \mathfrak{t}$ then $\mathfrak{r}$ would lie inside intersection of the $H(k)$-conjugates of $\mathfrak{t}$. The intersection of the $H(k)$-conjugates of T is the finite center Z_H that is étale since $\mathrm{char}(k) = 0$, and it coincides with such an intersection using finitely many $H(k)$-conjugates (due to the noetherian property of G), so the $H(k)$-conjugates of $\mathfrak{t}$ have intersection $\mathrm{Lie}(Z_H) = 0$. Thus, $\mathfrak{r}$ contains $\mathfrak{h}_a$ for some $a \in \Phi(H, T)$, so it contains the line $\mathfrak{h}_{-a}$ in the Ad_H-orbit of $\mathfrak{h}_a$. Thus, the solvable ideal $\mathfrak{r}$ contains the Lie subalgebra generated by $\mathfrak{h}_a$ and $\mathfrak{h}_{-a}$. But this subalgebra is the non-solvable $\mathfrak{sl}_2$, so we have reached a contradiction. □

Over general fields k there is an important class of smooth connected affine k-groups that always admit a Levi k-subgroup: parabolic k-subgroups in connected reductive k-groups. We will use the dynamic method to prove a relative version of this result. (See Corollary 5.2.5 for the notion of unipotent radical in parabolic subgroup schemes of reductive group schemes over any base.) To get started, we define Levi subgroups in the relative setting.

Definition 5.4.2. — Let $G \to S$ be a reductive group scheme, $P \subset G$ a parabolic subgroup. A *Levi subgroup* of P is a smooth closed S-subgroup $L \subset P$ such that $L \ltimes \mathscr{R}_u(P) \to P$ is an isomorphism. The functor $\underline{\mathrm{Lev}}(P)$ assigns to any S-scheme S' the set of Levi S'-subgroups of $P_{S'}$.

If $\lambda : \mathbf{G}_m \to G$ is a 1-parameter subgroup and P is the parabolic subgroup $P_G(\lambda)$ then $Z_G(\lambda)$ is a Levi subgroup because $\mathscr{R}_u(P) = U_G(\lambda)$. To construct Levi S-subgroups more generally (at least when S is affine), we shall use the action of $\mathscr{R}_u(P)$ on $\underline{\mathrm{Lev}}(P)$. For this purpose, it is convenient to first construct a P-equivariant filtration of $\mathscr{R}_u(P)$ with vector bundle successive quotients:

Theorem 5.4.3. — *There is a descending filtration* $\mathscr{R}_u(P) =: U_1 \supset U_2 \supset \dots$ *by* $\underline{\mathrm{Aut}}_{P/S}$*-stable smooth closed* S*-subgroups such that*

(i) for all $s \in S$ *we have* $U_{i,s} = 1$ *if* $i > \dim \mathscr{R}_u(P_s)$*;*

(ii) $uu'u^{-1}u'^{-1} \in U_{i+j}$ *for any points* $u \in U_i$ *and* $u' \in U_j$ (*valued in an* S*-scheme*)*;*

(iii) each commutative S*-group* U_i/U_{i+1} *admits a unique* P*-equivariant* $\mathscr{O}_S$*-linear structure making it a vector bundle* (*so* U_i/U_{i+1} *is canonically identified with* $\mathrm{Lie}(U_i/U_{i+1})$ *respecting the actions of* P)*;*

(iv) the formation of $\{U_i\}$ *is compatible with base change on* S *and functorial with respect to isomorphisms in the pair* (G, P)*.*

This is [**SGA3**, XXVI, 2.1]; the dynamic method simplifies the proof.

Proof. — We first reduce to the adjoint semisimple case. Let $\mathrm{G}^{\mathrm{ad}} = \mathrm{G}/\mathrm{Z}_{\mathrm{G}}$; this contains the parabolic S-subgroup $\mathrm{P}/\mathrm{Z}_{\mathrm{G}}$. We claim that (a) the map $q : \mathrm{G} \twoheadrightarrow \mathrm{G}^{\mathrm{ad}}$ carries $\mathscr{R}_u(\mathrm{P})$ isomorphically onto $\mathscr{R}_u(\mathrm{P}/\mathrm{Z}_{\mathrm{G}})$, (b) $\mathrm{Z}_{\mathrm{G}} = \mathrm{Z}_{\mathrm{P}}$. Once these properties are proved, $\underline{\mathrm{Aut}}_{\mathrm{P}/\mathrm{S}}$ naturally acts on $\mathrm{P}/\mathrm{Z}_{\mathrm{G}}$ and hence the problem for (G, P) is reduced to the one for $(\mathrm{G}^{\mathrm{ad}}, \mathrm{P}/\mathrm{Z}_{\mathrm{G}})$.

To prove (a) and (b), we may work étale-locally on S so that $\mathrm{P} = \mathrm{P}_{\mathrm{G}}(\lambda)$ for some $\lambda : \mathbf{G}_{\mathrm{m}} \to \mathrm{G}$. Then $\mathrm{P} = \mathrm{Z}_{\mathrm{G}}(\lambda) \ltimes \mathrm{U}_{\mathrm{G}}(\lambda)$, so $\mathrm{Z}_{\mathrm{G}} \subset \mathrm{Z}_{\mathrm{G}}(\lambda) \subset \mathrm{P}$ and $\mathrm{U}_{\mathrm{G}}(\lambda) = \mathscr{R}_u(\mathrm{P})$. In particular, $\mathscr{R}_u(\mathrm{P}) \cap \mathrm{Z}_{\mathrm{G}} = 1$. The fibral isomorphism criterion (Lemma B.3.1) and behavior of dynamic constructions with respect to flat quotients over a field [**CGP**, Cor. 2.1.9] imply that q identifies $\mathrm{P}/\mathrm{Z}_{\mathrm{G}}$ with $\mathrm{P}_{\mathrm{G}^{\mathrm{ad}}}(q \circ \lambda)$ and carries $\mathscr{R}_u(\mathrm{P}) = \mathrm{U}_{\mathrm{G}}(\lambda)$ isomorphically onto $\mathrm{U}_{\mathrm{G}^{\mathrm{ad}}}(q \circ \lambda) = \mathscr{R}_u(\mathrm{P}/\mathrm{Z}_{\mathrm{G}})$ (since the map $\mathrm{U}_{\mathrm{G}}(\lambda)_s \to \mathrm{U}_{\mathrm{G}^{\mathrm{ad}}}(q \circ \lambda)_s$ is faithfully flat with trivial kernel for all $s \in \mathrm{S}$). This settles (a). To prove that $\mathrm{Z}_{\mathrm{P}} = \mathrm{Z}_{\mathrm{G}}$ we may localize on S so that P contains a Borel S-subgroup B of G that in turn contains a maximal torus T. Since $\mathrm{Z}_{\mathrm{G}}(\mathrm{T}) = \mathrm{T}$, so $\mathrm{Z}_{\mathrm{P}} \subset \mathrm{T}$, it suffices to show that any point of T centralizing $\mathscr{R}_u(\mathrm{B})$ is central in G. Passing to the case when (G, T) is split, the T-action on $\mathrm{Lie}(\mathscr{R}_u(\mathrm{B}))$ encodes all roots up to a sign. Hence, the centralizer of $\mathscr{R}_u(\mathrm{B})$ in T is contained in the intersection of the kernels of the roots. But this intersection is Z_{G} (Corollary 3.3.6(1)), so (b) is also proved.

Now we may and do assume G is adjoint semisimple (but otherwise arbitrary). We will make a construction satisfying the desired properties in the split case, show it is independent of all choices, and then use descent theory to settle the general (adjoint semisimple) case. Suppose $(\mathrm{G}, \mathrm{T}, \mathrm{M})$ is split and that there is a Borel subgroup $\mathrm{B} \subset \mathrm{P}$ containing T; this situation can always be achieved étale-locally on S. By Corollary 5.2.7(2) and Zariski localization on S we can arrange that there is a (unique) positive system of roots Φ^+ in $\Phi \subset \mathrm{M}$ such that $\Phi^+ = \Phi(\mathrm{B}_s, \mathrm{T}_s)$ for all $s \in \mathrm{S}$. The base $\Delta = \{a_i\}$ of Φ^+ is a basis of the root lattice $\mathbf{Z}\Phi$ that is equal to M (since G is adjoint semisimple). Let $\{\omega_i^\vee\}$ be the dual basis of $\mathrm{M}^\vee \subset \mathrm{X}_*(\mathrm{T})$. By Exercise 4.4.8(ii) (applied on geometric fibers) and Corollary 5.2.7(2), further Zariski localization brings us to the case that $\mathrm{P} = \mathrm{P}_{\mathrm{G}}(\lambda_{\mathrm{I}})$ for a (necessarily unique) subset $\mathrm{I} \subset \Delta$, with $\lambda_{\mathrm{I}} := \sum_{a_i \in \mathrm{I}} \omega_i^\vee$. The merit of this description of P is that λ_{I} is determined by additional group-theoretic data $(\mathrm{B}, \mathrm{T}, \mathrm{M})$ that exist étale-locally on S.

For each $a \in \Phi$, let $\mathrm{U}_a \subset \mathrm{G}$ be the corresponding root group for $(\mathrm{G}, \mathrm{T}, \mathrm{M})$. By Remark 5.2.6, the S-group $\mathrm{U} := \mathscr{R}_u(\mathrm{P}) = \mathrm{U}_{\mathrm{G}}(\lambda_{\mathrm{I}})$ is directly spanned in any order by the U_a's for a such that $\langle a, \lambda_{\mathrm{I}} \rangle \geqslant 1$. More specifically, by Proposition 5.1.16, for all $n \geqslant 1$ the root groups U_a for a satisfying $\langle a, \lambda_{\mathrm{I}} \rangle \geqslant n$ directly span (in any order) a normal smooth closed S-subgroup $\mathrm{U}_{\geqslant n}$ of U such that the successive quotients $\mathrm{U}_{\geqslant n}/\mathrm{U}_{\geqslant n+1}$ are commutative and the natural map

$$\prod_{\langle a,\lambda_{\mathrm{I}}\rangle=n} \mathrm{U}_a \to \mathrm{U}_{\geqslant n}/\mathrm{U}_{\geqslant n+1}$$

defined by multiplication in G is an isomorphism.

Lemma 5.4.4. — *For all $n \geqslant 1$, the subgroup $\mathrm{U}_{\geqslant n}$ is normal in P and the quotient $\mathrm{U}_{\geqslant n}/\mathrm{U}_{\geqslant n+1}$ has a unique P-equivariant $\mathscr{O}_{\mathrm{S}}$-linear structure.*

Proof. — For all $a \in \Phi$, the construction of root groups (see Theorem 4.1.4) provides a canonical T-equivariant isomorphism $\exp_a : \mathbf{W}(\mathfrak{g}_a) \simeq \mathrm{U}_a$ (where $\mathbf{W}(\mathscr{E})$ is the additive S-group scheme underlying a vector bundle $\mathscr{E}$ on S). The equivariance implies that $\exp_a$ identifies the $\mathbf{G}_{\mathrm{m}}$-action on U_a via λ_{I}-conjugation with the $\mathbf{G}_{\mathrm{m}}$-scaling on the line bundle $\mathfrak{g}_a$ via $t^{\langle a,\lambda_{\mathrm{I}}\rangle}$. In particular, the canonical isomorphism

$$\mathbf{W}(\prod_{\langle a,\lambda_{\mathrm{I}}\rangle=n} \mathfrak{g}_a) \simeq \mathrm{U}_{\geqslant n}/\mathrm{U}_{\geqslant n+1}$$

defines a vector bundle structure on the target under which the $\mathbf{G}_{\mathrm{m}}$-action via λ_{I}-conjugation corresponds to multiplication by t^n, so $\mathrm{U}_{\geqslant n}/\mathrm{U}_{\geqslant n+1}$ is the schematic centralizer in $\mathrm{U}/\mathrm{U}_{\geqslant n+1}$ of the action by $\mu_n \subset \mathbf{G}_{\mathrm{m}}$ via λ_{I}-conjugation. Via *descending* induction on n, this characterizes the subgroups $\mathrm{U}_{\geqslant n}$ of $\mathrm{U} = \mathrm{U}_{\mathrm{G}}(\lambda_{\mathrm{I}})$ solely in terms of λ_{I}.

By (5.1.5), for all $n, m \geqslant 1$ and points $u \in \mathrm{U}_{\geqslant n}$ and $u' \in \mathrm{U}_{\geqslant m}$, we have $uu'u^{-1}u'^{-1} \in \mathrm{U}_{\geqslant n+m}$. Thus, conjugation by $\mathrm{U} = \mathrm{U}_{\geqslant 1}$ on the normal subgroups $\mathrm{U}_{\geqslant n}$ and $\mathrm{U}_{\geqslant n+1}$ induces the trivial action on $\mathrm{U}_{\geqslant n}/\mathrm{U}_{\geqslant n+1}$. The preceding characterization of the $\mathrm{U}_{\geqslant n}$'s in terms of λ_{I} implies that $\mathrm{Z}_{\mathrm{G}}(\lambda_{\mathrm{I}})$ normalizes each $\mathrm{U}_{\geqslant n}$, so the subgroup $\mathrm{P} = \mathrm{Z}_{\mathrm{G}}(\lambda_{\mathrm{I}}) \ltimes \mathrm{U}$ normalizes each $\mathrm{U}_{\geqslant n}$. Likewise, the vector bundle structure constructed on $\mathrm{U}_{\geqslant n}/\mathrm{U}_{\geqslant n+1}$ is *uniquely* characterized by identifying λ_{I}-conjugation with scaling against t^n because the only additive automorphisms of $\mathbf{W}(\mathscr{O}^r) = \mathbf{G}_{\mathrm{a}}^r$ centralizing t^n-scaling for all $t \in \mathbf{G}_{\mathrm{m}}$ are the linear automorphisms. Hence, this vector bundle structure on $\mathrm{U}_{\geqslant n}/\mathrm{U}_{\geqslant n+1}$ commutes with $\mathrm{Z}_{\mathrm{G}}(\lambda_{\mathrm{I}})$-conjugation and so more generally commutes with conjugation against $\mathrm{P} = \mathrm{Z}_{\mathrm{G}}(\lambda_{\mathrm{I}}) \ltimes \mathrm{U}$.

The P-equivariant vector bundle structure just built on each $\mathrm{U}_{\geqslant n}/\mathrm{U}_{\geqslant n+1}$ is *unique*. Indeed, any such structure identifies conjugation against $\lambda_{\mathrm{I}} : \mathbf{G}_{\mathrm{m}} \to \mathrm{T} \subset \mathrm{P}$ with a linear action of $\mathbf{G}_{\mathrm{m}}$ that has to be scaling against t^n since we can read off the action on the Lie algebra (as any *linear* action of $\mathbf{G}_{\mathrm{m}}$ on a vector bundle $\mathscr{E}$ is encoded in a weight space decomposition, and canonically $\mathrm{Lie}(\mathbf{W}(\mathscr{E})) \simeq \mathscr{E}$ as vector bundles). $\square$

The S-subgroups $\mathrm{U}_i := \mathrm{U}_{\geqslant i}$ for $i \geqslant 1$ constitute a descending filtration with the desired properties (ii) and (iii) for (G, P) except that we have only shown the U_i's are stable under P-conjugation rather than under the action of $\underline{\mathrm{Aut}}_{\mathrm{P/S}}$ on $\mathrm{U} = \mathscr{R}_u(\mathrm{P})$. Suppose $(\mathrm{B}', \mathrm{T}', \mathrm{M}')$ is another such triple over S, so $\mathrm{B/U}$ and B'/U are Borel subgroups of the reductive S-group P/U. Since $\mathrm{P} \to \mathrm{P/U}$ is a smooth surjection, if S' is an S-scheme then any point in $(\mathrm{P/U})(\mathrm{S}')$ lifts to P over an étale cover of S'. Hence, by Theorem 5.2.11(2) (applied to the reductive P/U) it follows that étale-locally on S we can use P-conjugation to bring B' to B. Once we have arranged that $\mathrm{B}' = \mathrm{B}$, Theorem 3.2.6 (applied to B) allows us to arrange by suitable B-conjugation over an étale cover of S that $\mathrm{T}' = \mathrm{T}$. Zariski-locally over S, this torus equality identifies M' with M.

The $\mathrm{U}_{\geqslant n}$'s are normalized by P and the preceding constructions with them are uniquely characterized via the cocharacter λ_{I} of G uniquely determined by P and the triple $(\mathrm{B}, \mathrm{T}, \mathrm{M})$ with $\mathrm{B} \subset \mathrm{P}$, so the $\mathrm{U}_{\geqslant n}$'s are *independent* of $(\mathrm{B}, \mathrm{T}, \mathrm{M})$. Hence, by descent theory we obtain the descending filtration $\{\mathrm{U}_i\}$ with the desired properties (ii) and (iii) in the general case, as the independence of all choices ensures stability under the entire automorphism functor of P. Property (iv) holds by construction, and property (i) holds provided that the set of values $\langle \lambda_{\mathrm{I}}, a\rangle \geqslant 1$ for $a \in \Phi^+$ is an interval in $\mathbf{Z}$ beginning at 1 (as that ensures the largest such value is at most $\#\Phi_{\lambda_{\mathrm{I}} \geqslant 1} = \dim \mathscr{R}_u(\mathrm{P}_s)$). Since $\langle \lambda_{\mathrm{I}}, a\rangle \in \{0, 1\}$ for all $a \in \Delta$ by definition of λ_{I}, we just need to recall a general property of root systems: any $a \in \Phi^+ - \Delta$ has the form $b + c$ with $b \in \Delta$ and $c \in \Phi^+$ (see [**SGA3**, XXI, 3.1.2] or [**Bou2**, Ch. VI, § 1.6, Prop. 19]). ☐

Proposition 5.4.5. — *For every maximal torus* $\mathrm{T} \subset \mathrm{P}$ *there is a unique Levi* S-*subgroup* $\mathrm{L} \subset \mathrm{P}$ *containing* T.

Proof. — By the uniqueness, standard limit arguments and étale descent let us assume S is strictly henselian local. Thus, T is split and $\Phi := \Phi(\mathrm{G}, \mathrm{T})$ is a root system in $\mathrm{M} := \mathrm{Hom}_{\mathrm{S}}(\mathrm{T}, \mathbf{G}_{\mathrm{m}})$. By Corollary 5.2.7 there exists $\lambda \in \mathrm{X}_*(\mathrm{T}) = \mathrm{M}^\vee$ so that $\mathrm{P} = \mathrm{P}_{\mathrm{G}}(\lambda) = \mathrm{Z}_{\mathrm{G}}(\lambda) \ltimes \mathrm{U}_{\mathrm{G}}(\lambda)$. Since $\mathrm{U}_{\mathrm{G}}(\lambda) = \mathscr{R}_u(\mathrm{P})$ and $\mathrm{Z}_{\mathrm{G}}(\lambda) \to \mathrm{P}/\mathscr{R}_u(\mathrm{P})$ is an isomorphism, so $\mathrm{Z}_{\mathrm{G}}(\lambda)$ is a Levi S-subgroup, it suffices to prove there is only one Levi subgroup L of P containing T.

The subset $\Phi(\mathrm{L}, \mathrm{T})$ of the parabolic set of roots $\Phi(\mathrm{P}, \mathrm{T})$ consists of those $a \in \Phi(\mathrm{P}, \mathrm{T})$ such that $-a \in \Phi(\mathrm{P}, \mathrm{T})$. Indeed, since $\mathrm{L} \to \mathrm{P}/\mathscr{R}_u(\mathrm{P})$ is an isomorphism, it is equivalent to show that $\Phi(\mathscr{R}_u(\mathrm{P}), \mathrm{T})$ is the set of $a \in \Phi(\mathrm{P}, \mathrm{T})$ such that $-a \notin \Phi(\mathrm{P}, \mathrm{T})$; this latter assertion has nothing to do with L. Since $\Phi = -\Phi$ and

$$\Phi(\mathrm{Z}_{\mathrm{G}}(\lambda), \mathrm{T}) = \{a \in \Phi \mid \langle a, \lambda\rangle = 0\}, \;\; \Phi(\mathrm{U}_{\mathrm{G}}(\lambda), \mathrm{T}) = \{a \in \Phi \mid \langle a, \lambda\rangle > 0\},$$

we obtain the desired descriptions of sets of roots. In particular, the subset $\Phi(\mathrm{L}, \mathrm{T}) \subset \Phi$ is determined by the pair (P, T) without reference to L.

For all $a \in \Phi(\mathrm{L}, \mathrm{T}) \subset \Phi$, the root group U_a for the reductive group L satisfies the conditions that uniquely characterize the a-root group of (G, T).

Hence, by consideration of the open cell of (L, T) relative to a positive system of roots in $\Phi(\mathrm{L}, \mathrm{T})$, the group sheaf L is generated as a group sheaf by T and the root groups U_a of (G, T) for all $a \in \Phi(\mathrm{L}, \mathrm{T})$. (More generally, if $\mathscr{G} \to \mathrm{S}$ is a smooth group scheme with connected fibers and $\Omega \subset \mathscr{G}$ is an open subscheme with non-empty fibers over S then the smooth multiplication map $\mathscr{G} \times \mathscr{G} \to \mathscr{G}$ restricts to a smooth map $\Omega \times \Omega \to \mathscr{G}$ that is surjective since the geometric fibers $\mathscr{G}_{\overline{s}}$ are irreducible.) This is an explicit description of L in terms of data (such as $\Phi(\mathrm{L}, \mathrm{T}) \subset \Phi$) that depend only on (P, T). □

Corollary 5.4.6. — [**SGA3**, XXVI, 1.8] *The functor* $\underline{\mathrm{Lev}}(\mathrm{P})$ *of Levi subgroups of* P*, equipped with its natural* $\mathscr{R}_u(\mathrm{P})$*-action via conjugation in* G*, is represented by an* $\mathscr{R}_u(\mathrm{P})$*-torsor. In particular, any Levi* S*-subgroup* L *of* P *is its own schematic normalizer in* P*.*

Proof. — Since $\mathrm{U} := \mathscr{R}_u(\mathrm{P})$ is S-affine and Levi subgroups exist étale-locally on S (e.g., $\mathrm{L} = \mathrm{Z}_{\mathrm{G}}(\lambda)$ when there exists $\lambda : \mathbf{G}_{\mathrm{m}} \to \mathrm{G}$ such that $\mathrm{P} = \mathrm{P}_{\mathrm{G}}(\lambda)$), it suffices to show that the sheaf $\underline{\mathrm{Lev}}(\mathrm{P})$ is a U-torsor sheaf. Using general S, it suffices to show that if $\mathrm{L}, \mathrm{L}' \subset \mathrm{P}$ are Levi S-subgroups then $\mathrm{L}' = u\mathrm{L}u^{-1}$ for a unique $u \in \mathrm{U}(\mathrm{S})$. The uniqueness allows us to work étale-locally on S, so we may assume that L and L′ contain respective maximal S-tori T and T′. By further étale localization we may arrange that $\mathrm{T}' = p\mathrm{T}p^{-1}$ for some $p \in \mathrm{P}(\mathrm{S})$ (see Theorem 3.2.6 applied to P).

Writing $p = ug$ for unique $u \in \mathrm{U}(\mathrm{S})$ and $g \in \mathrm{L}(\mathrm{S})$, by replacing T with $g\mathrm{T}g^{-1}$ we may assume $\mathrm{T}' = u\mathrm{T}u^{-1}$. Thus, L′ and $u\mathrm{L}u^{-1}$ are Levi S-subgroups of P containing the same maximal torus T′, so $\mathrm{L}' = u\mathrm{L}u^{-1}$ by Proposition 5.4.5. It remains to prove uniqueness of u, which expresses the property that $\mathrm{N}_{\mathrm{P}}(\mathrm{L}) = \mathrm{L}$ (since $\mathrm{L} \ltimes \mathrm{U} = \mathrm{P}$). That is, if $uxu^{-1} \in \mathrm{L}$ for all $x \in \mathrm{L}$ then we wish to prove $u = 1$. Obviously $uxu^{-1}x^{-1} \in \mathrm{L}$ for all $x \in \mathrm{L}$, but $u(xu^{-1}x^{-1}) \in \mathrm{U}$, so the triviality of $\mathrm{L} \cap \mathrm{U}$ implies that $u = xux^{-1}$ for all $x \in \mathrm{L}$. In other words, $u \in \mathrm{Z}_{\mathrm{G}}(\mathrm{L})$. But $\mathrm{Z}_{\mathrm{G}}(\mathrm{L}) \subset \mathrm{Z}_{\mathrm{G}}(\mathrm{T}) = \mathrm{T} \subset \mathrm{L}$, so $u \in \mathrm{L} \cap \mathrm{U} = 1$. □

Remark 5.4.7. — If G is semisimple and simply connected then for every Levi S-subgroup L of a parabolic S-subgroup P, the semisimple derived group $\mathscr{D}(\mathrm{L})$ is also simply connected. To prove this fact, which we will never use but is important in practice, by working étale-locally on S we may assume $\mathrm{L} = \mathrm{Z}_{\mathrm{G}}(\lambda)$ for a closed subtorus $\lambda : \mathbf{G}_{\mathrm{m}} \hookrightarrow \mathrm{T}$ of a maximal S-torus $\mathrm{T} \subset \mathrm{G}$. Then we may apply Exercise 6.5.2(iv) to conclude.

The following result uses non-abelian degree-1 Čech cohomology for the étale topology with group sheaves. (A geometric interpretation of this cohomology via torsors is given in Exercise 2.4.11 for smooth affine group schemes, and when the well-known low-degree formalism in Galois cohomology over fields is expressed in terms of the étale topology rather than Galois groups then it easily adapts to construct a 6-term exact sequence of pointed sets associated to any short exact sequence of smooth affine group schemes.)

Corollary 5.4.8. — *If* S *is affine then* P *admits a Levi* S*-subgroup* L *and the natural map* $j_{\mathrm{L}} : \mathrm{H}^1(\mathrm{S}_{\text{ét}}, \mathrm{L}) \to \mathrm{H}^1(\mathrm{S}_{\text{ét}}, \mathrm{P})$ *is an isomorphism.*

Proof. — Let $\mathrm{U} = \mathscr{R}_u(\mathrm{P})$, so $\underline{\mathrm{Lev}}(\mathrm{P})$ is represented by a U-torsor over S. The existence of a Levi S-subgroup L means exactly that this torsor is trivial. Hence, to find L it suffices to show that *every* U-torsor over S is trivial, which is to say that $\mathrm{H}^1(\mathrm{S}_{\text{ét}}, \mathrm{U}) = 1$. The descending filtration $\{\mathrm{U}_i\}$ provided by Theorem 5.4.3 reduces this to the vanishing of each $\mathrm{H}^1(\mathrm{S}_{\text{ét}}, \mathrm{U}_i/\mathrm{U}_{i+1})$ for vector bundles $\mathrm{U}_i/\mathrm{U}_{i+1}$. Such vanishing holds because S is affine.

Injectivity of j_{L} is clear since the composite $\mathrm{L} \to \mathrm{P} \to \mathrm{P}/\mathrm{U}$ is an isomorphism. For surjectivity it suffices to show that $f : \mathrm{H}^1(\mathrm{S}_{\text{ét}}, \mathrm{P}) \to \mathrm{H}^1(\mathrm{S}_{\text{ét}}, \mathrm{P}/\mathrm{U})$ is injective. Since P naturally acts on the short exact sequence

$$1 \to \mathrm{U} \to \mathrm{P} \to \mathrm{P}/\mathrm{U} \to 1$$

as well as on the descending filtration $\{\mathrm{U}_i\}$ of U respecting the vector bundle structure on each $\mathrm{U}_i/\mathrm{U}_{i+1}$, a representative Čech 1-cocycle c for $\xi \in \mathrm{H}^1(\mathrm{S}_{\text{ét}}, \mathrm{P})$ gives descent datum throughout to built an étale-twisted form

$$1 \to \mathrm{U}_c \to \mathrm{P}_c \to (\mathrm{P}/\mathrm{U})_c \to 1$$

and descending terminating filtration $\{\mathrm{U}_{c,i}\}$ of U_c consisting of smooth normal closed S-subgroups such that each $\mathrm{U}_{c,i}/\mathrm{U}_{c,i+1}$ is a vector bundle.

The choice of c provides an identification of sets $\mathrm{H}^1(\mathrm{S}_{\text{ét}}, \mathrm{P}) \simeq \mathrm{H}^1(\mathrm{S}_{\text{ét}}, \mathrm{P}_c)$ carrying the fiber of f through ξ over to the image of $\mathrm{H}^1(\mathrm{S}_{\text{ét}}, \mathrm{U}_c) \to \mathrm{H}^1(\mathrm{S}_{\text{ét}}, \mathrm{P}_c)$. Hence, it suffices to prove $\mathrm{H}^1(\mathrm{S}_{\text{ét}}, \mathrm{U}_c) = 1$. The descending filtration $\{\mathrm{U}_{c,i}\}$ of U_c reduces this to the vanishing of $\mathrm{H}^1(\mathrm{S}_{\text{ét}}, \mathrm{U}_{c,i}/\mathrm{U}_{c,i+1})$ for all i. Such vanishing holds because S is affine. □

To conclude our discussion of Levi subgroups, we use them to address the existence of dynamic descriptions of parabolic subgroups in the relative setting. First we provide motivation over a general field k. An ingredient in the Borel–Tits theory of relative root systems is that *any* parabolic k-subgroup Q of a connected reductive k-group G admits a dynamic description as $\mathrm{P}_{\mathrm{G}}(\lambda)$ for a 1-parameter k-subgroup $\lambda : \mathbf{G}_{\mathrm{m}} \to \mathrm{G}$ (see [**CGP**, Prop. 2.2.9] for a proof); any such λ is valued in $\mathrm{Z}_{\mathrm{G}}(\lambda) \subset \mathrm{P}_{\mathrm{G}}(\lambda) = \mathrm{Q}$. Since $\mathrm{Z}_{\mathrm{G}}(\lambda)$ is a Levi k-subgroup of Q, the $\mathscr{R}_u(\mathrm{Q})(k)$-conjugacy of all Levi k-subgroups of Q (Corollary 5.4.6) implies that *every* Levi k-subgroup $\mathrm{L} \subset \mathrm{Q}$ arises as $\mathrm{Z}_{\mathrm{G}}(\lambda)$ for some such λ.

The dynamic method produces parabolic subgroups Q and Levi subgroups $\mathrm{L} \subset \mathrm{Q}$ in reductive group schemes G over any base scheme S when 1-parameter subgroups are provided over S, so it is natural to ask if such pairs (Q, L) always arise in the form $(\mathrm{P}_{\mathrm{G}}(\lambda), \mathrm{Z}_{\mathrm{G}}(\lambda))$ for some $\lambda : \mathbf{G}_{\mathrm{m}} \to \mathrm{G}$ over S. The case of connected semi-local S (i.e., $\mathrm{S} = \mathrm{Spec}(\mathrm{A})$ for nonzero A with finitely many maximal ideals and no nontrivial idempotents) is addressed with affirmative results in [**SGA3**, XXVI, 6.10–6.14].

Over any S, if $\mathrm{Q} = \mathrm{P}_{\mathrm{G}}(\lambda)$ for some $\lambda : \mathbf{G}_{\mathrm{m}} \to \mathrm{G}$ then not only does Q admit a Levi S-subgroup, namely $\mathrm{Z}_{\mathrm{G}}(\lambda) \subset \mathrm{Q}$, but by Corollary 5.4.6 every Levi S-subgroup $\mathrm{L} \subset \mathrm{Q}$ has the form $\mathrm{Z}_{\mathrm{G}}(\mu)$ for some μ in the $\mathscr{R}_u(\mathrm{Q})(\mathrm{S})$-conjugacy class of λ. In [**G**, § 7.3] a deeper study of parabolic subgroups and their Levi subgroups is combined with the structure of automorphism schemes of reductive group schemes (developed in § 7 below) to show that for any connected S the dynamic method produces *all* pairs (Q, L) consisting of a parabolic S-subgroup $\mathrm{Q} \subset \mathrm{G}$ and Levi S-subgroup $\mathrm{L} \subset \mathrm{Q}$.

In particular, if S is a connected *affine* scheme then every parabolic S-subgroup Q in a reductive S-group G admits a dynamic description because such Q always admit a Levi S-subgroup (proved by non-dynamic means, such as the vanishing of cohomological obstructions as in the proof of Corollary 5.4.8). If we drop the affineness hypothesis then the cohomological proof of Corollary 5.4.8 breaks down and it can happen that the $\mathscr{R}_u(\mathrm{P})$-torsor $\underline{\mathrm{Lev}}(\mathrm{P})$ is nontrivial, so the parabolic S-subgroup P in G has *no* dynamic description. I am grateful to Edixhoven for suggesting the following counterexamples.

Example 5.4.9. — Let S be a scheme such that the group $\mathrm{H}^1(\mathrm{S}, \mathscr{O}_{\mathrm{S}}) = \mathrm{H}^1(\mathrm{S}_{\text{ét}}, \mathscr{O}_{\mathrm{S}})$ is nonzero, which is to say that S admits a nontrivial $\mathbf{G}_{\mathrm{a}}$-torsor U (for the étale topology, or equivalently for the Zariski topology). For example, S could be a smooth proper and geometrically connected curve with positive genus over a field k, or $\mathrm{S} = \mathbf{A}^2_k - \{(0,0)\}$. (Note that S is not affine.) We shall use U to make a nontrivial $\mathbf{P}^1$-bundle E over S admitting a section σ such that the automorphism scheme G of E is semisimple and the G-stabilizer P of σ is a parabolic S-subgroup with no Levi subgroup.

Let $\mathbf{G}_{\mathrm{a}}$ act on $\mathbf{P}^1_{\mathrm{S}}$ via the isomorphism $j : x \mapsto u(x) := \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$ onto the strictly upper-triangular subgroup of the S-group $\mathrm{PGL}_2 = \mathrm{Aut}_{\mathbf{P}^1_{\mathrm{S}}/\mathrm{S}}$. Consider the pushout $\mathrm{E} = \mathrm{U} \times^{\mathbf{G}_{\mathrm{a}}} \mathbf{P}^1_{\mathrm{S}}$ of U along the inclusion j of $\mathbf{G}_{\mathrm{a}}$ into PGL_2; by definition, this is the quotient of $\mathrm{U} \times \mathbf{P}^1_{\mathrm{S}}$ modulo the anti-diagonal $\mathbf{G}_{\mathrm{a}}$-action $x.(y,t) = (y + x, u(-x)(t))$. Informally, E is obtained by replacing the affine line $\mathbf{P}^1_{\mathrm{S}} - \{\infty\}$ with U. The $\mathbf{P}^1$-bundle $\mathrm{E} \to \mathrm{S}$ is equipped with an evident $j(\mathbf{G}_{\mathrm{a}})$-invariant section $\sigma \in \mathrm{E}(\mathrm{S})$ such that the S-scheme $\mathrm{E} - \sigma(\mathrm{S})$ is the $\mathbf{G}_{\mathrm{a}}$-torsor U, so $\mathrm{E} - \sigma(\mathrm{S}) \to \mathrm{S}$ has no global section. The construction of (E, σ) as a twisted form of $(\mathbf{P}^1, \infty)$ has no effect on the relative tangent line along the section, so the line bundle $\mathrm{T}_\sigma(\mathrm{E})$ over S is globally trivial.

Let G be the automorphism S-scheme of E; it is a Zariski-form of PGL_2 since the $\mathbf{G}_{\mathrm{a}}$-torsor U is trivial Zariski-locally on S. Let P be the G-stabilizer of $\sigma \in \mathrm{E}(\mathrm{S})$, so $\mathrm{P} \subset \mathrm{G}$ is clearly a Borel subgroup. The action of P on $\mathrm{T}_\sigma(\mathrm{E})$ defines a character $\chi : \mathrm{P} \to \mathbf{G}_{\mathrm{m}}$ whose kernel is seen to be $\mathscr{R}_u(\mathrm{P})$ by computing Zariski-locally over S. Thus, to show that P has no Levi S-subgroup it is equivalent to show that χ has no homomorphic section over S. Suppose there is such a section $\lambda : \mathbf{G}_{\mathrm{m}} \to \mathrm{P}$, so the natural map $\mathrm{U}_{\mathrm{G}}(-\lambda) - \{1\} \to (\mathrm{G}/\mathrm{P}) - \{1\} = \mathrm{E} - \sigma(\mathrm{S})$ is an isomorphism of S-schemes (as we see by working locally over S). In particular, every section in $\mathrm{U}_{\mathrm{G}}(-\lambda)(\mathrm{S})$ meets the identity section, so $\mathrm{U}_{\mathrm{G}}(-\lambda)$ cannot be isomorphic to $\mathbf{G}_{\mathrm{a}}$ as S-groups.

The action of $\mathbf{G}_{\mathrm{m}}$ on $\mathfrak{g} = \mathrm{Lie}(\mathrm{G})$ through $\mathrm{Ad}_{\mathrm{G}} \circ \lambda$ has weights $\{\pm a\}$ where $a(t) = t$, and the corresponding root groups $\mathrm{U}_{\pm a}$ are precisely $\mathrm{U}_{\mathrm{G}}(\pm\lambda)$. The weight space $\mathfrak{g}_{-a}$ is identified with the line bundle $\mathrm{T}_e(\mathrm{G}/\mathrm{P}) = \mathrm{T}_\sigma(\mathrm{E})$ that is globally trivial, so the isomorphism $\exp_{-a} : \mathbf{W}(\mathfrak{g}_{-a}) \simeq \mathrm{U}_{-a}$ (see Theorem 4.1.4) implies that $\mathrm{U}_{\mathrm{G}}(-\lambda) \simeq \mathbf{G}_{\mathrm{a}}$ as S-groups, contrary to what we saw above. Hence, there is no such λ, so P has no Levi S-subgroup.

5.5. Exercises. —

Exercise 5.5.1. — Let G be a smooth connected affine group over a field k.

(i) For a maximal k-torus T in G (see Remark A.1.2) and a smooth connected k-subgroup N in G that is normalized by T, prove that $\mathrm{T}\bigcap\mathrm{N}$ is a maximal k-torus in N (e.g., smooth and connected!). Show by example that $\mathrm{S}\bigcap\mathrm{N}$ can be disconnected for a non-maximal k-torus S. Hint: first analyze $\mathrm{Z_G(T)}\bigcap\mathrm{N}$ using $\mathrm{T}\ltimes\mathrm{N}$ to reduce to the case when T is central in G, and then pass to G/T.

(ii) Let H be a smooth connected normal k-subgroup of G, and P a parabolic k-subgroup. Prove $(\mathrm{P}_{\overline{k}}\bigcap\mathrm{H}_{\overline{k}})^0_{\mathrm{red}}$ is a parabolic subgroup of $\mathrm{H}_{\overline{k}}$, and use Theorem 1.1.9 (applied to H) to prove $\mathrm{P}\bigcap\mathrm{H}$ is connected (hint: work over $\overline{k}$).

(iii) For H as in (ii), by using that $\mathrm{Q}=\mathrm{N_H(Q)}$ scheme-theoretically for parabolic Q in H (Corollary 5.2.8), prove $\mathrm{P}\bigcap\mathrm{H}$ in (ii) is smooth and therefore parabolic in H. (Hint: when $k=\overline{k}$, prove $(\mathrm{P}\bigcap\mathrm{H})^0_{\mathrm{red}}$ is normal in P, hence in $\mathrm{P}\bigcap\mathrm{H}$.) In particular, prove that the scheme-theoretic intersection $\mathrm{B}\bigcap\mathrm{H}$ is a Borel k-subgroup of H for all Borel k-subgroups B of G.

Exercise 5.5.2. — This exercise generalizes Theorem 5.1.19 to the reductive case. Let G be a connected reductive group over a field k, Z its maximal central k-torus, and $\mathrm{G}'=\mathscr{D}(\mathrm{G})$ its semisimple derived group. Let $\{\mathrm{G}'_i\}$ be the k-simple factors of G'. Prove that they are precisely the minimal nontrivial normal smooth connected non-central k-subgroups of G, and that the multiplication homomorphism

$$\mathrm{Z}\times\prod\mathrm{G}'_i\to\mathrm{G}$$

is a central isogeny. (Keep in mind that if k is finite then $\mathrm{G}(k)$ is not Zariski-dense in G, so in general an argument is needed to prove that the G'_i are normal in G.) Also prove that the normal connected semisimple k-subgroups of G' are necessarily normal in G (the converse being obvious).

Exercise 5.5.3. — Let R be Dedekind with fraction field K, and G a connected reductive K-group. A reductive R-group scheme is *quasi-split* if it has a Borel subgroup over the base (see Definition 5.2.10).

(i) Show that $\mathrm{G}=\mathscr{G}_{\mathrm{K}}$ for a reductive group scheme $\mathscr{G}$ over a dense open $\mathrm{U}\subset\operatorname{Spec}\mathrm{R}$.

(ii) Assume R is a henselian (e.g., complete) discrete valuation ring and that $\mathrm{G}=\mathscr{G}_{\mathrm{K}}$ for a reductive R-group $\mathscr{G}$. Using $\mathrm{Bor}_{\mathscr{G}/\mathrm{R}}$, prove that if the special fiber $\mathscr{G}_0$ is split (resp. quasi-split) then so is G over K. What if R is not assumed to be henselian?

(iii) Using (i) and (ii), show that if G is a connected reductive group over a global field F then $\mathrm{G}_{\mathrm{F}_v}$ is quasi-split for all but finitely many places v of F.

Likewise show that any G-torsor over F admits an F_v-point for all but finitely many v. See Exercise 7.3.5 for analogues with the property of being split.

Exercise 5.5.4. — For a split adjoint semisimple group (G, T, M) over a non-empty scheme S, Example 5.3.9 used that any base Δ for Φ is a basis for M. Using Corollary 3.3.6, explain why Δ being a basis for M *characterizes* the adjoint property for G.

Exercise 5.5.5. — This exercise develops an important special case of Exercise 3.4.5, the group of "norm-1 units" in a central simple algebra.

(i) Linear derivations of a matrix algebra over a field are precisely the inner derivations (i.e., $x \mapsto yx - xy$ for some y); see [**DF**, Ch. 3, Exer. 30] for a proof based on a clever application of the Skolem–Noether theorem. Combining this with length-induction on artin local rings, prove the Skolem–Noether theorem for $\mathrm{Mat}_n(\mathrm{R})$ for any artin local ring R (i.e., all R-algebra automorphisms of $\mathrm{Mat}_n(\mathrm{R})$ are conjugation by a unit). Deduce $\mathrm{PGL}_n \simeq \underline{\mathrm{Aut}}_{\mathrm{Mat}_n/\mathbf{Z}}$.

(ii) Let A be a central simple algebra with dimension n^2 over a field k. Build an affine k-scheme I of finite type such that naturally in k-algebras R,

$$\mathrm{I}(\mathrm{R}) = \mathrm{Isom}_{\mathrm{R}\text{-alg}}(\mathrm{A}_\mathrm{R}, \mathrm{Mat}_n(\mathrm{R})).$$

Note that $\mathrm{I}(\overline{k})$ is non-empty. Prove I is smooth by checking the infinitesimal criterion for $\mathrm{I}_{\overline{k}}$ with the help of (i). Deduce that $\mathrm{A}_\mathrm{K} \simeq \mathrm{Mat}_n(\mathrm{K})$ for a finite *separable* extension K/k.

(iii) By (ii), we can choose a finite Galois extension K/k and a K-algebra isomorphism $\theta : \mathrm{A}_\mathrm{K} \simeq \mathrm{Mat}_n(\mathrm{K})$, and by Skolem–Noether this is unique up to conjugation by a unit. Prove that for any choice of θ, the determinant map transfers to a multiplicative map $\underline{\mathrm{A}}_\mathrm{K} \to \mathbf{A}^1_\mathrm{K}$ which is independent of θ. Deduce that it is $\mathrm{Gal}(\mathrm{K}/k)$-equivariant, and so descends to a multiplicative map $\mathrm{Nrd}_{\mathrm{A}/k} : \underline{\mathrm{A}} \to \mathbf{A}^1_k$ which "becomes" the determinant over *any* extension F/k for which $\mathrm{A}_\mathrm{F} \simeq \mathrm{Mat}_n(\mathrm{F})$. Prove that $\mathrm{Nrd}^n_{\mathrm{A}/k} = \mathrm{N}_{\mathrm{A}/k}$ (explaining the name *reduced norm* for $\mathrm{Nrd}_{\mathrm{A}/k}$), and conclude that $\underline{\mathrm{A}}^\times = \mathrm{Nrd}^{-1}_{\mathrm{A}/k}(\mathbf{G}_\mathrm{m})$.

(iv) Let $\mathrm{SL}(\mathrm{A}) = \ker(\mathrm{Nrd}_{\mathrm{A}/k} : \underline{\mathrm{A}}^\times \to \mathbf{G}_\mathrm{m})$ (denoted $\mathrm{SL}_{m,\mathrm{D}}$ if $\mathrm{A} = \mathrm{Mat}_m(\mathrm{D})$ for a central division algebra D over k). Prove that its formation commutes with any extension of the ground field, and that it becomes isomorphic to SL_n over $\overline{k}$. In particular, SL(A) is a connected semisimple k-group that is the derived group of the connected reductive $\underline{\mathrm{A}}^\times$. (In contrast, $\ker \mathrm{N}_{\mathrm{A}/k}$ is non-smooth whenever $\mathrm{char}(k)|n$ and is usually disconnected.)

(v) Using the preceding constructions and Galois descent, generalize the bijective correspondence in Exercise 4.4.6(ii) to central simple algebras over any field (possibly finite).

Exercise 5.5.6. — This exercise builds on Exercise 5.5.5 to prove a special case of a conjugacy result of Borel and Tits for maximal split tori in connected reductive groups over fields. Let A be a central simple algebra over a field k, T a k-torus in $\underline{\mathrm{A}}^{\times}$ containing $\mathrm{Z}_{\underline{\mathrm{A}}^{\times}} = \mathbf{G}_{\mathrm{m}}$, and A_{T} the corresponding étale commutative k-subalgebra of A (with $\dim_k \mathrm{A}_{\mathrm{T}} = \dim \mathrm{T}$) as in Exercise 4.4.6.

(i) Prove that SL(A) is k-anisotropic if and only if A is a division algebra.

(ii) Prove that the centralizer $\mathrm{B}_{\mathrm{T}} = \mathrm{Z}_{\mathrm{A}}(\mathrm{A}_{\mathrm{T}})$ is a semisimple k-algebra with center A_{T}.

(iii) If T is k-split, prove $\mathrm{A}_{\mathrm{T}} \simeq k^r$ and that the simple factors B_i of B_{T} are *central* simple k-algebras.

(iv) Assume T is k-split. Using (iii), prove T is maximal as a k-split torus in $\underline{\mathrm{A}}^{\times}$ if and only if the (central!) simple factors B_i of B_{T} are *division algebras*.

(v) Fix an isomorphism $\mathrm{A} \simeq \mathrm{End}_{\mathrm{D}}(\mathrm{V})$ for a right module V over a central division algebra D, and consider $(\mathrm{T}, \{\mathrm{B}_i\})$ as in (iv), so V is a left A-module and $\mathrm{V} = \prod \mathrm{V}_i$ with *nonzero* left B_i-modules V_i. If T is maximal as a k-split torus in $\underline{\mathrm{A}}^{\times}$, prove V_i has rank 1 over B_i and D, so $\mathrm{B}_i \simeq \mathrm{D}$. Using D-bases, deduce that *all maximal k-split tori in $\underline{\mathrm{A}}^{\times}$ are $\underline{\mathrm{A}}^{\times}(k)$-conjugate.*

Exercise 5.5.7. — In the proof of Theorem 5.3.5, we used that any flat affine group scheme G of finite type over a Dedekind domain R occurs as a closed subgroup of some GL_n over R.

(i) Prove the analogous result over fields by adapting whatever is your favorite proof for *smooth* affine groups over fields.

(ii) Make your argument in (i) work over R (for flat affine groups of finite type) by using that any finitely generated torsion-free R-module is projective (and hence a direct summand of a finite free R-module).

Exercise 5.5.8. — Let G be a reductive group over a scheme S. Show that if P is a parabolic subgroup of G then $\mathrm{Z}_{\mathrm{G}} \subset \mathrm{P}$ and that $\mathrm{P} \mapsto \mathrm{P}/\mathrm{Z}_{\mathrm{G}}$ is a bijective correspondence between the sets of parabolic subgroups of G and of $\mathrm{G}^{\mathrm{ad}} = \mathrm{G}/\mathrm{Z}_{\mathrm{G}}$, with inverse given by the formation of inverse images under the quotient map $\mathrm{G} \to \mathrm{G}^{\mathrm{ad}}$. Construct natural isomorphisms $\mathrm{Par}_{\mathrm{G}/\mathrm{S}} \simeq \mathrm{Par}_{\mathrm{G}^{\mathrm{ad}}/\mathrm{S}}$ and $\mathrm{Bor}_{\mathrm{G}/\mathrm{S}} \simeq \mathrm{Bor}_{\mathrm{G}^{\mathrm{ad}}/\mathrm{S}}$.

Exercise 5.5.9. — Let G be a reductive group over a non-empty scheme S, and $\mathrm{Z} \subset \mathrm{Z}_{\mathrm{G}}$ a flat central closed subgroup scheme (so Z is of multiplicative type). This exercise addresses splitting properties for G/Z given splitting hypotheses on G and Z.

(i) Prove that the smooth S-affine quotient $\mathrm{G}' = \mathrm{G}/\mathrm{Z}$ is a reductive S-group, and that if $\mathrm{T} \subset \mathrm{G}$ is a maximal torus (so $\mathrm{Z} \subset \mathrm{Z}_{\mathrm{G}} \subset \mathrm{Z}_{\mathrm{G}}(\mathrm{T}) = \mathrm{T}$) then so is $\mathrm{T}' := \mathrm{T}/\mathrm{Z} \subset \mathrm{G}'$. Give an example over a field in which T' is split, T is non-split, and Z_{G} is not a direct factor of T.

(ii) Consider a split triple (G, T, M), and assume Z is split, so $\mathrm{X}(\mathrm{Z}) = \overline{\mathrm{M}}_{\mathrm{S}}$ for a quotient $\overline{\mathrm{M}} = \mathrm{M}/\mathrm{M}'$ of M. For each $a \in \Phi \subset \mathrm{M}$ (so $a|_{\mathrm{Z}_{\mathrm{G}}} = 1$), let a'

denote the induced character of $\mathrm{T}' = \mathrm{T}/\mathrm{Z}$. Prove that $\mathrm{X}(\mathrm{T}') = \mathrm{M}'_{\mathrm{S}}$ inside $\mathrm{X}(\mathrm{T}) = \mathrm{M}_{\mathrm{S}}$, and that $a' \in \mathrm{M}'$ for all a.

(iii) For $(\mathrm{G}, \mathrm{T}, \mathrm{M})$ as in (ii), prove that the natural map $\mathrm{U}_a \to \mathrm{U}'_{a'}$ between root groups is an isomorphism (hint: fibral isomorphism criterion). Deduce that the line bundle $\mathfrak{g}'_{a'}$ on S is *globally trivial*, and that $(\mathrm{G}', \mathrm{T}', \mathrm{M}')$ is split as in Definition 5.1.1 (note Example 5.1.2).

Exercise 5.5.10. — Let U be a smooth connected unipotent group over a field k. If k is perfect then U is split (i.e., admits a composition series with successive quotients k-isomorphic to $\mathbf{G}_{\mathrm{a}}$), by [**Bo91**, 15.5(ii)] or [**SGA3**, XVII, 4.1.3]. Now assume $\mathrm{char}(k) = 0$.

(i) Let $\mathrm{U}_n \subset \mathrm{GL}_n$ be the smooth connected unipotent k-subgroup of strictly upper-triangular matrices, so the Lie subalgebra $\mathrm{Lie}(\mathrm{U}_n) \subset \mathfrak{gl}_n = \mathrm{Mat}_n(k)$ consists of nilpotent matrices. Equip $\mathrm{Lie}(\mathrm{U}_n)$ with the "Baker–Campbell–Hausdorff" (BCH) group law; this law is algebraic rather than formal, due to uniform control on the nilpotence. Prove the k-scheme map $\exp : \mathrm{Lie}(\mathrm{U}_n) \to \mathrm{U}_n$ is a k-group isomorphism and that if $\mathrm{U} \subset \mathrm{U}_n$ is a *commutative* closed k-subgroup then $\exp(\mathrm{Lie}(\mathrm{U})) \subset \mathrm{U}$ and $\exp : \mathrm{Lie}(\mathrm{U}) \to \mathrm{U}$ is a k-group isomorphism.

(ii) Let U be a *split* unipotent k-group, so U arises as in (i) (by [**Bo91**, 15.4(i)]). Equip Lie(U) with the BCH group law. Prove there is a unique k-group isomorphism $\mathrm{U} \simeq \mathrm{Lie}(\mathrm{U})$ lifting the identity on Lie algebras. In particular, $\mathrm{U} \simeq \mathbf{G}_{\mathrm{a}}^r$ when U is commutative. (This conclusion fails if $\mathrm{char}(k) > 0$ due to k-groups of truncated Witt vectors W_r for $r \geqslant 2$, so the existence of linear structures in Theorem 5.4.3(iii) is remarkable in positive characteristic.)

6. Existence, Isomorphism, and Isogeny Theorems

6.1. Pinnings and main results. — In § 1.5 we introduced the notion of a *pinning* on a triple $(\mathrm{G}, \mathrm{T}, \mathrm{B})$ over an algebraically closed field k. The purpose of that concept was to "rigidify" the triple (eliminating the action of the adjoint torus) so that passage to the root datum loses no information concerning isomorphisms. The Isomorphism Theorem for split reductive group schemes over a non-empty scheme S requires a relative version of pinnings, and there is a generalization (the Isogeny Theorem) that incorporates isogenies. The purpose of this preliminary section is to develop several concepts related to pinnings and morphisms of root data. At the end of this section we state the Existence, Isomorphism, and Isogeny Theorems over any scheme $\mathrm{S} \neq \varnothing$.

Definition 6.1.1. — Let $(\mathrm{G}, \mathrm{T}, \mathrm{M})$ be a split reductive group over a non-empty scheme S, and let $\mathrm{R}(\mathrm{G}, \mathrm{T}, \mathrm{M}) = (\mathrm{M}, \Phi, \mathrm{M}^\vee, \Phi^\vee)$ be its associated root datum. A *pinning* on $(\mathrm{G}, \mathrm{T}, \mathrm{M})$ is a pair $(\Phi^+, \{\mathrm{X}_a\}_{a\in\Delta})$ consisting of a positive system of roots $\Phi^+ \subset \Phi$ (or equivalently, a base Δ of Φ) and trivializing sections $\mathrm{X}_a \in \mathfrak{g}_a(\mathrm{S})$ for each simple positive root $a \in \Delta$.

The 5-tuple $(\mathrm{G}, \mathrm{T}, \mathrm{M}, \Phi^+, \{\mathrm{X}_a\}_{a\in\Delta})$ is a *pinned* split reductive S-group.

Since Δ determines Φ^+, we will usually write $(\mathrm{G}, \mathrm{T}, \mathrm{M}, \{\mathrm{X}_a\}_{a\in\Delta})$ and suppress the explicit mention of Φ^+. In Exercise 6.5.1 a more "group-theoretic" definition of pinnings over S is given, replacing the trivializations of simple positive root spaces $\mathfrak{g}_a$ with suitable homomorphisms from SL_2 into $\mathscr{D}(\mathrm{Z}_\mathrm{G}(\mathrm{T}_a))$ for each $a \in \Delta$. Keep in mind that the definition of the "split" property for $(\mathrm{G}, \mathrm{T}, \mathrm{M})$ in Definition 5.1.1 includes the condition that the line bundles $\mathfrak{g}_a$ are free of rank 1, so a pinning $(\Phi^+, \{\mathrm{X}_a\}_{a\in\Delta})$ can be chosen for any $\Phi^+ \subset \Phi$.

Remark 6.1.2. — It may be tempting to expect that the choice of Φ^+ is "the same" as a choice of Borel subgroup of G containing T as in the classical case, but that it not true when S is disconnected (and we must allow the base scheme to be disconnected for descent theory arguments). More precisely, by Proposition 5.2.3 and Corollary 5.2.7(2), $\Phi^+ = \Phi(\mathrm{B}, \mathrm{T})$ for a unique Borel subgroup $\mathrm{B} \subset \mathrm{G}$ containing T, and the Borel subgroups of G containing T that we obtain by varying Φ^+ are precisely those B for which the subgroup

$$\Phi(\mathrm{B}, \mathrm{T}) \subset \mathrm{Hom}_{\text{S-gp}}(\mathrm{T}, \mathbf{G}_\mathrm{m}) = \Gamma(\mathrm{S}, \mathrm{X}(\mathrm{T})) = \Gamma(\mathrm{S}, \mathrm{M}_\mathrm{S})$$

lies inside the subgroup M of "constant sections". In particular, when S is disconnected there are Borel subgroups B' of G containing T that do not arise from any choice of $\Phi^+ \subset \Phi$. For this reason, in the relative theory we work throughout with a choice of Φ^+ rather than with a choice of B (although the two viewpoints coincide when S is connected, such as in the theory over a field, domain, or local ring).

There is an evident notion of *isomorphism* between pinned split reductive S-groups. The definition of *isogeny* incorporating pinnings (refining the notion for smooth S-affine S-groups as in Definition 3.3.9) requires care to account for Frobenius isogenies between root groups on geometric fibers in positive characteristic. As motivation, consider pinned split reductive S-groups

$$(\mathrm{G}', \mathrm{T}', \mathrm{M}', \{\mathrm{X}'_{a'}\}_{a' \in \Delta'}), \quad (\mathrm{G}, \mathrm{T}, \mathrm{M}, \{\mathrm{X}_a\}_{a \in \Delta})$$

over S and a quasi-finite surjective S-homomorphism $f : \mathrm{G}' \to \mathrm{G}$ such that $f(\mathrm{T}') \subset \mathrm{T}$. (In Proposition 6.1.10 we will show that f is necessarily finite and flat, hence an isogeny.) Note that $f : \mathrm{T}' \to \mathrm{T}$ is an isogeny, since the map

$$\mathrm{X}(f) : \mathrm{M}_{\mathrm{S}} = \mathrm{X}(\mathrm{T}) \to \mathrm{X}(\mathrm{T}') = \mathrm{M}'_{\mathrm{S}}$$

induces a finite-index inclusion of lattices on geometric fibers over S.

There is an open cover $\{\mathrm{U}_i\}$ of S such that the map induced by $\mathrm{X}(f)$ on U_i-sections carries M into M′. We now suppose (as may be achieved by working Zariski-locally on S) that the map induced by $\mathrm{X}(f)$ on global sections over S carries M into M′. In particular, we get an isomorphism $\mathrm{M}_{\mathbf{Q}} \simeq \mathrm{M}'_{\mathbf{Q}}$ between $\mathbf{Q}$-vector spaces.

By the classical theory on geometric fibers, for each $a' \in \Phi'$ the root group $(\mathrm{U}_{a'})_{\overline{s}}$ for $(\mathrm{G}'_{\overline{s}}, \mathrm{T}'_{\overline{s}})$ is carried isogenously by $f_{\overline{s}}$ onto the root group of $(\mathrm{G}_{\overline{s}}, \mathrm{T}_{\overline{s}})$ for a unique $a(s) \in \Phi$. Since $f_{\overline{s}}$ is a (possibly non-central) isogeny, every root in $\Phi(\mathrm{G}_{\overline{s}}, \mathrm{T}_{\overline{s}})$ arises in this way from a unique $a' \in \Phi'$. Each resulting map between root groups of the $\overline{s}$-fibers is identified with an endomorphism of $\mathbf{G}_{\mathrm{a}}$ having the form $x \mapsto cx^{q(s)}$ for some $c \in k(\overline{s})^{\times}$ and some integral power $q(s) \geqslant 1$ of the characteristic exponent of $k(s)$, due to the equivariance of $f_{\overline{s}} : (\mathrm{U}_{a'})_{\overline{s}} \to \mathrm{U}_{a(s)}$ with respect to the isogeny $\mathrm{T}'_{\overline{s}} \to \mathrm{T}_{\overline{s}}$. It follows that $\mathrm{X}(f_{\overline{s}})(a(s)) = q(s)a'$. Likewise, $\mathrm{X}_*(f_{\overline{s}})(a'^{\vee}) = q(s)a(s)^{\vee}$ by the construction of coroots in the classical theory. The map $\Phi' \to \Phi$ defined by $a' \mapsto a(s)$ has very weak dependence on s:

Lemma 6.1.3. — *For $a' \in \Phi'$, the associated function $s \mapsto a(s) \in \Phi(\mathrm{G}_{\overline{s}}, \mathrm{T}_{\overline{s}}) = \Phi$ is Zariski-locally constant in s.*

Proof. — We may assume that S is noetherian, so every pair of distinct points $\{s, \eta\}$ in S with s in the closure of η can be dominated by the spectrum of a discrete valuation ring. Since our problem is to prove a constancy result on the connected components of S, and every point s of S is in the closure of the generic point of each irreducible component of S through s, by pullback to the spectra of discrete valuation rings we may and do assume $\mathrm{S} = \operatorname{Spec} \mathrm{R}$ for a discrete valuation ring R. In this case we have to prove that $a(s) = a(\eta)$ in Φ, where s is the closed point of S and η is the generic point of S.

Viewing $a(s)$ as an S-homomorphism $\mathrm{T} \to \mathbf{G}_m$ (i.e., a global section of $\mathrm{X}(\mathrm{T})$) via the inclusion $\Phi \subset \mathrm{M}$, the saturation of $\mathbf{Z}a(s)$ in M defines a split

S-subtorus $\mathrm{T}_{a(s)}$ of relative codimension 1 in T. The isogeny $\mathrm{T}' \to \mathrm{T}$ between split S-tori must carry $\mathrm{T}'_{a'}$ into $\mathrm{T}_{a(s)}$ since we can check this on the fibers over a single geometric closed point of S (such as s). Likewise using a geometric generic point of S gives that $\mathrm{T}'_{a'}$ is carried into $\mathrm{T}_{a(\eta)}$. Hence, $a(\eta) = \pm a(s)$ (since $a(s)$ and $a(\eta)$ lie in the reduced root system Φ).

Consider the cocharacters $\lambda' = a'^{\vee} \in \mathrm{M}'^{\vee}$ and $\lambda = f \circ \lambda' = q(s)a(s)^{\vee} \in \mathrm{M}^{\vee}$. The map f carries $\mathrm{U}_{a'} := \mathrm{U}_{\mathrm{G}'}(a'^{\vee}) = \mathrm{U}_{\mathrm{G}'}(\lambda')$ into $\mathrm{U}_{\mathrm{G}}(\lambda) := \mathrm{U}_{\mathrm{G}}(q(s)a(s)^{\vee}) = \mathrm{U}_{\mathrm{G}}(a(s)^{\vee}) = \mathrm{U}_{a(s)}$ (see Theorem 4.1.7(1) and Proposition 4.1.10(2)), so passing to $\overline{\eta}$-fibers gives that the elements $a(\eta)$ and $a(s)$ in Φ have the same root groups for $(\mathrm{G}_{\overline{\eta}}, \mathrm{T}_{\overline{\eta}})$. Hence, $a(\eta) = a(s)$. $\square$

By Lemma 6.1.3, it is reasonable to impose the additional requirement on f that there exists a (necessarily unique) bijection $d : \Phi' \to \Phi$ and a prime power $q_{a'} \geqslant 1$ for each $a' \in \Phi'$ so that the map $\mathrm{M} \to \mathrm{M}'$ induced by $\mathrm{X}(f)$ carries $d(a')$ to $q_{a'}a'$ and the map induced by its dual $\mathrm{X}_*(f)$ carries $a'^{\vee}$ to $q_{a'}d(a')^{\vee}$, where the integer $q_{a'}$ is an integral power of the characteristic exponent of $k(s)$ for each $s \in \mathrm{S}$. Indeed, the preceding discussion shows that Zariski-locally on S, every quasi-finite surjection $(\mathrm{G}', \mathrm{T}') \to (\mathrm{G}, \mathrm{T})$ satisfies these conditions.

Since $\mathrm{X}(f)_{\mathbf{Q}}$ is an isomorphism and d is injective, the conditions $d(-a') \mapsto q_{-a'}(-a') = -q_{-a'}a'$ and $d(a') \mapsto q_{a'}a'$ force $d(-a')$ and $d(a')$ to be distinct linearly dependent elements of the reduced root system Φ, so $d(-a') = -d(a')$. Likewise, $q_{-a'} = q_{a'}$.

Lemma 6.1.4. — *The quasi-finite f carries the* S*-subgroup scheme $\mathrm{U}'_{a'} \subset \mathrm{G}'$ into the* S*-subgroup $\mathrm{U}_{d(a')} \subset \mathrm{G}$ via a homomorphism that is $\mathbf{G}_{\mathrm{m}}$-equivariant for respective conjugation against $a'^{\vee} = q_{a'}d(a')^{\vee}$ and $d(a')^{\vee}$. Moreover, if $q_{a'} = p^n$ with a prime p and $n > 0$ then $p = 0$ in $\mathscr{O}_{\mathrm{S}}$ (so $(-1)^{q_{a'}} = -1$ in $\mathscr{O}_{\mathrm{S}}$) and for all $a' \in \Phi'$ there exists a unique isomorphism of line bundles*

$$f_{a'} : (\mathfrak{g}'_{a'})^{\otimes q(a')} \simeq \mathfrak{g}_{d(a')}$$

such that $\mathrm{U}'_{a'} \to \mathrm{U}_{d(a')}$ is given by $\exp_{a'}(\mathrm{X}') \mapsto \exp_{d(a')}(f_{a'}(\mathrm{X}'^{\otimes q(a')}))$ for all $\mathrm{X}' \in \mathfrak{g}'_{a'}$.

Proof. — Since $d(a') \circ f = q_{a'}a'$, f carries $\mathrm{T}'_{a'}$ into (hence onto) $\mathrm{T}_{d(a')}$. Thus, f carries $\mathrm{Z}_{\mathrm{G}'}(\mathrm{T}'_{a'})$ into $\mathrm{Z}_{\mathrm{G}}(\mathrm{T}_{d(a')})$ via a quasi-finite surjection. But $f \circ a'^{\vee} = q_{a'}d(a')^{\vee}$, so we can pass to the induced map between semisimple derived groups by working with the rank-1 split tori $a'^{\vee}(\mathbf{G}_{\mathrm{m}})$ and $d(a')^{\vee}(\mathbf{G}_{\mathrm{m}})$. This brings us to the case that G' and G are semisimple with fibers of rank 1. For $\lambda' = a'^{\vee}$ and $\lambda = f \circ \lambda' = q_{a'}d(a')^{\vee}$ we have $\mathrm{U}'_{a'} = \mathrm{U}_{\mathrm{G}'}(a'^{\vee}) = \mathrm{U}_{\mathrm{G}'}(\lambda')$ and $\mathrm{U}_{d(a')} = \mathrm{U}_{\mathrm{G}}(d(a')^{\vee}) = \mathrm{U}_{\mathrm{G}}(\lambda)$ (see Theorem 4.1.7(1)), so f carries $\mathrm{U}'_{a'}$ into $\mathrm{U}_{d(a')}$ by Proposition 4.1.10(2).

By the definition of a split reductive S-group, the root spaces admit global trivializations (as line bundles on S). Choose such trivializations for (G, T) and $(\mathrm{G}', \mathrm{T}')$, so we get S-group isomorphisms $\mathrm{U}'_{a'} \simeq \mathbf{G}_{\mathrm{a}}$ and $\mathrm{U}_{d(a')} \simeq \mathbf{G}_{\mathrm{a}}$ that are respectively T'-equivariant and T-equivariant via the respective scaling actions on $\mathbf{G}_{\mathrm{a}}$ by a' and $d(a')$. In other words, the map induced by f between the root groups becomes a surjective endomorphism $f : \mathbf{G}_{\mathrm{a}} \to \mathbf{G}_{\mathrm{a}}$ that satisfies

$$f(a'(t')x) = d(a')(f(t'))f(x) = a'(t')^{q_{a'}} f(x)$$

for points t' of T' and x of $\mathbf{G}_{\mathrm{a}}$ over S. Since $a' : \mathrm{T}' \to \mathbf{G}_{\mathrm{m}}$ is an fppf covering, it follows that $f(ux) = u^{q_{a'}} f(x)$ for all points u of $\mathbf{G}_{\mathrm{m}}$ and x of $\mathbf{G}_{\mathrm{a}}$. By the relative schematic density of $\mathbf{G}_{\mathrm{m}}$ in $\mathbf{G}_{\mathrm{a}}$ over S, the same identity holds with u permitted to be any point of $\mathbf{G}_{\mathrm{a}}$.

Letting $q = q_{a'}$, we claim that $f(x) = cx^q$ for a unique unit c on the base (so the existence and uniqueness of $f_{a'}$ will follow). The uniqueness is clear, so we may work Zariski-locally on S for existence. Hence, we can assume that f is given by a polynomial map $x \mapsto c_0 + c_1 x + \cdots + c_n x^n$ for some integer $n \geqslant 0$ and some global functions $c_0, \ldots, c_n$ on the base. Since $f(0) = 0$ we have $c_0 = 0$, and the identity $f(ux) = u^q f(x)$ implies that $c_j u^j = u^q c_j$ for all j. If $j \neq q$ then fppf-locally on S there exists a unit u such that $u^{q-j} - 1$ is a unit, so $c_j = 0$ on S. Hence, $f(x) = cx^q$ for some c on S. The maps induced by f between root groups on geometric fibers over S are isogenies, so c is nowhere-vanishing, which is to say that c is a unit on S.

Finally, we have to show that $p = 0$ in $\mathscr{O}_{\mathrm{S}}$ if $q = p^n$ for a prime p and $n > 0$. The homomorphism property for f and unit property for c imply that $x \mapsto x^q$ is additive in $\mathscr{O}_{\mathrm{S}}$ (so $(-1)^q = -1$ in $\mathscr{O}_{\mathrm{S}}$). Assume $q = p^n$ for a prime p and $n > 0$, so $(x+y)^q - x^q - y^q$ involves the monomial $x^{p^{n-1}} y^{q-p^{n-1}}$ with a binomial coefficient whose p-adic ordinal is 1 and involves the monomial xy^{q-1} with a coefficient of q, so the greatest common divisor of all monomial coefficients (in $\mathbf{Z}$) is p. Hence, $p = 0$ in $\mathscr{O}_{\mathrm{S}}$ in such cases. $\square$

Proposition 6.1.5. — *In the setting of Lemma* 6.1.4*, the $\mathscr{O}_{\mathrm{S}}$-linear isomorphisms $f_{a'}$ and $f_{-a'}$ are dual relative to the canonical dualities for the pair $\mathfrak{g}'_{a'}, \mathfrak{g}'_{-a'}$ and the pair $\mathfrak{g}_{d(a')}, \mathfrak{g}_{-d(a')} = \mathfrak{g}_{d(-a')}$. Moreover, if X' is a trivializing section of $\mathfrak{g}_{a'}$ and $\mathrm{X} := f_{a'}(\mathrm{X}'^{\otimes q_{a'}})$ is the associated trivialization of $\mathfrak{g}_{d(a')}$ then $f(w_{a'}(\mathrm{X}')) = w_a(\mathrm{X})$ with $a := d(a')$.*

See Corollary 5.1.9 for the definition of $w_a(\mathrm{X})$ for any $a \in \Phi$ and any trivializing section X of $\mathfrak{g}_a$.

Proof. — By passing to derived groups of torus centralizers, we can reduce to the case of groups that are fiberwise semisimple of rank 1. Now consider such groups, so we can let $q = q_{a'} = q_{-a'}$ and $a = d(a')$ (so $-a = d(-a')$). The maps $f_{\pm a'}$ are $\mathscr{O}_{\mathrm{S}}$-linear, so by using the induced map between open cells and the unique characterization of the coroots (and the duality pairing between root spaces in (4.2.1)) a straightforward calculation gives that if local sections X' of $\mathfrak{g}'_{a'}$ and Y' of $\mathfrak{g}'_{-a'}$ satisfy $1+\mathrm{X}'\mathrm{Y}' \in \mathbf{G}_{\mathrm{m}}$ then

$$1 + f_{a'}(\mathrm{X}'^{\otimes q}) f_{-a'}(\mathrm{Y}'^{\otimes q}) = (1+\mathrm{X}'\mathrm{Y}')^q = 1 + (\mathrm{X}'\mathrm{Y}')^q.$$

By taking $\mathrm{Y}' := u\mathrm{X}'^{-1}$ for a unit u such that $1+u$ is a unit (as may be done fppf locally on S), the respective trivializing sections $f_{a'}(\mathrm{X}'^{\otimes q})$ and

$$f_{-a'}(\mathrm{Y}'^{\otimes q}) = f_{-a'}(u^q(\mathrm{X}'^{-1})^{\otimes q}) = u^q f_{-a'}((\mathrm{X}'^{-1})^{\otimes q})$$

of $\mathfrak{g}_a$ and $\mathfrak{g}_{-a}$ have pairing equal to u^q. Thus, the asserted duality compatibility between $f_{a'}$ and $f_{-a'}$ holds.

By definition, $w_{a'}(\mathrm{X}') = \exp_{a'}(\mathrm{X}')\exp_{-a'}(-\mathrm{Y}')\exp_{a'}(\mathrm{X}')$ where Y' is the trivialization of $\mathfrak{g}'_{-a'}$ linked to X'. The preceding discussion shows that $\mathrm{Y} := f_{-a'}(\mathrm{Y}'^{\otimes q})$ is the trivialization of $\mathfrak{g}_{-d(a')}$ linked to X, so (using that $(-1)^{q_{a'}} = -1$ in $\mathscr{O}_{\mathrm{S}}$) we have

$$f(w_{a'}(\mathrm{X}')) = \exp_{d(a')}(\mathrm{X})\exp_{d(-a')}(-\mathrm{Y})\exp_{d(a')}(\mathrm{X}) = w_{d(a')}(\mathrm{X})$$

since $d(-a') = -d(a')$. □

Definition 6.1.6. — Let $(\mathrm{G},\mathrm{T},\mathrm{M})$ and $(\mathrm{G}',\mathrm{T}',\mathrm{M}')$ be split reductive groups over a scheme $\mathrm{S} \neq \varnothing$. A quasi-finite surjection $f : (\mathrm{G}',\mathrm{T}') \to (\mathrm{G},\mathrm{T})$ is *compatible with the splittings* if there is a homomorphism $h : \mathrm{M} \to \mathrm{M}'$, bijection $d : \Phi' \to \Phi$, and function $q : \Phi' \to \mathbf{Z}_{\geqslant 1}$ valued in prime powers such that:

1. the induced map $\mathrm{X}(f) : \mathrm{M}_{\mathrm{S}} = \mathrm{X}(\mathrm{T}) \to \mathrm{X}(\mathrm{T}') = \mathrm{M}'_{\mathrm{S}}$ arises from h,
2. for all $a' \in \Phi'$, $h(d(a')) = q(a')a'$ and $h^\vee(a'^\vee) = q(a')d(a')^\vee$,
3. if $q(a') > 1$ is a power of a prime $p(a')$ then S is a $\mathbf{Z}/p(a')\mathbf{Z}$-scheme.

In the setting of the preceding definition, $h_{\mathbf{Q}}$ is an isomorphism (since each f_s is an isogeny) and f uniquely determines h, d, and q. We will sometimes write $f : (\mathrm{G}',\mathrm{T}',\mathrm{M}') \to (\mathrm{G},\mathrm{T},\mathrm{M})$ to denote that f is compatible with the splittings. In [**SGA3**, XXII, 4.2.1] it is only required that f is quasi-finite between the derived subgroups (or rather, this property between derived groups is a consequence of other conditions imposed there), in which case $h_{\mathbf{Q}}$ may be neither injective nor surjective when there are nontrivial central tori; we only consider quasi-finite surjective f. Our interest in f that are compatible with splittings is due to the following immediate consequence of the preceding considerations.

Proposition 6.1.7. — *For split reductive* $(\mathrm{G},\mathrm{T},\mathrm{M})$ *and* $(\mathrm{G}',\mathrm{T}',\mathrm{M}')$ *over a scheme* $\mathrm{S} \neq \varnothing$*, any quasi-finite surjection* $f : (\mathrm{G}',\mathrm{T}') \to (\mathrm{G},\mathrm{T})$ *is compatible with the splittings Zariski-locally on* S. *If* S *is connected then* f *is compatible with the splittings over* S.

We are now led to:

Definition 6.1.8. — Let $\mathrm{R}' = (\mathrm{X}', \Phi', \mathrm{X}'^\vee, \Phi'^\vee)$ and $\mathrm{R} = (\mathrm{X}, \Phi, \mathrm{X}^\vee, \Phi^\vee)$ be reduced root data, and p a prime or 1. A *p-morphism* $\mathrm{R}' \to \mathrm{R}$ is a triple (h, d, q) consisting of a homomorphism $h : \mathrm{X} \to \mathrm{X}'$, a bijection $d : \Phi' \to \Phi$, and a function $q : \Phi' \to \{p^n\}_{n \geqslant 0}$ such that

1. the induced map $h_{\mathbf{Q}}$ is an isomorphism (i.e., h is a finite-index injection),
2. for all $a' \in \Phi'$, $h(d(a')) = q(a')a'$ and $h^\vee(a'^\vee) = q(a')d(a')^\vee$,

For p-morphisms $(h, d, q) : \mathrm{R}' \to \mathrm{R}$ and $(h', d', q') : \mathrm{R}'' \to \mathrm{R}'$, the *composition* $\mathrm{R}'' \to \mathrm{R}$ is $(h' \circ h, d \circ d', (q \circ d') \cdot q')$. If S is a non-empty scheme, a p-morphism (h, d, q) is called a *$p(\mathrm{S})$-morphism* if S is a $\mathbf{Z}/p(a')\mathbf{Z}$-scheme whenever $q(a') > 1$ is a power of a prime $p(a')$.

Since the root data are reduced and q takes values in $\mathbf{Z}_{\geqslant 1}$, the condition $h(d(a')) = q(a')a'$ implies that h determines d and q. The notion of *isomorphism* between root data is the evident one, and clearly a p-morphism that is an isomorphism must be a 1-morphism. Note also that for a p-morphism $\mathrm{R}' \to \mathrm{R}$, the map $h : \mathrm{X} \to \mathrm{X}'$ goes in the "other" direction, whereas the map $d : \Phi' \to \Phi$ goes in the "same" direction. This is motivated by the examples arising from split reductive group schemes:

Example 6.1.9. — For a non-empty scheme S and quasi-finite surjection $f : (\mathrm{G}', \mathrm{T}', \mathrm{M}') \to (\mathrm{G}, \mathrm{T}, \mathrm{M})$ compatible with the splittings, the associated triple $\mathrm{R}(f) := (h, d, q)$ is a $p(\mathrm{S})$-morphism between the root data where either $p = 1$ or $\mathrm{char}(k(s)) = p > 1$ for all $s \in \mathrm{S}$, and this is compatible with composition and base change.

By Proposition 3.3.10, a quasi-finite surjection $f : (\mathrm{G}', \mathrm{T}') \to (\mathrm{G}, \mathrm{T})$ over S is a central isogeny (in the sense of Definition 3.3.9) if and only if $\ker f_{\overline{s}} \subset \mathrm{T}'_{\overline{s}}$ for all $s \in \mathrm{S}$ (since T' contains $\mathrm{Z}_{\mathrm{G}'}$ and any normal subgroup scheme of multiplicative type in a connected group scheme over a field is necessarily central). We claim that f is a central isogeny if and only if the $p(\mathrm{S})$-morphism $\mathrm{R}(f)$ of root data satisfies $q(a') = 1$ for all $a' \in \Phi'$.

Assume $q(a') > 1$ for some a', so $\mathrm{U}_{a'} \to \mathrm{U}_{d(a')}$ has kernel $\alpha_{q(a')} \neq 1$ (cf. proof of Lemma 6.1.4). This cannot be contained in $\mathrm{T}' = \mathrm{Z}_{\mathrm{G}'}(\mathrm{T}')$, so $\ker f$ is non-central in such cases. Conversely, if $q(a') = 1$ for all a' then f restricts to an isomorphism between corresponding root groups by Lemma 6.1.4. To prove that f is a central isogeny in such cases it suffices to check on geometric fibers (Proposition 3.3.10), so we may assume $\mathrm{S} = \mathrm{Spec}\, k$ for an algebraically closed field k. By looking at f between compatible open cells, the isomorphism condition between root groups forces $(\ker f)^0 \subset \mathrm{T}'$, so $(\ker f)^0$ is of multiplicative type. Thus, the normality of $(\ker f)^0$ in the smooth k-group G' forces centrality since G' is connected and any group of multiplicative type has an étale automorphism scheme. By the same reasoning, it suffices to show that $\ker f \subset \mathrm{T}'$, so we can replace G' with $\mathrm{G}'/(\ker f)^0$ to reduce to the case that $\ker f$ is étale. The normality then again implies centrality, so we are done.

In view of the characterization of central isogenies between split reductive S-group schemes in Example 6.1.9, a 1-morphism $(h, d, q) : \mathrm{R}' \to \mathrm{R}$ between reduced root data is also called a *central isogeny*. The condition that $q(a') = 1$ for all $a' \in \Phi'$ says precisely that $h(d(a')) = a'$ and $h^\vee(a'^\vee) = d(a')^\vee$ for all $a' \in \Phi'$, or equivalently h induces a bijection between the sets of roots (with inverse d) and $h^\vee$ induces a bijection between the sets of coroots (with inverse equal to the "dual" of d). In non-central cases, h does not carry Φ into Φ'.

Proposition 6.1.10. — *Let $f : \mathrm{G}' \to \mathrm{G}$ be a homomorphism between reductive group schemes. If f_s is an isogeny for all $s \in \mathrm{S}$ then f is an isogeny.*

Proof. — As we saw in the discussion preceding Proposition 3.3.10, f is necessarily a quasi-finite flat surjection and it suffices to show that $\ker f$ is S-finite. By limit arguments, we may assume that S is noetherian. In view of the quasi-finiteness of f, finiteness is equivalent to properness. Thus, by the valuative criterion we can assume that $\mathrm{S} = \operatorname{Spec} \mathrm{R}$ for a discrete valuation ring R, with fraction field K and residue field k. We may and do assume that R is strictly henselian (so G' is S-split).

Since R is henselian, we can apply the structure theorem for quasi-finite morphisms [**EGA**, IV_4, 18.5.11(a),(c)]: for any quasi-finite separated S-scheme X, there exists a unique open and closed subscheme $\mathrm{X}' \subset \mathrm{X}$ that is S-*finite* and satisfies $\mathrm{X}'_k = \mathrm{X}_k$. In particular, the formation of X' is functorial in X and compatible with products over S, so if X is an S-group then X' is an S-subgroup. Consider the unique open and closed finite S-subgroup $\mathrm{H}' \subset \ker f$ with special fiber $\ker f_k$ (so H' is also flat). Clearly $\ker f$ is finite if and only if $\mathrm{H}' = \ker f$.

We claim that H' is normal in G'; i.e., the closed subgroup $\mathrm{N}_{\mathrm{G}'}(\mathrm{H}') \subset \mathrm{G}'$ from Proposition 2.1.2 is equal to G'. Since G' is S-split, by consideration of an open cell we see that $\mathrm{G}'(\mathrm{S})$ is fiberwise dense in G' (because for any field F, infinite subset $\Sigma \subset \mathrm{F}$, and dense open $\Omega \subset \mathbf{A}^n_{\mathrm{F}}$, $\Omega \cap \Sigma^n$ is Zariski-dense in Ω). Thus, by [**EGA**, IV_3, 11.10.9], the set of sections $\mathrm{G}'(\mathrm{S})$ is relatively schematically dense in G' over S in the sense of [**EGA**, IV_3, 11.10.8, 11.10.2, 11.10.1(d)]. Hence, to prove $\mathrm{N}_{\mathrm{G}'}(\mathrm{H}') = \mathrm{G}'$ it suffices to check equality on R-points, which is to say that $\mathrm{G}'(\mathrm{S})$ normalizes H'. By the uniqueness of H', such normalizing follows from the normality of $\ker f$ in G'.

A robust theory of quotients of finitely presented S-affine schemes modulo the free action of a finite locally free S-group scheme is developed in [**SGA3**, V]; see especially [**SGA3**, V, § 2(a), Thm. 4.1(iv)]. In particular, if $\mathscr{G}$ is a finitely presented relatively affine group and N is a normal closed subgroup that is finite locally free over the base then the fppf quotient group sheaf $\mathscr{G}/\mathrm{N}$ is represented by a finitely presented relatively affine group and $\mathscr{G} \to \mathscr{G}/\mathrm{N}$ is fppf with kernel N, so it is an N-torsor for the fppf topology. As a special case, G'/H' exists as a reductive group scheme and $\mathrm{G}' \to \mathrm{G}'/\mathrm{H}'$ is finite fppf with kernel H'. We may replace G' with G'/H', so now f_k is an isomorphism.

Our problem is to show that f is an isomorphism, and it is equivalent to check this on the generic fiber. Let T' be a maximal torus in G', so it is split (as R is strictly henselian); fix an isomorphism $\mathrm{T}' \simeq \mathrm{D}_{\mathrm{S}}(\mathrm{M})$ for a finite free $\mathbf{Z}$-module M. The kernel of $f|_{\mathrm{T}'} : \mathrm{T}' \to \mathrm{G}$ is a quasi-finite closed subgroup of T' with trivial special fiber, and by Exercise 2.4.2 any quasi-finite closed subgroup of the torus T' is finite. Thus, the S-group $\ker(f|_{\mathrm{T}'})$ is S-finite with trivial special fiber, so it is trivial. It follows that $f|_{\mathrm{T}'}$ is a monomorphism, so it is a closed immersion since T' is of multiplicative type (Lemma B.1.3). Hence, we may and do also view T' as a maximal torus of G and f as a map $(\mathrm{G}', \mathrm{T}', \mathrm{M}) \to (\mathrm{G}, \mathrm{T}', \mathrm{M})$ between reductive S-groups that is compatible with the splittings (using the identity map on M). As in Example 6.1.9 (and the discussion preceding it), we get an induced $p(\mathrm{S})$-morphism $\mathrm{R}(f)$ between the root data. But this map of root data can be computed using any fiber, so working with the special fibers implies that $\mathrm{R}(f)$ is an isomorphism. In particular, $\mathrm{R}(f)$ is central. Passing to the (split) generic fibers, it follows that the isogeny f_{K} is a central (Example 6.1.9), so the degree of f_{K} can be read off from the restriction $\mathrm{T}'_{\mathrm{K}} \to \mathrm{T}_{\mathrm{K}}$. But $\mathrm{R}(f_{\mathrm{K}}) = \mathrm{R}(f)$ is an isomorphism, so f_{K} is an isomorphism. □

We now deduce some elementary properties of p-morphisms (h, d, q) between reduced root data that were established immediately above Lemma 6.1.4 for triples (h, d, q) that arise from quasi-finite surjections between split reductive group schemes. Namely, we claim that always $d(-a') = -d(a')$ and $q(-a') = q(a')$. To prove this, first note that by the isomorphism property for $h_{\mathbf{Q}}$, $d(a')$ and $d(-a')$ are linearly dependent in M. By injectivity of d, we have $d(a') \neq d(-a')$. Hence, $d(-a') = -d(a')$ for all $a' \in \Phi'$ (due to reducedness of the root data). The identity $h(d(-a')) = q(-a') \cdot (-a')$ then implies that $q(-a') = q(a')$ for all $a' \in \Phi'$.

Remark 6.1.11. — In [**SGA3**, XXI, 6.1.1, 6.8.1], the notions of *morphism* and *p-morphism* between reduced root data are defined (with an integer $p \geqslant 1$). In the definition of a *morphism* there, q is identically 1 and h is only required to be $\mathbf{Z}$-linear (so $\ker h$ may be nontrivial and $h_{\mathbf{Q}}$ may not be surjective); this is intended to encode homomorphisms between split reductive group schemes with an isogeny condition between the derived groups but no such condition between the maximal central tori. The p-morphisms in [**SGA3**, XXI, 6.8.1] are a variant on our notion of p-morphism in which p is any integer $\geqslant 1$ and h is only required to be $\mathbf{Z}$-linear.

Lemma 6.1.12. — *Let* $(h, d, q) : \mathrm{R}' \to \mathrm{R}$ *be a p-morphism between reduced root data, and let* ${\Phi'}^+$ *be a positive system of roots in* Φ'*, with* Δ' *its base of simple roots. Then* $\Phi^+ := d({\Phi'}^+)$ *is a positive system of roots in* Φ *and* $\Delta := d(\Delta')$ *is its base of simple roots.*

Proof. — Pick a linear form λ' on $\mathrm{X}'_{\mathbf{Q}}$ such that ${\Phi'}^+ = \Phi'_{\lambda'>0}$, and let $\lambda = \lambda' \circ h_{\mathbf{Q}}$. The relations $h(d(a')) = q(a')a'$ for $a' \in \Phi$ with $q(a') \in \mathbf{Q}_{>0}$ and the isomorphism property for $h_{\mathbf{Q}}$ imply that λ is non-vanishing on $d(\Phi')$ and that $\Phi_{\lambda>0} = d({\Phi'}^+) =: \Phi^+$, so indeed Φ^+ is a positive system of roots. It is likewise clear from the isomorphism property of $h_{\mathbf{Q}}$ that $d(\Delta')$ is a linearly independent set whose span $\mathbf{Q}_{\geqslant 0} \cdot d(\Delta')$ over $\mathbf{Q}_{\geqslant 0}$ satisfies

$$\Phi \subset \mathbf{Q}_{\geqslant 0} \cdot d(\Delta') \bigcup -\mathbf{Q}_{\geqslant 0} \cdot d(\Delta').$$

It then follows from elementary inductive arguments with reduced root systems (see [**SGA3**, XXI, 3.1.5], which avoids a reducedness hypothesis) that this forces $d(\Delta')$ to be the base of a positive system of roots in Φ. But clearly $d(\Delta') \subset \Phi^+$, so $d(\Delta')$ must be the set of simple roots of Φ^+. □

Proposition 6.1.13. — *Let* $(\mathrm{G}', \mathrm{T}', \mathrm{M}')$ *and* $(\mathrm{G}, \mathrm{T}, \mathrm{M})$ *be split reductive groups over a non-empty scheme* S*,* $f : (\mathrm{G}', \mathrm{T}', \mathrm{M}') \to (\mathrm{G}, \mathrm{T}, \mathrm{M})$ *an isogeny compatible with the splittings, and* $(h, d, q) := \mathrm{R}(f)$ *the associated* $p(\mathrm{S})$*-morphism between the root data. Let* ${\Phi'}^+$ *be a positive system of roots in* Φ'*,* Δ' *its base of simple roots,* $\Phi^+ := d({\Phi'}^+)$ *the associated positive system of roots in* Φ*, and* $\Delta = d(\Delta')$ *its base of simple roots.*

1. *If* $\mathrm{B}' \subset \mathrm{G}'$ *is the Borel subgroup containing* T' *that corresponds to* ${\Phi'}^+$ *and* $\mathrm{B} \subset \mathrm{G}$ *is the Borel subgroup containing* T *that corresponds to* Φ^+ *then* f *carries* B' *into* B.
2. *For a pinning* $\{\mathrm{X}'_{a'}\}_{a' \in \Delta'}$ *of* $(\mathrm{G}', \mathrm{T}', \mathrm{M}', \Delta')$ *and pinning of* $(\mathrm{G}, \mathrm{T}, \mathrm{M}, \Delta)$ *given by the sections* $\mathrm{X}_{d(a')} = f_{a'}({\mathrm{X}'_{a'}}^{\otimes q(a')})$*,* f *is uniquely determined by* $\mathrm{R}(f)$ *and the pinnings* $\{\mathrm{X}'_{a'}\}_{a' \in \Delta'}$ *and* $\{\mathrm{X}_a\}_{a \in \Delta}$*. In particular, if* G *is semisimple then an automorphism of* $(\mathrm{G}, \mathrm{T}, \mathrm{M}, \{\mathrm{X}_a\}_{a \in \Delta})$ *whose effect on* $\mathfrak{g}$ *is the identity on each* X_a $(a \in \Delta)$ *must be the identity.*

Proof. — Since f carries T' into T and carries $\mathrm{U}'_{a'}$ into $\mathrm{U}_{d(a')}$, part (1) is immediate from the equalities

$$\mathrm{B}' = \mathrm{T}' \times \prod_{a' \in {\Phi'}^+} \mathrm{U}'_{a'}, \ \ \mathrm{B} = \mathrm{T} \times \prod_{a \in \Phi^+} \mathrm{U}_a$$

respectively defined in G' and G via multiplication (using any enumeration of the sets of positive roots). To prove part (2), first note that for any $a' \in \Delta'$ the restriction $f : \mathrm{U}'_{a'} \to \mathrm{U}_{d(a')}$ is uniquely determined because

$$f(\exp_{a'}(c\mathrm{X}'_{a'})) = \exp_{d(a')}(f_{a'}((c\mathrm{X}'_{a'})^{\otimes q(a')})) = \exp_{d(a')}(c^{q(a')}\mathrm{X}_{d(a')})$$

for any $c \in \mathbf{G}_{\mathrm{a}}$. By Proposition 6.1.5 we have $f(w_{a'}(\mathrm{X}'_{a'})) = w_a(\mathrm{X}_a)$, so f is also uniquely determined on the global sections $w_{a'}(\mathrm{X}'_{a'})$ that represent simple reflections generating the Weyl group $\mathrm{W}(\Phi')$.

Since every $\mathrm{W}(\Phi')$-orbit in Φ' meets Δ', it follows that f is uniquely determined on $\mathrm{U}'_{a'}$ for every $a' \in \Phi'$. The constituent h in $\mathrm{R}(f)$ determines $f : \mathrm{T}' \to \mathrm{T}$, so we conclude that f is uniquely determined on the open cell Ω' of $(\mathrm{G}', \mathrm{T}', \mathrm{M}', \Phi'^{+})$. The relative schematic density of Ω' in G' then implies that f is uniquely determined. □

Definition 6.1.14. — Let $(\mathrm{G}', \mathrm{T}', \mathrm{M}', \{\mathrm{X}'_{a'}\}_{a' \in \Delta'})$ and $(\mathrm{G}, \mathrm{T}, \mathrm{M}, \{\mathrm{X}_a\}_{a \in \Delta})$ be pinned split reductive groups over a non-empty scheme S. An isogeny $f : (\mathrm{G}', \mathrm{T}', \mathrm{M}') \to (\mathrm{G}, \mathrm{T}, \mathrm{M})$ compatible with the splittings (in the sense of Definition 6.1.6) is *compatible with the pinnings* if the bijection $d : \Phi' \to \Phi$ arising from $\mathrm{R}(f)$ carries Δ' into Δ and $f(\exp_{a'}(\mathrm{X}'_{a'})) = \exp_{d(a')}(\mathrm{X}_{d(a')})$ for all $a' \in \Delta'$.

For any isogeny $f : (\mathrm{G}', \mathrm{T}', \mathrm{M}') \to (\mathrm{G}, \mathrm{T}, \mathrm{M})$ compatible with the splittings and a pinning $\{\mathrm{X}'_{a'}\}_{a' \in \Delta'}$ of $(\mathrm{G}', \mathrm{T}', \mathrm{M}')$, we have shown in Proposition 6.1.13(2) that there exists a unique pinning $\{\mathrm{X}_a\}_{a \in \Delta}$ of $(\mathrm{G}, \mathrm{T}, \mathrm{M})$ such that f is compatible with these pinnings. The crucial fact is that in such situations, the pinning-compatible f is *uniquely determined* by $\mathrm{R}(f)$, due to Proposition 6.1.13(2). In particular:

Corollary 6.1.15. — *Let* S *be a non-empty scheme, and consider the category of pinned split reductive groups* $(\mathrm{G}, \mathrm{T}, \mathrm{M}, \{\mathrm{X}_a\}_{a \in \Delta})$ *over* S*, using as morphisms the isogenies that are compatible with the splittings and pinnings. The functor*

$$(6.1.1) \qquad (\mathrm{G}, \mathrm{T}, \mathrm{M}, \{\mathrm{X}_a\}_{a \in \Delta}) \rightsquigarrow \mathrm{R}(\mathrm{G}, \mathrm{T}, \mathrm{M}) = (\mathrm{M}, \Phi, \mathrm{M}^{\vee}, \Phi^{\vee})$$

into the category of root data equipped with $p(\mathrm{S})$*-morphisms is faithful.*

Moreover, if $(\mathrm{G}', \mathrm{T}')$ *and* (G, T) *are reductive* S*-groups equipped with* (*possibly non-split*) *maximal tori and* $f, \mathrm{F} : (\mathrm{G}', \mathrm{T}') \rightrightarrows (\mathrm{G}, \mathrm{T})$ *are isogenies then* f *and* F *induce the same isogeny* $\mathrm{T}' \to \mathrm{T}$ *if and only if* $f = c_{\overline{t}} \circ \mathrm{F}$ *for some* $\overline{t} \in (\mathrm{T}/\mathrm{Z}_{\mathrm{G}})(\mathrm{S})$*, where* $c_{\overline{g}}$ *denotes the natural action of* $\overline{g} \in (\mathrm{G}/\mathrm{Z}_{\mathrm{G}})(\mathrm{S})$ *on* G *induced by conjugation. In such cases,* $\overline{t}$ *is unique.*

This result is essentially the content of [**SGA3**, XXIII, 1.9.1, 1.9.2] (except that we record the role of the $(\mathrm{T}/\mathrm{Z}_{\mathrm{G}})(\mathrm{S})$-action when we do not require splittings or pinnings).

Proof. — The faithfulness of (6.1.1) is immediate from Proposition 6.1.13(2), so it remains to address the assertion concerning the equality of $f, \mathrm{F} : \mathrm{T}' \rightrightarrows \mathrm{T}$ in the absence of splittings and pinnings. It is clear that $c_{\overline{t}} \circ \mathrm{F}$ and F induce the same isogeny from T' to T for any $\overline{t}$ (since the $\mathrm{T}/\mathrm{Z}_{\mathrm{G}}$-action on G is the identity on T), and also that $c_{\overline{t}} \circ \mathrm{F}$ uniquely determines $\overline{t}$ when F is given (since F is faithfully flat). To prove the existence of $\overline{t}$ when $f|_{\mathrm{T}'} = \mathrm{F}|_{\mathrm{T}'}$ in $\mathrm{Hom}_{\mathrm{S}\text{-gp}}(\mathrm{T}', \mathrm{T})$, we may work étale-locally on S due to the uniqueness. Hence, we can assume that $\mathrm{T}' = \mathrm{D}_{\mathrm{S}}(\mathrm{M}')$ and $\mathrm{T} = \mathrm{D}_{\mathrm{S}}(\mathrm{M})$ making $(\mathrm{G}', \mathrm{T}', \mathrm{M}')$ and $(\mathrm{G}, \mathrm{T}, \mathrm{M})$ split as well as making the common isogeny $f, \mathrm{F} : \mathrm{T}' \rightrightarrows \mathrm{T}$ arise from a homomorphism $h : \mathrm{M} \to \mathrm{M}'$ (so f and F are compatible with the splittings).

Choose a pinning $\{\mathrm{X}'_{a'}\}_{a' \in \Delta'}$ of $(\mathrm{G}', \mathrm{T}', \mathrm{M}')$. The induced $p(\mathrm{S})$-morphisms $\mathrm{R}(f), \mathrm{R}(\mathrm{F}) : \mathrm{R}(\mathrm{G}', \mathrm{T}', \mathrm{M}') \rightrightarrows \mathrm{R}(\mathrm{G}, \mathrm{T}, \mathrm{M})$ have the same h, so they coincide; let (h, d, q) denote this common $p(\mathrm{S})$-morphism. We get two pinnings of $(\mathrm{G}, \mathrm{T}, \mathrm{M})$ relative to $\Delta = d(\Delta')$, namely

$$\mathrm{X}_{d(a')} = f_{a'}({\mathrm{X}'_{a'}}^{\otimes q(a')}), \ \ \mathrm{Y}_{d(a')} = \mathrm{F}_{a'}({\mathrm{X}'_{a'}}^{\otimes q(a')})$$

for $a' \in \Delta'$.

The action of G on G via conjugation factors through an action of $\mathrm{G}/\mathrm{Z}_{\mathrm{G}}$ on G. Upon restricting this to an action of $\mathrm{T}/\mathrm{Z}_{\mathrm{G}}$ on a root group U_a for $a \in \Phi$, we recover that such $a : \mathrm{T} \to \mathbf{G}_{\mathrm{m}}$ factors through $\mathrm{T}/\mathrm{Z}_{\mathrm{G}}$. Let $\overline{a} : \mathrm{T}/\mathrm{Z}_{\mathrm{G}} \to \mathbf{G}_{\mathrm{m}}$ denote the character thereby obtained from a; these $\overline{a}$ are the elements of the root system $\overline{\Phi}$ in the root datum for the split group $(\mathrm{G}/\mathrm{Z}_{\mathrm{G}}, \mathrm{T}/\mathrm{Z}_{\mathrm{G}}, \overline{\mathrm{M}})$, where $\overline{\mathrm{M}} = \sum_{a \in \Phi} \mathbf{Z}a \subset \mathrm{M}$ is the possibly non-saturated subgroup corresponding to the quotient $\mathrm{T}/\mathrm{Z}_{\mathrm{G}}$ of the split torus T modulo the split multiplicative type subgroup Z_{G} (see Corollary 3.3.6(1)).

For each $a \in \Delta$ there exists a unique unit $u_a \in \mathbf{G}_{\mathrm{m}}(\mathrm{S})$ such that $\mathrm{X}_a = u_a \mathrm{Y}_a$, so the necessary and sufficient condition for $c_{\overline{t}} \circ \mathrm{F}$ and f to agree as isogenies compatible with splittings and pinnings (and hence to be equal) is that $\overline{a}(\overline{t}) = u_a$ for all $a \in \Delta$. Hence, it is necessary and sufficient to show that the subset $\overline{\Delta} \subset \overline{\Phi}$ corresponding to Δ is a $\mathbf{Z}$-basis of $\overline{\mathrm{M}}$. But this basis property is obvious because

$$\overline{\mathrm{M}} = \sum_{a \in \Phi} \mathbf{Z}a = \sum_{a \in \Delta} \mathbf{Z}a = \bigoplus_{a \in \Delta} \mathbf{Z}a$$

(due to the fact that Δ is a base for a positive system of roots for Φ). □

The following theorem records the main results proved in the rest of § 6.

Theorem 6.1.16. — *Let* S *be a non-empty scheme.*

1. (Isogeny Theorem) *For split reductive* $(\mathrm{G}', \mathrm{T}', \mathrm{M}')$ *and* $(\mathrm{G}, \mathrm{T}, \mathrm{M})$ *over* S, *any* $p(\mathrm{S})$*-morphism* $\mathrm{R}(\mathrm{G}', \mathrm{T}', \mathrm{M}') \to \mathrm{R}(\mathrm{G}, \mathrm{T}, \mathrm{M})$ *is induced by an isogeny* $f : (\mathrm{G}', \mathrm{T}', \mathrm{M}') \to (\mathrm{G}, \mathrm{T}, \mathrm{M})$ *compatible with the splittings, and* f *is unique up to the faithful action of* $(\mathrm{T}/\mathrm{Z}_{\mathrm{G}})(\mathrm{S})$ *on* G.
2. (Existence Theorem) *Every root datum is isomorphic to the root datum of a split reductive* S*-group.*

The Isogeny Theorem is a slight weakening of [**SGA3**, XXV, 1.1] (which allows S-group maps that are not isogenies between maximal central tori, and similarly on root data). The Existence Theorem is [**SGA3**, XXV, 1.2]. An immediate consequence of Theorem 6.1.16 and the arguments with the **Z**-basis Δ of $\mathrm{X}(\mathrm{T}/\mathrm{Z}_{\mathrm{G}})$ in the proof of Corollary 6.1.15 is:

***Theorem 6.1.17* (Isomorphism Theorem)**. — *Let* S *be a scheme that is non-empty. The functor*

$$(\mathrm{G},\mathrm{T},\mathrm{M},\{\mathrm{X}_a\}_{a\in\Delta}) \rightsquigarrow (\mathrm{R}(\mathrm{G},\mathrm{T},\mathrm{M}),\Delta) \tag{6.1.2}$$

from pinned split reductive S*-groups to based root data is an equivalence of categories when using isomorphisms as morphisms.*

In particular, every split reductive S*-group* $(\mathrm{G},\mathrm{T},\mathrm{M})$ *is uniquely determined up to isomorphism by its root datum, and every isomorphism of root data* $\mathrm{R}(\mathrm{G}',\mathrm{T}',\mathrm{M}') \simeq \mathrm{R}(\mathrm{G},\mathrm{T},\mathrm{M})$ *arises from an isomorphism* $f : (\mathrm{G}',\mathrm{T}',\mathrm{M}') \simeq (\mathrm{G},\mathrm{T},\mathrm{M})$ *compatible with the splittings, with* f *uniquely determined up to the faithful action of* $(\mathrm{T}/\mathrm{Z}_{\mathrm{G}})(\mathrm{S})$ *on* G.

In the classical theory, the Existence, Isomorphism, and Isogeny Theorems are proved over a general algebraically closed field (see [**Spr**, 9.6.2, 9.6.5, 10.1.1]). The traditional proof of the Existence Theorem in the classical theory builds a group from its open cell via delicate procedures guided by the Bruhat decomposition and the structure of the Dynkin diagram. The approach over a general base scheme is different, because "points" do not have the same geometric meaning in the relative theory as in the classical case. (For example, if R is a nonzero ring then $\mathbf{A}^1_{\mathrm{R}}$ is stratified by $\mathrm{Z} = \{0\}$ and $\mathrm{U} = \mathbf{G}_{\mathrm{m}}$ but $\mathrm{Z}(\mathrm{R}) \bigcup \mathrm{U}(\mathrm{R}) = \{0\} \bigcup \mathrm{R}^{\times} \neq \mathrm{R} = \mathbf{A}^1_{\mathrm{R}}(\mathrm{R})$ whenever R is not a field.) In place of arguments inspired by the Bruhat decomposition, Weil's theory of birational group laws will be used to prove the Existence Theorem over **Z** (from which the Existence Theorem is deduced in general via base change).

To prove the Existence Theorem over **Z**, we need to know the Existence Theorem over *some* algebraically closed field of characteristic 0, such as **C**. Based on such input, we will use the full faithfulness of (6.1.2) and descent to prove the Existence Theorem over **Q** via pinnings (to rigidify structures). Thus, the Isogeny Theorem will be proved before the Existence Theorem. Split reductive groups over **Q** will be "spread out" over **Z** via arguments with open cells, structure constants, and birational group laws.

We emphasize that the proof of the Isogeny Theorem over a general (non-empty) scheme S will not use the classical case as input, and the proof of the Existence Theorem over S will not require the classical case of the Existence Theorem in positive characteristic. Apart from a few simplifications via the dynamic method, our treatment of the proofs is just an exposition of the proof presented in [**SGA3**, XXIII, XXV].

6.2. The Isogeny Theorem. — In this section we prove the Isogeny Theorem (i.e., Theorem 6.1.16(1)) and record some consequences of the full faithfulness of (6.1.2). Fix a non-empty scheme S and split triples $(\mathrm{G}', \mathrm{T}', \mathrm{M}')$ and $(\mathrm{G}, \mathrm{T}, \mathrm{M})$ over S. Since the root spaces are trivial as line bundles, these admit pinnings. For any $p(\mathrm{S})$-morphism $\phi : \mathrm{R}' \to \mathrm{R}$ between the corresponding root data, we seek an isogeny $f : (\mathrm{G}', \mathrm{T}', \mathrm{M}') \to (\mathrm{G}, \mathrm{T}, \mathrm{M})$ compatible with the splittings such that $\mathrm{R}(f) = \phi$. (The uniqueness of f up to the action of $(\mathrm{T}/\mathrm{Z}_{\mathrm{G}})(\mathrm{S})$ on G is provided by Corollary 6.1.15.)

To construct f, we require criteria for the existence of an S-homomorphism f from a split reductive S-group $(\mathrm{G}, \mathrm{T}, \mathrm{M})$ to an S-group H when the restrictions of f to T and its associated root groups U_a are all specified. More precisely, suppose that $(\mathrm{G}, \mathrm{T}, \mathrm{M}, \{\mathrm{X}_a\}_{a \in \Delta})$ is a pinned split reductive S-group, and for $a \in \Delta$ let

$$n_a = w_a(\mathrm{X}_a) = \exp_a(\mathrm{X}_a) \exp_{-a}(-\mathrm{X}_a^{-1}) \exp_a(\mathrm{X}_a),$$

where X_a^{-1} is the trivialization of $\mathfrak{g}_{-a}$ linked to X_a. (In [**SGA3**], the element $n_a \in \mathrm{N}_{\mathrm{G}}(\mathrm{T})(\mathrm{S})$ is denoted as w_a.) For the open cell Ω arising from T and the positive system of roots Φ^+ with base Δ, the multiplication map $\Omega \times \Omega \to \mathrm{G}$ is fppf. Thus, any S-homomorphism $f : \mathrm{G} \to \mathrm{H}$ to an S-group scheme H is uniquely determined by its restriction to Ω, so f is uniquely determined by its restrictions

$$f_{\mathrm{T}} : \mathrm{T} \to \mathrm{H}, \ \ f_a : \mathrm{U}_a \to \mathrm{H}$$

for $a \in \Phi$. Since Φ is covered by the $\mathrm{W}(\Phi)$-orbits of elements of Δ, and the elements n_a represent the simple positive reflections that generate $\mathrm{W}(\Phi)$, instead of keeping track of the maps f_a for all $a \in \Phi^+$ it is enough to record the maps f_a for $a \in \Delta$ provided that we also record the images $h_a = f(n_a) \in \mathrm{H}(\mathrm{S})$ for $a \in \Delta$.

Note that $\mathrm{N}_{\mathrm{G}}(\mathrm{T})$ is the disjoint union of translates $n\mathrm{T}$ for a set of elements $n \in \mathrm{N}_{\mathrm{G}}(\mathrm{T})(\mathrm{S})$ representing $\mathrm{W}(\Phi)$, such as products of the elements $n_a \in \mathrm{N}_{\mathrm{G}}(\mathrm{T})(\mathrm{S})$ for $a \in \Delta$ (upon writing each $w \in \mathrm{W}(\Phi)$ as a word in the simple positive reflections). Thus, a first step towards an existence criterion for a homomorphism $f : \mathrm{G} \to \mathrm{H}$ recovering given maps on T and the U_a's ($a \in \Delta$) and given values $h_a = f(n_a) \in \mathrm{H}(\mathrm{S})$ is to settle the case when there is given a homomorphism $f_{\mathrm{N}} : \mathrm{N}_{\mathrm{G}}(\mathrm{T}) \to \mathrm{H}$ (instead of f_{T} and h_a's) and homomorphisms $f_a : \mathrm{U}_a \to \mathrm{H}$ for *all* roots $a \in \Phi$ (not just for $a \in \Delta$). Such a preliminary gluing criterion is provided by the following result [**SGA3**, XXIII, 2.1]:

Theorem 6.2.1. — *For* S*-homomorphisms* $f_{\mathrm{N}} : \mathrm{N}_{\mathrm{G}}(\mathrm{T}) \to \mathrm{H}$ *and* $f_a : \mathrm{U}_a \to \mathrm{H}$ *for* $a \in \Phi$*, there exists an* S*-homomorphism* $f : \mathrm{G} \to \mathrm{H}$ *extending* f_{N} *and the maps* f_a *if and only if the following three conditions hold:*

1. *For all* $a \in \Delta$ *and* $b \in \Phi$*,*

$$f_{\mathrm{N}}(n_a) f_b(u_b) f_{\mathrm{N}}(n_a)^{-1} = f_{s_a(b)}(n_a u_b n_a^{-1})$$

for all $u_b \in \mathrm{U}_b$.

2. *There exists an* S*-homomorphism* $\mathrm{Z_G(T}_a) \to \mathrm{H}$ *extending the triple* $(f_a, f_{-a}, f_{\mathrm{N}}|_{\mathrm{N}_{\mathrm{Z_G(T}_a)}(\mathrm{T})})$ *for all* $a \in \Delta$.
3. *For all distinct* $a, b \in \Delta$ *and the subgroup* $\mathrm{U}_{[a,b]}$ *directly spanned in any order by the groups* U_c *for*
$$c \in [a,b] := \{ia + jb \in \Phi \,|\, i,j \geqslant 0\},$$
there exists an S*-homomorphism* $\mathrm{U}_{[a,b]} \to \mathrm{H}$ *restricting to* f_c *on* U_c *for all* $c \in [a,b]$.

Remark 6.2.2. — The existence of $\mathrm{U}_{[a,b]}$ is a special case of Proposition 5.1.16.

The idea of the proof of sufficiency in Theorem 6.2.1 (necessity being obvious) is that since $f_{\mathrm{N}}|_{\mathrm{T}}$ and the f_a determine what f must be on the open cell $\Omega = \mathrm{U}_{-\Phi^+} \times \mathrm{T} \times \mathrm{U}_{\Phi^+}$, and the translates $n\Omega$ by products n among the $\{n_a\}_{a\in\Delta}$ cover G (Corollary 1.4.14 on geometric fibers), we just have to keep track of the homomorphism property when extending f across translates of Ω.

By using induction on word length in the Weyl group (relative to the simple positive reflections s_a for $a \in \Delta$), the base of the induction amounts to checking that for *some* enumeration of Φ^+, the S-morphism $f_{\mathrm{U}} : \mathrm{U} = \mathrm{U}_{\Phi^+} := \prod_{a\in\Phi^+} \mathrm{U}_a \to \mathrm{H}$ defined by $(u_a) \mapsto \prod_{a\in\Phi^+} f_a(u_a)$ is a homomorphism. Of course, once this is proved for some choice of enumeration, it follows that the enumeration of Φ^+ does not matter, as U is directly spanned in any order by the positive root groups. The case of $\mathrm{U}_{-\Phi^+}$ is also needed, but this will follow formally from the case of U_{Φ^+} by using a representative n for the long Weyl element w to swap Φ^+ and $-\Phi^+$ (since $f_{w(b)}(u) = f_{\mathrm{N}}(n) f_b(n^{-1}un) f_{\mathrm{N}}(n)^{-1}$ for $b \in \Phi^+$ and $u \in \mathrm{U}_{w(b)}$ due to hypothesis (1)).

For the convenience of the reader, we now sketch the proof of the base case for the induction (i.e., the homomorphism property for f_{U}). Fix a structure of ordered vector space on $\mathrm{M}_{\mathbf{Q}}$ so that Φ^+ is the associated positive system of roots (see the discussion following Definition 1.4.1, and note that $\mathrm{W}(\Phi)$ acts transitively on the set of positive systems of roots in Φ). Consider the resulting enumeration $c_0 < \cdots < c_m$ of Φ^+.

Lemma 6.2.3. — *The map* f_{U} *is an* S*-homomorphism when it is defined using* $\{c_j\}$.

This is [**SGA3**, XXIII, 2.1.4].

Proof. — For $i \geqslant 1$, consider the direct product scheme $\mathrm{U}_{\geqslant i} := \prod_{c\geqslant c_i} \mathrm{U}_c$ and the map $\mathrm{U}_{\geqslant i} \to \mathrm{U}$ of S-schemes defined by multiplication in strictly increasing order of the roots. By Proposition 5.1.16, this identifies $\mathrm{U}_{\geqslant i}$ with a closed S-subgroup of U that is moreover normalized by $\mathrm{U}_{c_{i-1}}$ when $i > 0$. The homomorphism property for f_{U} on $\mathrm{U} = \mathrm{U}_{\geqslant 0}$ will be proved by descending induction: for all i we claim that f_{U} restricts to a homomorphism on $\mathrm{U}_{\geqslant i}$. The case $i = m$ is obvious (as the restriction to $\mathrm{U}_{\geqslant m} = \mathrm{U}_{c_m}$ is f_{c_m}).

In general, if the result holds for some $i > 0$ then since $\mathrm{U}_{\geqslant i-1} = \mathrm{U}_{c_{i-1}} \ltimes \mathrm{U}_{\geqslant i}$, it is straightforward to use the homomorphism property for f_{U} on $\mathrm{U}_{\geqslant i}$ to reduce to verifying the identities

$$f_b(u_b)^{-1} f_a(u_a) f_b(u_b) \stackrel{?}{=} f_{\mathrm{U}}(u_b^{-1} u_a u_b) \tag{6.2.1}$$

for $b = c_{i-1}$, $a > c_{i-1} = b$, and all points u_a of U_a and u_b of U_b. Note that these identities make sense because $u_b^{-1} u_a u_b \in \mathrm{U}_{\geqslant i}$ since $a \geqslant c_i$ and $b = c_{i-1}$. To summarize, we have reduced ourselves to proving that for any $a, b \in \Phi^+$ with $a > b$, the identities (6.2.1) are satisfied.

For any $u_a \in \mathrm{U}_a$ and $u_b \in \mathrm{U}_b$, clearly $u_b^{-1} u_a u_b \in \mathrm{U}_{[a,b]}$. Thus, if $a \in \Delta$ and $b \in [a, b_0]$ for $b_0 \in \Delta - \{a\}$ then the desired identities are a consequence of hypothesis (3) of Theorem 6.2.1. To reduce the case of a general pair to these special cases, we use a result in the theory of root systems: there exists $w \in \mathrm{W}(\Phi)$ such that $w(a) \in \Delta$ and $w(b) \in [w(a), b_0]$ for some $b_0 \in \Delta$ necessarily distinct from $w(a)$; this follows from the transitivity of the $\mathrm{W}(\Phi)$-action on the set of positive systems of roots and [**SGA3**, XXI, 3.5.4] (whose main content is [**Bou2**, VI, § 1.7, Cor. 2], applied to the reverse lexicographical ordering on Φ^+ relative to an enumeration of the base Δ). This enables us to reduce to the settled special case $a \in \Delta$ and $b \in [a, b_0]$ with $b_0 \in \Delta - \{a\}$ provided that if $n \in \mathrm{N_G(T)(S)}$ represents $w \in \mathrm{W}(\Phi)$ then

$$f_{\mathrm{N}}(n) f_a(u_a) f_{\mathrm{N}}(n)^{-1} = f_{w(a)}(n u_a n^{-1})$$

for all $a \in \Phi$ and points $u_a \in \mathrm{U}_a$. The case $n = n_b$ for $b \in \Delta$ is exactly hypothesis (1) in Theorem 6.2.1. The general case is deduced from this via induction on word length in $\mathrm{W}(\Phi)$ relative to the simple positive reflections $\{s_c\}_{c \in \Delta}$ and applications of hypotheses (1) and (2) in Theorem 6.2.1. (We need (2) because representatives $n_c \in \mathrm{N_G(T)(S)}$ for the s_c's do not generate $\mathrm{N_G(T)(S)}$.) See [**SGA3**, XXIII, 2.1.3] for further details. □

Now we are ready to formulate the main criterion for constructing homomorphisms, building on Theorem 6.2.1. We will eventually obtain a criterion that reduces all difficulties to the case of groups with semisimple-rank at most 2. That is, the serious computational effort will only be required with low-rank groups. Keep in mind that we have explicitly described the split cases with semisimple-rank 1 in Theorem 5.1.8, at least Zariski-locally on the base. (The intervention of Zariski-localization can be removed from Theorem 5.1.8 by using a pinning and Proposition 6.1.13(2), but we do not need that minor improvement here.)

Let $(\mathrm{G}, \mathrm{T}, \mathrm{M}, \{\mathrm{X}_a\}_{a\in\Delta})$ be a pinned split reductive S-group, and let H be an S-group. For each $a \in \Delta$, let $n_a = w_a(\mathrm{X}_a)$. For S-homomorphisms $f_\mathrm{T} : \mathrm{T} \to \mathrm{H}$ and $f_a : \mathrm{U}_a \to \mathrm{H}$ and elements $h_a \in \mathrm{H}(\mathrm{S})$ for all $a \in \Delta$, we have seen that there exists at most one S-homomorphism $f : \mathrm{G} \to \mathrm{H}$ such that $f|_\mathrm{T} = f_\mathrm{T}$ and $f|_{\mathrm{U}_a} = f_a$ and $f(n_a) = h_a$ for all $a \in \Delta$. But when does f exist? There are some necessary conditions. For example, for all $a \in \Delta$ we must have

$$f_\mathrm{T}(t)f_a(u_a)f_\mathrm{T}(t)^{-1} = f_a(tu_at^{-1}),\ \ h_af_\mathrm{T}(t)h_a^{-1} = f_\mathrm{T}(s_a(t)) \tag{6.2.2}$$

for all points $t \in \mathrm{T}$ and $u \in \mathrm{U}_a$ (valued in a common S-scheme).

There are additional conditions imposed by relations in the Weyl group. More specifically, note that if $a, b \in \Delta$ then $(s_as_b)^{m_{ab}} = 1$ in $\mathrm{W}(\Phi)$ where $m_{ab} = m_{ba}$ is the ab-entry in the symmetric Cartan matrix (m_{ab} is the order of s_as_b in $\mathrm{W}(\Phi)$, so $m_{aa} = 1$). In particular, $t_{ab} := (n_an_b)^{m_{ab}} \in \mathrm{T}(\mathrm{S})$, so $t_{aa} = n_a^2 = a^\vee(-1)$. This yields the further necessary conditions

$$h_a^2 = f_\mathrm{T}(a^\vee(-1)),\ \ (h_ah_b)^{m_{ab}} = f_\mathrm{T}(t_{ab}) \tag{6.2.3}$$

for $a, b \in \Delta$ with $b \neq a$. The relation $(n_a \exp_a(\mathrm{X}_a))^3 = 1$ (Remark 5.1.10) yields the necessary condition

$$(h_af_a(\exp_a(\mathrm{X}_a)))^3 = 1 \tag{6.2.4}$$

for all $a \in \Delta$. Finally, if $a, b \in \Delta$ are distinct then for a homomorphism $f_{ab} : \mathrm{U}_{[a,b]} \to \mathrm{H}$ extending f_a on U_a and f_b on U_b to arise from an $f : \mathrm{G} \to \mathrm{H}$ of the desired type, the following conjugation relations must hold on the root groups U_c for $c \in [a, b]$. If $c \neq a$ then

$$h_af_{ab}(u_c)h_a^{-1} = f_{ab}(n_au_cn_a^{-1}). \tag{6.2.5}$$

(The right side makes sense because (i) n_a conjugates U_c to $\mathrm{U}_{s_a(c)}$, and (ii) $s_a(c) = c - \langle c, a^\vee\rangle a$ lies in $[a, b]$, due to $s_a(c)$ having a positive b-coefficient with a and b distinct elements of the base Δ of Φ^+.) Likewise, if $c \neq b$ then

$$h_bf_{ab}(u_c)h_b^{-1} = f_{ab}(n_bu_cn_b^{-1}). \tag{6.2.6}$$

Remarkably, the preceding necessary conditions for the existence of f, each of which only involves subgroups $\mathrm{Z}_\mathrm{G}(\mathrm{T}_a)$ of semisimple-rank 1 and subgroups $\mathrm{Z}_\mathrm{G}(\mathrm{T}_{ab})$ of semisimple-rank 2 (with T_{ab} denoting the unique torus of relative codimension-2 in T contained in $\ker a \bigcap \ker b$ for distinct $a, b \in \Phi^+$), are also sufficient. This is [**SGA3**, XXIII, 2.3]:

Theorem 6.2.4. — *If the conditions* (6.2.2), (6.2.3), *and* (6.2.4) *hold and for all distinct* $a, b \in \Delta$ *there exists a homomorphism* $f_{ab} : \mathrm{U}_{[a,b]} \to \mathrm{H}$ *extending* f_a *and* f_b *and satisfying* (6.2.5) *and* (6.2.6) *then a homomorphism* $f : \mathrm{G} \to \mathrm{H}$ *exists satisfying* $f|_\mathrm{T} = f_\mathrm{T}$, $f|_{\mathrm{U}_a} = f_a$ *for all* $a \in \Delta$, *and* $f(n_a) = h_a$ *for all* $a \in \Delta$.

Proof. — The proof is largely a systematic (and intricate) argument with word length in Weyl groups, bootstrapping from the given conditions to eventually establish the hypotheses in Theorem 6.2.1. This entails constructing f_{N} and the homomorphisms f_a for *all* roots $a \in \Phi$ (recovering the given homomorphisms for $a \in \Delta$).

An elementary argument (see [**SGA3**, XXIII, 2.3.1]) constructs the S-homomorphism $f_{\mathrm{N}} : \mathrm{N}_{\mathrm{G}}(\mathrm{T}) \to \mathrm{H}$ that extends f_{T} and carries n_a to h_a for all $a \in \Delta$; such an f_{N} is unique since products among the n_a's represent all elements of $\mathrm{W}(\Phi)$ (and $\mathrm{N}_{\mathrm{G}}(\mathrm{T})$ is covered by the left-translates of its open and closed subgroup T by any set of representatives of $\mathrm{W}(\Phi)$ in $\mathrm{N}_{\mathrm{G}}(\mathrm{T})(\mathrm{S})$; the closed S-subgroup $\mathrm{T} \subset \mathrm{N}_{\mathrm{G}}(\mathrm{T})$ is open because $\mathrm{W}_{\mathrm{G}}(\mathrm{T})$ is étale, by Proposition 3.2.8). The construction of well-defined maps f_a for all $a \in \Phi$ is harder, and rests on a lemma of Tits in the theory of root systems (stated as Exercise 21 in [**Bou2**, Ch. VI], and proved in [**SGA3**, XXI, 5.6]).

We refer to [**SGA3**, XXIII, 2.3.2–2.3.6] for the details, and sketch the main group-theoretic argument that establishes the requirement in Theorem 6.2.1(2). This requirement says that for each $a \in \Delta$ there exists an S-homomorphism $\mathrm{F}_a : \mathrm{Z}_{\mathrm{G}}(\mathrm{T}_a) \to \mathrm{H}$ such that $\mathrm{F}_a|_{\mathrm{T}} = f_{\mathrm{T}}$, $\mathrm{F}_a|_{\mathrm{U}_a} = f_a$, and $\mathrm{F}_a(n_a) = h_a$. (Such an F_a is visibly unique if it exists, since n_a-conjugation swaps the opposite root groups that appear in the open cell of $\mathrm{Z}_{\mathrm{G}}(\mathrm{T}_a)$ relative to its split maximal torus T and roots $\pm a$, and it satisfies $\mathrm{F}_a|_{\mathrm{U}_{-a}} = f_{-a}$ by *defining* $f_{-a}(u) = h_a^{-1} f_a(n_a u n_a^{-1}) h_a$ for $u \in \mathrm{U}_{-a}$.) The construction of F_a in [**SGA3**, XXIII, 2.3.2] rests on calculations in [**SGA3**, XX, 6.2] with an "abstract" split reductive group of semisimple-rank 1. The explicit classification of such split groups in Theorem 5.1.8 will now be used to simplify those calculations.

We may replace G with $\mathrm{Z}_{\mathrm{G}}(\mathrm{T}_a)$, so our problem becomes exactly the special case of G with semisimple-rank 1. In particular, $\Delta = \{a\}$, so the conditions (6.2.5) and (6.2.6) become vacuous and (6.2.3) only involves the first relation there. The uniqueness allows us to work étale-locally on S for existence, so the central torus direct factor as in Theorem 5.1.8 can be dropped and we are reduced to three special cases: (G, T) is either $(\mathrm{SL}_2, \mathrm{D})$, $(\mathrm{PGL}_2, \overline{\mathrm{D}})$, or $(\mathrm{GL}_2, \widetilde{\mathrm{D}}) = (\mathrm{SL}_2 \times^{\mu_2} \mathbf{G}_{\mathrm{m}}, \mathrm{D} \times^{\mu_2} \mathbf{G}_{\mathrm{m}})$. The third case trivially reduces to the first case (since $a^\vee(-1) \in \mathscr{D}(\mathrm{G})$ in general), and so does the second case since the natural degree-2 central isogeny $\mathrm{SL}_2 \to \mathrm{PGL}_2$ has kernel $\mu_2 = \mathrm{D}[2] \subset \mathrm{D}$. Hence, we may and do assume that $(\mathrm{G}, \mathrm{T}, \mathrm{M}) = (\mathrm{SL}_2, \mathrm{D}, \mathbf{Z})$ where the element $1 \in \mathbf{Z}$ goes over to the isomorphism $\mathrm{D} \simeq \mathbf{G}_{\mathrm{m}}$ inverse to $c \mapsto \mathrm{diag}(c, 1/c)$, and likewise we can arrange that a is the standard positive root (i.e., U_a is the strictly upper triangular subgroup of $\mathrm{G} = \mathrm{SL}_2$) and $\mathrm{X}_a = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \in \mathfrak{sl}_2$.

In this special case, the challenge is not to define f on the open cell Ω (it is clear what the unique possibility for that must be), nor how to define f on another translate of Ω that (together with Ω) covers SL_2. The hard part is to verify that one gets a globally well-defined S-morphism that is moreover a homomorphism. The calculations to verify this in [**SGA3**, XX, 6.2] are done in the absence of an explicit classification in semisimple-rank 1, and they become simpler for the explicit case we need, namely $(\mathrm{SL}_2, \mathrm{D})$ (e.g., the auxiliary parameters $u \in \mathrm{U}_a = \mathbf{G}_{\mathrm{a}}$ and $\widetilde{u} \in \mathrm{U}_{-a} = \mathbf{G}_{\mathrm{a}}$ there become 1). □

Remark 6.2.5. — One may wonder about a finer result beyond Theorem 6.2.4 that replaces (6.2.5) and (6.2.6) with a more explicit presentation of $\mathrm{U}_{[a,b]}$ in terms of "generators and relations". This viewpoint is systematically developed in [**SGA3**, XXIII, 2.6, 3.1.3, 3.2.8, 3.3.7, 3.4.10, 3.5], but it is not logically relevant to the proofs of the Isogeny, Isomorphism, or Existence Theorems, so we will say nothing more about it.

Example 6.2.6. — We can make explicit what Theorem 6.2.4 says concerning the construction of isogenies $f : (\mathrm{G}', \mathrm{T}', \mathrm{M}', \{\mathrm{X}'_{a'}\}) \to (\mathrm{G}, \mathrm{T}, \mathrm{M}, \{\mathrm{X}_a\})$ between pinned split reductive S-group with semisimple-rank 1 (so $\Phi(\mathrm{G}', \mathrm{T}') = \{\pm a'\}$ and $\Phi(\mathrm{G}, \mathrm{T}) = \{\pm a\}$) such that f is compatible with the pinnings and the splittings. We want f to restrict to an isogeny $\mathrm{T}' \to \mathrm{T}$ dual to a given finite-index lattice inclusion $h : \mathrm{M} \to \mathrm{M}'$ satisfying $h(a) = qa'$ for a prime power $q = p^n \geqslant 1$ such that $p = 0$ in $\mathscr{O}_{\mathrm{S}}$ if $q > 1$ (duality forces $h^\vee(a'^\vee) = qa^\vee$ due to being in the case of semisimple-rank 1), and we want f to restrict to an isogeny $f_{a'} : \mathrm{U}'_{a'} \to \mathrm{U}_a$ given by $\exp_{a'}(c\mathrm{X}'_{a'}) \mapsto \exp_a(c^q\mathrm{X}_a)$ (a homomorphism when $q > 1$ because we assume S is a $\mathbf{Z}/p\mathbf{Z}$-scheme in such cases).

We claim that for any such h and q there is a unique such f. Necessarily $\mathrm{R}(f)$ coincides with the $p(\mathrm{S})$-morphism (h, d, q) where $d(a') = a$ and $d(-a') = -a$, so our claim is exactly the Isogeny Theorem for all cases with semisimple-rank 1. (This case of the Isogeny Theorem is [**SGA3**, XXIII, 4.1.2], whose proof via [**SGA3**, XX, 4.1] rests on extensive calculations. The reason that we will be able to avoid those calculations is because we have Theorem 5.1.8.)

The key feature of semisimple-rank 1 is that the conditions in Theorem 6.2.4 become very concrete in such cases. First of all, (6.2.3) says $n_a^2 \overset{?}{=} f_{\mathrm{T}}(a'^\vee(-1))$, and this is automatic since

$$n_a^2 = a^\vee(-1) = a^\vee((-1)^q) = (a^\vee)^q(-1) = h^\vee(a'^\vee)(-1) = f_{\mathrm{T}}(a'^\vee(-1))$$

(we have used that $(-1)^q = -1$ in $\mathscr{O}_{\mathrm{S}}$, since p vanishes in $\mathscr{O}_{\mathrm{S}}$ when $q = p^n > 1$). Likewise, the second relation in (6.2.2) merely says that f_{T} intertwines inversion on T' and T, and the first relation in (6.2.2) is automatic since $h(a) = qa'$ (and $f_{\mathrm{T}} = \mathrm{D}_{\mathrm{S}}(h)$). The relations in (6.2.5) and (6.2.6) are vacuous in cases with semisimple-rank 1. Finally, (6.2.4) is automatic since it asserts $(w_a(\mathrm{X}_a)\exp_a(\mathrm{X}_a))^3 = 1$, which always holds (see Remark 5.1.10).

The necessary and sufficient conditions in Theorem 6.2.4 involve only reductive closed subgroups with semisimple-rank $\leqslant 2$, so we immediately deduce the following crucial existence criterion for homomorphisms [**SGA3**, XXIII, 2.4] that is expressed entirely in terms of closed reductive subgroups with such low semisimple-rank.

Corollary 6.2.7. — *Let* $(\mathrm{G}, \mathrm{T}, \mathrm{M})$ *be split reductive over* S *with semisimple-rank* $\geqslant 2$. *Let* Φ^+ *be a positive system of roots in* Φ, *and let* Δ *be the corresponding base of simple roots. For each* $a, b \in \Delta$, *let* $\mathrm{T}_{ab} \subset \mathrm{T}$ *be the unique subtorus of relative codimension-*2 *contained in* $\ker a \bigcap \ker b$ *when* $a \neq b$ *and let* $\mathrm{T}_{aa} = \mathrm{T}_a$.

For an S*-group* H *and given* S*-homomorphisms* $f_{ab} : \mathrm{Z}_{\mathrm{G}}(\mathrm{T}_{ab}) \to \mathrm{H}$ *for all* $a, b \in \Delta$, *assume* $f_{ab} = f_{ba}$ *and* $f_{ab}|_{\mathrm{Z}_{\mathrm{G}}(\mathrm{T}_a)} = f_{aa}$ *for all* $a, b \in \Delta$. *There is a unique* S*-homomorphism* $f : \mathrm{G} \to \mathrm{H}$ *such that* $f|_{\mathrm{Z}_{\mathrm{G}}(\mathrm{T}_{ab})} = f_{ab}$ *for all* $a, b \in \Delta$.

Now we prove the Isogeny Theorem (i.e., Theorem 6.1.16(1)). The main work is for groups with semisimple-rank 2.

Proof of Isogeny Theorem. — Let Δ' be the base of a positive system of roots Φ'^+ in Φ', so by Lemma 6.1.12 the set $\Delta := d(\Delta')$ is a base for a positive system of roots $\Phi^+ = d(\Phi'^+)$ in Φ. The triviality hypothesis on the root spaces allows us to choose pinnings $\{\mathrm{X}'_{a'}\}_{a' \in \Delta'}$ and $\{\mathrm{X}_a\}_{a \in \Delta}$. Corollary 6.1.15 shows that f is unique up to the action of $(\mathrm{T}/\mathrm{Z}_{\mathrm{G}})(\mathrm{S})$, so it remains to prove the existence of f.

For the split reductive S-groups G and G′, the given isogeny between their root data implies that their constant fibral semisimple-ranks are the same. The case of semisimple-rank 0 is trivial (as then G and G′ are tori; i.e., $\mathrm{T} = \mathrm{G}$ and $\mathrm{T}' = \mathrm{G}'$), and the case of semisimple-rank 1 is Example 6.2.6, so we now assume the common semisimple-rank of $(\mathrm{G}', \mathrm{T}')$ and (G, T) is $\geqslant 2$.

Step 1. We reduce to the case when G and G′ have semisimple-rank 2. For each pair of (possibly equal) elements $a', b' \in \Delta'$, let $a = d(a')$ and $b = d(b')$ in Δ and consider the closed reductive S-subgroups $\mathrm{Z}_{\mathrm{G}'}(\mathrm{T}'_{a'b'}) \subset \mathrm{G}'$ and $\mathrm{Z}_{\mathrm{G}}(\mathrm{T}_{ab}) \subset \mathrm{G}$ with respective maximal tori $\mathrm{T}' = \mathrm{D}_{\mathrm{S}}(\mathrm{M}')$ and $\mathrm{T} = \mathrm{D}_{\mathrm{S}}(\mathrm{M})$. In these S-subgroups the respective sets of roots $\Phi_{a'b'}$ and Φ_{ab} lie in the respective subsets $\Phi' \subset \mathrm{M}' - \{0\}$ and $\Phi \subset \mathrm{M} - \{0\}$, and the root spaces are free of rank 1 as line bundles since they are root spaces for $(\mathrm{G}', \mathrm{T}')$ and (G, T) respectively. Hence, we get root data

$$\mathrm{R}(\mathrm{Z}_{\mathrm{G}'}(\mathrm{T}'_{a'b'}), \mathrm{T}', \mathrm{M}') = (\mathrm{M}', \Phi'_{a'b'}, \mathrm{M}'^{\vee}, \Phi'^{\vee}_{a'b'})$$

and

$$\mathrm{R}(\mathrm{Z}_{\mathrm{G}}(\mathrm{T}_{ab}), \mathrm{T}, \mathrm{M}) = (\mathrm{M}, \Phi_{ab}, \mathrm{M}^{\vee}, \Phi^{\vee}_{ab})$$

equipped with positive systems of roots $\Phi'_{a'b'} \bigcap \Phi'^+$ and $\Phi_{ab} \bigcap \Phi^+$.

Since $a', b' \in \Delta'$ and $a, b \in \Delta$, the positive systems of roots $\Phi'_{a'b'} \bigcap \Phi'^{+}$ and $\Phi_{ab} \bigcap \Phi^{+}$ (with rank 1 when $a' = b'$ and rank 2 when $a' \neq b'$) have as their respective bases $\{a', b'\}$ and $\{a, b\}$ when $a' \neq b'$ and $\{a'\}$ and $\{a\}$ when $a' = b'$. Thus, the split reductive S-groups $\mathrm{Z}_{\mathrm{G}'}(\mathrm{T}'_{a'b'})$ and $\mathrm{Z}_{\mathrm{G}}(\mathrm{T}_{ab})$ with semisimple-rank $\leqslant 2$ are also pinned (using our pinnings for $(\mathrm{G}', \mathrm{T}', \mathrm{M}')$ and $(\mathrm{G}, \mathrm{T}, \mathrm{M})$). Concretely, $\Phi'_{a'b'}$ consists of the roots that are trivial on the torus $\mathrm{T}'_{a'b'}$, and likewise for Φ_{ab} using T_{ab} (as we may check on geometric fibers, using the classical theory). But $\mathrm{X}(\mathrm{T}'_{a'b'})$ is the quotient of $\mathrm{X}(\mathrm{T}')$ modulo the saturation of $\mathbf{Z}a' + \mathbf{Z}b'$, and likewise for $\mathrm{X}(\mathrm{T}_{ab})$ as a quotient of $\mathrm{X}(\mathrm{T})$ using $\mathbf{Z}a + \mathbf{Z}b$, so it follows from the definition of an isogeny of reduced root data that for the given $p(\mathrm{S})$-morphism of root data $\phi = (h, d, q)$ the map $h : \mathrm{X}(\mathrm{T}) \to \mathrm{X}(\mathrm{T}')$ induces a compatible map $\mathrm{X}(\mathrm{T}_{ab}) \to \mathrm{X}(\mathrm{T}'_{a'b'})$. Hence, ϕ restricts to a $p(\mathrm{S})$-morphism of root data

$$\phi_{a'b'} : \mathrm{R}(\mathrm{Z}_{\mathrm{G}'}(\mathrm{T}'_{a'b'}), \mathrm{T}', \mathrm{M}') \to \mathrm{R}(\mathrm{Z}_{\mathrm{G}}(\mathrm{T}_{ab}), \mathrm{T}, \mathrm{M})$$

for all (possibly equal) $a', b' \in \Delta'$. Note that $\phi_{a'b'} = \phi_{b'a'}$.

Now assume that all cases of semisimple-rank 2 are settled (as is true for all cases with semisimple-rank 1), so we obtain isogenies

$$f_{a'b'} : (\mathrm{Z}_{\mathrm{G}'}(\mathrm{T}'_{a'b'}), \mathrm{T}', \mathrm{M}') \to (\mathrm{Z}_{\mathrm{G}}(\mathrm{T}_{ab}), \mathrm{T}, \mathrm{M})$$

that are compatible with the splittings and satisfy $\mathrm{R}(f_{a'b'}) = \phi_{a'b'}$. By the proof of Corollary 6.1.15, we may and do replace such an $f_{a'b'}$ with its composition against the action of a *unique* $\overline{t} \in (\mathrm{T}/\mathrm{Z}_{\mathrm{Z}_{\mathrm{G}}(\mathrm{T}_{ab})})(\mathrm{S})$ so that $f_{a'b'}$ is also compatible with the pinnings $\{\mathrm{X}'_{a'}, \mathrm{X}'_{b'}\}$ and $\{\mathrm{X}_a, \mathrm{X}_b\}$ (by which we mean $\{\mathrm{X}'_{a'}\}$ and $\{\mathrm{X}_a\}$ when $a' = b'$). Hence, the equality $\mathrm{R}(f_{a'b'}) = \phi_{a'b'} = \phi_{b'a'} = \mathrm{R}(f_{b'a'})$ forces $f_{a'b'} = f_{b'a'}$. Likewise, if $a' \neq b'$ then $\phi_{a'b'}$ restricts to $\phi_{a'a'}$ on the root datum $\mathrm{R}(\mathrm{Z}_{\mathrm{G}'}(\mathrm{T}'_{a'}), \mathrm{T}', \mathrm{M}')$, so $f_{a'b'}|_{\mathrm{Z}_{\mathrm{G}'}(\mathrm{T}'_{a'})} = f_{a'a'}$. By Corollary 6.2.7, there is a unique isogeny $f : (\mathrm{G}', \mathrm{T}') \to (\mathrm{G}, \mathrm{T})$ inducing all $f_{a'b'}$, so f respects the splittings and the pinnings. This completes the reduction of the Isogeny Theorem to the case of semisimple-rank 2.

Step 2. Assume that G and G′ have semisimple-rank 2. By the classification of rank-2 reduced root systems, there are four possibilities for each root system: $\mathrm{A}_1 \times \mathrm{A}_1$ (e.g., $\mathrm{SL}_2 \times \mathrm{SL}_2$), A_2 (e.g., SL_3), B_2 (e.g., Sp_4), and G_2. In each of these four cases we will define a “universal” choice of trivialization of all positive root spaces in a manner that only depends on the based root system (Φ, Δ) and an enumeration $\xi : \{1, 2\} \simeq \Delta$.

[Since we have not yet proved the Existence Theorem, we may not yet know that there exists a pinned split reductive S-group whose root system has type G_2 (if we haven't constructed G_2 already by some other means; e.g., octonion algebras). This is not logically relevant, since at present we are only aiming to prove that *if* we are given a pair of pinned split reductive S-groups then we can relate $p(\mathrm{S})$-morphisms between their root data to isogenies between the S-groups. To prove the Isogeny Theorem we only need to consider each of the root systems of rank 2 that *might* occur, without constructing specific S-groups.]

Consider a pinned split reductive S-group $(\mathrm{G}, \mathrm{T}, \mathrm{M}, \{\mathrm{X}_a\}_{a\in\Delta})$ whose root system has rank 2. We use the pinning to define a specific representative $n_a := w_a(\mathrm{X}_a) \in \mathrm{N}_{\mathrm{G}}(\mathrm{T})(\mathrm{S})$ for the simple reflection $s_a \in \mathrm{W}(\Phi)$ for each $a \in \Delta$. If $c \in \Phi^+$ is a positive root (relative to Δ) then by [**Bou2**, VI, § 1.5, Prop. 15] there exists $w \in \mathrm{W}(\Phi)$ such that $w^{-1}(c) \in \Delta$, so for any sign $\varepsilon_c \in \{\pm 1\}$ and any product $n_w \in \mathrm{N}_{\mathrm{G}}(\mathrm{T})(\mathrm{S})$ among $\{n_a\}_{a\in\Delta}$ such that n_w represents w,

$$\mathrm{X}_{c,n_w} := \varepsilon_c \mathrm{Ad}(n_w)(\mathrm{X}_{w^{-1}(c)})$$

is a trivialization of $\mathfrak{g}_c$. This trivialization depends on both w and n_w, neither of which is determined by c, so X_{c,n_w} is not intrinsic even if we set $\varepsilon_c = 1$ (e.g., if $\mathrm{S} = \mathrm{Spec}\,\mathbf{Z}$ then there is an ambiguity from scaling by $\mathbf{Z}^\times = \{1, -1\}$).

In [**SGA3**, XXIII, 3.1–3.4] each of the four reduced rank-2 root systems Φ is considered separately, along with a choice of base Δ and an enumeration $\xi : \{1, 2\} \simeq \Delta$ (in order of increasing root length for B_2 and G_2). In each case, for every $c \in \Phi^+$ an explicit choice is made for the data: $\varepsilon_c \in \{\pm 1\}$, w satisfying $w^{-1}(c) \in \Delta$, and n_w as a product among $\{n_a\}_{a\in\Delta}$ to define a trivialization X_c of $\mathfrak{g}_c$. There is nothing canonical about the choices of ε_c, w, or n_w, but these choices are made only depending on the based root system (Φ, Δ) and enumeration $\xi : \{1, 2\} \simeq \Delta$. In this way, we obtain a trivialization $\{\mathrm{X}_c\}_{c\in\Phi^+}$ for the positive root spaces in each pinned split reductive group scheme $(\mathrm{G}, \mathrm{T}, \mathrm{M}, \{\mathrm{X}_a\}_{a\in\Delta})$ with semisimple-rank 2 when Δ is equipped with an enumeration (say in order of increasing root length for B_2 and G_2). In [**SGA3**, XXIII, 3.4.1(ii)], the signs $\{\varepsilon_c\}_{c\in\Phi^+}$ are taken to be 1 except for G_2.

[The enumeration $\xi : \{1, 2\} \simeq \Delta$ is most important for A_2, since for B_2 and G_2 the root lengths give an intrinsic distinction between the two elements of Δ, whereas for $\mathrm{A}_1 \times \mathrm{A}_1$ the only positive roots are the simple ones and their root groups commute. The effect of the choice of enumeration for A_2 is seen via the signs that break the symmetry in the formulas for $\{\mathrm{X}_c\}_{c\in\Phi^+}$ in [**SGA3**, XXIII, 3.2.1(ii)] if one swaps the order of enumeration of the two simple positive roots.]

Any two isomorphisms (compatible with splittings and pinnings) between pinned split reductive groups of semisimple-rank 2 that induce the same bijection between the Δ's must coincide on derived groups (Proposition 6.1.13(2)), and so coincide on the "universal" trivializations of all positive root spaces. The induced bijection between the Δ's can be controlled by demanding compatibility with the chosen enumeration of Δ. Thus, $\{\mathrm{X}_c\}_{c\in\Phi^+}$ is functorial with respect to isomorphisms between pinned split reductive S-groups with semisimple-rank 2 when we demand that the isomorphism be compatible with a fixed choice of enumeration of Δ. In this sense, the above choice of $\{\mathrm{X}_c\}_{c\in\Phi^+}$

is "universal" for any based reduced root system (Φ, Δ) of rank 2 *equipped with* an enumeration of Δ.

Step 3. The trivializations X_c will now be used to unambiguously define "structure constants" (global functions on S, to be precise) that encode the S-group law. Choose a pinned split reductive S-group $(G, T, M, \{X_a\}_{a\in\Delta})$. For all $c \in \Phi^+$, define $p_c : \mathbf{G}_a \simeq U_c$ via $p_c(x) = \exp_c(xX_c)$. For any $a \in \Delta$ and $c \in \Phi^+ - \{a\}$ there exists a unique unit $u(a, c) \in \mathbf{G}_m(S)$ defined by

$$\mathrm{Ad}_G(n_a)(X_c) = u(a, c)X_{s_a(c)}. \tag{6.2.7}$$

(Note that $s_a(c) \in \Phi^+$ because the Δ-expansion of each positive $c \neq a$ has some positive coefficient away from a and hence $s_a(c) = c - \langle c, a^\vee\rangle a$ does as well.) Likewise, by introducing some universal signs in the definitions of the coefficients in (5.1.5) we see that for distinct $b, b' \in \Phi^+$ and roots $ib + jb' \in \Phi$ with $i, j \geqslant 1$ there are unique $C_{i,j,b,b'} \in \mathbf{G}_a(S)$ such that

$$p_{b'}(y)p_b(x) = p_b(x)p_{b'}(y) \prod_{i,j} p_{ib+jb'}(C_{i,j,b,b'}x^i y^j),$$

where the product on the right side is taken relative to the ordering on Φ^+ defined by lexicographical order relative to the chosen enumeration of Δ. (Recall that for B_2 and G_2 we made the convention to enumerate Δ by putting the short root first. This choice is implicit in the formulas in [**SGA3**, XXIII, 3.3, 3.4].)

A priori, $u(a, c)$ and $C_{i,j,b,b'}$ may depend on $(G, T, M, \{X_a\}_{a\in\Delta})$ over S and the enumeration $\xi : \{1, 2\} \simeq \Delta$. We call these the *structure constants* for $(G, T, M, \{X_a\}_{a\in\Delta})$. (They are global functions on S). If we already knew the Isomorphism and Existence Theorems then the following lemma would be immediate, and the miracle at the heart of the Isogeny, Isomorphism, and Existence Theorems is that this lemma can be proved directly:

Lemma 6.2.8. — *For each based reduced root system (Φ, Δ) of rank 2 equipped with an enumeration of Δ, there are signs $u(a, c) \in \mathbf{Z}^\times$ and integers $C_{i,j,b,b'} \in \mathbf{Z}$ that induce the structure constants for every pinned split reductive group $(G, T, M, \{X_a\}_{a\in\Delta})$ with based root system (Φ, Δ) over any non-empty scheme S.*

Proof. — The idea of the proof is to exploit additional relations arising from the $N_G(T)(S)$-action on the root spaces and the $W(\Phi)$-action on Φ. For example, if $a, b \in \Phi$ are distinct with $b \neq -a$ and if $n \in N_G(T)(S)$ represents $w \in W(\Phi)$ satisfying $w(a) = b$ then for any trivializations X of $\mathfrak{g}_a$ and Y of $\mathfrak{g}_b$ the unit $u \in \mathbf{G}_m(S)$ defined by $\mathrm{Ad}_G(n)(X) = uY$ satisfies

$$nw_a(X)n^{-1} = b^\vee(u)w_b(Y) \tag{6.2.8}$$

(where $w_a(\mathrm{X}) = \exp_a(\mathrm{X})\exp_{-a}(-\mathrm{X}^{-1})\exp_a(\mathrm{X})$, and similarly for b using Y). An equivalent formulation is to say that $w_b(u\mathrm{Y}) = b^\vee(u)w_b(\mathrm{Y})$, which is the first identity in Corollary 5.1.9(2).

The relations (6.2.8) lead to nontrivial conditions on the units $u(a,c)$ in (6.2.7), and for each Φ these extra relations yield unique universal solutions $u(a,c) \in \mathbf{Z}^\times = \{1,-1\}$ that are the same for *all* $(\mathrm{G},\mathrm{T},\mathrm{M},\{\mathrm{X}_a\}_{a\in\Delta})$ with root system Φ over *any* non-empty S. The details are in [**SGA3**, XXIII, 3.1–3.4], working case-by-case depending on the rank-2 root system Φ.

Relations between root groups and the $\mathrm{W}(\Phi)$-action on Φ are given in [**SGA3**, XXIII, 3.1.1] by applying conjugation against each $n_a = w_a(\mathrm{X}_a)$ $(a \in \Delta)$ to the commutation relations among root groups. This allows one to uniquely solve for *some* of the $\mathrm{C}_{i,j,b,b'}$'s. Miraculously, these solutions arise from constants in $\mathbf{Z}$ that depend only on (Φ,Δ,ξ) and do *not* depend on S or $(\mathrm{G},\mathrm{T},\mathrm{M},\{\mathrm{X}_a\}_{a\in\Delta})$. There are more commutation relations for the remaining unknown coefficients $\mathrm{C}_{i,j,b,b'}$, and case-by-case arguments in [**SGA3**, XXIII, 3.1–3.4] depending only on (Φ,Δ,ξ) yield linear equations on the $\mathrm{C}_{i,j,b,b'}$'s over $\mathbf{Z}$ that depend only on (Φ,Δ,ξ) and *not* on $(\mathrm{G},\mathrm{T},\mathrm{M},\{\mathrm{X}_a\}_{a\in\Delta})$ (or S). Remarkably, these equations over $\mathbf{Z}$ admit a unique solution over any non-empty scheme S. The solution over S arises from the unique one over $\mathbf{Z}$. $\square$

Example 6.2.9. — Before we use Lemma 6.2.8 to complete (our sketch of) the proof of the Isogeny Theorem, we illustrate the lemma for G_2 (cf. [**SGA3**, XXIII, 3.4.1]). We have $\Delta = \{a_0, b_0\}$ with a_0 short and b_0 long, so

$$\Phi^+ = \{a_0, b_0, a_0+b_0, 2a_0+b_0, 3a_0+b_0, 3a_0+2b_0\}.$$

For any $(\mathrm{G},\mathrm{T},\mathrm{M},\{\mathrm{X}_a\}_{a\in\Delta})$ with root system G_2 over any non-empty scheme S, define a "universal" trivialization X_c of $\mathfrak{g}_c$ over S for every $c \in \Phi^+$ via

$$\begin{aligned}
\mathrm{X}_{a_0+b_0} &= \mathrm{Ad}_\mathrm{G}(n_{b_0})(\mathrm{X}_{a_0}),\\
\mathrm{X}_{2a_0+b_0} &= \mathrm{Ad}_\mathrm{G}(n_{a_0})(\mathrm{X}_{a_0+b_0}) = \mathrm{Ad}_\mathrm{G}(n_{a_0}n_{b_0})(\mathrm{X}_{a_0}),\\
\mathrm{X}_{3a_0+b_0} &= -\mathrm{Ad}_\mathrm{G}(n_{a_0})(\mathrm{X}_{b_0}),\\
\mathrm{X}_{3a_0+2b_0} &= \mathrm{Ad}_\mathrm{G}(n_{b_0})(\mathrm{X}_{3a_0+b_0}) = -\mathrm{Ad}_\mathrm{G}(n_{b_0}n_{a_0})(\mathrm{X}_{b_0})
\end{aligned}$$

(note the signs).

Universality of $u(a,c) \in \mathbf{G}_\mathrm{m}(\mathrm{S})$ $(a \in \Delta,\ c \in \Phi^+)$ is illustrated by the fact that necessarily $u(a_0, 2a_0+b_0) = -1$, $u(a_0, 3a_0+b_0) = 1$, and $u(b_0, 3a_0+2b_0) = -1$ in $\mathbf{G}_\mathrm{m}(\mathrm{S})$; that is, $\mathrm{Ad}_\mathrm{G}(n_{a_0})(\mathrm{X}_{2a_0+b_0}) = -\mathrm{X}_{a_0+b_0}$, $\mathrm{Ad}_\mathrm{G}(n_{a_0})(\mathrm{X}_{3a_0+b_0}) = \mathrm{X}_{b_0}$, and $\mathrm{Ad}_\mathrm{G}(n_{b_0})(\mathrm{X}_{3a_0+2b_0}) = -\mathrm{X}_{3a_0+b_0}$. Likewise, commutation relations among the parameterizations $p_c(x) = \exp_c(x\mathrm{X}_c)$ for the positive root groups involve universal constants in $\mathbf{Z}$ as the coefficients; e.g.,

$$p_{a_0+b_0}(y)p_{a_0}(x) = p_{a_0}(x)p_{a_0+b_0}(y)p_{2a_0+b_0}(2xy)p_{3a_0+b_0}(3x^2y)p_{3a_0+2b_0}(3xy^2),$$

$$p_{2a_0+b_0}(y)p_{a_0}(x) = p_{a_0}(x)p_{2a_0+b_0}(y)p_{3a_0+b_0}(3xy), \tag{6.2.9}$$

$$p_{3a_0+b_0}(y)p_{b_0}(x) = p_{b_0}(x)p_{3a_0+b_0}(y)p_{3a_0+2b_0}(-xy).$$

The universal coefficients in the commutation relations are ± 1 for $\mathrm{A}_1 \times \mathrm{A}_1$ and A_2, but coefficients in $\{\pm 2\}$ arise for B_2 and coefficients in $\{\pm 2, \pm 3\}$ arise for G_2. An interesting consequence is that for B_2 (resp. G_2) there are some root groups that commute in characteristic 2 (resp. characteristic 3) but not in any other characteristic. For example, (6.2.9) gives an "extra" commutation among root groups of G_2 in characteristic 3.

Step 4. Returning to the (sketch of the) proof of the Isogeny Theorem, recall that we have reduced the problem to groups of semisimple-rank 2. Lemma 6.2.8 implies that in such cases the "structure constants" describing both the adjoint action of the n_a's on the positive root spaces and the commutation relations among the positive root groups are absolute constants in $\mathbf{Z}$ that depend only on (Φ, Δ) (and our enumeration of Δ); they do not depend on the base scheme S or the pinned split reductive S-group with root system Φ.

To go further and prove the Isogeny Theorem, one first determines all p-morphisms among reduced semisimple root data of rank 2 (especially with p a prime rather than $p = 1$); this is an elementary combinatorial problem since we are only considering rank 2. Also, there is a variant on Corollary 6.2.7 given in [**SGA3**, XXIII, 2.5] (as an immediate consequence of Theorem 6.2.4) that provides an existence criterion for homomorphisms out of a pinned split reductive group of semisimple-rank 2. Combining this criterion with case-by-case arguments (depending on Φ and the associated universal structure constants in $\mathbf{Z}$), one builds isogenies between pinned split reductive S-groups realizing any $p(\mathrm{S})$-morphism between the reduced root data. The details are elegantly explained in [**SGA3**, XXIII, 4.1.3–4.1.8]. (The hardest cases are B_2 over $\mathbf{F}_2$-schemes with $q \in \{2^n\}_{n\geqslant 1}$ and G_2 over $\mathbf{F}_3$-schemes with $q \in \{3^n\}_{n\geqslant 1}$.) □

Remark 6.2.10. — The Isogeny Theorem underlies the classification of "exceptional" isogenies between connected semisimple groups over fields. More specifically, in characteristic 0 all isogenies are central, so let us focus on connected semisimple groups over a field k with characteristic $p > 0$. There are two evident classes of isogenies over k: central isogenies and Frobenius isogenies $\mathrm{F}_{\mathrm{G}/k} : \mathrm{G} \to \mathrm{G}^{(p)}$. It is natural to wonder if every isogeny is a composition among these. A map factors through a Frobenius isogeny on the source if and only if it induces the zero map on Lie algebras (as the infinitesimal k-subgroups of G that have full Lie algebra are those which contain $\ker \mathrm{F}_{\mathrm{G}/k}$, due to Theorem A.4.1), so it is equivalent to determine if there are non-central isogenies that are nonzero on Lie algebras. This problem can be reduced to the case $k = k_s$, so we can restrict attention to the split case, for which the Isogeny Theorem is applicable.

By passing to simply connected central covers (Exercise 6.5.2) and direct factors thereof, the problem becomes: when do there exist non-central p-morphisms between irreducible and reduced root data that do not factor through "multiplication by p"? (Here we are using the Isogeny Theorem and the fact that Frobenius isogenies in characteristic $p > 0$ correspond to "multiplication by p" on the root datum.) This is a purely combinatorial problem for each prime p, and by considering the classification of root systems the answer is affirmative if and only if $p \in \{2, 3\}$. Some explicit examples in characteristic 2 are classical in the theory of quadratic forms, namely the isogenies $\mathrm{SO}_{2n+1} \to \mathrm{Sp}_{2n}$ with infinitesimal non-central commutative kernel α_2^{2n} for $n \geqslant 1$; see [**PY06**, Lemma 2.2] for further discussion of these isogenies. For $n = 2$ this isogeny $\mathrm{SO}_5 \to \mathrm{Sp}_4$ gives rise to an exotic *endomorphism* of Sp_4 since $\mathrm{Spin}_5 = \mathrm{Sp}_4$ (as $\mathrm{B}_2 = \mathrm{C}_2$; see Example C.6.5), and this is a "square root" of the Frobenius isogeny over $\mathbf{F}_2$. Similarly, one gets an exotic endomorphism of F_4 in characteristic 2 and of G_2 in characteristic 3. These endomorphisms underlie the existence of the Suzuki and Ree groups in the classification of finite simple groups.

We end this section with some interesting applications of the full faithfulness of (6.1.2) that is a consequence of the proved Isogeny Theorem. (See [**SGA3**, XXIII, § 5] for a more extensive discussion.)

Proposition 6.2.11. — *Let* G *and* G′ *be reductive groups over a non-empty scheme* S.

1. *If* G *and* G′ *are isomorphic fpqc-locally on* S *then they are so étale-locally on* S.
2. *Assume that* G *and* G′ *are isomorphic étale-locally on* S*, that* S *is connected with* $\mathrm{Pic}(\mathrm{S}) = 1$*, and that* G *and* G′ *have respective split maximal tori* T *and* T′*. The pairs* (G, T) *and* $(\mathrm{G}', \mathrm{T}')$ *are isomorphic, as are the triples* $(\mathrm{G}, \mathrm{B}, \mathrm{T})$ *and* $(\mathrm{G}', \mathrm{B}', \mathrm{T}')$ *for Borel subgroups* $\mathrm{B} \supset \mathrm{T}$ *and* $\mathrm{B}' \supset \mathrm{T}'$*. If* $\mathrm{G}' = \mathrm{G}$ *and* S *is also affine (so* $\mathrm{H}^1(\mathrm{S}, \mathscr{O}_\mathrm{S}) = 0$*) then these isomorphisms can be chosen to arise from* $\mathrm{G}(\mathrm{S})$*-conjugation.*

Proof. — For (1) we may work étale-locally on S to reach the split case with the same root datum. Then we can apply the full faithfulness of (6.1.2). For (2), fix isomorphisms $\mathrm{T} \simeq \mathrm{D}_\mathrm{S}(\mathrm{M})$ and $\mathrm{T}' \simeq \mathrm{D}_\mathrm{S}(\mathrm{M}')$ for finite free $\mathbf{Z}$-modules M and M′. Since S is connected, so constant sheaves on S have only constant global sections, $\mathrm{M} = \mathrm{Hom}_{\text{S-gp}}(\mathrm{T}, \mathbf{G}_\mathrm{m})$ and similarly for M′ and T′. Likewise, the root spaces for (G, T) and $(\mathrm{G}', \mathrm{T}')$ are free of rank 1 since $\mathrm{Pic}(\mathrm{S}) = 1$. Thus, $(\mathrm{G}, \mathrm{T}, \mathrm{M})$ and $(\mathrm{G}', \mathrm{T}', \mathrm{M}')$ are split. In particular, Borel subgroups $\mathrm{B} \supset \mathrm{T}$ and $\mathrm{B}' \supset \mathrm{T}'$ do exist; choose such S-subgroups.

The connectedness of S ensures that the choices for B correspond bijectively to the positive systems of roots in $\Phi \subset \mathrm{M}$, and similarly for B′ (see Remark

6.1.2). Hence, by choosing suitable pinnings, the full faithfulness of (6.1.2) provides an isomorphism $(\mathrm{G}, \mathrm{B}, \mathrm{T}) \simeq (\mathrm{G}', \mathrm{B}', \mathrm{T}')$. In the special case $\mathrm{G}' = \mathrm{G}$ with S also affine, it remains to show that G(S) acts transitively on the set of pairs (B, T).

Let B be a Borel subgroup of G, so the orbit map $\mathrm{G} \to \mathrm{Bor}_{\mathrm{G}/\mathrm{S}}$ through B identifies $\mathrm{Bor}_{\mathrm{G}/\mathrm{S}}$ with G/B by Corollary 5.2.7(1) (since $\mathrm{N}_{\mathrm{G}}(\mathrm{B}) = \mathrm{B}$, by Corollary 5.2.8). The S-group B has a composition series whose successive quotients are $\mathbf{G}_{\mathrm{a}}$ and $\mathbf{G}_{\mathrm{m}}$, so the vanishing of both $\mathrm{H}^1(\mathrm{S}_{\text{ét}}, \mathbf{G}_{\mathrm{a}}) = \mathrm{H}^1(\mathrm{S}_{\mathrm{Zar}}, \mathscr{O}_{\mathrm{S}})$ (as S is affine) and $\mathrm{H}^1(\mathrm{S}_{\text{ét}}, \mathbf{G}_{\mathrm{m}}) = \mathrm{H}^1(\mathrm{S}_{\mathrm{Zar}}, \mathscr{O}_{\mathrm{S}}^{\times})$ (as $\mathrm{Pic}(\mathrm{S}) = 1$) implies that $\mathrm{H}^1(\mathrm{S}_{\text{ét}}, \mathrm{B}) = 1$ (concretely, every étale B-torsor over S is split). Hence, the B-torsor $\mathrm{G} \to \mathrm{G}/\mathrm{B}$ induces a surjection $\mathrm{G}(\mathrm{S}) \to (\mathrm{G}/\mathrm{B})(\mathrm{S})$, so G(S) acts transitively on the set of Borel S-subgroups of G.

As we saw above, any split maximal torus T in G lies in some Borel subgroup B. In view of the G(S)-conjugacy of Borel S-subgroups of G, it remains to show that any two split maximal tori $\mathrm{T}, \mathrm{T}' \subset \mathrm{G}$ contained in B are B(S)-conjugate. Any such tori are B-conjugate étale-locally on S (Proposition 2.1.2), so $\mathrm{Transp}_{\mathrm{B}}(\mathrm{T}, \mathrm{T}')$ is a torsor over $\mathrm{S}_{\text{ét}}$ for $\mathrm{N}_{\mathrm{B}}(\mathrm{T}) = \mathrm{B} \bigcap \mathrm{N}_{\mathrm{G}}(\mathrm{T}) = \mathrm{T}$ (the final equality due to $\mathrm{N}_{\mathrm{G}}(\mathrm{T})/\mathrm{T}$ being the finite constant group for $\mathrm{W}(\Phi)$, with $\mathrm{W}(\Phi)$ acting simply transitively on the set of $\mathrm{B} \supset \mathrm{T}$ on geometric fibers over S). But $\mathrm{T} = \mathbf{G}_{\mathrm{m}}^r$ and $\mathrm{Pic}(\mathrm{S}) = 1$, so we are done. □

The Existence Theorem over $\mathbf{C}$ is well-known in the classical theory (see Appendix D). As an application of the full faithfulness in (6.1.2), we now improve on the Existence Theorem over $\mathbf{C}$ by pushing it down to $\mathbf{Q}$. This will be an ingredient in the proof of the Existence Theorem in general.

Proposition 6.2.12. — *For each reduced root datum* R*, there exists a split connected reductive* $\mathbf{Q}$*-group* (G, T) *such that* $\mathrm{R}(\mathrm{G}, \mathrm{T}) \simeq \mathrm{R}$.

Proof. — Choose a split connected reductive $\mathbf{C}$-group $(\mathrm{G}, \mathrm{T}, \mathrm{M})$ having R as its root datum. Writing $\mathbf{C} = \varinjlim \mathrm{A}_i$ for finite type $\mathbf{Q}$-subalgebras A_i, the triple $(\mathrm{G}, \mathrm{T}, \mathrm{M})$ descends to a split reductive group scheme $(\mathscr{G}, \mathscr{T}, \mathrm{M})$ over Spec A for some finite type $\mathbf{Q}$-subalgebra $\mathrm{A} = \mathrm{A}_{i_0} \subset \mathbf{C}$. Its root datum is R, so by passing to the fiber at a closed point we find a split triple $(\mathrm{G}', \mathrm{T}', \mathrm{M})$ over a number field F with root datum isomorphic to R; fix this isomorphism.

By replacing F with a finite extension we may assume that F is Galois over $\mathbf{Q}$. We will now carry out Galois descent down to $\mathbf{Q}$ via the crutch of a pinning. Choose a positive system of roots Φ^+ in $\Phi = \Phi(\mathrm{G}', \mathrm{T}')$, and let Δ be the corresponding base. For each $a \in \Delta$, pick a basis X_a of the F-line $\mathfrak{g}'_a$, so we get a pinned split reductive group $(\mathrm{G}', \mathrm{T}', \mathrm{M}, \{\mathrm{X}_a\}_{a \in \Delta})$. We have a chosen isomorphism $\phi : \mathrm{R}(\mathrm{G}', \mathrm{T}', \mathrm{M}) \simeq \mathrm{R}$, and for all $\gamma \in \Gamma := \mathrm{Gal}(\mathrm{F}/\mathbf{Q})$ we get another pinned split reductive F-group

$$(\gamma^*(\mathrm{G}'), \gamma^*(\mathrm{T}'), \mathrm{M}, \{\gamma^*(\mathrm{X}_a)\}_{a \in \Delta}))$$

via the evident identifications $\Phi(\gamma^*(\mathrm{G}'), \gamma^*(\mathrm{T}')) \simeq \Phi(\mathrm{G}', \mathrm{T}') = \Phi \subset \mathrm{M} - \{0\}$ and $\mathrm{X}(\gamma^*(\mathrm{T}')) \simeq \mathrm{X}(\mathrm{T}') = \mathrm{M}$ (defined by functoriality of scalar extension along the Γ-action on F).

It is easy to check that the resulting isomorphisms of root data

$$\mathrm{R}(\mathrm{G}', \mathrm{T}', \mathrm{M}) \simeq \mathrm{R}(\gamma^*(\mathrm{G}'), \gamma^*(\mathrm{T}'), \mathrm{M}, \{\gamma^*(\mathrm{X}_a)\}_{a\in\Delta}) \tag{6.2.10}$$

satisfy the cocycle condition. By the full faithfulness of (6.1.2), the isomorphism (6.2.10) arises from a unique isomorphism between *pinned* split reductive F-groups

$$(\mathrm{G}', \mathrm{T}', \mathrm{M}, \{\mathrm{X}_a\}_{a\in\Delta}) \simeq (\gamma^*(\mathrm{G}'), \gamma^*(\mathrm{T}'), \mathrm{M}, \{\gamma^*(\mathrm{X}_a)\}_{a\in\Delta}),$$

and the uniqueness implies that these isomorphisms inherit the cocycle condition from that aspect of the isomorphisms of root data. Note that these isomorphisms between pinned groups use the identity automorphism on M, so they use the identity bijection on Δ. (That is, X_a is carried to $\gamma^*(\mathrm{X}_a)$.) Hence, by Galois descent we obtain a pinned *split* reductive $\mathbf{Q}$-group descending $(\mathrm{G}', \mathrm{T}', \mathrm{M}, \{\mathrm{X}_a\}_{a\in\Delta})$, and its root datum is clearly R. $\square$

6.3. Existence Theorem. — Let $\mathrm{R} = (\mathrm{M}, \Phi, \mathrm{M}^\vee, \Phi^\vee)$ be a reduced root datum. By base change, to prove the Existence Theorem for R (i.e., Theorem 6.1.16(2)) over an arbitrary non-empty scheme S it suffices to treat the case $\mathrm{S} = \operatorname{Spec} \mathbf{Z}$. By the following lemma, whose proof is a formal argument with root data, it suffices to consider only R that is semisimple and simply connected (i.e., Φ spans $\mathrm{M}_{\mathbf{Q}}$ over $\mathbf{Q}$, and $\Phi^\vee$ spans $\mathrm{M}^\vee$ over $\mathbf{Z}$) and such that the root system associated to R is irreducible.

Lemma 6.3.1. — *To prove the Existence Theorem over a non-empty scheme* S*, it suffices to treat semisimple root data* $(\mathrm{X}, \Phi, \mathrm{X}^\vee, \Phi^\vee)$ *that are simply connected and have associated root system* $(\mathrm{X}_{\mathbf{Q}}, \Phi)$ *that is irreducible.*

Proof. — The idea is to use a preliminary central isogeny of root data to separate the maximal central torus from the derived group, and then to treat tori and semisimple groups separately. Let $\mathrm{R} = (\mathrm{X}, \Phi, \mathrm{X}^\vee, \Phi^\vee)$ be a root datum, so X contains $\mathbf{Z}\Phi \bigoplus (\mathbf{Z}\Phi^\vee)^\perp$ with finite index, where the annihilator $(\mathbf{Z}\Phi^\vee)^\perp$ in X is saturated but $\mathbf{Z}\Phi$ may not be saturated. In general, if $\mathrm{L} \to \mathrm{L}'$ is an injective map between finite free $\mathbf{Z}$-modules, we write $\mathrm{L}_{\mathrm{sat}}$ to denote the saturation of L in L' (i.e., the kernel of $\mathrm{L}' \to (\mathrm{L}'/\mathrm{L})_{\mathbf{Q}}$). The natural map

$$\mathrm{X} \to (\mathrm{X}/(\mathbf{Z}\Phi)_{\mathrm{sat}}) \oplus (\mathrm{X}/(\mathbf{Z}\Phi^\vee)^\perp) =: \mathrm{X}'$$

is a finite-index inclusion that carries Φ onto a subset Φ' that lies in the second summand of X'. The $\mathbf{Z}$-dual of X' is naturally identified with the direct sum

$$\mathrm{X}'^\vee = (\mathbf{Z}\Phi)^\perp \oplus (\mathbf{Z}\Phi^\vee)_{\mathrm{sat}}$$

and $\Phi'^\vee$ is defined to be the image of $\Phi^\vee$ under inclusion into the second factor.

Clearly $R' := (X', \Phi', X'^{\vee}, \Phi'^{\vee})$ is a reduced root datum, and the natural isogeny $(h, d, q) : R' \to R$ is "central": $q(a') = 1$ for all $a' \in \Phi'$. If R' arises from a split reductive S-group (G', T') then the cokernel of $h : X \to X' = X(T')$ corresponds to a split finite multiplicative type S-subgroup $\mu \subset T'$ such that $X(T'/\mu) = X$ inside $X(T') = X'$. In particular, the centrality of (h, d, q) implies that all roots of (G', T') lie in $X(T'/\mu)$, which is to say μ is a central S-subgroup of G' (Corollary 3.3.6(1)). Hence, the central quotient $G := G'/\mu$ makes sense as a reductive S-group in which $T := T'/\mu$ is a split maximal torus. The inclusion $h : X \hookrightarrow X'$ carries Φ onto Φ' (as q is identically 1) and the dual map $X'^{\vee} \to X^{\vee}$ carries $\Phi'^{\vee}$ onto $\Phi^{\vee}$ (due to the unique characterization of coroots for a root system), so the root datum $R(G, T)$ is identified with $(X, \Phi, X^{\vee}, \Phi^{\vee})$. Now it suffices to treat R' instead of R.

Let $L = X/(\mathbf{Z}\Phi^{\vee})^{\perp}$, $L^{\vee} = (\mathbf{Z}\Phi^{\vee})_{\mathrm{sat}}$, $\Psi = \Phi \bmod (\mathbf{Z}\Phi^{\vee})^{\perp} \subset L$, and $\Psi^{\vee} = \Phi^{\vee} \subset L^{\vee}$, so $R'' := (L, \Psi, L^{\vee}, \Psi^{\vee})$ is a semisimple reduced root datum and

$$R' = R'' \oplus (X/(\mathbf{Z}\Phi)_{\mathrm{sat}}, \varnothing, (\mathbf{Z}\Phi)^{\perp}, \varnothing).$$

It suffices to treat the two summands separately (as we can then form the direct product of the corresponding split connected reductive groups). The second summand is trivially handled by using the split torus with character group $X/(\mathbf{Z}\Phi)_{\mathrm{sat}}$, so we may now focus our attention on R''. That is, we may assume that our root datum R is semisimple.

As in (1.3.2), we have $\mathbf{Z}\Phi \subset X \subset (\mathbf{Z}\Phi^{\vee})^*$. Let $X' = (\mathbf{Z}\Phi^{\vee})^*$, $\Phi' = \Phi$, $X'^{\vee} = \mathbf{Z}\Phi^{\vee}$, and $\Phi'^{\vee} = \Phi^{\vee}$, so $R' = (X', \Phi', X'^{\vee}, \Phi'^{\vee})$ is a root datum that is semisimple and simply connected. There is an evident central isogeny of root data $R' \to R$, so by repeating the central quotient construction above we see that the Existence Theorem for R over S is reduced to the Existence Theorem for R' over S. Thus, it remains to treat the case of semisimple root data that are simply connected. The equality $X^{\vee} = \mathbf{Z}\Phi^{\vee}$ ensures that the decomposition of the root system into its irreducible components is also valid at the level of the root datum. Hence, it remains to settle the case of semisimple root data that are simply connected and have an irreducible associated root system. This is precisely the case that we are assuming is established. $\square$

Fix a semisimple reduced root datum R that is simply connected. (We will not require irreducibility for R.) Proposition 6.2.12 provides a split reductive group (G, T, M) over $\mathbf{Q}$ with root datum R. This yields the Existence Theorem for R over some $\mathbf{Z}[1/N]$, but N might depend on R. The problem is to get the result over the entirety of $\operatorname{Spec} \mathbf{Z}$, not ignoring any small primes. The rest of § 6.3 is devoted to the construction of such a split $\mathbf{Z}$-group by a method that works uniformly across all (simply connected and semisimple) R. In view of the classification of irreducible and reduced root systems, it would suffice to exhibit an explicit example for each Killing–Cartan type.

For the classical types A_n ($n \geqslant 1$), B_n ($n \geqslant 3$), C_n ($n \geqslant 2$), and D_n ($n \geqslant 4$) we can use the $\mathbf{Z}$-groups SL_{n+1} ($n \geqslant 1$), Spin_{2n+1} ($n \geqslant 3$), Sp_{2n} ($n \geqslant 2$), and Spin_{2n} ($n \geqslant 4$) respectively. (To make sense of spin groups over $\mathbf{Z}$ and not just over $\mathbf{Z}[1/2]$, we need a characteristic-free viewpoint on non-degeneracy for quadratic spaces over rings. This is discussed in Appendix C.) Thus, the arguments that follow are only needed to handle the exceptional types E_6, E_7, E_8, F_4, and G_2. Even some of these types can be settled by direct construction (e.g., G_2 and F_4 can be handled by using octonion and Jordan algebras over $\mathbf{Z}$). Explicit constructions can require special care at small primes (e.g., residue characteristic 2 for spin groups and type F_4, and residue characteristics 2 and 3 for type G_2). The uniform approach below is insensitive to the peculiar demands of small primes or of specific irreducible root systems.

Before we take up the proof of the Existence Theorem, we need to digress and discuss the following concept:

Definition 6.3.2. — Let (G, T, M) be a split reductive group over a non-empty scheme S. A *Chevalley system* for (G, T, M) is a collection of trivializing sections $X_a \in \mathfrak{g}_a(S)$ for all $a \in \Phi$ so that

$$\mathrm{Ad}_G(w_a(X_a))(X_b) = \pm X_{s_a(b)}$$

for all $a, b \in \Phi$, where the sign ambiguity is global over S (possibly depending on a and b) and

$$w_c(X) := \exp_c(X)\exp_{-c}(-X^{-1})\exp_c(X) \in N_G(T)(S)$$

for every trivializing section X of $\mathfrak{g}_c$ and every $c \in \Phi$.

The existence of a Chevalley system is vacuous for semisimple-rank 0, and for semisimple-rank 1 we can build one by using any X_a whatsoever and defining $X_{-a} := X_a^{-1}$ to be the linked trivialization of $\mathfrak{g}_{-a}$ (this works, since $\mathrm{Ad}_G(w_a(X))(X) = -X^{-1}$ for any trivializing section X of $\mathfrak{g}_a$; see Corollary 5.1.9(3)). By setting $b = a$ and using that $s_a(a) = -a$, it likewise follows that for any Chevalley system $\{X_a\}_{a\in\Phi}$ we must have $X_{-a} = \pm X_a^{-1}$ (i.e., X_a and X_{-a} are linked, up to a global sign depending on a).

Example 6.3.3. — Chevalley systems are closely related to the notion of a "Chevalley basis" for a complex semisimple Lie algebra (cf. [**Hum72**, 25.1–25.2]). To explain this, consider a connected semisimple $\mathbf{C}$-group G equipped with maximal torus T, so $\mathfrak{g} := \mathrm{Lie}(G)$ is a semisimple Lie algebra and $\mathfrak{t} := \mathrm{Lie}(T)$ is a Cartan subalgebra. Fix a positive system of roots in $\Phi(G, T) = \Phi(\mathfrak{g}, \mathfrak{t})$, and let Δ be the corresponding set of simple roots, so the vectors $v_a = \mathrm{Lie}(a^\vee)(\partial_t|_{t=1})$ with $a \in \Delta$ are a basis of $\mathfrak{t}$. Let X_c be a basis of $\mathfrak{g}_c$ for each $c \in \Phi$, so the collection of v_a's and X_c's is a basis of $\mathfrak{g}$.

Let's introduce the associated "structure constants". For $a \in \Delta$ and $c \in \Phi$, we have $[v_a, X_c] = \mathrm{Ad}_G(v_a)(X_c) = \langle c, a^\vee\rangle X_c$ since conjugation by $a^\vee(t)$ on

$\mathrm{U}_c = \mathbf{G}_{\mathrm{a}}$ acts via scaling by $c(a^\vee(t)) = t^{\langle c,a^\vee\rangle}$. Consider $c, c' \in \Phi$ with $c' \neq \pm c$. There is a positive system of roots containing c and c', so by Proposition 5.1.14 the groups U_c and $\mathrm{U}_{c'}$ commute if $c + c' \notin \Phi$ (forcing $[\mathrm{X}_c, \mathrm{X}_{c'}] = 0$) and otherwise $[\mathrm{X}_c, \mathrm{X}_{c'}] = r(c, c')\mathrm{X}_{c+c'}$ for some $r(c, c') \in \mathbf{C}$. The special feature of $\{\mathrm{X}_c\}_{c\in\Phi}$ being a Chevalley system is that the numbers $r(c, c')$ are (nonzero) *integers* that are moreover determined up to sign by the root system; see Remark 6.3.5. This provides an explicit $\mathbf{Z}$-form for every complex semisimple Lie algebra, and Chevalley used this viewpoint to construct adjoint split semisimple $\mathbf{Z}$-groups (see Theorem 5.3.5, as well as [**Hum72**, 25.5, § 26]).

Proposition 6.3.4. — *Let* $(\mathrm{G}, \mathrm{T}, \mathrm{M}, \{\mathrm{X}_a\}_{a\in\Delta})$ *be a pinned split reductive group over a non-empty scheme* S. *There is a Chevalley system* $\{\mathrm{X}_c\}_{c\in\Phi}$ *extending the pinning, and each* X_c *is unique up to a global sign.*

For semisimple-rank 2, the main computations for the construction of a Chevalley system were carried out in the proof of the Isogeny Theorem, but more work is required even for semisimple-rank 2 (since the definition of a Chevalley system involves the adjoint action for $w_a(\mathrm{X}_a)$ for all $a \in \Phi$).

Proof. — Let $n_a = w_a(\mathrm{X}_a)$ for all $a \in \Delta$. Since every element of $\mathrm{W}(\Phi)$ is represented by a product among the elements of $\{n_a\}_{a\in\Delta}$, for any $c \in \Phi$ we can find such a product n representing an element $w \in \mathrm{W}(\Phi)$ so that $w(a) = c$ for some $a \in \Delta$ (i.e., $w^{-1}(c) \in \Delta$). Thus, $\mathrm{X}_c = \pm\mathrm{Ad}_{\mathrm{G}}(n)(\mathrm{X}_a)$ with a global sign ambiguity. This shows the uniqueness of each X_c up to a global sign.

To prove existence, we begin by running the uniqueness proof in reverse. For each $c \in \Phi$ not in Δ, choose some $w \in \mathrm{W}(\Phi)$ such that $w^{-1}(c) \in \Delta$. Pick a word $a_1 \cdots a_m$ in elements a_i of Δ so that $s_{a_1} \cdots s_{a_m} = w$ in $\mathrm{W}(\Phi)$. For the element $n := n_{a_1} \cdots n_{a_m} \in \mathrm{N}_{\mathrm{G}}(\mathrm{T})(\mathrm{S})$, use the isomorphism $\mathrm{Ad}_{\mathrm{G}}(n) : \mathfrak{g}_{w^{-1}(c)} \simeq \mathfrak{g}_c$ to define

$$\mathrm{X}_c := \mathrm{Ad}_{\mathrm{G}}(n)(\mathrm{X}_{w^{-1}(c)}).$$

It suffices to show that $\{\mathrm{X}_c\}_{c\in\Phi}$ is a Chevalley system. (The definition of the X_c's depends on the choice of w and the word $a_1 \cdots a_m$. However, once the proof is done, it will follow that changing these choices affects each X_c by at most a global sign.)

For $c \in \Phi$ and $b \in \Delta$, we need to prove that

$$\mathrm{Ad}_{\mathrm{G}}(n_b)(\mathrm{X}_c) = \pm\mathrm{X}_{s_b(c)}. \tag{6.3.1}$$

By definition, $\mathrm{X}_c = \mathrm{Ad}_{\mathrm{G}}(n_0)(\mathrm{X}_{a_0})$ for some $a_0 \in \Delta$ and product n_0 among the elements of $\{n_a\}_{a\in\Delta}$ so that n_0 represents an element $w_0 \in \mathrm{W}(\Phi)$ satisfying $w_0(a_0) = c$. Likewise, $\mathrm{X}_{s_b(c)} = \mathrm{Ad}_{\mathrm{G}}(n_1)(\mathrm{X}_{a_1})$ for some $a_1 \in \Delta$ and product n_1 among the elements of $\{n_a\}_{a\in\Delta}$ so that n_1 represents an element $w_1 \in \mathrm{W}(\Phi)$ satisfying $w_1(a_1) = s_b(c) = (s_b w_0)(a_0)$. Thus, $n := n_0^{-1} n_b^{-1} n_1$ represents some $w \in \mathrm{W}(\Phi)$ satisfying $w(a_1) = a_0$, and our problem is to show that $\mathrm{Ad}_{\mathrm{G}}(n)(\mathrm{X}_{a_1}) = \pm \mathrm{X}_{a_0}$ (as then applying $\mathrm{Ad}_{\mathrm{G}}(n_b n_0)$ to both sides will give

$$\mathrm{X}_{s_b(c)} = \mathrm{Ad}_{\mathrm{G}}(n_1)(\mathrm{X}_{a_1}) = \pm \mathrm{Ad}_{\mathrm{G}}(n_b)(\mathrm{Ad}_{\mathrm{G}}(n_0)(\mathrm{X}_{a_0})) = \pm \mathrm{Ad}_{\mathrm{G}}(n_b)(\mathrm{X}_c)$$

as desired).

Although $n = n_0^{-1} n_b^{-1} n_1$ is not written as a product among the $\{n_a\}_{a \in \Delta}$, due to the intervention of some inversions, these inversions can be absorbed into the sign ambiguity in (6.3.1). The reason is as follows. We have $n_a^{-1} = a^\vee(-1) n_a$ with $a^\vee(-1) \in \mathrm{T}[2]$, and for any $t \in \mathrm{T}$ and $a' \in \Delta$ the identity

$$t n_{a'} t^{-1} = t w_{a'}(\mathrm{X}_{a'}) t^{-1} = w_{a'}(a'(t) \mathrm{X}_{a'}) = a'^\vee(a'(t)) w_{a'}(\mathrm{X}_{a'}) = t^2 n_{a'}$$

(using Corollary 5.1.9(2)) implies that $\mathrm{T}[2]$ centralizes $n_{a'}$. In particular, for $n = n_0^{-1} n_b^{-1} n_1$ and $n_0^{\mathrm{opp}} := n_{a_1} \cdots n_{a_m}$ where $n_{a_1} \cdots n_{a_m} = n_0$ with $a_i \in \Delta$, the product $n' := n_0^{\mathrm{opp}} n_b n_1$ among the elements of $\{n_a\}_{a \in \Delta}$ lifts the *same* word in the involutions s_a as does n and we have $n = t n'$ with $t := \lambda(-1)$ for some $\lambda \in \mathrm{M}^\vee \in \mathrm{Hom}_{\text{S-gp}}(\mathbf{G}_{\mathrm{m}}, \mathrm{T})$. Thus, $\mathrm{Ad}_{\mathrm{G}}(n) = \mathrm{Ad}_{\mathrm{G}}(t) \circ \mathrm{Ad}_{\mathrm{G}}(n')$, and the effect of $\mathrm{Ad}_{\mathrm{G}}(t)$ on each $\mathfrak{g}_c$ is scaling by $c(t) = (-1)^{\langle c, \lambda \rangle} = \pm 1$.

To summarize, we are reduced to proving a general fact about words in the elements n_a: if $a, a' \in \Delta$ and $\{a_1, \dots, a_m\}$ is a sequence in Δ such that $(s_{a_m} \circ \cdots \circ s_{a_1})(a) = a'$ then

$$\mathrm{Ad}_{\mathrm{G}}(n_{a_m} \circ \cdots \circ n_{a_1})(\mathrm{X}_a) \stackrel{?}{=} \pm \mathrm{X}_{a'}$$

in $\mathfrak{g}(\mathrm{S})$, for some global sign ± 1. Note that this equality is obvious when the semisimple-rank is at most 1, and it is also obvious when the pinning extends to a Chevalley system. Thus, to settle it for all cases with semisimple-rank 2 we just need to construct *some* Chevalley system extending the pinning in every case. But the rank-2 calculations in the proof of Lemma 6.2.8 (for the aspect concerning the units $u(a,c)$) achieve exactly this! To be precise, those calculations construct a "positive" Chevalley system: a collection of trivializations $\{\mathrm{X}_c\}_{c \in \Phi^+}$ of the positive root spaces that extends the pinning and satisfies $\mathrm{Ad}_{\mathrm{G}}(n_a)(\mathrm{X}_c) = \pm \mathrm{X}_{s_a(c)}$ for any $a \in \Delta$ and $c \in \Phi - \{a\}$. Hence, to settle the case of semisimple-rank 2 we just need to extend any "positive" Chevalley system $\{\mathrm{X}_c\}_{c \in \Phi^+}$ to an actual Chevalley system.

Define $\mathrm{X}_{-c} = \mathrm{X}_c^{-1}$ for all $c \in \Phi^+$. We claim that $\{\mathrm{X}_c\}_{c \in \Phi}$ is a Chevalley system. This amounts to checking that $\mathrm{Ad}_{\mathrm{G}}(n_a)(\mathrm{X}_c) = \pm \mathrm{X}_{s_a(c)}$ for all $c \in \Phi$. The cases (a, c) and $(a, -c)$ are equivalent, by the functoriality of duality of opposite root spaces with respect to n_a-conjugation on G. Hence, we may assume $c \in \Phi^+$. The case $c \in \Phi^+ - \{a\}$ is known by hypothesis (as $s_a(c) \in \Phi^+$ for all $c \in \Phi^+ - \{a\}$), and $\mathrm{Ad}_{\mathrm{G}}(n_a)(\mathrm{X}_a) = -\mathrm{X}_a^{-1}$ by Corollary 5.1.9(2). This completes the argument for semisimple-rank 2.

For the case of semisimple-rank > 2, one needs to apply several results in the theory of root systems and make artful use of presentations of Weyl groups as reflection groups to ultimately reduce to the settled case of semisimple-rank 2. We refer to [**SGA3**, XXIII, 6.3] for the details, which rest on two ingredients: many of the case-by-case "universal" formulas established for pinned split reductive groups with a root system of rank 2 in [**SGA3**, XXIII, 3.1–3.4], and root system arguments from [**SGA3**, XXIII, 2.3] that were used in the proof of Theorem 6.2.4. □

Remark 6.3.5. — A very useful application of the existence of Chevalley systems is "Chevalley's rule" [**SGA3**, XXIII, 6.5] that computes – up to a sign – universal formulas for the structure constants in the Lie algebra of a split semisimple group scheme $(\mathrm{G}, \mathrm{T}, \mathrm{M})$ relative to a Chevalley system in the Lie algebra. Explicitly, if $\{\mathrm{X}_c\}_{c\in\Phi}$ is a Chevalley system for $(\mathrm{G}, \mathrm{T}, \mathrm{M})$ then $[\mathrm{X}_a, \mathrm{X}_b] = \pm(p(a, b) + 1)\mathrm{X}_{a+b}$ whenever $a, b, a + b \in \Phi$, where $p(a, b)$ is the greatest integer $z \geqslant 0$ such that $a - zb \in \Phi$. This result is proved by inspecting the universal constants in Lemma 6.2.8 for the commutation relations among positive root groups in pinned split reductive groups with semisimple-rank 2.

Returning to the proof of the Existence Theorem, we have reduced to the task of extending a split reductive $\mathbf{Q}$-group $(\mathrm{G}, \mathrm{T}, \mathrm{M})$ to a split reductive $\mathbf{Z}$-group. We have also seen that it suffices to treat cases in which the root datum $\mathrm{R} = (\mathrm{M}, \Phi, \mathrm{M}^\vee, \Phi^\vee)$ for $(\mathrm{G}, \mathrm{T}, \mathrm{M})$ is semisimple and simply connected (i.e., Φ spans $\mathrm{M}_{\mathbf{Q}}$ over $\mathbf{Q}$ and $\Phi^\vee$ spans $\mathrm{M}^\vee$ over $\mathbf{Z}$).

To construct the required split semisimple $\mathbf{Z}$-group extending $(\mathrm{G}, \mathrm{T}, \mathrm{M})$, choose a base Δ of Φ and a pinning $\{\mathrm{X}_a\}_{a\in\Delta}$ of $(\mathrm{G}, \mathrm{T}, \mathrm{M})$. Using Proposition 6.3.4, extend this to a Chevalley system $\{\mathrm{X}_c\}_{c\in\Phi}$. Since $\mathrm{X}_{-c} = \pm\mathrm{X}_c^{-1}$ for all c, by using sign changes if necessary we may and do arrange that $\mathrm{X}_{-c} = \mathrm{X}_c^{-1}$ for all $c \in \Phi$.

Lemma 6.3.6. — *Let* $\mathrm{T} = \mathrm{D}_\mathrm{S}(\mathrm{M})$ *be a split torus over a scheme* S *and let* U *be a smooth affine* S*-group with unipotent fibers on which* T *acts. Assume that* U *contains a finite collection of* T*-stable* S*-subgroups* $\mathrm{U}_i = \mathbf{W}(\mathscr{E}_i)$ $(i \in \mathrm{I})$ *on which* T *acts through nontrivial characters* $a_i \in \mathrm{M} \subset \mathrm{X}(\mathrm{T})$ *that are pairwise linearly independent.*

If the multiplication map $\prod \mathrm{U}_i \to \mathrm{U}$ *for one enumeration of* I *is an* S*-scheme isomorphism then it is so for any enumeration of* I*; i.e., the* U_i*'s directly span* U *in any order.*

Proof. — By the fibral isomorphism criterion it suffices to work on fibers, so we may assume $\mathrm{S} = \operatorname{Spec} k$ for a field k. The assertion is a special case of a general dynamical "direct spanning in any order" result in [**CGP**, 3.3.11]. □

Fix an enumeration $\{a_1, \ldots, a_r\}$ of Δ. Use this to define the lexicographical ordering on $\mathrm{M}_{\mathbf{Q}}$, so we get an ordering $\{c_0, \ldots, c_m\}$ of Φ^+. Identify the unipotent radical $\mathrm{U}_+ := \mathrm{U}_{\Phi^+}$ of the Borel subgroup $\mathrm{B} = \mathrm{T} \ltimes \mathrm{U}_+$ corresponding to Φ^+ with a direct product (as $\mathbf{Q}$-schemes) of the positive root groups U_c via the chosen ordering $\{c_j\}$ on Φ^+. Let $\mathrm{U}_- = \mathrm{U}_{-\Phi^+}$.

The coroots in $\Delta^\vee$ are a $\mathbf{Z}$-basis for the cocharacter group $\mathrm{M}^\vee$ of R, due to R being simply connected, so we get an isomorphism $\mathbf{G}_{\mathrm{m}}^{\Delta} \simeq \mathrm{T}$ over $\mathbf{Q}$ via $(t_a)_{a\in\Delta} \mapsto \prod a^\vee(t_a)$. For each $c \in \Phi$, use X_c to identify U_c with $\mathbf{G}_{\mathrm{a}}$ via $p_c : x \mapsto \exp_c(x\mathrm{X}_c)$, so the open cell $\Omega = \mathrm{U}_- \times \mathrm{T} \times \mathrm{U}_+$ in the pinned split group $(\mathrm{G}, \mathrm{T}, \mathrm{M}, \{\mathrm{X}_a\}_{a\in\Delta})$ over $\mathbf{Q}$ is identified with a product of $\mathbf{G}_{\mathrm{m}}$'s and $\mathbf{G}_{\mathrm{a}}$'s as a $\mathbf{Q}$-scheme:

$$\prod_{j=-m}^{0} \mathbf{G}_{\mathrm{a}} \times \prod_{a\in\Delta} \mathbf{G}_{\mathrm{m}} \times \prod_{i=0}^{m} \mathbf{G}_{\mathrm{a}} \simeq \mathrm{U}_- \times \mathrm{T} \times \mathrm{U}_+ = \Omega \tag{6.3.2}$$

via

$$((x'_j), (t_a)_{a\in\Delta}, (x_i)) \mapsto \prod_{j=-m}^{0} p_{-c_{-j}}(-x'_j) \cdot \prod_{a\in\Delta} a^\vee(t_a) \cdot \prod_{i=0}^{m} p_{c_i}(x'_i)$$

in which the product description for U_+ uses the ordering on Φ^+ and the one for U_- uses the opposite ordering on $-\Phi^+$. (The specific choice of ordering for the product description of $\mathrm{U}_\pm$ will eventually turn out not to matter, but we need to make some definite choice at the outset.)

Lemma 6.3.7. — *The isomorphisms* $\prod_{j=-m}^{0} \mathbf{G}_{\mathrm{a}} \simeq \mathrm{U}_-$ *and* $\prod_{i=0}^{m} \mathbf{G}_{\mathrm{a}} \simeq \mathrm{U}_+$ *of* $\mathbf{Q}$*-schemes as defined above carry the* $\mathbf{Q}$*-group structures on* $\mathrm{U}_\pm$ *over to* $\mathbf{Q}$*-group structures on* $\mathbf{G}_{\mathrm{a}}^{m+1}$ *that are defined over* $\mathbf{Z}$.

Proof. — Observe that via the given choice of ordering, the resulting $\mathbf{Z}$-structure $\mathscr{U}_+$ on U_+ admits an evident action by $\mathscr{T} := \mathbf{G}_{\mathrm{m}}^{\Delta}$ extending the natural action on U_+ over $\mathbf{Q}$. Hence, once U_+ is settled for the initial choice of ordering on Φ^+, by Lemma 6.3.6 the same holds for U_+ using *any* ordering on Φ^+ to define the identification of U_+ with the $\mathbf{Q}$-scheme $\mathbf{G}_{\mathrm{a}}^{m+1}$.

The element $n = n_{a_1} \cdots n_{a_r} \in \mathrm{N}_{\mathrm{G}}(\mathrm{T})(\mathbf{Q})$ represents the long Weyl element $w = s_{a_1} \cdots s_{a_r} \in \mathrm{W}(\Phi)$ relative to Δ, so n-conjugation carries U_+ to U_- but the bijection $\Phi^+ \simeq -\Phi^+$ defined by the w-action need not carry the ordering on Φ^+ to an easily described ordering on $-\Phi^+$ (since the bijection $\Delta \simeq -\Delta$ may be hard to understand). Regardless, since $\{\mathrm{X}_c\}$ is a Chevalley system, so $np_c(x)n^{-1} = \exp_{w(c)}(\mathrm{Ad}_{\mathrm{G}}(n)(x\mathrm{X}_c)) = p_{w(c)}(\pm x)$ for some universal sign (depending only on n, c, and the choice of Chevalley system), it follows that the result for U_- is a formal consequence of the result for U_+ (for all enumerations of Φ^+!). Hence, we may and do now focus on the case of U_+.

Consider the $\mathbf{Q}$-scheme isomorphism $\mathrm{U}_+ \simeq \mathbf{G}_{\mathrm{a}}^{m+1}$ as defined in (6.3.2); we shall use this to equip U_+ with a $\mathbf{Z}$-group structure. Clearly the identity section of U_+ is defined over $\mathbf{Z}$, and inversion on U_+ corresponds to reversing the order of multiplication and replacing each $p_c(x)$ with $p_c(-x)$, so it suffices to check that the multiplication law on U_+ is defined over $\mathbf{Z}$ (as then the inversion on U_+ is defined over $\mathbf{Z}$, and all of the group scheme diagrams commute over $\mathbf{Z}$ since they commute over $\mathbf{Q}$). In other words, we can focus on the multiplication law and not dwell on inversion.

The multiplication description of U_+ as a product of root groups over $\mathbf{Q}$ is defined relative to the lexicographical order on Φ^+ using some ordering of Δ. Thus, for each $1 \leqslant i \leqslant m$, the product $\mathrm{U}_{\geqslant c_i} := \mathrm{U}_{c_i} \cdots \mathrm{U}_{c_m}$ is a closed $\mathbf{Q}$-subgroup of U normalized by $\mathrm{U}_{c_{i-1}}$ (as we saw in the proof of Lemma 6.2.3) and this computes the direct product structure that we have built into the $\mathbf{Z}$-structure. Hence, it suffices to prove by descending induction on i that the $\mathbf{Q}$-group law on each $\mathrm{U}_{\geqslant c_i}$ is defined over $\mathbf{Z}$ (on the corresponding direct product of copies of $\mathbf{G}_{\mathrm{a}}$). The base case $i = m$ is obvious, and likewise the $\mathbf{Q}$-group structure on each $\mathrm{U}_c = \mathbf{G}_{\mathrm{a}}$ is visibly defined over $\mathbf{Z}$, so to carry out the induction it is enough to check that the conjugation action by $\mathrm{U}_{c_{i-1}}$ on $\mathrm{U}_{\geqslant c_i}$ is defined over $\mathbf{Z}$. We can also assume that the semisimple-rank is at least 2 (or else there is nothing to do).

More generally, for $c, c' \in \Phi^+$ with $c < c'$, consider the conjugation action by $p_c(x)$ on $p_{c'}(x')$, assuming that the group law on $\mathrm{U}_{\geqslant c'}$ is already known to be defined over $\mathbf{Z}$. By (5.1.5) we have

$$p_c(x)p_{c'}(x')p_c(x)^{-1} = \prod_{i \geqslant 0, j>0} p_{ic+jc'}(\mathrm{C}_{i,j,c,c'}x^i x'^j) \tag{6.3.3}$$

where the product (in U_+) is taken relative to the ordering on Φ^+ and the coefficients $\mathrm{C}_{i,j,c,c'}$ lie in $\mathbf{Q}$. This product lies in $\mathrm{U}_{\geqslant c'}$, and it suffices to prove that $\mathrm{C}_{i,j,c,c'} \in \mathbf{Z}$ for all (i, j). The first term on the right in (6.3.3) is $p_{c'}(\mathrm{C}_{0,1,c,c'}x')$ because $c' < ic + jc'$ in Φ^+ for all $i, j \geqslant 1$, and (5.1.5) implies that $\mathrm{C}_{0,1,c,c'} = 1$. Hence, our problem really concerns the commutator $p_{c'}(-x')p_c(x)p_{c'}(-x')^{-1}p_c(x)^{-1}$. We may now replace G with $\mathrm{Z}_{\mathrm{G}}(\mathrm{T}_{c,c'})$ to reduce to the case of semisimple-rank 2 (keeping in mind that the ordering on Φ^+ is immaterial once the full result is proved).

Now consider the case of pinned split groups of semisimple-rank 2. The Chevalley system extending a pinning is unique up to signs, so we can use whatever Chevalley system we like that extends an initial choice of pinning. By Lemma 6.2.8, when using the lexicographical ordering on Φ^+ relative to *some* choice of enumeration of Δ, the structure constants in the commutation relations for the positive root groups are in $\mathbf{Z}$. This establishes the result in the semisimple-rank 2 case for some choice of enumeration on Δ, and it is sufficient to prove the result for one such choice. □

The evident $\mathbf{Z}$-scheme $\Omega_{\mathbf{Z}}$ extending the $\mathbf{Q}$-scheme on the left side of (6.3.2) has generic fiber Ω, and it is a direct product scheme $\mathscr{U}_- \times \mathscr{T} \times \mathscr{U}_+$ with

$\mathscr{T} = \mathbf{G}_\mathrm{m}^\Delta$ a split $\mathbf{Z}$-torus having cocharacter group $\mathbf{Z}\Phi^\vee = \mathrm{M}^\vee$ and $\mathscr{U}_\pm$ a $\mathbf{Z}$-group extending $\mathrm{U}_\pm$ (by Lemma 6.3.7). For $c \in \Phi^+$ let $\widetilde{p}_c : \mathbf{G}_\mathrm{a} \to \mathscr{U}_+$ be the evident inclusion extending p_c over $\mathbf{Q}$, and likewise using $\mathscr{U}_-$ for $c \in -\Phi^+$, so each $\widetilde{p}_c$ is a closed immersion of $\mathbf{Z}$-groups (as we can check the homomorphism property over $\mathbf{Q}$); let $\mathscr{U}_c$ denote its image. By construction the $\mathbf{Z}$-group $\mathscr{U}_+$ is directly spanned in *some* order by the $\mathscr{U}_c$'s for $c \in \Phi^+$, and similarly for $\mathscr{U}_-$ using $c \in -\Phi^+$. Likewise, the T-action on Ω extends to a $\mathscr{T}$-action on $\Omega_\mathbf{Z}$ that normalizes $\mathscr{U}_\pm$, as this amounts to some factorization assertions for flat closed subschemes that can be checked over $\mathbf{Q}$; more explicitly, $a^\vee(t)\widetilde{p}_c(x)a^\vee(t)^{-1} = \widetilde{p}_c(t^{\langle c, a^\vee\rangle}x)$ over $\mathbf{Z}$ since this holds over $\mathbf{Q}$.

By Lemma 6.3.6, the $\mathbf{Z}$-group $\mathscr{U}_+$ is directly spanned in any order by the $\mathscr{U}_c$'s for $c \in \Phi^+$, and $\mathscr{U}_-$ is directly spanned in any order by the $\mathscr{U}_c$'s for $c \in -\Phi^+$. (This direct spanning in any order is [**SGA3**, XXV, 2.5], and the proof there uses a theory of Lazard from [**BIBLE**, 13.1] for torus actions on unipotent groups. We can bypass that theory because the proof of Lemma 6.3.6 rests on the dynamical results in [**CGP**, 3.3].)

We conclude that the smooth affine $\mathbf{Z}$-groups $\mathscr{U}_\pm$ extending $\mathrm{U}_\pm$ can be defined using arbitrary orderings on Φ^+ and $-\Phi^+$. Define the "identity section" $\widetilde{e} \in \Omega_\mathbf{Z}(\mathbf{Z})$ to correspond to the direct product of the identity sections of $\mathscr{U}_\pm$ and $\mathscr{T}$ (so this extends the identity section $e \in \Omega(\mathbf{Q}) \subset \mathrm{G}(\mathbf{Q})$).

The $\mathbf{Q}$-group structure on G defines a birational group law on Ω, and we seek to extend it to a "$\mathbf{Z}$-birational group law" on $\Omega_\mathbf{Z}$ (see Definition 6.3.10) in a manner that interacts well with the $\mathbf{Z}$-groups $\mathscr{T}$, $\mathscr{U}_+$, and $\mathscr{U}_-$. This proceeds in several steps. Guided by the Bruhat decomposition of $\mathrm{G}(\overline{\mathbf{Q}})$, the first step is to consider the effect on Ω by n_a-conjugation for all $a \in \Delta$. For any $n \in \mathrm{N}_\mathrm{G}(\mathrm{T})(\mathbf{Q})$, we have $n\mathrm{U}_c n^{-1} = \mathrm{U}_{w(c)}$ for $w \in \mathrm{W}(\Phi)$ represented by n. In the special case $n = n_a$ for $a \in \Delta$, we have $w(c) = s_a(c) \in \Phi^+$ for all $c \in \Phi^+ - \{a\}$, whereas $w(a) = -a$. Also, n_a normalizes T.

Lemma 6.3.8. — *For each $a \in \Delta$, there exists an open subscheme $\mathscr{V}_a \subset \Omega_\mathbf{Z}$ containing $\mathscr{T}$ and every $\mathscr{U}_c$ $(c \in \Phi)$ such that the automorphism $g \mapsto n_a g n_a^{-1}$ of* G *carries $(\mathscr{V}_a)_\mathbf{Q}$ into Ω. Moreover, $\mathscr{V}_a$ can be chosen so that the resulting map $(\mathscr{V}_a)_\mathbf{Q} \to \Omega$ extends to a $\mathbf{Z}$-morphism $f_a : \mathscr{V}_a \to \Omega_\mathbf{Z}$ restricting to an automorphism of the $\mathbf{Z}$-group $\mathscr{T}$ and carrying the $\mathbf{Z}$-group $\mathscr{U}_c$ isomorphically onto the $\mathbf{Z}$-group $\mathscr{U}_{s_a(c)}$ for all $c \in \Phi$. In particular, since $\mathscr{T} \subset \mathscr{V}_a$, $\widetilde{e}$ factors through $\mathscr{V}_a$ and $f_a(\widetilde{e}) = \widetilde{e}$.*

This result is [**SGA3**, XXV, 2.7].

Proof. — For $c \in \Phi$, the isomorphism $\mathrm{U}_c \simeq \mathrm{U}_{s_a(c)}$ defined by n_a-conjugation is given by

$$p_c(x) \mapsto p_{s_a(c)}(\mathrm{Ad}_\mathrm{G}(n_a)(x\mathrm{X}_c)) = p_{s_a(c)}(\pm x\mathrm{X}_{s_a(c)})$$

since $\{\mathrm{X}_c\}_{c\in\Phi}$ is a Chevalley system. Hence, this visibly extends to a $\mathbf{Z}$-group isomorphism $\mathscr{U}_c \simeq \mathscr{U}_{s_a(c)}$. Likewise, on the $\mathbf{Q}$-group $\mathrm{T} = \mathbf{G}_{\mathrm{m}}^{\Delta} = \mathrm{D}_{\mathbf{Q}}(\mathrm{M})$, the effect of n_a-conjugation is given by the $\mathbf{Z}$-group automorphism of $\mathscr{T} = \mathbf{G}_{\mathrm{m}}^{\Delta} = \mathrm{D}_{\mathbf{Z}}(\mathrm{M})$ induced by s_a.

By Lemma 6.3.6, the $\mathbf{Z}$-group $\mathscr{U}_+$ is directly spanned in any order by $\{\mathscr{U}_c\}_{c\in\Phi^+}$ and the $\mathbf{Z}$-group $\mathscr{U}_-$ is directly spanned in any order by $\{\mathscr{U}_c\}_{c\in-\Phi^+}$. Thus,

$$\Omega_{\mathbf{Z}} = \prod_{c\in\Phi^+-\{a\}} \mathscr{U}_{-c} \times \mathscr{U}_{-a} \times \mathscr{T} \times \mathscr{U}_a \times \prod_{c\in\Phi^+-\{a\}} \mathscr{U}_c \tag{6.3.4}$$

using some fixed choice of enumeration of $\Phi^+ - \{a\}$ in both products. Finally, since we arranged that $\mathrm{X}_{-c} = \mathrm{X}_c^{-1}$ for all $c \in \Phi$, n_a-conjugation swaps U_a and U_{-a} via negation on the standard coordinate of $\mathbf{G}_{\mathrm{a}}$ relative to the parameterizations p_a and p_{-a} (Corollary 5.1.9(3)). Thus, using the chosen enumeration of $\Phi^+ - \{a\}$ to define the order of multiplication, n_a-conjugation on Ω carries

$$\prod_{c\in\Phi^+-\{a\}} p_{-c}(x_{-c}) \cdot p_{-a}(x') \cdot t \cdot p_a(x) \cdot \prod_{c\in\Phi^+-\{a\}} p_c(x_c)$$

to

$$\prod_{c\in\Phi^+-\{a\}} p_{-s_a(c)}(\pm x_{-c}) \cdot p_a(-x') \cdot \mathrm{D}_{\mathbf{Z}}(s_a)(t) \cdot p_{-a}(-x) \cdot \prod_{c\in\Phi^+-\{a\}} p_{s_a(c)}(\pm x_c)$$

for some universal signs. The terms $p_a(-x')$ and $p_{-a}(-x)$ appear in the "wrong" places; we want to swap their positions (at the cost of changing the $\mathscr{T}$-component) so that we can make things work over $\mathbf{Z}$.

For $t' := \mathrm{D}_{\mathbf{Z}}(s_a)(t)$ we have $(-a)(t') = a(t)$, so $p_a(-x')t'p_{-a}(-x) = p_a(-x')p_{-a}(-xa(t))t'$. Since $\mathrm{X}_{-a} = \mathrm{X}_a^{-1}$, Theorem 4.2.6(1) gives that $p_a(-x')p_{-a}(-xa(t))$ lies in $\mathrm{U}_{-a} \times \mathrm{T} \times \mathrm{U}_a$ if and only if $1 + x'xa(t)$ is a unit. Under this unit hypothesis, (4.2.1) implies that $p_a(-x')p_{-a}(-xa(t))t'$ equals

$$p_{-a}\left(\frac{-xa(t)}{1+x'xa(t)}\right) a^{\vee}(1+x'xa(t))t'\, p_a\left(\frac{-x'a(t)}{1+x'xa(t)}\right).$$

Using the $\mathbf{Z}$-group law on $\mathscr{U}_{\pm}$, we conclude that for the open subscheme $\mathscr{V}_a \subset \Omega_{\mathbf{Z}}$ defined by the unit condition $1 + x_{-a}x_a a(t) \in \mathbf{G}_{\mathrm{m}}$ (using the coordinatization on $\Omega_{\mathbf{Z}}$ relative to the product decomposition (6.3.4)), n_a-conjugation on Ω carries $(\mathscr{V}_a)_{\mathbf{Q}}$ into Ω and the resulting map $(\mathscr{V}_a)_{\mathbf{Q}} \to \Omega$ extends to a $\mathbf{Z}$-morphism $f_a : \mathscr{V}_a \to \Omega_{\mathbf{Z}}$.

By definition it is clear that $\mathscr{V}_a$ contains $\mathscr{T}$ and every $\mathscr{U}_c$. The map f_a carries $\mathscr{T}$ into itself and $\mathscr{U}_c$ into $\mathscr{U}_{s_a(c)}$ for all $c \in \Phi$ because such factorization through flat closed subschemes can be checked over $\mathbf{Q}$ (where it is obvious). Likewise, the induced map $\mathscr{T} \to \mathscr{T}$ is a $\mathbf{Z}$-group involution and the induced map $\mathscr{U}_c \to \mathscr{U}_{s_a(c)}$ is a $\mathbf{Z}$-homomorphism with inverse given by the induced map $\mathscr{U}_{s_a(c)} \to \mathscr{U}_{s_a^2(c)} = \mathscr{U}_c$ up to scaling by $(-1)^{\langle c,a^\vee\rangle}$ since $n_a^2 = a^\vee(-1)$. $\square$

In addition to conjugation by the representatives n_a for the simple reflections $s_a \in \mathrm{W}(\Phi)$, we need to address conjugation by some representative n for the long Weyl element $w \in \mathrm{W}(\Phi)$ relative to Δ. Explicitly, $w = \prod_{a\in\Delta} s_a$ in $\mathrm{W}(\Phi)$ using multiplication taken in the order of *any* enumeration of Δ, but the product $\prod_{a\in\Delta} n_a \in \mathrm{N_G(T)}(\mathbf{Q})$ generally depends on the choice of enumeration of Δ (i.e., if we change the order of multiplication then the product in $\mathrm{N_G(T)}(\mathbf{Q})$ changes by a possibly nontrivial $\mathrm{T}(\mathbf{Q})$-multiplication). For our purposes it is only necessary to work with *some* enumeration of Δ, so we shall use the enumeration chosen earlier to define the lexicographical ordering on $\mathrm{M}_{\mathbf{Q}}$ (which defined our ordering on Φ^+ and $-\Phi^+$).

Let n denote the resulting product $\prod_{a\in\Delta} n_a \in \mathrm{N_G(T)}(\mathbf{Q})$, so n-conjugation on G restricts to an automorphism of T and swaps U_+ and U_-.

Lemma 6.3.9. — *For n as defined above, there are open subschemes $\mathscr{V}, \mathscr{V}' \subset \Omega_{\mathbf{Z}}$ containing $\mathscr{U}_\pm$ and $\mathscr{T}$ such that the automorphism $g \mapsto ngn^{-1}$ of G carries $\mathscr{V}_{\mathbf{Q}}$ and $\mathscr{V}'_{\mathbf{Q}}$ into Ω and the resulting maps $\mathscr{V}_{\mathbf{Q}}, \mathscr{V}'_{\mathbf{Q}} \rightrightarrows \Omega$ extend to $\mathbf{Z}$-morphisms*

$$f : \mathscr{V} \to \Omega_{\mathbf{Z}},\ \ f' : \mathscr{V}' \to \Omega_{\mathbf{Z}}$$

satisfying the following properties:

1. *$f|_{\mathscr{U}_\pm}$ factors through a $\mathbf{Z}$-group morphism onto $\mathscr{U}_\mp$, and similarly for f';*
2. *f and f' restrict to $\mathbf{Z}$-group endomorphisms of $\mathscr{T}$;*
3. *$f' \circ f|_{f^{-1}(\mathscr{V}')} : f^{-1}(\mathscr{V}') \to \Omega_{\mathbf{Z}}$ is the canonical open immersion.*

In particular, $f^{-1}(\mathscr{V}')$ is fiberwise dense in $\mathscr{V}$ and $\widetilde{e}$ factors through $\mathscr{V}$ and $\mathscr{V}'$, with $f(\widetilde{e}) = \widetilde{e}$ and $f'(\widetilde{e}) = \widetilde{e}$.

This result is [**SGA3**, XXV, 2.8] (except that we include some $\mathbf{Z}$-group compatibilities in the statement).

Proof. — By $\mathbf{Z}$-flatness considerations and the evident properties on the $\mathbf{Q}$-fiber, once we find $\mathscr{V}$ and $\mathscr{V}'$ containing $\mathscr{T}$ and $\mathscr{U}_\pm$ so that n-conjugation carries their $\mathbf{Q}$-fibers into Ω and the resulting maps $\mathscr{V}_{\mathbf{Q}} \to \Omega$ and $\mathscr{V}'_{\mathbf{Q}} \to \Omega$ extend to $\mathbf{Z}$-morphisms, the additional properties in (1), (2), and (3) are immediate. The actual construction of $\mathscr{V}$ and $\mathscr{V}'$ involves an inductive argument on word length in $\mathrm{W}(\Phi)$, with Lemma 6.3.8 used to carry out the induction. The identity that makes it work is $n_a^4 = 1$ in $\mathrm{N_G(T)}(\mathbf{Q})$ for all $a \in \Delta$ (since $n_a^2 = a^\vee(-1)$). We refer the reader to [**SGA3**, XXV, 2.8] for the details. $\square$

Now we bring in birational group laws. We refer the reader to [**BLR**, § 2.5, § 5.1–5.2] for an elegant general discussion of S-rational maps and S-birational group laws with smooth separated S-schemes, and here give just the basic definitions:

Definition 6.3.10. — Let S be a scheme, and $X, Y \rightrightarrows S$ be two smooth separated morphisms. For any fiberwise-dense open subschemes $\Omega, \Omega' \subset X$ declare two S-morphisms $f : \Omega \to Y$ and $f' : \Omega' \to Y$, to be equivalent if f and f' agree on an open subset that is S-dense in the sense of being fiberwise dense. An S-*rational map* from X to Y is an equivalence class of such maps, and an S-rational map is S-*birational* if some (equivalently, every) representative morphism $f : \Omega \to Y$ restricts to an isomorphism between S-dense open subschemes of X and Y.

An S-*birational group law* on a smooth separated S-scheme X is an S-rational map m from $X \times_S X$ to X such that (i) the S-rational maps $(x, x') \mapsto (x, m(x, x'))$ and $(x, x') \mapsto (m(x, x'), x')$ from $X \times_S X$ to $X \times_S X$ are S-birational (so m is S-dominant in the sense of carrying an S-dense open subset of $X \times_S X$ onto an S-dense open subset of X) and (ii) m is associative in the sense of S-dominant S-rational maps.

By artful use of f and f' from Lemma 6.3.9, one can put a **Z**-birational group law on $\Omega_{\mathbf{Z}}$:

Proposition 6.3.11. — *There are open subschemes $\mathscr{V}_1 \subset \Omega_{\mathbf{Z}} \times \Omega_{\mathbf{Z}}$ and $\mathscr{V}_2 \subset \Omega_{\mathbf{Z}}$ such that:*

1. *$\mathscr{U}_\pm, \mathscr{T} \subset \mathscr{V}_2$ and $\mathscr{U}_+ \times \mathscr{U}_+, \mathscr{U}_- \times \mathscr{U}_-, \mathscr{T} \times \mathscr{T}, \Omega_{\mathbf{Z}} \times \{\widetilde{e}\}, \{\widetilde{e}\} \times \Omega_{\mathbf{Z}} \subset \mathscr{V}_1$,*
2. *for the generic fibers $V_j := (\mathscr{V}_j)_{\mathbf{Q}}$, the multiplication $m : G \times G \to G$ carries V_1 into Ω and inversion $\iota : G \simeq G$ carries V_2 into Ω,*
3. *the induced maps $m : V_1 \to \Omega$ and $\iota : V_2 \to \Omega$ extend to **Z**-morphisms $m_{\mathbf{Z}} : \mathscr{V}_1 \to \Omega_{\mathbf{Z}}$ and $\iota_{\mathbf{Z}} : \mathscr{V}_2 \to \Omega_{\mathbf{Z}}$.*

*Moreover, the **Q**-birational group law (Ω, m) extends to a **Z**-birational group law $(\Omega_{\mathbf{Z}}, m_{\mathbf{Z}})$ with inverse $\iota_{\mathbf{Z}}$ and identity $\widetilde{e}$ that restricts to the **Z**-group laws on $\mathscr{U}_+$, $\mathscr{U}_-$, and $\mathscr{T}$.*

This is [**SGA3**, XXV, 2.9], except that the assertions concerning containments of $\mathscr{U}_\pm$ and $\mathscr{T}$ as well as compatibility with their **Z**-group structures are not mentioned there (but are immediate from inspecting the construction there and using known identities for maps between the **Q**-fibers).

Proof. — The motivation for the construction of $\mathscr{V}_1$ and $\mathscr{V}_2$ can be seen by considering the special case $G = SL_2$. In that special case, $\Omega_{\mathbf{Z}} = \mathbf{G}_a \times \mathbf{G}_m \times \mathbf{G}_a$ with the action of $\mathscr{T} = \mathbf{G}_m$ on $\mathscr{U}_\pm = \mathbf{G}_a$ given by $t.x = t^{\pm 2}x$. The Zariski-open condition $1 + x'xt^2 \in \mathbf{G}_m$ defines a suitable $\mathscr{V}_2 \subset \Omega_{\mathbf{Z}}$ for exactly the same reason that the condition $1+x'xa(t) \in \mathbf{G}_{\mathrm{m}}$ arose in the proof of Lemma 6.3.8. To find a Zariski-open condition on points

$$((x'_2, t_2, x_2), (x'_1, t_1, x_1)) \in \Omega_{\mathbf{Z}} \times \Omega_{\mathbf{Z}}$$

to define $\mathscr{V}_1$, we use Theorem 4.2.6(1) to see that the condition $1+x_2x'_1 \in \mathbf{G}_{\mathrm{m}}$ does the job.

The construction of $\mathscr{V}_1$ and $\mathscr{V}_2$ in general is given in [**SGA3**, XXV, 2.9], modeled on the case of SL_2. One uses Lemma 6.3.9 to overcome the absence in the general case of formulas as explicit as in the case of SL_2. The $\mathbf{Z}$-birational group law property amounts to an associativity identity that can be checked on the $\mathbf{Q}$-fiber, and likewise for the inversion and identity assertions for this birational group law. □

Now we are in position to apply results that promote birational group laws to group schemes. For a scheme S, if $\mathrm{X} \to \mathrm{S}$ is a smooth surjective separated map of finite presentation equipped with an S-birational group law m, a *solution* is a smooth separated S-group (X', m') of finite presentation equipped with an S-birational isomorphism between X and X′ that is compatible with m and m' (i.e., an S-isomorphism $f : \Omega \simeq \Omega'$ between fiberwise dense open subschemes $\Omega \subset \mathrm{X}$ and $\Omega' \subset \mathrm{X}'$ such that $m' \circ (f \times f) = f \circ m$ as S-rational maps from $\mathrm{X} \times_{\mathrm{S}} \mathrm{X}$ to X′). A preliminary result in the theory of S-birational group laws is that a solution is unique up to unique S-isomorphism (not just S-birationally) if it exists [**BLR**, 5.1/3]. This is proved by translation arguments, using that a smooth surjective map has many étale-local sections. We emphasize that it is *not* required that X is open in X′; i.e., we allow for the possibility that only some fiberwise dense open subscheme of X appears as an open subscheme of X′ (compatibly with the S-birational group laws).

For applications, it is useful to have a criterion to ensure that a given S-birational group law (X, m) occurs as an open subscheme of an S-group, with no shrinking of X required. To motivate the criterion, consider S-birational group laws that arise from fiberwise dense open subschemes of S-groups. Here are some properties that such S-birational group laws must satisfy:

Example 6.3.12. — Let $\mathrm{G} \to \mathrm{S}$ be a smooth separated S-group of finite presentation, and $\mathrm{X} \subset \mathrm{G}$ a fiberwise dense open subscheme. Then $\mathrm{U} := m_{\mathrm{G}}^{-1}(\mathrm{X}) \bigcap (\mathrm{X} \times_{\mathrm{S}} \mathrm{X})$ is the open domain of definition of the associated S-birational group law on X, and it is X-*dense* in $\mathrm{X} \times_{\mathrm{S}} \mathrm{X}$ in the sense that U is fiberwise dense relative to both projections $\mathrm{X} \times_{\mathrm{S}} \mathrm{X} \rightrightarrows \mathrm{X}$. Indeed, for any geometric point $\overline{s}$ of S and $\overline{x} \in \mathrm{X}(\overline{s})$, the $\overline{x}$-fibers of $\mathrm{U}_{\overline{s}}$ under the projections $\mathrm{X}_{\overline{s}} \times \mathrm{X}_{\overline{s}} \rightrightarrows \mathrm{X}_{\overline{s}}$ are the open overlaps $\mathrm{X}_{\overline{s}} \bigcap (\overline{x}^{-1} \cdot \mathrm{X}_{\overline{s}})$ and $\mathrm{X}_{\overline{s}} \bigcap (\mathrm{X}_{\overline{s}} \cdot \overline{x}^{-1})$ in $\mathrm{G}_{\overline{s}}$ that are dense in $\mathrm{X}_{\overline{s}}$.

Moreover, the universal left and right "translation" maps $\mathrm{U} \rightrightarrows \mathrm{X}\times_{\mathrm{S}}\mathrm{X}$ defined by $u = (x_1, x_2) \mapsto (x_1, m(x_1, x_2)), (m(x_1, x_2), x_2)$ are open immersions with X-dense image because they are obtained by restriction to $\mathrm{U} \subset \mathrm{G} \times_{\mathrm{S}} \mathrm{G}$ of the universal translation maps $(g, g') \mapsto (gg', g'), (g, gg')$ that are automorphisms of the S-scheme $\mathrm{G} \times_{\mathrm{S}} \mathrm{G}$.

Motivated by Example 6.3.12, an S-birational group law (X, m) is called *strict* if there exists an open subscheme Ω of the domain of definition of m in $\mathrm{X} \times_{\mathrm{S}} \mathrm{X}$ such that Ω is X-dense and the maps $\Omega \rightrightarrows \mathrm{X} \times_{\mathrm{S}} \mathrm{X}$ defined by $(x, x') \mapsto (x, m(x, x'))$ and $(x, x') \mapsto (m(x, x'), x')$ are open immersions whose respective images in $\mathrm{X} \times_{\mathrm{S}} \mathrm{X}$ are X-dense. (This is an equivalent formulation of the definition of a "group germ" in [**SGA3**, XVIII, 3.1].) We have just seen in Example 6.3.12 that strictness is a necessary condition for a solution (X', m') to an S-birational group law (X, m) not to require any shrinking of X; i.e., it is necessary in order that X be open in a solution X' (as S-birational groups).

Remarkably, strictness is also sufficient for the existence of a solution (X', m') containing the (X, m) as a fiberwise dense open subscheme. This deep result is essentially [**SGA3**, XVIII, 3.7, 3.13(iii)], except that X' is built there only as an fppf sheaf of groups, resting on the special case of strictly henselian local S for which X' is built as a scheme. In fact, using [**Ar74**, Cor. 6.3], this construction can be reinterpreted to obtain that X' is an fppf (and hence smooth) algebraic space group. The construction of this solution X' as a scheme is given in [**BLR**, 5.2/3] for the cases that S is the spectrum of a separably closed field or strictly henselian discrete valuation ring.

By [**BLR**, 6.6/1] the algebraic space X' is always a scheme, but we are only interested in the special case that S is Dedekind (namely, $\mathrm{S} = \operatorname{Spec} \mathbf{Z}$). One can establish the result over such S without any (implicit) use of algebraic spaces as follows. First consider the "local" version: $\mathrm{S} = \operatorname{Spec} \mathrm{R}$ for R a field or discrete valuation ring. By using the known solution over a strict henselization of any such R, a solution can be built over R by descent arguments; see [**BLR**, 6.5/2]. In particular, for a general Dedekind scheme S we get a solution over the generic points (though in the cases of interest over $\mathbf{Z}$ we are even given a solution G over $\mathbf{Q}$). This "spreads out" to a solution over a dense open subscheme V of S. By limit considerations, the general Dedekind case reduces to local versions of the problem at the finitely many closed points of $\mathrm{S} - \mathrm{V}$, which are instances of the settled local case that R is a discrete valuation ring.

To summarize, the $\mathbf{Z}$-birational group $\Omega_{\mathbf{Z}}$ is fiberwise dense in a smooth finite type and separated $\mathbf{Z}$-group $\mathrm{G}_{\mathbf{Z}}$ once we verify:

Proposition 6.3.13. — *The $\mathbf{Z}$-birational group law on $\Omega_{\mathbf{Z}}$ is strict.*

Proof. — Consider $\mathscr{V}_1$ as in Proposition 6.3.11. The universal left and right "translation" morphisms $\mathscr{V}_1 \rightrightarrows \Omega_{\mathbf{Z}} \times \Omega_{\mathbf{Z}}$ defined by

$$(\omega, \omega') \mapsto (\omega, m_{\mathbf{Z}}(\omega, \omega')), \ \ (m_{\mathbf{Z}}(\omega, \omega'), \omega')$$

are maps between smooth separated $\mathbf{Z}$-schemes of finite type, and on the $\mathbf{Q}$-fibers they are open immersions since V_1 is a non-empty open in the smooth connected $\mathbf{Q}$-group G. Thus, these maps are birational. We claim that these $\mathbf{Z}$-maps are étale on an open subscheme $\mathscr{W}$ of $\mathscr{V}_1$ that contains $\{\widetilde{e}\} \times \Omega_{\mathbf{Z}}$ and $\Omega_{\mathbf{Z}} \times \{\widetilde{e}\}$. The case of the universal left "translation" will be treated, and right "translation" goes similarly.

In view of $\mathbf{Z}$-smoothness for the source and target, it suffices to check that the tangent maps at $(x, \widetilde{e}(s))$ and $(\widetilde{e}(s), x)$ are isomorphisms for every geometric point s of $\operatorname{Spec} \mathbf{Z}$ and closed point $x \in \Omega_s := (\Omega_{\mathbf{Z}})_s$. Restriction to the respective open neighborhoods

$$\mathscr{V}'_{1,s} = (\mathscr{V}_1)_s \bigcap (\Omega_s \times \{\widetilde{e}(s)\}) \subset \Omega_s, \ \ \mathscr{V}''_{1,s} = (\mathscr{V}_1)_s \bigcap (\{x\} \times \Omega_s) \subset \Omega_s$$

of x and $\widetilde{e}(s)$ in Ω_s gives maps $\mathscr{V}'_{1,s}, \mathscr{V}''_{1,s} \rightrightarrows \Omega_s \times \Omega_s$ that are respectively the diagonal map and the "left translation" by x into the slice $\{x\} \times \Omega_s$ (on an open domain containing $\widetilde{e}(s)$). Since $\widetilde{e}$ is an identity for the $\mathbf{Z}$-birational group law, this latter map at $(x, \widetilde{e}(s))$ is the canonical inclusion on the tangent space. Hence, by using the canonical decomposition

$$\operatorname{Tan}_{(x, \widetilde{e}(s))}((\Omega_{\mathbf{Z}} \times \Omega_{\mathbf{Z}})_s) = \operatorname{Tan}_x(\Omega_s) \bigoplus \operatorname{Tan}_{\widetilde{e}(s)}(\Omega_s)$$

we deduce the isomorphism property for the tangent map at $(x, \widetilde{e}(s))$ since $\mathrm{V} \bigoplus \mathrm{V}$ is the direct sum of the diagonal and $\{0\} \bigoplus \mathrm{V}$ for any vector space V. The same argument works at the points $(\widetilde{e}(s), x)$.

Pick an open subscheme $\mathscr{W} \subset \mathscr{V}_1$ containing $\Omega_{\mathbf{Z}} \times \{\widetilde{e}\}$ and $\{\widetilde{e}\} \times \Omega_{\mathbf{Z}}$ on which the universal left and right "translations" are étale maps to $\Omega_{\mathbf{Z}} \times \Omega_{\mathbf{Z}}$, so these define a pair of maps $\mathscr{W} \rightrightarrows \Omega_{\mathbf{Z}} \times \Omega_{\mathbf{Z}}$ that are birational, separated, and quasi-finite. By Zariski's Main Theorem [**EGA**, III_1,4.4.9], any birational, separated, and quasi-finite map between connected normal noetherian schemes is an open immersion. These open immersions $\mathscr{W} \rightrightarrows \Omega_{\mathbf{Z}} \times \Omega_{\mathbf{Z}}$ have $\Omega_{\mathbf{Z}}$-dense images since $\{\widetilde{e}\} \times \Omega_{\mathbf{Z}}, \Omega_{\mathbf{Z}} \times \{\widetilde{e}\} \subset \mathscr{W}$ and the fibers of $\Omega_{\mathbf{Z}} \to \operatorname{Spec} \mathbf{Z}$ are irreducible. □

Since $\Omega_{\mathbf{Z}}$ is now a fiberwise dense open subscheme of a unique smooth and separated $\mathbf{Z}$-group $\mathrm{G}_{\mathbf{Z}}$ of finite type (compatibly with $\mathbf{Z}$-birational group laws), $\mathrm{G}_{\mathbf{Z}} \to \operatorname{Spec} \mathbf{Z}$ has connected fibers. By the uniqueness of solutions to birational group laws, the inclusion $\Omega \hookrightarrow (\mathrm{G}_{\mathbf{Z}})_{\mathbf{Q}}$ extends to an isomorphism $\mathrm{G} \simeq (\mathrm{G}_{\mathbf{Z}})_{\mathbf{Q}}$, so we can view $\mathrm{G}_{\mathbf{Z}}$ as a $\mathbf{Z}$-model for G. The section $\widetilde{e} \in \Omega_{\mathbf{Z}}(\mathbf{Z}) \subset \mathrm{G}_{\mathbf{Z}}(\mathbf{Z})$ is the identity section, as this holds over $\mathbf{Q}$ and equalities of maps between separated flat $\mathbf{Z}$-schemes can be checked over $\mathbf{Q}$. Likewise, the immersions

$\mathscr{T} \to \mathrm{G}_{\mathbf{Z}}$ and $\mathscr{U}_{\pm} \to \mathrm{G}_{\mathbf{Z}}$ are $\mathbf{Z}$-homomorphisms, since we can check over $\mathbf{Q}$. We wish to avoid using the deep Theorem 5.3.5, so we do not yet know if these immersions are closed immersions, nor if $\mathrm{G}_{\mathbf{Z}}$ is affine.

To prove that the smooth separated finite type group $\mathrm{G}_{\mathbf{Z}}$ is semisimple, we will first show that its geometric fibers over $\mathrm{Spec}\,\mathbf{Z}$ (which we know are connected) are semisimple, and then deduce from this that $\mathrm{G}_{\mathbf{Z}}$ is affine. The key to fibral semisimplicity is:

Lemma 6.3.14. — *For every geometric point s of* $\mathrm{Spec}\,\mathbf{Z}$ *and every $c \in \Phi$, the fibral subgroups $(\mathscr{U}_c)_s$ and $(\mathscr{U}_{-c})_s$ in* $(\mathrm{G}_{\mathbf{Z}})_s$ *generate a subgroup that contains the nontrivial torus $c_s^\vee(\mathbf{G}_{\mathrm{m}})$. In particular, the subgroup of* $(\mathrm{G}_{\mathbf{Z}})_s$ *generated by the two unipotent subgroups $(\mathscr{U}_{\pm c})_s = \mathbf{G}_{\mathrm{a}}$ is not unipotent.*

Proof. — Identify $\mathscr{U}_{\pm c}$ with $\mathbf{G}_{\mathrm{a}}$ via $\widetilde{p}_{\pm c}$. We claim that if x, x' are points of $\mathbf{G}_{\mathrm{a}}$ such that $1 + x'x \in \mathbf{G}_{\mathrm{m}}$ and $(\widetilde{p}_c(x'), \widetilde{p}_{-c}(x)) \in \mathscr{V}_1$ then

$$\widetilde{p}_c(x')\widetilde{p}_{-c}(x) = \widetilde{p}_{-c}\left(\frac{x}{1+x'x}\right) c^\vee(1+x'x)\widetilde{p}_c\left(\frac{x'}{1+x'x}\right) \tag{6.3.5}$$

in the group law of $\mathrm{G}_{\mathbf{Z}}$. The two sides of (6.3.5) are scheme morphisms $(\mathscr{U}_c \times \mathscr{U}_{-c})_{1+x'x} \rightrightarrows \mathrm{G}_{\mathbf{Z}}$ using the group law on $\mathrm{G}_{\mathbf{Z}}$, so to prove their equality it suffices (by separatedness and flatness of $\mathrm{G}_{\mathbf{Z}}$ over $\mathbf{Z}$) to check over $\mathbf{Q}$. Via the identification $(\mathrm{G}_{\mathbf{Z}})_{\mathbf{Q}} \simeq \mathrm{G}$, we conclude via (4.2.1).

The direct product scheme $\mathscr{U}_c \times \mathscr{U}_{-c}$ inside $\Omega_{\mathbf{Z}} \times \Omega_{\mathbf{Z}}$ meets the open neighborhood $\mathscr{V}_1$ of $(\widetilde{e}, \widetilde{e})$, and imposing the additional condition "$1 + xx' \in \mathbf{G}_{\mathrm{m}}$" defines a fiberwise dense open subscheme on which (6.3.5) holds. Hence, by separatedness and smoothness considerations, it follows that (6.3.5) holds on the open subscheme of $\mathscr{U}_c \times \mathscr{U}_{-c} = \mathbf{A}^2_{\mathbf{Z}}$ where $1 + x'x \in \mathbf{G}_{\mathrm{m}}$. Now pass to s-fibers. □

The proof that $\mathrm{G}_{\mathbf{Z}}$ is affine will use some structural input on the fibers, so we first address the fibral structure:

Proposition 6.3.15. — *For geometric points $s \in$* $\mathrm{Spec}\,\mathbf{Z}$, $(\mathrm{G}_{\mathbf{Z}})_s$ *is semisimple and $\mathscr{T}_s$ is a maximal torus. In particular, the fibers of* $\mathrm{G}_{\mathbf{Z}}$ *are affine.*

Proof. — By construction, $\mathrm{G}_{\mathbf{Z}}$ contains the torus $\mathscr{T} = \mathrm{D}_{\mathbf{Z}}(\mathrm{M})$ as an S-subgroup (which we have not yet shown to be closed, as $\mathrm{G}_{\mathbf{Z}}$ is not yet shown to be affine, so we cannot apply Lemma B.1.3 and we wish to avoid using the deep Theorem 5.3.5). Consider the action by this torus on

$$\mathfrak{g} := \mathrm{Lie}(\mathrm{G}_{\mathbf{Z}}) = \mathrm{Tan}_{\widetilde{e}}(\Omega_{\mathbf{Z}}) = \mathrm{Lie}(\mathscr{T}) \oplus \bigoplus_{c\in\Phi} \mathrm{Lie}(\mathscr{U}_c).$$

These direct summands are stable under the adjoint action of $\mathscr{T}$, with $\mathrm{Lie}(\mathscr{T})$ centralized by the action and the line subbundle $\mathrm{Lie}(\mathscr{U}_c)$ acted upon through through the character $c \in \Phi \subset \mathrm{M} - \{0\}$. It follows that under the $\mathscr{T}$-action on

$\mathfrak{g}$, the trivial weight space is the subbundle $\mathrm{Lie}(\mathscr{T})$ and the set of nontrivial weights consists of the "constant sections" $c \in \Phi \subset \mathrm{M} - \{0\}$, with the c-weight space equal to $\mathrm{Lie}(\mathscr{U}_c)$ for $c \in \Phi$.

Now pass to fibers at a geometric point s of $\operatorname{Spec} \mathbf{Z}$. The s-group $\mathrm{G}_s := (\mathrm{G}_{\mathbf{Z}})_s$ contains the s-subgroups $\mathrm{U}_{c,s} := (\mathscr{U}_c)_s \simeq \mathbf{G}_{\mathrm{a}}$ and $\mathrm{T}_s := \mathscr{T}_s = \mathrm{D}_s(\mathrm{M})$ such that T_s normalizes each $\mathrm{U}_{c,s} = \mathbf{G}_{\mathrm{a}}$ via the action $t.x = c(t)x$, so in any commutative quotient C_s of G_s modulo a closed normal subgroup scheme, the image of $\mathrm{U}_{c,s}$ is trivial. Then (6.3.5) implies that for each $c \in \Phi$ the image of $c^\vee(\mathbf{G}_{\mathrm{m}})_s$ in this quotient is also trivial. But the subgroups $\mathrm{U}_{c,s}$ and $c^\vee(\mathbf{G}_{\mathrm{m}})$ for $c \in \Phi$ generate G_s (as $\Omega_{\mathbf{Z}}$ is a fiberwise dense open subscheme of $\mathrm{G}_{\mathbf{Z}}$ whose direct product decomposition as a $\mathbf{Z}$-scheme is given by *multiplication* in the $\mathbf{Z}$-group law on $\mathrm{G}_{\mathbf{Z}}$, as we can check over $\mathbf{Q}$), so $\mathrm{C}_s = 1$.

We have proved that G_s has no nontrivial commutative quotient modulo a normal closed subgroup scheme. It is a theorem of Chevalley [**Chev60**] (and Barsotti [**Bar**] and Rosenlicht [**Ro**, Thm. 16]) that every smooth connected group H over an algebraically closed field is an extension of an abelian variety by a connected linear algebraic group. (See [**Con02**] for a modern exposition of Chevalley's proof, and [**BSU**, § 2] and [**Mi13**] for modern expositions of Rosenlicht's proof.) In particular, if such a group H has no nontrivial commutative quotient modulo a normal closed subgroup scheme then H is a linear algebraic group! Hence, G_s is a connected linear algebraic group that is equal to its own derived subgroup. In particular, if it is reductive then it must be semisimple.

It remains to show that G_s is reductive and T_s is a maximal torus in G_s. The inclusion $\mathrm{T}_s \hookrightarrow \mathrm{Z}_{\mathrm{G}_s}(\mathrm{T}_s)$ between *connected* smooth linear algebraic groups is an equality on Lie algebras, so it is an isomorphism. Hence, T_s is a maximal torus in G_s. To establish the reductivity, we first require a dynamic characterization of $\mathrm{U}_{c,s}$ in terms of the T_s-action on G_s for each $c \in \Phi$.

Fix $c \in \Phi$. For the codimension-1 subtorus $\mathrm{T}'_s = ((\ker c)^0_s)_{\mathrm{red}}$ in T_s, the smooth connected centralizer $\mathrm{Z}_{\mathrm{G}_s}(\mathrm{T}'_s)$ has Lie algebra that is the T'_s-centralizer in $\mathfrak{g}_s$, which is the span of the weight spaces for the T_s-weights that are trivial on T'_s. Hence, $\mathrm{Lie}(\mathrm{Z}_{\mathrm{G}_s}(\mathrm{T}'_s))$ is the span of $\mathrm{Lie}(\mathrm{T}_s)$ and $\mathrm{Lie}(\mathrm{U}_{\pm c,s})$ since Φ is a reduced root system in $\mathrm{M} = \mathrm{X}(\mathrm{T}_s)$. Consider the smooth connected unipotent subgroup $\mathrm{U}_{\mathrm{Z}_{\mathrm{G}_s}(\mathrm{T}'_s)}(c^\vee)$. This clearly has Lie algebra $\mathrm{Lie}(\mathrm{U}_{c,s})$, so the inclusion $\mathrm{U}_{c,s} \subset \mathrm{U}_{\mathrm{Z}_{\mathrm{G}}(\mathrm{T}'_s)}(c^\vee)$ between smooth connected linear algebraic groups is an equality on Lie algebras. In other words, we have established the "dynamic" characterization

$$\mathrm{U}_{c,s} = \mathrm{U}_{\mathrm{Z}_{\mathrm{G}_s}(\mathrm{T}'_s)}(c^\vee) \tag{6.3.6}$$

in terms of the T_s-action on G_s.

Via the inclusion $\exp_c : \mathbf{G}_{\mathrm{a}} = \mathscr{U}_c \hookrightarrow \mathrm{G}_{\mathbf{Z}}$, define $n_c = \exp_c(1) \in \mathrm{G}_{\mathbf{Z}}(\mathbf{Z})$. By Corollary 5.1.9(3) applied to $((\mathrm{G}_{\mathbf{Z}})_{\mathbf{Q}}, \mathrm{T}, \mathrm{M})$, n_c-conjugation carries $(\mathscr{U}_c)_{\mathbf{Q}}$ into

$(\mathscr{U}_{-c})_{\mathbf{Q}}$ and hence carries $\mathscr{U}_c$ into $\mathscr{U}_{-c}$. The resulting immersion $n_c\mathscr{U}_c n_c^{-1} \subset \mathscr{U}_{-c}$ is an equality for fibral reasons, so $n_c \mathrm{U}_{c,s} n_c^{-1} = \mathrm{U}_{-c,s}$ for all c.

Now suppose that $\mathrm{U} := \mathscr{R}_u(\mathrm{G}_s)$ is nontrivial. By normality in G_s, the nonzero subspace $\mathrm{Lie}(\mathrm{U})$ in $\mathfrak{g}_s$ has a weight space decomposition relative to the T_s-action. Since $\mathrm{U} \bigcap \mathrm{T}_s = 1$ (due to the unipotence of U), passing to Lie algebras gives $\mathrm{Lie}(\mathrm{U}) \bigcap \mathrm{Lie}(\mathrm{T}_s) = 0$. Hence, all T_s-weights that occur on $\mathrm{Lie}(\mathrm{U})$ are nontrivial, so for some $c_0 \in \Phi$ the 1-dimensional c_0-weight space $\mathrm{Lie}(\mathrm{U}_{c_0,s})$ is contained in $\mathrm{Lie}(\mathrm{U})$.

The intersection $\mathrm{U} \bigcap \mathrm{Z}_{\mathrm{G}_s}(\mathrm{T}'_s)$ is smooth and connected since

$$\mathrm{T}'_s \times (\mathrm{U} \bigcap \mathrm{Z}_{\mathrm{G}_s}(\mathrm{T}'_s)) = \mathrm{Z}_{\mathrm{T}'_s \ltimes \mathrm{U}}(\mathrm{T}'_s),$$

so by applying Proposition 4.1.10(1) to $\mathrm{H} = \mathrm{U} \bigcap \mathrm{Z}_{\mathrm{G}_s}(\mathrm{T}'_s)$ equipped with its action by $c_0^\vee(\mathbf{G}_{\mathrm{m}})$ we conclude via the dynamic description (6.3.6) of $\mathrm{U}_{c_0,s}$ that $\mathrm{U} \bigcap \mathrm{U}_{c_0,s}$ is smooth and connected. But $\mathrm{Lie}(\mathrm{U} \bigcap \mathrm{U}_{c_0,s}) = \mathrm{Lie}(\mathrm{U}) \bigcap \mathrm{Lie}(\mathrm{U}_{c_0,s}) = \mathrm{Lie}(\mathrm{U}_{c_0,s})$, so the inclusion $\mathrm{U} \bigcap \mathrm{U}_{c_0,s} \hookrightarrow \mathrm{U}_{c_0,s}$ between smooth connected groups is an equality on Lie algebras. This forces $\mathrm{U} \bigcap \mathrm{U}_{c_0,s} = \mathrm{U}_{c_0,s}$, so $\mathrm{U}_{c_0,s} \subset \mathrm{U}$ inside G_s. Thus, the subgroup $\mathrm{U}_{-c_0,s} = n_{c_0} \mathrm{U}_{c_0,s} n_{c_0}^{-1}$ is contained in U by normality of U in G_s. But the subgroups $\mathrm{U}_{c_0,s}$ and $\mathrm{U}_{-c_0,s}$ of U generate a non-unipotent subgroup of G_s (by Lemma 6.3.14), which is absurd since U is unipotent. Hence, $\mathrm{U} = 1$. ☐

Proposition 6.3.16. — *The* $\mathbf{Z}$*-group* $\mathrm{G}_{\mathbf{Z}}$ *is affine with the split torus* $\mathscr{T}$ *as a maximal torus.*

Proof. — Consider the adjoint action $\mathrm{Ad}_{\mathrm{G}_{\mathbf{Z}}} : \mathrm{G}_{\mathbf{Z}} \to \mathrm{GL}(\mathfrak{g})$. The fibral semisimplicity in Proposition 6.3.15 implies that $\ker \mathrm{Ad}_{\mathrm{G}_{\mathbf{Z}}}$ has finite geometric fibers (by the classical theory), so $\mathrm{Ad}_{\mathrm{G}_{\mathbf{Z}}}$ is quasi-finite. But it is a general fact that any quasi-finite homomorphism between separated flat groups of finite type over a Dedekind base is necessarily an affine morphism; see [**SGA3**, XXV, § 4] for the proof (which is based on a clever translation argument). Hence, $\mathrm{Ad}_{\mathrm{G}_{\mathbf{Z}}}$ is an affine morphism, so $\mathrm{G}_{\mathbf{Z}}$ inherits affineness from $\mathrm{GL}(\mathfrak{g})$.

Since $\mathrm{G}_{\mathbf{Z}}$ is affine, $\mathscr{T}$ is a closed subgroup of $\mathrm{G}_{\mathbf{Z}}$ (as for any multiplicative type subgroup of a smooth affine group, by Lemma B.1.3). The maximality of this torus in geometric fibers was proved in Proposition 6.3.15. ☐

The $\mathbf{Q}$-fiber of $(\mathrm{G}_{\mathbf{Z}}, \mathscr{T}, \mathrm{M})$ is the triple $(\mathrm{G}, \mathrm{T}, \mathrm{M})$ whose root datum is the original R of interest, so Proposition 6.3.16 completes the proof of the Existence Theorem (since the root spaces for $(\mathrm{G}_{\mathbf{Z}}, \mathscr{T}, \mathrm{M})$ are free of rank 1 by construction, or because $\mathrm{Pic}(\mathbf{Z}) = 1$).

6.4. Applications of Existence and Isomorphism Theorems. — Chevalley groups were originally defined to be the output of a certain explicit construction over $\mathbf{Z}$ given in [**Chev61**] for any split connected semisimple

Q-group descending a given connected semisimple **C**-group. Turning the history around, we define a *Chevalley group* to be a reductive group scheme G over **Z** that admits a maximal torus T over **Z**. By Corollary B.3.6, all tori over **Z** are split (as Spec **Z** is normal and connected with no nontrivial connected finite étale cover). Thus, since any line bundle over Spec **Z** is trivial, all Chevalley groups are necessarily split. These are precisely the **Z**-groups constructed by Chevalley, at least in the semisimple case, due to:

Proposition 6.4.1. — *A Chevalley group is determined up to isomorphism by its associated reduced root datum, and every such root datum arises in this way. Two Chevalley groups are isomorphic over* **Z** *if and only if they are isomorphic over* **C**.

Proof. — The bijectivity assertion between sets of isomorphism classes is the combination of the Isomorphism and Existence Theorems over **Z**. By the Isomorphism Theorem, the equivalence between **Z**-isomorphism and **C**-isomorphism is immediate (as root data do not "know" the base scheme). □

Remark 6.4.2. — Let R be a semisimple root datum, Φ its underlying root system, and $\mathfrak{g}$ a split Lie algebra over **Q** with root system Φ. Chevalley initially proved the Existence Theorem over **Z** for adjoint R by making an explicit construction inside the automorphism algebra of a Lie algebra over **Z** generated by a Chevalley basis of $\mathfrak{g}$. From the viewpoint of [**SGA3**], this approach "works" due to Theorem 5.3.5 and Remark 6.3.5. In his 1961 Bourbaki report [**Chev61**], Chevalley removed the adjoint condition on R by working with a split semisimple **Q**-group $(\mathrm{G}, \mathrm{T}, \mathrm{M})$ having root datum R rather than with the Lie algebra $\mathfrak{g}$ over **Q** having root system Φ.

Chevalley's idea was to pick a faithful representation (V, ρ) of G over **Q** and use a Chevalley system $\{\mathrm{X}_a\}_{a\in\Phi}$ to construct a lattice Λ in V so that the schematic closure $\mathscr{G}$ of G in $\mathrm{GL}(\Lambda)$ has an "open cell" structure over **Z** extending one on G. The fibral connectedness and semisimplicity properties of $\mathscr{G}$ were analyzed via the open cell structure.

Example 6.4.3. — Here is a useful application of Chevalley groups. Let S be a connected non-empty scheme, and $(\mathrm{G}, \mathrm{T}, \mathrm{M})$ a split reductive group over S. Since $\mathrm{W}_{\mathrm{G}}(\mathrm{T}) = \mathrm{W}(\Phi)_{\mathrm{S}}$, the short exact sequence of S-groups

$$1 \to \mathrm{T} \to \mathrm{N}_{\mathrm{G}}(\mathrm{T}) \to \mathrm{W}_{\mathrm{G}}(\mathrm{T}) \to 1 \tag{6.4.1}$$

induces an exact sequence of groups

$$1 \to \mathrm{Hom}(\mathrm{M}, \mathbf{G}_{\mathrm{m}}(\mathrm{S})) \to \mathrm{N}_{\mathrm{G}}(\mathrm{T})(\mathrm{S}) \to \mathrm{W}(\Phi) \to 1,$$

where surjectivity holds on the right because $\mathrm{W}(\Phi)$ is generated by reflections s_a $(a \in \Phi)$ that are induced by elements $w_a(\mathrm{X}_a) \in \mathrm{N}_{\mathrm{G}}(\mathrm{T})(\mathrm{S})$ for any $\mathscr{O}_{\mathrm{S}}$-basis

X_a of $\mathfrak{g}_a$. We claim that (6.4.1) is the central pushout of an exact sequence

$$1 \to \mathrm{T}[2] \to \widetilde{\mathrm{W}}_{\mathrm{G}}(\mathrm{T}) \to \mathrm{W}_{\mathrm{G}}(\mathrm{T}) \to 1 \tag{6.4.2}$$

for a finite flat S-subgroup $\widetilde{\mathrm{W}}_{\mathrm{G}}(\mathrm{T}) \subset \mathrm{N}_{\mathrm{G}}(\mathrm{T})$ such that $\widetilde{\mathrm{W}}_{\mathrm{G}}(\mathrm{T})(\mathrm{S})$ is carried onto the finite group $\mathrm{W}(\Phi)$ of constant sections. In particular, if $\mathrm{S} = \operatorname{Spec} \mathrm{R}$ for a domain R (or more generally, if $\mu_2(\mathrm{S})$ is finite) then $\mathrm{N}_{\mathrm{G}}(\mathrm{T})(\mathrm{S})$ contains a *finite* subgroup mapping onto $\mathrm{W}(\Phi)$.

The construction of (6.4.2) rests on a choice of pinning, or more specifically on a choice of **Z**-descent of $(\mathrm{G}, \mathrm{T}, \mathrm{M})$. In other words, it suffices to make the construction when $\mathrm{S} = \operatorname{Spec} \mathbf{Z}$ (i.e., for Chevalley groups), and in such cases we claim that the finite flat **Z**-subgroup $\widetilde{\mathrm{W}}_{\mathrm{G}}(\mathrm{T}) \subset \mathrm{N}_{\mathrm{G}}(\mathrm{T})$ is *canonical*; it does not depend on a pinning (and is called the *Tits group* for (G, T)). Since $\mathbf{G}_{\mathrm{m}}(\mathbf{Z}) = \mu_2(\mathbf{Z})$, on **Z**-points the diagram (6.4.1) yields a short exact sequence

$$1 \to \mathrm{T}[2](\mathbf{Z}) \to \mathrm{N}_{\mathrm{G}}(\mathrm{T})(\mathbf{Z}) \to \mathrm{W}(\Phi) \to 1,$$

so $\mathrm{N}_{\mathrm{G}}(\mathrm{T})(\mathbf{Z})$ is *finite*. Hence, we can define $\widetilde{\mathrm{W}}_{\mathrm{G}}(\mathrm{T})$ to be the finite flat **Z**-subgroup of $\mathrm{N}_{\mathrm{G}}(\mathrm{T})$ generated by the $\mathrm{N}_{\mathrm{G}}(\mathrm{T})(\mathbf{Z})$-translates of $\mathrm{T}[2]$. (Translates of $\mathrm{T}[2]$ by representatives for distinct elements of $\mathrm{W}(\Phi)$ are disjoint inside $\mathrm{N}_{\mathrm{G}}(\mathrm{T})$ due to (6.4.1) over **Z** since $\mathrm{W}_{\mathrm{G}}(\mathrm{T}) = \mathrm{W}(\Phi)_{\mathbf{Z}}$.) Obviously the inclusion $\widetilde{\mathrm{W}}_{\mathrm{G}}(\mathrm{T})(\mathbf{Z}) \subset \mathrm{N}_{\mathrm{G}}(\mathrm{T})(\mathbf{Z})$ is an equality.

Explicitly, $\widetilde{\mathrm{W}}_{\mathrm{G}}(\mathrm{T})(\mathbf{Z})$ contains the elements $n_a = w_a(\mathrm{X}_a)$ for any $a \in \Phi$ and trivializing section X_a of $\mathfrak{g}_a$ (well-defined up to a sign). This same description gives the pinning-dependent construction over a general non-empty base S (over which unit scaling on the pinning may go beyond sign changes, thereby making the Tits group depend on the pinning). See [**Ti66b**, § 4.6] and [**Ti66c**, § 2.8] for further discussion in the simply connected semisimple case (so that the Chevalley group is determined by the root system), where $\widetilde{\mathrm{W}}_{\mathrm{G}}(\mathrm{T})(\mathbf{Z})$ is called the *extended Weyl group*; in [**Bou3**, IX, § 4, Exer. 12(d)] there is an interpretation via compact Lie groups.

Now we turn our attention to a relative version (and refinement) of the decomposition of a connected semisimple group over a field k into an "almost direct product" of its k-simple factors. In Theorem 5.1.19 we canonically described every nontrivial connected semisimple group G over a field k as a central isogenous quotient of a product of k-simple semisimple subgroups G_i. For the simply connected central covers $\widetilde{\mathrm{G}}_i \to \mathrm{G}_i$, each $\widetilde{\mathrm{G}}_i$ is k-simple (since G_i is) and the composite map

$$\prod \widetilde{\mathrm{G}}_i \to \prod \mathrm{G}_i \to \mathrm{G}$$

is a central isogeny (due to Corollary 3.3.5). Thus, this map identifies $\prod \widetilde{\mathrm{G}}_i$ with the simply connected central cover of G.

The problem of classifying all possible G over k is thereby largely reduced to the case of k-simple G that are simply connected. We wish to explain why the absolutely simple case (over finite separable extensions of k) is the most important case. This rests on:

Proposition 6.4.4. — *Let* $\mathrm{G} \to \mathrm{S}$ *be a fiberwise nontrivial semisimple group with simply connected fibers over a non-empty scheme* S. *There is a finite étale cover* $\mathrm{S}' \to \mathrm{S}$ *and a semisimple group* $\mathrm{G}' \to \mathrm{S}'$ *with simply connected and absolutely simple fibers such that* G *is* S*-isomorphic to the Weil restriction* $\mathrm{R}_{\mathrm{S}'/\mathrm{S}}(\mathrm{G}')$.

The pair $(\mathrm{S}'/\mathrm{S}, \mathrm{G}')$ *is uniquely determined up to unique* S*-isomorphism in the following sense: if* $(\mathrm{S}''/\mathrm{S}, \mathrm{G}'')$ *is another such pair then every* S*-group isomorphism* $\mathrm{R}_{\mathrm{S}'/\mathrm{S}}(\mathrm{G}') \simeq \mathrm{R}_{\mathrm{S}''/\mathrm{S}}(\mathrm{G}'')$ *arises from a unique pair* (α, f) *consisting of an* S*-isomorphism* $\alpha : \mathrm{S}' \simeq \mathrm{S}''$ *and group isomorphism* $f : \mathrm{G}' \simeq \mathrm{G}''$ *over* α.

Proof. — In view of the uniqueness assertions, by étale descent we may work étale-locally on S. Thus, we can assume that G is split, say with a split maximal torus $\mathrm{T} = \mathrm{D}_{\mathrm{S}}(\mathrm{M})$ whose root spaces $\mathfrak{g}_a$ are free of rank 1. The semisimple root datum $\mathrm{R}(\mathrm{G}, \mathrm{T}, \mathrm{M})$ is *simply connected*, so it decomposes as a direct product $\prod_{i \in \mathrm{I}} \mathrm{R}_i$ of simply connected root data R_i whose underlying root systems are irreducible (and $\mathrm{I} \neq \varnothing$). By the Existence Theorem there exists a split semisimple S-group $(\mathrm{G}_i, \mathrm{T}_i, \mathrm{M}_i)$ whose root datum is R_i. The geometric fibers of $\mathrm{G}_i \to \mathrm{S}$ are simply connected and simple (Corollary 5.1.18). By the Isomorphism Theorem, $\mathrm{G} \simeq \prod \mathrm{G}_i$. For $\mathrm{S}' = \coprod_{i \in \mathrm{I}} \mathrm{S}$ and the S′-group $\mathrm{G}' = \coprod \mathrm{G}_i$ we have $\mathrm{R}_{\mathrm{S}'/\mathrm{S}}(\mathrm{G}') = \prod \mathrm{G}_i = \mathrm{G}$.

It remains to prove the asserted unique description of isomorphisms (so then the preceding construction in the split case does indeed settle the general case, via descent theory). Consider two pairs $(\mathrm{S}'/\mathrm{S}, \mathrm{G}')$ and $(\mathrm{S}''/\mathrm{S}, \mathrm{G}'')$ and an S-group isomorphism $\varphi : \mathrm{R}_{\mathrm{S}'/\mathrm{S}}(\mathrm{G}') \simeq \mathrm{R}_{\mathrm{S}''/\mathrm{S}}(\mathrm{G}'')$. We seek to show that φ arises from a unique pair (α, f). The uniqueness allows us to work étale-locally on S for existence, so we can assume that $\mathrm{S} = \operatorname{Spec} \mathrm{A}$ for a strictly henselian local ring A. Then S′ and S″ are each a non-empty finite disjoint union of copies of S, so the assertion can be reformulated as follows: if $\{\mathrm{G}'_i\}$ and $\{\mathrm{G}''_j\}$ are non-empty finite collections of semisimple A-groups with simply connected and absolutely simple fibers then any A-group isomorphism

$$\varphi : \prod \mathrm{G}'_i \simeq \prod \mathrm{G}''_j$$

arises from a unique pair $(\alpha, \{f_i\})$ consisting of a bijection $\alpha : \mathrm{I} \simeq \mathrm{J}$ and A-group isomorphisms $f_i : \mathrm{G}'_i \simeq \mathrm{G}''_{\alpha(i)}$.

The uniqueness of α is immediate from passage to the special fiber, and then the uniqueness of $\{f_i\}$ is clear. To prove the existence of (α, f), we shall use the crutch of maximal tori over the strictly henselian local ring A. Let

$\mathrm{T}'_i \subset \mathrm{G}'_i$ be a maximal torus, so $\mathrm{T}' := \prod \mathrm{T}'_i$ is a maximal torus in $\prod \mathrm{G}'_i$. Over the strictly henselian local ring A, all maximal tori in a reductive A-group H are H(A)-conjugate to each other (Theorem 3.2.6). Hence, the product construction of maximal tori in $\prod \mathrm{G}'_i$ gives all maximal tori. The same applies to $\prod \mathrm{G}''_j$, so $\varphi(\mathrm{T}') = \prod \mathrm{T}''_j$ for maximal tori $\mathrm{T}''_j \subset \mathrm{G}''_j$.

We claim that the isomorphism $\varphi : \prod \mathrm{T}'_i \simeq \prod \mathrm{T}''_j$ arises from a bijection $\alpha : \mathrm{I} \simeq \mathrm{J}$ and collection of isomorphisms $h_i : \mathrm{T}'_i \simeq \mathrm{T}''_{\alpha(i)}$. Since these tori are split over the local A, it is equivalent to verify the assertion on the special fiber. But over the residue field we can appeal to Theorem 5.1.19 to identify the absolutely simple special fibers $\{(\mathrm{G}'_i)_0\}$ and $\{(\mathrm{G}''_j)_0\}$ with the simple factors of the respective product groups $\prod(\mathrm{G}'_i)_0$ and $\prod(\mathrm{G}''_j)_0$. Hence, φ_0 must permute these factors according to some bijection $\alpha : \mathrm{I} \simeq \mathrm{J}$ and carry $(\mathrm{G}'_i)_0$ isomorphically onto $(\mathrm{G}''_{\alpha(i)})_0$. This latter isomorphism must carry $(\mathrm{T}'_i)_0$ isomorphically onto $(\mathrm{T}''_{\alpha(i)})_0$, and these torus isomorphisms (together with α) do the job.

Having built $\alpha : \mathrm{I} \simeq \mathrm{J}$ and $h_i : \mathrm{T}'_i \simeq \mathrm{T}''_{\alpha(i)}$ compatible with φ, we can relabel the indices so that φ is an isomorphism of A-groups $\prod \mathrm{G}'_i \simeq \prod \mathrm{G}''_i$ carrying the A-subgroup T'_i isomorphically to the A-subgroup T''_i for each i. It remains to show that the A-subgroup G'_{i_0} is carried isomorphically to the A-subgroup G''_{i_0} for each i_0. We will do this via an intrinsic characterization of G'_{i_0} in terms of $\{\mathrm{T}'_i\}$ and $\mathrm{G}' = \prod \mathrm{G}'_i$. The centralizer of $\prod_{i \neq i_0} \mathrm{T}'_i$ in G' is $\mathrm{G}'_{i_0} \times \prod_{i \neq i_0} \mathrm{T}'_i$, so the derived group of this centralizer is G'_{i_0}. A similar description applies to G''_{i_0} in terms of $\{\mathrm{T}''_i\}$ and $\mathrm{G}'' = \prod \mathrm{G}''_i$, so we are done. □

Remark 6.4.5. — The assertions in Proposition 6.4.4 remain true, with the same proof, when "simply connected" is replaced by "adjoint" everywhere. The key point is that an "adjoint" root datum is a direct product of irreducible ones, as in the simply connected case. The existence of $(\mathrm{S}'/\mathrm{S}, \mathrm{G}')$ can fail more generally (when the semisimple root data for the geometric fibers of G are neither simply connected nor adjoint), as is well-known over fields. For instance, if k'/k is a nontrivial finite separable extension then there is no such pair for $\mathrm{G} = \mathrm{R}_{k'/k}(\mathrm{SL}_n)/\mu_n$ for any $n > 1$.

Example 6.4.6. — Let k be a field and $\mathrm{G} \neq 1$ a connected semisimple k-group that is simply connected. Proposition 6.4.4 provides a canonical isomorphism $\mathrm{G} \simeq \mathrm{R}_{k'/k}(\mathrm{G}')$ for a unique pair $(k'/k, \mathrm{G}')$ consisting of a nonzero finite étale k-algebra k' and a semisimple k'-group G' such that all fibers of $\mathrm{G}' \to \operatorname{Spec} k'$ are connected, simply connected, and absolutely simple. Using the decomposition into factor fields $k' = \prod k'_i$ and letting G'_i denote the k'_i-fiber of G', we have $\mathrm{G} \simeq \prod_i \mathrm{R}_{k'_i/k}(\mathrm{G}'_i)$. By Example 5.1.20, these factors are k-simple. In particular, G is k-simple if and only if it has the form $\mathrm{R}_{k'/k}(\mathrm{G}')$

for a finite separable extension k'/k and a connected semisimple k'-group G' that is *absolutely simple* and simply connected.

The final part of Proposition 6.4.4 shows that the pair $(k'/k, \mathrm{G}')$ is canonically attached to G (not merely up to k-isomorphism) in the sense that it is uniquely functorial with respect to k-isomorphisms in such k-groups G. An extrinsic characterization of k'/k is given in Exercise 6.5.8.

Example 6.4.7. — Let G be a semisimple $\mathbf{Z}$-group with simply connected fibers. By Proposition 6.4.4, $\mathrm{G} \simeq \mathrm{R}_{\mathrm{A}/\mathbf{Z}}(\mathrm{G}')$ for a nonzero finite étale $\mathbf{Z}$-algebra A and a semisimple A-group G' whose fibers are simply connected and absolutely simple. By Minkowski's theorem, $\mathrm{A} = \prod \mathrm{A}_i$ with $\mathrm{A}_i = \mathbf{Z}$. Thus, if G_i denotes the restriction of G' over $\operatorname{Spec} \mathrm{A}_i = \operatorname{Spec} \mathbf{Z}$ then $\mathrm{R}_{\mathrm{A}/\mathbf{Z}}(\mathrm{G}') = \prod \mathrm{G}_i$. Hence, to classify all such G one loses no generality by restricting attention to the case when G has absolutely simple fibers (i.e., an irreducible root system for the geometric fibers).

By the same reasoning, over any connected normal noetherian scheme S whatsoever, to classify semisimple S-groups G with simply connected fibers one can pass to the case of G with absolutely simple fibers at the cost of replacing S with some connected finite étale covers (namely, the connected components of S' as in Proposition 6.4.4).

6.5. Exercises. —

Exercise 6.5.1. — Let $(\mathrm{G}, \mathrm{T}, \mathrm{M})$ be a split semisimple group over a non-empty scheme S.

(i) Choose $a \in \Phi$. Build a homomorphism $\phi : \mathrm{SL}_2 \to \mathrm{Z}_{\mathrm{G}}(\mathrm{T}_a)$ satisfying $\phi(\mathrm{diag}(t, 1/t)) = a^\vee(t)$. Show any such ϕ is an isogeny onto $\mathscr{D}(\mathrm{Z}_{\mathrm{G}}(\mathrm{T}_a))$ carrying the strictly upper-triangular subgroup U^+ isomorphically onto U_a. [It is insufficient to replace $\mathrm{Z}_{\mathrm{G}}(\mathrm{T}_a)$ with G. Indeed, consider a simply connected split semisimple $(\mathrm{G}, \mathrm{T}, \mathrm{M})$ having orthogonal roots a and b for which $a^\vee + b^\vee$ is a coroot, such as Sp_4 with a a long positive root and b the negative of the other long positive root. Writing $c^\vee = a^\vee + b^\vee$, $\mathrm{G}'_a := \mathscr{D}(\mathrm{Z}_{\mathrm{G}}(\mathrm{T}_a)) = \mathrm{SL}_2$ commutes with $\mathrm{G}'_b = \mathrm{SL}_2$, and the subgroup $\mathrm{G}'_a \times \mathrm{G}'_b \subset \mathrm{G}$ contains $c^\vee(\mathbf{G}_{\mathrm{m}})$ as the "diagonal" in $a^\vee(\mathbf{G}_{\mathrm{m}}) \times b^\vee(\mathbf{G}_{\mathrm{m}})$ but it meets $\mathrm{U}_{\pm c}$ trivially. The diagonal map $\phi : \mathrm{SL}_2 \hookrightarrow \mathrm{SL}_2 \times \mathrm{SL}_2 = \mathrm{G}'_a \times \mathrm{G}'_b \subset \mathrm{G}$ restricts to $\mathrm{diag}(t, 1/t) \mapsto c^\vee(t)$ but carries U^+ "diagonally" into $\mathrm{U}_a \times \mathrm{U}_b$ rather than into U_c.]

Prove $\phi \mapsto \mathrm{Lie}(\phi)(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix})$ is a bijection between the set of such ϕ and the set of global bases X_a of $\mathfrak{g}_a = \mathrm{Lie}(\mathrm{U}_a)$, and that either all such ϕ are isomorphisms or all have the central μ_2 as kernel. (Hint: First solve the problem over a field. Then use the self-contained computation with open cells indicated in Example 7.1.8 to show SL_2 and PGL_2 have no nontrivial S-automorphism that is the identity on the standard upper triangular Borel subgroup.)

(ii) Under the dictionary in (i), we can pass from a choice of X_a to a choice of ϕ, and then to an element $\mathrm{X}_{-a} := \mathrm{Lie}(\phi)(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}) \in \mathfrak{g}_{-a}(\mathrm{S})$. Prove that X_{-a} is a nowhere-vanishing section of $\mathfrak{g}_{-a}$, and is the trivialization of $\mathfrak{g}_{-a}$ dual to X_a via the canonical perfect pairing between $\mathfrak{g}_a$ and $\mathfrak{g}_{-a}$ in Theorem 4.2.6.

(iii) Use (i) to give a group-theoretic definition of pinnings that does not mention bases for root spaces; cf. [**CGP**, A.4.12].

Exercise 6.5.2. — Using the Existence and Isomorphism Theorems, the equivalence in Exercise 1.6.13(ii) among definitions for "simply connected" in the classical case is now proved. A semisimple group over a non-empty scheme S is *simply connected* when its geometric fibers are simply connected (recovering the definition in Example 5.1.7 in the split case).

(i) For any semisimple S-group G, prove the existence and uniqueness of a central isogeny $\pi : \widetilde{\mathrm{G}} \to \mathrm{G}$ from a semisimple S-group $\widetilde{\mathrm{G}}$ that is simply connected. By "uniqueness" we mean that for any two such pairs $(\widetilde{\mathrm{G}}, \pi)$ and $(\widetilde{\mathrm{G}}', \pi')$ there exists a *unique* isomorphism $\widetilde{\mathrm{G}} \simeq \widetilde{\mathrm{G}}'$ carrying π to π'. We call $(\widetilde{\mathrm{G}}, \pi)$ the *simply connected central cover* of G.

(ii) For a semisimple S-group G, prove that $\widetilde{\mathrm{G}}/\mathrm{Z}_{\widetilde{\mathrm{G}}} \to \mathrm{G}/\mathrm{Z}_{\mathrm{G}}$ is an isomorphism and $\mathrm{G} \to \mathrm{G}/\mathrm{Z}_{\mathrm{G}}$ is uniquely covered by an isomorphism between the simply connected central covers of G and $\mathrm{G}/\mathrm{Z}_{\mathrm{G}}$. Deduce that the isomorphism classes of $\widetilde{\mathrm{G}}$ and $\mathrm{G}/\mathrm{Z}_{\mathrm{G}}$ determine each other.

(iii) For a simply connected semisimple S-group G, prove any central extension of G by a group H of multiplicative type is uniquely split. (Hint: Reduce to split H. Any central pushout along an inclusion of H into a torus is reductive and so admits a "derived group" that is semisimple.) Deduce that any homomorphism between semisimple S-groups uniquely lifts to one between simply connected central covers (so $(\widetilde{\mathrm{G}}, \pi)$ is uniquely functorial in G).

(iv) Let T' be a torus in a simply connected semisimple S-group G. Prove that $\mathscr{D}(\mathrm{Z}_{\mathrm{G}}(\mathrm{T}'))$ is simply connected. (Hint: pass to $\mathrm{S} = \mathrm{Spec}\, k$ for $k = \overline{k}$ and find $\lambda \in \mathrm{X}_*(\mathrm{T}')$ so that $\mathrm{Z}_{\mathrm{G}}(\mathrm{T}') = \mathrm{Z}_{\mathrm{G}}(\lambda)$. Pick a maximal torus $\mathrm{T} \supset \mathrm{T}'$ and closed Weyl chamber in $\mathrm{X}_*(\mathrm{T})_{\mathbf{Q}}$ containing λ, with associated bases Δ and $\Delta^\vee$, so $\langle a, \lambda\rangle \geqslant 0$ for all $a \in \Delta$. Prove that $\mathrm{Z}_{\mathrm{G}}(\lambda)$ supports precisely the T-roots spanned over $\mathbf{Q}$ by $\Delta' = \{a' \in \Delta \,|\, \langle a', \lambda\rangle = 0\}$. Deduce that $\mathrm{X}_*(\mathscr{D}(\mathrm{Z}_{\mathrm{G}}(\mathrm{T}')) \bigcap \mathrm{T})$ is spanned over $\mathbf{Z}$ by ${\Delta'}^\vee$.) Using $\mathrm{T}' = (\ker a)^0_{\mathrm{red}}$ for $a \in \Phi(\mathrm{G}, \mathrm{T})$, show by example that the analogue with "adjoint" replacing "simply connected" is false.

Exercise 6.5.3. — Is Proposition 4.3.1 valid when SL_2 is replaced by any semisimple group scheme that is simply connected as in Exercise 6.5.2?

Exercise 6.5.4. — If G is a connected semisimple group over a field k and if $\mathscr{L} \to \mathrm{G}$ is a $\mathbf{G}_{\mathrm{m}}$-torsor with a chosen basis $e' \in \mathscr{L}(e)$, prove that $\mathscr{L}$ admits a unique structure of central extension of G by $\mathbf{G}_{\mathrm{m}}$ with identity e'. (The proof imitates the case of abelian varieties after passing to the split case and using the triviality of line bundles on the open cell, together with a classic result of Chevalley that any pointed map from a smooth connected k-group to $(\mathbf{G}_{\mathrm{m}}, 1)$ is a homomorphism.) Does this generalize to semisimple group schemes G over a non-empty scheme S when a trivialization of $\mathscr{L}$ along $e \in \mathrm{G}(\mathrm{S})$ is given?

Exercise 6.5.5. — Let G be a connected linear algebraic group over an algebraically closed field k of characteristic 0. Avoiding classification theorems, prove G is semisimple if and only if Lie(G) is semisimple. (Hint for "$\Rightarrow$": Let $\mathfrak{r}$ be a solvable Lie ideal. By the Levi–Malcev Theorem [**Bou1**, I, 6.8, Thm. 5], semisimplicity is equivalent to the vanishing of such $\mathfrak{r}$. Prove $\mathfrak{r}$ is Ad_{G}-stable, and consider its weights under the action of a maximal torus of G. The details are given at the end of the proof of Proposition 5.4.1.)

Exercise 6.5.6. — Recall Lang's Theorem [**Bo91**, 16.5(i)]: if a connected linear algebraic group G over a finite field k acts on a finite type k-scheme V such that $\mathrm{G}(\overline{k})$ is transitive on $\mathrm{V}(\overline{k})$ then $\mathrm{V}(k) \neq \varnothing$. Apply it to $\mathrm{Tor}_{\mathrm{G}'/k}$ and $\mathrm{Bor}_{\mathrm{G}'/k}$ for $\mathrm{G}' = \mathrm{G}/\mathscr{R}_u(\mathrm{G})$ to prove that G contains a (geometrically) maximal k-torus and a Borel k-subgroup.

Exercise 6.5.7. — Let $S' \to S$ be a finite étale cover of schemes, G' a reductive S'-group, and $G := R_{S'/S}(G')$. Prove that G is a reductive S-group, and adapt the method of proof of Proposition 6.4.4 to show that $T' \mapsto R_{S'/S}(T')$ is a bijection between the sets of maximal tori of G' and of G. Do likewise for Borel subgroups, and use this to construct natural isomorphisms $\mathrm{Tor}_{G/S} \simeq R_{S'/S}(\mathrm{Tor}_{G'/S'})$ and $\mathrm{Bor}_{G/S} \simeq R_{S'/S}(\mathrm{Bor}_{G'/S'})$.

Exercise 6.5.8. — Let k be a field and G a connected semisimple k-group that is k-simple and simply connected. By Example 6.4.6, $G \simeq R_{k'/k}(G')$ for a unique $(k'/k, G')$ where k'/k is finite separable and the simply connected semisimple k'-group G' is absolutely simple.

(i) For an extension L/k, prove G_L is L-split if and only if $k' \otimes_k L$ is a split L-algebra (i.e., power of L) and $G' \otimes_{k',j'} L$ is L-split for all k-embeddings $j' : k' \to L$. (Hint: Exercise 6.5.7.)

(ii) Prove that if k is a global field and v is a place of k such that G_{k_v} is k_v-split then v is totally split in k'/k. Deduce that a k-simple connected semisimple k-group that is split at all finite places must be absolutely simple.

(ii) Define a natural action by $\mathrm{Gal}(k_s/k)$ on the set of k_s-simple factors of G_{k_s}, and relate k' to the open subgroup that preserves one of these factors.

7. Automorphism scheme

7.1. Structure of automorphisms. — Many arithmetic properties of non-degenerate quadratic spaces (V, q) of rank $\geqslant 3$ over number fields are encoded in terms of the corresponding connected semisimple group $\mathrm{SO}(q)$ (or the disconnected orthogonal group $\mathrm{O}(q)$). All such groups become isomorphic over $\overline{\mathbf{Q}}$, so the problem of classifying quadratic forms can be related to the problem of classifying connected semisimple groups with a fixed geometric isomorphism type (equivalently, with a fixed root datum for its geometric fiber). The classification of connected semisimple $\mathbf{Q}$-groups sharing a common isomorphism type over $\overline{\mathbf{Q}}$ is controlled by the structure of automorphism groups of split connected semisimple groups over number fields.

To see the link between automorphism groups and classification problems, consider the situation over a general field k. Let G be a connected reductive k-group, so G_{k_s} admits a splitting and pinning. This gives rise to a reduced root datum R whose isomorphism class is intrinsic to G (and can be computed over any separably closed extension of k). Let G_0 be the split connected reductive k-group with the same root datum (so G_0 is unique up to k-isomorphism). Since G and G_0 become isomorphic over a finite Galois extension of k (as both are split over k_s with the same root datum, and hence are isomorphic over k_s), we view G as a "k_s/k-form" of G_0. To classify the possibilities for G, we use:

Lemma 7.1.1. — *Let* R *be a reduced root datum, and* G_0 *a connected reductive k-group with root datum* R *over k_s. The set of k-isomorphism classes of connected reductive k-groups* G *whose associated root datum over k_s is isomorphic to* R *is in natural bijection with the Galois cohomology set* $\mathrm{H}^1(k_s/k, \mathrm{Aut}((\mathrm{G}_0)_{k_s}))$.

Proof. — Choose G and a finite Galois extension K/k so that there is a K-group isomorphism $\varphi : \mathrm{G_K} \simeq (\mathrm{G}_0)_\mathrm{K}$. Typically this isomorphism is not defined over k (i.e., it does not descend to a k-group isomorphism $\mathrm{G} \simeq \mathrm{G}_0$), and to measure this possible failure we examine how φ interacts with the canonical K-isomorphisms $\gamma^*(\mathrm{G_K}) \simeq \mathrm{G_K}$ for $\gamma \in \mathrm{Gal}(\mathrm{K}/k)$ that encode the k-descent G of $\mathrm{G_K}$, as well as the analogous K-isomorphisms for G_0 (where $\gamma^*(\mathrm{X})$ denotes the base change of a K-scheme X through scalar extension by $\gamma : \mathrm{K} \simeq \mathrm{K}$). That is, for each γ we get a K-isomorphism

$$c_\gamma : (\mathrm{G}_0)_\mathrm{K} \simeq \gamma^*((\mathrm{G}_0)_\mathrm{K}) \overset{\gamma^*(\varphi)^{-1}}{\simeq} \gamma^*(\mathrm{G_K}) \simeq \mathrm{G_K} \overset{\varphi}{\simeq} (\mathrm{G}_0)_\mathrm{K},$$

and it is straightforward to check that the cocycle condition

$$c_{\gamma'\gamma} = c_{\gamma'} \circ (\gamma'.c_\gamma)$$

holds in $\mathrm{Aut}((\mathrm{G}_0)_\mathrm{K})$, where $c \mapsto \gamma'.c$ denotes the natural action of $\gamma' \in \mathrm{Gal}(\mathrm{K}/k)$ on $\mathrm{Aut}((\mathrm{G}_0)_\mathrm{K})$ through scalar extension along $\gamma' : \mathrm{K} \simeq \mathrm{K}$ (combined with the descent isomorphism $\gamma'^*((\mathrm{G}_0)_{\mathrm{K}}) \simeq (\mathrm{G}_0)_{\mathrm{K}}$ defined by the k-structure G_0 on the K-group $(\mathrm{G}_0)_{\mathrm{K}}$.

The choice of (K, φ) defines the function $c : \mathrm{Gal}(\mathrm{K}/k) \to \mathrm{Aut}((\mathrm{G}_0)_{\mathrm{K}})$ via $\gamma \mapsto c_\gamma$, and in terms of the non-abelian cohomology conventions in [**Ser97**, I, § 5.1] this lies in the set $\mathrm{Z}^1(\mathrm{K}/k, \mathrm{Aut}((\mathrm{G}_0)_{\mathrm{K}}))$ of 1-cocycles on $\mathrm{Gal}(\mathrm{K}/k)$ with coefficients in the $\mathrm{Gal}(\mathrm{K}/k)$-group $\mathrm{Aut}((\mathrm{G}_0)_{\mathrm{K}})$. This 1-cocycle depends on φ, but the other choices are precisely $a \circ \varphi$ for $a \in \mathrm{Aut}((\mathrm{G}_0)_{\mathrm{K}})$, which leads to the 1-cocycle $\gamma \mapsto a \circ c_\gamma \circ (\gamma.a)^{-1}$. As we vary through all such a, these cocycles vary through precisely the ones that are cohomologous to c. In particular, the cohomology class $[c] \in \mathrm{H}^1(\mathrm{K}/k, \mathrm{Aut}((\mathrm{G}_0)_{\mathrm{K}}))$ is independent of φ. Note that $c = 1$ as a function precisely when φ is defined over k, and more generally c is a coboundary $\gamma \mapsto a^{-1} \circ \gamma.a$ for $a \in \mathrm{Aut}((\mathrm{G}_0)_{\mathrm{K}})$ precisely when $a \circ \varphi$ is defined over k.

Since Galois descent is effective for affine schemes, $\mathrm{H}^1(\mathrm{K}/k, \mathrm{Aut}((\mathrm{G}_0)_{\mathrm{K}}))$ is identified with the pointed set of isomorphism classes of k-groups G such that there exists a K-group isomorphism $\mathrm{G}_{\mathrm{K}} \simeq (\mathrm{G}_0)_{\mathrm{K}}$. This is a special case of the general formalism of "twisted forms" as in [**Ser97**, III] (i.e., it has nothing to do with G_0 being a connected reductive k-group), and for a k-embedding $\mathrm{K} \to \mathrm{K}'$ into another finite Galois extension of k the resulting inflation map of pointed sets

$$\mathrm{H}^1(\mathrm{K}/k, \mathrm{Aut}((\mathrm{G}_0)_{\mathrm{K}})) \to \mathrm{H}^1(\mathrm{K}'/k, \mathrm{Aut}((\mathrm{G}_0)_{\mathrm{K}'}))$$

relaxes existence of a K-isomorphism (to $(\mathrm{G}_0)_{\mathrm{K}}$) to existence of a K′-isomorphism (to $(\mathrm{G}_0)_{\mathrm{K}'}$). Passing to the direct limit over all K/k inside k_s/k gives the result. □

Remark 7.1.2. — If we fix a separable closure k_s/k then we can define a *canonical* based root datum attached to any (possibly non-split) connected reductive k-group G as follows. For each pair (T, B) in G_{k_s} consisting of a (geometrically) maximal k_s-torus T and a Borel k_s-subgroup B that contains it, we get an associated based root datum R(G, T, B). If (T′, B′) is another choice then by Proposition 6.2.11(2) there exists $g \in \mathrm{G}(k_s)$ such that g-conjugation carries (T, B) to (T′, B′). The choice of g is unique up to $\mathrm{N}_{\mathrm{G}}(\mathrm{B})(k_s) \bigcap \mathrm{N}_{\mathrm{G}}(\mathrm{T})(k_s) = \mathrm{B}(k_s) \bigcap \mathrm{N}_{\mathrm{G}}(\mathrm{T})(k_s) = \mathrm{T}(k_s)$, so the induced isomorphism $\mathrm{R}(\mathrm{G}, \mathrm{T}, \mathrm{B}) \simeq \mathrm{R}(\mathrm{G}, \mathrm{T}', \mathrm{B}')$ of based root data is independent of the choice of g. More specifically, we get *canonical* isomorphisms among all of the based root data R(G, T, B) as we vary (T, B), and these isomorphisms are compatible with respect to composition.

By forming the (inverse or direct) limit along this system of isomorphisms, we get "the" based root datum of G, to be denoted $(\mathrm{R}(\mathrm{G}), \Delta)$. This generally depends on k_s/k when G is not k-split. That is, if $\gamma \in \mathrm{Gal}(k_s/k)$ and (T, B) is a pair in G_{k_s}, then the isomorphism $\mathrm{R}(\mathrm{G}_{k_s}, \mathrm{T}, \mathrm{B}) \simeq \mathrm{R}(\mathrm{G}_{k_s}, \gamma^*(\mathrm{T}), \gamma^*(\mathrm{B}))$

defined via scalar extension along γ and the k-structure G on G_{k_s} may not coincide with the effect of conjugation by an element of $G(k_s)$ that carries (T, B) to $(\gamma^*(T), \gamma^*(B))$. Put in other terms, we have just defined a natural action of $\gamma \in \mathrm{Gal}(k_s/k)$ on $(R(G), \Delta)$ via

$$(R(G), \Delta) = R(G_{k_s}, T, B) \simeq R(G_{k_s}, \gamma^*(T), \gamma^*(B)) = (R(G), \Delta);$$

this is independent of the choice of (T, B) and may be nontrivial. It is the "$*$-action" of $\mathrm{Gal}(k_s/k)$ on $\mathrm{Dyn}(\Phi)$ that appears in the Borel–Tits approach to classifying isotropic connected semisimple groups over fields (modulo the "anisotropic kernel").

When k_s/k is understood from context, $(R(G), \Delta)$ is called the *based root datum* for G. By the Existence and Isomorphism Theorems, up to unique k-isomorphism there exists a unique pinned split connected reductive k-group $(G_0, T_0, M_0, \{X_a\}_{a\in\Delta})$ with based root datum $(R(G), \Delta)$. As we vary through the different choices for the base, the resulting pinned split k-groups are canonically isomorphic to each other. This all depends on k_s/k except when G is k-split. The dependence on k_s/k can be eliminated by working with suitable étale k-schemes, such as the finite étale k-scheme associated to Δ equipped with the above action of $\mathrm{Gal}(k_s/k)$; i.e., the "scheme of Dynkin diagrams" $\mathrm{Dyn}(G)$; see Example 7.1.11.)

The preceding considerations can be adapted to a general base scheme. To explain this, it is convenient to introduce some notation.

Definition 7.1.3. — If $G \to S$ is a group scheme, its *automorphism functor* $\underline{\mathrm{Aut}}_{G/S}$ on the category of S-schemes is $\underline{\mathrm{Aut}}_{G/S} : S' \rightsquigarrow \mathrm{Aut}_{S'\text{-gp}}(G_{S'})$. A representing object (if one exists) is denoted $\mathrm{Aut}_{G/S}$ and is called the *automorphism scheme* of G.

The automorphism functor is a sheaf for the fppf (and even fpqc) topology, and we will see that representability and structural properties of the automorphism functor in the case of split reductive G are extremely useful in the classification of all reductive G. As an example, we will show that $\mathrm{Aut}_{SL_2/S}$ exists and is identified with PGL_2 (see Theorem 7.1.9(3)). Note that in general $\underline{\mathrm{Aut}}_{G/S}(S)$ is the automorphism group of G in the usual sense, which we also denote as $\mathrm{Aut}(G)$ (as for any category).

Consider a reductive group G over a *connected* non-empty scheme S. Connectedness of the base ensures that all geometric fibers have the same reduced root datum R (see the proof of Lemma 6.1.3), so it makes sense to define G_0 to be the split reductive S-group with root datum R. (This defines G_0 uniquely up to S-isomorphism, by the Isomorphism Theorem.) The S-groups G and G_0 become isomorphic étale-locally on S, due to the Isomorphism Theorem and the existence étale-locally on S of splittings and pinnings for $G \to S$.

For any sheaf of groups $\mathscr{F}$ on $\mathrm{S}_{\text{ét}}$, we define the set $\mathrm{Z}^1(\mathrm{S}'/\mathrm{S}, \mathscr{F})$ of *Čech 1-cocycles* to consist of those $\xi \in \mathscr{F}(\mathrm{S}' \times_{\mathrm{S}} \mathrm{S}')$ such that $p_{13}^*(\xi) = p_{23}^*(\xi) p_{12}^*(\xi)$. A pair of such 1-cocycles ξ and ξ' are *cohomologous*, denoted $\xi \sim \xi'$, if there exists $g \in \mathscr{F}(\mathrm{S}')$ such that $\xi' = p_2^*(g)\xi p_1^*(g)^{-1}$. It is straightforward to check that $\sim$ is an equivalence relation. The quotient set by this relation is denoted $\mathrm{H}^1(\mathrm{S}'/\mathrm{S}, \mathscr{F})$, and it has evident functoriality in S' over S. In concrete terms, $\mathrm{Z}^1(\mathrm{S}'/\mathrm{S}, \mathscr{F})$ is the set of descent data relative to $\mathrm{S}' \to \mathrm{S}$ on the étale sheaf of sets $\mathscr{F}|_{\mathrm{S}'_{\text{ét}}}$ equipped with its right $\mathscr{F}$-translation action. Thus, by effective descent for étale sheaves we see that the set of 1-cocycles is naturally identified with the set of isomorphism classes of pairs $(\mathscr{E}, \theta)$ consisting of a right $\mathscr{F}$-torsor $\mathscr{E}$ on $\mathrm{S}_{\text{ét}}$ and an element $\theta \in \mathscr{E}(\mathrm{S}')$.

The relation $\sim$ encodes the property that two $\mathscr{F}$-torsors on $\mathrm{S}_{\text{ét}}$ are isomorphic. Hence, $\mathrm{H}^1(\mathrm{S}'/\mathrm{S}, \mathscr{F})$ is the set of isomorphism classes of right $\mathscr{F}$-torsors on $\mathrm{S}_{\text{ét}}$ that admit a section over S'. This interpretation shows that functoriality with respect to S-maps $\mathrm{S}'' \to \mathrm{S}'$ turns the pointed sets $\mathrm{H}^1(\mathrm{S}'/\mathrm{S}, \mathscr{F})$ into a directed system relative to the partial order of refinement among covers $\mathrm{S}' \to \mathrm{S}$ (i.e., these transition maps do not depend on the specific S-maps between such covers). Thus, it makes sense to form the direct limit

$$\check{\mathrm{H}}^1(\mathrm{S}_{\text{ét}}, \mathscr{F}) := \varinjlim_{\mathrm{S}'/\mathrm{S}} \mathrm{H}^1(\mathrm{S}'/\mathrm{S}, \mathscr{F}). \tag{7.1.1}$$

Example 7.1.4. — Let G_0 be a reductive S-group. The set $\mathrm{Z}^1(\mathrm{S}'/\mathrm{S}, \mathrm{Aut}_{\mathrm{G}_0/\mathrm{S}})$ is identified with the set of étale descent data on $(\mathrm{G}_0)_{\mathrm{S}'}$ relative to $\mathrm{S}' \to \mathrm{S}$, and such descent data are effective since $(\mathrm{G}_0)_{\mathrm{S}'}$ is S'-affine. The equivalence relation $\sim$ encodes that two descent data have isomorphic S-descents, so $\check{\mathrm{H}}^1(\mathrm{S}'/\mathrm{S}, \mathrm{Aut}_{\mathrm{G}_0/\mathrm{S}})$ is identified (functorially in S' over S) with the set of S-isomorphism classes of reductive S-groups G such that $\mathrm{G}_{\mathrm{S}'} \simeq (\mathrm{G}_0)_{\mathrm{S}'}$. The dictionary relating right $\mathrm{Aut}_{\mathrm{G}_0/\mathrm{S}}$-torsors over $\mathrm{S}_{\text{ét}}$ and S-forms G of G_0 is that to G we associate the right $\mathrm{Aut}_{\mathrm{G}_0/\mathrm{S}}$-torsor $\mathrm{Isom}(\mathrm{G}_0, \mathrm{G})$ classifying group scheme isomorphisms from G_0 to G over S-schemes.

For split G_0 with root datum R, $\mathrm{H}^1(\mathrm{S}'/\mathrm{S}, \mathrm{Aut}_{\mathrm{G}_0/\mathrm{S}})$ is the set of isomorphism classes of reductive S-groups that split over S' and have geometric fibers with root datum R. Hence, the set of isomorphism classes of reductive S-groups G having geometric fibers with root datum isomorphic to R is naturally identified with the étale cohomology set

$$\mathrm{H}^1(\mathrm{S}_{\text{ét}}, \underline{\mathrm{Aut}}_{\mathrm{G}_0/\mathrm{S}}) := \varinjlim_{\mathrm{S}'/\mathrm{S}} \check{\mathrm{H}}^1(\mathrm{S}'/\mathrm{S}, \underline{\mathrm{Aut}}_{\mathrm{G}_0/\mathrm{S}}),$$

where we vary through a cofinal set of étale covers $\mathrm{S}' \to \mathrm{S}$. (See Exercise 7.3.3(i) and Exercise 2.4.11(i)) for further discussion.) When S is connected we only need to compute the root datum on a single fiber, due to Lemma 6.1.3.

The classical theory over an algebraically closed field k does not address representability properties of the automorphism functor on k-algebras, but it does suggest that the automorphism group over k should be viewed as having a "geometric" structure. More specifically (by Proposition 1.5.1, the identification (1.5.2), and Proposition 1.5.5), if G is a simply connected (and connected) semisimple group over an algebraically closed field k and Φ is its root system then $\mathrm{Aut}(\mathrm{G})$ is naturally the group of k-points of

$$(\mathrm{G}/\mathrm{Z}_{\mathrm{G}}) \rtimes \mathrm{Aut}(\mathrm{Dyn}(\Phi)), \tag{7.1.2}$$

where the semi-direct product structure rests on a choice of splitting and pinning. If we want to use (7.1.2) as anything deeper than a bookkeeping device (e.g., exploit that $\mathrm{G}/\mathrm{Z}_{\mathrm{G}}$ is connected), we should prove representability of $\underline{\mathrm{Aut}}_{\mathrm{G}/k}$ and not just make a construction that has the "right" geometric points. This will be done in Theorem 7.1.9.

As a prelude to the case of a general base scheme, now consider the problem of describing $\underline{\mathrm{Aut}}_{\mathrm{G}/k}(k) = \mathrm{Aut}(\mathrm{G})$ for a split connected reductive group G over a general field k. This goes beyond the classical arguments related to (7.1.2) over algebraically closed fields because maps such as $\mathrm{G}(k) \to (\mathrm{G}/\mathrm{Z}_{\mathrm{G}})(k)$ can fail to be surjective when $k \neq \overline{k}$ (e.g., $\mathrm{SL}_n(k) \to \mathrm{PGL}_n(k)$ for any field k such that $k^\times$ is not n-divisible).

Let $(\mathrm{G}, \mathrm{T}, \mathrm{M})$ be a split connected reductive group over a field k. The action of G on itself by conjugation factors through an action of $\mathrm{G}/\mathrm{Z}_{\mathrm{G}}$ on G. This identifies the adjoint quotient $\mathrm{G}/\mathrm{Z}_{\mathrm{G}}$ with a subfunctor of the automorphism functor $\underline{\mathrm{Aut}}_{\mathrm{G}/k}$, so $(\mathrm{G}/\mathrm{Z}_{\mathrm{G}})(k)$ is a subgroup of $\underline{\mathrm{Aut}}_{\mathrm{G}/k}(k) = \mathrm{Aut}(\mathrm{G})$. Note that this works even when $\mathrm{G}(k) \to (\mathrm{G}/\mathrm{Z}_{\mathrm{G}})(k)$ fails to be surjective, and that it defines a normal subgroup of $\mathrm{Aut}(\mathrm{G})$. (See Exercise 7.3.2.)

Due to the canonicity of the associated based root datum $(\mathrm{R}(\mathrm{G}), \Delta)$ as in Remark 7.1.2, $\mathrm{Aut}(\mathrm{G})$ naturally acts on $(\mathrm{R}(\mathrm{G}), \Delta)$. Here is how it works in concrete terms, by identifying $\mathrm{Aut}(\mathrm{G})/(\mathrm{G}/\mathrm{Z}_{\mathrm{G}})(k)$ with $\mathrm{Aut}(\mathrm{R}(\mathrm{G}), \Delta)$. Pick a positive system of roots Φ^+ in $\Phi = \Phi(\mathrm{G}, \mathrm{T})$, with base denoted Δ. Equivalently, choose a Borel subgroup B in G containing T (with $\Phi(\mathrm{B}, \mathrm{T}) = \Phi^+$). Let ϕ be an arbitrary automorphism of G, so $\mathrm{T}' := \phi(\mathrm{T})$ is a *split* (geometrically) maximal k-torus of G and $\mathrm{B}' := \phi(\mathrm{B})$ is a Borel k-subgroup of G containing T'. By Proposition 6.2.11(2), there exists $g \in \mathrm{G}(k)$ such that $\mathrm{T}' = g\mathrm{T}g^{-1}$ and $\mathrm{B}' = g\mathrm{T}g^{-1}$. Thus, by composing ϕ with an automorphism arising from $\mathrm{G}(k)$ we can arrange that $\phi(\mathrm{T}) = \mathrm{T}$ and $\phi(\Phi^+) = \Phi^+$. These additional requirements are preserved under composition of ϕ with the action of an element $g \in (\mathrm{G}/\mathrm{Z}_{\mathrm{G}})(k)$ if and only if $g \in (\mathrm{T}/\mathrm{Z}_{\mathrm{G}})(k)$, since $\mathrm{B}/\mathrm{Z}_{\mathrm{G}} = \mathrm{N}_{\mathrm{G}/\mathrm{Z}_{\mathrm{G}}}(\mathrm{B}/\mathrm{Z}_{\mathrm{G}})$ and $\mathrm{N}_{\mathrm{G}/\mathrm{Z}_{\mathrm{G}}}(\mathrm{T}/\mathrm{Z}_{\mathrm{G}}) \bigcap (\mathrm{B}/\mathrm{Z}_{\mathrm{G}}) = \mathrm{T}/\mathrm{Z}_{\mathrm{G}}$.

The automorphism of the based root datum $(\mathrm{R}(\mathrm{G}, \mathrm{T}), \Delta)$ induced by this ϕ is unaffected by composing ϕ with the action of $(\mathrm{T}/\mathrm{Z}_{\mathrm{G}})(k)$ since the $\mathrm{T}/\mathrm{Z}_{\mathrm{G}}$-action on G is trivial on T. The resulting action via ϕ on the set Δ has more structure: it is an automorphism of the Dynkin diagram $\mathrm{Dyn}(\Phi)$. This can be nontrivial:

Example 7.1.5. — Let $\mathrm{G} = \mathrm{SL}_n$, and let $w \in \mathrm{G}(k)$ be the anti-diagonal matrix whose entries alternate $1, -1, 1, -1, \dots$ beginning in the upper right (so $ww^\top = 1$). The automorphisms $g \mapsto (g^\top)^{-1}$ and $g \mapsto wgw^{-1}$ of G swap the upper triangular and lower triangular Borel subgroups while preserving the diagonal torus D. The composite automorphism $\iota : g \mapsto w(g^\top)^{-1}w^{-1}$ is an involution of G that preserves D and the upper triangular Borel subgroup, inducing the involution of the Dynkin diagram when $n > 2$.

Returning to a general split (G, T) over k, let $\{\mathrm{X}_a\}_{a\in\Delta}$ be a pinning of $(\mathrm{G}, \mathrm{T}, \mathrm{B})$. For $\mathrm{X}'_a := \mathrm{Lie}(\phi)(\mathrm{X}_{\phi^{-1}(a)})$ we get another pinning $\{\mathrm{X}'_a\}_{a\in\Delta}$. Clearly $\mathrm{X}'_a = c_a\mathrm{X}_a$ for a unique $(c_a) \in (k^\times)^\Delta$. Since Δ is a $\mathbf{Z}$-basis of the character group $\mathbf{Z}\Phi$ of the split maximal torus $\mathrm{T}/\mathrm{Z}_\mathrm{G} \subset \mathrm{G}/\mathrm{Z}_\mathrm{G}$ (Corollary 3.3.6(1)), we have $\mathrm{T}/\mathrm{Z}_\mathrm{G} \simeq \prod_{a\in\Delta}\mathbf{G}_\mathrm{m}$ via $t \bmod \mathrm{Z}_\mathrm{G} \mapsto (a(t))_{a\in\Delta}$. Hence, $(\mathrm{T}/\mathrm{Z}_\mathrm{G})(k)$ acts simply transitively on the set of all pinnings of $(\mathrm{G}, \mathrm{T}, \mathrm{B})$, so by composing ϕ with a *unique* automorphism arising from $(\mathrm{T}/\mathrm{Z}_\mathrm{G})(k)$ we can arrange that $\mathrm{Lie}(\phi)(\mathrm{X}_a) = \mathrm{X}_{\phi(a)}$ for all $a \in \Delta$ (i.e., $\mathrm{X}'_a = \mathrm{X}_a$ for all $a \in \Delta$).

Let $\Theta = \mathrm{Aut}(\mathrm{R}(\mathrm{G}, \mathrm{T}), \Delta)$ be the automorphism group of the based root datum, so by Corollary 6.1.15 the conditions we have imposed on ϕ relative to $(\mathrm{T}, \mathrm{B}, \{\mathrm{X}_a\}_{a\in\Delta})$ make it determined *uniquely* by its image in Θ. That is, we have identified $\mathrm{Aut}(\mathrm{G})/(\mathrm{G}/\mathrm{Z}_\mathrm{G})(k)$ with a subgroup of Θ. In fact, this subgroup inclusion is an equality, or equivalently *every* element of Θ arises from a unique ϕ satisfying the conditions imposed relative to $(\mathrm{T}, \mathrm{B}, \{\mathrm{X}_a\}_{a\in\Delta})$. This is exactly the content of the precise form of the Isomorphism Theorem given in Theorem 6.1.17. In other words, the choice of pinning defines an injective homomorphism $\Theta \hookrightarrow \mathrm{Aut}(\mathrm{G})$ with image equal to the automorphism group of $(\mathrm{G}, \mathrm{T}, \mathrm{B}, \{\mathrm{X}_a\}_{a\in\Delta})$, so Θ is carried to a subgroup of $\mathrm{Aut}(\mathrm{G})$ that preserves T and B and maps isomorphically onto $\mathrm{Aut}(\mathrm{G})/(\mathrm{G}/\mathrm{Z}_\mathrm{G})(k)$.

We have just constructed an isomorphism of groups

$$(\mathrm{G}/\mathrm{Z}_\mathrm{G})(k) \rtimes \mathrm{Aut}(\mathrm{R}(\mathrm{G}, \mathrm{T}), \Delta) \simeq \mathrm{Aut}(\mathrm{G}). \tag{7.1.3}$$

This depends on the choice of pinning, as well as the split hypothesis on G. (Although the inclusion of $(\mathrm{G}/\mathrm{Z}_\mathrm{G})(k)$ as a normal subgroup of $\mathrm{Aut}(\mathrm{G})$ is available without any split hypothesis on G, in the non-split case there may be no homomorphic section to $\mathrm{Aut}(\mathrm{G}) \to \mathrm{Aut}(\mathrm{G})/(\mathrm{G}/\mathrm{Z}_\mathrm{G})(k)$; see Example 7.1.12.) Since the preceding discussion is compatible with extension of the ground field, we likewise have $(\mathrm{G}/\mathrm{Z}_\mathrm{G})(\mathrm{K}) \rtimes \mathrm{Aut}(\mathrm{R}(\mathrm{G}, \mathrm{T}), \Delta) \simeq \mathrm{Aut}(\mathrm{G}_\mathrm{K}) = \underline{\mathrm{Aut}}_{\mathrm{G}/k}(\mathrm{K})$ for any extension field K/k. To summarize, we have proved:

Proposition 7.1.6. — *Let* $(\mathrm{G}, \mathrm{T}, \mathrm{M}, \{\mathrm{X}_a\}_{a\in\Delta})$ *be a pinned split connected reductive group over a field* k*. The group* $\mathrm{Aut}(\mathrm{G})$ *is naturally an extension of*

$\mathrm{Aut}(\mathrm{R}(\mathrm{G},\mathrm{T}),\Delta)$ *by* $(\mathrm{G}/\mathrm{Z}_{\mathrm{G}})(k)$*, and the pinning naturally splits this extension as a semi-direct product.*

Remark 7.1.7. — By Proposition 1.5.1, if (R,Δ) is a semisimple based root datum with root system Φ then $\mathrm{Aut}(\mathrm{R},\Delta) \subset \mathrm{Aut}(\mathrm{Dyn}(\Phi))$, with equality when R is simply connected or adjoint or when the "fundamental group" $(\mathbf{Z}\Phi^{\vee})^{*}/\mathbf{Z}\Phi$ is cyclic (which includes all irreducible reduced root systems except D_{2n}, $n \geqslant 2$).

Example 7.1.8. — The root system $\Phi = \mathrm{A}_{n-1}$ ($n \geqslant 2$) has cyclic "fundamental group" $(\mathbf{Z}\Phi^{\vee})^{*}/(\mathbf{Z}\Phi) \simeq \mathbf{Z}/n\mathbf{Z}$ and $\mathrm{Aut}(\mathrm{Dyn}(\Phi))$ is equal to $\mathbf{Z}/2\mathbf{Z}$ when $n > 2$ (and $\{1\}$ when $n = 2$). Thus, $\mathrm{Aut}(\mathrm{SL}_n) = \mathrm{PGL}_n(k) \rtimes (\mathbf{Z}/2\mathbf{Z})$ for $n > 2$, with the factor $\mathbf{Z}/2\mathbf{Z}$ generated by the involution $g \mapsto (g^{\top})^{-1}$ (due to Example 7.1.5). This involution is available for SL_2, but in that case it is inner (arising from conjugation by $w = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$).

[The equality $\mathrm{Aut}_k(\mathrm{SL}_2) = \mathrm{PGL}_2(k)$ for $n = 2$ can be proved in elementary terms, as follows. Using k-rational conjugacy results, it suffices to show that an automorphism f of SL_2 that is the identity on the upper triangular Borel subgroup B^{+} is the identity on the open cell Ω^{+} (and hence is the identity). Certainly f preserves the opposite Borel subgroup B^{-} relative to the diagonal torus, and so preserves its unipotent radical U^{-}. The effect on U^{-} must be $\left(\begin{smallmatrix} 1 & 0 \\ x & 1 \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} 1 & 0 \\ cx & 1 \end{smallmatrix}\right)$ for some $c \in k^{\times}$. But the two standard open cells Ω^{+} and Ω^{-} have intersection defined by $1 + xy \neq 0$ inside $\mathbf{G}_{\mathrm{a}} \times \mathbf{G}_{\mathrm{m}} \times \mathbf{G}_{\mathrm{a}} = \Omega^{+}$ via

$$(y,t,x) \mapsto \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} \begin{pmatrix} t & 0 \\ 0 & 1/t \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix},$$

so we must have that the open loci $1 + xy \neq 0$ and $1 + cxy \neq 0$ on $\mathbf{G}_{\mathrm{a}} \times \mathbf{G}_{\mathrm{a}}$ coincide. This forces $c = 1$, as desired.]

In Example 7.1.10 we will compute the section $\mathbf{Z}/2\mathbf{Z} = \mathrm{Aut}(\mathrm{Dyn}(\Phi)) \hookrightarrow \mathrm{Aut}(\mathrm{SL}_n)$ associated to a "standard" choice of splitting and pinning for SL_n. This section carries the diagram involution to ι from Example 7.1.5. In particular, the "coordinate dependence" of the involution ι reflects the fact that the semi-direct product structure in (7.1.3) is not intrinsic to G or even to (G,T): it depends on the choice of the splitting and pinning.

The same conclusions apply to $\mathrm{Aut}(\mathrm{PGL}_n)$, as well as to $\mathrm{Aut}(\mathrm{SL}_n/\mu)$ for any k-subgroup $\mu \subset \mu_n$. This illustrates that automorphisms of a connected semisimple k-group lift uniquely to automorphisms of the simply connected central cover (Exercise 6.5.2(iii)).

The preceding arguments suggest that in the split case we should represent $\underline{\mathrm{Aut}}_{\mathrm{G}/k}$ by a smooth k-group with identity component $\mathrm{G}/\mathrm{Z}_{\mathrm{G}}$ and constant component group $\mathrm{Aut}(\mathrm{R},\Delta)$, where $\mathrm{R} = \mathrm{R}(\mathrm{G},\mathrm{T})$. In particular, if

G is semisimple then $\underline{\mathrm{Aut}}_{\mathrm{G}/k}$ should be represented by a linear algebraic k-group having constant component group that is a subgroup of the finite group $\mathrm{Aut}(\mathrm{Dyn}(\Phi))$. This can be made rather concrete: define the $\mathrm{G}/\mathrm{Z}_{\mathrm{G}}$-group isomorphism

$$\alpha^0 : (\mathrm{G}/\mathrm{Z}_{\mathrm{G}}) \times \mathrm{G} \to (\mathrm{G}/\mathrm{Z}_{\mathrm{G}}) \times \mathrm{G}$$

by passage to the Z_{G}-quotient on the first factor relative to the map of G-group schemes $(g, g') \mapsto (g, gg'g^{-1})$, and use translation against the injection $\mathrm{Aut}(\mathrm{R}, \Delta) \hookrightarrow \mathrm{Aut}(\mathrm{G})$ (defined by a choice of pinning) to obtain a "universal automorphism"

$$\alpha : ((\mathrm{G}/\mathrm{Z}_{\mathrm{G}}) \rtimes \mathrm{Aut}(\mathrm{R}, \Delta)) \times \mathrm{G} \simeq ((\mathrm{G}/\mathrm{Z}_{\mathrm{G}}) \rtimes \mathrm{Aut}(\mathrm{R}, \Delta)) \times \mathrm{G}$$

of group schemes over $(\mathrm{G}/\mathrm{Z}_{\mathrm{G}}) \rtimes \mathrm{Aut}(\mathrm{R}, \Delta)$. For $\xi \in ((\mathrm{G}/\mathrm{Z}_{\mathrm{G}}) \rtimes \mathrm{Aut}(\mathrm{R}, \Delta))(\mathrm{K})$ (with a field extension K/k), the fiber map α_ξ is the automorphism corresponding to ξ via the analogue of (7.1.3) for K-valued points.

This construction defines a morphism of k-group functors

$$(\mathrm{G}/\mathrm{Z}_{\mathrm{G}}) \rtimes \mathrm{Aut}(\mathrm{R}, \Delta) \to \underline{\mathrm{Aut}}_{\mathrm{G}/k}, \tag{7.1.4}$$

and the classical approach would end here since this map has been constructed to be bijective on points valued in any extension field K/k (which is sufficient for many applications). For problems related to deforming automorphisms or working over k-algebras that are not fields, it is useful to go beyond field-valued points and prove that (7.1.4) is an isomorphism of functors on k-algebras.

Now we turn to the general case. Let G be a reductive S-group, with $\mathrm{S} \neq \varnothing$. For the maximal central torus Z and semisimple derived group $\mathrm{G}' = \mathscr{D}(\mathrm{G})$, the multiplication map $\mathrm{Z} \times \mathrm{G}' \to \mathrm{G}$ is a central isogeny (Corollary 5.3.3). The kernel μ of this isogeny is a finite S-group of multiplicative type, so an automorphism of G is "the same" as a pair of automorphisms of G' and Z that coincide on μ. By Exercise 7.3.1, $\underline{\mathrm{Aut}}_{\mu/\mathrm{S}}$ is a finite étale S-group and $\underline{\mathrm{Aut}}_{\mathrm{Z}/\mathrm{S}}$ is represented by a separated étale S-group that is an étale form of $\mathrm{GL}_r(\mathbf{Z})$ where r is the rank of Z (locally constant on S). Hence, to understand properties of $\underline{\mathrm{Aut}}_{\mathrm{G}/\mathrm{S}}$, the real content is in the semisimple case. Here is the main result.

Theorem 7.1.9. — *Let* G *be a reductive group over a non-empty scheme* S.

1. *The functor* $\underline{\mathrm{Aut}}_{\mathrm{G}/\mathrm{S}}$ *is represented by a separated and smooth* S*-group* $\mathrm{Aut}_{\mathrm{G}/\mathrm{S}}$ *that fits into a short exact sequence*

$$1 \to \mathrm{G}/\mathrm{Z}_{\mathrm{G}} \to \mathrm{Aut}_{\mathrm{G}/\mathrm{S}} \to \mathrm{Out}_{\mathrm{G}/\mathrm{S}} \to 1$$

where: $\mathrm{G}/\mathrm{Z}_{\mathrm{G}}$ *is closed in* $\mathrm{Aut}_{\mathrm{G}/\mathrm{S}}$, $\mathrm{Out}_{\mathrm{G}/\mathrm{S}}$ *is a separated étale* S*-group that is locally constant for the étale topology on* S*. If* S *is locally noetherian and normal then every connected component of* $\mathrm{Out}_{\mathrm{G}/\mathrm{S}}$ *is* S*-finite.*

2. *The* S*-group* $\mathrm{Out}_{\mathrm{G}/\mathrm{S}}$ *has finite geometric fibers if and only if the maximal central torus of* G *has rank* $\leqslant 1$*, in which case* $\mathrm{Out}_{\mathrm{G}/\mathrm{S}}$ *is* S*-finite and* $\mathrm{Aut}_{\mathrm{G}/\mathrm{S}}$ *is* S*-affine.*
3. *For pinned split reductive* $(\mathrm{G}, \mathrm{T}, \mathrm{M}, \{\mathrm{X}_a\}_{a\in\Delta})$ *over* S*,* $\mathrm{Out}_{\mathrm{G}/\mathrm{S}}$ *is identified with the constant* S*-group associated to* $\mathrm{Aut}(\mathrm{R}(\mathrm{G}, \mathrm{T}, \mathrm{M}), \Delta)$*. Moreover, the pinning defines a semi-direct product splitting of* S*-groups*

$$\mathrm{Aut}_{\mathrm{G}/\mathrm{S}} \simeq (\mathrm{G}/\mathrm{Z}_{\mathrm{G}}) \rtimes \mathrm{Aut}(\mathrm{R}(\mathrm{G}, \mathrm{T}, \mathrm{M}), \Delta)_{\mathrm{S}}.$$

Proof. — Étale descent is effective for schemes that are separated and étale over the base. (Indeed, one can reduce to the finite type case, which is [**SGA1**, IX, 4.1]. Alternatively, the descent trivially exists as an algebraic space, and any algebraic space that is separated and locally quasi-finite over a scheme is a scheme [**LMB**, A.2].) Also, any fppf group-sheaf extension E of an S-group scheme H by an fpqc S-affine group scheme is necessarily representable since E is an algebraic space that is affine over H and hence is a scheme. (Algebraic spaces are not needed to prove that E is a scheme; one just has to use the effectivity of fpqc descent for schemes affine over the base.) Thus, for the proof of (1) apart from the final assertion it suffices to work étale-locally on S. It is likewise enough to work étale-locally on S for the proof of (2).

For the proof of the entire theorem apart from the final assertion in (1), it is now enough to work with a pinned split group $(\mathrm{G}, \mathrm{T}, \mathrm{M}, \{\mathrm{X}_a\}_{a\in\Delta})$. Thus, we have the inclusion of group sheaves $(\mathrm{G}/\mathrm{Z}_{\mathrm{G}}) \rtimes \mathrm{Aut}(\mathrm{R}(\mathrm{G}, \mathrm{T}, \mathrm{M}), \Delta)_{\mathrm{S}} \subset \mathrm{Aut}_{\mathrm{G}/\mathrm{S}}$ for the fppf topology on the category of S-schemes. Hence, to prove that this is an equality on S′-points for S-schemes S′ it suffices to work Zariski-locally on S′. It suffices to treat local rings on S′ (as finite presentation then permits us to "spread out" from local rings to Zariski-open subschemes of S′).

The arguments leading up to the proof of (7.1.3) were written to apply verbatim at the level of S′-points over any local S-scheme S′, upon noting that Proposition 6.2.11(2) is applicable to any local scheme. To conclude, we just need to make three observations: (i) the semi-direct product structure in (3) implies that $\mathrm{G}/\mathrm{Z}_{\mathrm{G}}$ is closed in $\mathrm{Aut}_{\mathrm{G}/\mathrm{S}}$, (ii) a constant S-group is S-finite if and only if it has finite geometric fibers, (iii) a based root datum (R, Δ) has finite automorphism group if and only if the underlying root datum $\mathrm{R} = (\mathrm{M}, \Phi, \mathrm{M}^{\vee}, \Phi^{\vee})$ satisfies $\dim \mathrm{M}_{\mathbf{Q}}/\mathbf{Q}\Phi \leqslant 1$ (since $\mathrm{GL}_1(\mathbf{Z})$ is finite).

It remains to prove the final assertion in (1). More generally, let S be a locally noetherian scheme that is normal, and let $\mathrm{E} \to \mathrm{S}$ be an S-scheme that becomes constant étale-locally on S. We claim that every connected component of E is S-finite. We may assume that S is connected, so it is irreducible. Let C be a connected component of E, so C is normal and hence irreducible. Thus, if U is a non-empty open subscheme of S then C_{U} is (non-empty and hence) a connected component of E_{U}. Our problem is therefore

Zariski-local on S, so we can assume S is affine. We may choose an étale covering $\{S_i \to S\}$ with connected affine S_i such that E_{S_i} is a constant S_i-scheme. The open images of the S_i's cover S, and we can replace S with each of those separately. That is, we may assume there is an étale cover $S' \to S$ with connected affine S' such that $E_{S'}$ is a constant S'-scheme. The open and closed subscheme $C_{S'}$ inside $E_{S'}$ must be a union of connected components and hence a disjoint union of copies of S'. Since the map $C \to S$ is finite if and only if $C_{S'} \to S'$ is finite, it suffices to show that $C_{S'}$ has only finitely many connected components. For this purpose we may pass to the fiber over the generic point of S', or equivalently work over the generic point of S. Now we may assume $S = \mathrm{Spec}(K)$ for a field K, so the étale K-scheme E clearly has K-finite connected components. □

Example 7.1.10. — Let $(G, T, M, \{X_a\}_{a\in\Delta})$ be a pinned split semisimple S-group whose root system $\Phi \subset M$ is irreducible and not D_{2n} $(n \geqslant 2)$. By Proposition 1.5.1 we have $\mathrm{Aut}(R(G,T),\Delta) = \mathrm{Aut}(\mathrm{Dyn}(\Phi))$, so $\mathrm{Aut}_{G/S} = (G/Z_G) \rtimes \mathrm{Aut}(\mathrm{Dyn}(\Phi))_S$.

We make this explicit for $G = \mathrm{SL}_n$ with the diagonal torus $T = D_S(M)$ for $M = \mathbf{Z}^n/\mathrm{diag}(\mathbf{Z})$, Δ corresponding to the upper triangular Borel subgroup B, and the standard pinning $\{X_a\}_{a\in\Delta}$. If $n = 2$ then the diagram is a point and so the automorphism scheme of SL_2 is $\mathrm{SL}_2/\mu_2 = \mathrm{PGL}_2$ with its evident action.

If $n \geqslant 3$ then we claim that the associated section $\mathrm{Aut}(\mathrm{Dyn}(\Phi))_S \to \mathrm{Aut}_{\mathrm{SL}_n}$ carries the diagram involution φ to the involution ι of SL_n from Example 7.1.5. Since ι is an involution that preserves (T, B) and induces the nontrivial involution on $\mathrm{Dyn}(\Phi)$, we just have to check that its effect on root groups is a permutation of the X_a's (without the intervention of signs). The standard root group U_{ij} for $i < j$ is carried by ι to $U_{n+1-j,n+1-i}$, and relative to the standard parameterizations $\mathbf{G}_a \simeq U_c$ the isomorphism $U_{ij} \simeq U_{n+1-j,n+1-i}$ goes over to the automorphism $x \mapsto (-1)^{1+j-i}x$. Thus, for $j = i+1$ we get $\iota(X_a) = X_{\varphi(a)}$ as desired.

Example 7.1.11. — Let $(G, T, M, \{X_a\}_{a\in\Delta})$ be a pinned split reductive S-group, with $\Phi \subset M$ the set of roots. The composite map $\mathrm{Out}_{G/S} \simeq \mathrm{Aut}(R(G,T,M),\Delta)_S \to \mathrm{Aut}(\mathrm{Dyn}(\Phi))_S$ into the constant group associated to the finite automorphism group of the Dynkin diagram of Φ induces a map

$$H^1(S_{\text{ét}}, \mathrm{Aut}_{G/S}) \to H^1(S_{\text{ét}}, \mathrm{Aut}(\mathrm{Dyn}(\Phi))_S)$$

into the pointed set of S-twisted forms of the Dynkin diagram. More specifically, if G' is an S-form of G then its class $[G']$ in $H^1(S_{\text{ét}}, \mathrm{Aut}_{G/S})$ gives rise to a finite étale S-scheme $\mathrm{Dyn}(G')$, the "scheme of Dynkin diagrams" for G'. The edges with multiplicity on geometric fibers are encoded by a finite étale closed subscheme $\mathrm{Edge}(G') \subset \mathrm{Dyn}(G') \times \mathrm{Dyn}(G')$ disjoint from the diagonal (no loops) and an S-map $\mathrm{Dyn}(\mathrm{G}') \to \{1,2,3\}_{\mathrm{S}}$ that assigns "squared length". These S-schemes $\mathrm{Dyn}(\mathrm{G}')$ and $\mathrm{Edge}(\mathrm{G}')$ are constant when $[\mathrm{G}']$ arises from $\mathrm{H}^1(\mathrm{S}_{\text{ét}}, \mathrm{G}/\mathrm{Z}_{\mathrm{G}})$ (i.e., inner twisting of the split form G). See [**SGA3**, XXIV, § 3] for further details.

If G is not k-split then there may not be a k-group section to $\mathrm{Aut}_{\mathrm{G}/k} \to \mathrm{Out}_{\mathrm{G}/k}$, in contrast with the split case in Theorem 7.1.9(3), as even on k-points this map can be non-surjective. Here are some examples.

Example 7.1.12. — Let A be a central simple algebra of rank n^2 over a field k, with an integer $n > 2$. (Class field theory provides many such A that are central division algebras when k is a global or non-archimedean local field.) Since $\mathrm{A}_{k_s} \simeq \mathrm{Mat}_n(k_s)$, by the Skolem–Noether theorem the k-group $\mathrm{G} = \mathrm{SL}(\mathrm{A})$ of units of reduced-norm 1 in A (Exercise 5.5.5(iv)) is an inner form of $\mathrm{G}_0 = \mathrm{SL}_n$. Since $\mathrm{Out}_{\mathrm{G}/k}$ is a k-form of the constant k-group $\mathrm{Out}_{\mathrm{G}_0/k} = (\mathbf{Z}/2\mathbf{Z})_k$ that has no nontrivial k-forms (since $\mathrm{Aut}(\mathbf{Z}/2\mathbf{Z}) = 1$), $\mathrm{Out}_{\mathrm{G}/k} = (\mathbf{Z}/2\mathbf{Z})_k$. We claim that the map $\mathrm{Aut}_{\mathrm{G}/k} \to \mathrm{Out}_{\mathrm{G}/k} = (\mathbf{Z}/2\mathbf{Z})_k$ is not surjective on k-points (and so has no k-group section) if and only if A is not 2-torsion in $\mathrm{Br}(k)$. Equivalently, we claim that G admits a k-automorphism that is not "geometrically inner" if and only if $\mathrm{A} \simeq \mathrm{A}^{\mathrm{opp}}$ as k-algebras.

To prove the implication "$\Leftarrow$", we first note that $\mathrm{G}^{\mathrm{opp}} = \mathrm{SL}(\mathrm{A}^{\mathrm{opp}})$, so a k-algebra isomorphism $\mathrm{A} \simeq \mathrm{A}^{\mathrm{opp}}$ induces a k-isomorphism $\mathrm{G} \simeq \mathrm{G}^{\mathrm{opp}}$ that is the identity on the common center (whose finite étale Cartier dual is geometrically cyclic order n). Thus, composing with inversion $\mathrm{G}^{\mathrm{opp}} \simeq \mathrm{G}$ yields a k-automorphism of G that is inversion on the center. But since $n > 2$, inversion on the dual-to-cyclic center is nontrivial. Hence, we have obtained a k-automorphism of G that is not geometrically inner.

For the converse implication "$\Rightarrow$", assume that G admits a k-automorphism $f : \mathrm{G} \to \mathrm{G}$ that is not geometrically inner. We claim that f acts as inversion on the finite center Z_{G}, and that it uniquely extends via gluing with inversion on the central $\mathbf{G}_{\mathrm{m}}$ in $\underline{\mathrm{A}}^{\times}$ to yield a k-automorphism of $\underline{\mathrm{A}}^{\times}$ whose composition with inversion uniquely extends to a k-algebra isomorphism $\mathrm{A} \simeq \mathrm{A}^{\mathrm{opp}}$ (as desired). In view of the uniqueness statements we may (by Galois descent) extend the ground field to k_s, so $\mathrm{A} = \mathrm{Mat}_n$ and $\mathrm{G} = \mathrm{SL}_n$. Since $\mathrm{G}/\mathrm{Z}_{\mathrm{G}} = \underline{\mathrm{A}}^{\times}/\mathbf{G}_{\mathrm{m}}$, so $(\mathrm{G}/\mathrm{Z}_{\mathrm{G}})(k) = \mathrm{A}^{\times}/k^{\times}$, it is harmless to precompose f with the effect on G of an inner automorphism of A. Thus, we can focus on the case of a single k-automorphism of SL_n that is not geometrically inner (as any two are related through composition against an inner automorphism of A over k). Hence, we may assume that f is the transpose-inverse automorphism of SL_n (here we use that $n > 2$). This is inversion on the central μ_n, which visibly uniquely glues with inversion on the central $\mathbf{G}_{\mathrm{m}}$ to define a k-automorphism of GL_n (namely, transpose-inverse). By the Zariski-density of GL_n in Mat_n, its composition with inversion uniquely extends to an algebra anti-automorphism of Mat_n (namely, transpose).

Example 7.1.13. — The automorphism scheme has an application in the non-split case over $\mathbf{R}$. Let G be an $\mathbf{R}$-anisotropic connected semisimple $\mathbf{R}$-group. (Such G correspond precisely to the connected compact Lie groups via $\mathrm{G} \rightsquigarrow \mathrm{G}(\mathbf{R})$, and $\mathrm{G}(\mathbf{R})$ is a maximal compact subgroup of $\mathrm{G}(\mathbf{C})$; see Theorem D.2.4 and Proposition D.3.2.) The $\mathbf{R}$-group $\mathrm{Aut}_{\mathrm{G}/\mathbf{R}}$ has identity component $\mathrm{G}/\mathrm{Z}_\mathrm{G}$, so it is also $\mathbf{R}$-anisotropic. In [**Ser97**, III, 4.5, Ex. (b)], the anisotropicity of $\mathrm{Aut}_{\mathrm{G}/\mathbf{R}}$ is used to recover E. Cartan's classification of $\mathbf{R}$-descents of a connected semisimple $\mathbf{C}$-group: the set of $\mathbf{R}$-descents of $\mathrm{G}_\mathbf{C}$ (up to isomorphism) is in bijection with the set of conjugacy classes of involutions ι of the maximal compact subgroup $\mathrm{G}(\mathbf{R})$ of $\mathrm{G}(\mathbf{C})$.

This can be pushed a step further: the map $\iota \mapsto \iota_\mathbf{C}$ from conjugacy classes of involutions of $\mathrm{G}(\mathbf{R})$ to conjugacy classes of involutions of the "complexification" $\mathrm{G}_\mathbf{C}$ of $\mathrm{G}(\mathbf{R})$ is a bijection (apply [**Ser97**, III, 4.5, Thm. 6] to the anisotropic $\mathrm{Aut}_{\mathrm{G}/\mathbf{R}}$), so the $\mathbf{R}$-descents H of a connected semisimple $\mathbf{C}$-group $\mathscr{G}$ are classified up to isomorphism by conjugacy classes of involutions of $\mathscr{G}$. This can be made more concrete by using Mostow's description [**Mos**, § 6] of maximal compact subgroups of $\mathrm{H}(\mathbf{R})$ in terms of maximal compact subgroups of $\mathscr{G}(\mathbf{C})$: the conjugacy class of involutions of $\mathscr{G}$ corresponding to H contains $(\theta_\mathrm{K})_\mathbf{C}$, where θ_K is the Cartan involution of H associated to a maximal compact subgroup K of $\mathrm{H}(\mathbf{R})$. Equivalently, if $\mathscr{G}_0$ is the split $\mathbf{R}$-descent of $\mathscr{G}$ then the descent datum on $\mathscr{G}$ relative to the $\mathbf{R}$-structure H corresponds to the involution $g \mapsto (\theta_\mathrm{K})_\mathbf{C}(\overline{g})$ of the real Lie group $\mathscr{G}(\mathbf{C}) = \mathscr{G}_0(\mathbf{C})$ for any maximal compact subgroup $\mathrm{K} \subset \mathrm{H}(\mathbf{R})$.

7.2. Cohomological approach to forms. — As an application of the structure of the automorphism scheme, we can gain some insight into the cohomological description (in Lemma 7.1.1) of the set of isomorphism classes of forms of a given connected reductive group over a field. We begin with the situation over a field because the relevant cohomology is Galois cohomology, which is more concrete than étale cohomology. Later we will generalize to the case of an arbitrary connected non-empty base scheme.

Let G be a connected reductive group over a field k, so for the split k-form G_0 of G the k-group $\mathrm{Aut}_{\mathrm{G}/k}$ is a k-form of $\mathrm{Aut}_{\mathrm{G}_0/k}$. We shall explicitly describe how to build this k-form as an instance of "inner cocycle-twisting" from [**Ser97**, I, § 5.3]. Fix a k_s-isomorphism $\varphi : \mathrm{G}_{k_s} \simeq (\mathrm{G}_0)_{k_s}$, so we get a k_s-isomorphism

$$\xi : (\mathrm{Aut}_{\mathrm{G}/k})_{k_s} = \mathrm{Aut}_{\mathrm{G}_{k_s}/k_s} \simeq \mathrm{Aut}_{(\mathrm{G}_0)_{k_s}/k_s} = (\mathrm{Aut}_{\mathrm{G}_0/k_s})_{k_s}$$

via $f \mapsto \varphi \circ f \circ \varphi^{-1}$. The obstruction to $\mathrm{Gal}(k_s/k)$-equivariance for ξ is expressed by the following identity for $\gamma \in \mathrm{Gal}(k_s/k)$:

$$\xi(\gamma^*(f)) = (\varphi \circ \gamma^*(\varphi)^{-1}) \circ \gamma^*(\xi(f)) \circ (\varphi \circ \gamma^*(\varphi)^{-1})^{-1}.$$

Thus, for the 1-cocycle $c : \gamma \mapsto \varphi \circ \gamma^*(\varphi)^{-1}$ in $\mathrm{Z}^1(k_s/k, \mathrm{Aut}((\mathrm{G}_0)_{k_s}))$ whose cohomology class classifies G as a form of G_0 in Lemma 7.1.1, the k-form $\mathrm{Aut}_{\mathrm{G}/k}$ of $\mathrm{Aut}_{\mathrm{G}_0/k}$ is obtained through inner twisting by the 1-cocycle c.

Using cocycle-twisting notation as in [**Ser97**, I, § 5.3, Ex. 2], we obtain an isomorphism

$$\mathrm{Aut}_{\mathrm{G}/k} = {}_c\,\mathrm{Aut}_{\mathrm{G}_0/k}\,. \tag{7.2.1}$$

This identification depends on c, not just on the cohomology class of c. (Replacing φ with $\psi \circ \varphi$ for $\psi \in \mathrm{Aut}((\mathrm{G}_0)_{k_s})$ has the effect of replacing c with a cohomologous 1-cocycle $c' : \gamma \mapsto \psi \circ c_\gamma \circ (\gamma^*(\psi))^{-1}$, and ψ^{-1} descends to a k-isomorphism ${}_c\,\mathrm{Aut}_{\mathrm{G}_0/k} \simeq {}_{c'}\,\mathrm{Aut}_{\mathrm{G}_0/k}$ respecting the identifications with $\mathrm{Aut}_{\mathrm{G}/k}$.) Since this twisting process is defined through cocycles valued in inner automorphisms of the group $\mathrm{Aut}_{\mathrm{G}_0/k}(k_s)$, which in turn preserve *normal* k_s-subgroups of $(\mathrm{Aut}_{\mathrm{G}_0/k})_{k_s}$, we have the compatible equalities of normal k-subgroups $\mathrm{G}/\mathrm{Z}_\mathrm{G} = {}_c(\mathrm{G}_0/\mathrm{Z}_{\mathrm{G}_0})$ and quotients $\mathrm{Out}_{\mathrm{G}/k} = {}_c\mathrm{Out}_{\mathrm{G}_0/k}$ (where we are now twisting by 1-cocycles valued in the automorphism functors of the k-groups $\mathrm{G}_0/\mathrm{Z}_{\mathrm{G}_0}$ and $\mathrm{Out}_{\mathrm{G}_0/k}$).

In general, if $\mathscr{G}$ is a smooth affine k-group then a continuous 1-cocycle $a : \mathrm{Gal}(k_s/k) \to \mathrm{Aut}(\mathscr{G}_{k_s})$ defines a k-form ${}_a\mathscr{G}$ of $\mathscr{G}$ whose isomorphism class as a k-group only depends on the class of a in $\mathrm{H}^1(k_s/k, \mathrm{Aut}(\mathscr{G}_{k_s}))$. It is an important fact that if a lifts to a 1-cocycle $\widetilde{a} \in \mathrm{Z}^1(k_s/k, \mathscr{G}(k_s))$ then a choice of such $\widetilde{a}$ defines a bijection of sets $t_{\widetilde{a}} : \mathrm{H}^1(k_s/k, {}_a\mathscr{G}) \simeq \mathrm{H}^1(k_s/k, \mathscr{G})$ functorially in the pair $(\mathscr{G}, \widetilde{a})$ and carrying the base point to the class of $\widetilde{a}$ [**Ser97**, I, § 5.3, Prop. 35bis]. This can be described in terms of the more conceptual language of torsors (say for the étale topology, though it is equivalent to use the fppf topology since $\mathscr{G}$ is smooth): if Y is a right $\mathscr{G}$-torsor whose isomorphism class in $\mathrm{H}^1(k, \mathscr{G})$ is represented by $\widetilde{a}$ then ${}_a\mathscr{G}$ is the automorphism scheme $\underline{\mathrm{Aut}}_{\mathscr{G}}(\mathrm{Y})$ of the right $\mathscr{G}$-torsor Y and $t_{\widetilde{a}}^{-1}$ carries the class of a right $\mathscr{G}$-torsor X to the class of the right $\underline{\mathrm{Aut}}_{\mathscr{G}}(\mathrm{Y})$-torsor $\underline{\mathrm{Isom}}(\mathrm{X}, \mathrm{Y})$. See [**Ser97**, I, 5.3] or [**Con11**, App. B] for further details.

It is important to distinguish between the images in $\mathrm{H}^1(k_s/k, \mathrm{Aut}(\mathscr{G}_{k_s}))$ of $\mathrm{H}^1(k, \mathscr{G})$ and $\mathrm{H}^1(k, \mathscr{G}/\mathrm{Z}_{\mathscr{G}})$. The k-groups ${}_a\mathscr{G}$ arising from a in the image of $\mathrm{H}^1(k, \mathscr{G}/\mathrm{Z}_{\mathscr{G}})$ are called *inner forms* of $\mathscr{G}$ (because $\mathscr{G}/\mathrm{Z}_{\mathscr{G}}$ is considered to be the scheme of "inner automorphisms" of $\mathscr{G}$), and the inner forms arising from a in the image of $\mathrm{H}^1(k, \mathscr{G})$ are called *pure inner forms*. In general, many inner forms are not pure inner forms, and the map $\mathrm{H}^1(k, \mathscr{G}) \to \mathrm{H}^1(k, \mathscr{G}/\mathrm{Z}_{\mathscr{G}})$ is neither injective nor surjective. Pure inner forms play an important role in local harmonic analysis and the local Langlands correspondence; see [**GGP**,

§ 2] for some examples with classical groups. We have seen above that passage to a pure inner form does not change the degree-1 Galois cohomology (as a set). The same is not true for passage to general inner forms:

Example 7.2.1. — For $n > 1$ we have $\mathrm{H}^1(k, \mathrm{SL}_n) = 1$ [**Ser79**, X, § 1, Cor.]. Thus, SL_n has no nontrivial pure inner forms. The quotient $\mathrm{SL}_n/\mathrm{Z}_{\mathrm{SL}_n}$ is PGL_n, and $\mathrm{H}^1(k, \mathrm{PGL}_n)$ classifies rank-n^2 central simple k-algebras A. The associated inner forms of SL_n are the k-groups SL(A) (see Exercise 5.5.5(iv)).

For a rank-n^2 central division algebra D over k, we claim that the k-form $\mathrm{SL}_{1,\mathrm{D}} := \mathrm{SL}(\mathrm{D})$ of SL_n can have *nontrivial* degree-1 Galois cohomology (in which case $\mathrm{H}^1(k, \mathrm{SL}_{1,\mathrm{D}})$ has no bijection with $\mathrm{H}^1(k, \mathrm{SL}_n)$). To see this, consider the exact sequence of k-groups

$$1 \to \mathrm{SL}_{1,\mathrm{D}} \to \underline{\mathrm{D}}^\times \overset{\mathrm{Nrd}}{\to} \mathbf{G}_\mathrm{m} \to 1.$$

The pointed set $\mathrm{H}^1(k, \underline{\mathrm{D}}^\times)$ is trivial: it classifies étale sheaves of rank-1 left modules over the quasi-coherent sheaf D on $\mathrm{Spec}\, k$, and the only such object up to isomorphism is D (due to effectivity of étale descent for quasi-coherent sheaves and the freeness of finitely generated D-modules). Thus, $\mathrm{H}^1(k, \mathrm{SL}_{1,\mathrm{D}})$ is trivial if and only if the reduced norm map $\nu_\mathrm{D} : \mathrm{D}^\times \to k^\times$ is surjective. Already with $n = 2$, among quaternion division algebras over $\mathbf{Q}$ one finds many examples where such surjectivity fails. For example, if $\mathrm{D}_\mathbf{R} \simeq \mathbf{H}$ (there are many such D, by class field theory) then ν_D does not hit $\mathbf{Q}_{<0}$. (The surjectivity of the reduced norm map for all D over k is closely related to k being a C_r-field with $r \leqslant 2$; see [**Ser97**, II, 4.5].)

Returning to (7.2.1), twisting against the $\mathrm{Aut}_{\mathrm{G}_0/k}(k_s)$-valued 1-cocycle c and using (7.2.1) defines a bijection of sets

$$\mathrm{H}^1(k, \mathrm{Aut}_{\mathrm{G}/k}) \simeq \mathrm{H}^1(k, \mathrm{Aut}_{\mathrm{G}_0/k}).$$

By construction this carries the trivial point to the class of G in $\mathrm{H}^1(k, \mathrm{Aut}_{\mathrm{G}_0/k})$, and it induces a compatible bijection $\mathrm{H}^1(k, \mathrm{Out}_{\mathrm{G}/k}) \simeq \mathrm{H}^1(k, \mathrm{Out}_{\mathrm{G}_0/k})$ (viewing c with values in the automorphism functor of $\mathrm{Out}_{\mathrm{G}_0/k} = \mathrm{Aut}(\mathrm{R}, \Delta)_k$). That is, we have a commutative diagram of exact sequences of pointed sets

$$\begin{tikzcd} \mathrm{H}^1(k, \mathrm{G}/\mathrm{Z}_\mathrm{G}) \arrow[r] & \mathrm{H}^1(k, \mathrm{Aut}_{\mathrm{G}/k}) \arrow[r] \arrow[d, "\simeq"] & \mathrm{H}^1(k, \mathrm{Out}_{\mathrm{G}/k}) \arrow[d, "\simeq"] \\ \mathrm{H}^1(k, \mathrm{G}_0/\mathrm{Z}_{\mathrm{G}_0}) \arrow[r] & \mathrm{H}^1(k, \mathrm{Aut}_{\mathrm{G}_0/k}) \arrow[r] & \mathrm{H}^1(k, \mathrm{Out}_{\mathrm{G}_0/k}) \end{tikzcd} \tag{7.2.2}$$

We do *not* claim to construct a map between the leftmost terms in (7.2.2) making a commutative square. But the commutativity and exactness imply that the fiber of $\mathrm{H}^1(k, \mathrm{Aut}_{\mathrm{G}_0/k}) \to \mathrm{H}^1(k, \mathrm{Out}_{\mathrm{G}_0/k})$ through [G] is identified with the image of $\mathrm{H}^1(k, \mathrm{G}/\mathrm{Z}_\mathrm{G}) \to \mathrm{H}^1(k, \mathrm{Aut}_{\mathrm{G}/k})$; cf. [**Ser97**, I, § 5.5, Prop. 39, Cor. 2]. The image of $\mathrm{H}^1(k, \mathrm{G}/\mathrm{Z}_\mathrm{G})$ in $\mathrm{H}^1(k, \mathrm{Aut}_{\mathrm{G}/k})$ consists of the k-forms of G obtained by twisting by 1-cocycles valued in $(\mathrm{G}/\mathrm{Z}_\mathrm{G})(k_s)$, which we called the *inner forms* of G. We have proved:

Proposition 7.2.2. — *The fibers of* $\mathrm{H}^1(k, \mathrm{Aut}_{\mathrm{G}_0/k}) \to \mathrm{H}^1(k, \mathrm{Out}_{\mathrm{G}_0/k})$ *are precisely the classes that are inner forms of each other.*

The commutativity of (7.2.2) also gives that $\mathrm{H}^1(k, \mathrm{Aut}_{\mathrm{G}/k}) \to \mathrm{H}^1(k, \mathrm{Out}_{\mathrm{G}/k})$ is *surjective*, since the analogous surjectivity holds for G_0 (due to a k-group section to $\mathrm{Aut}_{\mathrm{G}_0/k} \to \mathrm{Out}_{\mathrm{G}_0/k}$ defined by a choice of pinning).

If k is finite then $\mathrm{H}^1(k, \mathrm{G}/\mathrm{Z}_\mathrm{G})$ is a single point, due to Lang's Theorem [**Bo91**, 16.5(i)]. In equivalent terms, the fibers of $\mathrm{Aut}_{\mathrm{G}/k} \to \mathrm{Out}_{\mathrm{G}/k}$ over k-points are $\mathrm{G}/\mathrm{Z}_\mathrm{G}$-torsors and hence all have a k-point (as $\mathrm{G}/\mathrm{Z}_\mathrm{G}$ is a smooth connected k-group). Let's push this case further:

Example 7.2.3. — Let k be a finite field, and let $(\mathrm{G}_0, \mathrm{T}_0, \mathrm{M})$ be a split connected reductive k-group. Let R be the root datum and Δ a base of the associated root system, so $\Theta := \mathrm{Aut}(\mathrm{R}, \Delta)$ is identified with the constant component group $\mathrm{Out}_{\mathrm{G}_0/k}$ of $\mathrm{Aut}_{\mathrm{G}_0/k}$. The natural surjective map $\mathrm{H}^1(k, \mathrm{Aut}_{\mathrm{G}_0/k}) \to \mathrm{H}^1(k, \Theta_k)$ is bijective because the fiber through each class $[\mathrm{G}]$ is an image of the set $\mathrm{H}^1(k, \mathrm{G}/\mathrm{Z}_\mathrm{G})$ that consists of a single point.

Since Θ_k is a constant group and $\mathrm{Gal}(k_s/k) = \widehat{\mathbf{Z}}$ with generator given by Frob_k, the cohomology set $\mathrm{H}^1(k, \Theta_k)$ consists of conjugacy classes of elements $\theta \in \Theta = \mathrm{Aut}(\mathrm{R}, \Delta)$. Hence, the isomorphism classes of k-forms of G_0 are classified by the conjugacy classes in $\Theta = \mathrm{Aut}(\mathrm{R}, \Delta)$. In more concrete terms, if we fix a pinning $\{\mathrm{X}_a\}_{a \in \Delta}$ and identify $\Theta = \mathrm{Aut}(\mathrm{R}, \Delta)$ with $\mathrm{Aut}(\mathrm{G}_0, \mathrm{T}_0, \mathrm{M}, \{\mathrm{X}_a\}_{a \in \Delta})$ via Theorem 6.1.17 then to each $\theta \in \Theta$ we associate the k-form G_θ of G_0 obtained by replacing the Frob_k-action on $\mathrm{G}_0(k_s)$ with the usual action followed by the action of θ on G_0 via the pinning.

For example, if G_0 is semisimple and either simply connected or adjoint (or Φ is irreducible with cyclic fundamental group) then the set of isomorphism classes of k-forms of G_0 is in bijective correspondence with the set of conjugacy classes in $\mathrm{Aut}(\mathrm{Dyn}(\Phi))$. Since a connected semisimple k-group is split if and only if its simply connected central cover is split, we conclude that a connected semisimple k-group with irreducible root system Φ over k_s is necessarily split whenever $\mathrm{Dyn}(\Phi)$ has no nontrivial automorphisms. This holds for types A_1, B, C, E_7, E_8, F_4, and G_2. (For type A_1, this expresses the fact that $\mathbf{P}^1$ has no nontrivial forms over the finite field k.)

Among the simply connected semisimple k-groups of a fixed type equal to either A_n with $n \geqslant 2$, E_6, or D_n with $n \geqslant 5$, there is exactly one nontrivial k-form since $\mathrm{Aut}(\mathrm{Dyn}(\Phi)) = \mathbf{Z}/2\mathbf{Z}$ for such Φ. For example, in type A_n $(n \geqslant 2)$ a non-split simply connected form is $\mathrm{SU}_{n+1}(k'/k)$, where k'/k is a quadratic extension. With D_4 there are *two* nontrivial k-forms since the group $\mathrm{Aut}(\mathrm{Dyn}(D_4)) = \mathfrak{S}_3$ has two nontrivial conjugacy classes.

Remark 7.2.4. — Let $S = \mathrm{Spec}\, R$ for a henselian local ring R with finite residue field k (e.g., the valuation ring of a non-archimedean local field). The classification in Example 7.2.3 via conjugacy classes in Θ works over S because $\pi_1(S) = \pi_1(\mathrm{Spec}\, k)$ and every smooth surjection $X \to S$ acquires a section over the connected finite étale cover $S' \to S$ corresponding to any finite extension k'/k such that $X_k(k') \neq \varnothing$.

Consider a reductive group G over a non-empty scheme S. The *inner forms* of G are the forms of G that correspond to the image of $H^1(S_{\text{ét}}, G/Z_G)$ in $H^1(S_{\text{ét}}, \mathrm{Aut}_{G/S})$. If all geometric fibers of G have the same root datum R (as occurs when S is connected) and if G_0 denotes the split reductive S-group with root datum R, then the inner forms of G correspond to the fiber through the class [G] of G under the map

$$H^1(S_{\text{ét}}, \mathrm{Aut}_{G_0/S}) \to H^1(S_{\text{ét}}, \mathrm{Out}_{G_0/S}).$$

Indeed, when $S = \mathrm{Spec}\, k$ for a field k this is Proposition 7.2.2, and a variant of the method of proof works in general (Exercise 7.3.8).

Example 7.2.5. — The inner forms of SL_n over a field k are the k-groups SL(A) for central simple k-algebras A of rank n^2 (see Exercise 5.5.5(iv) for SL(A)). Indeed, SL(A) is the image under $f : H^1(k, \mathrm{PGL}_n) \to H^1(k, \mathrm{Aut}_{\mathrm{SL}_n})$ of the class of A in $H^1(k, \mathrm{PGL}_n) \subset \mathrm{Br}(k)[n]$. Since $\ker f = 1$, so $\mathrm{SL}(A) \not\simeq \mathrm{SL}_n$ when A is not split, the uniqueness in Proposition 7.2.12 below implies that SL(A) is not quasi-split over k when A is not a matrix algebra over k.

An automorphism $\phi \in \mathrm{Aut}_{G/S}(S')$ is *inner* if it arises from $(G/Z_G)(S')$ (such an automorphism may not arise from $G(S')$!). Since the S-group $\mathrm{Aut}_{G/S}/(G/Z_G) = \mathrm{Out}_{G/S}$ is étale and separated over S, so an element of $\mathrm{Out}_{G/S}(S)$ is trivial if and only if it is so on geometric fibers over S, an automorphism of G is inner if and only if it is so on geometric fibers over S. Here is an interesting application:

Proposition 7.2.6. — *Let* $G \to S$ *be a reductive group scheme, and* $H \to S$ *a group scheme with connected fibers. Any action of* H *on* G *must be through inner automorphisms; i.e., it is the composition of a unique* S*-homomorphism* $H \to G/Z_G$ *followed by the natural action of* G/Z_G *on* G.

Proof. — To give an action of H on G is to give an S-homomorphism $H \to \mathrm{Aut}_{G/S}$, so we just have to show that the composite map $H \to \mathrm{Out}_{G/S}$ is trivial. Since $\mathrm{Out}_{G/S}$ is locally constant over S, it suffices to work on geometric fibers over S, where the result is clear since each fiber H_s is connected. □

We now discuss the classification of forms in the semisimple case over a general connected non-empty scheme S. Fix a semisimple reduced root datum R, and let G_0 be the split semisimple S-group with root datum R. For any semisimple S-group G whose geometric fibers have root datum R, the simply connected central cover $\widetilde{G}$ is a form of $\widetilde{G}_0$ with root datum R^{sc} equal to the simply connected "cover" of R (using $M = (\mathbf{Z}\Phi^\vee)^*$). Likewise, the adjoint quotient G/Z_G is a form of G_0/Z_{G_0} with root datum R^{ad} equal to the adjoint "quotient" of R (using $M = \mathbf{Z}\Phi$). The kernel of $\widetilde{G}_0 \to G_0/Z_{G_0} = \widetilde{G}_0/Z_{\widetilde{G}_0}$ is the finite multiplicative type group $Z_{\widetilde{G}_0} = D_S(\Pi_0)$ where Π_0 is the "fundamental group" $(\mathbf{Z}\Phi^\vee)^*/\mathbf{Z}\Phi$. The kernel $\mu = \ker(\widetilde{G} \to G) \subset Z_{\widetilde{G}}$ is a form of $\mu_0 := \ker(\widetilde{G}_0 \to G_0) \subset D_S(\Pi_0)$.

The problem of classifying the possibilities for G falls into two parts: classify the forms of $\widetilde{G}_0$, and then for each such form determine if the twisting process applied to $D_S(\Pi_0)$ preserves the subgroup μ_0. This second part is always affirmative when Π_0 is cyclic, such as for irreducible Φ not of type D_{2n}. Also, preservation of μ_0 can be studied on a single geometric fiber and is always a purely combinatorial problem since the action of the S-subgroup $G_0/Z_{G_0} = \widetilde{G}_0/Z_{\widetilde{G}_0} \subset \mathrm{Aut}_{\widetilde{G}_0/S}$ has no effect on the center of $\widetilde{G}_0$.

It follows that we lose little of the real content of the classification problem in the semisimple case by focusing on simply connected G, so now consider such G. The root datum decomposes as a direct product according to the irreducible components of the root system Φ. Setting aside the combinatorial problem of permutations of irreducible components of Φ in the twisting process (handled in practice using Weil restriction through a suitable finite étale covering), we likewise lose little generality by assuming that Φ is irreducible. Then the automorphism group of the based root datum coincides with the automorphism group of the Dynkin diagram (i.e., no problems arise for D_{2n} as in Example 1.5.2), so we obtain:

Corollary 7.2.7. — *Let Φ be an irreducible reduced root system, and* S *a connected non-empty scheme. The set of isomorphism classes of simply connected and semisimple* S*-groups with root system Φ on geometric fibers is in natural bijection with*

$$\mathrm{H}^1(\mathrm{S}_{\acute{e}t}, (\mathrm{G}_0/\mathrm{Z}_{\mathrm{G}_0}) \rtimes \mathrm{Aut}(\mathrm{Dyn}(\Phi))_{\mathrm{S}}), \tag{7.2.3}$$

where $(G_0, T_0, M, \{X_a\}_{a\in\Delta})$ is the pinned split simply connected and semisimple S*-group with root system Φ.*

The cohomological description of forms via (7.2.3) is useful in multiple ways. Firstly, if we make many constructions of forms then the cohomological viewpoint can be helpful for proving that all possibilities have been exhausted. Secondly, cohomology provides an efficient mechanism for understanding the conceptual meaning of invariants that enter into a classification theorem, such as auxiliary Galois extensions that occur in a construction (e.g., the local invariants that arise in the Hasse–Minkowski theorem for non-degenerate quadratic forms over global fields).

Here is an example that illustrates the usefulness of the fact that the automorphism scheme of a semisimple group scheme is smooth and affine.

Example 7.2.8. — Let F be a global field, and Σ a non-empty finite set of places of F containing the archimedean places. The étale cohomology set $\mathrm{H}^1(\mathscr{O}_{\mathrm{F},\Sigma}, \mathscr{G})$ is finite for any smooth affine $\mathscr{O}_{\mathrm{F},\Sigma}$-group $\mathscr{G}$ with reductive fibers such that the order of the component group of $\mathscr{G}_{\overline{\mathrm{F}}}$ is not divisible by char(F) [**GM**, 5.1, § 7]. (The hypothesis of reductive fibers can easily be removed when $\mathrm{char(F)} = 0$, but to do so when $\mathrm{char(F)} > 0$ requires a local-global finiteness result [**Con11**, 1.3.3(i)] which rests on the structure theory of pseudo-reductive groups.) In particular, if G_0 is a split semisimple $\mathscr{O}_{\mathrm{F},\Sigma}$-group whose root system Φ is irreducible then $\mathrm{H}^1(\mathscr{O}_{\mathrm{F},\Sigma}, \mathrm{Aut}_{\mathrm{G}_0/\mathscr{O}_{\mathrm{F},\Sigma}})$ is finite provided that char(F) does not divide the order of $\mathrm{Aut}(\mathrm{Dyn}(\Phi))$. By inspecting the list of Dynkin diagrams, the restriction on char(F) only arises when $\mathrm{char(F)} \in \{2, 3\}$.

[This restriction on the characteristic is genuine: when G_0 is simply connected and its connected Dynkin diagram has an automorphism of order $p = \mathrm{char(F)} \in \{2, 3\}$ then the infinitely many degree-p Artin–Schreier extensions of F unramified outside Σ give rise to infinitely many pairwise non-isomorphic $\mathscr{O}_{\mathrm{F},\Sigma}$-forms of G_0. There is a similar infinitude phenomenon over local function fields of characteristic 2 or 3. Examples include special unitary groups associated to quadratic Galois extensions F'/F in characteristic 2.]

We conclude that as long as $\mathrm{char(F)} \neq 2, 3$, up to isomorphism there are only *finitely many* semisimple $\mathscr{O}_{\mathrm{F},\Sigma}$-groups with a given irreducible root datum over $\overline{\mathrm{F}}$. Beware that this is *not* saying anything about the number of (isomorphism classes of) connected semisimple F-groups arising as generic fibers of $\mathscr{O}_{\mathrm{F},\Sigma}$-forms of G_0. Indeed, since $\mathrm{Aut}_{\mathrm{G}_0/\mathscr{O}_{\mathrm{F},\Sigma}}$ is not $\mathscr{O}_{\mathrm{F},\Sigma}$-proper, if $\mathscr{G}$ and $\mathscr{G}'$ are $\mathscr{O}_{\mathrm{F},\Sigma}$-forms of G_0 then the Isom-scheme $\mathrm{Isom}(\mathscr{G}, \mathscr{G}')$ is not $\mathscr{O}_{\mathrm{F},\Sigma}$-proper and hence may have an F-point that does not extend to an $\mathscr{O}_{\mathrm{F},\Sigma}$-point. This is illustrated in the next example.

Example 7.2.9. — Let Φ be an irreducible and reduced root system such that $\mathrm{Aut}(\mathrm{Dyn}(\Phi)) = 1$ (i.e., types A_1, B, C, E_7, E_8, F_4, G_2). Let G_0 be the split simply connected semisimple S-group with root system Φ. The cohomology set in Corollary 7.2.7 is $\mathrm{H}^1(\mathrm{S}_{\text{ét}}, \mathrm{G}_0/\mathrm{Z}_{\mathrm{G}_0})$, so we get an exact sequence of pointed sets

$$\mathrm{H}^1(\mathrm{S}, \mathrm{Z}_{\mathrm{G}_0}) \to \mathrm{H}^1(\mathrm{S}, \mathrm{G}_0) \to \mathrm{H}^1(\mathrm{S}, \mathrm{Aut}_{\mathrm{G}_0/\mathrm{S}}) \to \mathrm{H}^2(\mathrm{S}, \mathrm{Z}_{\mathrm{G}_0})$$

(using the fppf topology if Z_{G_0} is not smooth; see Exercise 7.3.4 for the effect on $H^1(S, G_0)$ and $H^1(S, \mathrm{Aut}_{G_0/S})$ when passing from the étale topology to the fppf topology). For example, if $S = \mathrm{Spec}\,\mathbf{Z}$ then Z_{G_0} is a product of various μ_n's and $H^2(\mathbf{Z}, \mu_n) = H^2(\mathbf{Z}, \mathbf{G}_m)[n]$ (using fppf cohomology). But $H^2(\mathbf{Z}, \mathbf{G}_m)$ is the same whether we use fppf or étale topologies [**Gr68**, 11.7], so since $\mathbf{Z}$ has only one archimedean place, the group $H^2(\mathbf{Z}, \mathbf{G}_m)$ vanishes by global class field theory (see [**Mi80**, III, Ex. 2.22(f)]). Likewise, since $\mathrm{Pic}(\mathbf{Z}) = 1$ and $\mathbf{Z}^\times = \{\pm 1\}$, we have $H^1(\mathbf{Z}, Z_{G_0}) = \Pi_0/(2)$ and

$$H^1(\mathbf{Z}, \mathrm{Aut}_{G_0/S}) = H^1(\mathbf{Z}, G_0)/H^1(\mathbf{Z}, Z_{G_0})$$

(the right side denotes the pointed quotient set arising from the translation action on G_0 by its central subgroup scheme Z_{G_0}). One can give a Weil-style adelic description of $H^1(\mathbf{Z}, G_0)$ when partitioned according to the isomorphism class of the $\mathbf{Q}$-fiber; see [**Con14**, Rem. 7.1] and references therein for $\mathbf{Z}$-models with an $\mathbf{R}$-anisotropic $\mathbf{Q}$-fiber.

The case $S = \mathrm{Spec}\,K$ for a global or non-archimedean local field K works out nicely: since G_0 is simply connected, $H^1(K, G_0)$ is rather small and entirely understood (by work of Kneser, Harder, Bruhat–Tits, and Chernousov), and the contribution from $H^2(K, Z_{G_0}) \subset \mathrm{Br}(K)$ is well-understood. This illustrates the general fact that the classification of connected semisimple groups over a field is intimately tied up with the Galois cohomological properties of the field (e.g., the structure of the Brauer group).

To prove more definitive classification results over a field, especially over arithmetically interesting fields, an entirely different approach is required. One has to use the finer structure theory of Borel–Tits that involves rational conjugacy for maximal k-split tori and minimal parabolic k-subgroups, relative root systems, and the"$*$-action" of $\mathrm{Gal}(k_s/k)$ on the Dynkin diagram (see Remark 7.1.2). This gives a classification of k-groups "modulo the k-anisotropic groups" (whose structure depends on special features of k).

As we noted in Remark 7.1.2, the $*$-action of $\mathrm{Gal}(k_s/k)$ on the diagram can be refined to an action on the based root datum $(R(G), \Delta)$. If $(G_0, T_0, M_0, \{X_a\}_{a \in \Delta_0})$ denotes the pinned split connected reductive k-group with $(R(G_0), \Delta_0) = (R(G), \Delta)$ then the homomorphism

$$\mathrm{Gal}(k_s/k) \to \mathrm{Aut}((R(G), \Delta)) = \mathrm{Out}_{G_0/k}(k)$$

defines a class in $H^1(k, \mathrm{Out}_{G_0/k})$. Using Proposition 7.2.2, the image of this class under the natural section to $H^1(k, \mathrm{Aut}_{G_0/k}) \to H^1(k, \mathrm{Out}_{G_0/k})$ is the quasi-split inner form of G, in view of the proof of uniqueness of this inner form in Proposition 7.2.12. See the tables at the end of [**Ti66a**] and [**Spr**] for the classification of k-forms for general as well as special fields k. (Examples without this extra technology are also discussed in [**Ser97**, III, 1.4, 2.2–2.3, 3.2] and [**Ser97**, III, App. 2, § 3].) See [**PS**] for a relativization over arbitrary semilocal rings.

Example 7.2.10. — The semi-direct product structure of $\mathrm{Aut}_{\mathrm{G}_0/\mathrm{S}}$ in Theorem 7.1.9(3) is generally destroyed under passage to a form of G_0 that is not quasi-split (Example 7.1.12), and this creates difficulties in any attempt to explicitly describe the degree-1 cohomology. For example, if $\mathrm{G}_0 = \mathrm{SL}_n$ over a field k with $n > 2$, the exact sequence

$$1 \to \mathrm{PGL}_n \to \mathrm{Aut}_{\mathrm{SL}_n/k} \to \mathbf{Z}/2\mathbf{Z} \to 1$$

induces an exact sequence of pointed sets

$$\mathrm{H}^1(k, \mathrm{PGL}_n) \to \mathrm{H}^1(k, \mathrm{Aut}_{\mathrm{SL}_n/k}) \to \mathrm{H}^1(k, \mathbf{Z}/2\mathbf{Z}). \tag{7.2.4}$$

The map on the right must be surjective, since the automorphism scheme splits as a semi-direct product over k, and likewise the map on the left has trivial kernel (as a map of pointed sets) though it is not generally injective (Example 7.1.12). In Example 7.2.5 we addressed the image of $\mathrm{H}^1(k, \mathrm{PGL}_n)$.

For any form G of SL_n there is an associated class in $\mathrm{H}^1(k, \mathbf{Z}/2\mathbf{Z})$ via (7.2.4). This class is trivial precisely when G comes from $\mathrm{H}^1(k, \mathrm{PGL}_n)$, which is to say that it is one of the norm-1 unit groups as in Example 7.2.5. Let us focus on the case when the class in $\mathrm{H}^1(k, \mathbf{Z}/2\mathbf{Z})$ is nontrivial, so it corresponds to a quadratic separable extension field k'/k. The fiber in $\mathrm{H}^1(k, \mathrm{Aut}_{\mathrm{SL}_n/k})$ over the class $[k']$ of k'/k in $\mathrm{H}^1(k, \mathbf{Z}/2\mathbf{Z})$ contains a unique quasi-split form G. We wish to describe it explicitly (and then its inner forms will exhaust the entire fiber, by Proposition 7.2.2). The group $\mathrm{G}_{k'}$ is identified with SL_n in such a way that for any commutative k-algebra A, the subgroup

$$\mathrm{G}(\mathrm{A}) \subset \mathrm{G}(k' \otimes_k \mathrm{A}) = \mathrm{SL}_n(k' \otimes_k \mathrm{A}) = \mathrm{R}_{k'/k}(\mathrm{SL}_n)(\mathrm{A})$$

consists of the points $g' \in \mathrm{SL}_n(k' \otimes_k \mathrm{A})$ satisfying $g' = \iota(\overline{g}')$ where ι is as in Example 7.1.10 and $c' \mapsto \overline{c}'$ is the nontrivial k-automorphism of k'. Since ι is an involution, for w as in Example 7.1.5 the condition $g' = \iota(\overline{g}')$ says exactly

$$\overline{g}' = \iota(g') = w({g'}^{\top})^{-1}w^{-1} = w^{-1}({g'}^{\top})^{-1}w$$

(the final equality uses that $w^{-1} = \pm w$). This means that g' preserves the non-degenerate $(-1)^{n+1}$-hermitian pairing $h : {k'}^n \times {k'}^n \to k'$ defined by $h(x', y') = {x'}^{\top} w \overline{y}'$, so $\mathrm{G} = \mathrm{SU}(h)$ (see Exercise 4.4.5). Note that h has an isotropic subspace of the maximal possible dimension, $\lfloor n/2 \rfloor$, due to the matrix for w.

The k-group $\mathrm{G}' = \mathrm{SU}_n(k'/k)$ defined using the hermitian $\mathrm{H}(x', y') = {x'}^{\top}\overline{y}'$ on ${k'}^n$ is an inner form of G. Detecting when G' is distinct from G (equivalently: quasi-split) is an arithmetic problem. For example, if k is finite then $\mathrm{G}' \simeq \mathrm{G}$ (all connected semisimple groups over finite fields are quasi-split) whereas if $k = \mathbf{R}$ then $\mathrm{G}' \not\simeq \mathrm{G}$ (since $\mathrm{G}'(\mathbf{R})$ is compact).

It is a very useful fact (for the classification of forms and other purposes) that over any non-empty scheme S every reductive S-group G has an inner form that is quasi-split (see Definition 5.2.10). Put another way, every G can be obtained from a quasi-split reductive S-group via inner twisting:

Proposition 7.2.11. — *Let* S *be a non-empty scheme,* $(\mathrm{G}_0, \mathrm{T}_0, \mathrm{M})$ *a split reductive* S*-group with root datum* R*, and* G *a reductive* S*-group whose geometric fibers have root datum* R*. There exists a quasi-split inner form* G' *of* G *such that* $\mathrm{Aut}_{\mathrm{G}'/\mathrm{S}} \to \mathrm{Out}_{\mathrm{G}'/\mathrm{S}}$ *has an* S*-group section, some Borel subgroup* $\mathrm{B}' \subset \mathrm{G}'$ *contains a maximal torus* T' *of* G'*, and* $\mathrm{T}'/\mathrm{Z}_{\mathrm{G}'} \simeq \mathrm{R}_{\mathrm{S}'/\mathrm{S}}(\mathbf{G}_{\mathrm{m}})$ *for a finite étale cover* $\mathrm{S}' \to \mathrm{S}$*.*

Proof. — Let Δ be the base of a positive system of roots $\Phi^+ \subset \Phi$, and let B_0 be the Borel subgroup containing T_0 that corresponds to Φ^+. Define $\Theta = \mathrm{Aut}(\mathrm{R}, \Delta)$. Choose a pinning $\{\mathrm{X}_a\}_{a \in \Delta}$; this defines an S-group section to $\mathrm{Aut}_{\mathrm{G}_0/\mathrm{S}} \to \mathrm{Out}_{\mathrm{G}_0/\mathrm{S}} \simeq \Theta_{\mathrm{S}}$ which (by construction) lands inside the subgroup scheme of automorphisms of G_0 that *preserve* T_0 and B_0.

Consider the cohomology class $[\mathrm{G}] \in \mathrm{H}^1(\mathrm{S}_{\text{ét}}, \mathrm{Aut}_{\mathrm{G}_0/\mathrm{S}})$ classifying G. If ξ denotes its image in $\mathrm{H}^1(\mathrm{S}_{\text{ét}}, \Theta_{\mathrm{S}})$, then the section $\Theta \to \mathrm{Aut}(\mathrm{G}_0)$ defined by the pinning carries ξ to a class $[\mathrm{G}'] \in \mathrm{H}^1(\mathrm{S}_{\text{ét}}, \mathrm{Aut}_{\mathrm{G}_0/\mathrm{S}})$ in the fiber over ξ. By Exercise 7.3.8, all classes in the same fiber of the map $\mathrm{H}^1(\mathrm{S}_{\text{ét}}, \mathrm{Aut}_{\mathrm{G}_0/\mathrm{S}}) \to \mathrm{H}^1(\mathrm{S}_{\text{ét}}, \mathrm{Out}_{\mathrm{G}_0/\mathrm{S}})$ are inner forms of each other, so G' is an inner form of G. Since the class of G' in $\mathrm{H}^1(\mathrm{S}_{\text{ét}}, \mathrm{Aut}_{\mathrm{G}_0/\mathrm{S}})$ is represented by a 1-cocycle with values in the subgroup of $\mathrm{Aut}_{\mathrm{G}_0/\mathrm{S}}$ that preserves T_0 and B_0, G' admits a maximal torus T' and Borel subgroup B' containing T'. This cocycle is valued in Θ_{S}, so the twisting process that constructs $\mathrm{Aut}_{\mathrm{G}'/\mathrm{S}}$ as a form of $\mathrm{Aut}_{\mathrm{G}_0/\mathrm{S}}$ also twists Θ_{S} into an S-subgroup of $\mathrm{Aut}_{\mathrm{G}'/\mathrm{S}}$ that maps isomorphically onto the twist $\mathrm{Out}_{\mathrm{G}'/\mathrm{S}}$ of $\mathrm{Out}_{\mathrm{G}_0/\mathrm{S}}$.

To describe the maximal torus $\mathrm{T}'/\mathrm{Z}_{\mathrm{G}'}$ in the adjoint quotient $\mathrm{G}'/\mathrm{Z}_{\mathrm{G}'}$, we may pass to the case when G_0 is of adjoint type (due to the functoriality of the formation of adjoint central quotients). Thus, the twisting process is done against a cocycle representing some $\xi \in \mathrm{H}^1(\mathrm{S}_{\text{ét}}, \Theta_{\mathrm{S}})$ for $\Theta = \mathrm{Aut}(\mathrm{Dyn}(\Phi)) \subset \mathrm{Aut}(\Delta)$. Since $\mathrm{T}_0 = \prod_{a \in \Delta} \mathbf{G}_{\mathrm{m}}$ via $t \mapsto (a(t))$ (as G_0 is adjoint), we have $\mathrm{T}' = \mathrm{R}_{\mathrm{S}'/\mathrm{S}}(\mathbf{G}_{\mathrm{m}})$ where $\mathrm{S}' \to \mathrm{S}$ is the twist of $\mathrm{S} \times \Delta$ corresponding to the image of ξ in $\mathrm{H}^1(\mathrm{S}_{\text{ét}}, \mathrm{Aut}(\Delta))$. □

When S is semi-local, we have a uniqueness result:

Proposition 7.2.12. — *For semi-local* S*, up to isomorphism every reductive* S*-group* G *admits a unique quasi-split inner form* G'*.*

Proof. — We may assume G admits a Borel subgroup B satisfying the properties in Proposition 7.2.11. Letting G' be a quasi-split inner form of G, with B' a Borel subgroup, we seek to prove that $G' \simeq G$. Consider a class $\xi \in H^1(S_{ét}, G/Z_G)$ which twists G into G'. Since Borel subgroups are conjugate étale-locally on the base (Corollary 5.2.13), we can arrange that ξ is valued in $N_{G/Z_G}(B/Z_G) = B/Z_G$. Hence, it suffices to prove that $H^1(S_{ét}, B/Z_G) = 1$.

By Theorem 5.4.3, the unipotent radical $U = \mathscr{R}_u(B/Z_G) \simeq \mathscr{R}_u(B)$ has a composition series whose successive quotients are vector groups, and the torus $(B/Z_G)/U$ has the form $R_{S'/S}(\mathbf{G}_m)$ for a finite étale cover $S' \to S$ (so S' is semi-local). The vanishing of $H^1(S_{ét}, B/Z_G)$ is therefore reduced to the vanishing of $H^1(S_{ét}, \mathscr{V})$ for vector groups $\mathscr{V}$ and the vanishing of $H^1(S_{ét}, R_{S'/S}(\mathbf{G}_m))$. The higher vector group cohomology vanishes because it agrees with the Zariski cohomology and S is affine. The Weil restriction through S'/S is a finite pushforward for the étale sites, so by a degenerating Leray spectral sequence

$$H^1(S_{ét}, R_{S'/S}(\mathbf{G}_m)) \simeq H^1(S'_{ét}, \mathbf{G}_m) = \mathrm{Pic}(S')$$

(the final equality by descent theory). Since S' is semi-local, $\mathrm{Pic}(S') = 1$. □

Remark 7.2.13. — The only role of semi-locality for the affine S in the proof of Proposition 7.2.12 is to ensure that every finite étale cover of S has trivial Picard group. Thus, the conclusion also holds (with the same proof) for $S = \operatorname{Spec} \mathbf{Z}$ and $S = \mathbf{A}^n_k$ for a field k of characteristic 0.

As a final application of our work with the degree-1 cohomology of automorphism schemes, we establish a useful result in the structure theory of reductive groups over local fields. Let K be a non-archimedean local field, and R its ring of integers. Let G be a reductive group scheme over $S = \operatorname{Spec} R$. Bruhat–Tits theory shows that the compact open subgroup G(R) in G(K) is maximal as a compact subgroup; these are called *hyperspecial* maximal compact subgroups of G(K). Such a subgroup of the topological group G(K) is defined in terms of a specified reductive R-group model of the K-group G_K, and so one may wonder if there are several such R-models of G_K for which the associated compact open subgroups are not related through G(K)-conjugacy.

There are generally several G(K)-conjugacy classes of such subgroups, due to the fact that $G(K) \to G^{ad}(K)$ may not be surjective, where $G^{ad} := G/Z_G$ is the adjoint central quotient of G. To understand this, consider an automorphism $f : G_K \simeq G_K$ and the (maximal) compact open subgroup $f(G(R))$ in G(K). Is this G(K)-conjugate to G(R)? By Theorem 7.1.9, $\mathrm{Aut}_{G/R}$ has open and closed relative identity component G^{ad} with $\mathrm{Out}_{G/R} := \mathrm{Aut}_{G/R}/G^{ad}$ a disjoint union of finite étale R-schemes. In particular, $\mathrm{Out}_{G/R}(R) = \mathrm{Out}_{G/R}(K)$. The map $\mathrm{Aut}(G) \to \mathrm{Out}_{G/R}(R)$ is surjective because the obstruction to surjectivity lies in the cohomology set $H^1(S_{ét}, G^{ad})$ that vanishes (due to Lang's theorem over the finite residue field and the smoothness of G^{ad}-torsors over the henselian local R).

Thus, the image of $f \in \mathrm{Aut}(\mathrm{G_K})$ in $\mathrm{Out}_{\mathrm{G_K/K}}(\mathrm{K}) = \mathrm{Out}_{\mathrm{G/R}}(\mathrm{R})$ lifts to $\mathrm{Aut}(\mathrm{G})$, so at the cost of pre-composing f with an R-automorphism of G (which is harmless) we may arrange that f has trivial image in $\mathrm{Out}_{\mathrm{G_K/K}}(\mathrm{K})$; i.e., f arises from the action of a point $\overline{g} \in \mathrm{G^{ad}(K)}$. Whether or not $f(\mathrm{G(R)})$ is G(K)-conjugate to G(R) depends only on the image of $\overline{g}$ in $\mathrm{G(K)}\backslash\mathrm{G^{ad}(K)}/\mathrm{G^{ad}(R)}$. In this split case the obstruction to G(K)-conjugacy can be made rather explicit:

Proposition 7.2.14. — *If* G *is* R*-split then the "$\overline{g}$-conjugate" of* G(R) *inside* G(K) *is* not G(K)*-conjugate to* G(R) *whenever* $\overline{g} \notin \mathrm{G(K)G^{ad}(R)}$ *inside* $\mathrm{G^{ad}(K)}$*. Equivalently,*

$$\mathrm{G^{ad}(K)/G(K)G^{ad}(R)}$$

labels the set of G(K)*-conjugacy classes within the* $\mathrm{G^{ad}(K)}$*-orbit of* G(R).

Proof. — Assume $f(\mathrm{G(R)}) = h\mathrm{G(R)}h^{-1}$ for some $h \in \mathrm{G(K)}$. Letting $\overline{h} := h \bmod (\mathrm{Z_G})_{\mathrm{K}}$, we see that the action of $\overline{h}^{-1}\overline{g} \in \mathrm{G^{ad}(K)}$ carries G(R) onto itself. Letting T be a split maximal R-torus of $\mathrm{G^{ad}}$, the Iwasawa decomposition (from the split case of Bruhat–Tits theory, due to Iwahori and Matsumoto) says $\mathrm{G^{ad}(K)} = \mathrm{G^{ad}(R)T(K)G^{ad}(R)}$, so

$$\overline{h}^{-1}\overline{g} = \gamma t \gamma'$$

for $\gamma, \gamma' \in \mathrm{G^{ad}(R)}$ and $t \in \mathrm{T(K)}$. Thus, t-conjugation on G(K) carries G(R) onto itself.

For each root $b \in \Phi(\mathrm{G^{ad}}, \mathrm{T})$, the root group scheme $\mathrm{U}_b \simeq \mathbf{G}_{\mathrm{a}}$ over R is preserved by the T-action. More precisely, scaling by $b(t) \in \mathrm{K}^\times$ on $\mathrm{U}_b(\mathrm{K}) = \mathrm{K}$ carries $\mathrm{U}_b(\mathrm{R}) = \mathrm{R}$ onto itself, so $b(t) \in \mathrm{R}^\times$. The roots span the cocharacter group of the R-split T since $\mathrm{G^{ad}}$ is of adjoint type, so $t \in \mathrm{T(R)}$ and hence $\overline{h}^{-1}\overline{g} \in \mathrm{G^{ad}(R)}$. In other words, $\overline{g} \in \mathrm{G(K)G^{ad}(R)}$. $\square$

Example 7.2.15. — Consider $\mathrm{G} = \mathrm{SL}_n$ with $n > 1$, so

$$\mathrm{G^{ad}(K)/G(K)G^{ad}(R)} = \mathrm{K}^\times/(\mathrm{K}^\times)^n\mathrm{R}^\times = \mathbf{Z}/n\mathbf{Z}.$$

In this case the Iwasawa decomposition used in the proof of the preceding proposition is entirely elementary, and the proposition provides (at least) n distinct conjugacy classes of hyperspecial maximal compact subgroups of $\mathrm{SL}_n(\mathrm{K})$. In concrete terms, consider the R-group SL_n identified as an R-structure of the K-group SL_n by composing the natural generic fiber identification with conjugation by elements of $\mathrm{PGL}_n(\mathrm{K})$ whose "determinants" in $\mathrm{K}^\times/(\mathrm{K}^\times)^n$ have "valuations mod n" that vary through all classes in $\mathbf{Z}/n\mathbf{Z}$. The subgroups of $\mathrm{SL}_n(\mathrm{K})$ arising from the R-points of these various R-structures are pairwise *non-conjugate*, but by construction they are all related to each other through the action on $\mathrm{SL}_n(\mathrm{K})$ by the subgroup $\mathrm{PGL}_n(\mathrm{K})$ of $\mathrm{Aut}(\mathrm{SL}_n)$.

In general, without an R-split hypothesis on G, we shall next prove that the failure of hyperspecial maximal compact subgroups to be G(K)-conjugate is entirely explained by the gap between $G^{ad}(K)$ and G(K). We do not make this more precise (i.e., prove a version of Proposition 7.2.14 without a split hypothesis), as that gets involved with the general Iwasawa decomposition for G(K) and so requires more substantial input from Bruhat–Tits theory when G is not R-split.

Let G and G′ be reductive R-groups. For an isomorphism $f : G_K \simeq G'_K$ between their generic fibers (so G(R) and G′(R) may be viewed as subgroups of G(K)), what is the obstruction to extending f to an R-group isomorphism $G \simeq G'$? Obviously we wish to permit at least the ambiguity of G(K)-conjugation on the source (permitting the same on the target is superfluous, as f induces a bijection between K-points), and the preceding considerations suggest that it is more natural to permit the ambiguity of the action of $G^{ad}(K)$ on the source. We aim to prove:

Theorem 7.2.16. — *By composing f with the action of a suitable element of $G^{ad}(K)$, it extends to an R-group isomorphism $G \simeq G'$. Equivalently, the natural map*

$$\mathrm{Isom}(G, G') \to \mathrm{Isom}(G_K, G'_K)/G^{ad}(K)$$

is surjective. In particular, G and G′ are abstractly isomorphic.

Proof. — The key point is first to prove the final assertion in the theorem: the R-groups G and G′ are abstractly isomorphic. Since G and G′ are R-forms for the étale topology, the R-isomorphism class of G′ is classified by an element in the pointed set $H^1(R_{ét}, \mathrm{Aut}_{G/R})$, where the automorphism scheme $\mathrm{Aut}_{G/R}$ fits into a short exact sequence (for the étale topology)

$$1 \to G^{ad} \to \mathrm{Aut}_{G/R} \to \mathrm{Out}_{G/R} \to 1.$$

The R-scheme $\mathrm{Out}_{G/R}$ becomes constant over a finite étale cover of R, so every $\mathrm{Out}_{G/R}$-torsor E for the étale topology over R becomes constant over an étale cover of R and hence is a disjoint union of finite étale R-schemes (by the argument given in the proof of the final assertion in Theorem 7.1.9(1)). Thus, E(R) = E(K), so the natural restriction map $H^1(R_{ét}, \mathrm{Out}_{G/R}) \to H^1(K_{ét}, \mathrm{Out}_{G_K/K})$ has trivial kernel.

The class $[G'] \in H^1(R_{ét}, \mathrm{Aut}_{G/R})$ has trivial image in $H^1(K_{ét}, \mathrm{Aut}_{G_K/K})$, so its image in $H^1(R_{ét}, \mathrm{Out}_{G/R})$ is also trivial due to the commutative diagram

$$\begin{array}{ccc} H^1(R_{ét}, \mathrm{Aut}_{G/R}) & \longrightarrow & H^1(R_{ét}, \mathrm{Out}_{G/R}) \\ \downarrow & & \downarrow \\ H^1(K_{ét}, \mathrm{Aut}_{G_K/K}) & \longrightarrow & H^1(K_{ét}, \mathrm{Out}_{G_K/K}) \end{array}$$

It follows that $[\mathrm{G}']$ lies in the image of $\mathrm{H}^1(\mathrm{R}_{\text{ét}}, \mathrm{G}^{\text{ad}})$. But this latter cohomology set is trivial by Lang's theorem and smoothness considerations (since R is henselian with finite residue field). Hence, G and G' are R-isomorphic.

Now we can rename G' as G and recast our problem in terms of K-automorphisms of G_{K}: we claim that

$$\mathrm{Aut}(\mathrm{G})\mathrm{G}^{\text{ad}}(\mathrm{K}) = \mathrm{Aut}(\mathrm{G}_{\mathrm{K}}).$$

Pick $f \in \mathrm{Aut}(\mathrm{G}_{\mathrm{K}})$, so its image in $\mathrm{Out}_{\mathrm{G}_{\mathrm{K}}/\mathrm{K}}(\mathrm{K}) = \mathrm{Out}_{\mathrm{G}/\mathrm{R}}(\mathrm{R})$ lifts to some $\mathrm{F} \in \mathrm{Aut}(\mathrm{G})$. Thus, $\mathrm{F}_{\mathrm{K}}^{-1} \circ f$ has trivial image in $\mathrm{Out}_{\mathrm{G}_{\mathrm{K}}/\mathrm{K}}(\mathrm{K})$ and hence arises from $\mathrm{G}^{\text{ad}}(\mathrm{K})$. This says $f \in \mathrm{Aut}(\mathrm{G})\mathrm{G}^{\text{ad}}(\mathrm{K})$. □

As an illustration, we conclude that the number of $\mathrm{SL}_n(\mathrm{K})$-conjugacy classes of hyperspecial maximal compact subgroups in $\mathrm{SL}_n(\mathrm{K})$ (relative to the K-group structure SL_n) is *exactly* n, with $\mathrm{PGL}_n(\mathrm{K})$ acting transitively on this set. More generally, it follows from the theorem that if G is a split reductive R-group then the set of G(K)-conjugacy classes of hyperspecial maximal compact subgroups in G(K) is acted upon transitively by $\mathrm{G}^{\text{ad}}(\mathrm{K})$, with $\mathrm{G}^{\text{ad}}(\mathrm{K})/\mathrm{G}(\mathrm{K})\mathrm{G}^{\text{ad}}(\mathrm{R})$ naturally labeling the set of these G(K)-conjugacy classes (by considering the $\mathrm{G}^{\text{ad}}(\mathrm{K})$-orbit of the hyperspecial maximal compact subgroup $\mathrm{G}(\mathrm{R}) \subset \mathrm{G}(\mathrm{K})$).

7.3. Exercises. —

Exercise 7.3.1. — Let T be a group of multiplicative type over a scheme S. Prove that the automorphism functor $\underline{\mathrm{Aut}}_{\mathrm{T/S}}$ on S-schemes is represented by a separated étale S-group, and that if $\mathrm{T} = \mathrm{D_S(M)}$ for a finitely generated **Z**-module M then this functor is represented by the constant S-group associated to $\mathrm{Aut(M)^{opp}}$.

Deduce that if T is normal in an S-group G with connected fibers then T is central in G. (If $\mathrm{G} \to \mathrm{S}$ is smooth and affine then Theorem 2.3.1 yields another proof of this fact.)

Exercise 7.3.2. — Let G be a smooth group scheme over a field k.

(i) Prove that the image of $\mathrm{G}(k)$ in $(\mathrm{G/Z_G})(k)$ is a normal subgroup and that the quotient by this image is a subgroup of the commutative fppf cohomology group $\mathrm{H}^1(k, \mathrm{Z_G})$ (so $(\mathrm{G/Z_G})(k)/\mathrm{G}(k)$ is *commutative*). Make this explicit when $\mathrm{G} = \mathrm{SL}_n$, and see Exercise 7.3.4.

(ii) The G-action on itself through conjugation factors through an action by $\mathrm{G/Z_G}$ on G. Explain how this identifies $(\mathrm{G/Z_G})(k)$ with a subgroup of Aut(G) even when $(\mathrm{G/Z_G})(k)$ contains points not lifting to $\mathrm{G}(k)$. Make it explicit for $\mathrm{G} = \mathrm{SL}_n$ via $\mathrm{SL}_n \hookrightarrow \mathrm{GL}_n$.

(iii) Show that the k-automorphisms of G arising from $(\mathrm{G/Z_G})(k)$ are precisely the k-automorphisms which become inner on $\mathrm{G}(\overline{k})$. Deduce that $(\mathrm{G/Z_G})(k)$ is normal in Aut(G), and over every separably closed field $k \neq \overline{k}$ give an example in which the action on $\mathrm{G}(k)$ by some element of $(\mathrm{G/Z_G})(k)$ is not inner in the sense of abstract group theory.

Exercise 7.3.3. — Let G be a reductive group over a connected scheme S, with R the common root datum of the geometric fibers.

(i) Explain in terms of descent theory why the set of isomorphism classes of reductive S-groups with root datum R on geometric fibers is in natural bijection with the set $\mathrm{H}^1(\mathrm{S_{\acute{e}t}}, \mathrm{Aut_{G/S}})$. The classes arising from $\mathrm{H}^1(\mathrm{S_{\acute{e}t}}, \mathrm{G/Z_G})$ are called *inner forms* of G.

(ii) Let $\mathrm{G_0}$ be the unique split reductive S-group with root datum R. Prove that the set of (isomorphism classes of) inner forms of G is identified with the fiber through the class of G for $h : \mathrm{H}^1(\mathrm{S_{\acute{e}t}}, \mathrm{Aut_{G_0/S}}) \to \mathrm{H}^1(\mathrm{S_{\acute{e}t}}, \mathrm{Out_{G_0/S}}) = \mathrm{H}^1(\mathrm{S_{\acute{e}t}}, \Theta_{\mathrm{S}})$, where $\Theta = \mathrm{Aut}(\mathrm{R}(\mathrm{G_0}, \mathrm{T_0}), \Delta)$.

(iii) Show $\mathrm{H}^1(\mathrm{S_{\acute{e}t}}, \mathrm{G_0^{ad}}) \to \mathrm{H}^1(\mathrm{S_{\acute{e}t}}, \mathrm{Aut_{G_0/S}})$ has trivial kernel but via Example 7.1.12 it is *not* injective for $\mathrm{G_0} = \mathrm{SL}_n$ over a field k admitting a rank-n^2 central simple algebra not in $\mathrm{Br}(k)[2]$.

(iv) For $n \geqslant 2$, prove the forms of SO_{2n+1} (type B_n) over S are the groups $\mathrm{SO}(q)$ for non-degenerate (V, q) of rank $2n+1$ over S (see Proposition C.3.14 for a refinement). What happens for SO_{2n} with $n \geqslant 4$ (type D_n)?

Exercise 7.3.4. — Let S be scheme, and G a smooth S-affine S-group. Use descent theory to prove that the natural map $\mathrm{H}^1(\mathrm{S}_{\text{ét}}, \mathrm{G}) \to \mathrm{H}^1(\mathrm{S}_{\text{fppf}}, \mathrm{G})$ is bijective. If you are familiar with algebraic spaces, prove the same result without the affineness restriction on G. (For a remarkable generalization to higher cohomology in the commutative case, see [**Gr68**, 11.7].)

Exercise 7.3.5. — Let G be a connected semisimple k-group for a global field k.

(i) If G is an inner form of a k-split group, prove G_{k_v} is split for all but finitely many v.

(ii) If G is *not* an inner form of a k-split group, prove G_{k_v} is non-split for a set of v with positive Dirichlet density. Make this explicit for $\mathrm{SU}_n(k'/k)$ defined as in Example 7.2.10.

Exercise 7.3.6. — Let $(\mathrm{G}, \mathrm{T}, \mathrm{M})$ be split semisimple over a connected non-empty scheme S. For each positive system of roots $\Phi^+ \subset \Phi$ (with base Δ) and choice of pinning $\{\mathrm{X}_a\}_{a\in\Delta}$, let $\{\mathrm{X}_{-a}\}_{a\in\Delta}$ be the linked pinning for $(\mathrm{G}, \mathrm{T}, -\Phi^+)$ (i.e., $\mathrm{X}_{-a} = \mathrm{X}_a^{-1}$ is the dual trivialization of $\mathfrak{g}_{-a}$).

(i) Use the Isomorphism Theorem and the functoriality of duality between $\mathfrak{g}_a$ and $\mathfrak{g}_{-a}$ to construct an *involution* of G that restricts to inversion on T and swaps Φ^+ and $-\Phi^+$. Prove that up to a $(\mathrm{G}/\mathrm{Z}_\mathrm{G})(\mathrm{S})$-conjugation, this involution of G is independent of the choice of T, Φ^+, and pinning (hint: Proposition 6.2.11(2) and Corollary 3.3.6(1)). Deduce the existence of a canonical $(\mathrm{G}/\mathrm{Z}_\mathrm{G})(\mathrm{S})$-conjugacy class of involutions of G (called *Chevalley involutions*). What are the Chevalley involutions of SL_n? Of Sp_{2n}?

(ii) Prove the Chevalley involutions are inner (i.e., arise from $(\mathrm{G}/\mathrm{Z}_\mathrm{G})(\mathrm{S})$) precisely when the long element of $\mathrm{W}(\Phi)$ (relative to a choice of Φ^+) acts via negation on M.

(iii) Via the dictionary in Example 7.1.13, show that if $\mathrm{S} = \operatorname{Spec} \mathbf{R}$ then the Chevalley involutions of G are the Cartan involutions. See [**AV**] for more on Chevalley involutions.

Exercise 7.3.7. — This exercise considers groups with no maximal torus or no Borel subgroup.

(i) Prove that if S is a connected normal noetherian scheme with $\pi_1(\mathrm{S}) = 1$ and if V is a vector bundle on S of rank $r > 1$ that is not a direct sum of line bundles then $\mathrm{SL}(\mathrm{V})$ is a semisimple S-group with no maximal S-torus. Show that an example of such (S, V) is $\mathrm{S}_0 = \mathbf{P}^2_k$ for any separably closed field k and $\mathrm{V}_0 = \mathrm{Tan}_{\mathrm{S}_0/k}$. (Hint: $\dim \mathrm{V}_0(\mathrm{S}_0) = 8$.)

(ii) Let $\mathrm{K} = \mathbf{Q}(\sqrt{5})$, and let D be the quaternion division K-algebra ramified at precisely the two archimedean places. Use gluing methods as in Remark 5.1.5 to extend the anisotropic K-group $\mathrm{G} = \mathrm{SL}(\mathrm{D})$ to a semisimple $\mathscr{O}_\mathrm{K}$-group

$\mathscr{G}$. Prove that $\mathscr{G}$ contains no maximal $\mathscr{O}_{\mathrm{K}}$-torus. (Hint: show that K has no quadratic extension unramified at all finite places.)

(iii) Let (L, q) be a positive-definite even unimodular lattice with rank $\geqslant 3$. Prove that the semisimple $\mathbf{Z}$-group $\mathrm{SO}(q)$ has no maximal $\mathbf{Z}$-torus.

(iv) Over $\mathbf{R}$ or any non-archimedean local field k, prove that a nontrivial connected semisimple k-group G has no Borel k-subgroup if $\mathrm{G}(k)$ is compact. Verify the compactness for $\mathrm{G} = \mathrm{SL}(\mathrm{D})$ with any central division algebra D over k.

Exercise 7.3.8. — Prove that Proposition 7.2.2 remains valid over any non-empty scheme S.

Exercise 7.3.9. — Let G be reductive over $\mathrm{S} \neq \varnothing$ with every $\mathrm{G}_{\overline{s}}$ having root datum R. Choose a base Δ of R. Let $(\mathrm{G}_0, \mathrm{T}_0, \mathrm{M})$ be split reductive over S with associated root datum R, and let $\mathrm{B}_0 \subset \mathrm{G}_0$ the Borel subgroup containing T_0 that corresponds to the choice of Δ.

(i) Using Proposition 6.2.11 and Corollary 5.2.8, show $\mathrm{Aut}_{(\mathrm{G}_0,\mathrm{B}_0)/\mathrm{S}} = (\mathrm{B}_0/\mathrm{Z}_{\mathrm{G}_0}) \rtimes \mathrm{Aut}(\mathrm{R}, \Delta)$.

(ii) If $\mathrm{B} \subset \mathrm{G}$ is a Borel subgroup, classify (G, B) up to isomorphism by a class in the cohomology set $\mathrm{H}^1(\mathrm{S}_{\text{ét}}, (\mathrm{B}_0/\mathrm{Z}_{\mathrm{G}_0}) \rtimes \mathrm{Aut}(\mathrm{R}, \Delta))$ and prove that G splits if it is an inner form of G_0 and $\mathrm{H}^1(\mathrm{S}, \mathscr{O}_{\mathrm{S}}) = 0$ and $\mathrm{Pic}(\mathrm{S}) = 1$.

(iii) Using the properness of $\mathrm{Bor}_{\mathrm{G}/\mathrm{S}}$, prove that if S is connected and Dedekind then $\mathrm{Bor}_{\mathrm{G}/\mathrm{S}}(\mathrm{S}) \neq \varnothing$ when the generic fiber G_η is split (hint: valuative criterion).

(iv) If R is Dedekind with fraction field K and $\mathrm{Pic}(\mathrm{R}) = 1$, show that a reductive R-group G is split if G_{K} is split.

Exercise 7.3.10. — This exercise demonstrates the necessity of the triviality of the Picard group in Exercise 7.3.9(iv). Let R be a domain with fraction field K.

(i) Prove that the automorphism functor of the $\mathbf{Z}$-scheme Mat_n (viewed as a functor valued in associative rings) is represented by an affine $\mathbf{Z}$-group scheme $\mathrm{Aut}_{\mathrm{Mat}_n/\mathbf{Z}}$ of finite type, and show that the resulting natural map of $\mathbf{Z}$-groups $f : \mathrm{PGL}_n \to \mathrm{Aut}_{\mathrm{Mat}_n/\mathbf{Z}}$ is bijective on artinian points (hint: Exercise 5.5.5(i)). Deduce that f is an isomorphism.

(ii) Using (i) and Theorem 7.1.9(3), the set $\mathrm{H}^1(\mathrm{Spec}(\mathrm{R})_{\text{ét}}, \mathrm{PGL}_2)$ parameterizes R-forms of SL_2 as well as R-forms of Mat_2. For an R-form A of Mat_2 (i.e., a rank-4 Azumaya algebra over R), show that the cohomology class of A coincides with that of the R-group $\mathrm{SL}_1(\mathrm{A})$ of units of reduced norm 1 (as an R-form of SL_2). For rank-2 vectors bundles V and W over R, deduce that $\mathrm{SL}(\mathrm{V}) \simeq \mathrm{SL}(\mathrm{W})$ as R-groups if and only if $\mathrm{End}(\mathrm{V}) \simeq \mathrm{End}(\mathrm{W})$ as R-algebras.

(iii) Let V and W be vector bundles over R with the same finite rank $n > 0$. Show that the natural map of Zariski sheaves of sets

$$q : \mathscr{I}som(\mathrm{V}, \mathrm{W}) \to \mathscr{I}som(\mathscr{E}nd(\mathrm{V}), \mathscr{E}nd(\mathrm{W}))$$

over $\operatorname{Spec} \mathrm{R}$ (defined by $\varphi \mapsto (\mathrm{T} \mapsto \varphi \circ \mathrm{T} \circ \varphi^{-1})$) corresponds to pushout of right torsors along $\mathrm{GL}(\mathrm{V}) \to \mathrm{PGL}(\mathrm{V})$. Deduce that q is a $\mathbf{G}_m$-torsor for the Zariski topology via the $\mathbf{G}_m$-action on W (which is invisible on $\mathscr{E}nd(\mathrm{W})$!).

(iv) Let $f : \mathrm{End}_{\mathrm{R}}(\mathrm{V}) \simeq \mathrm{End}_{\mathrm{R}}(\mathrm{W})$ be an R-algebra isomorphism, with V and W as in (iii). Viewing f in $\Gamma(\mathrm{Spec}(\mathrm{R}), \mathscr{I}som(\mathscr{E}nd(\mathrm{V}), \mathscr{E}nd(\mathrm{W})))$, use the $\mathbf{G}_m$-torsor $q^{-1}(f)$ over $\mathrm{Spec}(\mathrm{R})$ to construct an invertible R-module L such that f is induced by an isomorphism $\mathrm{V} \simeq \mathrm{L} \otimes_{\mathrm{R}} \mathrm{W}$.

(v) Using (ii) and (iv), for an invertible R-module J such that $\mathrm{SL}(\mathrm{R} \oplus \mathrm{J}) \simeq \mathrm{SL}_2$ as R-groups show that $\mathrm{J} \simeq \mathrm{L}^{\otimes 2}$ for an invertible R-module L. (Note that $\mathrm{SL}(\mathrm{R} \oplus \mathrm{J})$ has an evident split maximal R-torus, but its root spaces are not globally free when J is not.) Deduce that if K is a number field with even class number then $\mathrm{SL}_{2,\mathrm{K}}$ extends to a semisimple $\mathscr{O}_{\mathrm{K}}$-group that is not isomorphic to $\mathrm{SL}_{2,\mathscr{O}_{\mathrm{K}}}$ but splits Zariski-locally over $\mathscr{O}_{\mathrm{K}}$.

Appendix A
Grothendieck's theorem on tori

A.1. Motivation and definitions. — In the early days of the theory of linear algebraic groups, the ground field was assumed to be algebraically closed (as in work of Chevalley). The needs of Lie theory, number theory, and finite group theory (such as finite simple groups of Lie type) led to the development (independently by Borel, Satake, and Tits) of a theory of connected reductive groups over any perfect field (using Galois-theoretic techniques to deduce results from the algebraically closed case). Problems over local and global function fields motivated the elimination of the perfectness assumption by Borel and Tits [**BoTi**]. The initial breakthrough that made it possible to work over an arbitrary field is the following result [**SGA3**, XIV, 1.1]:

***Theorem A.1.1* (Grothendieck).** — *Any smooth connected affine group* G *over a field* k *contains a* k*-torus* T *such that* $\mathrm{T}_{\overline{k}}$ *is maximal in* $\mathrm{G}_{\overline{k}}$.

The hardest case of the proof is when k is imperfect, and it was for this purpose that Grothendieck's scheme-theoretic ideas were essential, at first. (In [**SGA3**, XIV, 1.5(d)] there is a second scheme-theoretic proof for infinite k, using the scheme of maximal tori for general smooth connected affine groups over a field (Exercise 3.4.8); this is unirational [**SGA3**, XIV, 6.1], and unirational varieties over infinite fields have rational points.) Borel and Springer eliminated the use of schemes via Lie-theoretic methods (see [**Bo91**, 18.2(i)]); this amounts to working with certain infinitesimal group schemes in disguise, as we shall see. The aim of this appendix is to give a scheme-theoretic interpretation of their argument.

Remark A.1.2. — As an application of Theorem A.1.1, we now show via torus-centralizer arguments that if K/k is any extension field and if $\mathrm{T} \subset \mathrm{G}$ is a k-torus not contained in a strictly larger one then T_{K} is not contained in a strictly larger K-torus of G_{K}. In particular, taking $\mathrm{K} = \overline{k}$, it follows that over a field k, "maximality" for k-tori in the geometric sense of Definition 3.2.1 is the same as maximality in the k-rational sense of containment of k-tori; i.e., there is no ambiguity about the meaning of the phrase "maximal k-torus", and all such tori have the same dimension (due to conjugacy over $\overline{k}$). (One can also consider the same problem over artin local S; see [**SGA3**, XIV, 1.4].) The common dimension of the maximal k-tori is sometimes called the *reductive rank* of G because it coincides with the same invariant for the reductive quotient $\mathrm{G}_{\overline{k}}/\mathscr{R}_u(\mathrm{G}_{\overline{k}})$.

The equivalence of maximality for T over k and for T_{K} over K was mentioned in the Introduction, and we now deduce it from Theorem A.1.1. For any field extension K/k, a torus of G_{K} containing T_{K} must lie in the closed subgroup scheme $Z_{G_K}(T_K) = Z_G(T)_K$, but $Z_G(T)$ is also smooth (see Corollary 1.2.4 or Lemma 2.2.4), so we may replace G with $Z_G(T)$ to reduce to the special case that T is *central* in G. Then we can pass to the affine quotient k-group G/T to arrive at the case $T = 1$.

A smooth connected affine group over an algebraically closed field contains no nontrivial torus if and only if it is unipotent, so we are reduced to the following: if a smooth connected affine group G over a field k contains no nontrivial k-torus then must it be unipotent? The general problem, for an arbitrary smooth connected affine k-group, is the existence of *one* k-torus that is maximal over $\overline{k}$; i.e., a k-torus that is maximal in the sense of Definition 3.2.1. This is exactly Theorem A.1.1. It therefore follows from this theorem that if the only such k-torus is the trivial one then the group must be unipotent, as desired.

Remark A.1.3. — Beware that if $k \neq k_s$ then typically there are *many* $G(k)$-conjugacy classes of maximal k-tori. For example, if $G = GL_n$ then by Exercise 4.4.6(i) the set of maximal k-tori in G is in bijective correspondence with the set of maximal finite étale commutative k-subalgebras of $\mathrm{Mat}_n(k)$. In particular, two maximal k-tori are $G(k)$-conjugate if and only if the corresponding maximal finite étale commutative k-subalgebras of $\mathrm{Mat}_n(k)$ are $GL_n(k)$-conjugate. Hence, if such k-subalgebras are not abstractly k-isomorphic then the corresponding maximal k-tori are not $G(k)$-conjugate. For example, non-isomorphic degree-n finite separable extension fields of k yield such algebras.

Remark A.1.4. — By the classical theory, $G_{\overline{k}}$ has no nontrivial tori if and only if $G_{\overline{k}}$ is unipotent. Hence, Grothendieck's theorem implies that every smooth connected affine k-group is *either* unipotent or contains a nontrivial k-torus. If all k-tori in G are *central* then for a maximal k-torus T the quotient G/T is unipotent (as $(G/T)_{\overline{k}} = G_{\overline{k}}/T_{\overline{k}}$ contains no nontrivial torus). Hence, in such cases G is solvable. Thus, in the non-solvable case there are always k-tori S whose scheme-theoretic centralizer $Z_G(S)$ (which is smooth, by Corollary 1.2.4, and connected by Theorem 1.1.19(1)) has lower dimension than G. This is useful for induction arguments based on dimension.

Definition A.1.5. — For a maximal k-torus T in a smooth connected affine k-group G, the associated *Cartan k-subgroup* $C \subset G$ is $C = Z_G(T)$, the scheme-theoretic centralizer.

Cartan k-subgroups are always smooth (Corollary 1.2.4) and connected (by the classical theory). Since T is central in its Cartan C, it follows that T is the *unique* maximal k-torus in C. (Indeed, if there exists another then the k-subgroup it generates along with the central $T \subset C$ would be a bigger k-torus.)

We have $C_{\overline{k}} = Z_{G_{\overline{k}}}(T_{\overline{k}})$ since the formation of scheme-theoretic centralizers commutes with base change, and over $\overline{k}$ all maximal tori are conjugate. Hence, over $\overline{k}$ the Cartan subgroups are conjugate, so the dimension of a Cartan k-subgroup is both independent of the choice of Cartan k-subgroup and invariant under extension of the ground field. This dimension is called the *nilpotent rank* of G in [**SGA3**, XII, 1.0], and the *rank* of G in [**Bo91**, 12.2]. For a connected reductive group, the Cartan subgroups are precisely the maximal tori.

Remark A.1.6. — It is a difficult theorem that for any smooth connected affine group G over any field k, among all k-*split* tori in G the maximal ones (with respect to inclusion) are *rationally* conjugate, i.e. conjugate under $G(k)$. This is [**Bo91**, 20.9(i)] for reductive G. The general case was announced without proof by Borel and Tits, and is proved in [**CGP**, C.2.3]. The dimension of a maximal *split* k-torus is thus also an invariant, called the k-*rank* of G (and is of much interest in the reductive case).

A.2. Start of proof of main result. — For the proof of Theorem A.1.1, we proceed by induction on $\dim G$, the case $\dim G \leqslant 1$ being trivial. Thus, we now assume Theorem A.1.1 is known in all dimensions $< \dim G$. We will largely focus on the case when k is *infinite*, which ensures that the subset $k^n \subset \mathbf{A}^n_{\overline{k}}$ is Zariski-dense, and thus $\mathfrak{g} = \mathrm{Lie}(G)$ is Zariski-dense in $\mathfrak{g}_{\overline{k}}$. The case of finite k requires a completely different argument; see Exercise 6.5.6.

We first treat the case when $G_{\overline{k}}$ has a *central* maximal torus S. (The argument in this case will work over all k, even finite fields.) Since all maximal tori are $G(\overline{k})$-conjugate, there exists a *unique* maximal $\overline{k}$-torus $S \subset G_{\overline{k}}$. Our problem is to produce one defined over k. This is rather elementary over perfect fields via Galois descent, but here is a uniform method based on group schemes that applies over all fields (and the technique will be useful later).

Let $Z = Z^0_G$, the identity component of the scheme-theoretic center of G. (See Proposition 1.2.3, Lemma 2.2.4, or Exercise 2.4.4(ii) for the existence of Z_G.) Since the formation of the center and its identity component commute with extension of the ground field (see Exercise 1.6.5), we have $S \subset (Z_{\overline{k}})_{\mathrm{red}}$ as a maximal torus in the smooth commutative affine $\overline{k}$-group $(Z_{\overline{k}})_{\mathrm{red}}$. By the structure of smooth connected commutative affine $\overline{k}$-groups, it follows that $(Z_{\overline{k}})_{\mathrm{red}} = S \times U$ for a smooth connected unipotent $\overline{k}$-group U. For any n not divisible by $\mathrm{char}(k)$, consider the torsion subgroup $Z[n]$. This is a commutative affine k-group of finite type. The derivative of $[n] : Z \to Z$ at 0 is $n : \mathrm{Lie}(Z) \to \mathrm{Lie}(Z)$ (as for any commutative k-group scheme), so $\mathrm{Lie}(Z[n])$ is killed by $n \in k^\times$. Thus $\mathrm{Lie}(Z[n]) = 0$, so $Z[n]$ is finite étale over k.

This implies that

$$Z[n]_{\overline{k}} = Z_{\overline{k}}[n] \supset (Z_{\overline{k}})_{\mathrm{red}}[n] \supset Z[n]_{\overline{k}},$$

so $Z[n]_{\overline{k}} = (Z_{\overline{k}})_{\mathrm{red}}[n]$. Since U is unipotent, $U[n] = 0$. Hence, $Z[n]_{\overline{k}} = S[n]$.

Set $H = (\overline{\bigcup_n Z[n]})^0 \subset G$, where the union is taken over n not divisible by $\mathrm{char}(k)$. This is a connected closed k-subgroup scheme of G.

Lemma A.2.1. — *The k-group* H *is a torus descending* S*.*

Proof. — By Galois descent, the formation of H commutes with scalar extension to k_s, so we can assume $k = k_s$. Hence, the finite étale groups $Z[n]$ are constant, so H is the identity component of the Zariski closure of a set of k-points. It follows that the formation of H commutes with any further extension of the ground field, so

$$H_{\overline{k}} = (\overline{\bigcup_n Z_{\overline{k}}[n]})^0 = (\overline{\bigcup_n S[n]})^0 = S$$

where the final equality uses that in any k-torus, the collection of n-torsion subgroups for n not divisible by $\mathrm{char}(k)$ is dense (as we see by working over $\overline{k}$ and checking for $\mathbf{G}_{\mathrm{m}}$ by hand). □

Now we turn to the hard case, when $G_{\overline{k}}$ does not have a central maximal torus. In particular, there must exist a non-central

$$S = \mathbf{G}_{\mathrm{m}} \hookrightarrow G_{\overline{k}}. \tag{A.2.1}$$

We will handle these cases via induction on $\dim G$. (The case $\dim G = 1$ is trivial.)

Lemma A.2.2. — *It suffices to prove* G *contains a nontrivial k-torus* M*.*

Proof. — Suppose there exists a nontrivial k-torus $M \subset G$. Consider $Z_G(M)$, which is a smooth connected k-subgroup of G. The maximal tori of $Z_G(M)_{\overline{k}} = Z_{G_{\overline{k}}}(M_{\overline{k}})$ must have the same dimension as those of $G_{\overline{k}}$, as can be seen by considering one containing $M_{\overline{k}}$. Thus, if we can find a k-torus in $Z_G(M)$ that remains maximal as such after extension of the ground field to $\overline{k}$ then the $\overline{k}$-fiber of such a torus must also be maximal in $G_{\overline{k}}$ for dimension reasons. Hence, it suffices to prove Theorem A.1.1 for $Z_G(M)$.

Consider $Z_G(M)/M$. Since M was assumed to be nontrivial, this has strictly smaller dimension (even if $Z_G(M) = G$, which might have happened). Hence, by dimension induction, there exists a k-torus $\overline{T} \subset Z_G(M)/M$ which is geometrically maximal. Let T be the scheme-theoretic preimage of $\overline{T}$ in $Z_G(M)$. Since M is smooth and connected, the quotient map $Z_G(M) \to Z_G(M)/M$ is smooth, so T is a smooth connected closed k-subgroup of G. It sits in a short exact sequence of k-groups

$$1 \to M \to T \to \overline{T} \to 1.$$

Since M and $\overline{T}$ are tori and T is smooth and connected, by the structure theory for connected solvable $\overline{k}$-groups it follows that T is a torus. The quotient

$T_{\overline{k}}/M = \overline{T}_{\overline{k}}$ is a maximal torus in $(Z_G(M)/M)_{\overline{k}}$, so $T_{\overline{k}}$ is a maximal torus in $Z_G(M)_{\overline{k}}$. Hence, $T_{\overline{k}}$ is also maximal as a torus in $G_{\overline{k}}$. □

Now we need to find such an M. The idea for infinite k is to use $\mathrm{Lie}(S) = \mathfrak{gl}_1 \subset \mathfrak{g}_{\overline{k}}$ (with S as in (A.2.1)) and the Zariski-density of $\mathfrak{g}$ in $\mathfrak{g}_{\overline{k}}$ (infinite k!) to create a nonzero $X \in \mathfrak{g}$ that is "semisimple" and such that the k-subgroup $Z_G(X)^0 \subset G$ is a lower-dimensional smooth subgroup in which the maximal $\overline{k}$-tori are maximal in $G_{\overline{k}}$. (We define $Z_G(X)$ to be the schematic G-stabilizer of X under Ad_G.) Then a geometrically maximal k-torus in $Z_G(X)^0$ will do the job. (Below we will define what we mean by "semisimple" for elements of $\mathfrak{g}_{\overline{k}}$ relative to $G_{\overline{k}}$. This is a Lie-theoretic version of Jordan decomposition for linear algebraic groups.) The motivation is that whereas it is hard to construct tori over k, it is much easier to use Zariski-density arguments in $\mathfrak{g}_{\overline{k}}$ to create semisimple elements in $\mathfrak{g}$. Those will serve as a substitute for tori to carry out a centralizer trick and apply dimension induction.

Remark A.2.3. — Beware that non-centrality of S in $G_{\overline{k}}$ does not imply non-centrality of Lie(S) in $\mathfrak{g}_{\overline{k}}$ when $\mathrm{char}(k) > 0$. For example, if S is the diagonal torus in SL_2 and $\mathrm{char}(k) = 2$ then Lie(S) coincides with the Lie algebra of the central μ_2, so it is central in $\mathfrak{sl}_2(k)$ (as is also clear by inspection).

A.3. The case of infinite k. — Now we assume k is infinite, but otherwise arbitrary. Consider the following hypothesis:

$(\star)$ there exists a non-central semisimple element $X \in \mathfrak{g}$.

To make sense of this, we now define the concept "non-central, semisimple" in $\mathfrak{g}$. The definition of "semisimple" will involve G. This is not surprising, since the same 1-dimensional Lie algebra k arises for both $\mathbf{G}_a$ and $\mathbf{G}_{\mathrm{m}}$, and in the first case we want to declare all elements of the Lie algebra to be nilpotent (since unipotent subgroups of GL_N have all elements in their Lie algebra nilpotent inside $\mathfrak{gl}_N$, by the Lie–Kolchin theorem) and in the latter case we want to declare all elements of the Lie algebra to be semisimple.

We now briefly digress for a review of Lie algebras of smooth linear algebraic groups G over general fields k. The *center* of a Lie algebra $\mathfrak{g}$ is the kernel of the adjoint action

$$\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}), X \mapsto [X, -].$$

In [**Bo91**, § 4.1–§ 4.4], a general "Jordan decomposition" is constructed as follows in $\mathfrak{g}_{\overline{k}}$, with $\mathfrak{g} = \mathrm{Lie}(G)$. Choose a closed k-subgroup inclusion $G \hookrightarrow GL_N$, and consider the resulting inclusion of Lie algebras $\mathfrak{g} \hookrightarrow \mathfrak{gl}_N$ over k. For any $X \in \mathfrak{g}_{\overline{k}}$ we have an additive Jordan decomposition $X = X_s + X_n$ in $\mathfrak{gl}_N(\overline{k}) = \mathrm{Mat}_N(\overline{k})$. In particular, $[X_s, X_n] = 0$. The elements $X_s, X_n \in \mathfrak{gl}_N(\overline{k})$ lie in $\mathfrak{g}_{\overline{k}}$ and as such are independent of the chosen embedding $G \hookrightarrow GL_N$ (proved similarly to the construction of Jordan decomposition in $G(\overline{k})$). Likewise, the decomposition $X = X_s + X_n$ is functorial in G (not $\mathfrak{g}$!), so $\mathrm{ad}(X_s) = \mathrm{ad}(X)_s$ and $\mathrm{ad}(X_n) = \mathrm{ad}(X)_n$.

Definition A.3.1. — An element $X \in \mathfrak{g}$ is *semisimple* if $X = X_s$ and *nilpotent* if $X = X_n$.

Remark A.3.2. — Note that we are *not* claiming that $\mathrm{ad}(X)$ alone detects the semisimplicity or nilpotence, nor that the definition is being made intrinsically to $\mathfrak{g}$. The definitions of semisimplicity and nilpotence involve the k-group G. By definition, these concepts are preserved under passage from $\mathfrak{g}$ to $\mathfrak{g}_{\overline{k}}$ (and as with algebraic groups, the Jordan components of $X \in \mathfrak{g}$ are generally only rational over the perfect closure of k). One can develop versions of these concepts intrinsically to the Lie algebra, but we do not discuss it; see [**Sel**, V.7.2].

If $p = \mathrm{char}(k) > 0$, then upon choosing a faithful representation $\rho : G \hookrightarrow \mathrm{GL}_N$, the resulting inclusion $\mathfrak{g} \hookrightarrow \mathfrak{gl}_N$ makes the p-power map $A \mapsto A^p$ on $\mathfrak{gl}_N = \mathrm{Mat}_N(k)$ (*not* the p-power map on matrix entries) induce the structure of a *p-Lie algebra* on $\mathfrak{g}$. This is a map of sets $\mathfrak{g} \to \mathfrak{g}$, denoted $X \mapsto X^{[p]}$, that satisfies $(cX)^{[p]} = c^p X^{[p]}$, $\mathrm{ad}(X^{[p]}) = \mathrm{ad}(X)^p$ (a computation in $\mathfrak{gl}_N$), and a certain identity for $(X+Y)^{[p]} - X^{[p]} - Y^{[p]}$. It has the intrinsic description $D \mapsto D^p$ from the space of left-invariant derivations to itself (so it is independent of ρ), and is functorial in G. For further details on p-Lie algebras, see [**Bo91**, §3.1], [**CGP**, A.7] (especially [**CGP**, A.7.13]), and [**SGA3**, $\mathrm{VII_A}$, §5]; e.g., the interaction of p-Lie algebra structures and scalar extension is addressed in [**SGA3**, $\mathrm{VII_A}$, 5.3.3bis].

Remark A.3.3. — In characteristic $p > 0$, if $X \in \mathfrak{g}$ is nilpotent then $X^{[p^r]} = 0$ for $r \gg 0$. This is very important below, and follows from a computation in the special case of $\mathfrak{gl}_N$.

Returning to our original problem over infinite k, let us verify hypothesis $(\star)$ in characteristic zero. The non-central $S = \mathbf{G}_m \hookrightarrow G_{\overline{k}}$ gives an action of S on $\mathfrak{g}_{\overline{k}}$ (via the adjoint action of $G_{\overline{k}}$ on $\mathfrak{g}_{\overline{k}}$), and this decomposes as a direct sum of weight spaces

$$\mathfrak{g}_{\overline{k}} = \bigoplus \mathfrak{g}_{\chi_i}.$$

The S-action is described by the weights n_i, where $\chi_i(t) = t^{n_i}$.

Lemma A.3.4. — *There is at least one nontrivial weight.*

Proof. — The centralizer $Z_{G_{\overline{k}}}(S)$ is a smooth (connected) subgroup of $G_{\overline{k}}$, and $\mathrm{Lie}(Z_{G_{\overline{k}}}(S)) = \mathfrak{g}_{\overline{k}}^S$ by Proposition 1.2.3. Thus, if S acts trivially on $\mathfrak{g}_{\overline{k}}$ then $Z_{G_{\overline{k}}}(S)$ has Lie algebra with full dimension, forcing $Z_{G_{\overline{k}}}(S) = G_{\overline{k}}$ by smoothness, connectedness, and dimension reasons. This says that S is central in $G_{\overline{k}}$, which is contrary to our hypotheses on S. $\square$

If we choose a $\overline{k}$-basis Y for Lie(S) then the element $Y \in \mathfrak{g}_{\overline{k}}$ is semisimple since any $G_{\overline{k}} \hookrightarrow GL_N$ carries S into a torus and hence carries Lie(S) onto a semisimple subalgebra of $\mathfrak{gl}_N = \mathrm{Mat}_N(k)$. By Lemma A.3.4, some weight is nonzero. Thus, in characteristic zero (or more generally if $\mathrm{char}(k) \nmid n_i$ for some i) we know moreover that ad(Y) is nonzero. Hence, Y is semisimple and in characteristic 0 is *non-central*.

This does not establish ($\star$) when $\mathrm{char}(k) = 0$, since $Y \in \mathfrak{g}_{\overline{k}}$ and we seek a non-central semisimple element of $\mathfrak{g}$. To remedy this, consider the characteristic polynomial $f(X, t)$ of ad(X) for "generic" $X \in \mathfrak{g}$, as a polynomial in $k[\mathfrak{g}^*][t]$. Working in $\overline{k}[\mathfrak{g}^*][t] = \overline{k}[\mathfrak{g}^*_{\overline{k}}][t]$, the existence of the non-central semisimple element Y as established above (when $\mathrm{char}(k) = 0$) shows that $f(X, t) \neq t^{\dim \mathfrak{g}}$. In other words, there are lower-order (in t) coefficients in $k[\mathfrak{g}^*]$ which are nonzero as functions on $\mathfrak{g}_{\overline{k}}$. The subset $\mathfrak{g} \subset \mathfrak{g}_{\overline{k}}$ is Zariski-dense (as k is infinite), so there exists $X \in \mathfrak{g}$ such that $f(X, t) \in k[t]$ is not equal to $t^{\dim \mathfrak{g}}$. In particular, ad(X) is not nilpotent, so $\mathrm{ad}(X)_s$ is nonzero. Since $\mathrm{ad}(X_s) = \mathrm{ad}(X)_s \neq 0$, X_s is noncentral and semisimple in $\mathfrak{g}_{\overline{k}}$. When k is perfect, such as a field of characteristic 0, the Jordan decomposition is rational over the ground field. Thus, X_s satisfies the requirements in ($\star$).

A.4. Hypothesis ($\star$) for G implies the existence of a nontrivial k-torus. — Now we assume there exists $X \in \mathfrak{g}$ that is noncentral and semisimple. We will show (for infinite k of any characteristic) that there exists a smooth closed k-subgroup $G' \subset G$ *which is a proper subgroup* (and hence $\dim G' < \dim G$) such that $\mathfrak{g}' := \mathrm{Lie}(G')$ contains a nonzero semisimple element of $\mathfrak{g}$. This implies that $G'_{\overline{k}}$ is not unipotent (for if it were unipotent then its Lie algebra would be nilpotent inside $\mathfrak{g}_{\overline{k}}$). By dimension induction, G' contains a geometrically maximal k-torus. Since $G'_{\overline{k}}$ is not unipotent, this means G' (and hence G!) contains a *nontrivial* k-torus, which is all we need (by Lemma A.2.2).

Granting ($\star$), it is very easy to finish the proof, as follows. Consider the scheme-theoretic stabilizer $Z_G(X)$ of X (for the action $\mathrm{Ad}_G : G \to GL(\mathfrak{g})$). By Cartier's theorem in characteristic 0, or the semisimplicity of X and [**Bo91**, 9.1] in any characteristic, $Z_G(X)$ is smooth. We must have $Z_G(X) \neq G$. Indeed, assume $Z_G(X) = G$, so $\mathrm{Ad}_G(g)(X) = X$ for all $g \in G$. By differentiating, $\mathrm{ad}(X) = 0$ on $\mathfrak{g}$. But X is non-central in $\mathfrak{g}$, so this is a contradiction. Thus $Z_G(X)$ is a smooth subgroup of G distinct from G, and its Lie algebra contains the nonzero semisimple X. This does the job as required above, so we are done in characteristic 0, and are also done in characteristic $p > 0$ if the specific G under consideration satisfies ($\star$).

For the remainder of the proof, assume $\mathrm{char}(k) = p > 0$. We will make essential use of p-Lie algebras, due to two facts: (i) p-Lie subalgebras $\mathfrak{h} \subset \mathfrak{g}$ are in functorial bijection with infinitesimal k-subgroup schemes $\mathrm{H} \subset \mathrm{G}$ that have height $\leqslant 1$ (meaning $a^p = 0$ for all nilpotent functions a on H) via $\mathrm{H} \mapsto \mathrm{Lie}(\mathrm{H})$, and (ii) $\mathfrak{h}$ is commutative if and only if H is commutative. The idea behind the proofs of these facts is to imitate classical Lie-theoretic arguments by using Taylor series truncated in degrees $< p$. This makes it possible to reconstruct H from Lie(H) via the functor $\mathfrak{h} \rightsquigarrow \mathrm{Spec}\,\mathrm{U}_p(\mathfrak{h})^*$, where $\mathrm{U}_p(\mathfrak{h})$ denotes the *restricted universal enveloping algebra* (carrying the p-operation on $\mathfrak{h}$ over to the p-power map on the associative algebra $\mathrm{U}_p(\mathfrak{h})$; see [**SGA3**, $\mathrm{VII_A}$, 5.3]). A precise statement is in [**SGA3**, $\mathrm{VII_A}$, 7.2. 7.4]:

Theorem A.4.1. — *For a commutative* $\mathbf{F}_p$*-algebra* B*, the* p*-Lie algebra functor* $\mathrm{H} \rightsquigarrow \mathrm{Lie}_p(\mathrm{H})$ *is an equivalence between the category of finite locally free* B*-group schemes whose augmentation ideal is killed by the* p*-power map and the category of finite locally free* p*-Lie algebras over* B*.*

In particular, if k *is a field of characteristic* $p > 0$ *and* G *is a* k*-group scheme of finite type, then for any* H *of height* $\leqslant 1$ *the* p*-Lie algebra functor defines a bijection*

$$\begin{aligned}\mathrm{Hom}_k(\mathrm{H}, \mathrm{G}) = \mathrm{Hom}_k(\mathrm{H}, \ker \mathrm{F}_{\mathrm{G}/k}) &\simeq \mathrm{Hom}(\mathrm{Lie}_p(\mathrm{H}), \mathrm{Lie}_p(\ker \mathrm{F}_{\mathrm{G}/k})) \\ &= \mathrm{Hom}(\mathrm{Lie}_p(\mathrm{H}), \mathrm{Lie}_p(\mathrm{G})).\end{aligned}$$

In this result, $\mathrm{F}_{\mathrm{G}/k} : \mathrm{G} \to \mathrm{G}^{(p)}$ denotes the relative Frobenius morphism, discussed in Exercise 1.6.8 over fields and in [**CGP**, A.3] over $\mathbf{F}_p$-algebras. (For GL_n it is the p-power map on matrix entries, and in general it is functorial in G.) By Nakayama's Lemma, a map $f : \mathrm{H} \to \mathrm{G}$ from an infinitesimal H is a closed immersion if and only if $\mathrm{Lie}(f)$ is injective.

Remark A.4.2. — The proof of Theorem A.4.1 rests on general arguments with p-Lie algebras in [**SGA3**, $\mathrm{VII_A}$, § 4-§ 5], and a key ingredient is that the natural identification of $\mathfrak{h}$ with $\mathrm{Tan}_e(\mathrm{Spec}\,\mathrm{U}_p(\mathfrak{h})^*)$ respects the p-Lie algebra structures. This compatibility rests on a functorial description of the p-Lie algebra structure arising from a group scheme (and the explicit description of $\mathrm{U}_p(\mathfrak{h})$). Such a functorial description is proved in [**CGP**, A.7.5, A.7.13] (and is proved in related but more abstract terms in [**SGA3**, $\mathrm{VII_A}$, § 6]).

In the special case of *commutative* k-groups whose augmentation ideal is killed by the p-power map, an elementary proof of the equivalence with finite-dimensional *commutative* p-Lie algebras over k is given in the proof of the unique Theorem in [**Mum**, § 14] via a method which works over any field (even though [**Mum**] always assumes the ground field is algebraically closed). But beware that the commutative case is not enough for us, since we need the final bijection among Hom's in Theorem A.4.1, and that rests on using the k-group scheme $\ker \mathrm{F}_{\mathrm{G}/k}$ which is generally non-commutative.

As an illustration of the usefulness of Theorem A.4.1, we now give an alternative construction of a nontrivial k-torus when $\mathrm{char}(k) = p > 0$ and $(\star)$ holds, bypassing the smoothness of $\mathrm{Z}_{\mathrm{G}}(\mathrm{X})$. This also provides an opportunity to present some arguments that will be useful in our treatment of the cases when $(\star)$ fails.

Let $\mathfrak{h} = \mathrm{Span}_k(\{\mathrm{X}^{[p^i]}\}_{i\geqslant 0})$. This is manifestly closed under the map $v \mapsto v^{[p]}$. Moreover, the $\mathrm{X}^{[p^i]}$ all commute with one another. (Proof: use an embedding $\mathfrak{g} \hookrightarrow \mathfrak{gl}_n$ arising from a k-group inclusion of G into GL_n, and the description of the p-operation on $\mathfrak{gl}_n = \mathrm{Mat}_n(k)$ as $\mathrm{A} \mapsto \mathrm{A}^p$.) Thus $\mathfrak{h}$ is a commutative p-Lie subalgebra of $\mathfrak{g}$. A linear combination of commuting semisimple operators is semisimple. Moreover the pth power of a nonzero semisimple operator is *nonzero*. Hence, $v \mapsto v^{[p]}$ has trivial kernel on $\mathfrak{h}$. It is a general fact in Frobenius-semilinear algebra that if V is a finite-dimensional vector space over a perfect field F of characteristic p and if $\phi : \mathrm{V} \to \mathrm{V}$ is a Frobenius-semilinear endomorphism then there exists a unique decomposition $\mathrm{V} = \mathrm{V}_{\mathrm{ss}} \bigoplus \mathrm{V}_{\mathrm{n}}$ such that ϕ is nilpotent on V_{n} and there is a basis of "ϕ-fixed vectors" ($\phi(v) = v$) for $(\mathrm{V}_{\mathrm{ss}})_{\overline{\mathrm{F}}}$. We will only need this over an algebraically closed field, in which case it is proved in the Corollary at the end of [**Mum**, § 14].

Lemma A.4.3. — *The scheme-theoretic centralizer* $\mathrm{Z}_{\mathrm{G}}(\mathfrak{h}) \subset \mathrm{G}$ *of* $\mathfrak{h} \subset \mathfrak{g}$ *under* Ad_{G} *is smooth.*

Proof. — We may assume $k = \overline{k}$, as smoothness can be detected over $\overline{k}$ and the formation of scheme theoretic centralizers commutes with base change. Now using Theorem A.4.1, let $\mathrm{H} \subset \mathrm{G}$ be the infinitesimal k-subgroup scheme of $\ker \mathrm{F}_{\mathrm{G}/k}$ whose Lie algebra is $\mathfrak{h} \subset \mathfrak{g}$.

As observed above, $\mathfrak{h}$ splits as a direct sum of $(\cdot)^{[p]}$-eigenlines,

$$\mathfrak{h} = \bigoplus k\mathrm{X}_i, \qquad \mathrm{X}_i^{[p]} = \mathrm{X}_i.$$

Thus, H is a power of the order-p infinitesimal commutative k-subgroup corresponding to the p-Lie algebra kv with $v^{[p]} = v$. But there are only *two* 1-dimensional p-Lie algebras over k: the one with $v^{[p]} = 0$ and the one with $v^{[p]} = v$ for some k-basis v. (Indeed, if $v^{[p]} = cv$ for some $c \in k^{\times}$ then by replacing v with $w = av$ where $a^{p-1} = c$ we get $w^{[p]} = w$.) Hence, there are exactly two commutative infinitesimal order-p groups over an algebraically closed field, so the non-isomorphic μ_p and α_p must be these two possibilities. That is, H is a power of either μ_p or α_p.

We claim that H is a power of μ_p. To prove this, we will use the p-Lie algebra structure. The embeddings $\alpha_p \hookrightarrow \mathbf{G}_a$ and $\mu_p \hookrightarrow \mathbf{G}_{\mathrm{m}}$ induce isomorphisms on

p-Lie algebras, and the nonzero invariant derivations on $\mathbf{G}_a$ and $\mathbf{G}_{\mathrm{m}}$ are given by ∂_t and $t\partial_t$ respectively. Taking p-th powers of these derivations computes the $(\cdot)^{[p]}$-map on them, and clearly $\partial_t^p = 0$ and $(t\partial_t)^p = t\partial_t$. Hence, the p-operation on α_p vanishes and on μ_p is non-vanishing. Thus, the condition $v^{[p]} = v$ forces $\mathrm{H} = \mu_p^{\mathrm{N}}$ for some N.

By Lemma 2.2.4, $\mathrm{Z_G(H)}$ is smooth. To conclude the proof, it will suffice to show that the evident inclusion $\mathrm{Z_G(H)} \subset \mathrm{Z_G}(\mathfrak{h})$ as k-subgroup schemes of G is an equality. Theorem A.4.1 provides more: if R is any k-algebra, then the p-Lie functor defines a bijective correspondence between the sets of R-group maps $\mathrm{H_R} \to \mathrm{G_R}$ and p-Lie algebra maps $\mathfrak{h}_{\mathrm{R}} \to \mathfrak{g}_{\mathrm{R}}$. Hence, by Yoneda's lemma, $\mathrm{Z_G(H)} = \mathrm{Z_G}(\mathfrak{h})$ because to check this equality of k-subgroup schemes of G it suffices to compare R-points for arbitrary k-algebras R. □

As in the characteristic zero case, since $\mathfrak{h}$ contains noncentral elements of $\mathfrak{g}$, it follows that $\mathrm{Z_G}(\mathfrak{h}) \neq \mathrm{G}$. And as we saw above, this guarantees the existence of a nontrivial k-torus in G, by dimension induction applied to the smooth identity component $\mathrm{Z_G}(\mathfrak{h})^0$ that is *nontrivial* (since it contains the infinitesimal $\mathrm{H} \neq 1$)

We have already completed the proof of Theorem A.1.1 in characteristic zero, since $(\star)$ always holds in characteristic 0, and more generally we have completed it over any k whatsoever for G that satisfy $(\star)$ when the conclusion of Theorem A.1.1 is known over k in all lower dimensions (as we may always assume, since we argue by induction on $\dim \mathrm{G}$).

A.5. The case $\mathrm{char}(k) = p > 0$ **and** $(\star)$ **fails.** — Now the idea is to find a central infinitesimal k-subgroup $\mathrm{M} \subset \mathrm{G}$ such that G/M satisfies $(\star)$. We will lift the result from G/M back to G when such an M exists, and if no such M exists then we will use a different method to find a nontrivial k-torus in G.

Lemma A.5.1. — *Regardless of whether or not* $(\star)$ *holds (but still assuming, as we have been, that* $\mathrm{G}_{\overline{k}}$ *has a noncentral* $\mathbf{G}_{\mathrm{m}}$*), there exists a nonzero semisimple element* $\mathrm{X} \in \mathfrak{g}$*.*

Proof. — Arguing as at the end of § A.3, and *using the infinitude of the field* k, there exists $\mathrm{X}_0 \in \mathfrak{g}$ such that $\mathrm{ad}(\mathrm{X}_0)$ is not nilpotent. Consider the additive Jordan decomposition $\mathrm{X}_0 = (\mathrm{X}_0)_s + (\mathrm{X}_0)_n$ in $\mathfrak{g}_{\overline{k}}$ as a sum of commuting semisimple and nilpotent elements. These components of X_0 are defined over the perfect closure of k, by Galois descent. For $r \gg 0$ we see that $\mathrm{X} := \mathrm{X}_0^{[p^r]} = ((\mathrm{X}_0)_s)^{[p^r]}$. This is nonzero and semisimple, and if $r \gg 0$ then $\mathrm{X} \in \mathfrak{g}$ (since $(\mathrm{X}_0)_s$ is rational over the perfect closure of k). □

Obviously if $(\star)$ fails for G then every semisimple element of $\mathfrak{g}$ is central. Assume this is the case. Set

$$\mathfrak{m} = \mathrm{Span}_k(\text{all semisimple } \mathrm{X} \in \mathfrak{g}) \subset \mathfrak{g};$$

this is nonzero due to Lemma A.5.1. Since all semisimple elements of $\mathfrak{g}$ are central, $\mathfrak{m}$ is a commutative Lie subalgebra of $\mathfrak{g}$. The pth power of a semisimple element of $\mathrm{Mat}_{\mathrm{N}}(\overline{k})$ is semisimple, so $\mathfrak{m}$ is $(\cdot)^{[p]}$-stable. Thus, $\mathfrak{m}$ is a p-Lie subalgebra, so we can exponentiate it to an infinitesimal commutative subgroup $\mathrm{M} \subset \ker \mathrm{F}_{\mathrm{G}/k}$ by Theorem A.4.1. A linear combination of commuting semisimple elements in $\mathrm{Mat}_{\mathrm{N}}(\overline{k})$ is semisimple, so $\mathfrak{m}$ consists only of semisimple elements. This implies that $(\cdot)^{[p]}$ has vanishing kernel on $\mathfrak{m}_{\overline{k}}$. Thus, as in the proof of Lemma A.4.3, $\mathrm{M}_{\overline{k}} = \mu_p^{\mathrm{N}}$ for some $\mathrm{N} > 0$.

Lemma A.5.2. — *The k-subgroup scheme* M *in* G *is central.*

Proof. — Let $\mathrm{V} \subset \mathfrak{g}_{k_s}$ be the k_s-span of all semisimple central elements of $\mathfrak{g}_{k_s}$. Clearly we have $\mathfrak{m}_{k_s} \subset \mathrm{V}$. Let $\Gamma = \mathrm{Gal}(k_s/k)$. Since V is Γ-stable, by Galois descent we have $\mathrm{V} = (\mathrm{V}^{\Gamma})_{k_s}$. Since $\mathrm{V}^{\Gamma} \subset \mathfrak{m}$, this gives $\mathrm{V} = \mathfrak{m}_{k_s}$. By inspection, it is clear that V is stable under the action of $\mathrm{G}(k_s)$ on $\mathfrak{g}_{k_s}$. But $\mathrm{G}(k_s)$ is Zariski-dense in G_{k_s}, so G_{k_s} preserves $\mathrm{V} = \mathfrak{m}_{k_s} \subset \mathfrak{g}_{k_s}$ under the adjoint action. Hence, G preserves $\mathfrak{m}$, so M is normal in G. Thus, it is central by Theorem 2.3.1 or Exercise 7.3.1. □

Now consider the central purely inseparable k-isogeny $\pi : \mathrm{G} \to \mathrm{G}' := \mathrm{G}/\mathrm{M}$. Note that G' is smooth and connected of the same dimension as G, and even contains a non-central torus $\pi_{\overline{k}}(\mathrm{S})$ over $\overline{k}$ (as π is bijective on $\overline{k}$-points). Does G' satisfy $(\star)$? If it does not, then we can run through the same procedure all over again to get a nontrivial central $\mathrm{M}' \subset \mathrm{G}'$ such $\mathrm{M}'_{\overline{k}} \simeq \mu_p^{\mathrm{N}'}$, and can then consider the composite purely inseparable k-isogeny

$$\mathrm{G} \to \mathrm{G}/\mathrm{M} = \mathrm{G}' \to \mathrm{G}'/\mathrm{M}'.$$

This is not so bad, since the kernel E of this composite map is necessarily an infinitesimal multiplicative type subgroup, by the following lemma, so it is *central* in G due to normality and Theorem 2.3.1 (or Exercise 7.3.1):

Lemma A.5.3. — *If*

$$1 \to \mathrm{M}' \to \mathrm{E} \to \mathrm{M} \to 1$$

is a short exact sequence of finite k-group schemes with M *and* M' *multiplicative infinitesimal k-groups then so is* E*; in particular,* E *is commutative.*

Proof. — We may assume $k = \overline{k}$. The infinitesimal nature of M and M' implies that $\mathrm{E}(k) = 1$, so E is infinitesimal (hence connected). The normality of M' in E implies that the conjugation action of E on M' is classified by a k-group homomorphism from the connected E to the étale automorphism group scheme of M'. This classifying map must be trivial, so M' is *central* in E. Since $\mathrm{M} = \mathrm{E}/\mathrm{M}'$ is commutative, the functorial commutator $\mathrm{E} \times \mathrm{E} \to \mathrm{E}$ factors through a k-scheme morphism

$$[\cdot, \cdot] : \mathrm{M} \times \mathrm{M} = (\mathrm{E}/\mathrm{M}') \times (\mathrm{E}/\mathrm{M}') \to \mathrm{M}'$$

which is seen to be bi-additive by thinking about $\mathrm{M} = \mathrm{E}/\mathrm{M}'$ in terms of fppf quotient sheaves. In other words, this bi-additive pairing corresponds (in two ways!) to a k-group homomorphism

$$\mathrm{M} \to \underline{\mathrm{Hom}}(\mathrm{M}, \mathrm{M}'),$$

where the target is the affine finite type k-scheme classifying group scheme homomorphisms (over k-algebras). By Cartier duality, this Hom-scheme is étale, so the map to it from M must be *trivial.* This shows that E has trivial commutator, so E is commutative.

With commutativity of E established, we apply Cartier duality $\mathbf{D}(\cdot)$ to our original short exact sequence. This duality operation is contravariant and preserves exact sequences (since it is order-preserving and carries right-exact sequences to left-exact sequences), so we get an exact sequence

$$1 \to \mathbf{D}(\mathrm{M}) \to \mathbf{D}(\mathrm{E}) \to \mathbf{D}(\mathrm{M}') \to 1.$$

The outer terms are finite constant groups of p-power order, so the middle one must be too. Hence, $\mathrm{E} = \mathbf{D}(\mathbf{D}(\mathrm{E}))$ is multiplicative, as desired. $\square$

Returning to our setup of interest, by Lemma A.5.3 the composite isogeny

$$\mathrm{G} \to \mathrm{G}/\mathrm{M} = \mathrm{G}' \to \mathrm{G}'/\mathrm{M}'$$

is a quotient by a central multiplicative infinitesimal k-group. Now we're in position to wrap things up in positive characteristic (when k is infinite, arguing by induction on $\dim \mathrm{G}$).

First, we handle the case when the above process keeps going on *forever.* This provides a *strictly increasing* sequence $\mathrm{M}_1 \subset \mathrm{M}_2 \subset \dots$ of central multiplicative infinitesimal k-subgroups of G. This is all happening inside the k-subgroup scheme Z_{G}, so it forces Z_{G} to not be finite (as otherwise there would be an upper bound on the k-dimensions of the coordinate rings of the M_j's). Since $(\mathrm{Z}_{\mathrm{G}})^0_{\overline{k}}/((\mathrm{Z}_{\mathrm{G}})^0_{\overline{k}})_{\mathrm{red}}$ is a finite infinitesimal group scheme, for large enough j the map to this from $(\mathrm{M}_j)_{\overline{k}}$ must have nontrivial kernel. In other words, the smooth connected commutative group $((\mathrm{Z}_{\mathrm{G}})^0_{\overline{k}})_{\mathrm{red}}$ contains a nontrivial infinitesimal subgroup that is multiplcative. The group $((\mathrm{Z}_{\mathrm{G}})^0_{\overline{k}})_{\mathrm{red}}$ therefore cannot be unipotent (since a smooth unipotent group cannot contain μ_p), so it must contain a nontrivial torus! We conclude by the same argument with $\mathrm{Z}_{\mathrm{G}}[n]$'s as in § A.2 (using n not divisible by $\mathrm{char}(k) = p$) that Z_{G} contains a nontrivial k-torus, so we win.

There remains the more interesting case when the preceding process does eventually stop, so we wind up with a central quotient map

$$\mathrm{G} \to \mathrm{G}/\mathrm{M}$$

by a multiplicative infinitesimal k-subgroup M such that G/M satisfies $(\star)$; beware that now $\mathrm{M}_{\overline{k}}$ is merely a product of several $\mu_{p^{n_i}}$'s, not necessarily a power of μ_p. We therefore get a nontrivial k-torus T in G/M, so if $\mathrm{E} \subset \mathrm{G}$ denotes its scheme-theoretic preimage then there exists a short exact sequence of k-group schemes

$$1 \to \mathrm{M} \to \mathrm{E} \to \mathrm{T} \to 1$$

with M central in E. We will be done (for infinite k) if any such E contains a nontrivial k-torus. This is the content of the following lemma.

Lemma A.5.4. — *For any field k of characteristic $p > 0$ and short exact sequence of k-groups*

$$1 \to \mathrm{M} \to \mathrm{E} \to \mathrm{T} \to 1$$

with a central multiplicative infinitesimal k-subgroup M *in* E *and a nontrivial k-torus* T*, there exists a nontrivial k-torus in* E.

Proof. — Certainly $\mathrm{E}_{\overline{k}}$ is connected, since T and M are connected. The commutator map on E factors through a bi-additive pairing $\mathrm{T} \times \mathrm{T} \to \mathrm{M}$ which is trivial since T is smooth and M is infinitesimal. Hence, E is commutative. The map $\mathrm{E}_{\overline{k}} \to \mathrm{T}_{\overline{k}}$ is bijective on $\overline{k}$-points, so $(\mathrm{E}_{\overline{k}})_{\mathrm{red}}$ is a nontrivial smooth connected commutative $\overline{k}$-group. It is therefore a direct product of a torus and a smooth connected unipotent group, and the unipotent part must be trivial (since $\mathrm{T}_{\overline{k}}$ is a torus). Hence, $(\mathrm{E}_{\overline{k}})_{\mathrm{red}}$ is a nontrivial torus. Since E is commutative, the identity component of the Zariski-closure of the k-subgroup schemes $\mathrm{E}[n]$ for n not divisible by p is a k-torus T' in E such that $\mathrm{T}'_{\overline{k}} \to (\mathrm{E}_{\overline{k}})_{\mathrm{red}}$ is surjective, so $\mathrm{T}' \neq 1$. $\square$

Appendix B

Groups of multiplicative type

B.1. Basic definitions and properties. — For a scheme S and finitely generated $\mathbf{Z}$-module M, we define $\mathrm{D_S(M)}$ to be the S-group $\mathrm{Spec}(\mathscr{O}_\mathrm{S}[\mathrm{M}])$ representing the functor $\underline{\mathrm{Hom}}_{\text{S-gp}}(\mathrm{M_S}, \mathbf{G}_\mathrm{m})$ of characters of the constant S-group $\mathrm{M_S}$. (This is denoted $\mathrm{D(M)_S}$ in [**Oes**, I, 5.1]; it is the base change to S of the analogous group scheme over $\mathrm{Spec}\,\mathbf{Z}$. See [**Oes**, I, 5.2] for the proof that it represents the functor of characters of $\mathrm{M_S}$.)

Definition B.1.1. — A group scheme $\mathrm{G} \to \mathrm{S}$ is *of multiplicative type* if there is an fppf covering $\{\mathrm{S}_i\}$ of S such that $\mathrm{G}_{\mathrm{S}_i} \simeq \mathrm{D}_{\mathrm{S}_i}(\mathrm{M}_i)$ for a finitely generated abelian group M_i for each i.

By fppf descent, a multiplicative type S-group G is faithfully flat and finitely presented over S. Such a G is *split* (or *diagonalizable*) over S if $\mathrm{G} \simeq \mathrm{D_S(M)}$ for some M; see [**Oes**, I] for the theory of such groups, and [**Oes**, I, 5.2] for the anti-equivalence between the categories of split S-groups of multiplicative type and constant commutative S-groups with finitely generated geometric fibers.

In [**Oes**] the basic theory of multiplicative type groups $\mathrm{G} \to \mathrm{S}$ is developed under weaker conditions (following [**SGA3**]): M is not required to be finitely generated, G is fpqc and affine over S but may not be of finite presentation, and diagonalizability is required only fpqc-locally on S. The proofs of all results in [**Oes**] that we cite below work verbatim under our finiteness restrictions, due to our insistence on fppf-local diagonalizability in the definition. (In Corollary B.4.2(1) we show that fpqc-local diagonalizability recovers Definition B.1.1 for fppf group schemes. Thus, our multiplicative type groups are precisely those of [**Oes**] and [**SGA3**] with finite type structural morphism; we never use this.)

Remark B.1.2. — In [**Oes**, I, 5.4] it is noted that for any fppf closed S-subgroup H of $\mathrm{D_S(M)}$ there is a unique partition $\{\mathrm{S_N}\}_{\mathrm{N}\subset\mathrm{M}}$ of S into pairwise disjoint open and closed subschemes $\mathrm{S_N}$ indexed by the subgroups $\mathrm{N} \subset \mathrm{M}$, with $\mathrm{H}|_{\mathrm{S_N}} = \mathrm{D_S(M/N)}$ inside $\mathrm{D_S(M)}$. The special case $\mathrm{S} = \mathrm{Spec}\,k$ for a field k is addressed there (M may not be torsion-free, so $\mathrm{D}_k(\mathrm{M})$ may be disconnected or non-smooth), and the general case is addressed in [**Oes**, II, § 1.5, Rem. 3, 4]. A direct proof over fields under our finiteness hypotheses on M is given in Exercise 2.4.1, and to settle the case of arbitrary S we may use "spreading out" considerations (under our finiteness hypotheses) to reduce to S that is local, and even noetherian. This case is deduced from the field case in the proof of Corollary B.3.3 (which provides a generalization for all S-groups of multiplicative type).

An interesting consequence of this description of all such H is the general fact that if $G \to S$ is a group scheme of multiplicative type then there is a closed subtorus $T \subset G$ that is maximal in the sense that it (i) contains all closed subtori of G and (ii) retains this property after *arbitrary* base change. Such a T is unique if it exists, so by fppf descent it suffices to treat the case $G = D_S(M)$ for a finitely generated abelian group M. Let $M' = M/M_{tor}$ denote the maximal torision-free quotient of M, and define $T = D_S(M') \subset D_S(M) = G$. Clearly it suffices to show that every closed subtorus of G is contained in T, and we may assume S is non-empty. Working Zariski-locally on S, the above description reduces our task to considering closed subtori of the form $D_S(M/N)$ for a subgroup N in M. For any such N, the torus property for $D_S(M/N)$ forces M/N to be torsion-free (since $S \neq \varnothing$). Hence, M/N is dominated by M' as quotients of M, so we are done.

We refer the reader to [**Oes**, II, 2.1] for several notions of "local triviality" for multiplicative type groups (isotriviality, quasi-isotriviality, etc.) For many applications, it is important that multiplicative type groups are split étale-locally on the base (quasi-isotriviality). This will be proved in Proposition B.3.4 and rests on the following lemma.

Lemma B.1.3. — *Let* G *be an* S*-affine* S*-group scheme of finite presentation and* H *an* S*-group of multiplicative type. Any monic homomorphism* $j : H \to G$ *is necessarily a* closed immersion.

This lemma is useful in constructions with fiberwise maximal tori in reductive group schemes, and eliminates ambiguity about the meaning of "subgroup of multiplicative type" for homomorphisms from groups of multiplicative type: working sheaf-theoretically (or in terms of group functors) with monomorphisms is equivalent to working algebro-geometrically with closed immersions.

Proof. — This result is [**SGA3**, IX, 2.5] without finite presentation hypotheses, and it is also a special case of [**SGA3**, VIII, 7.13(b)] (relaxing affineness to separatedness). Here is an alternative direct argument under our hypotheses.

We may pass to the case of noetherian S, and it suffices to show that j is proper (since a finitely presented proper monomorphism is a closed immersion [**EGA**, IV_3, 8.11.5]). By the valuative criterion for properness, we may express the problem in terms of points valued in a discrete valuation ring R and its fraction field K over S, so we may reduce to the case that $S = \operatorname{Spec} R$ and it is harmless to make a (necessarily faithfully flat) local injective base change $R \to R'$ to another discrete valuation ring. Thus, we can assume that R is henselian. We claim that such an R' may be found so that $H_{R'}$ is split. Pick an fppf cover $S' \to S$ that splits H. As for any fppf cover of an affine scheme, there is an affine flat quasi-finite surjection $S'' \to S$ admitting a map $S'' \to S'$ over S [**EGA**, IV_4, 17.16.2]. Since R is henselian local, by [**EGA**, IV_4, 18.5.11(c)] the affine flat quasi-finite cover S'' of S contains an open and closed local subscheme that is S-finite. This subscheme is finite flat over S and non-empty, so by the Krull–Akizuki theorem [**Mat**, Thm. 11.7] the normalization of its underlying reduced scheme has the form $\mathrm{Spec}\,R'$ for a discrete valuation ring R'. Moreover, by design $H_{R'}$ is split, so by renaming R' as R, now H is split.

Letting k be the residue field of R, the maps j_K and j_k are monomorphisms between affine finite type group schemes over a field, so they are closed immersions (apply Remark 1.1.4 on geometric generic and special fibers). Let H' be the schematic closure of H_K in G. This is an R-flat closed subgroup since R is Dedekind, and it is commutative since the K-group $H'_K = H_K$ is commutative. We may replace G with H', so now G is commutative. For any $n \geqslant 1$, the monomorphism $H[n] \to G$ is proper since $H[n]$ is finite, so it is a closed immersion.

It is harmless to pass to $G/H[n]$ and $H/H[n]$. (See [**SGA3**, VIII, 5.1; IX, 2.3] for a discussion of the existence and properties of quotients by the free action of a group of multiplicative type on a finitely presented relatively affine scheme. This is simpler than the general theory of quotients by the free action of a finite locally free group scheme as developed in [**SGA3**, V, § 2(a), Thm. 4.1].) More specifically, for n divisible by the order of the torsion subgroup of the finitely generated abelian group that is "dual" to the split H, geometric fibers of $H/H[n]$ are smooth (i.e., tori). Thus, by passing to $G/H[n]$ and $H/H[n]$ we may assume $H \simeq \mathbf{G}_m^r$ for some $r \geqslant 0$.

We can assume that k is algebraically closed and $H \neq 1$. Let $T = \mathbf{G}_m$ be the first factor of $H = \mathbf{G}_m^r$ and let T' be the closure of T_K in G. If we can prove that $T' = T$ then we may pass to the quotient by T and induct on r. Thus, we may assume $H = \mathbf{G}_m$, so G also has 1-dimensional fibers. In particular, the closed immersion j_k must identify H_k with $(G_k)^0_{\mathrm{red}}$. For $N > 0$ relatively prime to the order of the finite group scheme G_k/H_k, a snake lemma argument shows that $N : G_k \to G_k$ is a quotient map with kernel $H_k[N]$, so it is flat. Hence, by the fibral flatness criterion, $N : G \to G$ is flat, so $G[N]$ is flat and hence $G[N] = H[N]$ as closed subschemes of G due to the equality of their generic fibers in $G_K = H_K$.

The translation action by $H = \mathbf{G}_m$ on G defines a $\mathbf{Z}$-grading $\bigoplus_{n \in \mathbf{Z}} A_n$ of the coordinate ring A of G [**Oes**, III, 1.5]. (Explicitly, this grading extends the natural one on the coordinate ring A_K of $G_K = H_K = \mathbf{G}_m$.) The quotient map $A \twoheadrightarrow \mathscr{O}(G[N])$ between flat R-modules is injective on each A_n because it is so over K (as $G_K = \mathbf{G}_m$). It follows that each A_n is *finitely generated* over R, and thus is finite free of rank 1 because we can compute the rank over K. The map $A \to \mathscr{O}(H) = R[t, 1/t]$ respects the $\mathbf{Z}$-gradings and induces a surjection on special fibers (as j_k is a closed immersion), so the induced maps between the rank-1 graded parts are isomorphisms. Hence, $A \to \mathscr{O}(H)$ is an isomorphism. □

Remark B.1.4. — Beware that even over a discrete valuation ring, there are examples of monic homomorphisms f between smooth affine groups with connected fibers such that f is not an immersion! See [**SGA3**, XVI, 1.1(c)] for some explicit examples (and [**SGA3**, VIII, § 7] for further discussion). The contrast with the case $\mathrm{S} = \operatorname{Spec} k$ for a field k is that in such cases homomorphisms between smooth affine S-groups are always faithfully flat onto a closed image (due to the "locally closed" property of G-orbits over fields, which has no good analogue in comparable generality in the relative case).

Lemma B.1.5. — *Let k be a field. A k-group* H *of multiplicative type splits over a finite separable extension of k.*

Proof. — By direct limit considerations, it suffices to prove that H_{k_s} is split. Thus, we may assume $k = k_s$ and aim to prove that $\mathrm{H} \simeq \mathrm{D}_k(\mathrm{M})$ for a finitely generated abelian group M. Since any fppf cover of $\operatorname{Spec} k$ acquires a rational point over a finite extension k'/k, there is a finitely generated abelian group M such that H and $\mathrm{D}_k(\mathrm{M})$ become isomorphic over a finite extension k' of k.

The functor $\mathrm{I} = \underline{\mathrm{Isom}}(\mathrm{H}, \mathrm{D}_k(\mathrm{M}))$ is an fppf sheaf on the category of k-schemes, so it is an Aut(M)-torsor because this can be checked upon restriction to the category of k'-schemes. More specifically, the restriction of I over k' is represented by a constant k'-scheme, and that constant scheme is equipped with an evident descent datum (arising from I) relative to k'/k. Although $\mathrm{I}_{k'}$ is generally not affine, nor even quasi-compact, the descent to k is easily checked to be effective because (i) fppf descent is effective for affine schemes and (ii) in our descent problem the "equivalence classes" inside $\mathrm{I}_{k'}$ are open and closed subschemes that are *k-finite*. Hence, I is represented by a k-scheme that is non-empty and étale (as it becomes constant and non-empty over k'), so I is constant (since k is separably closed). □

B.2. Hochschild cohomology. — The key to the infinitesimal properties of groups of multiplicative type is the vanishing of their higher Hochschild cohomology over an affine base. A general introduction to Hochschild cohomology $\mathrm{H}^i(\mathrm{G}, \mathscr{F})$ for flat affine group schemes $\mathrm{G} \to \mathrm{S}$ acting linearly on quasi-coherent $\mathscr{O}_\mathrm{S}$-modules $\mathscr{F}$ over an affine S is given in [**Oes**, III, § 3]. Hochschild cohomology is a scheme-theoretic version of ordinary group cohomology, and is explained (with proofs) in [**Oes**, III, § 3]. Some applications below require a more "functorial" description of Hochschild cohomology, so we now review the basic setup using a variation on the formulation given in [**Oes**, III,§ 3].

Let M be a commutative group functor on the category of S-schemes, equipped with an action by an S-group scheme G. A case of much interest is the functor $\mathrm{S}' \rightsquigarrow \Gamma(\mathrm{S}', \mathscr{F}_{\mathrm{S}'})$ arising from a *quasi-coherent* G*-module*, by which we mean a quasi-coherent $\mathscr{O}_\mathrm{S}$-module $\mathscr{F}$ equipped with a linear G-action;

see [**Oes**, III, 1.2]. (This amounts to an $\mathscr{O}_{S'}$-linear action of $G(S')$ on $\mathscr{F}_{S'}$ functorially in S'. By consideration of the "universal point" $\mathrm{id}_G : G \to G$, it is equivalent to specifying an $\mathscr{O}_G$-linear automorphism of $\mathscr{F}_G$.) For $n \geqslant 0$, let $C^n(G, M)$ be the abelian group of natural transformations of set-valued functors $c : G^n \to M$ on the category of S-schemes (i.e., compatible systems of maps of sets $G(T)^n \to M(T)$, or equivalently the abelian group $M(G^n)$). For example, if $S = \operatorname{Spec} A$ is affine and $G = \operatorname{Spec}(B)$ is affine and $\mathscr{F} = \widetilde{V}$ for an A-linear B-comodule V then the element $v \otimes b_1 \otimes \cdots \otimes b_n \in V \otimes_A B^{\otimes n} = \Gamma(G^n, \mathscr{F}_{G^n})$ corresponds to $c : G^n \to \mathscr{F}$ defined functorially by $(g_1, \ldots, g_n) \mapsto (\prod g_i^*(b_i))v_{S'}$ on S'-valued points for any S-scheme S'.

The groups $C^n(G, M) = M(G^n)$ form a complex $C^\bullet(G, M)$ in the habitual manner, by defining $(\mathrm{d}c)(g_0, \ldots, g_n)$ to be

$$g_0.c(g_1, \ldots, g_n) + \sum_{i=1}^{n} (-1)^i c(g_0, \ldots, g_{i-1}g_i, \ldots, g_n) + (-1)^{n+1} c(g_0, \ldots, g_{n-1}).$$

We define the Hochschild cohomology

$$H^n(G, M) = H^n(C^\bullet(G, M)).$$

For example, $H^0(G, M)$ is the group $M(S)^G$ of $m \in M(S)$ that are G-invariant in the sense that $g.m_{S'} = m_{S'}$ in $M(S')$ for any $S' \to S$ and $g \in G(S')$. More concretely, G-invariance means that the pullback $m_G \in M(G)$ is invariant under the universal point $\mathrm{id}_G \in G(G)$ (not to be confused with the G-pullback of the identity section $e \in G(S)$).

Example B.2.1. — Let $j : S_0 \hookrightarrow S$ be a closed immersion of schemes. For any quasi-coherent sheaf $\mathscr{F}_0$ on S_0 equipped with a G_0-action, and the associated quasi-coherent sheaf $\mathscr{F} := j_*(\mathscr{F}_0)$ on S with G-action via $G \to j_*(G_0)$, naturally $H^*(G, \mathscr{F}) \simeq H^*(G_0, \mathscr{F}_0)$.

Proposition B.2.2. — *If* $f : S' = \operatorname{Spec} A' \to \operatorname{Spec} A = S$ *is a flat map of affine schemes and* G *is* S*-affine then for any quasi-coherent* G*-module* $\mathscr{F}$ *the natural map* $A' \otimes_A H^n(G, \mathscr{F}) \to H^n(G_{A'}, \mathscr{F}_{A'})$ *is an isomorphism for all* $n \geqslant 0$.

Proof. — Each G^n is affine, so the natural map of complexes

$$A' \otimes_A C^\bullet(G, \mathscr{F}) \to C^\bullet(G_{S'}, \mathscr{F}_{S'})$$

is identified in degree n with the natural map $A' \otimes_A V_{G^n} \to (A' \otimes_A V)_{G^n_{A'}}$ where $\widetilde{V} = \mathscr{F}$. This map is visibly an isomorphism. Passing to homology in degree n recovers the map of interest as an isomorphism because the exact functor $A' \otimes_A (\cdot)$ commutes with the formation of homology in the evident manner. $\square$

Assume G is S-affine and S-*flat*, and that S is also affine (so each G^n is affine and S-flat). The functor $\mathscr{F} \rightsquigarrow C^n(G, \mathscr{F})$ on quasi-coherent G-modules is exact for each n, so $H^\bullet(G, \mathscr{F})$ is δ-functorial in such $\mathscr{F}$ via the snake lemma. We shall prove that the category of quasi-coherent G-modules has enough injectives and the derived functor of $\mathscr{F} \rightsquigarrow \mathscr{F}(S)^G$ on this category is $H^\bullet(G, \cdot)$.

For S-affine G, there is a right adjoint "Ind" to the forgetful functor from quasi-coherent G-modules to quasi-coherent $\mathscr{O}_S$-modules. Explicitly, for any S-scheme S′ and quasi-coherent $\mathscr{O}_S$-module $\mathscr{G}$, $\mathrm{Ind}(\mathscr{G})(S') := \Gamma(G_{S'}, \mathscr{G}_{G_{S'}})$ equipped with the left $G(S')$-action $(g.f)(x) = f(xg)$ for points x of $G_{S'}$. In other words, as in [**Oes**, III, 3.2, Ex.], $\mathrm{Ind}(\mathscr{G})$ is the quasi-coherent pushforward of $\mathscr{G}_G$ along the affine map $G \to S$, equipped with a natural $\mathscr{O}_S$-linear G-action. The G-action amounts to an $\mathscr{O}_G$-linear automorphism of the pullback of $\mathrm{Ind}(\mathscr{G})$ along $G \to S$. By general nonsense, Ind carries injectives to injectives, and monomorphisms to monomorphisms. Moreover, the adjunction morphism $\mathscr{F} \to \mathrm{Ind}(\mathscr{F})$ is given by $m \mapsto (g \mapsto g.m)$; it has trivial kernel since $f \mapsto f(1)$ is a retraction. Thus, the category of quasi-coherent G-modules on S has enough injectives (since the category of quasi-coherent $\mathscr{O}_S$-modules does; recall that S is affine).

Lemma B.2.3. — *For affine* S *and* S-*flat* S-*affine* G, *the* δ-*functor* $H^\bullet(G, \cdot)$ *on quasi-coherent* G-*modules is the right derived functor of* $\mathscr{F} \rightsquigarrow \mathscr{F}(S)^G$.

Proof. — An elegant argument in [**Oes**, III, 3.2, Rem.] (inspired by the proof of acyclicity of induced modules for ordinary group cohomology) shows that $H^i(G, \mathrm{Ind}(\mathscr{G})) = 0$ for any quasi-coherent $\mathscr{O}_S$-module $\mathscr{G}$ and $i > 0$. For any injective $\mathscr{F}$ in the category of quasi-coherent G-modules, the inclusion $\mathscr{F} \to \mathrm{Ind}(\mathscr{F})$ splits off $\mathscr{F}$ as a G-equivariant direct summand. But $\mathrm{Ind}(\mathscr{F})$ is acyclic for Hochschild cohomology, so $\mathscr{F}$ is as well. We conclude that $H^\bullet(G, \cdot)$ is erasable on the category of quasi-coherent G-modules, so on this category it is the desired right derived functor. □

Remark B.2.4. — A consequence of Lemma B.2.3 (as in [**Oes**, III, 3.3]) is that for any *multiplicative type* group G over an affine $S = \mathrm{Spec}\, A$, $H^i(G, \cdot) = 0$ on quasi-coherent objects for $i > 0$. Indeed, by Proposition B.2.2 it suffices to prove this after a faithfully flat affine base change on A, so we may assume $G = D_S(M)$. Thus, quasi-coherent G-modules are "the same" as M-graded quasi-coherent $\mathscr{O}_S$-modules $\mathscr{F} = \bigoplus_{m \in M} \mathscr{F}_m$ [**Oes**, III, 1.5], so $H^0(G, \mathscr{F}) = \mathscr{F}_0(S)$. This is exact in such $\mathscr{F}$, so its higher derived functors vanish.

An important application of Hochschild cohomology is the construction of obstructions to deformations of homomorphisms between group schemes. This is inspired by the use of low-degree group cohomology to classify group extensions, and comes out as follows.

Proposition B.2.5. — *Consider a scheme* S *and short exact sequence of group sheaves*

$$1 \to \mathrm{M} \to \mathrm{G} \to \mathrm{G}' \to 1$$

for the fpqc topology, with M *commutative. Use this exact sequence to make* G' *act on* M *via the left* G*-conjugation action on* M.

Let H *be an* S*-group scheme, and fix an* S*-homomorphism* $f' : \mathrm{H} \to \mathrm{G}'$*. Make* H *act on* M *through composition with* f'*. Assume that* f' *admits a lifting to an* H*-valued point of* G*.*

There is a canonically associated class $c(f') \in \mathrm{H}^2(\mathrm{H}, \mathrm{M})$ *that vanishes if and only if* f' *lifts to an* S*-homomorphism* $f : \mathrm{H} \to \mathrm{G}$*. If such an* f *exists, the set of such lifts taken up to conjugation by* M(S) *on* G *is a principal homogeneous space for the group* $\mathrm{H}^1(\mathrm{H}, \mathrm{M})$*.*

The result holds with the same proof using any topology for which representable functors are sheaves (e.g., Zariski, étale, fppf). See [**SGA3**, III, 1.2.2] for further generality.

Proof. — Fix an $f \in \mathrm{G}(\mathrm{H})$ lifting f'. The obstruction to f being a homomorphism is the vanishing of the map of $c = c_f \in \mathrm{M}(\mathrm{H} \times \mathrm{H})$ defined by $c(h_0, h_1) = f(h_0h_1)f(h_1)^{-1}f(h_0)^{-1}$. The action of any h_0 on $c(h_1, h_2)$ is induced by $f(h_0)$-conjugation, so

$$\begin{aligned} h_0.c(h_1, h_2) &= f(h_0)(f(h_1h_2)f(h_2)^{-1}f(h_1)^{-1})f(h_0)^{-1} \\ &= c(h_0, h_1h_2)^{-1}c(h_0h_1, h_2)c(h_0, h_1) \\ &= c(h_0h_1, h_2) - c(h_0, h_1h_2) + c(h_0, h_1). \end{aligned}$$

In other words, the element $c_f \in \mathrm{M}(\mathrm{H} \times \mathrm{H})$ is a Hochschild 2-cocycle (where M is equipped with its natural H-action through f'). Fixing one choice of f, all choices have exactly the form $m \cdot f : h \mapsto m(h) \cdot f(h)$ for $m \in \mathrm{M}(\mathrm{H})$, and the value on (h_0, h_1) for the associated 2-cocycle $c_{m \cdot f}$ is

$$\begin{aligned} &m(h_0h_1)f(h_0h_1)f(h_1)^{-1}f(h_0)^{-1}(f(h_0)m(h_1)f(h_0)^{-1})^{-1}m(h_0)^{-1} = \\ &\qquad\qquad m(h_0h_1) + c_f(h_0, h_1) - h_0.m(h_1) - m(h_0). \end{aligned}$$

Thus, the class of c_f in $\mathrm{H}^2(\mathrm{H}, \mathrm{M})$ only depends on f' and not the choice of f, and as we vary through all f this 2-cocycle exhausts exactly the members of its cohomology class. Thus, this class vanishes if and only if we can choose f so that $c_f = 0$, which is to say that f is an S-homomorphism.

Assume that an S-homomorphism f lifting f' exists. Fix one such choice of f. The preceding calculation shows that the possible choices for f as an S-homomorphism are precisely $h \mapsto m(h)f(h)$ where m is a Hochschild 1-cocycle on H with values in M. Applying conjugation to such an f by some $m_0 \in \mathrm{M}(\mathrm{S})$ replaces f with the lifting

$$h \mapsto m_0 \cdot f(h) \cdot m_0^{-1} = m_0 \cdot (f(h)m_0^{-1}f(h)^{-1}) \cdot f(h) = (m_0 - h.m_0) \cdot f(h)$$

due to the definition of the H-action on M (through the G′-action on M induced by G-conjugation). It follows that $\mathrm{H}^1(\mathrm{H}, \mathrm{M})$ acts simply transitively on the set of M(S)-conjugacy classes of S-homomorphisms f lifting f'. □

Corollary B.2.6. — *Let* $\mathrm{G} \to \mathrm{S}$ *be a smooth group over an affine scheme* $\mathrm{S} = \operatorname{Spec} \mathrm{A}$*, and let* H *be an affine* S*-group. Let* J *be a square-zero ideal in* A*,* $\mathrm{A}_0 = \mathrm{A}/\mathrm{J}$*,* $\mathrm{S}_0 = \operatorname{Spec} \mathrm{A}_0$*,* $\mathrm{G}_0 = \mathrm{G} \bmod \mathrm{J}$*, and* $\mathrm{H}_0 = \mathrm{H} \bmod \mathrm{J}$*. Fix an* S_0*-homomorphism* $f_0 : \mathrm{H}_0 \to \mathrm{G}_0$*, and let* H_0 *act on* $\mathrm{Lie}(\mathrm{G}_0)$ *via* $\mathrm{Ad}_{\mathrm{G}_0} \circ f_0$*.*

There is a canonically associated class $c(f_0) \in \mathrm{H}^2(\mathrm{H}_0, \mathrm{Lie}(\mathrm{G}_0) \otimes \mathrm{J})$ *whose vanishing is necessary and sufficient for* f_0 *to lift to an* S*-homomorphism* $f : \mathrm{H} \to \mathrm{G}$*. If such an* f *exists, the set of such lifts taken up to conjugation by* $\ker(\mathrm{G}(\mathrm{S}) \to \mathrm{G}(\mathrm{S}_0))$ *is a principal homogeneous space for the group* $\mathrm{H}^1(\mathrm{H}_0, \mathrm{Lie}(\mathrm{G}_0) \otimes \mathrm{J})$*.*

See [**SGA3**, III, 2.2, 2.3] for generalizations. By Remark B.2.4, $c(f_0) = 0$ if H is of multiplicative type.

Proof. — Let $i : \mathrm{S}_0 \to \mathrm{S}$ be the canonical closed immersion, so $i_*(\mathrm{G}_0)$ is the group functor $\mathrm{S}' \rightsquigarrow \mathrm{G}_0(\mathrm{S}'_0)$ on S-schemes. An S_0-homomorphism $f_0 : \mathrm{H}_0 \to \mathrm{G}_0$ corresponds to an S-group functor homomorphism $f'_0 : \mathrm{H} \to i_*(\mathrm{G}_0)$. An S-homomorphism $f : \mathrm{H} \to \mathrm{G}$ lifts f'_0 if and only if $f \bmod \mathrm{J} = f_0$. Thus, we focus on lifting f'_0.

By the smoothness of G, the natural homomorphism $q : \mathrm{G} \to i_*(\mathrm{G}_0)$ is surjective for the Zariski topology (and hence for any finer topology). More specifically, since G is smooth and $\mathrm{H}_0 \hookrightarrow \mathrm{H}$ is defined by a square-zero ideal on the affine scheme H, f_0 admits a lifting f'_0 through q as a scheme morphism. It also follows from the smoothness of G that $\ker q$ is the group functor on S-schemes associated to the quasi-coherent $\mathscr{O}_{\mathrm{S}_0}$-module $\mathrm{Lie}(\mathrm{G}_0) \otimes \mathrm{J}$. (This uses that $\mathrm{J}\mathscr{O}_{\mathrm{G}} = \mathrm{J} \otimes_{\mathrm{A}_0} \mathscr{O}_{\mathrm{G}_0}$, a consequence of the A-flatness of G.) The conjugation action by G on the commutative $\ker q = \mathrm{Lie}(\mathrm{G}_0) \otimes \mathrm{J}$ factors through an action by $i_*(\mathrm{G}_0)$, and this "is" the adjoint action of G_0 on $\mathrm{Lie}(\mathrm{G}_0)$. Thus, Proposition B.2.5 applies to the exact sequence

$$1 \to \mathrm{Lie}(\mathrm{G}_0) \otimes \mathrm{J} \to \mathrm{G} \to i_*(\mathrm{G}_0) \to 1.$$

Quasi-coherence of the kernel implies $\mathrm{H}^i(\mathrm{H}, \mathrm{Lie}(\mathrm{G}_0) \otimes \mathrm{J}) = \mathrm{H}^i(\mathrm{H}_0, \mathrm{Lie}(\mathrm{G}_0) \otimes \mathrm{J})$ via Example B.2.1. □

Via induction, the preceding corollary immediately yields:

Corollary B.2.7. — *Let* $\mathrm{S} = \mathrm{Spec}(\mathrm{A})$ *and* $\mathrm{S}_0 = \mathrm{Spec}(\mathrm{A}/\mathrm{J})$ *for an ideal* J *of* A *such that* $\mathrm{J}^{n+1} = 0$ *for some* $n \geqslant 0$*. An* S*-homomorphism* $f : \mathrm{H} \to \mathrm{G}$ *from a multiplicative type* S*-group* H *to an arbitrary* S*-group scheme* G *is trivial if its restriction* f_0 *over* S_0 *is trivial.*

This rigidity property is [**SGA3**, IX, Cor. 3.5].

B.3. Deformation theory. — To relativize results established over fields, it is important to have "fibral criteria" for properties of morphisms (such as flatness, smoothness, etc.) as well as deformation-theoretic results concerning the obstructions to lifting problems. In the direction of fibral criteria, we often need the "fibral isomorphism criterion":

Lemma B.3.1. — *Let $h : \mathrm{Y} \to \mathrm{Y}'$ be a map between locally finitely presented schemes over a scheme* S*, and assume that* Y *is* S*-flat. If h_s is an isomorphism for all $s \in \mathrm{S}$ then h is an isomorphism.*

Proof. — This is part of [**EGA**, IV_4, 17.9.5] (or see Exercise 3.4.3). □

Theorem B.3.2. — *Let $(\mathrm{A}, \mathfrak{m})$ be a complete local noetherian ring with residue field k, $\mathrm{S} = \operatorname{Spec} \mathrm{A}$, G an affine* S*-group of finite type, and* H *an* S*-group of multiplicative type that splits over a finite étale cover of* S*. Let $\mathrm{S}_n = \operatorname{Spec} \mathrm{A}/\mathfrak{m}^{n+1}$.*

1. *The natural map*

$$\mathrm{Hom}_{\mathrm{S}\text{-gp}}(\mathrm{H}, \mathrm{G}) \to \varprojlim \mathrm{Hom}_{\mathrm{S}_n\text{-gp}}(\mathrm{H}_{\mathrm{S}_n}, \mathrm{G}_{\mathrm{S}_n}) \tag{B.3.1}$$

   is bijective.

2. *If* G *is* S*-flat and the special fiber G_0 is of multiplicative type then the map*

$$\mathrm{Hom}_{\mathrm{S}\text{-gp}}(\mathrm{H}, \mathrm{G}) \to \mathrm{Hom}_{k\text{-gp}}(\mathrm{H}_0, \mathrm{G}_0)$$

   is bijective and any isomorphism $j_0 : \mathrm{H}_0 \simeq \mathrm{G}_0$ uniquely lifts to an open and closed immersion $j : \mathrm{H} \to \mathrm{G}$.

The splitting hypothesis on H is temporary in the sense that it will be shown to always hold (see Proposition B.3.4). Also, the flatness hypothesis on G in (2) cannot be dropped: for an integer $d > 1$ let $\mathrm{H} = (\mathbf{Z}/d\mathbf{Z})_{\mathrm{R}}$ over a discrete valuation ring R such that $d \in \mathrm{R}^{\times}$ and let G be the reduced closed complement of the open non-identity locus in the generic fiber.

Proof. — Let $\mathrm{S}' = \operatorname{Spec} \mathrm{A}'$ be a finite étale cover of S that splits H. We may assume S' is connected and Galois over S, so A' is a complete local noetherian ring with maximal ideal $\mathfrak{m}\mathrm{A}'$ and descent from S' to S can be expressed in terms of actions by $\Gamma = \mathrm{Aut}(\mathrm{S}'/\mathrm{S})$. The same Γ works for descent from S'_n to S_n. Thus, if the analogue of (B.3.1) over S' is bijective then Γ-equivariance considerations show that (B.3.1) is bijective. The same holds for part (2). We therefore may and do assume $\mathrm{H} = \mathrm{D}_{\mathrm{S}}(\mathrm{M})$ for a finitely generated abelian group M. Upon expressing the problem in (1) in terms of maps of Hopf algebras, it is solved by a clever use of the interaction between completions and tensor products beyond the module-finite setting. See the proof of [**SGA3**, IX, 7.1] for this important calculation.

We now prove part (2). In such cases, by (1) the bijectivity of "passage to the special fiber" on homomorphisms is reduced to showing that for all $n \geqslant 1$ the reduction map

$$\mathrm{Hom}_{\mathrm{S}_n\text{-gp}}(\mathrm{D}_{\mathrm{S}_n}(\mathrm{M}), \mathrm{G}_{\mathrm{S}_n}) \to \mathrm{Hom}_{k\text{-gp}}(\mathrm{D}_k(\mathrm{M}), \mathrm{G}_0)$$

is bijective. That is, upon renaming S as S_n, we may assume A is an artin local ring. By descent we may replace A with a finite étale extension so that $\mathrm{G}_0 \simeq \mathrm{D}_k(\mathrm{N})$ for a finitely generated abelian group N. Since G is an *infinitesimal* flat deformation of $\mathrm{D}_k(\mathrm{N})$, by the deformation theory of split multiplicative type groups we claim that G must be of multiplicative type and even that $\mathrm{G} \simeq \mathrm{D}_{\mathrm{S}}(\mathrm{N})$.

To be precise, by induction on n it suffices to show that if J is a square-zero ideal in a ring R and $\mathrm{G} = \mathrm{Spec}\,\mathrm{A}$ is an fppf affine R-group such that over $\mathrm{R}_0 = \mathrm{R}/\mathrm{J}$ there is a group isomorphism $f_0 : \mathrm{G}_0 := \mathrm{G} \bmod \mathrm{J} \simeq \mathrm{D}_{\mathrm{R}_0}(\mathrm{N})$ for a finitely generated abelian group N then f_0 lifts to an R-group isomorphism $f : \mathrm{G} \simeq \mathrm{D}_{\mathrm{R}}(\mathrm{N})$. Since $\mathrm{H}^2(\mathrm{G}_0, \cdot) = 0$ on quasi-coherent G_0-modules (as G_0 is of multiplicative type), by Corollary B.2.6 the map f_0 lifts to an S-homomorphism f. But any such lift must be an isomorphism because f_0 is an isomorphism and the source and target of f are fppf over S. (A more explicit construction of f is given in [**Oes**, IV, § 1], directly building an obstruction in a degree-2 Hochschild cohomology group for the multiplicative type G_0.) The desired bijectivity in (2) now follows via duality for diagonalizable groups (of finite type).

Finally, with general complete local noetherian A, it remains to show that if G_0 is of multiplicative type and G is flat then for any $j_0 : \mathrm{D}_k(\mathrm{M}) \simeq \mathrm{G}_0$ the unique S-homomorphism $j : \mathrm{D}_{\mathrm{S}}(\mathrm{M}) \to \mathrm{G}$ lifting j_0 is an open and closed immersion. The preceding argument over S_n's shows that $j_n := j \bmod \mathfrak{m}^{n+1}$ is an isomorphism for all $n \geqslant 0$, so the map induced by j between formal completions along the identity is flat (by [**Mat**, 22.3(1)⇔(5)]). Thus, j is flat near the identity section. But j_s must be flat for all $s \in \mathrm{S}$ since a homomorphism between finite type groups over a field is flat if it is flat near the identity (use translation considerations on a geometric fiber), so by the fibral flatness criterion j is flat.

Consider the kernel $\mathrm{K} = \ker j$, a flat closed S-subgroup of $\mathrm{D}_{\mathrm{S}}(\mathrm{M})$ with $\mathrm{K}_0 = 0$. It suffices to prove $\mathrm{K} = 0$. Indeed, then j will be a monomorphism and so even a closed immersion (Lemma B.1.3), and a flat closed immersion between noetherian schemes is an open immersion. To prove that the flat S-group K is trivial, observe that each fiber K_s ($s \in \mathrm{S}$) is of multiplicative type, due to being a closed subgroup scheme of $\mathrm{D}_s(\mathrm{M})$ (see Exercise 2.4.1). For $n \geqslant 1$, the n-torsion $\mathrm{K}[n]$ is a finite S-group scheme (as it is closed in $\mathrm{D}_{\mathrm{S}}(\mathrm{M})[n] = \mathrm{D}_{\mathrm{S}}(\mathrm{M}/n\mathrm{M})$) and it has special fiber $\mathrm{K}[n]_0 = \mathrm{K}_0[n] = 0$, so by Nakayama's Lemma we have $\mathrm{K}[n] = 0$. Hence, for each $s \in \mathrm{S}$ the torsion

$K_s[n]$ vanishes for all $n \geqslant 1$, so the multiplicative type group K_s over $k(s)$ vanishes [**Oes**, II, 3.2]. In other words, the identity section $e : S \to K$ is an isomorphism on fibers over S, so it is an isomorphism by the fibral isomorphism criterion (Lemma B.3.1). $\square$

Corollary B.3.3. — *Let* S *be a scheme and* H' *an* S*-group of multiplicative type. Any fppf closed subgroup* $H \subset H'$ *is of multiplicative type.*

Proof. — Passing to an fppf cover, we may assume that $H' = D_S(M)$ for a finitely generated abelian group M. We may reduce to the case when $S = \operatorname{Spec} A$ for a ring A that is noetherian, and then local; let k be the residue field. Exercise 2.4.1 provides a subgroup $N \subset M$ such that $H_k = D_k(M/N)$ inside $D_k(M)$. We claim that $H = D_S(M/N)$ inside $D_S(M)$. It suffices to check this equality of closed subschemes after the fpqc base change to $\operatorname{Spec} \widehat{A}$, so we may assume A is complete. Thus, by Theorem B.3.2 the isomorphism $D_k(M/N) \simeq H_k$ uniquely lifts to an abstract S-homomorphism $j : D_S(M/N) \to H$ that is moreover an open and closed immersion. The composition of j with the inclusion $H \hookrightarrow H' = D_S(M)$ is an S-homomorphism $D_S(M/N) \to D_S(M)$ that reduces to the canonical inclusion over the closed point, so by duality for diagonalizable groups it is the canonical inclusion over the entire connected base S. In other words, j is a containment inside $H' = D_S(M)$; i.e., H contains $D_S(M/N)$ as an open and closed subscheme inside $D_S(M)$. Now we can pass to the quotients by $D_S(M/N)$ to reduce to the case that H_k is the trivial k-group and the identity section of H is an open and closed immersion. In this case we will prove that H is the trivial S-group. Since the fppf group scheme $H \to S$ has an identity section that is an open and closed immersion, its fibers are étale, so H is S-étale.

We claim that $H \to S$ is killed by some integer $n > 0$. Since S is noetherian and $H \to S$ is quasi-finite, there is an $n > 0$ that is a multiple of all fiber-degrees for $H \to S$. Thus, n kills each finite étale fiber group H_s. Since $\Delta_{H/S} : H \to H \times_S H$ is an open and closed immersion (as $H \to S$ is étale and separated), the pullback of $\Delta_{H/S}$ under $([n], 0) : H \rightrightarrows H \times_S H$ is an open and closed subscheme of H. But this open and closed subscheme has been seen to contain all fibers, so it coincides with H. Hence, n kills the S-group H.

We conclude that the *closed* subgroup $H \subset H'$ is contained in the S-finite $H'[n]$, so the S-étale H is S-finite and therefore $H \to S$ has constant fiber rank (by connectedness of S). This rank must be 1, due to triviality of H_k. But the identity section $e : S \to H$ is an open and closed immersion, so it is surjective and therefore an isomorphism as desired. $\square$

Proposition B.3.4. — *Let* $H \to S$ *be an* S*-group scheme that becomes diagonalizable* (*of finite type*) *fppf-locally on* S. *Then* H *is diagonalizable* *étale-locally on* S*; i.e.,* H *is quasi-isotrivial. More specifically, the functor* $\mathrm{H} \rightsquigarrow \underline{\mathrm{Hom}}_{\mathrm{S}\text{-gp}}(\mathrm{H}, \mathbf{G}_{\mathrm{m}})$ *is an anti-equivalence between the category of* S*-groups of multiplicative type and the category of locally constant abelian étale sheaves on* S *whose geometric fibers are finitely generated abelian groups.*

If $\mathrm{S} = \operatorname{Spec} \mathrm{A}$ *for a henselian local* A *then* H *splits over a finite étale cover of* S*.*

This result is used very often, generally without comment. For example, it implies that the splitting hypothesis on H in Theorem B.3.2 is always satisfied.

Proof. — The final assertion concerning local henselian S is a consequence of the rest because any étale cover of such an S has a refinement that is finite étale over S (due to the equivalent characterizations of henselian local schemes in [**EGA**, IV_4, 18.5.11(a),(c)]).

Now using general S, the group H is commutative of finite presentation (by fppf descent). By standard limit arguments (including the descent of quasi-compact fppf coverings through limits in the base), we may assume $\mathrm{S} = \operatorname{Spec} \mathrm{A}$ is local noetherian, and even strictly henselian. Consider the special fiber H_s, a group scheme of multiplicative type over the separably closed $k(s)$. By Lemma B.1.5, there is an isomorphism $j_s : \mathrm{H}_s \simeq \mathrm{D}_s(\mathrm{M})$ for a finitely generated abelian group M. Let $\widehat{\mathrm{S}} = \operatorname{Spec} \widehat{\mathrm{A}}$. The map j_s lifts to an open and closed immersion of $\widehat{\mathrm{S}}$-groups $j : \mathscr{H} = \mathrm{D}_{\widehat{\mathrm{S}}}(\mathrm{M}) \hookrightarrow \widehat{\mathrm{H}} := \mathrm{H}_{\widehat{\mathrm{S}}}$ (see Theorem B.3.2). We will prove that j is an isomorphism, and then descend it to an isomorphism $\mathrm{D}_{\mathrm{S}}(\mathrm{M}) \simeq \mathrm{H}$.

The fppf-local hypothesis on H is preserved by base change, so there is an fppf cover $\mathrm{S}' \to \widehat{\mathrm{S}}$ such that $\mathrm{H}_{\mathrm{S}'} \simeq \mathrm{D}_{\mathrm{S}'}(\mathrm{M}')$ for a finitely generated abelian group M'. Localize S' at a point over s, so $\mathrm{S}' \to \widehat{\mathrm{S}}$ is a local flat map (hence fpqc). The map $j_{\mathrm{S}'} : \mathrm{D}_{\mathrm{S}'}(\mathrm{M}) \to \mathrm{D}_{\mathrm{S}'}(\mathrm{M}')$ must arise from a map $u : \mathrm{M}' \to \mathrm{M}$ since M and M' are finitely generated (and S' is connected), and passage to the special fiber implies that u is an isomorphism (since $\mathrm{D}_s(u_s) = j_s$). Hence, $j_{\mathrm{S}'} = \mathrm{D}_{\mathrm{S}'}(u)$ is an isomorphism, so j is an isomorphism (by fpqc descent).

Any descent of j to an S-homomorphism $\mathrm{D}_{\mathrm{S}}(\mathrm{M}) \to \mathrm{H}$ is necessarily an isomorphism (by fpqc descent), so it suffices to prove that the natural map

$$\mathrm{Hom}_{\mathrm{S}\text{-gp}}(\mathrm{D}_{\mathrm{S}}(\mathrm{M}), \mathrm{H}) \to \mathrm{Hom}_{\widehat{\mathrm{S}}\text{-gp}}(\mathrm{D}_{\widehat{\mathrm{S}}}(\mathrm{M}), \mathrm{H}_{\widehat{\mathrm{S}}})$$

is bijective. Injectivity is clear, and for surjectivity we will use fpqc descent for morphisms (inspired by the proof of [**SGA3**, X, 4.3]).

Since H splits over an fppf covering of S, the map $n : \mathrm{H} \to \mathrm{H}$ is finite flat for each $n \geqslant 1$ because this can be checked over an fppf covering where H becomes diagonalizable. Hence, each $\mathrm{H}[n]$ is a commutative *finite* S-group of multiplicative type with étale Cartier dual. Consider a map $\widehat{f} : \mathrm{D}_{\widehat{\mathrm{S}}}(\mathrm{M}) \to \mathrm{H}_{\widehat{\mathrm{S}}}$. For each $n \geqslant 1$, the induced map $\widehat{f_n}$ between n-torsion subgroups uniquely descends to an S-homomorphism $f_n : \mathrm{D}_{\mathrm{S}}(\mathrm{M})[n] \to \mathrm{H}[n]$ because we can apply

Cartier duality and use that $E \rightsquigarrow E_{\widehat{S}}$ is an equivalence between the categories of finite étale schemes over S and $\widehat{S}$ (as A is henselian local). To descend $\widehat{f}$ to an S-homomorphism, by fpqc descent it is equivalent to check the equality of the pullback maps

$$p_1^*(\widehat{f}), p_2^*(\widehat{f}) : D_{\widehat{S}\times_S \widehat{S}}(M) \rightrightarrows H_{\widehat{S}\times_S \widehat{S}}.$$

A homomorphism from a multiplicative type group to a separated group scheme is determined by its restrictions to the n-torsion subgroups for all $n \geqslant 1$ [**Oes**, II, 3.2]. By applying this over the (typically non-noetherian!) base $\widehat{S} \times_S \widehat{S}$, the desired equality $p_1^*(\widehat{f}) = p_2^*(\widehat{f})$ is reduced to the same with $D_S(M)$ replaced by $D_S(M)[n] = D_S(M/nM)$ for every $n \geqslant 1$. Thus, we may assume that M is killed by some $n \geqslant 1$. Since $\widehat{f_n}$ descends, we are done. □

Corollary B.3.5. — *Let* $S = \operatorname{Spec} A$ *for a complete local noetherian ring* $(A, \mathfrak{m})$ *with residue field* k. *Let* G *be a smooth affine* S*-group and* H *an* S*-group of multiplicative type. Any homomorphism* $f_0 : H_0 \to G_0$ *between special fibers lifts to an* S*-homomorphism* $f : H \to G$*, and if* $f' : H \to G$ *is another such lift then* f *and* f' *are conjugate under* $\ker(G(S) \to G(k))$.

Moreover, if f_0 *is a closed immersion then so is any such* f.

This corollary is a special case of [**SGA3**, IX, 7.3].

Proof. — Since G is affine, the natural map $G(S) \to \varprojlim G(S_n)$ is bijective. The smoothness of G implies that $G(S_{n+1}) \to G(S_n)$ is surjective for all $n \geqslant 0$. In view of the bijectivity of (B.3.1) (which is applicable, due to Proposition B.3.4), to prove the existence of f and its uniqueness up to G(S)-conjugacy it suffices to show that for each $n \geqslant 1$ the maps

$$\mathrm{Hom}_{S_{n+1}\text{-gp}}(H_{S_{n+1}}, G_{S_{n+1}}) \to \mathrm{Hom}_{S_n\text{-gp}}(H_{S_n}, G_{S_n})$$

are surjective and each (necessarily non-empty) fiber is a single orbit under conjugation by $\ker(G(S_{n+1}) \to G(S_n))$.

Let $J_n = \mathfrak{m}^{n+1}/\mathfrak{m}^{n+2}$. Since G is S-smooth, H is S-affine, and $S_n \hookrightarrow S_{n+1}$ is defined by the square-zero ideal J_n, by Corollary B.2.6 the obstructions to surjectivity lie in degree-2 Hochschild cohomology for H_{S_n} with coefficients in a quasi-coherent H_{S_n}-module over S_n. Since H_{S_n} is of multiplicative type, this cohomology vanishes (Remark B.2.4). Applying Corollary B.2.6 once more, the obstruction to the transitivity of the conjugation action of $\ker(G(S_{n+1}) \to G(S_n))$ on the non-empty set of homomorphisms $H_{S_{n+1}} \to G_{S_{n+1}}$ lifting a given homomorphism $f_n : H_{S_n} \to G_{S_n}$ lies in a degree-1 Hochschild cohomology group for H_{S_n} with quasi-coherent coefficients, so again the obstruction vanishes.

Finally, we assume f_0 is a closed immersion and aim to show that f is a closed immersion. By Lemma B.1.3, it suffices to show that $\ker f = 1$.

Applying Nakayama's Lemma to the augmentation ideal of the kernel (viewed as a finitely generated module over the coordinate ring of $\ker f$), it suffices to check that $\ker(f_s) = 1$ for all $s \in \mathrm{S}$. The case when s is the closed point is our hypothesis, so assume s is not the closed point. By [**EGA**, II, 7.1.7], there is a complete discrete valuation ring R and a map $\mathrm{Spec}(\mathrm{R}) \to \mathrm{S}$ carrying the closed point to the closed point and the generic point to s. Via base change along such maps, we may assume A is a discrete valuation ring, say with residue field k and fraction field K. We need to prove that $\ker(f_{\mathrm{K}}) = 1$.

Let $\mathrm{H}' \subset \mathrm{H}$ be the schematic closure of $\ker(f_{\mathrm{K}})$ in H. This is an A-flat closed subgroup scheme of H, so H' is of multiplicative type by Corollary B.3.3. By A-flatness of H', the map $f|_{\mathrm{H}'} : \mathrm{H}' \to \mathrm{G}$ vanishes since it vanishes over K. Thus, $\mathrm{H}' \subset \ker f$. But $\ker(f_k) = 1$, so $\mathrm{H}'_k = 1$. Hence, $\mathrm{H}' = 1$ since H' is of multiplicative type, so $\ker(f_{\mathrm{K}}) = 1$ as desired. □

As an illustration, if $\mathrm{G} = \mathrm{GL}_n$ and $\mathrm{H}_0 = \mathbf{G}_{\mathrm{m}}$ then Corollary B.3.5 just says that a decomposition of $k(s)^n$ into a direct sum of subspaces can be lifted to a decomposition of A^n into a direct sum of finite free submodules.

Corollary B.3.6. — *Let* S *be a normal scheme. Every* S*-group* $\mathrm{H} \to \mathrm{S}$ *of multiplicative type is locally isotrivial: there is a Zariski-open cover* $\{\mathrm{U}_i\}$ *of* S *such that each* $\mathrm{H}|_{\mathrm{U}_i}$ *is isotrivial* (*i.e., splits over a finite étale cover of* U_i). *If* S *is irreducible* (*e.g., connected and locally noetherian*) *then* $\mathrm{H} \to \mathrm{S}$ *is isotrivial.*

In particular, for irreducible normal S *and a geometric point* $\overline{s}$ *of* S, *the functor* $\mathrm{H} \rightsquigarrow \underline{\mathrm{Hom}}_{\mathrm{S}\text{-gp}}(\mathrm{H}, \mathbf{G}_{\mathrm{m}})_{\overline{s}}$ *is an anti-equivalence from the category of multiplicative type* S*-groups to the category of discrete* $\pi_1(\mathrm{S}, \overline{s})$*-modules that are finitely generated as abelian groups.*

As an example, for connected locally noetherian normal S, the category of S-tori is anti-equivalent to the category of discrete $\pi_1(\mathrm{S}, \overline{s})$-representations on $\mathbf{Z}$-lattices (generalizing the classical case $\mathrm{S} = \mathrm{Spec}\, k$ for a field k). For instance, all $\mathbf{Z}$-tori are split because $\pi_1(\mathrm{Spec}\,\mathbf{Z}) = 1$ (Minkowski). Although a connected normal scheme is irreducible in the locally noetherian case, irreducibility can fail in the non-noetherian affine case; see Exercise 2.4.12.

Proof. — We call an étale sheaf on a scheme *isotrivial* if it becomes constant over a finite étale cover. By Proposition B.3.4, our task is to prove that if $\mathscr{F}$ is a finite type locally constant abelian étale sheaf on a normal scheme S then: $\mathscr{F}$ is isotrivial if S is irreducible, and in general $\mathscr{F}$ is isotrivial Zariski-locally on S. Each $\mathrm{Spec}\,\mathscr{O}_{\mathrm{S},s}$ is normal and irreducible, and any finite étale cover of $\mathrm{Spec}\,\mathscr{O}_{\mathrm{S},s}$ spreads out to a finite étale cover of an open neighborhood of s in S. Thus, by the local constancy and finite type hypotheses on $\mathscr{F}$, the isotriviality Zariski-locally on S is reduced to the isotriviaity for each $\mathscr{F}|_{\mathrm{Spec}\,\mathscr{O}_{\mathrm{S},s}}$.

Now we may and do assume that S is irreducible. Every connected finite étale S-scheme S′ is also irreducible. Indeed, the generic points of S′ lie over the unique generic point of the irreducible S, so there are only finitely many of them. Hence, there are only finitely many irreducible components of S′. These components are pairwise disjoint (since S′ is normal), so by *finiteness* each is open and closed. By connectedness, S′ must therefore be irreducible.

Since étale maps are open for the Zariski topology, and $\mathscr{F}$ becomes constant over the constituents of an étale cover of S, the isomorphism type of the geometric stalk $\mathscr{F}_{\overline{s}}$ is locally constant in s for the Zariski topology on S. Thus, by connectedness of S, there exists a finitely generated abelian group M so that $\mathscr{F}_{\mathrm{S}'} \simeq \mathrm{M}_{\mathrm{S}'}$ for some étale cover $\mathrm{S}' \to \mathrm{S}$. For each $n \geqslant 1$, we have $(\mathscr{F}/n\mathscr{F})_{\mathrm{S}'} \simeq \underline{\mathrm{M}/n\mathrm{M}}_{\mathrm{S}'}$. The constant S′-group $(\mathrm{M}/n\mathrm{M})_{\mathrm{S}'}$ is finite étale (especially *affine*) over S′, so the descent datum on it relative to $\mathrm{S}' \to \mathrm{S}$ arising from $\mathscr{F}/n\mathscr{F}$ is effective. Hence, $\mathscr{F}/n\mathscr{F}$ is represented by a finite étale S-group $\mathrm{G}_n \to \mathrm{S}$. Since a finite étale map has open and closed image, the connectedness of S and constancy of the fiber degree of a finite étale S-scheme E forces any such E to "disconnect" at most finitely many times; i.e., E is a disjoint union of finitely many *connected* finite étale S-schemes. By choosing a single connected Galois finite étale cover of S that dominates all connected components of such an E, we can split E using a connected finite étale cover $\mathrm{S}' \to \mathrm{S}$.

Fix $n \geqslant 3$ divisible by the exponent of $\mathrm{M}_{\mathrm{tor}}$ and apply base change to a connected finite étale cover $\mathrm{S}' \to \mathrm{S}$ that splits G_n, so G_n is a constant S-group. We shall prove that $\mathscr{F}$ is constant. First consider the special case that $\mathrm{S} = \operatorname{Spec} k$ for a field k. The category of étale abelian sheaves on S is the category of discrete $\mathrm{Gal}(k_s/k)$-modules, so $\mathscr{F}$ is identified with an action on M by a finite Galois group $\mathrm{Gal}(k'/k)$. To prove that $\mathscr{F}$ is constant we need to prove the triviality of this action. The constancy of $\mathscr{F}/n\mathscr{F}$ implies that the $\mathrm{Gal}(k'/k)$-action factors through a finite subgroup of $\Gamma = \ker(\mathrm{Aut}(\mathrm{M}) \to \mathrm{Aut}(\mathrm{M}/n\mathrm{M}))$, so it suffices to prove that Γ is torsion-free.

Recall the classical fact that for any integer $d > 0$, the kernel of $\mathrm{GL}_d(\mathbf{Z}) \to \mathrm{GL}_d(\mathbf{Z}/n\mathbf{Z})$ is torsion-free for $n \geqslant 3$. (This is most efficiently proved by observing that a torsion element in the kernel has eigenvalues that are roots of unity, and considering p-adic logarithms for $p|n$.) Since $\mathrm{M}' := \mathrm{M}/\mathrm{M}_{\mathrm{tor}} \simeq \mathbf{Z}^d$ for some d, and $\mathrm{M}'/n\mathrm{M}'$ is a quotient of $\mathrm{M}/n\mathrm{M}$, any finite-order $\gamma \in \Gamma$ is trivial on M′; i.e., $\gamma(m) = m + h(m)$ for some $h : \mathrm{M} \to \mathrm{M}_{\mathrm{tor}}$. The hypothesis $\gamma \equiv \mathrm{id} \bmod n\mathrm{M}$ implies that h is valued in $n\mathrm{M} \bigcap \mathrm{M}_{\mathrm{tor}} \subset (n\mathrm{M})_{\mathrm{tor}}$. Non-canonically $\mathrm{M} \simeq \mathrm{M}' \oplus \mathrm{M}_{\mathrm{tor}}$, so $n\mathrm{M}$ is torsion-free. This forces $h = 0$, so $\gamma = \mathrm{id}$ as desired. For general S, we conclude that $\mathscr{F}_s$ over $\operatorname{Spec} k(s)$ is constant for each $s \in \mathrm{S}$.

Next consider the case $\mathrm{S} = \operatorname{Spec} \mathrm{A}$ for an integrally closed *local* domain A. Let K be the fraction field of A. Fix an isomorphism $f_\eta : \mathrm{M}_{\mathrm{K}} \simeq \mathscr{F}_{\mathrm{K}}$. Let

$\mathrm{S}' \to \mathrm{S}$ be an étale cover such that $\mathrm{M}_{\mathrm{S}'} \simeq \mathscr{F}|_{\mathrm{S}'}$. We may assume $\mathrm{S}' = \operatorname{Spec} \mathrm{A}'$ is affine, so A' has only finitely many minimal primes (due to the finiteness of S'_{K}). The finitely many irreducible components of S' are pairwise disjoint, so each is open. Hence, there are only finitely many connected components of S' and they are irreducible. At least one of these has non-empty special fiber, and so must cover S (as its open image in the *local* S contains the closed point and so is full). Hence, we may assume A' is a domain, so S'_{K} is the generic point η' of S'. By the connectedness of S' and constancy of $\mathscr{F}_{\mathrm{S}'}$, $(f_\eta)_{\eta'}$ uniquely extends to an S'-isomorphism $\theta : \mathrm{M}_{\mathrm{S}'} \simeq \mathscr{F}_{\mathrm{S}'}$. For $\mathrm{S}'' = \mathrm{S}' \times_{\mathrm{S}} \mathrm{S}'$, the descent datum $p_1^*(\mathscr{F}_{\mathrm{S}'}) \simeq p_2^*(\mathscr{F}_{\mathrm{S}'})$ is identified via θ with an S''-group isomorphism $\varphi : \mathrm{M}_{\mathrm{S}''} \simeq \mathrm{M}_{\mathrm{S}''}$ whose restriction over $\mathrm{S}''_{\mathrm{K}} = \operatorname{Spec}(\mathrm{K}' \otimes_{\mathrm{K}} \mathrm{K}')$ is the *identity* map (due to the descent of $\theta_{\mathrm{S}'_{\mathrm{K}}} = \theta_{\eta'}$ to f_η). But S'' is a disjoint union of finitely many connected components, each of which meets $\mathrm{S}''_{\mathrm{K}}$, so φ is the identity. Thus, f_η extends to an isomorphism $\mathrm{M}_{\mathrm{S}} \simeq \mathscr{F}$.

Finally, we treat the general irreducible case. Fix an isomorphism $f_\eta : \mathrm{M}_\eta \simeq \mathscr{F}_\eta$. Applying the preceding over the local rings of S implies that $\mathscr{F}$ is constant Zariski-locally on S. Let $\{\mathrm{U}_i\}$ be a covering of the irreducible S by non-empty open subschemes so that there are isomorphisms $f_i : \mathrm{M}_{\mathrm{U}_i} \simeq \mathscr{F}|_{\mathrm{U}_i}$. Each U_i is connected, so we may uniquely choose each f_i such that $(f_i)_\eta = f_\eta$. The overlaps $\mathrm{U}_i \bigcap \mathrm{U}_j$ are irreducible with generic point η, so f_i and f_j coincide over $\mathrm{U}_i \bigcap \mathrm{U}_j$. Hence, the f_i glue to an isomorphism $\mathrm{M}_{\mathrm{S}} \simeq \mathscr{F}$. □

Remark B.3.7. — A scheme S is *unibranched* if $\operatorname{Spec} \mathscr{O}^{\mathrm{h}}_{\mathrm{S},s}$ is irreducible for all $s \in \mathrm{S}$. For irreducible S, it is equivalent that $\mathrm{S}_{\mathrm{red}}$ has normalization $\mathrm{S}' \to \mathrm{S}_{\mathrm{red}}$ that is radiciel (in which case the integral surjective morphism $\mathrm{S}' \to \mathrm{S}$ is radiciel). Pullback along a radiciel integral surjection defines an equivalence between étale sites (by [**SGA4**, VIII, 1.1], which reduces to the finitely presented case treated in [**SGA1**, IX, 4.10]), so by the anti-equivalence in Proposition B.3.4 we see that Corollary B.3.6 is valid with "normal" replaced by "unibranched". This fact is also noted at the end of [**Oes**, II, 2.1]. (See [**Oes**, IV, § 2] for an elegant general discussion of the "topological invariance" of the theory of multiplicative type groups without finiteness hypotheses.)

In Corollary B.3.3 we saw that any fppf closed subgroup scheme of a multiplicative type group scheme is of multiplicative type, so in particular every fppf closed subgroup scheme of a torus is of multiplicative type. Using the anti-equivalence between group schemes of multiplicative type and locally constant étale abelian sheaves with finitely generated stalks (Proposition B.3.4), we obtain a converse result:

Proposition B.3.8. — *Let* $\mathrm{H} \to \mathrm{S}$ *be a group scheme of multiplicative type. There exists an* S*-torus* T *such that* H *is a closed* S*-subgroup of* T.

Proof. — As a preliminary step, we check that H is uniquely an extension of a *finite* S-group of multiplicative type by an S-torus. Via the anti-equivalence in Proposition B.3.4, under which tori correspond to locally constant finitely generated étale abelian sheaves with torsion-free stalks, it is equivalent to show that any locally constant finitely generated étale abelian sheaf $\mathscr{F}$ on $\mathrm{S}_{\text{ét}}$ contains a unique locally constant finitely generated étale abelian subsheaf $\mathscr{F}' \subset \mathscr{F}$ such that the stalks of $\mathscr{F}'$ are torsion groups and the stalks of $\mathscr{F}/\mathscr{F}'$ are torsion-free. In view of the local constancy condition, it is clear that the subsheaf $\mathscr{F}_{\text{tor}}$ of locally torsion sections of $\mathscr{F}$ is the unique possibility for $\mathscr{F}'$ and that it works.

Now consider the unique short exact sequence of fppf S-groups

$$0 \to \mathrm{T} \to \mathrm{H} \to \mathrm{H}' \to 0 \tag{B.3.2}$$

with T an S-torus and H′ finite over S (necessarily of multiplicative type). The isomorphism class of stalks $\mathrm{H}'_{\overline{s}}$ at geometric points $\overline{s}$ of S is Zariski-locally constant on S, so by passing to the constituents of a disjoint union decomposition of S we may assume the isomorphism class of $\mathrm{H}'_{\overline{s}}$ is the same for all $\overline{s}$. Hence, H′ is killed by an integer $n > 0$.

By the snake lemma applied to the multiplication-by-n endomorphism of (B.3.2) and the fppf surjectivity of $n : \mathrm{T} \to \mathrm{T}$, we obtain a short exact sequence of finite multiplicative type S-groups

$$0 \to \mathrm{T}[n] \to \mathrm{H}[n] \to \mathrm{H}' \to 0. \tag{B.3.3}$$

If $\mathrm{S}' \to \mathrm{S}$ is a finite étale cover and $j' : \mathrm{H}_{\mathrm{S}'} \hookrightarrow \mathrm{T}'$ is an inclusion into an S′-torus then H is an S-subgroup of an S-torus: we compose the canonical inclusion $\mathrm{H} \hookrightarrow \mathrm{R}_{\mathrm{S}'/\mathrm{S}}(\mathrm{H}_{\mathrm{S}'})$ with the inclusion $\mathrm{R}_{\mathrm{S}'/\mathrm{S}}(j')$ of $\mathrm{R}_{\mathrm{S}'/\mathrm{S}}(\mathrm{H}_{\mathrm{S}'})$ into the S-group $\mathrm{R}_{\mathrm{S}'/\mathrm{S}}(\mathrm{T}')$ that is an S-torus (as $\mathrm{S}' \to \mathrm{S}$ is finite étale). Thus, it is harmless to make a base change to a finite étale cover S′ of S (which we promptly rename as S) so that the terms in (B.3.3) have constant Cartier dual.

For any two finite $\mathbf{Z}/n\mathbf{Z}$-modules M and M′, any homomorphism $f : \mathrm{M}_{\mathrm{S}} \to \mathrm{M}'_{\mathrm{S}}$ between the associated constant S-groups defines a disjoint-union decomposition $\mathrm{S} = \coprod \mathrm{S}_\phi$ indexed by the elements ϕ of the finite group $\mathrm{Hom}(\mathrm{M}, \mathrm{M}')$ via the condition that $f|_{\mathrm{S}_\phi}$ arises from ϕ. Thus, by passing to the constituents of such a decomposition we may assume that (B.3.3) arises from applying Cartier duality to a short exact sequence

$$0 \to \mathrm{M}' \to \mathrm{M} \to \mathrm{M}'' \to 0$$

of finite $\mathbf{Z}/n\mathbf{Z}$-modules in which M″ is *free* over $\mathbf{Z}/n\mathbf{Z}$. This latter short exact sequence splits, so (B.3.3) also splits and hence H′ lifts to an S-subgroup of $\mathrm{H}[n] \subset \mathrm{H}$. It follows that $\mathrm{H} \simeq \mathrm{T} \times \mathrm{H}'$, so we may replace H with H′ to reduce to the case when H is S-finite. Passing to a further finite étale cover of S reduces us to the case when H has constant Cartier dual, so $\mathrm{H} \simeq \mathrm{D}_{\mathrm{S}}(\mathrm{M})$ for a finite abelian group M. By choosing a surjection $F \twoheadrightarrow M$ with F a finitely generated free $\mathbf{Z}$-module we get an S-group inclusion of $H = D_S(M)$ into the S-group $D_S(F)$ that is an S-torus. □

B.4. Fibral criteria. — For a finite type group scheme G_0 over a field k and any extension field K/k, G_0 is of multiplicative type if and only if $(G_0)_K$ is. Indeed, this is a problem involving compatible algebraic closures of k and K, so it suffices to prove that $G_0 \simeq D_k(M)$ for a finitely generated abelian group M if and only if $(G_0)_K \simeq D_K(M)$. But $D_K(M) = (D_k(M))_K$, so the equivalence is clear (as G_0 and $D_k(M)$ are finite type over k); see the proof of Proposition 3.2.2 for an illustration of the general "spreading out and specialization" technique for descending results from an algebraically closed field to an algebraically closed subfield. We conclude that when we consider whether or not the fibers of an fppf S-affine group scheme $G \to S$ are of multiplicative type, it does not matter if we consider the actual fibers G_s or associated geometric fibers $G_{\overline{s}}$ for geometric points $\overline{s}$ of S.

Theorem B.4.1. — *Let* S *be a scheme and* H *an fppf* S*-affine* S*-group. Then* H *is of multiplicative type if and only if its geometric fibers are of multiplicative type and the order of* $H_s[n]$ *is locally constant in* s *for each* $n \geqslant 1$*. The fibral torsion condition can be dropped if* $H \to S$ *has connected fibers; e.g., tori.*

Before we prove Theorem B.4.1, we note that the torsion-order hypothesis holds if we assume that the "type" of each H_s (i.e., the isomorphism class of the character group of the geometric fiber at s) is locally constant in s. Local constancy of the "type" is used in the formulation of the fibral criterion in [**SGA3**, X, 4.8]. In the absence of a fibral connectedness condition it is necessary to impose some local constancy hypothesis on the type or at least its torsion levels, even if we assume H is commutative.

For example, suppose $S = \operatorname{Spec} R$ for a discrete valuation ring R and $d > 1$ is an integer not divisible be the residue characteristic. Consider the fppf affine R-group H obtained by removing the closed non-identity locus from the special fiber of the constant group $(\mathbf{Z}/d\mathbf{Z})_R$. This is quasi-finite flat and not finite flat (due to jumping of fiber-degree), so it is not of multiplicative type but its fibers are of multiplicative type.

Proof. — The necessity is clear, and for the proof of sufficiency we know that each fiber H_s is of multiplicative type since we have already noted that the "algebraic" geometric fibers $H_{\overline{s}}$ (i.e., using an algebraic closure of $k(s)$) must be of multiplicative type.

For any $s \in S$, any étale cover of $\operatorname{Spec} \mathscr{O}_{S,s}$ admits an affine refinement and an affine étale cover of $\operatorname{Spec} \mathscr{O}_{S,s}$ spreads out to an étale cover over an open neighborhood of s in S. Hence, since multiplicative type groups split over an étale cover, it suffices to treat the case when $\mathrm{S} = \operatorname{Spec} \mathrm{A}$ for a local ring A. We may also assume that A is strictly henselian (since a strict henselization of A is a directed union of local-étale extensions of A). Let k be the separably closed residue field of A. By Lemma B.1.5 applied to the special fiber, there is an isomorphism $j_k : \mathrm{D}_k(\mathrm{M}) \simeq \mathrm{H}_k$ for a finitely generated abelian group M.

We claim that j_k lifts to an A-homomorphism $j : \mathrm{D}_{\mathrm{A}}(\mathrm{M}) \to \mathrm{H}$ that is an open and closed immersion. Granting this, we prove that j is an isomorphism as follows. If $\mathrm{H} \to \mathrm{S}$ has connected fibers then j is fiberwise surjective and hence an isomorphism (as it is an open immersion). Suppose instead that the order of $\mathrm{H}_s[n]$ is locally constant in $s \in \mathrm{S}$ for each $n \geqslant 1$. By the connectedness of S this order must be constant, so $\mathrm{D}_s(\mathrm{M})[n]$ and $\mathrm{H}_s[n]$ have the same order for all $s \in \mathrm{S}$ due to comparison of orders of the special fibers. Thus, the open and closed immersion $j_s : \mathrm{D}_s(\mathrm{M}) \hookrightarrow \mathrm{H}_s$ between multiplicative type groups at each $s \in \mathrm{S}$ is an isomorphism on n-torsion for all $n \geqslant 1$, so j_s is an isomorphism [**Oes**, II, 3.2]. The open immersion j is therefore surjective, so it is an isomorphism.

To construct j lifting j_k, we only use the weaker hypothesis that H_k is of multiplicative type. The reason for weakening the hypothesis is that if we express A as a directed union of strictly henselian local *noetherian* subrings $\{\mathrm{A}_i\}$ (with local inclusion maps) then a descent of H to an fppf affine A_i-group H_i for large i inherits the multiplicative type hypothesis for its special fiber but the same for other fibers seems hard to control in the limit process. In this way, we may assume that A is also noetherian. By Theorem B.3.2, the isomorphism $j_k : \mathrm{D}_k(\mathrm{M}) \simeq \mathrm{H}_k$ uniquely lifts to an open and closed immersion

$$\widehat{j} : (\mathrm{D}_{\mathrm{A}}(\mathrm{M}))_{\widehat{\mathrm{A}}} = \mathrm{D}_{\widehat{\mathrm{A}}}(\mathrm{M}) \hookrightarrow \mathrm{H}_{\widehat{\mathrm{A}}}.$$

In particular, $\widehat{j}$ is an isomorphism between infinitesimal special fibers (so H has commutative infinitesimal fibers). Using fpqc descent for morphisms as at the end of the proof of Proposition B.3.4, we shall prove that $\widehat{j}$ descends to an A-morphism $j : \mathrm{D}_{\mathrm{A}}(\mathrm{M}) \to \mathrm{H}$; such a descent is necessarily a homomorphism and an open and closed immersion (thereby establishing what we need), by general results on fpqc descent for properties of morphisms.

To perform descent for $\widehat{j}$, it suffices to show that the two pullback homomorphisms

$$p_1^*(\widehat{j}), p_2^*(\widehat{j}) : \mathrm{D}_{\mathrm{A}}(\mathrm{M})_{\widehat{\mathrm{A}} \otimes_{\mathrm{A}} \widehat{\mathrm{A}}} \rightrightarrows \mathrm{H}_{\widehat{\mathrm{A}} \otimes_{\mathrm{A}} \widehat{\mathrm{A}}}$$

over the (typically non-noetherian) ring $\widehat{\mathrm{A}} \otimes_{\mathrm{A}} \widehat{\mathrm{A}}$ coincide. Arguing via [**Oes**, II, 3.2] as near the end of the proof of Proposition B.3.4, it suffices to check equality on n-torsion for each $n \geqslant 1$, which is exactly descent for the restriction $\widehat{j}_n$ of $\widehat{j}$ to $\mathrm{D}_{\mathrm{A}}(\mathrm{M}/n\mathrm{M})_{\widehat{\mathrm{A}}}$ for each $n \geqslant 1$.

The scheme morphism $f_n : \mathrm{H} \to \mathrm{H}$ given by $h \mapsto h^n$ is quasi-finite flat (as may be checked on fibers, using that each H_s is of multiplicative type) and a homomorphism on infinitesimal special fibers, so $\mathrm{H}[n] := f_n^{-1}(1)$ is a quasi-finite flat closed subscheme of H that is a subgroup scheme on infinitesimal special fibers. By the structure theorem for quasi-finite separated morphisms [**EGA**, IV_4, 18.5.11(a),(c)], any quasi-finite separated scheme over a henselian local base Z is uniquely the disjoint union of a Z-*finite* open and closed subscheme and an open and closed subscheme with empty special fiber. Let $\mathrm{H}[n]_{\mathrm{fin}}$ denote the resulting "finite part" of $\mathrm{H}[n]$. This closed subscheme of H is a subgroup scheme on infinitesimal special fibers, so it is a subgroup scheme of H (as the preimage of $\mathrm{X} := \mathrm{H}[n]_{\mathrm{fin}}$ under $\mathrm{X} \times \mathrm{X} \hookrightarrow \mathrm{H} \times \mathrm{H} \to \mathrm{H}$ contains all infinitesimal special fibers of the A-finite $\mathrm{X} \times \mathrm{X}$ and hence exhausts $\mathrm{X} \times \mathrm{X}$).

The finite flat group schemes $\mathrm{H}[n]_{\mathrm{fin}}$ are *commutative* because we may check on the infinitesimal special fibers (where even H becomes commutative). Their Cartier duals are étale, as this may be checked on the special fiber (due to the openness of the étale locus, or by more direct arguments). Moreover, since the formation of the "finite part" commutes with local henselian base change, we have $(\mathrm{H}[n]_{\mathrm{fin}})_{\widehat{\mathrm{A}}} = (\mathrm{H}_{\widehat{\mathrm{A}}}[n])_{\mathrm{fin}}$. Since $\mathrm{D_A}(\mathrm{M}/n\mathrm{M})$ is A-finite, $\widehat{j}_n$ factors through $(\mathrm{H}_{\widehat{\mathrm{A}}}[n])_{\mathrm{fin}} = (\mathrm{H}[n]_{\mathrm{fin}})_{\widehat{\mathrm{A}}}$. But $\mathrm{D_A}(\mathrm{M}/n\mathrm{M})$ and $\mathrm{H}[n]_{\mathrm{fin}}$ are finite flat commutative A-groups whose Cartier duals are étale (and hence *constant*), so any homomorphism between their special fibers uniquely lifts. The same holds over $\widehat{\mathrm{A}}$, so comparison through the common special fibers over A and $\widehat{\mathrm{A}}$ implies that $\widehat{j}_n$ descends to an A-homomorphism $\mathrm{D_A}(\mathrm{M}/n\mathrm{M}) \to \mathrm{H}[n]_{\mathrm{fin}} \subset \mathrm{H}$ for each $n \geqslant 1$. This completes the proof that $\widehat{j}$ descends, and hence the proof that H is of multiplicative type. $\square$

Corollary B.4.2. — *Let* S *be a scheme,* $\mathrm{H} \to \mathrm{S}$ *an* S*-affine fppf group scheme.*

1. *If* H *becomes multiplicative type fpqc-locally on* S *then it is of multiplicative type.*
2. *Consider a short exact sequence*

$$1 \to \mathrm{H}' \to \mathrm{H} \to \mathrm{H}'' \to 1$$

of fppf S*-affine* S*-groups with* H' *and* H'' *of multiplicative type. If each* H_s *is either connected or commutative then* H *is of multiplicative type.*

Assertion (1) shows that the notion of "multiplicative type" used in [**SGA3**] (with fpqc-local triviality) coincides with the notion that we are using (with fppf-local triviality) in the finite type case. Also, there must be some fibral hypothesis on H in (2), since the finite constant group over a field of characteristic 0 associated to a non-commutative solvable group of order p^3 for a prime p is an extension of one multiplicative type group by another.

Proof. — The validity of (1) is immediate from Theorem B.4.1: the fpqc-locality hypothesis ensures that H is commutative and each H_s is of multiplicative type, so $H[n]$ is quasi-finite over S for all $n \geqslant 1$. For each $n \geqslant 1$, the locus of points $s \in S$ where $H_s[n]$ has a given order is open since Zariski-locally this holds over an fpqc cover (and fpqc maps are topologically quotient maps). Hence, we have local constancy in s for the order of $H_s[n]$ for each $n \geqslant 1$.

For the proof of (2) we may assume $S = \operatorname{Spec} A$ for a strictly henselian local noetherian ring A at the cost of only knowing that the special fiber H_k of H (rather than every fiber H_s) is connected or commutative. Now $H' = D_S(M')$ and $H'' = D_S(M'')$ for finitely generated abelian groups M' and M'' (Proposition B.3.4). Granting that the special fiber H_k is of multiplicative type, we may conclude as follows. By (1), we may assume A is complete. The k-group H_k has the form $D_k(M)$ for an abelian group M that is an extension of M' by M'' (dual to the given exact sequence on special fibers). By Theorem B.3.2(2), the isomorphism $D_k(M) \simeq H_k$ uniquely lifts to a homomorphism $j : D_S(M) \to H$ that is moreover an open and closed immersion. The composition of j with the canonical homomorphism $D_S(M') \hookrightarrow D_S(M)$ must be the inclusion $D_S(M') = H' \hookrightarrow H$ because the two maps agree on special fibers (and we can appeal to the uniqueness for lifting in Theorem B.3.2). In other words, the open and closed subgroup $D_S(M) \subset H$ contains $H' = D_S(M')$. This forces $D_S(M) = H$ because passing to quotients by $D_S(M')$ gives a chain of inclusions

$$D_S(M)/D_S(M') \subset H/D_S(M') \subset H'' = D_S(M'')$$

whose composition is induced by the identification of M' with M/M''.

It remains to prove (2) when $S = \operatorname{Spec} k$ for a field k, and we may assume k is algebraically closed. First we show that if H is connected (so H'' is also connected) then H must be commutative. The automorphism functor of H' is represented by a constant k-group (via duality for diagonalizable finite type k-groups), so the conjugation action on H' by the connected H must be trivial. That is, H' is central in H. The commutator morphism for H therefore factors through a bi-additive pairing $\beta : H'' \times H'' \to H'$ since for points a_1, a_2, b of H (valued in an S-scheme) the centrality of a commutator $a_2 b a_2^{-1} b^{-1} \in H'$ in H implies

$$(a_1 b a_1^{-1} b^{-1})(a_2 b a_2^{-1} b^{-1}) = a_1 (a_2 b a_2^{-1} b^{-1}) b a_1^{-1} b^{-1} = (a_1 a_2) b (a_1 a_2)^{-1} b^{-1}.$$

The vanishing of β is equivalent to the commutativity of H. To prove that β vanishes it suffices to prove that the only homomorphism of group functors $f : H'' \to \underline{\mathrm{Hom}}_{k\text{-gp}}(H'', H')$ is the trivial one. This Hom-functor is represented by a constant k-group since H'' and H' are diagonalizable finite type k-groups and H'' is connected, so f must vanish.

Now working with commutative H in general, note that the subgroup $T := H^0_{\mathrm{red}}$ is smooth and connected without $\mathbf{G}_a$ as a subgroup, so it is a torus by the classical theory. Let $\mathrm{T}'' \subset \mathrm{H}''^0_{\mathrm{red}}$ be the image of T and let $\mathrm{G}' = \mathrm{H}' \bigcap \mathrm{T}$, so $1 \to \mathrm{H}'/\mathrm{G}' \to \mathrm{H}/\mathrm{T} \to \mathrm{H}''/\mathrm{T}'' \to 1$ is a short exact sequence of finite commutative k-group schemes. Both finite quotients $\mathrm{H}''/\mathrm{T}''$ and H'/G' are of multiplicative type (since H' and H'' are; use Exercise 2.4.1), so by Cartier duality for finite commutative k-groups we see that H/T has *étale* Cartier dual. Double duality then implies that H/T is of multiplicative type. In other words, H is a commutative affine finite type extension of $\mathrm{D}_k(\mathrm{M})$ by a torus T for a finite abelian group M. It suffices to show that any such extension splits, which is to say that the abelian group $\mathrm{Ext}^1_k(\mathrm{D}_k(\mathrm{M}), \mathrm{T})$ of isomorphism classes of such k-group extensions vanishes. Bi-additivity of this Ext-functor reduces the problem to the case $\mathrm{T} = \mathbf{G}_{\mathrm{m}}$ and $\mathrm{M} = \mathbf{Z}/n\mathbf{Z}$ for an integer $n \geqslant 1$.

If E is a commutative affine finite type k-group extension of μ_n by $\mathbf{G}_{\mathrm{m}}$, the fppf surjectivity of $[n] : \mathbf{G}_{\mathrm{m}} \to \mathbf{G}_{\mathrm{m}}$ implies (via the snake lemma in the abelian category of commutative fppf group sheaves over k) that the sequence

$$1 \to \mu_n \to \mathrm{E}[n] \to \mu_n \to 1 \tag{B.4.1}$$

is *short exact* for the fppf topology. It suffices to show that this splits. Applying Cartier duality to (B.4.1), the dual of $\mathrm{E}[n]$ must be étale, so it is the constant k-group corresponding to an *n-torsion* finite abelian group that is an extension of $\mathbf{Z}/n\mathbf{Z}$ by $\mathbf{Z}/n\mathbf{Z}$. Any such exact sequence of $\mathbf{Z}/n\mathbf{Z}$-modules splits, so by double duality the sequence (B.4.1) also splits. $\square$

Appendix C
Orthogonal group schemes

C.1. Basic definitions and smoothness results. — Let V be a vector bundle of constant rank $n \geqslant 1$ over a scheme S, and let L be a line bundle on S. A *quadratic form* $q : \mathrm{V} \to \mathrm{L}$ is a map of sheaves of sets such that $q(cv) = c^2 q(v)$ and the symmetric map $\mathrm{B}_q : \mathrm{V} \times \mathrm{V} \to \mathrm{L}$ defined by

$$\mathrm{B}_q(x, y) = q(x + y) - q(x) - q(y)$$

is $\mathscr{O}_{\mathrm{S}}$-bilinear. We call any such $(\mathrm{V}, \mathrm{L}, q)$ a *line bundle-valued quadratic form*. Using local trivializations of V and L, $q(x) = \sum_{i \leqslant j} a_{ij} x_i x_j$. There is an evident notion of base change for line bundle-valued quadratic forms.

Assume q is fiberwise non-zero over S, so the zero scheme $(q = 0) \subset \mathbf{P}(\mathrm{V}^*)$ (which is well-posed without assuming L to be trivial) is an S-flat hypersurface with fibers of dimension $n - 2$ (understood to be empty when $n = 1$). By Exercise 1.6.10 (and trivial considerations when $n = 1$), this is S-smooth precisely when for each $s \in \mathrm{S}$ one of the following holds: (i) B_{q_s} is non-degenerate and either $\mathrm{char}(k(s)) \neq 2$ or $\mathrm{char}(k(s)) = 2$ with n even, (ii) the defect $\delta_{q_s} := \dim \mathrm{V}_s^{\perp}$ is 1, $q_s|_{\mathrm{V}_s^{\perp}} \neq 0$, and $\mathrm{char}(k(s)) = 2$ with n odd. (By Exercise 1.6.10, $\delta_{q_s} \equiv \dim \mathrm{V}_s \bmod 2$ when $\mathrm{char}(k(s)) = 2$.) In such cases we say $(\mathrm{V}, \mathrm{L}, q)$ is *non-degenerate* (the terminology *ordinary* is used in [**SGA7**, XII, § 1]). Case (ii) is the "defect-1" case at s.

A *quadratic space* is a pair (V, q) where $q : \mathrm{V} \to \mathscr{O}_{\mathrm{S}}$ is a non-degenerate $\mathscr{O}_{\mathrm{S}}$-valued quadratic form. (When we need to consider pairs (V, q) with q possibly not non-degenerate, we may call (V, q) a *possibly degenerate* quadratic space.) The evident notion of isomorphism among the non-degenerate line bundle-valued quadratic forms $(\mathrm{V}, \mathscr{O}_{\mathrm{S}}, q)$ corresponds to the classical notion of *similarity* (compatibility up to a unit scaling on the form) for quadratic spaces. We will usually work with quadratic spaces rather than non-degenerate line bundle-valued quadratic forms below, at least once we begin needing to consider Clifford algebras, due to difficulties with Clifford constructions for line bundle-valued q when there is not a given trivialization of L.

Remark C.1.1. — Our notion of "non-degeneracy" is frequently called "regularity" or "semi-regularity" (especially for odd n when 2 is not a unit on S); see [**Knus**, IV, 3.1]. In the study of quadratic forms q over a domain A, such as the ring of integers in a number field or a discrete valuation ring, the phrase "non-degenerate" is often used to mean "non-degenerate over the fraction field". Indeed, non-degeneracy over A in the sense defined above is rather more restrictive, since for even n it says that the discriminant is a global unit and for odd n it says that the "half-discriminant" (see Proposition C.1.4) is a global unit. Non-degenerate examples over $\mathbf{Z}$ (in our restrictive sense) include the quadratic spaces arising from even unimodular lattices, such as the E_8 and Leech lattices.

For a non-degenerate line bundle-valued quadratic form (V, L, q), clearly the functor

$$S' \rightsquigarrow \{g \in GL(V_{S'}) \mid q_{S'}(gx) = q_{S'}(x) \text{ for all } x \in V_{S'}\}$$

on S-schemes is represented by a finitely presented closed S-subgroup $O(q)$ of $GL(V)$. We call it the *orthogonal group* of (V, L, q). This has bad properties without a non-degeneracy hypothesis, and is μ_2 if $n = 1$. Define the *naive special orthogonal group* to be

$$SO'(q) := \ker(\det : O(q) \to \mathbf{G}_m)$$

(so $SO'(q) = 1$ if $n = 1$); we say "naive" because this is the wrong notion for non-degenerate (V, L, q) when n is even and 2 is not a unit on S. The *special orthogonal group* $SO(q)$ will be defined shortly in a characteristic-free way, using Clifford algebras when n is even. (The distinction between even and odd n when defining $SO(q)$ in terms of $O(q)$ is natural, because we will see that $O(q)/SO(q)$ is μ_2 for odd n but $(\mathbf{Z}/2\mathbf{Z})_S$ for even n. Also, if $n \geqslant 3$ then $SO(q)_{\overline{s}}$ will be connected semisimple of type B_m for $n = 2m+1$ and type D_m for $n = 2m$.)

Definition C.1.2. — Let $S = \operatorname{Spec} \mathbf{Z}$. The *standard split* quadratic form q_n on $V = \mathbf{Z}^n$ is as follows, depending on the parity of $n \geqslant 1$:

$$q_{2m} = \sum_{i=1}^{m} x_{2i-1}x_{2i}, \quad q_{2m+1} = x_0^2 + \sum_{i=1}^{m} x_{2i-1}x_{2i} \tag{C.1.1}$$

(so $q_1 = x_0^2$). We define $O_n = O(q_n)$ and $SO'_n = SO'(q_n)$.

It is elementary to check that $(\mathbf{Z}^n, q_n)$ is non-degenerate. We do not define a notion of "split" for general line bundle-valued non-degenerate q because for odd n this turns out not to be an interesting concept except essentially for cases when L is trivial.

Remark C.1.3. — In some references (e.g., [**Bo69**, 11.16]) the quadratic forms

$$x_1x_{2m} + \cdots + x_mx_{m+1}, \ \ x_0x_{2m} + \cdots + x_{m-1}x_{m+1} + x_m^2$$

are preferred over (C.1.1), since for such q the split group $SO(q)$ admits a Borel subgroup contained in the upper triangular Borel subgroup of SL_n.

There is a convenient characterization of non-degeneracy for (V, L, q) when L and V are globally free, using the unit condition on values of a polynomial associated to q (depending on the parity of n), as follows. Suppose there is a chosen isomorphism $L \simeq \mathscr{O}_S$ and ordered $\mathscr{O}_S$-basis $\mathbf{e} = \{e_1, \ldots, e_n\}$ of V (as may be arranged by Zariski-localization on S). Let $[\mathrm{B}_q]_{\mathbf{e}}$ be the matrix $(\mathrm{B}_q(e_i, e_j))$ that computes B_q relative to $\mathbf{e}$. The determinant $\mathrm{disc}_{\mathbf{e}}(q) = \det([\mathrm{B}_q]_{\mathbf{e}})$ is the *discriminant* of q relative to $\mathbf{e}$ (and the chosen trivialization of L). If $\mathbf{e}'$ is a second ordered $\mathscr{O}_{\mathrm{S}}$-basis of V then $\mathrm{disc}_{\mathbf{e}'}(q) = u^2\mathrm{disc}_{\mathbf{e}}(q)$ for the unit u given by the determinant of the matrix that converts $\mathbf{e}'$-coordinates into $\mathbf{e}$-coordinates. In more intrinsic terms, $\mathrm{disc}_{\mathbf{e}}(q)$ computes the induced linear map $\wedge^n(\mathrm{V}) \to \wedge^n(\mathrm{V}^*)$ arising from the linear map $\mathrm{V} \to \mathrm{V}^*$ defined by $v \mapsto \mathrm{B}_q(v, \cdot) = \mathrm{B}_q(\cdot, v)$ (when using the bases $e_1 \wedge \cdots \wedge e_n$ and $e_1^* \wedge \cdots \wedge e_n^*$). The condition $\mathrm{disc}_{\mathbf{e}}(q) \in \mathbf{G}_{\mathrm{m}}(\mathrm{S})$ is independent of the choice of $\mathbf{e}$, and it expresses exactly the property that B_q is a perfect pairing on V. Hence, if n is even *or* if n is odd and 2 is a unit on S then the condition $\mathrm{disc}_{\mathbf{e}}(q) \in \mathbf{G}_{\mathrm{m}}$ is equivalent to the non-degeneracy of q.

To handle the case of odd n in a characteristic-free way, it is convenient to introduce a modification of the discriminant that was independently discovered by Grothendieck and M. Kneser. This involves a "universal" construction:

Proposition C.1.4. — *Let $n \geqslant 1$ be odd and $\mathrm{Q} = \sum_{i \leqslant j} \mathrm{A}_{ij} x_i x_j \in \mathbf{Z}[\mathrm{A}_{ij}][x_1, \ldots, x_n]$ the universal quadratic form in n variables.*

1. *The polynomial* $\mathrm{disc}'(\mathrm{Q}) := (1/2)\mathrm{disc}(\mathrm{Q}) \in (1/2)\mathbf{Z}[\mathrm{A}_{ij}]$ *lies in* $\mathbf{Z}[\mathrm{A}_{ij}]$.
2. *Over the ring $\mathbf{Z}[\mathrm{A}_{ij}, \mathrm{C}_{hk}][1/\det(\mathrm{C}_{hk})]$ consider the universal linear change of coordinates $x_h = \sum_k \mathrm{C}_{kh} x_k'$ dual to the universal change of basis $e_h = \sum_k \mathrm{C}_{hk} e_k'$. The quadratic form Q' in $x_1', \ldots, x_n'$ obtained from $\mathrm{Q}(x_1, \ldots, x_n)$ satisfies* $\mathrm{disc}'(\mathrm{Q}') = \mathrm{disc}'(\mathrm{Q}) \det(\mathrm{C}_{hk})^2$.
3. *If* R *is any commutative ring and $q = \sum_{i \leqslant j} a_{ij} x_i x_j$ is a quadratic form in n variables over* R *then* $\mathrm{disc}'(q) := (\mathrm{disc}'(\mathrm{Q}))(a_{ij})$ *changes by a unit square under linear change of variables and it lies in* $\mathrm{R}^\times$ *if and only if q is non-degenerate over* $\mathrm{Spec}\,\mathrm{R}$.

We call $\mathrm{disc}'(q)$ the *half-discriminant* of q (since when $2 \in \mathrm{R}^\times$, it is $(1/2)\mathrm{disc}(q)$). Computation of the half-discriminant when $2 \notin \mathrm{R}^\times$ (especially when 2 is a zero-divisor in R) requires lifting q to a ring in which 2 is not a zero-divisor (e.g., if $\mathrm{R} = \mathbf{F}_2$ then we can work with a lift of q over $\mathbf{Z}$), but it will nonetheless be theoretically useful. The non-degeneracy of q_{2m+1} in characteristic 2 shows that for odd n the universal half-discriminant in characteristic 2 is not identically zero.

Proof. — For (1) it suffices to show that over the fraction field $\mathbf{F}_2(\mathrm{A}_{ij})$ of $\mathbf{Z}[\mathrm{A}_{ij}]/(2)$ the quadratic form Q has vanishing discriminant. Over any field k of characteristic 2, a quadratic form q on a finite-dimensional vector space V satisfies $\delta_q \equiv \dim \mathrm{V} \bmod 2$ (Exercise 1.6.10), so the defect δ_q is positive when $\dim \mathrm{V}$ is odd. Since $\delta_q > 0$ precisely when $\mathrm{disc}(q) = 0$, part (1) is proved. Part

(2) is obvious, since we can multiply both sides by 2 to reduce to the known case of the usual discriminant.

To prove part (3), note that the unit square aspect follows from (2). Thus, we may assume $R = k$ is an algebraically closed field and it is harmless to apply a linear change of variables. The case $\mathrm{char}(k) \neq 2$ is trivial, so we can assume $\mathrm{char}(k) = 2$. First we show that $\mathrm{disc}'(q) \in k^\times$ if q is non-degenerate. For such q, by Exercise 1.6.10(ii) we may apply a linear change of variables so that $q = x_0^2 + \sum_{i=1}^m x_{2i-1}x_{2i}$ with $n = 2m+1$. This arises by scalar extension from the quadratic form q_n over $\mathbf{Z}$ given by the same formula, so $\mathrm{disc}'(q)$ is the image of $\mathrm{disc}'(q_n) \in \mathbf{Z}$. Clearly $\mathrm{disc}(q_n) = 2(-1)^m$, so $\mathrm{disc}'(q_n) = (-1)^m$. Hence, $\mathrm{disc}'(q) = (-1)^m \in k^\times$.

For the converse, we assume q is degenerate and shall prove that $\mathrm{disc}'(q) = 0$ in k. In accordance with the definition of $\mathrm{disc}'(q)$, we can compute it by working with a lift of q over the ring $W(k)$ of Witt vectors (in which 2 is not a zero divisor). It is harmless to first apply a preliminary linear change of variables over k. Consider the defect space $V^\perp$ for the alternating B_q. This has odd dimension (since n is odd and B_q induces a symplectic form on $V/V^\perp$), and $q|_{V^\perp}$ is the square of a linear form (since k is algebraically closed of characteristic 2). The degeneracy of q implies that either $\dim V^\perp \geqslant 3$ or $\dim V^\perp = 1$ with $q|_{V^\perp} = 0$. Either way, $q|_{V^\perp}$ can be expressed in terms of fewer than $\dim V^\perp$ variables relative to a suitable basis, so q can be expressed in fewer than n variables after a linear change of coordinates. Hence, there is a lift $\widetilde{q}$ of q to a quadratic form over $W(k)$ that can be expressed in fewer than n variables after a linear change of coordinates over $W(k)$, so the discriminant $\mathrm{disc}(\widetilde{q})$ attached to $(W(k)^n, \widetilde{q})$ vanishes (as we may check over the field $W(k)[1/2]$ of characteristic 0). Since $\mathrm{disc}'(q)$ is the reduction of $\mathrm{disc}'(\widetilde{q}) = (1/2)\mathrm{disc}(\widetilde{q}) = 0$, we are done. □

It is important to note that the discriminant and half-discriminant are attached to a line bundle-valued quadratic form (V, L, q) equipped with global bases of V and L, and not to a "bare" degree-2 homogeneous polynomial. For example, to define the discriminant (or half-discriminant) of the quadratic form $x_0^2 + x_1x_2$ over $\mathbf{Z}$, it is necessary to specify whether this is viewed as a quadratic form on $\mathbf{Z}^3$ or $\mathbf{Z}^n$ for some $n > 3$. The convenience of the half-discriminant is demonstrated by its role in the proof of:

Theorem C.1.5. — *Let (V, L, q) be a non-degenerate line bundle-valued quadratic form with V of rank $n \geqslant 1$ over a scheme S. The S-group $O(q)$ is smooth if and only if either n is even or n is odd with 2 a unit on S, and the S-group $SO'(q)$ is smooth if either n is odd or n is even with 2 a unit on S. These smooth groups have fibers of dimension $n(n-1)/2$.*

If n is even then over fields k of characteristic 2 the map $\det : \mathrm{O}(q) \to \mu_2$ is identifically 1 due to smoothness of $\mathrm{O}(q)$, so $\mathrm{SO}'(q) = \mathrm{O}(q)$ in such cases.

Proof. — We use the following *smoothness criterion*: if X and Y are S-schemes locally of finite presentation such that X is S-flat, then an S-morphism $f : \mathrm{X} \to \mathrm{Y}$ is smooth if and only if $f_s : \mathrm{X}_s \to \mathrm{Y}_s$ is smooth for all $s \in \mathrm{S}$. To prove the criterion it is only necessary to show that f is flat when each f_s is flat, and this follows from the S-flatness of X (as part of the fibral flatness criterion [**EGA**, IV_3, 11.3.10]).

Step 1. First we treat the case of orthogonal group schemes. The S-scheme $\mathrm{Quad}(\mathrm{V}, \mathrm{L})$ of L-valued quadratic forms on V is represented by a smooth S-scheme that is an affine space of relative dimension $n(n+1)/2$ Zariski-locally over S, and the subfunctor of such forms that are non-degenerate is represented by an open subscheme $\mathrm{Y} \subset \mathrm{Quad}(\mathrm{V}, \mathrm{L})$ given Zariski-locally over S by the non-vanishing of the discriminant or half-discriminant depending on the parity of n. Thus, $\mathrm{Y} \to \mathrm{S}$ is smooth and surjective (since q_n is non-degenerate over any field). There is an evident right action of $\mathrm{X} := \mathrm{GL}(\mathrm{V})$ on Y over S via $(\mathrm{Q}, g) \mapsto \mathrm{Q} \circ g$, and the orbit map $f : \mathrm{X} \to \mathrm{Y}$ through $q \in \mathrm{Y}(\mathrm{S})$ is surjective because over any algebraically closed field k the non-degenerate quadratic forms on k^n are a single $\mathrm{GL}_n(k)$-orbit (Exercise 1.6.10(ii),(iii)). The fiber of f over a geometric point of Y is the orthogonal group scheme for the corresponding quadratic form, so by the smoothness criterion we may assume $\mathrm{S} = \mathrm{Spec}\, k$ for an algebraically closed field k.

Now X and Y are k-smooth and irreducible with respective dimensions n^2 and $n(n+1)/2$, so the smoothness of f is equivalent to surjectivity of the maps $\mathrm{Tan}_g(\mathrm{X}) \to \mathrm{Tan}_{q \circ g}(\mathrm{Y})$ for $g \in \mathrm{X}(k)$. By homogeneity it is equivalent to verify such surjectivity for a single g, such as $g = 1$, and this in turn is equivalent to $\mathrm{Tan}_1(f)$ having kernel of dimension $n^2 - n(n+1)/2 = n(n-1)/2$. But $f(1) = q$ and $f^{-1}(q) = \mathrm{O}(q)$, so the kernel of $\mathrm{Tan}_1(f)$ is $\mathrm{Tan}_1(\mathrm{O}(q))$. Thus, the case of orthogonal group schemes is reduced to showing that for any quadratic form $q : \mathrm{V} \to k$, $\dim \mathrm{Tan}_1(\mathrm{O}(q)) = n(n-1)/2$ precisely when B_q is non-degenerate.

Computing with the algebra $k[\epsilon]$ of dual numbers, the subspace

$$\mathrm{Tan}_1(\mathrm{O}(q)) \subset \mathrm{Tan}_1(\mathrm{GL}(\mathrm{V})) = \mathrm{End}(\mathrm{V})$$

consists of $\mathrm{T} : \mathrm{V} \to \mathrm{V}$ such that $q((1 + \epsilon \mathrm{T})(v)) = q(v)$ on $\mathrm{V}_{k[\epsilon]}$. Since $q(v + \epsilon \mathrm{T}(v)) = q(v) + \epsilon \mathrm{B}_q(\overline{v}, \mathrm{T}(\overline{v}))$ for $\overline{v} = v \bmod \epsilon$, the necessary and sufficient condition on T is that $\mathrm{B}_q(x, \mathrm{T}(x)) = 0$ for all $x \in \mathrm{V}$, which is to say that the bilinear form $\mathrm{B}_q(\cdot, \mathrm{T}(\cdot))$ on V is alternating. Let $\mathrm{V}' = \mathrm{V}/\mathrm{V}^{\perp}$, so B_q induces a non-degenerate bilinear form B'_q on V'. When $\mathrm{B}_q(v, \mathrm{T}(w))$ is alternating it is skew-symmetric, so in such cases T must preserve $\mathrm{V}^{\perp}$ (because $\mathrm{B}_q(v, \mathrm{T}(w)) = 0$ for $v \in \mathrm{V}^{\perp}$ and any $w \in \mathrm{V}$).

Thus, $\mathrm{Tan}_1(\mathrm{O}(q))$ is the space of T that preserve $\mathrm{V}^\perp$ and whose induced endomorphism T' of V' makes $\mathrm{B}'_q(v', \mathrm{T}'(w'))$ alternating. Since T' determines T up to precisely translation by $\mathrm{Hom}(\mathrm{V}, \mathrm{V}^\perp)$, by the non-degeneracy of B'_q on V' we obtain a short exact sequence

$$0 \to \mathrm{Hom}(\mathrm{V}, \mathrm{V}^\perp) \to \mathrm{Tan}_1(\mathrm{O}(q)) \to \mathrm{Alt}^2(\mathrm{V}/\mathrm{V}^\perp) \to 0$$

(the second map is $\mathrm{T} \mapsto \mathrm{B}'_q(\cdot, \mathrm{T}'(\cdot))$). Letting $\delta = \dim \mathrm{V}^\perp$ denote the defect, we find that

$$\dim \mathrm{Tan}_1(\mathrm{O}(q)) = n\delta + (n-\delta)(n-\delta-1)/2 = n(n-1)/2 + (\delta^2+\delta)/2.$$

This coincides with $n(n-1)/2$ precisely when $\delta = 0$.

Step 2. Now consider the case of $\mathrm{SO}'(q)$. By Zariski-localization on S, we may assume $\mathrm{L} = \mathscr{O}_\mathrm{S}$ and may choose an ordered basis of V. For any quadratic form Q on $\mathrm{V}_{\mathrm{S}'}$ for an S-scheme S', relative to the chosen basis let $\mathrm{D}(\mathrm{Q})$ denote $\mathrm{disc}(\mathrm{Q})$ when n is even and $\mathrm{disc}'(\mathrm{Q})$ when n is odd. For any S-scheme S' and any quadratic form Q on $\mathrm{V}_{\mathrm{S}'}$ we have $\mathrm{D}(\mathrm{Q}(gx)) = (\det g)^2 \mathrm{D}(\mathrm{Q}(x))$ for $g \in \mathrm{GL}(\mathrm{V}_{\mathrm{S}'})$. Indeed, this is obvious when n is even and follows from reduction to the $\mathbf{Z}$-flat universal case for the half-discriminant when n is odd. In particular, orthogonal group schemes for non-degenerate line bundle-valued quadratic forms $(\mathrm{V}, \mathrm{L}, q)$ lie in $\det^{-1}(\mu_2)$. Since the determinant map $\mathrm{O}(q) \to \mu_2$ restricts to the identity on the central μ_2 when n is odd, we get $\mathrm{O}(q) = \mu_2 \times \mathrm{SO}'(q)$ for odd n. This settles the case of odd n when 2 is a unit, but below we will give a characteristic-free argument for odd n that does not ignore characteristic 2.

Let $\mathrm{X}' = \mathrm{SL}(\mathrm{V})$ and let Y' be the scheme of non-degenerate L-valued quadratic forms Q on V such that $\mathrm{D}(\mathrm{Q}) = \mathrm{D}(q)$, so X' acts on Y'. Let $f' : \mathrm{X}' \to \mathrm{Y}'$ be the orbit map through $q \in \mathrm{Y}'(\mathrm{S})$. Note that $\mathrm{SO}'(q) = f'^{-1}(q)$, so this is S-smooth *provided* that f' is smooth. We claim that f' is surjective. It suffices to treat the case $\mathrm{S} = \mathrm{Spec}\, k$ for an algebraically closed field k and to work with k-points. By Exercise 1.6.10, Proposition C.1.4(3), and the behavior of $\mathrm{D}(\mathrm{Q})$ under a linear change of coordinates on V, any non-degenerate Q on k^n with $\mathrm{D}(\mathrm{Q}) = \mathrm{D}(q)$ has the form $\mathrm{Q} = q \circ g$ for some $g \in \mathrm{GL}_n(k)$ satisfying $(\det g)^2 = 1$. Writing $n = 2m$ or $n = 2m+1$, $q = q_n \circ \gamma$ with $\mathrm{D}(q) = (\det \gamma)^2(-1)^m$ and $\mathrm{Q} = q_n \circ \gamma'$ with $\mathrm{D}(\mathrm{Q}) = (\det \gamma')^2(-1)^m$. Hence, $\det \gamma' = \pm \det \gamma$, so it suffices to check that $\mathrm{O}(\mathrm{Q})(k)$ contains an element with determinant -1. This is obvious by direct inspection of q_n depending on the parity of n (see Exercise 1.6.10(ii),(iii)).

Since X' is S-flat (even smooth), f' is smooth if it is so between geometric fibers over S. We shall now show that f' is smooth when $\mathrm{S} = \mathrm{Spec}\, k$ for an algebraically closed field k provided that either n is odd or n is even with $\mathrm{char}(k) \neq 2$. It is harmless to scale q by some $c \in k^\times$, so we may now assume $\mathrm{D}(q) = \mathrm{D}(q_n)$. Thus, q_n lies in the level set of q, so by surjectivity of the orbit map f' through q it suffices for the proof of smoothness of f' to assume $q = q_n$. Since f' is either everywhere smooth or nowhere smooth (by homogeneity), we shall check smoothness holds at $g = 1 \in \mathrm{SL}_n(k) = \mathrm{X}'(k)$.

Step 3. Assume n is odd. We claim that the level set Y' is smooth (so smoothness of f' is equivalent to surjectivity on tangent spaces). By transitivity of the SL(V)-action, this amounts to checking that the equation $\mathrm{D}(\mathrm{Q}) = \mathrm{D}(q_n)$ defining Y' in the smooth scheme Y of all non-degenerate quadratic forms on V is non-constant to first order at q_n. That is, relative to suitable linear coordinates on V, we claim the polynomial $\mathrm{D}(q_n+\epsilon\mathrm{Q})-\mathrm{D}(q_n)$ in a varying $\mathrm{Q} \in \mathrm{Quad}(k^n)$ is nonzero. For $\mathrm{Q} = cx_0^2$ we have $\mathrm{D}(q_n+\epsilon\mathrm{Q})-\mathrm{D}(q_n) = \pm\epsilon c$ by base change from the universal case over $\mathbf{Z}[\mathrm{C}]$ (the sign depends on the parity of $(n-1)/2$). This proves the smoothness of Y'.

Since $\mathrm{X}' = \mathrm{SL(V)}$ is a smooth hypersurface in $\mathrm{X} = \mathrm{GL(V)}$, the difference in tangent space dimensions at $1 \in \mathrm{X}'(k)$ and $q_n \in \mathrm{Y}'(k)$ is $n^2 - n(n+1)/2 = n(n-1)/2$. Thus, surjectivity of f' on tangent spaces is equivalent to the kernel $\mathrm{Tan}_1(\mathrm{SO}'(q))$ of $\mathrm{Tan}_1(f')$ having dimension $n(n-1)/2$. But $\mathrm{O}(q) = \mu_2 \times \mathrm{SO}'(q)$ due to the oddness of n, so we want $\mathrm{Tan}_1(\mathrm{O}(q))$ to have dimension $\dim \mathrm{Tan}_1(\mu_2)+n(n-1)/2$; i.e., $n(n-1)/2$ when $\mathrm{char}(k) \neq 2$ and $1+n(n-1)/2$ in the defect-1 case in characteristic 2. These were both established in the analysis of smoothness for orthogonal group schemes.

Suppose $n = 2m$ is even and $\mathrm{char}(k) \neq 2$. We can diagonalize q as a sum of squares of all variables, so the smoothness proof for Y' when n is odd carries over. The surjectivity on tangent spaces can also be proved exactly as for odd n, so f' is smooth as desired. $\square$

An alternative proof of the smoothness of orthogonal group schemes for even n is to use Lemma C.2.1 below to pass to the case of q_n, for which a direct counting of equations does the job. This alternative method is worked out in [**DG**, III, § 5, 2.3], based on an equation-counting smoothness criterion in [**DG**, II, § 5, 2.7]. We have chosen to use the preceding argument based on another smoothness criterion because it treats even and odd n on an equal footing (and adapts to other groups; see [**GY**]).

C.2. Clifford algebras and special orthogonal groups. — Let $q : \mathrm{V} \to \mathscr{O}_\mathrm{S}$ be a possibly degenerate quadratic space, with V of rank $n \geqslant 1$. The *Clifford algebra* $\mathrm{C}(\mathrm{V}, q)$ is the quotient of the tensor algebra of V by the relations $x^{\otimes 2} = q(x)$ for local sections x of V. This has a natural $\mathbf{Z}/2\mathbf{Z}$-grading (as a direct sum of an "even" part and "odd" part) via the $\mathbf{Z}$-grading on the tensor algebra. By considering expansions relative to a local basis of V we see that $\mathrm{C}(\mathrm{V}, q)$ is a finitely generated $\mathscr{O}_\mathrm{S}$-module. For $q = 0$, this is the exterior algebra of V. We are primarily interested in non-degenerate q.

There are versions of the Clifford construction for (possibly degenerate) line bundle-valued q, but numerous complications arise; see [**Au**, 1.8], [**BK**], and [**PS**, § 4] for further discussion. An alternative reference on Clifford algebras and related quadratic invariants is [**Knus**, IV], at least for an affine base.

Lemma C.2.1. — *If* (V, q) *is a quadratic space of rank* $n \geqslant 1$ *over a scheme* S *then it is isomorphic to* $(\mathscr{O}_{\mathrm{S}}^{n}, q_n)$ *fppf-locally on* S*. If* n *is even or* 2 *is a unit on* S *then it suffices to use the étale topology rather than the fppf topology.*

Keep in mind that a "quadratic space" is understood to be non-degenerate (in the fiberwise sense) unless we say otherwise.

Proof. — In [**SGA7**, XII, Prop. 1.2] the smoothness of $(q = 0)$ is used to prove the following variant by induction on n: étale-locally on S, q becomes isomorphic to q_n when n is even and to $ux_0^2 + q_{2m}$ for some unit u when $n = 2m+1$ is odd. Once the induction is finished, we are done when n is even and we need to extract a square root of u when n is odd. This requires working fppf-locally for odd n when 2 is not a unit on the base. □

Lemma C.2.1 is useful for reducing problems with general (non-degenerate) quadratic spaces to the case of q_n over $\mathbf{Z}$, as we shall now see. In what follows, we fix a quadratic space (V, q) of rank $n \geqslant 1$ over a scheme S.

Proposition C.2.2. — *Assume* n *is even. The* $\mathscr{O}_{\mathrm{S}}$*-algebra* $\mathrm{C}(\mathrm{V}, q)$ *and its even part* $\mathrm{C}_0(\mathrm{V}, q)$ *are respectively isomorphic, fppf-locally on* S*, to* $\mathrm{Mat}_{2^{n/2}}(\mathscr{O})$ *and a product of two copies of* $\mathrm{Mat}_{2^{(n/2)-1}}(\mathscr{O})$*, with the left* $\mathrm{C}_0(\mathrm{V}, q)$*-module* $\mathrm{C}_1(\mathrm{V}, q)$ *free of rank* 1 *Zariski-locally on* S*.*

In particular, $\mathrm{C}(\mathrm{V}, q)$ *and the* $\mathrm{C}_j(\mathrm{V}, q)$ *are vector bundles and the quasi-coherent centers of* $\mathrm{C}(\mathrm{V}, q)$ *and* $\mathrm{C}_0(\mathrm{V}, q)$ *are respectively equal to* $\mathscr{O}_{\mathrm{S}}$ *and a rank-*2 *finite étale* $\mathscr{O}_{\mathrm{S}}$*-algebra* Z_q*. Moreover, the natural map* $\mathrm{V} \to \mathrm{C}(\mathrm{V}, q)$ *is a subbundle inclusion and* $\mathrm{C}_0(\mathrm{V}, q)$ *is the centralizer of* Z_q *in* $\mathrm{C}(\mathrm{V}, q)$*.*

It follows that $\mathrm{C}(\mathrm{V}, q)$ is an Azumaya algebra over S (and $\mathrm{C}_0(\mathrm{V}, q)$ is an Azumaya algebra over a degree-2 finite étale cover of S). The notion of Azumaya algebra will only arise in Example C.6.3.

Proof. — By Lemma C.2.1 we may assume that V admits a basis $\{e_i\}$ identifying q with q_n. In $\mathrm{C}(\mathrm{V}, q)$ we have $v'v = -vv' + \mathrm{B}_q(v, v')$ for $v, v' \in \mathrm{V}$, so $\mathrm{C}(\mathrm{V}, q)$ is spanned by the 2^n products $e_{\mathrm{J}} = e_{j_1} \cdots e_{j_h}$ for subsets $\mathrm{J} = \{j_1, \ldots, j_h\} \subset \{1, \ldots, n\}$ (with $j_1 < \cdots < j_h$, and $e_{\varnothing} = 1$ for $h = 0$). Thus, if we construct a surjection from $\mathrm{C}(\mathrm{V}, q)$ onto $\mathrm{Mat}_{2^{n/2}}(\mathscr{O})$ then it must be an isomorphism (and the e_{J} must be a basis of $\mathrm{C}(\mathrm{V}, q)$).

Since n is even, there are complementary isotropic free subbundles $\mathrm{W}, \mathrm{W}' \subset \mathrm{V}$ of rank $n/2$, in perfect duality via B_q; e.g., $\mathrm{W} = \mathrm{span}\{e_{2i-1}\}_{1 \leqslant i \leqslant n/2}$ and $\mathrm{W}' = \mathrm{span}\{e_{2i}\}_{1 \leqslant i \leqslant n/2}$. Let $\mathrm{A} := \mathrm{C}(\mathrm{W}, q|_{\mathrm{W}}) = \mathrm{C}(\mathrm{W}, 0) = \wedge^{\bullet}(\mathrm{W})$, with even and odd parts $A_+ = \bigoplus \wedge^{2j}(W)$ and $A_- = \bigoplus \wedge^{2j+1}(W)$, so A is a vector bundle of rank $2^{n/2}$. The endomorphism algebra $\mathscr{E}nd(A)$ has a $\mathbf{Z}/2\mathbf{Z}$-grading: an endomorphism of $A = A_+ \bigoplus A_-$ is *even* if it carries $A_\pm$ into $A_\pm$ and *odd* if it carries $A_\pm$ into $A_\mp$. We will construct a surjective algebra homomorphism $\rho : C(V, q) \twoheadrightarrow \mathscr{E}nd(A) \simeq \mathrm{Mat}_{2^{n/2}}(\mathscr{O})$ respecting the $\mathbf{Z}/2\mathbf{Z}$-grading, so ρ is an isomorphism that carries $C_0(V, q)$ onto $\mathscr{E}nd(A_+) \times \mathscr{E}nd(A_-)$ and carries $C_1(V, q)$ onto $\mathscr{H}om(A_+, A_-) \oplus \mathscr{H}om(A_-, A_+)$, completing the proof since A_+ and A_- visibly have the *same* rank, as one sees via the binomial expansion of $0 = (1-1)^{n/2}$. (In [**KO**] there is given a generalization of the isomorphism ρ for certain non-split (V, q). Also, there is a simpler direct proof of the invertibility of $C_1(V, q)$ as a left $C_0(V, q)$-module, namely via right multiplication by a local section of V on which q is unit-valued; this is the argument used in [**Knus**, IV, 7.5.2].)

To build ρ, by the universal property of the Clifford algebra we just need to define a linear map $L : W \bigoplus W' = V \to \mathscr{E}nd(A)$ such that $L(v) \circ L(v)$ is multiplication by $q(v)$. For $w \in W$ and $w' \in W'$, define $L(w) = w \wedge (\cdot)$ and define $L(w')$ to be the contraction operator

$$\delta_{w'} : w_1 \wedge \cdots \wedge w_j \mapsto \sum_{i=1}^{j} (-1)^{i-1} B_q(w_i, w') w_1 \wedge \ldots \widehat{w_i} \cdots \wedge w_j$$

(the unique anti-derivation of A coinciding with $B_q(\cdot, w')$ on $W \subset A_-$). Induction on j gives $\delta_{w'} \circ \delta_{w'} = 0$, and clearly $L(w) \circ L(w) = 0$, so

$$L(w + w') \circ L(w + w') = L(w) \circ L(w') + L(w') \circ L(w).$$

This is multiplication by $B_q(w, w') = q(w + w')$ because

$$\begin{aligned} (L(w') \circ L(w))(x) = \delta_{w'}(w \wedge x) &= \delta_{w'}(w)x - w \wedge \delta_{w'}(x) \\ &= B_q(w, w')x - (L(w) \circ L(w'))(x). \end{aligned}$$

The resulting map of algebras $\rho : C(V, q) \to \mathscr{E}nd(A)$ respects the $\mathbf{Z}/2\mathbf{Z}$-gradings since $w \wedge (\cdot)$ and $\delta_{w'}$ are odd endomorphisms. To prove ρ is surjective it suffices to check on fibers. The maps ρ_s are isomorphisms by the classical theory over fields (see the proof of [**Chev97**, II.2.1], or [**Knus**, IV, 2.1.1]). □

Remark C.2.3. — In the special case $q = q_{2m}$, the computation of the center Z_q of $C_0(V, q)$ as $\mathscr{O}_S \times \mathscr{O}_S$ via an explicit description of a fiberwise nontrivial idempotent z is given in [**DG**, III, 5.2.4]. (Generalizations for non-split q are given in [**Knus**, IV, 2.3.1, 4.8.5].) We now give such a calculation. Consider the standard basis $\{e_1, \ldots, e_{2m}\}$, so $\{e_{2i-1}, e_{2i}\}$ for $1 \leqslant i \leqslant m$ is a collection of pairwise orthogonal bases of standard hyperbolic planes. To compute Z_q in terms of products among the e_i's, we may and do work over $\mathbf{Z}$.

In $\mathrm{C}_0(\mathrm{V},q)$, for each such pair $\{e,e'\}=\{e_{2i-1},e_{2i}\}$ we have

$$1=q(e+e')=(e+e')^2=e^2+(ee'+e'e)+e'^2=ee'+e'e, \tag{C.2.1}$$

so $(ee')^2=e(e'e)e'=e(1-ee')e'=ee'$. Hence, the elements $w_i=e_{2i-1}e_{2i}$ pairwise commute (signs cancel in pairs) and $(1-2w_i)^2=1$, so the product $w:=\prod_{i=1}^m(1-2w_i)$ satisfies $w^2=1$. If we define $w_i'=e_{2i}e_{2i-1}=1-w_i$ then $1-2w_i'=-(1-2w_i)$, so $w':=\prod_{i=1}^m(1-2w_i')$ is equal to $(-1)^m w$. Direct calculation shows that e_j commutes with $1-2w_i$ when $j\notin\{2i-1,2i\}$ whereas e_j anti-commutes with $1-2w_i$ if $j\in\{2i-1,2i\}$, so e_j anti-commutes with w. Hence, all $e_je_{j'}$ with $j<j'$ commute with w, so w is central in $\mathrm{C}_0(\mathrm{V},q)$.

Over $\mathbf{Z}[1/2]$ we define

$$z=(1/2)(1-w)=\sum_i w_i-2\sum_{i<i'}w_iw_{i'}+4\sum_{i<i'<i''}w_iw_{i'}w_{i''}+\cdots+(-2)^{m-1}\prod_{i=1}^m w_i.$$

By inspection, z lies in $\mathrm{C}_0(\mathrm{V},q)$ over $\mathbf{Z}$ and modulo every prime it is distinct from 0 and 1. Hence, $\mathbf{Z}\oplus\mathbf{Z}z$ is a direct summand of $\mathrm{C}_0(\mathrm{V},q)$ that is a central subalgebra, so it is Z_q. Moreover, by computing over $\mathbf{Z}[1/2]$ we see that $z^2=z$, so z is a fiberwise nontrivial idempotent of Z_q (i.e., $\mathrm{Z}_q=\mathbf{Z}\cdot z\times\mathbf{Z}\cdot(1-z)$). Note that for $z':=(1/2)(1-w')$ we have $z'=z$ when m is even and $z'=1-z$ when m is odd. Also, $ze_j+e_jz=e_j$ for all j, so $zv=v(1-z)$ for all $v\in\mathrm{V}$. Thus, if σ denotes the nontrivial involution of Z_q (given by swapping the idempotents z and $1-z$) then $\zeta v=v\sigma(\zeta)$ for all $v\in\mathrm{V}$ and $\zeta\in\mathrm{Z}_q$.

A straightforward calculation that we omit shows that if $m\geqslant 2$ then the expansion of e_1z and e_2z in the basis of products of e_i's contains a term of the form $e_1e_3e_4$ and $-e_1e_3e_4$ respectively. Thus, if $m\geqslant 2$ then an element $a+bz\in\mathrm{Z}_q$ (with $a,b\in\mathscr{O}_\mathrm{S}$) satisfies $\mathrm{V}\cdot(a+bz)\subset\mathrm{V}$ inside $\mathrm{C}(\mathrm{V},q)$ if and only if $b=0$. Hence, by fppf descent from the split case we conclude in general (without any split hypothesis on (V,q)) that if $m\geqslant 2$ and $\zeta\in\mathrm{Z}_q^\times$ then $\zeta\mathrm{V}\zeta^{-1}=\mathrm{V}$ if and only if $\sigma(\zeta)/\zeta\in\mathscr{O}_\mathrm{S}^\times$, or equivalently $\zeta^2\in\mathscr{O}_\mathrm{S}^\times$. That is, for $m\geqslant 2$ the normalizer of V in the S-torus $\mathrm{R}_{\mathrm{Z}_q/\mathrm{S}}(\mathbf{G}_\mathrm{m})$ via its conjugation action on $\mathrm{C}(\mathrm{V},q)$ is the subgroup of points ζ such that $\zeta^2\in\mathbf{G}_\mathrm{m}$. In contrast, if $m=1$ then the entire group $\mathrm{R}_{\mathrm{Z}_q/\mathrm{S}}(\mathbf{G}_\mathrm{m})$ preserves V, either by computations in the split case or the observation that when V has rank 2 the subbundle inclusions $\mathrm{V}\subset\mathrm{C}_1(\mathrm{V},q)$ and $\mathrm{Z}_q\subset\mathrm{C}_0(\mathrm{V},q)$ are equalities for rank reasons.

The structure of $\mathrm{C}(\mathrm{V},q)$ and $\mathrm{C}_0(\mathrm{V},q)$ for odd n is opposite that for even n:

Proposition C.2.4. — *Assume $n=2m+1$ is odd. The even part $\mathrm{C}_0(\mathrm{V},q)$ is isomorphic, fppf-locally on* S, *to $\mathrm{Mat}_{2^m}(\mathscr{O})$. The center Z_q of $\mathrm{C}(\mathrm{V},q)$ is a $\mathbf{Z}/2\mathbf{Z}$-graded finite locally free $\mathscr{O}_\mathrm{S}$-module of rank 2 whose degree-0 part Z_q^0 is $\mathscr{O}_\mathrm{S}$ and whose degree-1 part Z_q^1 is an invertible sheaf that satisfies $\mathrm{Z}_q^1\otimes\mathrm{Z}_q^1\simeq$*

$Z_q^0 = \mathscr{O}_S$ *via multiplication* (*so* Z_q *is locally generated as an* $\mathscr{O}_S$*-algebra by the square root of a unit, and hence over* $S[1/2]$ *it is finite étale of rank* 2).

The natural $\mathbf{Z}/2\mathbf{Z}$*-graded multiplication map* $Z_q \otimes_{\mathscr{O}_S} C_0(V, q) \to C(V, q)$ *is an isomorphism. In particular,* $C(V, q)$ *is a locally free* $\mathscr{O}_S$*-module of rank* 2^n *that is isomorphic to* $\mathrm{Mat}_{2^m}(Z_q)$ *as a* Z_q*-algebra fppf-locally on* S. *Moreover,* $V \to C(V, q)$ *is a subbundle inclusion.*

This result implies that $C_0(V, q)$ is an Azumaya algebra over S whereas $C(V, q)$ is an Azumaya algebra over a degree-2 finite fppf cover of S.

Proof. — First consider the case $n = 1$ (i.e., $m = 0$), so by Zariski localization $q = ux^2$ for a unit u on $S = \operatorname{Spec} R$. Then $C(V, q) = R[z]/(z^2 - u) = R \oplus Rz$ with degree-0 part R and degree-1 part Rz. This settles all of the assertions in this case, so now we may and do assume $n \geqslant 3$ (i.e., $m \geqslant 1$).

The assertions imply that the center is a local direct summand, so via an fppf base change and Lemma C.2.1 it suffices to treat the case $q = q_n$ over any S. This is the orthogonal direct sum of x_0^2 and q_{2m}, and we can use the known results for rank $2m \geqslant 2$ from Proposition C.2.2. More specifically, if (V', q') and (V'', q'') are (possibly degenerate) quadratic spaces over a scheme S then the Clifford algebra of their orthogonal sum $(V' \bigoplus V'', q' \perp q'')$ is naturally isomorphic as a $\mathbf{Z}/2\mathbf{Z}$-graded algebra to the "super-graded tensor product" $C(V', q') \otimes' C(V'', q'')$ which is the ordinary tensor product module equipped with the algebra structure defined by the requirement $(1 \otimes a'')(a' \otimes 1) = (-1)^{\deg(a')\deg(a'')}(a' \otimes a'')$ for homogeneous a' and a''. (See [**Knus**, IV, 1.3.1].) Hence, $C(q_n) = C(x_0^2) \otimes' C(q_{2m})$, so $C(q_n)$ is a free $\mathscr{O}_S$-module of rank $2^{1+2m} = 2^n$. In particular, the even and odd parts of $C(V, q) = C(q_n)$ are locally free of finite rank and $V \to C(V, q)$ is a subbundle inclusion.

Multiplication by the standard basis vector e_0 swaps even and odd parts of $C(V, q)$ (since $e_0^2 = q(e_0) = 1$), so each part has rank $2^{n-1} = 2^{2m}$. For any w in the span W of $\{e_1, \ldots, e_{2m}\}$ we have $(e_0w)(e_0w) = -e_0^2w^2 = -q_{2m}(w)$. Thus, $w \mapsto e_0w \in C_0(q_n)$ extends to a homomorphism $f : C(W, -q_{2m}) \to C_0(q_n)$. The map f is fiberwise injective since geometric fibers of $C(W, -q_{2m})$ are simple algebras with rank 2^{2m} (in fact, Mat_{2^m}), so f is an isomorphism on fibers over S. The algebra $C(W, -q_{2m})$ is a locally free module of rank 2^{2m} (in fact, fppf-locally on the base it is $\mathrm{Mat}_{2^m}(\mathscr{O})$, by Proposition C.2.2), so f is an isomorphism of algebras. In particular, $C_0(V, q)$ has the asserted matrix algebra structure fppf-locally on S.

Consider the element $z = e_0 \prod_{i=1}^m (1 - 2e_{2i-1}e_{2i}) \in C_1(V, q)$. Computing as in Remark C.2.3, if $j > 0$ then e_j anti-commutes with $\prod_{i=1}^m (1 - 2e_{2i-1}e_{2i})$ by (C.2.1), yet such e_j also anti-commutes with e_0, so e_j commutes with z. It is likewise clear that z commutes with e_0 (since the anti-commutation of e_0 with both e_{2i-1} and e_{2i} implies that e_0 commutes with $e_{2i-1}e_{2i}$ for all i), so z commutes with every e_j ($j \geqslant 0$). Likewise, $z^2 = 1$. Thus, $\{1, z\}$ spans a $\mathbf{Z}/2\mathbf{Z}$-graded central subalgebra Z of $\mathrm{C}(\mathrm{V}, q)$ that is a subbundle of rank 2 with odd part spanned by z, so $\mathrm{Z}^1 \otimes \mathrm{Z}^1 \simeq \mathscr{O}_\mathrm{S}$ via multiplication. Since z is a unit, the natural $\mathbf{Z}/2\mathbf{Z}$-graded algebra map $\mathrm{Z} \otimes \mathrm{C}_0(\mathrm{V}, q) \to \mathrm{C}(\mathrm{V}, q)$ defined by multiplication is fiberwise injective and hence (for rank reasons) an isomorphism. But $\mathrm{C}_0(\mathrm{V}, q)$ has trivial center, so Z is the center of $\mathrm{C}(\mathrm{V}, q)$. □

Assume n is even. The natural action of $\mathrm{O}(q)$ on $\mathrm{C}(\mathrm{V}, q)$ preserves the $\mathbf{Z}/2\mathbf{Z}$-grading and hence induces an action of $\mathrm{O}(q)$ on $\mathrm{C}_0(\mathrm{V}, q)$, so we obtain an action of $\mathrm{O}(q)$ on the finite étale center Z_q of $\mathrm{C}_0(\mathrm{V}, q)$. The automorphism scheme $\underline{\mathrm{Aut}}_{\mathrm{Z}_q/\mathscr{O}_\mathrm{S}}$ is uniquely isomorphic to $(\mathbf{Z}/2\mathbf{Z})_\mathrm{S}$ since Z_q is finite étale of rank 2 over $\mathscr{O}_\mathrm{S}$. Thus, for even n we get a homomorphism

$$\mathrm{D}_q : \mathrm{O}(q) \to (\mathbf{Z}/2\mathbf{Z})_\mathrm{S} \tag{C.2.2}$$

compatible with isomorphisms in the quadratic space (V, q), and its formation commutes with any base change on S. This is the *Dickson invariant*, and it is discussed in detail in [**Knus**, IV, § 5]. (The Dickson invariant for even n goes back to Dickson [**Di**, p. 206] over finite fields, and Arf [**Arf**] over general fields of characteristic 2.) By Remark C.2.3, $\mathrm{D}_q(-1) = 0$.

Remark C.2.5. — We have defined the S-group $\mathrm{O}(q) \subset \mathrm{GL}(\mathrm{V})$ for any non-degenerate line bundle-valued quadratic form $(\mathrm{V}, \mathrm{L}, q)$, but the definition of the Dickson invariant D_q involves the Clifford algebra $\mathrm{C}(\mathrm{V}, q)$ that we have only defined and studied for quadratic forms rather than for general line bundle-valued q.

Below we explain how to define the Dickson invariant $\mathrm{D}_q : \mathrm{O}(q) \to (\mathbf{Z}/2\mathbf{Z})_\mathrm{S}$ for any non-degenerate $(\mathrm{V}, \mathrm{L}, q)$ by working Zariski-locally on S to trivialize L. The main point, going back to the thesis of W. Bichsel, is that we can always define what should be the "even Clifford algebra" $\mathrm{C}_0(\mathrm{V}, \mathrm{L}, q)$ equipped with its $\mathrm{O}(q)$-action for line bundle-valued q, despite the lack of a definition of $\mathrm{C}(\mathrm{V}, \mathrm{L}, q)$ for such general q (cf. [**Au**, 1.8] and references therein). We will also define the "odd Clifford module" $\mathrm{C}_1(\mathrm{V}, \mathrm{L}, q)$ as an $\mathrm{O}(q)$-equivariant left $\mathrm{C}_0(\mathrm{V}, \mathrm{L}, q)$-module that is Zariski-locally (on S) free of rank 1 and naturally contains V as an $\mathrm{O}(q)$-equivariant subbundle (similarly to the case $\mathrm{L} = \mathscr{O}_\mathrm{S}$ as in Proposition C.2.2).

The construction of $\mathrm{C}_0(\mathrm{V}, \mathrm{L}, q)$, explained below, will be compatible with base change on S and naturally recover the usual even Clifford algebra when q takes values in the trivial line bundle (and similarly for $\mathrm{C}_1(\mathrm{V}, \mathrm{L}, q)$ encoding the left module structure over the even part). Granting this, the center Z_q of $\mathrm{C}_0(\mathrm{V}, \mathrm{L}, q)$ is therefore a finite étale $\mathscr{O}_\mathrm{S}$-algebra of degree 2 on which $\mathrm{O}(q)$ naturally acts (cf. [**Au**, Def. 1.12] and [**BK**, Thm. 3.7(2)]). Such a general Z_q is often called the *discriminant algebra* (or *discriminant extension*). (If $n = 2$ it recovers the classical notion of discriminant, by Proposition C.3.15; we do not use this.) The resulting homomorphism $\mathrm{O}(q) \to \underline{\mathrm{Aut}}_{\mathrm{Z}_q/\mathscr{O}_\mathrm{S}} = (\mathbf{Z}/2\mathbf{Z})_\mathrm{S}$ defines D_q in general. (In particular, $\mathrm{D}_q = \mathrm{D}_{uq}$ for units u and quadratic spaces (V, q), via the natural equality $\mathrm{O}(q) = \mathrm{O}(uq)$ inside $\mathrm{GL}(\mathrm{V})$.)

To build $\mathrm{C}_0(\mathrm{V}, \mathrm{L}, q)$, we show that for any quadratic space (V, q) and $u \in \mathbf{G}_\mathrm{m}(\mathrm{S})$, there is a natural $\mathscr{O}_\mathrm{S}$-algebra isomorphism $h_{u,q} : \mathrm{C}_0(\mathrm{V}, q) \simeq \mathrm{C}_0(\mathrm{V}, uq)$ that satisfies the following properties: it is compatible with base change on S, it respects the actions by the common subgroup $\mathrm{O}(q) = \mathrm{O}(uq)$ inside $\mathrm{GL}(\mathrm{V})$, and it is multiplicative in u in the sense that $h_{u',uq} \circ h_{u,q} = h_{u'u,q}$ for units u and u'. Once such isomorphisms are in hand, we can globally build $\mathrm{C}_0(\mathrm{V}, \mathrm{L}, q)$ equipped with its $\mathrm{O}(q)$-action for general non-degenerate line bundle-valued quadratic forms via Zariski-gluing.

The existence of such isomorphisms $h_{u,q}$ is provided over an affine base by [**Knus**, IV, 7.1.1, 7.1.2] via direct work that avoids base change. Here is an alternative construction via descent theory. Consider an fppf cover S' of S such that the pullback unit $u' = u|_{\mathrm{S}'}$ has the form $u' = a^2$ for a unit a on S'. Let $(\mathrm{V}', q') = (\mathrm{V}, q)_{\mathrm{S}'}$, so there is an isomorphism of quadratic spaces $(\mathrm{V}', q') \simeq (\mathrm{V}', u'q')$ defined by $v' \mapsto a^{-1}v'$. This induces a graded isomorphism of Clifford algebras $f'_a : \mathrm{C}(\mathrm{V}', q') \simeq \mathrm{C}(\mathrm{V}', u'q')$ that is clearly equivariant for the actions of the common subgroup $\mathrm{O}(q') = \mathrm{O}(u'q') \subset \mathrm{GL}(\mathrm{V}')$. Changing the choice of a amounts to multiplying a by some $\zeta \in \mu_2(\mathrm{S}')$, and $f'_{\zeta a} = f'_a \circ [\zeta]$ where $[\zeta]$ is the graded automorphism of $\mathrm{C}(\mathrm{V}', q')$ induced by the automorphism of (V', q') defined by $v' \mapsto \zeta v'$. Since $\zeta^2 = 1$, $[\zeta]$ induces the identity on $\mathrm{C}_0(\mathrm{V}', q')$, so the restriction of f'_a to the even parts is *independent* of the choice of a. Consequently, denoting this isomorphism on the even parts as f', we see that the pullbacks $\mathrm{pr}_1^*(f')$ and $\mathrm{pr}_2^*(f')$ over $\mathrm{S}'' = \mathrm{S}' \times_\mathrm{S} \mathrm{S}'$ coincide, so f' descends to an algebra isomorphism $h_{u,q} : \mathrm{C}_0(\mathrm{V}, q) \simeq \mathrm{C}_0(\mathrm{V}, uq)$. By fppf descent from S', $h_{u,q}$ is equivariant for the actions of $\mathrm{O}(q) = \mathrm{O}(uq)$, compatible with base change on S, and multiplicative in u.

To construct the $\mathrm{O}(q)$-equivariant left $\mathrm{C}_0(\mathrm{V}, \mathrm{L}, q)$-module $\mathrm{C}_1(\mathrm{V}, \mathrm{L}, q)$, we need to modify the procedure used for the even part because the restriction of f'_a to the odd part generally depends on the choice of square root a of u. More precisely, for a point ζ of μ_2, the effect of $[\zeta]$ on the odd part of $\mathrm{C}(\mathrm{V}, q)$ for $\mathscr{O}_\mathrm{S}$-valued q is multiplication by ζ rather than the identity map. Thus, letting m_a denote multiplication by the $\mathscr{O}_{\mathrm{S}'}^\times$-valued a, we replace f'_a with $\mathrm{F}_a := m_a \circ f'_a = f'_a \circ m_a$. Clearly $\mathrm{F}_{\zeta a} = \zeta^2 \mathrm{F}_a = \mathrm{F}_a$ for any μ_2-valued ζ, so F_a depends only on $a^2 = u$ and hence by using F_a we can carry out the descent and verify the desired properties. (For $\mathrm{L} = \mathscr{O}_\mathrm{S}$ the effect of F_a on the subbundle $\mathrm{V} \subset \mathrm{C}_1(\mathrm{V}, q)$ is multiplication by $a \cdot 1/a = 1$, so for general L we naturally find V as an $\mathrm{O}(q)$-equivariant subbundle of $\mathrm{C}_1(\mathrm{V}, \mathrm{L}, q)$.)

To analyze properties of the Dickson invariant for even n, it is convenient to introduce a certain closed subgroup of the S-group $\mathrm{C}(\mathrm{V}, q)^\times$ of units of the

Clifford algebra. The following definition for even n will later be generalized to odd n, for which some additional complications arise.

Definition C.2.6. — For even n, the *Clifford group* $\mathrm{GPin}(q)$ is the closed S-subgroup scheme

$$\mathrm{GPin}(q) := \{u \in \mathrm{C}(\mathrm{V}, q)^\times \mid u\mathrm{V}u^{-1} = \mathrm{V}\} \tag{C.2.3}$$

of units of the Clifford algebra that normalize V inside $\mathrm{C}(\mathrm{V}, q)$.

Remark C.2.7. — Assume $n = 2m$ is even. The explicit computations with Z_q in the split case in Remark C.2.3 show that if $n \geqslant 4$ then $\mathrm{GPin}(q) \bigcap \mathrm{R}_{\mathrm{Z}_q/\mathrm{S}}(\mathbf{G}_\mathrm{m})$ is the subgroup of points of the rank-2 torus $\mathrm{R}_{\mathrm{Z}_q/\mathrm{S}}(\mathbf{G}_\mathrm{m})$ whose square lies in $\mathbf{G}_\mathrm{m}$ whereas if $n = 2$ then $\mathrm{R}_{\mathrm{Z}_q/\mathrm{S}}(\mathbf{G}_\mathrm{m}) \subset \mathrm{GPin}(q)$. The case $n \geqslant 4$ with $\mathrm{S} = \mathrm{Spec}\, k$ for a field k is [**KMRT**, III, 13.16].

Consider (V, q) with even rank n. Since the center of $\mathrm{C}(\mathrm{V}, q)$ is $\mathscr{O}_\mathrm{S}$, an elementary calculation (given in [**Chev97**, II.3.2] over fields, but working verbatim over any S) shows that every point of $\mathrm{GPin}(q)$ is "locally homogeneous": Zariski-locally on the base, it is in either the even or odd part of $\mathrm{C}(\mathrm{V}, q)$. (This is false for odd n, in view of the structure of Z_q in Proposition C.2.4.) Thus, the S-group $\mathrm{GPin}(q)$ agrees with the "Clifford group" as defined in [**Knus**, IV, 6.1]. For odd n we will have to force local homogeneity into the definition of the Clifford group in order to get the right notion for such n; see § C.4.

There is a natural action by $\mathrm{GPin}(q)$ on V via conjugation, and the resulting homomorphism $\mathrm{GPin}(q) \to \mathrm{GL}(\mathrm{V})$ lands in $\mathrm{O}(q)$ because for $v \in \mathrm{V}$ and $u \in \mathrm{GPin}(q)$ we have $q(uvu^{-1}) = (uvu^{-1})^2 = uv^2u^{-1} = q(v)$ in $\mathrm{C}(\mathrm{V}, q)$. However, this action has a drawback: the intervention of an unpleasant sign for the conjugation action by the dense open non-vanishing locus $\mathrm{U} = \{q \neq 0\} \subset \mathrm{V}$. To be precise, any $u \in \mathrm{U}$ satisfies $u^2 = q(u) \in \mathbf{G}_\mathrm{m}$ in $\mathrm{C}(\mathrm{V}, q)$, so $u \in \mathrm{C}(\mathrm{V}, q)^\times$ and $uv + vu = \mathrm{B}_q(u, v)$ for any $v \in \mathrm{V}$. Hence,

$$uvu^{-1} = -v + (\mathrm{B}_q(u, v)/q(u))u \in \mathrm{V}, \tag{C.2.4}$$

so $\mathrm{U} \subset \mathrm{GPin}(q)$ but the conjugation action on V by any $u \in \mathrm{U}$ is the *negative* of reflection through u relative to q. Thus, following Atiyah–Bott–Shapiro [**ABS**, § 3] (and [**Knus**, IV, 6.1]), we define a representation

$$\pi_q : \mathrm{GPin}(q) \to \mathrm{GL}(\mathrm{V})$$

by

$$\pi_q(u)(v) = (-1)^{\deg_q(u)} uvu^{-1} \tag{C.2.5}$$

where $\deg_q : \mathrm{GPin}(q) \to (\mathbf{Z}/2\mathbf{Z})_\mathrm{S}$ is the restriction of the degree on the subsheaf $\mathrm{C}(\mathrm{V}, q)_{\mathrm{lh}}$ of locally homogenous sections of the $\mathbf{Z}/2\mathbf{Z}$-graded Clifford algebra $\mathrm{C}(\mathrm{V}, q)$.

In other words, π_q is the twist of the conjugation action on V against the S-homomorphism $\mathrm{GPin}(q) \to \mu_2$ defined by $u \mapsto (-1)^{\deg_q(u)}$. Since quadratic forms are μ_2-invariant, the representation π_q lands in $\mathrm{O}(q)$ as well, so π_q extends to an action of the Clifford group $\mathrm{GPin}(q)$ on the entire $\mathbf{Z}/2\mathbf{Z}$-graded algebra $\mathrm{C}(\mathrm{V}, q)$. This action coincides with the ordinary conjugation action when restricted to the intersection of $\mathrm{GPin}(q)$ with the even subalgebra $\mathrm{C}_0(\mathrm{V}, q)$. (The representation $u : x \mapsto (-1)^{\deg_q(u)} uxu^{-1}$ on $\mathrm{C}(\mathrm{V}, q)$ by the group $\mathrm{C}(\mathrm{V}, q)^{\times}_{\mathrm{lh}}$ of locally homogeneous units is denoted as $u \mapsto i_u$ in [**Knus**, IV, 6.1]. We will not use it.)

By Remark C.2.3, $\mathrm{Z}_q \subset \mathrm{C}_0(\mathrm{V}, q)$. Thus, the $\mathrm{GPin}(q)$-action on $\mathrm{C}(\mathrm{V}, q)$ via π_q on V restricts to ordinary conjugation on Z_q. But $\mathrm{C}_0(\mathrm{V}, q)$ is the centralizer of Z_q in $\mathrm{C}(\mathrm{V}, q)$ (Proposition C.2.2), so $\mathrm{D}_q \circ \pi_q : \mathrm{GPin}(q) \to (\mathbf{Z}/2\mathbf{Z})_{\mathrm{S}}$ computes the restriction to $\mathrm{GPin}(q)$ of the $\mathbf{Z}/2\mathbf{Z}$-grading of $\mathrm{C}(\mathrm{V}, q)$ (for even n).

Proposition C.2.8. — *Assume n is even. The map $\pi_q : \mathrm{GPin}(q) \to \mathrm{O}(q)$ is a smooth surjection with kernel $\mathbf{G}_{\mathrm{m}}$, and $\mathrm{D}_q : \mathrm{O}(q) \to (\mathbf{Z}/2\mathbf{Z})_{\mathrm{S}}$ is a smooth surjection. In particular, the* S-*affine* S-*group* $\mathrm{GPin}(q)$ *is smooth.*

Proof. — Since $\mathrm{O}(q)$ is smooth (as n is even; see Theorem C.1.5), to prove D_q is a smooth surjection it suffices to check surjectivity on geometric fibers. We have already noted that $\mathrm{D}_q \circ \pi_q$ computes the degree on locally homogeneous sections of $\mathrm{GPin}(q)$, so for the assertion concerning D_q it suffices to check that on geometric fibers $\mathrm{GPin}(q)$ does not consist entirely of even elements. But as we saw above, the Zariski-dense open locus $\{q \neq 0\} \subset \mathrm{V}$ viewed in the odd part of $\mathrm{C}(\mathrm{V}, q)$ consists of units that lie in the Clifford group $\mathrm{GPin}(q)$.

Now we turn to the assertion that π_q is a smooth surjection with kernel $\mathbf{G}_{\mathrm{m}}$. The kernel of $\mathrm{D}_q \circ \pi_q$ consists of the points of $\mathrm{GPin}(q)$ in the even part of $\mathrm{C}(\mathrm{V}, q)$, and this even part acts on V through ordinary conjugation under π_q, so $\ker \pi_q$ is the intersection of $\mathrm{GPin}(q)$ with the part of $\mathrm{C}_0(\mathrm{V}, q)$ that centralizes V inside $\mathrm{C}(\mathrm{V}, q)$. But V generates $\mathrm{C}(\mathrm{V}, q)$ as an algebra, so the centralizer of V inside $\mathrm{C}(\mathrm{V}, q)$ is the center of $\mathrm{C}(\mathrm{V}, q)$. This center is $\mathscr{O}_{\mathrm{S}}$ since n is even, so $\ker \pi_q = \mathbf{G}_{\mathrm{m}}$. By smoothness of this kernel, to show π_q is a smooth surjection it suffices to prove π_q is surjective fppf-locally on the base.

By applying a preliminary base change on S and renaming the new base as S, it suffices to show that for any $g \in \mathrm{O}(q)(\mathrm{S})$, fppf-locally on S there exists a point u of $\mathrm{GPin}(q)$ satisfying $\pi_q(u) = g$. Define the sign $\varepsilon = (-1)^{\mathrm{D}_q(g)}$ that is Zariski-locally constant on S. Consider the automorphism $[\varepsilon g]$ of the $\mathscr{O}_{\mathrm{S}}$-algebra $\mathrm{C}(\mathrm{V}, q)$ induced by $\varepsilon g \in \mathrm{O}(q)(\mathrm{S})$. We claim that this automorphism is inner, fppf-locally (even Zariski-locally) on S. Since the quotient S-group $\mathrm{C}(\mathrm{V}, q)^{\times}/\mathbf{G}_{\mathrm{m}}$ is a subfunctor of the automorphism scheme of the algebra $\mathrm{C}(\mathrm{V}, q)$ via conjugation, it suffices to show that this quotient coincides with the automorphism scheme. The problem is fppf-local on S, so we can assume

$q = q_{2m}$, in which case $\mathrm{C}(\mathrm{V}, q)$ is a matrix algebra over $\mathscr{O}_{\mathrm{S}}$. But for any $\mathrm{N} \geqslant 1$, the natural map of finite type $\mathbf{Z}$-groups $\mathrm{PGL}_{\mathrm{N}} \to \underline{\mathrm{Aut}}(\mathrm{Mat}_{\mathrm{N}})$ is an isomorphism ("relative Skolem–Noether") by Exercise 5.5.5(i) applied on artinian points, so $\mathrm{C}(\mathrm{V}, q)^{\times}/\mathbf{G}_{\mathrm{m}} = \underline{\mathrm{Aut}}(\mathrm{C}(\mathrm{V}, q))$.

Since $\mathbf{G}_{\mathrm{m}}$-torsors for the fppf topology are automatically torsors for the Zariski topology, we may now arrange by Zariski localization on S that there exists $u \in \mathrm{C}(\mathrm{V}, q)^{\times}$ such that $[\varepsilon g](x) = uxu^{-1}$ for all $x \in \mathrm{C}(\mathrm{V}, q)$. Setting $x = v \in \mathrm{V}$ gives $[\varepsilon g](x) = \varepsilon g(v)$, so $u \in \mathrm{GPin}(q)$ and $\pi_q(u) = (-1)^{\deg_q(u)}\varepsilon g$. But $\deg_q = \mathrm{D}_q \circ \pi_q$ and $\mathrm{D}_q(-1) = 0$, so $\deg_q(u) = \mathrm{D}_q(g)$. Hence, $\varepsilon = (-1)^{\deg_q(u)}$, so $\pi_q(u) = g$. $\square$

Remark C.2.9. — Our preceding study of the structure of Clifford algebras provides representations of the S-group of units $\mathrm{C}_0(\mathrm{V}, q)^{\times}$ that underlies the half-spin and spin representations of spin groups (see Remark C.4.11).

Consider (V, q) with even rank $n \geqslant 2$, and suppose there are complementary isotropic subbundles W, W' of rank $n/2$. These are in perfect duality via B_q and hence Zariski-locally on S can be put into the form that was considered in the proof of Proposition C.2.2. In that proof we showed for $\mathrm{A}_+ := \oplus \wedge^{2j}(\mathrm{W})$ and $\mathrm{A}_- := \oplus \wedge^{2j+1}(\mathrm{W})$ that naturally $\mathrm{C}_0(\mathrm{V}, q) \simeq \mathscr{E}nd(\mathrm{A}_+) \times \mathscr{E}nd(\mathrm{A}_-)$. Hence, each of $\mathrm{A}_{\pm}$ are equipped with a natural representation of the S-group $\mathrm{C}_0(\mathrm{V}, q)^{\times}$. The same argument identifies $\mathrm{C}(\mathrm{V}, q)$ with $\mathscr{E}nd(\mathrm{A}_+ \oplus \mathrm{A}_-)$.

Suppose instead that V has odd rank $n \geqslant 1$ and that V admits a pair of isotropic subbundles W and W′ of rank $(n-1)/2$ in perfect duality under B_q. Non-degeneracy on fibers implies (via Zariski-local considerations over S) that $\mathrm{L} := \mathrm{W}^{\perp} \cap \mathrm{W}'^{\perp}$ is a line subbundle of V on whose local generators the values of q are units and for which $\mathrm{L} \oplus \mathrm{W} \oplus \mathrm{W}' = \mathrm{V}$. By Zariski-localizing to acquire a trivialization e_0 of L, the proof of Proposition C.2.4 shows that $x \mapsto e_0 x$ defines an isomorphism $\mathrm{C}(\mathrm{W} \oplus \mathrm{W}', \lambda q) \simeq \mathrm{C}_0(\mathrm{V}, q)$ for $\lambda := q(e_0) \in \mathscr{O}(\mathrm{S})^{\times}$. The preceding treatment of even rank identifies $\mathrm{C}(\mathrm{W} \oplus \mathrm{W}', q)$ with $\mathscr{E}nd(\mathrm{A})$ where the vector bundle A is the exterior algebra of W. Thus, this provides a representation of the S-group $\mathrm{C}_0(\mathrm{V}, q)^{\times}$ on A.

We can finally define special orthogonal groups, depending on the parity of n (and using (C.2.2) and Remark C.2.5).

Definition C.2.10. — Let $(\mathrm{V}, \mathrm{L}, q)$ be a non-degenerate line bundle-valued quadratic form with V of rank $n \geqslant 1$ over a scheme S. The *special orthogonal group* $\mathrm{SO}(q)$ is $\mathrm{SO}'(q) = \ker(\det|_{\mathrm{O}(q)})$ when n is odd and $\ker \mathrm{D}_q$ when n is even (with D_q as in (C.2.2)). For any $n \geqslant 1$, $\mathrm{SO}_n := \mathrm{SO}(q_n)$.

By definition, $\mathrm{SO}(q)$ is a closed subgroup of $\mathrm{O}(q)$, and it is also an open subscheme of $\mathrm{O}(q)$ when n is even. (In contrast, SO_{2m+1} is not an open subscheme of O_{2m+1} over $\mathbf{Z}$ because $\mathrm{O}(q) = \mathrm{SO}(q) \times \mu_2$ for odd n via the central $\mu_2 \subset \mathrm{GL}(\mathrm{V})$, and over $\operatorname{Spec} \mathbf{Z}$ the identity section of μ_2 is not an open immersion.)

The group $\mathrm{SO}'(q)$ is not of any real interest when n is even and 2 is not a unit on the base (and we will show that it coincides with $\mathrm{SO}(q)$ in all other cases). The only reason we defined $\mathrm{SO}'(q)$ for all n is because it is the first thing that comes to mind when trying to generalize the theory over $\mathbf{Z}[1/2]$ to work over $\mathbf{Z}$. We will see that SO'_{2m} is not $\mathbf{Z}_{(2)}$-flat. (Example: Consider $m = 1$ and $\mathrm{S} = \operatorname{Spec} \mathbf{Z}_{(2)}$. We have $\mathrm{O}_2 = \mathbf{G}_{\mathrm{m}} \rtimes (\mathbf{Z}/2\mathbf{Z})$ using inversion on $\mathbf{G}_{\mathrm{m}}$ for the semi-direct product, and $\mathrm{SO}_2 = \mathbf{G}_{\mathrm{m}}$, whereas SO'_2 is the reduced closed subscheme of O_2 obtained by removing the open non-identity component in the generic fiber.)

Here are the main properties we shall prove for the "good" groups associated to quadratic spaces (V, q).

Theorem C.2.11. — *The group* $\mathrm{SO}(q)$ *is smooth with connected fibers of dimension* $n(n-1)/2$. *In particular,* $\mathrm{SO}(cq) = \mathrm{SO}(q)$ *for* $c \in \mathscr{O}(\mathrm{S})^{\times}$. *The functorial center of* $\mathrm{SO}(q)$ *is trivial for odd* n *and is the central* $\mu_2 \subset \mathrm{O}(q)$ *for even* $n > 2$. *For* $n \geqslant 3$, *the functorial center of* $\mathrm{O}(q)$ *is the central* μ_2.

The smoothness and relative dimension aspects are immediate from Theorem C.1.5 since $\mathrm{SO}(q) = \mathrm{SO}'(q)$ for odd n and $\mathrm{SO}(q)$ is an open and closed subgroup of $\mathrm{O}(q)$ for even n (as the Dickson invariant $\mathrm{D}_q : \mathrm{O}(q) \to (\mathbf{Z}/2\mathbf{Z})_{\mathrm{S}}$ is a smooth surjection for such n, by Proposition C.2.8). The problem is to analyze the fibral connectedness (so $\mathrm{SO}(cq) = \mathrm{SO}(q)$) and the center.

Remark C.2.12. — Via the Dickson invariant D_q, for even n we have $\mathrm{O}(q)/\mathrm{SO}(q) = (\mathbf{Z}/2\mathbf{Z})_{\mathrm{S}}$, so Theorem C.2.11 implies that $\#\pi_0(\mathrm{O}(q)_s) = 2$ for all $s \in \mathrm{S}$. In contrast, for odd n multiplication against the central $\mu_2 \subset \mathrm{O}(q)$ defines an isomorphism $\mu_2 \times \mathrm{SO}(q) \simeq \mathrm{O}(q)$ because $\det : \mathrm{O}(q) \to \mathbf{G}_{\mathrm{m}}$ factors through μ_2 (due to the half-discriminant, as we saw in the proof of Theorem C.1.5). Thus, if n is odd then $\mathrm{O}(q)$ is fppf over S with $\mathrm{O}(q)/\mathrm{SO}(q) = \mu_2$ (so $\mathrm{O}(q)_s$ is connected and non-smooth when $\operatorname{char}(k(s)) = 2$).

We first analyze even rank, and then we analyze odd rank. In each case, fibers in residue characteristic 2 are treated by a special argument.

C.3. Connectedness and center. — Let $(\mathrm{V}, \mathrm{L}, q)$ be a non-degenerate line bundle-valued quadratic form with V of even rank $n \geqslant 2$. We first seek to understand the connectedness properties of the fibers of $\mathrm{SO}(q) \to \mathrm{S}$.

Proposition C.3.1. — *If* n *is even then* $\mathrm{SO}(q) \to \mathrm{S}$ *has connected fibers.*

Proof. — We proceed by induction on the even n, and we can assume that $\mathrm{S} = \operatorname{Spec} k$ for an algebraically closed field k. Without loss of generality, $q = q_n$. In view of the surjectivity of the Dickson invariant, it is equivalent to show that $\mathrm{O}(q)$ has exactly 2 (equivalently, at most 2) connected components. Since $q_2 = xy$, clearly $\mathrm{O}_2 = \mathbf{G}_{\mathrm{m}} \coprod \mathbf{G}_{\mathrm{m}}\iota$ for $\iota = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Now assume $n \geqslant 4$ and that the result is known for $n-2$.

Since q is not a square (as $n > 1$), it is straightforward to check that the smooth affine hypersurface $\mathrm{H} = \{q = 1\}$ is irreducible. The points in $\mathrm{H}(k)$ correspond to isometric embeddings $(k, x^2) \hookrightarrow (\mathrm{V}, q)$. By Witt's extension theorem [**Chev97**, I.4.1], if $\mathrm{Q} : \mathrm{W} \to \mathrm{K}$ is a quadratic form on a finite-dimensional vector space over a field K and B_{Q} is non-degenerate (so $\dim \mathrm{W}$ is even when $\mathrm{char}(\mathrm{K}) = 2$) then $\mathrm{O}(\mathrm{Q})(\mathrm{K})$ acts transitively on the set of isometric embeddings of a fixed (possibly degenerate) quadratic space into (W, Q). Hence, $\mathrm{O}(q)(k)$ acts transitively on $\mathrm{H}(k)$, so the orbit map $\mathrm{O}(q) \to \mathrm{H}$ through $e_1 + e_2 \in \mathrm{H}(k)$ is surjective with stabilizer $\mathrm{G}' := \mathrm{Stab}_{e_1+e_2}(\mathrm{O}(q))$ that preserves the orthogonal complement $\mathrm{V}' := (e_1+e_2)^{\perp} = k(e_1 - e_2) + \sum_{i \geqslant 3} ke_i$.

By Theorem C.1.5, $\dim \mathrm{O}(q) = \dim \mathrm{SO}'(q) = n(n-1)/2$, so

$$\dim \mathrm{G}' = \dim \mathrm{O}(q) - \dim \mathrm{H} = n(n-1)/2 - (n-1) = \dim \mathrm{O}_{n-1} = \dim \mathrm{Sp}_{n-2},$$

(for the final equality, which we will use when $\mathrm{char}(k) = 2$, note that n is even). Since H is connected, the identity component $\mathrm{O}(q)^0$ also acts transitively on H. Hence, $\#\pi_0(\mathrm{O}(q)) \leqslant \#\pi_0(\mathrm{G}')$, so it suffices to show that G' has at most 2 connected components.

The kernel of the action map $\mathrm{G}' \to \mathrm{GL}(\mathrm{V}')$ consists of $g' \in \mathrm{GL}(\mathrm{V})$ fixing $e_1 + e_2$ and $\{e_1 - e_2, e_3, \dots, e_n\}$ pointwise and preserving q. Thus, if $\mathrm{char}(k) \neq 2$ then $g' = 1$, and if $\mathrm{char}(k) = 2$ then preserving $(\sum_{i \geqslant 3} ke_i)^{\perp} = ke_1 + ke_2$ and stabilizing $e_1 + e_2$ and q implies $\ker(\mathrm{G}' \to \mathrm{GL}(\mathrm{V}')) = \mathbf{Z}/2\mathbf{Z}$, generated by the automorphism of V swapping e_1 and e_2 and fixing $e_3, \dots, e_n$. For this reason, we shall argue separately depending on whether or not $\mathrm{char}(k) = 2$.

Suppose $\mathrm{char}(k) \neq 2$, so $\{e_1 + e_2, e_1 - e_2\}$ is an orthogonal basis of $ke_1 + ke_2 = \mathrm{W}^{\perp}$ where $\mathrm{W} := \sum_{i \geqslant 3} ke_i$, with $q(e_1 \pm e_2) = \pm 1$. The restriction $q' := q|_{\mathrm{V}'}$ is given by the formula $q(c(e_1 - e_2) + w) = -c^2 + q(w)$ for $w \in \mathrm{W}$, and the inclusion $\mathrm{G}' \hookrightarrow \mathrm{GL}(\mathrm{V}')$ has image exactly $\mathrm{O}(q')$ since relative to the basis $\{e_1 + e_2, e_1 - e_2, e_3, \dots, e_n\}$ of V we identify (V, q) with the orthogonal direct sum of $(k(e_1 + e_2), x^2)$ and (V', q').

Since $\mathrm{B}_{q'}$ is non-degenerate (as $\mathrm{char}(k) \neq 2$), Witt's extension theorem is applicable to the hypersurface $\mathrm{H}' = \{q' = -1\}$ in V' that contains $e_1 - e_2$ and is irreducible (as $n - 1 > 1$), so G'^0 acts transitively on H'. Thus, $\#\pi_0(\mathrm{G}')$ is at most the number of connected components of the stabilizer of $e_1 - e_2$ in $\mathrm{G}' = \mathrm{O}(q')$. But since $\mathrm{char}(k) \neq 2$, this stabilizer is the orthogonal group of the q'-orthogonal space in V' to $e_1 - e_2$. This orthogonal group is identified with $\mathrm{O}(q'|_{\mathrm{W}}) = \mathrm{O}(q|_{\mathrm{W}})$. Since $q|_{\mathrm{W}} = q_{n-2}$, we conclude by induction when $\mathrm{char}(k) \neq 2$.

Now assume $\mathrm{char}(k) = 2$, so $e_1 - e_2$ spans the defect line ℓ of the non-degenerate quadratic space (V', q') of dimension $n-1$. The action of G' on V' preserves q', so its induced action on V' defines an action on V'/ℓ preserving the induced symmetric bilinear form $\overline{\mathrm{B}}_{q'}$ that is alternating since $\mathrm{char}(k) = 2$. Since the line ℓ is the defect space $\mathrm{V}'^{\perp}$, $\overline{\mathrm{B}}_{q'}$ is non-degenerate and hence symplectic. The induced homomorphism

$$h : \mathrm{O}(q') \to \mathrm{Sp}(\mathrm{V}'/\ell, \overline{\mathrm{B}}_{q'}) \simeq \mathrm{Sp}_{n-2}$$

to a symplectic group has kernel that is seen by calculation to be $\mu_2 \ltimes \alpha_2^{n-1}$, where μ_2 acts on the Frobenius kernel $\alpha_2^{n-1} \subset \mathbf{G}_{\mathrm{a}}^{n-1}$ by the usual diagonal scaling action. But $\dim \mathrm{O}_{n-1} = \dim \mathrm{Sp}_{n-2}$ and symplectic groups over fields are *connected* (proved by a fibration argument using lower-dimensional symplectic spaces, or see [**Bo91**, 23.3] for another proof), so h is surjective for dimension reasons.

We saw above that the restriction map $\mathrm{G}' \to \mathrm{GL}(\mathrm{V}')$ lands in $\mathrm{O}(q')$ and has kernel $\mathbf{Z}/2\mathbf{Z}$, so since the restriction map $\mathrm{O}(q') \to \mathrm{Sp}(\overline{\mathrm{B}}_{q'})$ in characteristic 2 has infinitesimal kernel, the composite homomorphism $f : \mathrm{G}' \to \mathrm{Sp}(\overline{\mathrm{B}}_{q'}) = \mathrm{Sp}_{n-2}$ has finite kernel with 2 geometric points. But $\dim \mathrm{G}' = \dim \mathrm{Sp}_{n-2}$ and symplectic groups are connected, so f must be surjective for dimension reasons and $\#\pi_0(\mathrm{G}') \leqslant 2$. $\square$

Corollary C.3.2. — *Assume n is even. Let $f_{\mathrm{S}} : (\mathbf{Z}/2\mathbf{Z})_{\mathrm{S}} \to \mu_2$ be the unique* S*-homomorphism satisfying $f(1) = -1$. The determinant map* $\det : \mathrm{O}(q) \to \mu_2$ *coincides with $f_{\mathrm{S}} \circ \mathrm{D}_q$. In particular,* det *kills* $\mathrm{SO}(q)$.

The inclusion $\mathrm{SO}(q) \subset \ker(\det) = \mathrm{SO}'(q)$ *is an equality over* $\mathrm{S}[1/2]$*, and* $\mathrm{SO}'(q) \hookrightarrow \mathrm{O}(q)$ *is an equality on fibers at points in characteristic* 2.

Proof. — By Lemma C.2.1, we may pass to the case $q = q_n$ over $\mathbf{Z}$. The equality $\det = f_{\mathbf{Z}} \circ \mathrm{D}_{q_n}$ can then be checked over $\mathbf{Q}$, where it is immediate from the connectedness of SO_n over $\mathbf{Q}$ and the nontriviality of det on O_n over $\mathbf{Q}$. Since $f_{\mathbf{Z}[1/2]}$ is an isomorphism, we get the equality of SO_n and SO'_n over $\mathbf{Z}[1/2]$. Over $\mathbf{F}_2$, the smooth group O_n must be killed by the determinant map into the infinitesimal μ_2, so $\mathrm{SO}'_n = \mathrm{O}_n$ over $\mathbf{F}_2$. $\square$

Remark C.3.3. — Assume n is even. Consider the element $g \in \mathrm{O}_n(\mathbf{Z})$ that swaps e_1 and e_2 while leaving the other e_i invariant. The section $\mathrm{D}_q(g)$ of the constant $\mathbf{Z}$-group $\mathbf{Z}/2\mathbf{Z}$ is equal to 1 mod 2 since it suffices to check this on a single geometric fiber (and at any fiber away from characteristic 2 it is clear, as SO_n coincides with SO'_n over $\mathbf{Z}[1/2]$). Thus, the Dickson invariant $\mathrm{D}_q : \mathrm{O}(q) \to (\mathbf{Z}/2\mathbf{Z})_{\mathrm{S}}$ splits as a semi-direct product when $q = q_n$.

The induced map $\mathrm{H}^1(\mathrm{S}_{\text{ét}}, \mathrm{O}_n) \to \mathrm{H}^1(\mathrm{S}_{\text{ét}}, \mathbf{Z}/2\mathbf{Z})$ assigns to every non-degenerate (V, q) of rank n over S (taken up to isomorphism) a degree-2 finite étale cover of S. Consideration of étale Čech 1-cocycles and the definition of

D_q shows that this double cover corresponds to the quadratic étale center Z_q of $\mathrm{C}_0(\mathrm{V}, \mathrm{L}, q)$. If S is a $\mathbf{Z}[1/2]$-scheme (so $(\mathbf{Z}/2\mathbf{Z})_{\mathrm{S}} = \mu_2$) and $\mathrm{Pic}(\mathrm{S}) = 1$ then it recovers the class in $\mathrm{H}^1(\mathrm{S}, \mu_2) = \mathbf{G}_{\mathrm{m}}(\mathrm{S})/\mathbf{G}_{\mathrm{m}}(\mathrm{S})^2$ of a unit c such that the induced quadratic form on the top exterior power $\det \mathrm{V}$ of V is locally cx^2. (Concretely, if V is globally free then this is represented by $\det[\mathrm{B}_q]$, where $[\mathrm{B}_q]$ is the matrix of B_q relative to a basis for V; see [**Knus**, IV, 4.1.1, 5.3.2] for affine S.) If S is an $\mathbf{F}_2$-scheme then it recovers the *pseudo-discriminant*, also called the *Arf invariant* when $\mathrm{S} = \mathrm{Spec}\, k$ for a field $k/\mathbf{F}_2$; see [**Knus**, IV, 4.7] for an explicit formula when S is affine and V is globally free.

Remark C.3.4. — For even n, the $\mathbf{Z}$-group SO'_n turns out to be reduced but not $\mathbf{Z}$-flat (due to problems at the prime 2). The failure of flatness is a consequence of the more precise observation that the open and closed subscheme $\mathrm{SO}_n \hookrightarrow \mathrm{SO}'_n$ has complement equal to the non-identity component of $(\mathrm{O}_n)_{\mathbf{F}_2}$. To prove these assertions (which we will never use), first note that Corollary C.3.2 gives the result over $\mathbf{Z}[1/2]$, as well as the topological description of the $\mathbf{F}_2$-fiber. It remains to show that SO'_n is reduced.

It is harmless to pass to the quotient by the *smooth* normal subgroup SO_n (since reducedness asecends through smooth surjections), so under the identification of $\mathrm{O}_n/\mathrm{SO}_n$ with the constant group $\mathbf{Z}/2\mathbf{Z}$ via the Dickson invariant we see that $\mathrm{G} := \mathrm{SO}'_n/\mathrm{SO}_n$ is identified with the kernel of the unique homomorphism of $\mathbf{Z}$-groups $f : \mathbf{Z}/2\mathbf{Z} \to \mu_2$ sending 1 to -1. As a map from the $\mathbf{Z}$-group $(\mathbf{Z}/2\mathbf{Z})_{\mathbf{Z}} = \mathrm{Spec}\, \mathbf{Z}[t]/(t^2 - t)$ to $\mu_2 = \mathrm{Spec}\, \mathbf{Z}[\zeta]/(\zeta^2 - 1)$, it is given by $\zeta - 1 \mapsto -2t$ on coordinate rings, so the kernel G is $\mathrm{Spec}\, \mathbf{Z}[t]/(-2t, t^2 - t)$. This is the disjoint union of the identity section and a single $\mathbf{F}_2$-point in the $\mathbf{F}_2$-fiber.

Assume n is odd, so $\mathrm{SO}(q) = \mathrm{SO}'(q)$. This is smooth by Theorem C.1.5. As in Remark C.2.12, by consideration of the half-discriminant, the morphism $\det : \mathrm{O}(q) \to \mathbf{G}_{\mathrm{m}}$ factors through μ_2. Thus, by the oddness of n, the determinant splits off the central $\mu_2 \subset \mathrm{O}(q)$ as a direct factor: $\mathrm{O}(q) = \mu_2 \times \mathrm{SO}(q)$. Hence, $\mathrm{SO}(q) \to \mathrm{S}$ has fibers of dimension $n(n-1)/2$ by Theorem C.1.5. Fibral connectedness will be proved by induction on the odd n:

Proposition C.3.5. — *Let* (V, q) *be a quadratic space over a field* k*, with* $n = \dim \mathrm{V}$ *odd. The group* $\mathrm{SO}(q)$ *is connected.*

Proof. — We may assume k is algebraically closed and $q = q_n$. The case $n = 1$ is trivial, so we assume $n \geqslant 3$. We treat characteristic 2 separately from other characteristics, due to the appearance of the defect space $\mathrm{V}^{\perp} = ke_0$ in characteristic 2.

First assume $\mathrm{char}(k) \neq 2$, so the symmetric bilinear form B_q is non-degenerate and $\mu_2 = (\mathbf{Z}/2\mathbf{Z})_k$. Since $\mu_2 \times \mathrm{SO}(q) = \mathrm{O}(q)$, $\mathrm{O}(q)$ has at least 2 connected components. It has exactly 2 such components if and only if

$\mathrm{SO}(q)$ is connected. Since $n > 1$, the hypersurface $\mathrm{H} = \{q = 1\}$ is irreducible, and exactly as in the proof of Proposition C.3.1 we may apply Witt's extension theorem (valid for odd n since $\mathrm{char}(k) \neq 2$) to deduce that the action of $\mathrm{O}(q)$ on H is transitive. The orthogonal complement V' of e_0 is spanned by $\{e_1, \ldots, e_{n-1}\}$ since $\mathrm{char}(k) \neq 2$, and it is preserved by $\mathrm{Stab}_{e_0}(\mathrm{O}(q))$. It is straightforward to check that the action of this stabilizer on V' defines an isomorphism onto $\mathrm{O}(q|_{\mathrm{V}'}) \simeq \mathrm{O}_{n-1}$. This group has 2 connected components by Proposition C.3.1, so by irreducibility of H it follows that $\mathrm{O}(q)$ has at most 2 connected components (hence exactly 2 such components). This settles the case $\mathrm{char}(k) \neq 2$.

Now assume $\mathrm{char}(k) = 2$. Since $\mathrm{O}(q) = \mu_2 \times \mathrm{SO}(q)$, the connectedness of $\mathrm{SO}(q)$ is equivalent to the connectedness of $\mathrm{O}(q)$. The non-vanishing defect space obstructs induction using the action on H, so instead we use the quotient $\overline{\mathrm{V}} := \mathrm{V}/\mathrm{V}^{\perp} = \mathrm{V}/ke_0$ by the defect line $\mathrm{V}^{\perp}$ rather than use a hyperplane as above. As in the proof of Proposition C.3.1, B_q on V induces a symplectic form $\overline{\mathrm{B}}_q$ on $\overline{\mathrm{V}}$, yielding a natural map $\mathrm{O}(q) \to \mathrm{Sp}(\overline{\mathrm{V}}, \overline{\mathrm{B}}_q) \simeq \mathrm{Sp}_{n-1}$ that is surjective with infinitesimal kernel, so the connectedness of symplectic groups implies the connectedness of $\mathrm{O}(q)$. $\square$

Remark C.3.6. — Assume n is odd. As in the proof of Proposition C.3.1, if $\mathrm{char}(k) = 2$ then there is a surjective homomorphism $h : \mathrm{O}(q) \to \mathrm{Sp}(\overline{\mathrm{V}}, \overline{\mathrm{B}}_q)$ with $(\ker h)_{\overline{k}} = \mu_2 \ltimes \alpha_2^{n-1}$. This kernel meets the kernel $\mathrm{SO}(q)$ of the determinant map on $\mathrm{O}(q)$ in α_2^{n-1} over $\overline{k}$, so by smoothness of $\mathrm{SO}(q)$ we obtain a purely inseparable isogeny $\mathrm{SO}(q) \to \mathrm{Sp}(\overline{\mathrm{V}}, \overline{\mathrm{B}}_q)$ with kernel that is a form of α_2^{n-1} (and hence is isomorphic to α_2^{n-1}, as α_p^{N} has automorphism scheme GL_{N}). This "unipotent isogeny" is a source of many phenomena related to algebraic groups in characteristic 2 (e.g., see [**CP**, A.3]).

Special orthogonal groups in $2m+1$ variables are type B_m (see Proposition C.3.10) and symplectic groups in $2m$ variables are type C_m; these types are distinct for $m \geqslant 3$ (and they coincide for $m = 1, 2$; see Example C.6.2 and Example C.6.5 respectively.) In characteristics distinct from 2 and 3 there are no isogenies between (absolutely simple) connected semisimple groups of different types. In characteristic 2 we have just built "exceptional" isogenies between B_m and C_m for all $m \geqslant 3$. See [**SGA3**, XXI, 7.5] for further details.

Remark C.3.7. — For even n, by definition $\mathrm{SO}(q)$ is the kernel of the action of $\mathrm{O}(q)$ on the degree-2 finite étale center Z_q of $\mathrm{C}_0(\mathrm{V}, \mathrm{L}, q)$. For odd n and $\mathscr{O}_{\mathrm{S}}$-valued q there is a similar description of $\mathrm{SO}(q)$: it is the kernel of the action of $\mathrm{O}(q)$ on the center Z_q of the entire Clifford algebra $\mathrm{C}(\mathrm{V}, q)$. (Triviality of the action on an appropriate commutative rank-2 subalgebra of the Clifford algebra is the unified definition of $\mathrm{SO}(q)$ for all n in [**Knus**, Ch. IV, § 5].)

Since $\mathrm{O}(q) = \mu_2 \times \mathrm{SO}(q)$ for odd n, and μ_2 acts by ordinary scaling on the line bundle $\mathrm{Z}_q^1 = \mathrm{Z}_q \cap \mathrm{C}_1(\mathrm{V}, q)$ (immediate from the explicit description of Z_q in the proof of Proposition C.2.4 for $q = q_n$, to which the general case may be reduced), to justify this description of $\mathrm{SO}(q)$ for odd n it suffices to check triviality of the $\mathrm{SO}(q)$-action on Z_q. By working fppf-locally on S we may assume $q = q_n$, so it is enough to treat q_n over $\mathbf{Z}$. But SO_n is $\mathbf{Z}$-flat, so to verify triviality of the $\mathbf{Z}$-homomorphism from SO_n into the automorphism scheme of the rank-2 algebra Z_q it suffices to to work over $\mathbf{Z}[1/2]$. Now Z_q is a quadratic étale algebra, so its automorphism scheme is $\mathbf{Z}/2\mathbf{Z}$, which admits no nontrivial homomorphism from a smooth group scheme with connected fibers.

The remaining task for $\mathrm{SO}(q)$ and $\mathrm{O}(q)$ is to determine the functorial center if $n \geqslant 3$ (as the case $n \leqslant 2$ is easy to analyze directly). For odd $n \geqslant 3$, the central μ_2 in $\mathrm{O}(q)$ has trivial intersection with $\mathrm{SO}'(q)$, and hence with $\mathrm{SO}(q)$. If $n > 2$ is even then the central μ_2 is contained in $\mathrm{SO}'(q)$, and we claim that it also lies in $\mathrm{SO}(q)$. In other words, for even n we claim that the Dickson invariant $\mathrm{D}_q : \mathrm{O}(q) \to (\mathbf{Z}/2\mathbf{Z})_\mathrm{S}$ kills the central μ_2. It suffices to treat the case of $q = q_n$ over $\mathbf{Z}$, in which case we just need to show that the *only* homomorphism of $\mathbf{Z}$-groups $\mu_2 \to \mathbf{Z}/2\mathbf{Z}$ is the trivial one. By $\mathbf{Z}$-flatness, to prove such triviality it suffices to check after localization to $\mathbf{Z}_{(2)}$. But over the local base $\operatorname{Spec} \mathbf{Z}_{(2)}$ the scheme μ_2 is connected and thus it must be killed by a homomorphism into a constant group.

Proposition C.3.8. — *Assume $n \geqslant 3$. The functorial center of* $\mathrm{SO}(q)$ *is represented by μ_2 in the central $\mathbf{G}_\mathrm{m} \subset \mathrm{GL}(\mathrm{V})$ when n is even, and it is trivial when n is odd.*

Proof. — By Lemma C.2.1, it suffices to treat q_n over $\mathrm{S} = \operatorname{Spec} k$ for any ring k. We will use a method similar to the treatment of $\mathrm{Z}_{\mathrm{Sp}_{2n}}$ in Exercise 2.4.6: exhibit a specific torus T that we show to be its own centralizer in $\mathrm{G} := \mathrm{SO}'(q)$ (so T is its own centralizer in $\mathrm{SO}(q)$) and then we will look for the center inside this T. We will also show that the functorial center of $\mathrm{SO}(q)$ coincides with that of G.

Suppose $n = 2m$, so relative to some ordered basis $\{e_1, e_1', \ldots, e_m, e_m'\}$ we have $q = \sum_{i=1}^m x_i x_i'$. In this case we identify GL_1^m with a k-subgroup T of $\mathrm{SO}'(q)$ via

$$j : (t_1, \ldots, t_m) \to (t_1, 1/t_1, \ldots, t_m, 1/t_m).$$

The action by T on $k^n = k^{2m}$ has each standard basis line as a weight space for a collection of $2m$ fiberwise distinct characters over $\operatorname{Spec} k$. Hence, $\mathrm{Z}_{\mathrm{GL}_n}(\mathrm{T})$ is the diagonal torus in GL_n, so clearly $\mathrm{Z}_{\mathrm{O}(q_n)}(\mathrm{T}) = \mathrm{T}$ and hence $\mathrm{Z}_\mathrm{G}(\mathrm{T}) = \mathrm{T}$.

Next, assume $n = 2m+1$ for $m \geqslant 1$. Pick a basis $\{e_0, e_1, e_1', \dots, e_m, e_m'\}$ relative to which

$$q = x_0^2 + \sum_{i=1}^{m} x_i x_i'.$$

If we define T in the same way (using the span of $e_1, e_1', \dots, e_m, e_m'$) then the same analysis gives the same result: T is its own scheme-theoretic centralizer in $\mathrm{SO}'(q)$. The point is that there is no difficulty created by e_0 because we are requiring the determinant to be 1. (If we try the same argument with $\mathrm{O}(q)$ then the centralizer of T is $\mu_2 \times \mathrm{T}$.)

We are now in position to identify the center of $\mathrm{SO}'(q)$ for general $n \geqslant 3$. First we assume $n \geqslant 4$ (i.e., $m \geqslant 2$). In terms of the ordered bases as above, consider the automorphism g_i obtained by swapping the ordered pairs (e_i, e_i') and (e_1', e_1) for $1 < i \leqslant m$. (Such i exist precisely because $m \geqslant 2$.) These automorphisms g_i lie in $\mathrm{SO}'(q)$ since the determinant is $(-1) \cdot (-1) = 1$, and a point of T centralizes g_i if and only if $t_1 = t_i$. Letting i vary, we conclude that the center of $\mathrm{SO}'(q)$ is contained in the "scalar" subgroup $\mathbf{G}_{\mathrm{m}} \hookrightarrow \mathrm{T}$ given by $t_1 = \cdots = t_m$. This obviously holds when $n = 3$ as well.

Letting λ denote the common value of the t_j, to constrain it further we consider more points of $\mathrm{SO}'(q)$ that it must centralize. First assume $m \geqslant 2$. Consider the automorphism f of V which acts on the plane $ke_i \bigoplus ke_i'$ by the matrix $w = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ for exactly *two* values $i_0, i_1 \in \{1, \dots, m\}$ (and leaves all other basis vectors invariant), so $\det f = 1$. Clearly f preserves q, so f lies in $\mathrm{SO}'(q)(k)$. But f-conjugation of $t \in \mathrm{T}$ viewed in $\mathrm{SO}'(q)$ (or $\mathrm{GL}(\mathrm{V})$) swaps the entries t_i and $1/t_i$ for $i \in \{i_0, i_1\}$. Thus, the centralizing property forces $\lambda \in \mu_2$, so if $n \geqslant 4$ is even then $\mathrm{Z}_{\mathrm{SO}'(q)}$ is contained in the central μ_2 in $\mathrm{GL}(\mathrm{V})$. This inclusion is an equality for even $n \geqslant 4$, since the central $\mu_2 \subset \mathrm{GL}(\mathrm{V})$ is contained in $\mathrm{SO}'(q)$ for even n. If instead $n \geqslant 4$ is odd then $\mathrm{SO}'(q) = \mathrm{SO}(q)$ by definition and we have shown that its center viewed inside GL_n lies in the subgroup $\mu_2 \hookrightarrow \mathbf{G}_{\mathrm{m}}^n$ defined by the inclusion $\zeta \mapsto (1, \zeta, \zeta, \dots \zeta)$. The automorphism

$$(x_0, x_1, x_2, x_3, \dots, x_n) \mapsto (x_0 + x_1, x_1, -2x_0 - x_1 + x_2, x_3, \dots, x_n)$$

arising from $\mathrm{SO}(x_0^2 + x_1 x_2)$ lies in $\mathrm{SO}(q)$ and centralizing this forces $\zeta = 1$, so the center is trivial. This completes our analysis for odd $n \geqslant 5$.

Next, we prove that $\mathrm{Z}_{\mathrm{SO}_3} = 1$. The action of PGL_2 on $\mathfrak{sl}_2$ via conjugation defines an isomorphism $\mathrm{PGL}_2 \simeq \mathrm{SO}_3$; see the self-contained calculations in Example C.6.2. By Exercise 2.4.6(ii) the scheme-theoretic center of PGL_r is trivial for any $r \geqslant 2$ (and for PGL_2 it can be verified by direct calculation), so SO_3 has trivial center.

We have settled the case of odd $n \geqslant 3$, and for even $n \geqslant 4$ we have proved that $\mathrm{SO}'(q)$ has functorial center μ_2 that also lies in $\mathrm{SO}(q)$. It remains to show, assuming $n \geqslant 4$ is even, that the functorial center of $\mathrm{SO}(q)$ is no larger than this μ_2. We may and do assume $q = q_n$. The torus T constructed above in SO'_n lies in the open and closed subgroup SO_n for topological reasons, and $\mathrm{Z}_{\mathrm{SO}_n}(\mathrm{T}) = \mathrm{T}$ since T has been shown to be its own centralizer in SO'_n. Thus, it suffices to show that the central μ_2 is the kernel of the adjoint action of T on $\mathrm{Lie}(\mathrm{SO}_n) = \mathrm{Lie}(\mathrm{O}_n)$. The determination of the weight space decomposition for T acting on $\mathrm{Lie}(\mathrm{O}_n)$ for even n is classical, so the kernel is seen to be the diagonal μ_2 for such n. □

Corollary C.3.9. — *For $n \geqslant 2$, the functorial center of $\mathrm{O}(q)$ is represented by the central μ_2.*

Proof. — The case $n = 2$ is handled directly, so assume $n \geqslant 3$. If n is odd then the identification $\mathrm{O}(q) = \mu_2 \times \mathrm{SO}(q)$ yields the result since $\mathrm{SO}(q)$ has trivial functorial center for such n. Now suppose that n is even. In this case the open and closed subgroup $\mathrm{SO}(q)$ contains the central μ_2 as its functorial center. To prove that μ_2 is the functorial center of $\mathrm{O}(q)$ we again pass to the case $q = q_n$. It suffices to check that the diagonal torus T in SO_n is its own centralizer in O_n, and this was shown in the proof of Proposition C.3.8. □

Proposition C.3.10. — *For $m \geqslant 1$, the smooth affine **Z**-group SO_{2m+1} is adjoint semisimple and it contains a split maximal torus $\mathrm{T} \subset \mathrm{SO}_{2m+1}$ defined by*

$$(t_1, \ldots, t_m) \mapsto \mathrm{diag}(1, t_1, 1/t_1, \ldots, t_m, 1/t_m).$$

The root system $\Phi(\mathrm{SO}_{2m+1}, \mathrm{T})$ is B_m.

*For $m \geqslant 2$, the smooth affine **Z**-group SO_{2m} is semisimple and it contains a split maximal torus $\mathrm{T} \subset \mathrm{SO}_{2m}$ defined by*

$$(t_1, \ldots, t_m) \mapsto \mathrm{diag}(t_1, 1/t_1, \ldots, t_m, 1/t_m).$$

The diagonal $\mu_2 \subset \mathrm{T}$ is the schematic center of SO_{2m}, and $\Phi(\mathrm{SO}_{2m}, \mathrm{T})$ is D_m.

We use the convention that $\mathrm{B}_1 = \mathrm{A}_1$ and $\mathrm{D}_2 = \mathrm{A}_1 \times \mathrm{A}_1$.

Proof. — Let $n = 2m + 1$ and $2m$ in these respective cases, so $n \geqslant 3$ and we are studying $\mathrm{SO}_n \subset \mathrm{GL}_n$. The smoothness of SO_n follows from Theorem C.1.5 (as we have noted immediately after the statement of Theorem C.2.11), and the fibral connectedness is Proposition C.3.1 for even n and Proposition C.3.5 for odd n. The structure of the center is given by Proposition C.3.8. Clearly T is a split torus in SO_n, and its maximality on geometric fibers was shown in the proof of Proposition C.3.8.

The remaining problem is to show that over an algebraically closed field k of any characteristic (including characteristic 2), the smooth connected affine group SO_n is semisimple with the asserted type for its root system. The cases $n = 3, 4$ can be handled by direct arguments (given in a self-contained manner in Examples C.6.2 and C.6.3), so we may restrict attention to $n \geqslant 5$ (i.e., $m \geqslant 2$ for odd n and $m \geqslant 3$ for even n). Since smoothness and connectedness are known, as is the dimension, it is straightforward to directly compute the weight space decomposition for T on $\mathrm{Lie}(\mathrm{SO}_n)$ and so to verify reductivity by the general technique in Exercise 1.6.16(i), using constructions given in Exercise 1.6.15. This method also shows that $\mathrm{X(T)}_{\mathbf{Q}}$ is spanned by the roots, so SO_n is semisimple, and an inspection of the roots shows that the root system is of the desired type (depending on the parity of $n \geqslant 5$). These calculations are left to the reader. $\square$

As an application of the basic properties of orthogonal and special orthogonal group schemes, we now prepare to prove an interesting fact in the *global* theory of quadratic forms. Observe that for any scheme S, the group Pic(S) acts on the set of isomorphism classes of non-degenerate line bundle-valued quadratic forms $(\mathrm{V}, \mathrm{L}, q)$ with V of a fixed rank $n \geqslant 1$: the class of a line bundle L' on S carries the isomorphism class of $q : \mathrm{V} \to \mathrm{L}$ to the isomorphism class of $q_{\mathrm{L}'} : \mathrm{V} \otimes \mathrm{L}' \to \mathrm{L} \otimes \mathrm{L}'^{\otimes 2}$ (defined by $v \otimes \ell' \mapsto q(v) \otimes \ell'^{\otimes 2}$). We seek to understand the orbits under this action, called *projective similarity classes*. In the special case $\mathrm{Pic(S)} = 1$ we can use the language of quadratic spaces (V, q) and this becomes the consideration of similarity classes: (V', q') is in the same *similarity class* as (V, q) if there is a linear isomorphism $f : \mathrm{V}' \simeq \mathrm{V}$ such that $q \circ f = uq'$ for a unit u on S. (Projective similarity classes can be interpreted via algebras with involution; see [**Au**, 3.1] for literature references.)

We wish to classify projective similarity classes in terms of another invariant. If $\mathrm{Pic(S)} = 1$ this amounts to classifying similarity classes of quadratic spaces over S, and over a field k it is an interesting arithmetic problem (different from classifying isomorphism classes of quadratic spaces over k when $k^\times$ contains non-squares). Under the evident identification $\mathrm{GL}(\mathrm{V} \otimes \mathrm{L}') = \mathrm{GL(V)}$ it is easy to check that $\mathrm{SO}(q_{\mathrm{L}'}) = \mathrm{SO}(q)$, so the isomorphism class of $\mathrm{SO}(q)$ is the same across all members of a projective similarity class. Our aim (achieved in Proposition C.3.14) is to show that if $n \neq 2$ then the isomorphism class of $\mathrm{SO}(q)$ *determines* the projective similarity class of $(\mathrm{V}, \mathrm{L}, q)$; the case $n = 2$ exhibits more subtle behavior, as we shall see.

To analyze the general problem, it is convenient to introduce an appropriate group: the *orthogonal similitude group*

$$\mathrm{GO}(q) \subset \mathrm{GL(V)}$$

is the closed subgroup of linear automorphisms of V preserving $q : \mathrm{V} \to \mathrm{L}$ up to an automorphism of L. (The automorphism of L is uniquely determined since q is fiberwise non-zero.) The group $\mathrm{GO}(q)$ contains $\mathrm{O}(q)$ and the central $\mathbf{G}_{\mathrm{m}}$, with $\mathrm{GO}(q)/\mathbf{G}_{\mathrm{m}} = \mathrm{O}(q)/\mu_2$ as fppf quotient sheaves. This S-affine quotient group is denoted $\mathrm{PGO}(q)$ and is called the *projective similitude group*.

Inside $\mathrm{GL}(\mathrm{V})$, the intersection of $\mathbf{G}_{\mathrm{m}}$ and $\mathrm{SO}(q)$ is trivial if n is odd and is μ_2 if n is even (by Theorem C.2.11 if $n \neq 2$ and by inspection if $n = 2$), so the fppf subgroup sheaf of $\mathrm{GO}(q)$ generated by $\mathrm{SO}(q)$ and $\mathbf{G}_{\mathrm{m}}$ is identified with $\mathbf{G}_{\mathrm{m}} \times \mathrm{SO}(q)$ for odd n and with $(\mathbf{G}_{\mathrm{m}} \times \mathrm{SO}(q))/\mu_2$ for even n; this smooth S-subgroup of $\mathrm{GO}(q)$ is denoted $\mathrm{GSO}(q)$. By Theorem C.2.11, for all $n \geqslant 3$ the group $\mathrm{GSO}(q)$ is smooth with center $\mathbf{G}_{\mathrm{m}}$ and the quotient $\mathrm{GSO}(q)/\mathbf{G}_{\mathrm{m}}$ is the adjoint semisimple $\mathrm{SO}(q)/\mathrm{Z}_{\mathrm{SO}(q)}$ (so $\mathrm{GSO}(q)/\mathbf{G}_{\mathrm{m}}$ is sometimes denoted $\mathrm{PGSO}(q)$).

For odd n the equality $\mathrm{SO}(q) \times \mu_2 = \mathrm{O}(q)$ implies $\mathrm{GSO}(q) = \mathrm{GO}(q)$. For even n, $\mathrm{GSO}(q)$ is an open and closed normal S-subgroup of $\mathrm{GO}(q)$ satisfying $\mathrm{GO}(q)/\mathrm{GSO}(q) = \mathrm{O}(q)/\mathrm{SO}(q) = (\mathbf{Z}/2\mathbf{Z})_{\mathrm{S}}$. This follows immediately from the description of $\mathrm{PGO}(q)$ as $\mathrm{O}(q)/\mu_2$ and the observation (for even n) that the central μ_2 in $\mathrm{O}(q)$ lies in $\mathrm{SO}(q)$. Put another way, for even n, the isomorphism of fppf group sheaves $\mathrm{GO}(q)/\mathrm{GSO}(q) \simeq (\mathbf{Z}/2\mathbf{Z})_{\mathrm{S}}$ defines a quotient map of S-group schemes

$$\mathrm{GD}_q : \mathrm{GO}(q) \to (\mathbf{Z}/2\mathbf{Z})_{\mathrm{S}} \tag{C.3.1}$$

whose open and closed kernel is $\mathrm{GSO}(q)$.

Remark C.3.11. — For even n, the quotient map GD_q in (C.3.1) clearly extends $\mathrm{D}_q : \mathrm{O}(q) \to (\mathbf{Z}/2\mathbf{Z})_{\mathrm{S}}$. We now give a "Clifford" construction of this map on $\mathrm{GO}(q)$. The main point is that the $\mathrm{O}(q)$-action on the algebra $\mathrm{C}_0(\mathrm{V}, \mathrm{L}, q)$ from Remark C.2.5 (which exists even in the absence of $\mathrm{C}(\mathrm{V}, \mathrm{L}, q)$ for line bundle-valued q) naturally extends to a $\mathrm{GO}(q)$-action, and similarly on the left $\mathrm{C}_0(\mathrm{V}, \mathrm{L}, q)$-module $\mathrm{C}_1(\mathrm{V}, \mathrm{L}, q)$ (extending the natural $\mathrm{GO}(q)$-action on the subbundle $\mathrm{V} \subset \mathrm{C}_1(\mathrm{V}, \mathrm{L}, q)$).

To build this $\mathrm{GO}(q)$-action, first observe that if $g \in \mathrm{GO}(q)$ has action on V that intertwines with the action on L by a unit u then g induces an isomorphism $(\mathrm{V}, q, \mathrm{L}) \simeq (\mathrm{V}, uq, \mathrm{L})$ that is the *identity* on L. Thus, if L is trivial then g induces an isomorphism $\mathrm{C}(\mathrm{V}, q) \simeq \mathrm{C}(\mathrm{V}, uq)$, so if moreover $u = a^2$ then composing this isomorphism with the isomorphism $\mathrm{C}(\mathrm{V}, uq) \simeq \mathrm{C}(\mathrm{V}, q)$ defined by $v \mapsto av$ yields a composite isomorphism $[g] : \mathrm{C}_0(\mathrm{V}, q) \simeq \mathrm{C}_0(\mathrm{V}, q)$ that is independent of a and hence *multiplicative* in such g. Feeding this into the descent procedure used in Remark C.2.5 (to handle the possibility that u is not a square on S and L may not be globally trivial), we thereby obtain a natural $\mathrm{GO}(q)$-action on $\mathrm{C}_0(\mathrm{V}, \mathrm{L}, q)$ without any triviality requirement on L. Similarly we get a compatible $\mathrm{GO}(q)$-action on the $\mathrm{C}_0(\mathrm{V}, \mathrm{L}, q)$-module $\mathrm{C}_1(\mathrm{V}, \mathrm{L}, q)$ that restricts to the usual action on the subbundle V.

These actions visibly extend the $\mathrm{O}(q)$-actions, so in particular we get an action of $\mathrm{GO}(q)$ on the center Z_q of $\mathrm{C}_0(\mathrm{V}, \mathrm{L}, q)$ that extends the $\mathrm{O}(q)$-action on Z_q. Thus, we obtain an S-homomorphism $\mathrm{GO}(q) \to \mathrm{Aut}_{\mathrm{Z}_q/\mathrm{S}} = (\mathbf{Z}/2\mathbf{Z})_{\mathrm{S}}$ extending the Dickson invariant D_q on $\mathrm{O}(q)$. This recovers (C.3.1) because

$\mathrm{GSO}(q)$ has no nontrivial S-homomorphism to $(\mathbf{Z}/2\mathbf{Z})_{\mathrm{S}}$ for fibral connectedness reasons and $(\mathbf{Z}/2\mathbf{Z})_{\mathrm{S}}$ has no nontrivial automorphism. A consequence of the identification $\mathrm{GSO}(q) = \ker \mathrm{GD}_q$ is that $\mathrm{GSO}(q)$ acts Z_q-linearly on $\mathrm{C}_1(\mathrm{V}, \mathrm{L}, q)$.

Lemma C.3.12. — *The* S*-affine* S*-groups* $\mathrm{GO}(q)$ *and* $\mathrm{PGO}(q)$ *are smooth, with geometric fibers that are connected when n is odd and have two connected components when n is even. In general, if $n \geqslant 1$ is odd then* $\mathrm{PGO}(q)$ *coincides with the adjoint group* $\mathrm{SO}(q)$ *and if $n \neq 2$ is even then* $\mathrm{PGO}(q)$ *is an extension of* $(\mathbf{Z}/2\mathbf{Z})_{\mathrm{S}}$ *by the adjoint quotient* $\mathrm{SO}(q)/\mu_2 = \mathrm{PGSO}(q)$.

Proof. — Since $\mathrm{GO}(q)$ is an fppf $\mathbf{G}_{\mathrm{m}}$-torsor over $\mathrm{PGO}(q)$, it suffices to study $\mathrm{PGO}(q) = \mathrm{O}(q)/\mu_2$. If n is odd then this is $\mathrm{SO}(q)$ since the determinant on $\mathrm{O}(q)$ splits off the central μ_2 as a direct factor for such n, and if n is even then $\mathrm{O}(q)/\mu_2$ is an extension of $(\mathbf{Z}/2\mathbf{Z})_{\mathrm{S}}$ by $\mathrm{SO}(q)/\mu_2$. Theorem C.2.11 provides the required properties of $\mathrm{SO}(q)$ for $n \geqslant 3$ to complete the proof. $\square$

Our interest in the orthogonal similitude group is twofold. First, for the smooth group $\mathrm{GO}_n = \mathrm{GO}(q_n)$, the étale cohomology set $\mathrm{H}^1(\mathrm{S}_{\text{ét}}, \mathrm{GO}_n)$ is the set of isomorphism classes of rank-n non-degenerate line bundle-valued quadratic forms over S. To prove this, observe that any q is an fppf form of q_n (by Lemma C.2.1), so $(\mathrm{V}, \mathrm{L}, q)$ is an fppf form of $(\mathscr{O}_{\mathrm{S}}^n, \mathscr{O}_{\mathrm{S}}, q_n)$. Thus, the S-scheme $\mathrm{Isom}((\mathscr{O}_{\mathrm{S}}^n, \mathscr{O}_{\mathrm{S}}, q_n), (\mathrm{V}, \mathrm{L}, q))$ is an fppf right GO_n-torsor over S, and by the smoothness of GO_n its torsors for the fppf topology are actually trivialized étale-locally on the base (i.e., the Isom-scheme inherits smoothness from GO_n, so it admits sections étale-locally on S). The assignment of this Isom-scheme therefore defines the desired bijection (since GO_n represents the automorphism functor of $(\mathscr{O}_{\mathrm{S}}^n, \mathscr{O}_{\mathrm{S}}, q_n)$). In [**Au**] this bijection is used to systematically transfer properties of $\mathrm{GO}(q)$ (such as short exact sequences) into global structural results concerning quadratic forms valued in line bundles.

The second reason for our interest in orthogonal similitude groups is that the central subgroup $\mathbf{G}_{\mathrm{m}}$ inside GO_n induces an action by $\mathrm{H}^1(\mathrm{S}_{\text{ét}}, \mathbf{G}_{\mathrm{m}}) = \mathrm{Pic}(\mathrm{S})$ on $\mathrm{H}^1(\mathrm{S}_{\text{ét}}, \mathrm{GO}_n)$ that is precisely the natural twisting action of $\mathrm{Pic}(\mathrm{S})$ on the set of isomorphism classes of rank-n non-degenerate line bundle-valued quadratic forms over S. Thus, these orbits are the projective similarity classes, so we seek to classify the orbits of the $\mathrm{H}^1(\mathrm{S}_{\text{ét}}, \mathbf{G}_{\mathrm{m}})$-action on $\mathrm{H}^1(\mathrm{S}_{\text{ét}}, \mathrm{GO}_n)$. This is a useful viewpoint because of an interesting interpretation of the quotient group $\mathrm{PGO}(q) = \mathrm{GO}(q)/\mathbf{G}_{\mathrm{m}}$ that is due to Dieudonné (over fields with characteristic not equal to 2) and which we now recall.

Since $\mathrm{GO}(q)$ is generated for the fppf topology by $\mathrm{O}(q)$ and the central subgroup $\mathbf{G}_{\mathrm{m}}$, the normality of $\mathrm{SO}(q)$ in $\mathrm{O}(q)$ implies that $\mathrm{SO}(q)$ is normal in $\mathrm{GO}(q)$. The conjugation action of $\mathrm{GO}(q)$ on its normal subgroup $\mathrm{SO}(q)$ defines a homomorphism of group functors $\mathrm{GO}(q) \to \underline{\mathrm{Aut}}_{\mathrm{SO}(q)/\mathrm{S}}$. Since the maximal central torus in $\mathrm{SO}(q)$ is trivial when $n \geqslant 3$ and has rank 1 when $n = 2$, by Theorem 7.1.9 the automorphism functor of $\mathrm{SO}(q)$ is represented by a smooth S-affine S-group $\mathrm{Aut}_{\mathrm{SO}(q)/\mathrm{S}}$ that is an extension of a finite étale group by the adjoint quotient $\mathrm{SO}(q)/\mathrm{Z}_{\mathrm{SO}(q)}$. A geometric fiber of this étale group is the outer automorphism group $\mathrm{Out}(\mathrm{SO}_n)$ of SO_n. Since the central $\mathbf{G}_{\mathrm{m}}$ in $\mathrm{GO}(q)$ acts trivially on $\mathrm{SO}(q)$, we arrive at an S-homomorphism

$$h_q : \mathrm{PGO}(q) \to \mathrm{Aut}_{\mathrm{SO}(q)/\mathrm{S}}.$$

***Lemma C.3.13* (Dieudonné).** — *If $n \neq 2$ then h_q is an isomorphism.*

See [**Dieu**] for a classical treatment away from characteristic 2, and [**KMRT**, VI, 26.12, 26.15, 26.17] for a treatment over all fields. (Strictly speaking, for even $n \geqslant 4$, [**KMRT**] considers automorphisms of the adjoint quotient $\mathrm{SO}(q)/\mu_2$ rather than of $\mathrm{SO}(q)$ itself. This has no effect on the automorphism group, due to the structure of the outer automorphism group for even $n \geqslant 4$, as we shall see in the proof below.)

Proof. — Since h_q is a map between smooth S-affine S-groups, it suffices to prove the isomorphism property on geometric fibers. Hence, we may assume $\mathrm{S} = \mathrm{Spec}(k)$ for an algebraically closed field k. In particular, $q = q_n$. The case $n = 1$ is trivial, so assume $n \geqslant 3$.

First consider odd $n \geqslant 3$, so $\mathrm{SO}(q)$ is semisimple with trivial center and $\mathrm{O}(q) = \mu_2 \times \mathrm{SO}(q)$. The assertion in this case is that $\mathrm{SO}(q)$ is its own automorphism scheme. Such an equality holds for any connected semisimple k-group with trivial center and no nontrivial diagram automorphisms (by Theorem 7.1.9). For $m \geqslant 1$ we know that SO_{2m+1} has type B_m and trivial center (see Proposition C.3.8), and by inspection (treating $m = 1$ separately) the B_m diagram has no nontrivial diagram automorphisms.

Now assume n is even, so $n = 2m$ with $m \geqslant 2$ and $\mathrm{O}(q)/\mathrm{SO}(q) = \mathbf{Z}/2\mathbf{Z}$. The group SO_n is semisimple of type D_m (with $\mathrm{D}_2 = \mathrm{A}_1 \times \mathrm{A}_1$), its center $\mathrm{Z}_{\mathrm{SO}(q)}$ is equal to the central μ_2 inside $\mathrm{O}(q)$ (by Proposition C.3.8), and $\mathrm{O}(q)/\mu_2$ is an extension of $\mathrm{O}(q)/\mathrm{SO}(q) = \mathbf{Z}/2\mathbf{Z}$ by $\mathrm{SO}(q)/\mathrm{Z}_{\mathrm{SO}(q)}$. By Theorem 7.1.9, h_q identifies $\mathrm{SO}(q)/\mathrm{Z}_{\mathrm{SO}(q)}$ with $\mathrm{Aut}^0_{\mathrm{SO}(q)/k}$ and moreover the component group of $\mathrm{Aut}_{\mathrm{SO}(q)/k}$ (which is visibly $\mathrm{Out}(\mathrm{SO}_{2m})$) is identified with the automorphism group of the based root datum attached to SO_{2m}.

Any point in $\mathrm{O}(q)(k) - \mathrm{SO}(q)(k)$ acts on $\mathrm{SO}(q)(k)$ by a non-inner automorphism (since $\mathrm{Z}_{\mathrm{O}(q)} = \mu_2 \subset \mathrm{SO}(q)$; see Corollary C.3.9), so $\ker h_q = 1$ and our problem is to show that $\#\mathrm{Out}(\mathrm{SO}_{2m}) \leqslant 2$. But $\mathrm{Out}(\mathrm{SO}_{2m})$ is the automorphism group of the based root datum for the semisimple group SO_{2m}, so it is a subgroup of the automorphism group Γ_m of the D_m diagram. By inspection $\#\Gamma_m = 2$ for all $m \geqslant 2$ except for $m = 4$, so we are done except if $m = 4$.

Finally, consider the case $m = 4$, so $\Gamma_m \simeq \mathfrak{S}_3$ has order 6. We just have to rule out the possibility that the action of the entire group Γ_4 on the pinned simply connected central cover G of SO_8 descends to an action on SO_8. For any $m \geqslant 2$, the center of SO_{2m} has order 2 and the fundamental group Π_m of the D_m root system has order 4, so the simply connected central cover of SO_{2m} has degree $4/2 = 2$ over SO_{2m}. Hence, $\ker(\mathrm{G} \twoheadrightarrow \mathrm{SO}_8)$ is a subgroup of Z_G of order 2 and we just have to show that the Γ_4-action on Z_G does not preserve this subgroup. The Cartier dual of Z_G is the fundamental group Π_m that is $\mathbf{Z}/2 \times \mathbf{Z}/2\mathbf{Z}$ for even m. Thus, it suffices to observe by inspection that the action by $\Gamma_4 = \mathfrak{S}_3$ on the 2-dimensional $\mathbf{F}_2$-vector space Π_4 is transitive on the set of three $\mathbf{F}_2$-lines, so it does not preserve any of these lines. □

The isomorphism in Lemma C.3.13 is the key input into the proof of our desired result away from rank 2:

Proposition C.3.14. — *Let* $(\mathrm{V}, \mathrm{L}, q)$ *be a non-degenerate line bundle-valued quadratic form with* V *of rank* $n \neq 2$ *over a scheme* S*. The projective similarity class of* $(\mathrm{V}, \mathrm{L}, q)$ *is determined by the isomorphism class of the* S*-group* $\mathrm{SO}(q)$*.*

Proof. — If $n = 1$ then $\mathrm{SO}(q) = 1$ and so we need to prove that there is a single projective similiarity class. Since V is a line bundle, we can twist by its dual to arrive at a non-degenerate line bundle-valued quadratic form $(\mathscr{O}_\mathrm{S}, \mathrm{L}', q')$. The non-degeneracy implies that $q'(1)$ is a trivializing section of L', under which q' becomes $x \mapsto x^2$. Hence, for $n = 1$ there is indeed only one projective similarity class.

Now we assume $n \geqslant 3$. It is straightforward to check that the induced map

$$\mathrm{H}^1(\mathrm{S}_{\text{ét}}, \mathrm{GO}_n) \to \mathrm{H}^1(\mathrm{S}_{\text{ét}}, \mathrm{Aut}_{\mathrm{SO}_n/\mathrm{S}})$$

carries the Pic(S)-orbit of $(\mathrm{V}, \mathrm{L}, q)$ to the isomorphism class of the étale form $\mathrm{SO}(q)$ of SO_n. Thus, our problem is to show that the fibers of this map are precisely the Pic(S)-orbits.

By Lemma C.3.13, since $n \geqslant 3$ we obtain an exact sequence of smooth S-affine S-groups

$$1 \to \mathbf{G}_\mathrm{m} \to \mathrm{GO}(q) \to \mathrm{Aut}_{\mathrm{SO}(q)/\mathrm{S}} \to 1.$$

Consider the induced map of pointed sets

$$f_q : \mathrm{H}^1(\mathrm{S}_{\text{ét}}, \mathrm{GO}(q)) \to \mathrm{H}^1(\mathrm{S}_{\text{ét}}, \mathrm{Aut}_{\mathrm{SO}(q)/\mathrm{S}}).$$

As we saw in the discussion preceding Lemma C.3.13, for any rank-n non-degenerate line bundle-valued quadratic form $(\mathrm{V}', \mathrm{L}', q')$ over S, the scheme $\mathrm{Isom}(q', q)$ of isomorphisms from $(\mathrm{V}', \mathrm{L}', q')$ to $(\mathrm{V}, \mathrm{L}, q)$ is an étale left $\mathrm{GO}(q)$-torsor whose isomorphism class over S determines the isomorphism class of $(\mathrm{V}', \mathrm{L}', q')$. Hence, the source of f_q is the set of isomorphism classes of such

(V', L', q') of rank n over S. Likewise, the target of f_q is the set of étale forms of $SO(q)$ as an S-group.

The map f_q carries the étale $GO(q)$-torsor $\mathrm{Isom}(q', q)$ to the class of $SO(q')$ as an étale form of $SO(q)$. This proves that $\ker f_q$ is the set of isomorphism classes of those (V', L', q') for which $SO(q') \simeq SO(q)$ as S-groups. But $\ker f_q$ is the image of $H^1(S_{\mathrm{fppf}}, \mathbf{G}_m) = \mathrm{Pic}(S)$ induced by the central inclusion $\mathbf{G}_m \to GO(q)$. This image is the Pic(S)-orbit of (the distinguished point) (V, L, q) for the natural action of Pic(S) on the set of isomorphism classes of line bundle-valued quadratic forms, which is to say the projective similarity class of (V, L, q), so we are done. □

We conclude § C.3 by considering the remaining case $n = 2$, which exhibits entirely different behavior. In particular, we will see that usually the isomorphism class of $SO(q)$ badly fails to determine the projective similarity class of (V, L, q) (except when S is local). By inspection (see Example C.6.1), $GO_2 = (\mathbf{Z}/2\mathbf{Z})_S \ltimes GSO_2$ with GSO_2 a torus of rank 2 that contains two split rank-1 subtori having intersection μ_2: the torus SO_2 and the "scalar" torus $\mathbf{G}_m \subset GL_2$. Our description of GO_2 shows that in general if $n = 2$ then the following properties hold: $GSO(q)$ is a torus of rank 2 that coincides with $SO(q) \times^{\mu_2} \mathbf{G}_m$, the group $SO(q)$ is a rank-1 torus whose automorphism scheme is uniquely isomorphic to $(\mathbf{Z}/2\mathbf{Z})_S$, the quotient $PGSO(q)$ of $GSO(q)$ is also a rank-1 torus, and the map $GO(q) \to \mathrm{Aut}_{SO(q)/S} = (\mathbf{Z}/2\mathbf{Z})_S$ induced by conjugation is the quotient modulo $GSO(q)$. In particular, it coincides with the enhanced Dickson invariant GD_q from (C.3.1) and Remark C.3.11 (for $n = 2$).

Proposition C.3.15. — *Let* (V, L, q) *be non-degenerate of rank* 2.

1. *There is a natural isomorphism* $GSO(q) \simeq R_{Z_q/S}(\mathbf{G}_m)$ *extending the natural inclusion on* $\mathbf{G}_m$ *and under which* $SO(q)$ *is identified with the group of norm*-1 *units. Moreover, the coordinate ring over* $\mathscr{O}_S$ *of the finite étale zero scheme* $(q = 0) \subset \mathbf{P}(V^*)$ *is naturally isomorphic to* Z_q.
2. *The class of* $SO(q)$ *in* $H^1(S_{\text{ét}}, \mathrm{Aut}_{SO_2/S}) = H^1(S_{\text{ét}}, \mathbf{Z}/2\mathbf{Z})$ *corresponds to the étale double cover* Z_q. *In particular, for non-degenerate* (V, L, q) *and* (V', L', q') *of rank* 2, $SO(q') \simeq SO(q)$ *if and only if* $Z_q \simeq Z_{q'}$.

This result is due to Kneser [**Kne**, § 6, Prop. 2] (also see [**BK**, 6.1], [**Au**, 5.2]). Note that the existence of an abstract S-group isomorphism as in (1) is not Zariski-local on S. Thus, the case of general S does not formally follow from the case of affine S.

Proof. — Granting (1), let us deduce (2). The second assertion in (2) reduces immediately to the first. For the first assertion in (2) we use the canonical identification of $SO(q)$ as a norm-1 torus in (1) to reduce to showing that a degree-2 finite étale cover $E \to S$ is uniquely determined up to unique isomorphism by the norm-1 subtorus $\mathrm{T_E}$ inside $\mathrm{R_{E/S}}(\mathbf{G}_{\mathrm{m}})$. More precisely, if $\mathrm{E}' \to \mathrm{S}$ is another such cover then we claim that the natural map $\mathrm{Isom_S}(\mathrm{E}',\mathrm{E}) \to \mathrm{Isom_{S\text{-}gp}}(\mathrm{T_{E'}},\mathrm{T_E})$ is bijective and that every rank-1 torus T over S arises in the form $\mathrm{T_E}$ for some E. For $\mathrm{E} = \mathrm{S}\coprod\mathrm{S}$ the torus $\mathrm{T_E}$ is the "$\mathbf{G}_{\mathrm{m}}$-hyperbola" of points $(t, 1/t)$ inside $\mathbf{G}_{\mathrm{m}}^2$. By descent theory, it suffices to prove the bijectivity result for isomorphisms when E and E$'$ are split covers. Thus, upon identifying the hyperbola $\mathrm{T_{S\coprod S}}$ with $\mathbf{G}_{\mathrm{m}}$ via projection to the first factor, it suffices to check that the natural map $\mathrm{Aut_S}(\mathrm{S}\coprod\mathrm{S}) \to \mathrm{Aut_{S\text{-}gp}}(\mathbf{G}_{\mathrm{m}})$ is bijective. By "spreading out" arguments we may assume S is local, so both automorphism groups have size 2 and the bijectivity is obvious.

To prove the first assertion in (1), note that since V has rank 2, the subbundle inclusion $\mathrm{V} \hookrightarrow \mathrm{C_1}(\mathrm{V},\mathrm{L},q)$ is an isomorphism. Likewise, the $\mathscr{O}_{\mathrm{S}}$-algebra $\mathrm{C_0}(\mathrm{V},\mathrm{L},q)$ has rank 2 and so coincides with its quadratic étale center Z_q. Left multiplication by $\mathrm{Z}_q = \mathrm{C_0}(\mathrm{V},\mathrm{L},q)$ must preserve $\mathrm{C_1}(\mathrm{V},\mathrm{L},q) = \mathrm{V}$ and thereby makes V into a Z_q-module. As such, V is an invertible Z_q-module due to the general invertibility (Zariski-locally on S) of $\mathrm{C_1}(\mathrm{V},\mathrm{L},q)$ as a left $\mathrm{C_0}(\mathrm{V},\mathrm{L},q)$ for any even rank (see Proposition C.2.2). In Remark C.3.11 we defined compatible actions of $\mathrm{GO}(q)$ on $\mathrm{C_0}(\mathrm{V},\mathrm{L},q)$ and on the left $\mathrm{C_0}(\mathrm{V},\mathrm{L},q)$-module $\mathrm{C_1}(\mathrm{V},\mathrm{L},q)$ (extending the natural action on $\mathrm{V} \subset \mathrm{C_1}(\mathrm{V},\mathrm{L},q)$), so in our rank-2 setting this recovers the usual action of $\mathrm{GO}(q)$ on V and shows that it is semilinear over an action on the $\mathscr{O}_{\mathrm{S}}$-algebra Z_q. But the subgroup $\mathrm{GSO}(q)$ acts trivially on Z_q (as we saw in Remark C.3.11 for any even rank), so the natural $\mathrm{GSO}(q)$-action on V is linear over the invertible Z_q-module structure.

Letting T_q denote the rank-2 torus $\mathrm{R}_{\mathrm{Z}_q/\mathrm{S}}(\mathbf{G}_{\mathrm{m}})$, the multiplication action by Z_q on V defines a closed immersion

$$\mathrm{T}_q := \mathrm{R}_{\mathrm{Z}_q/\mathrm{S}}(\mathbf{G}_{\mathrm{m}}) \to \mathrm{GL}(\mathrm{V})$$

extending the natural inclusion on $\mathbf{G}_{\mathrm{m}}$. This identifies T_q with the functor of Z_q-linear automorphisms of V, so the Z_q-linearity of the natural $\mathrm{GSO}(q)$-action on V thereby shows that $\mathrm{GSO}(q) \subset \mathrm{T}_q$ as closed S-tori inside $\mathrm{GL}(\mathrm{V})$ containing the "scalar" torus $\mathbf{G}_{\mathrm{m}}$. The tori $\mathrm{GSO}(q)$ and T_q have rank 2, so fibral considerations show that $\mathrm{GSO}(q) = \mathrm{T}_q$ inside $\mathrm{GL}(\mathrm{V})$.

To show that this identification of $\mathrm{GSO}(q)$ with T_q identifies $\mathrm{SO}(q)$ with the norm-1 torus inside T_q, we claim more specifically that the norm map $\mathrm{T}_q \to \mathbf{G}_{\mathrm{m}}$ is identified with the restriction to $\mathrm{GSO}(q)$ of the similitude character $\mathrm{GO}(q) \to \mathbf{G}_{\mathrm{m}}$ (giving the $\mathbf{G}_{\mathrm{m}}$-scaling action of $\mathrm{GO}(q)$ on L that intertwines through q with the action on V). To prove this equality of characters of an S-torus it suffices to work on geometric fibers over S, so we can assume $\mathrm{S} = \mathrm{Spec}(k)$ for an algebraically closed field k and $q = q_2$. (See [**Knus**, V, 2.5.2] for a direct argument over any ring.) Thus, $\mathrm{V} = k^2$ with standard ordered basis $\{e, e'\}$, $\mathrm{L} = k$, and $q(x, y) = xy$. By Remark C.2.3 (with $m = 1$),

$$\mathrm{C}(q_2) = k \oplus (ke \oplus ke') \oplus kee'$$

with $ee' + e'e = 1$ and $e^2 = 0 = e'^2$. Clearly $\mathrm{Z}_q = \mathrm{C}_0(\mathrm{V}, q_2) = k \oplus kee'$ with ee' and $e'e$ orthogonal idempotents in Z_q whose sum is 1. Since

$$ee'(e) = e(1 - ee') = e, \;\; ee'(e') = 0,$$

the left multiplication action by Z_q on $\mathrm{V} = ke \oplus ke'$ carries ee' to $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. This action therefore defines an isomorphism of Z_q onto the diagonal étale subalgebra of $\mathrm{Mat}_2(k)$, whose unit group is exactly GSO_2. The norm character on the group $\mathrm{T}_q(k) = \mathrm{Z}_q^{\times}$ of diagonal elements in $\mathrm{GL}_2(k)$ is the restriction of the determinant on GL_2, and the determinant also clearly restricts to the homothety character on $k^{\times} \cdot \mathrm{O}_2(k) = \mathrm{GO}_2(k)$.

Returning to the relative setting over a general base S, it remains to naturally identify Z_q with the coordinate ring of the finite étale zero scheme E_q of q in $\mathbf{P}(\mathrm{V}^*)$. This is a problem of identifying degree-2 finite étale covers of S, so as we saw in the reduction of (2) to (1) it is enough to identify the torus of norm-1 units on E_q with the torus of norm-1 units in Z_q. This latter torus has already been identified with $\mathrm{SO}(q)$, so we just need to construct a natural isomorphism between $\mathrm{SO}(q)$ and the S-group of norm-1 units on E_q. By étale descent it suffices to construct such an isomorphism when q is split *provided* that the isomorphism is natural in the sense that it is compatible with base change and functorial with respect to isomorphisms in (V, q). Now we may assume $q = xy$ for fiberwise independent linear forms x, y on V, and we need to naturally identify $\mathrm{SO}(q)$ with the group of norm-1 units on E_q.

By working over the local rings of S and using unique factorization over the residue field at the closed point, it is easy to verify that any two such factorizations of q are related Zariski-locally on S through a combination of swapping x and y as well as multiplying them by reciprocal units. In particular, the unordered pair of complementary line subbundles $\ell^* = \mathscr{O}_{\mathrm{S}}x$ and $\ell'^* = \mathscr{O}_{\mathrm{S}}y$ in V^* is intrinsic, so likewise for the associated unordered dual pair of line subbundles $\ell = \mathscr{O}_{\mathrm{S}}x^*$ and $\ell' = \mathscr{O}_{\mathrm{S}}y^*$ in V. The zero-scheme E_q is the union of the disjoint sections $0 := (x = 0) = \mathbf{P}(\ell'^*)$ and $\infty := (y = 0) = \mathbf{P}(\ell^*)$ in $\mathbf{P}(\mathrm{V})$, so a unit u on E_q amounts to a pair of units $u(0), u(\infty)$ on S and the condition that u is a norm-1 unit is that $u(0)u(\infty) = 1$. Consider the linear automorphism $[u]$ of V that is multiplication by $u(0)$ on ℓ' and multiplication by $u(\infty)$ on ℓ. The formation of $[u]$ is obviously functorial with respect to isomorphisms in the pair (V, q) (given that q is split as above), and the map $u \mapsto [u]$ is visibly an isomorphism from the unit group of E_q onto $\mathrm{GSO}(q)$ carrying the norm-1 unit group onto $\mathrm{SO}(q)$. $\square$

The exact sequence

$$1 \to \mathbf{G}_{\mathrm{m}} \to \mathrm{GSO}(q) \to \mathrm{PGSO}(q) \to 1 \tag{C.3.2}$$

of S-tori induces an exact sequence of commutative groups

$$\ldots \stackrel{\delta_1}{\to} \mathrm{Pic}(\mathrm{S}) \to \mathrm{H}^1(\mathrm{S}_{\text{ét}}, \mathrm{GSO}(q)) \to \mathrm{H}^1(\mathrm{S}_{\text{ét}}, \mathrm{PGSO}(q)) \stackrel{\delta_2}{\to} \mathrm{Br}(\mathrm{S}),$$

where $\mathrm{Br}(\mathrm{S}) := \mathrm{H}^2(\mathrm{S}_{\text{ét}}, \mathbf{G}_{\mathrm{m}})$ is the cohomological Brauer group. (The image of δ_2 lands in the subset of classes represented by a rank-4 Azumaya algebra, due to the compatibility of (C.3.2) with the analogous exact sequence that expresses GL(V) as a central extension of PGL(V) by $\mathbf{G}_{\mathrm{m}}$.) By Proposition C.3.15(1) and the exactness of finite pushforward for the étale topology, δ_2 has image $\ker(\mathrm{Br}(\mathrm{S}) \to \mathrm{Br}(\mathrm{Z}_q))$. Thus, in general there is an exact sequence

$$0 \to \mathrm{H}^1(\mathrm{S}_{\text{ét}}, \mathrm{GSO}(q))/\mathrm{Pic}(\mathrm{S}) \to \mathrm{H}^1(\mathrm{S}_{\text{ét}}, \mathrm{PGSO}(q)) \to \mathrm{Br}(\mathrm{S}) \to \mathrm{Br}(\mathrm{Z}_q).$$

By Proposition C.3.15(2) and the explicit description of GD_q immediately above Proposition C.3.15, the pointed set $\ker \mathrm{H}^1(\mathrm{GD}_q)$ classifies isomorphism classes of $(\mathrm{V}', \mathrm{L}', q')$ whose associated special orthogonal group is isomorphic to $\mathrm{SO}(q)$. There is a natural surjection of pointed sets $\mathrm{H}^1(\mathrm{S}_{\text{ét}}, \mathrm{GSO}(q)) \twoheadrightarrow \ker \mathrm{H}^1(\mathrm{GD}_q)$ that intertwines the Pic(S)-action on the source with projective similarity on the target, so the set of projective similarity classes in $\ker \mathrm{H}^1(\mathrm{GD}_q)$ is a quotient of the subgroup

$$\mathrm{H}^1(\mathrm{S}_{\text{ét}}, \mathrm{GSO}(q))/\mathrm{Pic}(\mathrm{S}) \subset \mathrm{H}^1(\mathrm{S}_{\text{ét}}, \mathrm{PGSO}(q)).$$

By Proposition C.3.15(1), $\mathrm{H}^1(\mathrm{S}_{\text{ét}}, \mathrm{GSO}(q))$ is naturally identified with $\mathrm{Pic}(\mathrm{Z}_q)$, which vanishes when S is local (as then Z_q is affine and semi-local). Thus, we obtain:

Corollary C.3.16. — *Assume* S *is local. The set* $\ker \mathrm{H}^1(\mathrm{GD}_q)$ *consists of a single projective similarity class for any* $(\mathrm{V}, \mathrm{L}, q)$ *with* V *of rank* 2*. In particular, the similitude class of a binary quadratic form* (V, q) *over* S *is determined by the isomorphism class of* $\mathrm{SO}(q)$*, or equivalently by the isomorphism class of the discriminant scheme* $(q = 0) \subset \mathbf{P}(\mathrm{V}^*)$*.*

This corollary shows that Proposition C.3.14 is also valid for $n = 2$ when $\mathrm{S} = \mathrm{Spec}(\mathrm{R})$ for a local ring R (e.g., any field).

An interesting situation over general S is when (C.3.2) is split-exact, as we shall soon see. Split-exactness occurs over local S if $\mathrm{GSO}(q)$ is a split torus, since an inclusion between split tori always splits off globally as a direct factor over local S (with a split torus complement too), as we see via duality from the elementary analogue for a surjection $\mathbf{Z}_{\mathrm{S}}^{\mathrm{N}} \twoheadrightarrow \mathbf{Z}_{\mathrm{S}}^{\mathrm{N}'}$ between constant $\mathrm{S}_{\text{ét}}$-groups. Here is a characterization in rank 2 for when $\mathrm{GSO}(q)$ is a split torus:

Lemma C.3.17. — *Assume* V *has rank* 2*. The* S*-torus* $\mathrm{GSO}(q)$ *is split if and only if* Z_q *is globally split as a quadratic étale* $\mathscr{O}_{\mathrm{S}}$*-algebra.*

Note that Z_q can be globally split even when V not globally free, so generally q cannot be identified with q_2 when Z_q is split.

Proof. — By Proposition C.3.15(1), the problem is to show that the quadratic étale $\mathscr{O}_S$-algebra Z_q is globally split if and only if the S-torus of units $R_{Z_q/S}(\mathbf{G}_m)$ is S-split. (If S is not normal noetherian then it is *not* sufficient for the subtorus of norm-1 units to be S-split, as there can be non-split S-tori T that are an extension of $\mathbf{G}_m$ by $\mathbf{G}_m$. Indeed, it suffices to build a nontrivial $\mathbf{Z}$-torsor E over S and then use the subgroup of upper triangular unipotent elements in $GL_2(\mathbf{Z})$ to build such a T using E. The nodal cubic is an irreducible non-normal noetherian S admitting a nontrivial $\mathbf{Z}$-torsor.)

The isomorphism class of the $\mathscr{O}_S$-algebra Z_q corresponds to an element of the pointed set $H^1(S_{ét}, \mathbf{Z}/2\mathbf{Z}) = H^1(S_{ét}, \mathbf{Z}^\times)$, and the isomorphism class of the S-torus $GSO(q)$ corresponds to an element of the pointed set $H^1(S_{ét}, GL_2(\mathbf{Z}))$. The determinant map $GL_2(\mathbf{Z}) \to \mathbf{Z}^\times$ has a section σ given by

$$-1 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and it is straightforward to check that the map

$$H^1(\sigma) : H^1(S_{ét}, \mathbf{Z}^\times) \to H^1(S_{ét}, GL_2(\mathbf{Z}))$$

carries the class of Z_q to the class of $R_{Z_q/S}(\mathbf{G}_m)$. Thus, we may conclude by noting that $H^1(\sigma)$ has trivial kernel (since $H^1(\det)$ provides a retraction). $\square$

Now assume (with $n = 2$) that (C.3.2) is split-exact (e.g., $GSO(q)$ is split), so the connecting maps δ_1 and δ_2 vanish. This yields an isomorphism

$$\mathrm{Pic}(Z_q)/\mathrm{Pic}(S) = H^1(S_{ét}, GSO(q))/\mathrm{Pic}(S) \simeq H^1(S_{ét}, PGSO(q)),$$

so the pointed set $H^1(S_{ét}, PGSO(q))$ parameterizes (possibly with some repetitions when $\mathrm{Pic}(S) \neq 1$) the set of projective similarity classes among the set of isomorphism classes of (V', L', q') for which $SO(q') \simeq SO(q)$. Typically there are many projective similarity classes among such (V', L', q'), in contrast with the case $n \neq 2$.

Overall, for non-local S it is hard to give a simple interpretation of when two non-degenerate binary quadratic forms (V, L, q) and (V', L', q') lie in the same projective similarity class. However, by Proposition C.3.15(2), there is a nice interpretation of when (V, L, q) and (V', L', q') have isomorphic special orthogonal groups (replacing the answer via projective similarity classes for rank $\neq 2$): it is equivalent to the discriminant algebras Z_q and Z'_q being isomorphic. Understanding the global geometry of the double covers of S associated to these quadratic étale algebras is a rather nontrivial problem when these covers are not split.

C.4. Spin groups and related group schemes. — Let (V, q) be a non-degenerate quadratic space of rank $n \geqslant 1$ (so q is valued in $\mathscr{O}_{\mathrm{S}}$). We shall use unit groups of Clifford algebras to "explicitly" construct a degree-2 central extension $\mathrm{Spin}(q)$ of $\mathrm{SO}(q)$ by μ_2 that is the simply connected central cover when $n \geqslant 3$. For even n the Clifford algebra has center $\mathscr{O}_{\mathrm{S}}$ (Proposition C.2.2) whereas for odd n the center is a finite locally free $\mathscr{O}_{\mathrm{S}}$-algebra of rank 2 that is non-étale in characteristic 2 (Proposition C.2.4). Thus, we shall first consider even n. In such cases we will use the Clifford group $\mathrm{GPin}(q)$ from Definition C.2.6 to construct $\mathrm{Spin}(q) \to \mathrm{SO}(q)$ by relativizing arguments over fields in [**Chev97**, II.3.5]. (In [**Chev97**, II.3.5], spin groups are called *reduced Clifford groups.*)

Assume $n = 2m \geqslant 2$ is *even*. The S-group $\mathrm{GPin}(q)$ is a central extension of $\mathrm{O}(q)$ by $\mathbf{G}_{\mathrm{m}}$ (Proposition C.2.8), so $\mathrm{GSpin}(q) := \mathrm{GPin}(q)^0$ is a central extension of $\mathrm{SO}(q)$ by $\mathbf{G}_{\mathrm{m}}$ and

$$\mathrm{GPin}(q)/\mathrm{GSpin}(q) \simeq \mathrm{O}(q)/\mathrm{SO}(q) = (\mathbf{Z}/2\mathbf{Z})_{\mathrm{S}}. \tag{C.4.1}$$

In particular, $\mathrm{GSpin}(q)$ is open and closed in $\mathrm{GPin}(q)$, so it is S-smooth and S-affine. Since $\mathrm{GSpin}(q)$ is an extension of $\mathrm{SO}(q)$ by $\mathbf{G}_{\mathrm{m}}$, its fibers are connected reductive. Thus, $\mathrm{GSpin}(q)$ is a reductive S-group.

Via the link between D_q in (C.2.2) and the $\mathbf{Z}/2\mathbf{Z}$-grading on $\mathrm{GPin}(q)$ as noted immediately before Proposition C.2.8, we see that $\mathrm{GSpin}(q) = \mathrm{C}_0(\mathrm{V}, q) \bigcap \mathrm{GPin}(q)$. For this reason, $\mathrm{GSpin}(q)$ is called the *even Clifford group.* If Z_q denotes the quadratic étale center of $\mathrm{C}_0(\mathrm{V}, q)$ then by Remark C.2.7 the S-group $\mathrm{GSpin}(q)$ contains the S-torus $\mathrm{R}_{\mathrm{Z}_q/\mathrm{S}}(\mathbf{G}_{\mathrm{m}})$ when $n = 2$ whereas if $n \geqslant 4$ then $\mathrm{GSpin}(q) \bigcap \mathrm{R}_{\mathrm{Z}_q/\mathrm{S}}(\mathbf{G}_{\mathrm{m}})$ is the subgroup of points of $\mathrm{R}_{\mathrm{Z}_q/\mathrm{S}}(\mathbf{G}_{\mathrm{m}})$ whose square lies in $\mathbf{G}_{\mathrm{m}}$ (so it is a commutative extension of μ_2 by $\mathbf{G}_{\mathrm{m}}$).

For $n = 2$, the containment $\mathrm{R}_{\mathrm{Z}_q/\mathrm{S}}(\mathbf{G}_{\mathrm{m}}) \subset \mathrm{GSpin}(q)$ of smooth S-groups is an equality for fibral connectedness and dimension reasons. When $n = 2m \geqslant 4$ the S-group $\mathrm{SO}(q)$ is semisimple (with geometric fibers of type D_m, where $\mathrm{D}_2 := \mathrm{A}_1 \times \mathrm{A}_1$), so the derived group (in the sense of Theorem 5.3.1)

$$\mathrm{Spin}(q) := \mathscr{D}(\mathrm{GSpin}(q))$$

is a central extension of $\mathrm{SO}(q)$ by an S-subgroup $\mu \subset \mathbf{G}_{\mathrm{m}}$ that must be finite fppf (Proposition 6.1.10). Since we saw that $\mathrm{GSpin}(q)$ is an extension of $\mathrm{SO}(q)$ by $\mathbf{G}_{\mathrm{m}}$, it follows that $\mathrm{GSpin}(q) = \mathbf{G}_{\mathrm{m}} \times^{\mu} \mathrm{Spin}(q)$.

Lemma C.4.1. — *For even $n \geqslant 4$, the intersection $\mu = \mathbf{G}_{\mathrm{m}} \bigcap \mathrm{Spin}(q)$ is equal to $\mu_2 \subset \mathbf{G}_{\mathrm{m}}$. The central extension $\mathrm{Spin}(q)$ of $\mathrm{SO}(q)$ by μ_2 is the simply connected central cover, and $\mathrm{Z}_{\mathrm{Spin}(q)}$ is a form of μ_4 if $n \equiv 2 \bmod 4$ and is a form of $\mu_2 \times \mu_2$ if $4|n$.*

Proof. — Consider the *main anti-involution* α of $\mathrm{C}(\mathrm{V}, q)$ induced by the anti-involution of the tensor algebra of V via $v_1 \otimes \cdots \otimes v_j \mapsto v_j \otimes \cdots \otimes v_1$. Note that α restricts to the identity on V and the identity on the center $\mathscr{O}_{\mathrm{S}}$. (The effect of α on the rank-2 étale central $\mathscr{O}_{\mathrm{S}}$-algebra Z_q in $\mathrm{C}_0(\mathrm{V}, q)$ is the identity when $4|n$ and is the unique fiberwise nontrivial algebra automorphism when $n \equiv 2 \bmod 4$. We will not need this, but to prove it we just need to consider $q = q_n$ over $\mathbf{Z}$. The decomposition $\mathrm{Z}_{q_n} = \mathbf{Z}z \times \mathbf{Z}(1-z)$ in Remark C.2.3 for a fiberwise nontrivial idempotent z described over $\mathbf{Z}[1/2]$ in terms of an explicit element w shows that $\alpha(z) = 1 - z$ when $n \equiv 2 \bmod 4$ and $\alpha(z) = z$ when $4|n$ because $\alpha(w) = (-1)^{n/2}w$. This argument holds for $n = 2$ as well.)

For $u \in \mathrm{GPin}(q)$, the operator $h = \pi_q(u) \in \mathrm{O}(q)$ as in (C.2.5) satisfies $uv = \varepsilon_u h(v)u$ in $\mathrm{C}(\mathrm{V}, q)$ for all $v \in \mathrm{V}$, where $\varepsilon_u = (-1)^{\deg_q(u)}$ as in (C.2.5). Applying α gives $-v\alpha(u) = \varepsilon_u \alpha(u)(-h(v))$. Hence, $\alpha(u)uv = \varepsilon_u \alpha(u)h(v)u = v\alpha(u)u$, so the point $\alpha(u)u$ in $\mathrm{GPin}(q)$ centralizes V in $\mathrm{C}(\mathrm{V}, q)$ and thus is central in $\mathrm{C}(\mathrm{V}, q)$. In other words, $\alpha(u)u \in \mathbf{G}_{\mathrm{m}}$ (so $\alpha(u)u = u\alpha(u)$, as $u^{-1}\alpha(u)u = \alpha(u)uu^{-1} = \alpha(u)$). Since α is an anti-automorphism, so

$$\alpha(uu')uu' = \alpha(u')\alpha(u)uu' = (\alpha(u)u)(\alpha(u')u'),$$

the map $u \mapsto \alpha(u)u$ is an S-homomorphism

$$\nu_q : \mathrm{GPin}(q) \to \mathbf{G}_{\mathrm{m}} \tag{C.4.2}$$

(called the *Clifford norm*) whose restriction to the central $\mathbf{G}_{\mathrm{m}}$ is $t \mapsto t^2$. The Clifford norm must kill the semisimple $\mathrm{Spin}(q)$, so $\mu := \mathbf{G}_{\mathrm{m}} \bigcap \mathrm{Spin}(q) \subset \mu_2$. We claim that this inclusion between finite fppf S-groups is an isomorphism. It suffices to check on fibers.

For $\mathrm{S} = \mathrm{Spec}\, k$ with a field k we must rule out the possibility $\mathbf{G}_{\mathrm{m}} \cap \mathrm{Spin}(q) = 1$. If this happens then $\pi_q : \mathrm{Spin}(q) \to \mathrm{SO}(q)$ is an isomorphism. This map is equivariant for the natural actions by $\mathrm{O}(q) = \mathrm{GPin}(q)/\mathbf{G}_{\mathrm{m}}$ on each side, so if π_q were an isomorphism then $\mathrm{Spin}(q)$ would contain a μ_2 that is centralized by $\mathrm{GPin}(q)$ (lifting $\mathrm{Z}_{\mathrm{O}(q)} = \mu_2 \subset \mathrm{SO}(q)$) and is not contained in the central $\mathbf{G}_{\mathrm{m}} \subset \mathrm{C}(\mathrm{V}, q)$. But $\mathrm{GPin}(q)$ generates the k-algebra $\mathrm{C}(\mathrm{V}, q)$ (since $\mathrm{GPin}(q)$ contains the Zariski-dense open $\mathrm{U} = \{q \neq 0\} \subset \mathrm{V}$) and this algebra has center k, so $\mathbf{G}_{\mathrm{m}} = \mathrm{Z}_{\mathrm{GPin}(q)}$. Hence, no such μ_2 subgroup can exist.

Returning to the relative setting, we conclude that for even $n = 2m \geqslant 4$, the semisimple S-group $\mathrm{Spin}(q)$ is a central extension of $\mathrm{SO}(q)$ by μ_2. Since $\#\mathrm{Z}_{\mathrm{Spin}(q)} = 2 \cdot \#\mathrm{Z}_{\mathrm{SO}(q)} = 4$ and the fundamental group for the root system D_m ($m \geqslant 2$) has order 4, $\mathrm{Spin}(q)$ must be simply connected. Thus, $\mathrm{Spin}(q)$ is the simply connected central cover of $\mathrm{SO}(q)$. The structure of $\mathrm{Z}_{\mathrm{Spin}(q)}$ can be read off from the fact that it is a form of the Cartier dual of the quotient P/Q of the weight lattice modulo the root lattice for the root system D_m ($m = n/2 \geqslant 2$). $\square$

Remark C.4.2. — For odd n, the definitions of α and ν_q as in the preceding proof carry over but Z_q is a rank-2 finite flat $\mathscr{O}_{\mathrm{S}}$-algebra not contained in the even part. This leads to some complications when n is odd, as we discuss soon.

In [**ABS**, Def. 1.8, (3.7)] and [**Knus**, IV, 6.1] the Clifford norm ν_q is defined by replacing α with the unique anti-involution α^- extending negation on V. The signless α and associated Clifford norm that we use agree with [**Chev97**, II.3.5], [**Fr**, App. I], [**O**, § 55], and [**Sch**, Ch. 9, § 3], and also implicitly with [**Ser84**] (see Remark C.5.4). Both sign conventions for the definition of the Clifford norm yield the same restriction to the even part of the Clifford algebra.

By construction, the central pushout of $\mathrm{Spin}(q)$ along $\mu_2 \hookrightarrow \mathbf{G}_{\mathrm{m}}$ is the relative identity component $\mathrm{GSpin}(q)$ of $\mathrm{GPin}(q)$ inside $\mathrm{C}(\mathrm{V}, q)^{\times}$. Since $\mathrm{GSpin}(q) = \mathbf{G}_{\mathrm{m}} \times^{\mu_2} \mathrm{Spin}(q)$, we see that $\mathrm{Spin}(q)$ is the kernel of the Clifford norm $\nu_q : \mathrm{GSpin}(q) \to \mathbf{G}_{\mathrm{m}}$. This description of $\mathrm{Spin}(q)$ via the Clifford norm on $\mathrm{GSpin}(q)$ makes sense even when $n = 2$ (whereas the "derived group" definition of the spin group for even $n \geqslant 4$ is not suitable, due to the commutativity of $\mathrm{GSpin}(q) = \mathrm{R}_{\mathrm{Z}_q/\mathrm{S}}(\mathbf{G}_{\mathrm{m}})$ when $n = 2$), so we use it as the *definition* of $\mathrm{Spin}(q)$ when $n = 2$; direct calculation with q_2 shows that if $n = 2$ then the Clifford norm $\nu_q : \mathrm{GSpin}(q) = \mathrm{R}_{\mathrm{Z}_q/\mathrm{S}}(\mathbf{G}_{\mathrm{m}}) \to \mathbf{G}_{\mathrm{m}}$ arises from the $\mathscr{O}_{\mathrm{S}}$-algebra norm on $\mathscr{O}_{\mathrm{Z}_q}$. Thus, if $n = 2$ then $\mathrm{Spin}(q)$ is the group of norm-1 units in the rank-2 torus $\mathrm{R}_{\mathrm{Z}_q/\mathrm{S}}(\mathbf{G}_{\mathrm{m}})$ and the resulting map $\mathrm{Spin}(q) \to \mathrm{SO}(q)$ is a degree-2 isogeny with kernel $\mu_2 \subset \mathbf{G}_{\mathrm{m}}$, so Lemma C.4.1 holds for $n = 2$. Likewise, by inspection we have $\mathrm{GSpin}(q) = \mathbf{G}_{\mathrm{m}} \times^{\mu_2} \mathrm{Spin}(q)$ when $n = 2$ as well.

Now we turn to spin groups and other related groups for *odd* $n = 2m+1 \geqslant 1$. As in the case of even n, we abuse notation by letting $\mathrm{C}(\mathrm{V}, q)^{\times}$ denote the "S-group scheme of units of $\mathrm{C}(\mathrm{V}, q)$". Recall from the discussion following Definition C.2.6 that for even n, all points of $\mathrm{GPin}(q)$ are locally homogeneous. For odd n the same definition "makes sense" but will not be used as the definition of $\mathrm{GPin}(q)$ in such cases because for such a definition (applied to odd n) local homogeneity generally fails, as we can already see with the $\mathbf{Z}/2\mathbf{Z}$-graded center. The definition of the $\mathrm{GPin}(q)$ for odd n will require the insertion of a local homogeneity condition that is automatically satisfied for even n.

The Zariski-closed subgroup scheme $\mathrm{C}(\mathrm{V}, q)^{\times}_{\mathrm{lh}}$ of locally homogeneous units in $\mathrm{C}(\mathrm{V}, q)^{\times}$ meets the unit group $\mathrm{Z}_q^{\times} = \mathrm{R}_{\mathrm{Z}_q/\mathrm{S}}(\mathbf{G}_{\mathrm{m}})$ in the group $(\mathrm{Z}_q^{\times})_{\mathrm{lh}}$ of locally homogeneous units of Z_q. The S-group $(\mathrm{Z}_q^{\times})_{\mathrm{lh}}$ is an extension of $(\mathbf{Z}/2\mathbf{Z})_{\mathrm{S}}$ by $\mathbf{G}_{\mathrm{m}}$, consisting of sections that locally lie in either $\mathbf{G}_{\mathrm{m}}$ or the $\mathbf{G}_{\mathrm{m}}$-torsor of local generators of the degree-1 line of Z_q. Thus, the isomorphism $\mathrm{Z}_q \otimes_{\mathscr{O}_{\mathrm{S}}} \mathrm{C}_0(\mathrm{V}, q) = \mathrm{C}(\mathrm{V}, q)$ from Proposition C.2.4 shows that $\mathrm{C}(\mathrm{V}, q)^{\times}_{\mathrm{lh}}$ is an extension of $(\mathbf{Z}/2\mathbf{Z})_{\mathrm{S}}$ by $\mathrm{C}_0(\mathrm{V}, q)^{\times}$ for the Zariski topology. In particular, $\mathrm{C}(\mathrm{V}, q)^{\times}_{\mathrm{lh}}$ is smooth with relative identity component $\mathrm{C}_0(\mathrm{V}, q)^{\times}$ that is a form of GL_{2^m} ($n = 2m + 1$). Note that $\mathrm{Z}_q^{\times}$ is a rank-2 torus over $\mathrm{S}[1/2]$ but it has non-reductive fibers at points in characteristic 2. Local homogeneity for $(\mathrm{Z}_q^\times)_{\mathrm{lh}}$ eliminates the intervention of such non-reductivity in what follows.

Definition C.4.3. — For odd $n \geqslant 1$, the *naive Clifford group* over S is the closed subgroup

$$\mathrm{GPin}'(q) = \{u \in \mathrm{C}(\mathrm{V}, q)^\times \mid u\mathrm{V}u^{-1} = \mathrm{V}\}$$

of $\mathrm{C}(\mathrm{V}, q)^\times$, and the *Clifford group* over S is

$$\mathrm{GPin}(q) = \mathrm{GPin}'(q) \bigcap \mathrm{C}(\mathrm{V}, q)^\times_{\mathrm{lh}} = \mathrm{GPin}'(q) \bigcap \mathrm{C}(\mathrm{V}, q)_{\mathrm{lh}}.$$

Remark C.4.4. — Note that for even n, Definition C.4.3 makes sense and we have seen in the discussion after Definition C.2.6 that the two resulting groups coincide. For odd n they are fiberwise distinct (see Lemma C.4.5). It is $\mathrm{GPin}'(q)$ rather than $\mathrm{GPin}(q)$ that is called the *Clifford group* in [**Chev97**, II, § 3]. For odd n both groups will yield the same "Spin" and "GSpin" groups, but the associated "Pin groups" (see § C.5) will only agree when $n \equiv 1 \bmod 4$ and S is a $\mathbf{Z}[1/2]$-scheme (see Remark C.5.3). In [**Chev97**, II, § 3], the case of odd n is ruled out of consideration in characteristic 2 essentially by definition and Pin groups are not considered.

By the same calculation as with even n before (C.2.4), for odd n and any point u of $\mathrm{GPin}'(q)$ the induced linear automorphism $v \mapsto uvu^{-1}$ of V lies in $\mathrm{O}(q)$, and moreover $\mathrm{Z}_q^\times \subset \ker(\mathrm{GPin}'(q) \to \mathrm{O}(q) = \mu_2 \times \mathrm{SO}(q))$.

Lemma C.4.5. — *For odd n, the inclusion $\mathrm{Z}_q^\times \times^{(\mathrm{Z}_q^\times)_{\mathrm{lh}}} \mathrm{GPin}(q) \subset \mathrm{GPin}'(q)$ is an equality. In particular, $\mathrm{GPin}'(q)/\mathrm{GPin}(q)$ is smooth with geometric fiber $\mathbf{G}_{\mathrm{m}}$ away from characteristic 2 and $\mathbf{G}_{\mathrm{a}}$ in characteristic 2.*

Proof. — Let u' be a point of $\mathrm{GPin}'(q)$ valued in some S-scheme S′, so we may rename S′ as S. Consider the global decomposition $u'_+ + u'_-$ of u' as a sum of even and odd parts in $\mathrm{C}(\mathrm{V}, q)$. We claim that Zariski-locally on S, at least one of u'_+ or u'_- is a unit in $\mathrm{C}(\mathrm{V}, q)$. To prove this we may and do assume $\mathrm{S} = \operatorname{Spec} k$ for an algebraically closed field k. By the classical theory over fields (see [**Chev97**, II.3.2]), $u' = zu$ where u is an element of $\mathrm{GPin}(q)$ and $z \in \mathrm{Z}_q$. It follows that $z \in \mathrm{Z}_q^\times$. Since $\mathrm{C}(\mathrm{V}, q) = \mathrm{Z}_q \otimes \mathrm{C}_0(\mathrm{V}, q)$, we can scale z and u by reciprocal elements of $\mathrm{Z}_q^1 - \{0\}$ if necessary to arrange that u has degree 0. In particular, $u'_\pm = (z_\pm)u$, where $z_\pm$ are the homogeneous components of z. But a nonzero homogeneous element of Z_q is a *unit*, and u lies in $\mathrm{GPin}(q)$, so one of $u_\pm$ is a unit in $\mathrm{C}(\mathrm{V}, q)$.

Over the original S, locally scale u' by a generator of Z_q^1 if necessary so that u'_+ is a unit. For every local section v of V, the local section $\widetilde{v} := u'vu'^{-1}$ of V satisfies $u'_+v + u'_-v = u'v = \widetilde{v}u' = \widetilde{v}u'_+ + \widetilde{v}u'_-$. Comparing odd-degree terms on both sides gives that $u'_+ \in \mathrm{GPin}(q)$, so we may assume $u'_+ = 1$.

Writing $u = 1 + x$ with x of degree 1, the relation $uv = \widetilde{v}u$ for $v \in \mathrm{V}$ and $\widetilde{v} := uvu^{-1} \in \mathrm{V}$ says $v + xv = \widetilde{v} + \widetilde{v}x$. Comparing odd-degree parts implies $\widetilde{v} = v$, so u centralizes V and hence is central in $\mathrm{C}(\mathrm{V}, q)$ (i.e., $u \in \mathrm{Z}_q^\times$). □

Fix *odd* $n \geqslant 1$. The locally constant degree $\deg_q : \mathrm{GPin}(q) \to (\mathbf{Z}/2\mathbf{Z})_{\mathrm{S}}$ has restriction to $(\mathrm{Z}_q^\times)_{\mathrm{lh}}$ that kills the degree-0 part $\mathbf{G}_{\mathrm{m}}$ and induces the unique isomorphism $(\mathrm{Z}_q^\times)_{\mathrm{lh}}/\mathbf{G}_{\mathrm{m}} \simeq (\mathbf{Z}/2\mathbf{Z})_{\mathrm{S}}$. Hence, the *even Clifford group*

$$\mathrm{GSpin}(q) := \ker(\deg_q) \subset \mathrm{GPin}(q)$$

is open and closed, with $\mathrm{GPin}(q)/\mathrm{GSpin}(q) = (\mathbf{Z}/2\mathbf{Z})_{\mathrm{S}}$. Since

$$(\mathrm{Z}_q^\times)_{\mathrm{lh}} \bigcap \mathrm{GSpin}(q) = \mathbf{G}_{\mathrm{m}},$$

we have

$$(\mathrm{Z}_q^\times)_{\mathrm{lh}} \times^{\mathbf{G}_{\mathrm{m}}} \mathrm{GSpin}(q) = \mathrm{GPin}(q). \tag{C.4.3}$$

In particular,

$$\mathrm{GPin}(q)/\mathrm{GSpin}(q) = (\mathrm{Z}_q^\times)_{\mathrm{lh}}/\mathbf{G}_{\mathrm{m}} = (\mathbf{Z}/2\mathbf{Z})_{\mathrm{S}}$$

for odd n, just as we saw for even n in (C.4.1).

For odd n, the calculation (C.2.4) carries over without change and so leads us to define $\pi_q : \mathrm{GPin}(q) \to \mathrm{GL}(\mathrm{V})$ by $\pi_q(u)(v) = (-1)^{\deg_q(u)} uvu^{-1}$ as for even n (so for $u \in \mathrm{V} \subset \mathrm{C}_1(\mathrm{V}, q)$ such that $q(u)$ is a unit, the automorphism $\pi_q(u) : \mathrm{V} \simeq \mathrm{V}$ is the reflection through u relative to q). This is the twist of the conjugation action by the μ_2-valued character given by exponentiating $\deg_q$. Since μ_2-scaling has no effect on q, we see that π_q is valued in $\mathrm{O}(q)$. Also, since $\mathrm{GSpin}(q) \subset \mathrm{C}_0(\mathrm{V}, q)$, the restriction of π_q to $\mathrm{GSpin}(q)$ is the action on V through conjugation inside $\mathrm{C}(\mathrm{V}, q)$.

Proposition C.4.6. — *Fix an odd $n \geqslant 1$.*

1. *The homomorphism $\pi_q : \mathrm{GSpin}(q) \to \mathrm{O}(q) = \mu_2 \times \mathrm{SO}(q)$ factors through $\mathrm{SO}(q)$ and defines a diagram of* S*-groups*

$$1 \to \mathbf{G}_{\mathrm{m}} \to \mathrm{GSpin}(q) \to \mathrm{SO}(q) \to 1$$

that is short exact for the Zariski topology on the category of S*-schemes. In particular,* $\mathrm{GSpin}(q)$ *is* S*-smooth with connected reductive fibers.*

2. *The natural map*

$$(\mathbf{Z}/2\mathbf{Z})_{\mathrm{S}} \times \mathrm{SO}(q) = ((\mathrm{Z}_q^\times)_{\mathrm{lh}}/\mathbf{G}_{\mathrm{m}}) \times (\mathrm{GSpin}(q)/\mathbf{G}_{\mathrm{m}}) \to \mathrm{GPin}(q)/\mathbf{G}_{\mathrm{m}}$$

is an isomorphism (*so* $\mathrm{GPin}(q)$ *is* S*-smooth*). *Composing its inverse with the canonical homomorphism $(\mathbf{Z}/2\mathbf{Z})_{\mathrm{S}} \to \mu_2$ recovers the homomorphism $\mathrm{GPin}(q)/\mathbf{G}_{\mathrm{m}} \to \mu_2 \times \mathrm{SO}(q) = \mathrm{O}(q)$ induced by π_q. In particular, over* $\mathrm{S}[1/2]$ *the* S*-group* $\mathrm{GPin}(q)$ *is an extension of* $\mathrm{O}(q)$ *by* $\mathbf{G}_{\mathrm{m}}$.

Keep in mind that the representation π_q of $\mathrm{GPin}(q)$ on V is the twist of the conjugation action by the quadratic character $u \mapsto (-1)^{\deg_q(u)}$.

Proof. — Since $\mathrm{GPin}(q)$ is defined in terms of $\mathrm{GPin}'(q)$, it will be convenient to formulate an analogue for (1) for $\mathrm{GPin}'(q)$ from which we shall deduce (1):

> $(*)$ The representation $\mathrm{GPin}'(q) \to \mathrm{GL}(\mathrm{V})$ defined by the conjugation action $u.v = uvu^{-1}$ on V inside $\mathrm{C}(\mathrm{V}, q)$ is valued in $\mathrm{SO}(q)$, and defines a diagram
> $$1 \to \mathrm{Z}_q^\times \to \mathrm{GPin}'(q) \to \mathrm{SO}(q) \to 1$$
> that is short exact for the Zariski topology on the category of S-schemes. (In particular, $\mathrm{GPin}'(q)$ is S-smooth with connected fibers that are reductive over $\mathrm{S}[1/2]$ and non-reductive in characteristic 2.)

Rather than directly show that $\mathrm{GPin}'(q)$ is carried into $\mathrm{SO}(q)$, we will first show that all points g of $\mathrm{SO}(q)$ Zariski-locally lift to $\mathrm{GPin}'(q)$ under the conjugation action on V (without the quadratic twist). Since $\mathrm{SO}(q) \subset \mathrm{O}(q)$, by the functoriality of Clifford algebras there exists a unique algebra automorphism $[g]$ of $\mathrm{C}(\mathrm{V}, q)$ extending g on V. Consider the induced automorphism of Z_q. We claim that this is trivial, due to the condition that g lies in $\mathrm{SO}(q)$. The construction $g \mapsto [g]|_{\mathrm{Z}_q}$ defines a homomorphism of S-groups $\mathrm{SO}(q) \to \underline{\mathrm{Aut}}_{\mathrm{Z}_q/\mathrm{S}}$ and we are claiming that this is trivial. By Lemma C.2.1 it suffices to consider $q = q_n$ over $\mathrm{S} = \mathrm{Spec}\,\mathbf{Z}$. Since SO_n is $\mathbf{Z}$-flat, it suffices to treat the problem over $\mathbf{Z}[1/2]$, or even over $\mathbf{Q}$. We have reduced to the case when Z_q is étale over S, so its automorphism scheme is $(\mathbf{Z}/2\mathbf{Z})_{\mathrm{S}}$. Since $\mathrm{SO}(q) \to \mathrm{S}$ is smooth with connected fibers, $\mathrm{Hom}_{\mathrm{S}\text{-gp}}(\mathrm{SO}(q), (\mathbf{Z}/2\mathbf{Z})_{\mathrm{S}}) = 1$.

Now returning to the original base S, since $[g]$ is an automorphism of the Z_q-algebra $\mathrm{C}(\mathrm{V}, q)$ that becomes a matrix algebra over Z_q fppf-locally on S, by the relative Skolem–Noether theorem (as in the proof of Proposition C.2.8) we may Zariski-localize on the base so that $[g]$ is inner. That is, there exists a unit u of $\mathrm{C}(\mathrm{V}, q)$ such that $[g](x) = uxu^{-1}$ for all points x of $\mathrm{C}(\mathrm{V}, q)$. Clearly u is a point of $\mathrm{GPin}'(q)$ that lifts g.

Since we have proved that all points of $\mathrm{SO}(q)$ Zariski-locally lift into $\mathrm{GPin}'(q)$, and $\mathrm{O}(q) = \mu_2 \times \mathrm{SO}(q)$, to show that $\mathrm{GPin}'(q)$ is carried into $\mathrm{SO}(q)$ it suffices to check that if $\zeta \in \mu_2(\mathrm{S}')$ for an S-scheme S' and it arises from $u \in \mathrm{GPin}'(q)(\mathrm{S}')$ then $\zeta = 1$. We may and do rename S' as S. The automorphism $v \mapsto \zeta v$ of V uniquely extends to an algebra automorphism f of $\mathrm{C}(\mathrm{V}, q)$ that is the identity on $\mathrm{C}_0(\mathrm{V}, q)$ and is multiplication by ζ on $\mathrm{C}_1(\mathrm{V}, q)$. By hypothesis, the inner automorphism $x \mapsto uxu^{-1}$ of $\mathrm{C}(\mathrm{V}, q)$ agrees with f on V, so it agrees with f (since V generates $\mathrm{C}(\mathrm{V}, q)$ as an algebra). In particular, f is trivial on the $\mathbf{Z}/2\mathbf{Z}$-graded central subalgebra Z_q. But the odd part Z_q^1 is an invertible sheaf on which f acts as multiplication by ζ, so $\zeta = 1$. This completes the proof of $(*)$.

The representation π_q of $\mathrm{GSpin}(q) = \mathrm{GPin}'(q) \bigcap \mathrm{C}_0(\mathrm{V}, q)^\times$ on V is via conjugation inside $\mathrm{C}(\mathrm{V}, q)$ (the quadratic twist is trivial in even degree), so since $\mathrm{Z}_q^\times$ has degree-0 part $\mathbf{G}_\mathrm{m}$ we can deduce (1) from $(*)$ once we show that the quotient map $\mathrm{GPin}'(q) \to \mathrm{SO}(q)$ in $(*)$ carries $\mathrm{GSpin}(q)$ onto $\mathrm{SO}(q)$ Zariski-locally on S. By Lemma C.4.5, $\mathrm{GPin}(q)$ is carried onto $\mathrm{SO}(q)$ under the Zariski-quotient map in $(*)$. Since scaling by a Zariski-local generator of Z_q^1 swaps $\mathrm{GSpin}(q)$ with the fiber of $\mathrm{GPin}(q) \to (\mathbf{Z}/2\mathbf{Z})_\mathrm{S}$ over 1, it follows via (C.4.3) that $\mathrm{GSpin}(q)$ is also carried onto $\mathrm{SO}(q)$ as required in (1).

The initial isomorphism assertion in (2) is now clear, and for the rest it remains to analyze the S-homomorphism

$$(\mathrm{Z}_q^\times)_\mathrm{lh} \xrightarrow{\pi_q} \mathrm{O}(q) = \mu_2 \times \mathrm{SO}(q).$$

The kernel contains $\mathbf{G}_\mathrm{m}$, so it induces an S-homomorphism

$$(\mathbf{Z}/2\mathbf{Z})_\mathrm{S} = (\mathrm{Z}_q^\times)_\mathrm{lh}/\mathbf{G}_\mathrm{m} \to \mu_2 \times \mathrm{SO}(q).$$

Our problem is to show that the second component of this map is trivial and that the first component is the canonical S-homomorphism $(\mathbf{Z}/2\mathbf{Z})_\mathrm{S} \to \mu_2$. In other words, we have to show that a local generator of Z_q^1 is carried to $-1 \in \mathrm{O}(q)(\mathrm{S})$ under π_q. But this is obvious, since π_q on $\mathrm{GPin}(q)$ is defined as the quadratic twist of conjugation inside $\mathrm{C}(\mathrm{V}, q)$ against $u \mapsto (-1)^{\deg_q(u)}$. $\square$

For odd n, Proposition C.4.6 shows that $\mathrm{GSpin}(q)$ is the relative identity component of the smooth S-group $\mathrm{GPin}(q)$ (which we took as the *definition* of $\mathrm{GSpin}(q)$ for even n), and our definition $\mathrm{GSpin}(q) = \mathrm{C}_0(\mathrm{V}, q) \bigcap \mathrm{GPin}(q)$ for odd n was earlier seen to be valid for even n. Hence, our descriptions of the relationships between $\mathrm{GPin}(q)$ and $\mathrm{GSpin}(q)$ for even and odd n are consistent.

The fibral connectedness of the naive Clifford group $\mathrm{GPin}'(q)$ for odd n forces us to use the grading if we wish to define $\mathrm{GSpin}'(q)$ for any $n \geqslant 1$: it is $\mathrm{C}_0(\mathrm{V}, q) \bigcap \mathrm{GPin}'(q)$. But this is $\mathrm{GSpin}(q)$, so it provides nothing new.

Corollary C.4.7. — *Assume $n \geqslant 1$ is odd. The subgroup* $\mathrm{GSpin}(q) \subset \mathrm{GPin}(q)$ *is a central extension of* $\mathrm{SO}(q)$ *by* $\mathbf{G}_\mathrm{m}$*, and*

$$\mathrm{GPin}(q) = (\mathrm{Z}_q^\times)_\mathrm{lh} \times^{\mathbf{G}_\mathrm{m}} \mathrm{GSpin}(q), \;\; \mathrm{GPin}'(q) = \mathrm{Z}_q^\times \times^{\mathbf{G}_\mathrm{m}} \mathrm{GSpin}(q).$$

This result is a relative version of [**Chev97**, II.3.2].

Proof. — Since $\mathrm{Z}_q^\times$ has degree-0 part $\mathbf{G}_\mathrm{m}$ and local bases of Z_q^1 are units, Lemma C.4.5 and Proposition C.4.6 yield the assertions immediately. $\square$

In the remainder of § C.4 we are primarily interested in odd n (e.g., to construct the simply connected central cover of $\mathrm{SO}(q)$ for odd $n \geqslant 3$), so for ease of notation through the end of § C.4 we denote by Z_q the center of the *entire* algebra $\mathrm{C}(\mathrm{V}, q)$ regardless of the parity of n. For even n this is *not* the center of $\mathrm{C}_0(\mathrm{V}, q)$ (in contrast with Proposition C.2.2), but since our main focus is on odd n this will not create confusion.

For any $n \geqslant 1$, the *main anti-involution* α of $\mathrm{C}(\mathrm{V}, q)$ is defined as in the proof of Lemma C.4.1: it is the unique anti-involution that restricts to the identity on V, which is to say that it carries $v_1 \cdots v_r$ to $v_r \cdots v_1$. For any $u \in \mathrm{GPin}'(q)$, the same calculation as in the proof of Lemma C.4.1 (but setting $\varepsilon_u = 1$ and $h(v) = uvu^{-1}$) shows that $\alpha(u)u \in \mathrm{Z}_q^{\times}$ and that the resulting S-morphism

$$\nu_q' : \mathrm{GPin}'(q) \to \mathrm{Z}_q^{\times}$$

given by $u \mapsto \alpha(u)u = u\alpha(u)$ is a homomorphism of S-groups; it is called the *unrestricted Clifford norm.* When n is even, so $\mathrm{Z}_q = \mathscr{O}_{\mathrm{S}}$, ν_q' restricts to the squaring map on $\mathrm{Z}_q^{\times}$. When n is odd, we have:

Lemma C.4.8. — *For odd $n \geqslant 1$, $\alpha : \mathrm{Z}_q \to \mathrm{Z}_q$ is the identity map when $n \equiv 1 \bmod 4$ and is the canonical algebra involution $z \mapsto \mathrm{Tr}_{\mathrm{Z}_q/\mathscr{O}_{\mathrm{S}}}(z) - z$ when $n \equiv 3 \bmod 4$. In particular, ν_q' is the squaring map on $\mathrm{Z}_q^{\times}$ when $n \equiv 1 \bmod 4$ and it is the restriction of the algebra norm $\mathrm{Z}_q \to \mathscr{O}_{\mathrm{S}}$ when $n \equiv 3 \bmod 4$.*

Proof. — By working fppf-locally on S, we may assume $q = q_n$ for $n = 2m+1$ with $m \geqslant 0$. Hence, if $\{e_0, \ldots, e_{2m}\}$ denotes the standard basis, then we saw in the proof of Proposition C.2.4 that $\mathrm{Z}_q = \mathscr{O}_{\mathrm{S}} \oplus \mathscr{O}_{\mathrm{S}} z_0 = \mathscr{O}_{\mathrm{S}}[t]/(t^2 - 1)$, where

$$z_0 = e_0 \prod_{i=1}^{m} (1 - 2e_{2i-1}e_{2i})$$

and $e_{2i-1}e_{2i} + e_{2i}e_{2i-1} = 1$. The main anti-involution α carries e_j to e_j, so $\alpha(1-2e_{2i-1}e_{2i}) = 1-2e_{2i}e_{2i-1} = 1-2(1-e_{2i-1}e_{2i}) = -(1-2e_{2i-1}e_{2i})$. Hence, $\alpha(z_0) = (-1)^m z_0$, so α is the identity on Z_q for even m and is the canonical involution $x \mapsto \mathrm{Tr}_{\mathrm{Z}_q/\mathscr{O}_{\mathrm{S}}}(x) - x$ for odd m. $\square$

The analogue of Lemma C.4.8 for even n was seen early in the proof of Lemma C.4.1: for even $n \geqslant 2$, the effect of α on the rank-2 finite étale center of $\mathrm{C}_0(\mathrm{V}, q)$ is the identity when $4|n$ and is the unique fiberwise nontrivial automorphism when $n \equiv 2 \bmod 4$.

We are primarily interested in the restriction ν_q of ν_q' to the subgroup $\mathrm{GPin}(q)$ of locally homogeneous points u for any $n \geqslant 1$. For such u, $\nu_q(u)$ is the product of the locally homogeneous points u and $\alpha(u)$ of the same degree, so $\nu_q : u \mapsto u\alpha(u)$ is valued in the degree-0 part $\mathbf{G}_{\mathrm{m}}$ of $\mathrm{Z}_q^{\times}$. This is the *Clifford norm*

$$\nu_q : \mathrm{GPin}(q) \to \mathbf{G}_{\mathrm{m}} \tag{C.4.4}$$

for any $n \geqslant 1$.

Remark C.4.9. — As in Remark C.4.2 with even n, for any $n \geqslant 1$ there is an anti-involution α^- of $\mathrm{C}(\mathrm{V}, q)_{\mathrm{lh}}$ defined similarly to α except with a sign twist on the odd part. This yields a variant ν_q^- of (C.4.4) that agrees with ν_q on the even part $\mathrm{GSpin}(q)$. Recall from (C.4.1) for even n and from Proposition C.4.6(2) for odd n that $\mathrm{GPin}(q)/\mathrm{GSpin}(q)$ is identified with $(\mathbf{Z}/2\mathbf{Z})_{\mathrm{S}}$ via the restriction of $\deg_q : \mathrm{C}(\mathrm{V}, q)_{\mathrm{lh}} \to (\mathbf{Z}/2\mathbf{Z})_{\mathrm{S}}$. It is the map ν_q^- that is used in [**ABS**, (3.7)] and [**Knus**, IV, § 6.1], whereas ν_q is used in [**Fr**, App. I]

We claim that ν_q^- is obtained from ν_q via multiplication against $(-1)^{\deg_q}$. To prove this claim we may work Zariski-locally on S so that there exists $v_0 \in \mathrm{V}(\mathrm{S})$ satisfying $q(v_0) \in \mathscr{O}(\mathrm{S})^\times$. The description of $\mathrm{GPin}(q)/\mathrm{GSpin}(q)$ implies that $\mathrm{GPin}(q)$ is generated (for the étale or fppf topolgies) by $\mathrm{GSpin}(q)$ and a single such v_0. But ν_q^- and ν_q agree on $\mathrm{GSpin}(q)$ since α and α^- agree on $\mathrm{C}_0(\mathrm{V}, q)^\times$, so comparing ν_q^- and ν_q up to the desired quadratic twist amounts to comparing their values on v_0. By definition, $\nu_q(v_0) = v_0^2 = q(v_0)$ whereas $\nu_q^-(v_0) = -q(v_0)$. Their ratio is $-1 = (-1)^{\deg_q(v_0)}$.

Now assume $n = 2m + 1 \geqslant 3$ (so $m \geqslant 1$). The group $\mathrm{SO}(q)$ is semisimple of type B_m, so the derived group

$$\mathrm{Spin}(q) := \mathscr{D}(\mathrm{GSpin}(q))$$

is a semisimple S-group that is a central extension of $\mathrm{SO}(q)$ by a finite fppf subgroup of $\mathbf{G}_{\mathrm{m}}$ (Proposition 6.1.10). The group $\mathrm{SO}(q)$ is adjoint since n is odd, and calculations with B_m show that the simply connected central cover of $\mathrm{SO}(q)$ has degree 2. We can now adapt arguments in the proof of Lemma C.4.1 to show that the central isogeny $\pi_q : \mathrm{Spin}(q) \to \mathrm{SO}(q)$ has kernel equal to $\mu_2 \subset \mathbf{G}_{\mathrm{m}}$, so this is the simply connected central cover:

Proposition C.4.10. — *Assume $n \geqslant 3$ is odd. The map π_q identifies* $\mathrm{Spin}(q)$ *with the simply connected central cover of* $\mathrm{SO}(q)$.

Proof. — By Lemma C.2.1, it suffices to treat the case $q = q_n$ over $\mathbf{Z}$. The finite multiplicative type kernel $\ker \pi_q$ is flat and so has constant degree that must be 1 or 2, so we just have to rule out the possibility of degree 1. Equivalently, it suffices to prove that over a field k of characteristic $\neq 2$ (or even just algebraically closed of characteristic 0), the *central* subgroup $\mu_2 \subset \mathbf{G}_{\mathrm{m}}$ in $\mathrm{GSpin}(q)$ is contained in the derived group $\mathrm{Spin}(q)$.

The first step is to reduce to the case $n = 3$. We have $n = 2m + 1 \geqslant 3$ and $q = q_n$. Let $\mathrm{V}_3 \subset \mathrm{V} = k^n$ be the span of $\{e_0, e_1, e_2\}$ (so $q|_{\mathrm{V}_3} = q_3$) and let V' be the span of $\{e_3, \ldots, e_{2m}\}$ (so $\mathrm{V}' = 0$ if $n = 3$). The inclusion $(\mathrm{V}_3, q_3) \hookrightarrow (\mathrm{V}, q)$ induces an injective homomorphism of $\mathbf{Z}/2\mathbf{Z}$-graded algebras $j : \mathrm{C}(q_3) \to \mathrm{C}(\mathrm{V}, q)$. Since V_3 is orthogonal to V', the *even* subalgebra $\mathrm{C}_0(q_3)$ centralizes $\mathrm{C}(\mathrm{V}', q|_{\mathrm{V}'}) \subset \mathrm{C}(\mathrm{V}, q)$ because for $i \in \{0, 1, 2\}$ and $i' > 2$ we have $e_i e_{i'} = -e_{i'} e_i$ (treating $i = 0$ separately from $i = 1, 2$, using that $q(e_0 + e_{i'}) = 1$

and $q(e_0) = 1$). Hence, a unit in $\mathrm{C}_0(q_3)^\times$ whose conjugation action on $\mathrm{C}_0(q_3)$ preserves V_3 is carried into the group $\mathrm{GSpin}(q)$ of even units in $\mathrm{C}(\mathrm{V}, q)$ whose conjugation action on $\mathrm{C}(\mathrm{V}, q)$ preserves V. In other words, j carries $\mathrm{GSpin}(q_3)$ into $\mathrm{GSpin}(q)$, so on derived groups it carries $\mathrm{Spin}(q_3)$ into $\mathrm{Spin}(q)$. But j carries $\mathbf{G}_\mathrm{m}$ to $\mathbf{G}_\mathrm{m}$ via the identity map, so to prove that $\mu_2 \subset \mathrm{Spin}(q)$ it suffices to treat the case of q_3. That is, we may and do now assume $n = 3$.

If $\mathrm{Spin}(q)$ does not contain the central μ_2 then the natural map $\mathrm{Spin}(q) \to \mathrm{SO}(q)$ is an isomorphism. The inverse would provide a section to $\mathrm{GSpin}(q) \to \mathrm{SO}(q)$ that identifies $\mathrm{GSpin}(q)$ with $\mathbf{G}_\mathrm{m} \times \mathrm{SO}(q)$. In particular, the kernel of the Clifford norm $\nu_q : \mathrm{GSpin}(q) \to \mathbf{G}_\mathrm{m}$ would equal $\mu_2 \times \mathrm{SO}(q)$, which is disconnected (since $\mathrm{char}(k) \neq 2$). Hence, it suffices to prove that the kernel of ν_q on $\mathrm{GSpin}(q)$ is irreducible.

For the standard basis $\{e_0, e_1, e_2\}$ of V_3, let $e = e_0e_1, e' = e_0e_2, e'' = e_1e_2$, so $\{1, e, e', e''\}$ is a k-basis of $\mathrm{C}_0(q_3)$. We claim that the inclusion $\mathrm{GSpin}(q_3) \subset \mathrm{C}_0(q_3)^\times$ between k-groups is an equality. Since $\mathrm{GSpin}(q_3)$ is smooth and connected of dimension 4 (a central extension of SO_3 by $\mathbf{G}_\mathrm{m}$), it suffices to show the same for $\mathrm{C}_0(q_3)^\times$. For any finite-dimensional associative algebra A over a field F, the associated F-group $\underline{\mathrm{A}}^\times$ of units is defined by the non-vanishing on the affine space $\underline{\mathrm{A}}$ of the determinant of the left multiplication action of A on itself. In particular, $\underline{\mathrm{A}}^\times$ is smooth and connected of dimension equal to $\dim_\mathrm{F} \mathrm{A}$. Hence, $\mathrm{C}_0(q_3)^\times$ is smooth and connected of dimension 4 as desired.

Computing as in Remark C.2.3, the algebra structure on $\mathrm{C}_0(q_3)$ is determined by the relations

$$e^2 = e'^2 = 0, e''^2 = e'', ee' = -e'', e'e = e'' - 1,$$
$$e'e'' = e', e''e' = 0, ee'' = 0, e''e = e,$$

and the restriction to $\mathrm{C}_0(q_3)$ of the main anti-involution α of $\mathrm{C}(q_3)$ is given by

$$u = t + ae + be' + ce'' \mapsto (t + c) - ae - be' - ce'' =: u^*.$$

Using the above relations, we compute that $\nu(u) := u^*u = ab + t^2 + ct$. By inspection, $\nu - 1$ is an irreducible polynomial in $k[t, a, b, c]$ and hence its non-unit restriction over the open subset $\mathrm{GSpin}(q_3)$ in the affine 4-space $\mathrm{C}_0(q_3)$ (which has zero locus $\mathrm{Spin}(q_3)$) has irreducible zero locus. □

Fix an odd $n \geqslant 1$. If $n \geqslant 3$ then $\mu_2 \subset \mathrm{Spin}(q)$ and this must exhaust the order-2 center, so $\mathrm{GSpin}(q) = \mathrm{Spin}(q) \times^{\mu_2} \mathbf{G}_\mathrm{m}$ (by Proposition C.4.6(1)) and $\mathrm{Spin}(q) = \ker(\nu_q|_{\mathrm{GSpin}(q)})$. Hence, Corollary C.4.7 gives

$$\mathrm{GPin}(q) = \mathrm{Spin}(q) \times^{\mu_2} (\mathrm{Z}_q^\times)_{\mathrm{lh}}, \ \ \mathrm{GPin}'(q) = \mathrm{Spin}(q) \times^{\mu_2} \mathrm{Z}_q^\times.$$

These hold if $n = 1$ by *defining* $\mathrm{Spin}(q) := \mu_2$ inside $\mathrm{GSpin}(q) = \mathbf{G}_\mathrm{m}$ if $n = 1$.

Remark C.4.11. — Using notation as in the setting of Remark C.2.9 (which is applicable whenever (V, q) is split), we obtain vector bundle representations of the even Clifford group $\mathrm{GSpin}(q) \subset \mathrm{C}_0(\mathrm{V}, q)$ on which the central $\mathbf{G}_{\mathrm{m}}$ acts as ordinary scaling: if n is even then use the vector bundles A_+ and A_-, and if n is odd then use the vector bundle A given by the exterior algebra of W. The kernel $\mathrm{Spin}(q)$ of the Clifford norm $\mathrm{GSpin}(q) \to \mathbf{G}_{\mathrm{m}}$ is a central extension of $\mathrm{O}(q)$ by μ_2. Thus, the central μ_2 acts by ordinary scaling (hence fiberwise nontrivially) for the restriction to $\mathrm{Spin}(q)$ of the action of $\mathrm{C}_0(\mathrm{V}, q)^\times$ on $\mathrm{A}_\pm$ when n is even and on A when n is odd. The actions on A_+ and A_- for even n are the *half-spin* representations of $\mathrm{Spin}(q)$, and the action on A for odd n is the *spin* representation of $\mathrm{Spin}(q)$; these do *not* factor through $\mathrm{SO}(q)$.

C.5. Pin groups and spinor norm. — For any $n \geqslant 1$, the *Pin group* $\mathrm{Pin}(q)$ is the kernel of the Clifford norm $\nu_q : \mathrm{GPin}(q) \to \mathbf{G}_{\mathrm{m}}$ as in (C.4.4). For odd $n \geqslant 1$, the *naive Pin group* $\mathrm{Pin}'(q)$ is the kernel of the unrestricted Clifford norm $\nu'_q : \mathrm{GPin}'(q) \to \mathrm{Z}_q^\times$ (so $\mathrm{Pin}(q) \subset \mathrm{Pin}'(q)$).

Remark C.5.1. — For the modified Clifford norm ν_q^- as in Remark C.4.9, its kernel is an S-subgroup $\mathrm{Pin}^-(q)$ of $\mathrm{GPin}(q)$ distinct from $\mathrm{Pin}(q)$ in general. For example, if S is over $\mathbf{Z}[1/2]$ and $v \in \mathrm{V}(\mathrm{S})$ satisfies $q(v) = 1$ then $v \in \mathrm{Pin}(q)(\mathrm{S})$ but $v \notin \mathrm{Pin}^-(q)(\mathrm{S})$ (since $-q(v) = -1 \neq 1$). Since ν_q and ν_q^- agree on the S-group $\mathrm{C}_0(\mathrm{V}, q)^\times$, we have $\mathrm{Pin}^-(q) \cap \mathrm{C}_0(\mathrm{V}, q) = \mathrm{Spin}(q)$. In [**ABS**, Thm. 3.11] and [**Knus**, IV, § 6.2] it is $\mathrm{Pin}^-(q)$ that is called the "Pin group" attached to (V, q) whereas in [**Fr**, App. I] the group $\mathrm{Pin}(q)$ is used (denoted there as $\widetilde{\mathrm{Pin}}(q)$ to avoid conflict with the notation in [**ABS**]).

Suppose n is even, so (as we saw early in § C.4) $\mathrm{GSpin}(q)$ is an open and closed subgroup of $\mathrm{GPin}(q)$. The kernel of the Clifford norm on $\mathrm{GSpin}(q)$ is the smooth S-group $\mathrm{Spin}(q)$ (as we saw below Remark C.4.2), so $\mathrm{Spin}(q)$ is an open and closed subgroup of $\mathrm{Pin}(q)$. But $\mathrm{GPin}(q)$ maps onto $\mathrm{O}(q)$ with kernel $\mathbf{G}_{\mathrm{m}}$, and ν_q carries this kernel onto $\mathbf{G}_{\mathrm{m}}$ via $t \mapsto t^2$, so the kernel $\mathrm{Pin}(q)$ of ν_q fits into an fppf central extension

$$1 \to \mu_2 \to \mathrm{Pin}(q) \to \mathrm{O}(q) \to 1. \tag{C.5.1}$$

The subgroup $\mathrm{Spin}(q) \subset \mathrm{Pin}(q)$ is compatibly a central extension of $\mathrm{SO}(q)$ by μ_2 (by Lemma C.4.1 for even $n \geqslant 4$, and direct arguments with the definition of $\mathrm{Spin}(q)$ when $n = 2$), so $\mathrm{Pin}(q)/\mathrm{Spin}(q) = \mathrm{O}(q)/\mathrm{SO}(q) = (\mathbf{Z}/2\mathbf{Z})_{\mathrm{S}}$ when n is even. Hence, $\mathrm{Pin}(q)$ is a smooth S-affine S-group for even n.

For odd n the analogous assertions are more subtle to prove, due to complications in characteristic 2. Suppose $n \geqslant 1$ is odd. By Lemma C.4.8, if $n \equiv 1 \bmod 4$ then

$$\mathrm{Pin}(q) = \mathrm{Spin}(q) \times^{\mu_2} (\mathrm{Z}_q^\times)_{\mathrm{lh}}[2], \quad \mathrm{Pin}'(q) = \mathrm{Spin}(q) \times^{\mu_2} \mathrm{Z}_q^\times[2], \tag{C.5.2}$$

(writing A[2] to denote the 2-torsion in A) whereas if $n \equiv 3 \bmod 4$ then

$$\mathrm{Pin}(q) = \mathrm{Spin}(q) \times^{\mu_2} \ker(\mathrm{N}_{\mathrm{Z}_q/\mathrm{S}}|_{(\mathrm{Z}_q^\times)_{\mathrm{lh}}}), \quad \mathrm{Pin}'(q) = \mathrm{Spin}(q) \times^{\mu_2} \ker(\mathrm{N}_{\mathrm{Z}_q/\mathrm{S}}|_{\mathrm{Z}_q^\times}).$$

Hence, if $n \equiv 1 \bmod 4$ then

$$\mathrm{Pin}(q)/\mathrm{Spin}(q) = (\mathrm{Z}_q^\times)_{\mathrm{lh}}[2]/\mu_2, \;\; \mathrm{Pin}'(q)/\mathrm{Spin}(q) = \mathrm{Z}_q^\times[2]/\mu_2,$$

whereas if $n \equiv 3 \bmod 4$ then

$$\mathrm{Pin}(q)/\mathrm{Spin}(q) = \ker(\mathrm{N}_{\mathrm{Z}_q/\mathrm{S}}|_{(\mathrm{Z}_q^\times)_{\mathrm{lh}}})/\mu_2, \;\; \mathrm{Pin}'(q)/\mathrm{Spin}(q) = \ker(\mathrm{N}_{\mathrm{Z}_q/\mathrm{S}}|_{\mathrm{Z}_q^\times})/\mu_2.$$

The norm map $\mathrm{N}_{\mathrm{Z}_q/\mathrm{S}} : \mathrm{Z}_q^\times \to \mathbf{G}_{\mathrm{m}}$ is an fppf surjection whose kernel is a torus over S[1/2] and has geometric fiber $\mathbf{G}_{\mathrm{a}}$ in characteristic 2. Hence, if $n \equiv 3 \bmod 4$ then $\mathrm{Pin}'(q)$ is smooth. We will see later that if $n \equiv 1 \bmod 4$ then $\mathrm{Pin}'(q_n)$ is not even $\mathbf{Z}$-flat (due to dimension-jumping of the fiber at $\mathrm{Spec}\,\mathbf{F}_2$).

Proposition C.5.2. — *Fix an odd $n \geqslant 1$. The quotient* $\mathrm{Pin}(q)/\mathrm{Spin}(q)$ *is identified with* $(\mathbf{Z}/2\mathbf{Z})_{\mathrm{S}}$ (*so* $\mathrm{Pin}(q)$ *is smooth*) *and there is a natural central extension*

$$1 \to \mu_2 \to \mathrm{Pin}(q) \to (\mathbf{Z}/2\mathbf{Z})_{\mathrm{S}} \times \mathrm{SO}(q) \to 1$$

whose restriction over S[1/2] *coincides with the restriction to* $\mathrm{Pin}(q)|_{\mathrm{S}[1/2]}$ *of* $\pi_q : \mathrm{GPin}(q) \to \mathrm{O}(q) = \mu_2 \times \mathrm{SO}(q)$ *via the unique* S[1/2]*-isomorphism* $\mu_2 \simeq (\mathbf{Z}/2\mathbf{Z})_{\mathrm{S}[1/2]}$

Together with our preceding considerations for even n, we conclude that $\mathrm{Pin}(q)/\mathrm{Spin}(q) = (\mathbf{Z}/2\mathbf{Z})_{\mathrm{S}}$ for *all* n (proved in [**Knus**, IV, (6.4.1)] for $\mathrm{Pin}^-(q)$) and $\pi_q : \mathrm{Pin}(q) \to \mathrm{O}(q) = \mu_2 \times \mathrm{SO}(q)$ is not an fppf quotient map on fibers in characteristic 2 when n is odd.

Proof. — First assume $n \equiv 1 \bmod 4$. The exact sequence

$$1 \to \mathbf{G}_{\mathrm{m}} \to (\mathrm{Z}_q^\times)_{\mathrm{lh}} \to (\mathbf{Z}/2\mathbf{Z})_{\mathrm{S}} \to 1$$

for the fppf topology (for any odd n) identifies $(\mathrm{Z}_q^\times)_{\mathrm{lh}}[2]/\mu_2$ with $(\mathbf{Z}/2\mathbf{Z})_{\mathrm{S}}$, so if $n \equiv 1 \bmod 4$ then $\mathrm{Pin}(q)/\mathrm{Spin}(q) = (\mathbf{Z}/2\mathbf{Z})_{\mathrm{S}}$ by (C.5.2).

Assume $n \equiv 3 \bmod 4$. Under the identification of $(\mathrm{Z}_{q_n}^\times)_{\mathrm{lh}}$ with $\mathbf{G}_{\mathrm{m}} \times (\mathbf{Z}/2\mathbf{Z})_{\mathrm{S}}$ as in the proof of Lemma C.4.8, $\mathrm{N}_{\mathrm{Z}_{q_n}/\mathrm{S}} : (\mathrm{Z}_{q_n}^\times)_{\mathrm{lh}} \to \mathbf{G}_{\mathrm{m}}$ is identified with the product of the squaring map on $\mathbf{G}_{\mathrm{m}}$ and the canonical map $(\mathbf{Z}/2\mathbf{Z})_{\mathrm{S}} \to \mu_2 \hookrightarrow \mathbf{G}_{\mathrm{m}}$, so $\ker(\mathrm{N}_{\mathrm{Z}_{q_n}/\mathrm{S}}|_{(\mathrm{Z}_{q_n}^\times)_{\mathrm{lh}}})$ is identified with the subgroup of $\mathbf{G}_{\mathrm{m}} \times (\mathbf{Z}/2\mathbf{Z})_{\mathrm{S}}$ consisting of (ζ, c) such that $\zeta^2 = (-1)^c$. Hence, the finite subgroup scheme

$$\mathrm{Pin}(q)/\mathrm{Spin}(q) \subset (\mu_4/\mu_2) \times (\mathbf{Z}/2\mathbf{Z})_{\mathrm{S}} \simeq \mu_2 \times (\mathbf{Z}/2\mathbf{Z})_{\mathrm{S}}$$

is (uniquely) isomorphic to $(\mathbf{Z}/2\mathbf{Z})_{\mathrm{S}}$ since $\mu_4/\mu_2 \simeq \mu_2$ via $\zeta \mapsto \zeta^2$. □

Remark C.5.3. — The analogous result for $\mathrm{Pin}'(q)$ in place of $\mathrm{Pin}(q)$ is that $\mathrm{Pin}'(q)/\mathrm{Spin}(q)$ has geometric fiber $\mathbf{G}_{\mathrm{a}}$ in characteristic 2 and restriction over $\mathrm{S}[1/2]$ that is $(\mathbf{Z}/2\mathbf{Z})_{\mathrm{S}}$ if $n \equiv 1 \bmod 4$ but is a rank-1 torus if $n \equiv 3 \bmod 4$. In particular, for odd n, $\mathrm{Pin}(q) = \mathrm{Pin}'(q)$ if and only if $n \equiv 1 \bmod 4$ and S is a $\mathbf{Z}[1/2]$-scheme.

To prove these assertions, for $n \equiv 1 \bmod 4$ the same exact sequence argument as in the preceding proof shows that $\mathrm{Pin}'(q)/\mathrm{Spin}(q) = \mathrm{Z}_q^{\times}[2]/\mu_2 = (\mathrm{Z}_q^{\times}/\mathbf{G}_{\mathrm{m}})[2]$. Over $\mathrm{S}[1/2]$ this is uniquely isomorphic to the constant group $(\mathbf{Z}/2\mathbf{Z})$ but its geometric fibers in characteristic 2 are $\mathbf{G}_{\mathrm{a}}$. For $n \equiv 3 \bmod 4$, the quotient $\mathrm{Pin}'(q)/\mathrm{Spin}(q) = \ker(\mathrm{N}_{\mathrm{Z}_q/\mathrm{S}}|_{\mathrm{Z}_q^{\times}})/\mu_2$ is a rank-1 torus over $\mathrm{S}[1/2]$ and has geometric fiber $\mathbf{G}_{\mathrm{a}}$ in characteristic 2 since Z_q is étale of rank 2 over $\mathrm{S}[1/2]$ and has geometric fiber algebra $k[x]/(x^2-1)$ in characteristic 2.

The jumping of fiber dimension implies that for $n \equiv 1 \bmod 4$ the naive Pin group $\mathrm{Pin}'_n := \mathrm{Pin}'(q_n)$ over $\mathbf{Z}$ is not flat at $\mathrm{Spec}\,\mathbf{F}_2$ (whereas we have seen that Pin'_n is smooth when $n \equiv 3 \bmod 4$).

Pin groups exhibit some subtleties under unit-scaling of q, as follows. For $c \in \mathscr{O}(\mathrm{S})^{\times}$, $\mathrm{C}(\mathrm{V}, cq)$ is not easily related to $\mathrm{C}(\mathrm{V}, q)$ when c is a non-square but the equality $\mathrm{O}(cq) = \mathrm{O}(q)$ inside $\mathrm{GL}(\mathrm{V})$ implies $\mathrm{SO}(cq) = \mathrm{SO}(q)$. Over the latter equality there is a unique isomorphism $\mathrm{Spin}(cq) \simeq \mathrm{Spin}(q)$ as simply connected central extensions by μ_2. However, if S is local and we assume n is even in case of residue characteristic 2 (so $\mathrm{Pin}(q)$ is a central extension of $\mathrm{O}(q) = \mathrm{O}(cq)$ by μ_2) then $\mathrm{Pin}(cq)$ and $\mathrm{Pin}(q)$ are never S-isomorphic as central extensions when c is not a square in $\mathscr{O}(\mathrm{S})^{\times}$. This rests on the *spinor norm*, as we explain in Example C.5.5.

Likewise, for S over $\mathbf{Z}[1/2]$, the groups $\mathrm{Pin}^-(q)$ and $\mathrm{Pin}(cq)$ are *never* isomorphic as central extensions of $\mathrm{O}(q) = \mathrm{O}(cq)$ by μ_2 (even though their relative identity components are uniquely isomorphic over $\mathrm{SO}(q) = \mathrm{SO}(cq)$ and for $c = -1$ they meet V in the same locus $\{q = -1\}$ of non-isotropic vectors inside the respective Clifford algebras $\mathrm{C}(\mathrm{V}, q)$ and $\mathrm{C}(\mathrm{V}, -q)$). It suffices to check this on geometric fibers, so consider $\mathrm{S} = \mathrm{Spec}(k)$ for a field k with $\mathrm{char}(k) \neq 2$. The reflection $r_v \in \mathrm{O}(q)(k)$ through non-isotropic v has preimages in $\mathrm{Pin}^-(q)(\overline{k})$ and $\mathrm{Pin}(cq)(\overline{k})$ respectively identified with $\{\pm v/\sqrt{-q(v)}\}$ and $\{\pm v/\sqrt{cq(v)}\}$ in the Clifford algebras $\mathrm{C}(\mathrm{V}, q)$ and $\mathrm{C}(\mathrm{V}, cq)$. The squares of the elements in these preimages are equal to -1 and 1 in $\mu_2(k)$ respectively, so elements of the preimages have respective orders 4 and 2. For example, if $\dim \mathrm{V} = 1$ then the k-groups $\mathrm{Pin}^-(q)$ and $\mathrm{Pin}(cq)$ are each finite étale of order 4, but the first of these is cyclic whereas the second is 2-torsion.

We will see below that over any field k, with n even if $\mathrm{char}(k) = 2$, the central extensions $\mathrm{Pin}^-(q)$ and $\mathrm{Pin}(-q)$ of $\mathrm{O}(q) = \mathrm{O}(-q)$ by μ_2 yield the *same* connecting homomorphism $\mathrm{O}(q)(k) \to \mathrm{H}^1(k, \mu_2)$. However, the preceding shows that these central extensions are *not* k-isomorphic when $\mathrm{char}(k) \neq 2$.

Remark C.5.4. — Suppose $(\mathrm{V}, q) = (k^n, \sum x_j^2)$ over a field k with $\mathrm{char}(k) \neq 2$. Assume $q(k^\times) \subset (k^\times)^2$, as when $k = \overline{k}$ or $k = \mathbf{R}$ with positive-definite q, so $\mathrm{Pin}(q)(k) \to \mathrm{O}(q)(k)$ is surjective (since $\mathrm{O}(q)(k)$ is generated by reflections r_v in non-isotropic vectors $v \in \mathrm{V}$ [**Chev97**, I.5.1], and $v/\sqrt{q(v)}$ is a lift of r_v with $\sqrt{q(v)} \subset k^\times$). Under the permutation representation $\mathfrak{S}_n \to \mathrm{O}(q)(k)$ of the symmetric group, the central extension $\mathrm{Pin}(q)(k)$ of $\mathrm{O}(q)(k)$ by $\mu_2(k)$ pulls back to a central extension E_n of $\mathfrak{S}_n$ by $\{1, -1\}$. Elements of E_n lying over a transposition visibly have order 2, and if $n \geqslant 4$ then elements of E_n lying over a product of two transpositions with disjoint support have order 4, so E_n is the central extension denoted as $\widetilde{\mathfrak{S}}_n$ in [**Ser84**, § 1.5].

By contrast, the surjective $\mathrm{Pin}^-(-q)(k) \to \mathrm{O}(-q)(k) = \mathrm{O}(q)(k)$ pulls back to a central extension of $\mathfrak{S}_n$ by $\{1, -1\}$ in which elements lying over any transposition have order 4. The quadratic form $-q$ on $\mathbf{R}^n$ is used in [**ABS**, § 2] because $\mathrm{Spin}(\mathbf{R}^n, -q)$ is the anisotropic $\mathbf{R}$-form of Spin_n, so [**ABS**] uses the Pin^- construction resting on the signed Clifford norm because $\mathrm{Pin}^-(-q)(\mathbf{R}) \to \mathrm{O}(-q)(\mathbf{R})$ is surjective whereas $\mathrm{Pin}(-q)(\mathbf{R}) \to \mathrm{O}(-q)(\mathbf{R})$ is not surjective (due to obstructions provided by the spinor norm, as explained below).

As a prelude to defining the spinor norm, observe that for (V, q) with rank $n \geqslant 1$ over a scheme S, if $v_0 \in \mathrm{V}(\mathrm{S})$ satisfies $q(v_0) = 1$ then $\nu_q(v_0) = v_0^2 = 1$. Hence, v_0 belongs to the group $\mathrm{Pin}(q)(\mathrm{S})$ of S-points of the kernel of the Clifford norm ν_q on $\mathrm{GPin}(q)$, and it lies over the reflection $r_{v_0} \in \mathrm{O}(q)(\mathrm{S})$.

More generally, assuming S is a $\mathbf{Z}[1/2]$-scheme when n is odd (but arbitrary when n is even), the structure on $\mathrm{Pin}(q)$ as a *central* extension of $\mathrm{O}(q)$ by μ_2 yields a connecting homomorphism (called the *spinor norm*)

$$\mathrm{sp}_q : \mathrm{O}(q)(\mathrm{S}) \to \mathrm{H}^1(\mathrm{S}, \mu_2)$$

that carries the reflection r_v through $v \in q^{-1}(\mathbf{G}_{\mathrm{m}}) \cap \mathrm{V}(\mathrm{S})$ to the class of the μ_2-torsor of square roots of $q(v)$ since over the fppf cover $\mathrm{S}' \to \mathrm{S}$ defined by $t^2 = q(v)$ we have $q(v/t) = 1$ and $r_{v/t} = r_v$. Via the equality $\mathrm{O}(cq) = \mathrm{O}(q)$ inside $\mathrm{GL}(\mathrm{V})$ for $c \in \mathscr{O}(\mathrm{S})^\times$, observe that $\mathrm{sp}_{cq}(r_v) = [c] \cdot \mathrm{sp}_q(r_v)$ where $[c]$ is the image of c under $\mathscr{O}(\mathrm{S})^\times/(\mathscr{O}(\mathrm{S})^\times)^2 \hookrightarrow \mathrm{H}^1(\mathrm{S}, \mu_2)$.

If $\mathrm{S} = \mathrm{Spec}(k)$ for a field k, with n even when $\mathrm{char}(k) = 2$, then $\mathrm{O}(q)(k)$ is generated by such reflections except if $k = \mathbf{F}_2$ with $\dim \mathrm{V} = 4$ [**Chev97**, I.5.1]. Hence, the condition $\mathrm{sp}_q(r_v) = q(v)$ determines sp_q (as $k^\times/(k^\times)^2 = 1$ when $k = \mathbf{F}_2$), and $\mathrm{sp}_{cq} = [c]^{\pi_0} \cdot \mathrm{sp}_q$ for $c \in k^\times$ where $\pi_0 : \mathrm{O}(q) \twoheadrightarrow \mathrm{O}(q)/\mathrm{SO}(q) = \mathbf{Z}/2\mathbf{Z}$ is projection to the component group.

Example C.5.5. — Suppose S is local, with n even in case of residue characteristic 2. Consider $c \in \mathscr{O}(\mathrm{S})^\times$ such that $\mathrm{Pin}(cq) \simeq \mathrm{Pin}(q)$ as central extensions of $\mathrm{O}(cq) = \mathrm{O}(q)$ by μ_2. Pick residually non-isotropic $v \in \mathrm{V}$, so $q(v) \in \mathscr{O}(\mathrm{S})^\times$. It follows that $\mathrm{sp}_{cq}(r_v) = \mathrm{sp}_q(r_v)$. But $\mathrm{sp}_{cq}(r_v) = [c] \cdot \mathrm{sp}_q(r_v)$, so c must be a square on S. Hence, if there exists such an isomorphism of Pin groups as central extensions then c is a square on S (and the converse is obvious).

Example C.5.6. — For fiberwise non-degenerate quadratic spaces (V, q) and (V', q') over S, the Clifford algebra of the orthogonal sum $(\mathrm{V} \oplus \mathrm{V}', q \perp q')$ is naturally isomorphic to the super-graded tensor product of $\mathrm{C}(\mathrm{V}, q)$ and $\mathrm{C}(\mathrm{V}', q')$ (as we noted in the proof of Proposition C.2.4). It follows that if $g \in \mathrm{O}(q)(\mathrm{S})$ and $g' \in \mathrm{O}(q')(\mathrm{S})$ then

$$\mathrm{sp}_{q\perp q'}(g \oplus g') = \mathrm{sp}_q(g)\mathrm{sp}_{q'}(g') \tag{C.5.3}$$

since $g \oplus g' = (g \oplus 1) \circ (1 \oplus g')$ and the inclusion $\mathrm{C}(\mathrm{V}, q) \hookrightarrow \mathrm{C}(\mathrm{V} \oplus \mathrm{V}', q \perp q')$ carries $\mathrm{Pin}(q)$ into $\mathrm{Pin}(q \perp q')$ over the inclusion $\mathrm{O}(q) \hookrightarrow \mathrm{O}(q \perp q')$.

For (V, q) over a field k with $\mathrm{char}(k) \neq 2$, there is a useful formula due to Zassenhaus [**Za**, § 2, Cor.] for spinor norms that does not require expressing $g \in \mathrm{O}(q)(k)$ in terms of reflections. Below we use the structure of Clifford algebras to establish such a formula. See Remark C.5.12 for a discussion of the case $\mathrm{char}(k) = 2$.

Before we state and prove Zassenhaus' result, it is convenient to recall some elementary properties of generalized eigenspaces for orthogonal transformations. Consider $g \in \mathrm{O}(q)(k)$, so the g-action on V is an automorphism that leaves invariant the associated symmetric bilinear form $\mathrm{B}_q(v, v') = q(v + v') - q(v) - q(v')$ (which is non-degenerate, as $\mathrm{char}(k) \neq 2$). For any $\lambda \in \overline{k}^\times$, the generalized λ-eigenspace $\mathrm{V}_{\overline{k}}(\lambda)$ of g on $\mathrm{V}_{\overline{k}}$ is orthogonal to $\mathrm{V}_{\overline{k}}(\mu)$ except possibly when $\lambda\mu = 1$. Indeed, suppose $\lambda^{-1} \neq \mu$ and choose large n so that the operator $(g - \lambda)^n$ on $\mathrm{V}_{\overline{k}}(\lambda)$ vanishes. Note that $g - \lambda^{-1}$ is invertible on $\mathrm{V}_{\overline{k}}(\mu)$, so any $v' \in \mathrm{V}_{\overline{k}}(\mu)$ can be written as $v' = (g - \lambda^{-1})^n(v'')$ for some $v'' \in \mathrm{V}_{\overline{k}}(\mu)$ and hence

$$\begin{aligned}
\mathrm{B}_q(v, v') = \mathrm{B}_q(v, (g - \lambda^{-1})^n(v'')) &= \mathrm{B}_q((g^{-1} - \lambda^{-1})^n(v), v'') \\
&= \mathrm{B}_q((\lambda g)^{-n}(\lambda - g)^n(v), v'') \\
&= 0.
\end{aligned}$$

Since $\mathrm{V}_{\overline{k}}$ is the direct sum of generalized eigenspaces for g, the non-degeneracy of B_q on V implies that $\mathrm{V}_{\overline{k}}(1/\lambda)$ and $\mathrm{V}_{\overline{k}}(\lambda)$ are in perfect duality under B_q for all λ (even if $\lambda = \pm 1$).

Letting $\mathrm{V}_0 \subset \mathrm{V}$ be the generalized -1-eigenspace for g and $\mathrm{V}' \subset \mathrm{V}$ be its g-stable B_q-orthogonal, we conclude that $\mathrm{V}_0 \oplus \mathrm{V}' \to \mathrm{V}$ is an isometry with q non-degenerate on each of V_0 and V'. Define $q_0 = q|_{\mathrm{V}_0}$, $q' = q|_{\mathrm{V}'}$, $g_0 = g|_{\mathrm{V}_0}$, and $g' = g|_{\mathrm{V}'}$. The compatibility (C.5.3) implies

$$\mathrm{sp}_q(g) = \mathrm{sp}_{q_0}(g_0)\mathrm{sp}_{q'}(g').$$

But the effect of $-g$ on V_0 is visibly *unipotent*, and the $k^\times/(k^\times)^2$-valued spinor norm kills unipotent elements of $\mathrm{O}(q)(k)$ because $\mathrm{char}(k) \neq 2$, so $\mathrm{sp}_{q_0}(g_0) = \mathrm{sp}_{q_0}(-1)$. Moreover, the element $g' \in \mathrm{O}(q')(k)$ does not have -1 as an eigenvalue, so the eigenvalues of g' aside from 1 occur in reciprocal pairs with generalized eigenspaces for $\lambda, 1/\lambda \in \overline{k}^\times - \{1,-1\}$ having the same dimension. Hence, $\det(g') = 1$, so $g' \in \mathrm{SO}(q')$. Note that $(\mathrm{V}', q', g') = (\mathrm{V}, q, g)$ when -1 is not an eigenvalue of g (recovering that every $g \in \mathrm{O}(q) - \mathrm{SO}(q)$ has -1 as an eigenvalue).

***Theorem C.5.7* (Zassenhaus).** — *Via the standard representation* $\mathrm{O}(q) \hookrightarrow \mathrm{GL}(\mathrm{V})$ *and the preceding notation,*

$$\mathrm{sp}_q(g) = \mathrm{disc}(q_0) \cdot \det((1+g')/2) \bmod (k^\times)^2.$$

In particular, if $\det(1+g) \neq 0$ *then* $\mathrm{sp}_q(g) = \det((1+g)/2) \bmod (k^\times)^2$.

In view of the preceding calculations, the proof of Theorem C.5.7 reduces to separately treating (V', q', g') and $(\mathrm{V}_0, q_0, -1)$. By renaming each of (V', q') and (V_0, q_0) as (V, q), this amounts to the general identities $\mathrm{sp}_q(-1) = \mathrm{disc}(q)$ and $\mathrm{sp}_q(g) = \det((1+g)/2) \bmod (k^\times)^2$ for any $g \in \mathrm{SO}(q)$ that does not have -1 as an eigenvalue.

Lemma C.5.8. — *For non-degenerate* (V, q) *over a field* k *with* $\mathrm{char}(k) \neq 2$, $\mathrm{sp}_q(-1) = \mathrm{disc}(q)$.

Proof. — Since $\mathrm{char}(k) \neq 2$, we can diagonalize q; i.e., (V, q) is an orthogonal sum of 1-dimensional non-degenerate quadratic spaces. Verifying $\mathrm{sp}_q(-1) = \mathrm{disc}(q)$ therefore reduces to showing that $\mathrm{sp}_{ax^2}(-1) = a \bmod (k^\times)^2$. The $\mathbf{Z}/2\mathbf{Z}$-graded k-algebra

$$\mathrm{C}(k, ax^2) = k[t]/(t^2 - a) = k \oplus kt$$

is commutative and on the algebraic group of homogeneous units the Clifford norm is the squaring map (since $\alpha|_{\mathrm{Z}_q}$ is the identity map when $n \equiv 1 \bmod 4$), on which the kernel $\mathrm{Pin}(ax^2)$ is the functor of points $\{c + c't\}$ where $2cc' = 0$ and $c^2 + ac'^2 = 1$. The fiber of $\pi_{ax^2} : \mathrm{Pin}(ax^2) \to \mathrm{O}(ax^2) = \mu_2$ over -1 is the functor of points $c't$ satisfying $c'^2 = a^{-1}$, so $\mathrm{sp}_{ax^2}(-1)$ is the μ_2-torsor over k classified by a^{-1}; i.e., $\mathrm{sp}_{ax^2}(-1) = a^{-1}(k^\times)^2 = a(k^\times)^2$. □

Now consider $g \in \mathrm{SO}(q)$ with -1 not an eigenvalue of g, so $\det(1+g) \neq 0$. We claim that $\mathrm{sp}_q(g)$ is represented by $\det((1+g)/2)$. The case $n = 1$ is trivial (as then $\mathrm{SO}(q) = 1$), so we assume for the rest of the proof of Theorem C.5.7 that $n \geqslant 2$. The cases of even and odd n will be treated by similar arguments with different details due to how the structure of $\mathrm{C}_0(\mathrm{V}, q)$ depends on the parity of n.

Step 1; Suppose $n = 2m$ is even (with $m \geqslant 1$). By Proposition C.2.2 the k-algebra $\mathrm{C}_0(\mathrm{V}, q)$ is central simple of rank 2^{n-2} over a degree-2 finite étale k-algebra k' (which is the center of $\mathrm{C}_0(\mathrm{V}, q)$), and the algebraic group of units $\mathrm{C}_0(\mathrm{V}, q)^\times$ contains $\mathrm{Spin}(q)$. If (V, q) is split then this k-algebra is a product of two copies of $\mathrm{Mat}_{2^{m-1}}(k)$, and the resulting two 2^{m-1}-dimensional representations of $\mathrm{Spin}(q)$ are the *half-spin representations* from Remark C.4.11. For $m = 1$ – i.e., $n = 2$ – and split (V, q), the group $\mathrm{Spin}(q)$ is a 1-dimesional split torus and these two 1-dimensional representations are the two faithful 1-dimensional representations of such a torus.

Assume instead that $n = 2m + 1$ is odd ($m \geqslant 1$). By Proposition C.2.4 the k-algebra $\mathrm{C}_0(\mathrm{V}, q)$ is central simple of rank 2^{n-1}, and its unit group contains $\mathrm{Spin}(q)$. If moreover (V, q) is split then this is a matrix algebra $\mathrm{Mat}_{2^m}(k)$, and the resulting 2^m-dimensional representation of $\mathrm{Spin}(q)$ is the *spin representation* from Remark C.4.11.

Step 2. We shall now reduce to the case when $\mathrm{C}_0(\mathrm{V}, q)$ is a matrix algebra over its center (though (V, q) might not be split). The reason for interest in this case is that if $\mathrm{C}_0(\mathrm{V}, q)$ is a matrix algebra over its center then we obtain a k-descent ρ of the direct sum of the half-spin representations of $\mathrm{Spin}(q)_{k_s}$ for even n (using a choice of k-basis of the center k' of $\mathrm{C}_0(\mathrm{V}, q)$) and a k-descent of the spin representation of $\mathrm{Spin}(q)_{k_s}$ for odd n.

We may certainly assume k is finitely generated over its prime field, so there exists a finitely generated $\mathbf{Z}[1/2]$-subalgebra $\mathrm{R} \subset k$ with fraction field k such that: (V, q) arises from a fiberwise non-degenerate quadratic space $(\mathscr{V}, \mathscr{Q})$ of rank n over R, $g \in \mathrm{SO}(\mathscr{Q})(\mathrm{R})$, and $\det(1+g) \in \mathrm{R}^\times$. Consider the spinor norm

$$\mathrm{sp}_{\mathscr{Q}} : \mathrm{SO}(\mathscr{Q})(\mathrm{R}) \to \mathrm{H}^1(\mathrm{R}, \mu_2).$$

By replacing R with $\mathrm{R}[1/r]$ for a suitable nonzero $r \in \mathrm{R}$ we may arrange that the image of $\mathrm{sp}_{\mathscr{Q}}(g)$ in $\mathrm{H}^1(\mathrm{R}, \mathrm{GL}_1) = \mathrm{Pic}(\mathrm{R})$ is trivial, so $\mathrm{sp}_{\mathscr{Q}}(g)$ lies in the subgroup $\mathrm{R}^\times/(\mathrm{R}^\times)^2 \subset \mathrm{H}^1(\mathrm{R}, \mu_2)$. To control this square class, at least after further Zariski-localization on R, we now reduce to working over finite fields.

If $\mathrm{char}(k) > 0$ then we may use generic smoothness over the perfect field $\mathbf{F}_p$ to find a nonzero $r' \in \mathrm{R}$ such that $\mathrm{R}[1/r']$ is $\mathbf{F}_p$-smooth. Likewise, if $\mathrm{char}(k) = 0$ then generic smoothness over $\mathbf{Q}$ provides a nonzero $r' \in \mathrm{R}$ such that $\mathrm{R}[1/r']$ is $\mathbf{Z}[1/\mathrm{N}]$-smooth for a sufficiently divisible integer $\mathrm{N} > 0$. Hence, in all cases we can replace R with a suitable $\mathrm{R}[1/r']$ to arrange that R is normal (i.e., integrally closed). It suffices to show that $\mathrm{sp}_{\mathscr{Q}}(g) = \det(1+g) \bmod (\mathrm{R}^\times)^2$ (as then localizing at the generic point will conclude the proof).

Functoriality of Kummer theory with respect to base change implies that for any closed point $\xi \in \mathrm{Spec}(\mathrm{R})$ and its finite residue field $\kappa = \kappa(\xi)$, the specialization $g(\xi) \in \mathrm{SO}(\mathscr{Q})(\kappa)$ has spinor norm in $\kappa^\times/(\kappa^\times)^2$ that is equal to the ξ-specialization of $\mathrm{sp}_{\mathscr{Q}}(g)$. We claim that a unit r in R is a square in R if and only if its ξ-specialization in $\kappa(\xi)^\times$ is a square for all ξ, in which case taking r to be a representative of $\det(1+g)^{-1}\mathrm{sp}_{\mathscr{Q}}(g)$ would reduce our task to the case of finite fields. More generally:

Lemma C.5.9. — *Let* X *be a connected normal* $\mathbf{Z}$*-scheme of finite type. If a finite étale cover* $\mathrm{X}' \to \mathrm{X}$ *has split fibers over all closed points then it is a split covering. In particular, if* X *is a* $\mathbf{Z}[1/2]$*-scheme and an element* $u \in \mathscr{O}(\mathrm{X})^{\times}$ *has square image in the residue field* $\kappa(x)$ *at every closed point* x *then* u *is a square in* $\mathscr{O}(\mathrm{X})$.

Proof. — The assertion for square roots of units follows from the rest by using the cover $\mathrm{X}' \to \mathrm{X}$ defined by $t^2 = u$. In general, let $d = \dim(\mathrm{X})$ and let m be the degree of X' over X. Consider the zeta functions $\zeta_{\mathrm{X}}(s)$ and $\zeta_{\mathrm{X}'}(s)$; these are absolutely and uniformly convergent products on $\mathrm{Re}(s) \geqslant d+\varepsilon$ for any $\varepsilon > 0$ [**Ser63**, § 1.3]. The zeta function of any $\mathbf{Z}$-scheme Y of finite type with dimension d has a meromorphic continuation to the half-plane $\mathrm{Re}(s) > d - 1/2$ with pole at $s = d$ equal to the number of d-dimensional irreducible components [**Ser63**, § 1.4, Cor. 1].

The normal noetherian X is connected and hence irreducible, so $\zeta_{\mathrm{X}}(s)$ has a simple pole at $s = d$. The hypotheses imply that $\zeta_{\mathrm{X}'} = \zeta_{\mathrm{X}}^m$ where $m = [\mathrm{X}' : \mathrm{X}]$, so X' has an mth-order pole at $s = d$ and hence has m irreducible components of dimension d. But these are also the connected components of X' (as X' is finite étale over the normal noetherian X), so by degree considerations each connected component of X' maps isomorphically onto X. □

Remark C.5.10. — The meromorphicity assertion used above is not proved in [**Ser63**], though it can be deduced from the Lang–Weil estimate for geometrically irreducible schemes over finite fields [**LW**, § 2, Cor. 2] (or from Weil's Riemann Hypothesis for curves) via a fibration technique. Here is a short proof via Deligne's mixedness bounds [**D**, Cor. 3.3.4].

Via stratification and dimension induction, we may assume Y is separated, irreducible, and reduced. If the function field F of Y has characteristic $p > 0$ and κ is the algebraic closure of $\mathbf{F}_p$ in F then Y is geometrically irreducible over κ. Thus, the top-degree cohomology $\mathrm{H}_c^{2d}(\mathrm{Y}_{\overline{\kappa}}, \mathbf{Q}_\ell)$ is $\mathbf{Q}_\ell(-d)$. The Grothendieck–Lefschetz formula for $\zeta_{\mathrm{Y}}(s)$ as a rational function in q^{-s} ($q = \#\kappa$) therefore has a factor of $1 - q^{d-s}$ in the denominator arising from the action of geometric Frobenius on the top-degree cohomology, and all other cohomological contributions in the denominator are non-vanishing for $\mathrm{Re}(s) > d - 1/2$ due to Deligne's bounds. This settles the case $\mathrm{char}(\mathrm{F}) > 0$.

Assume $\mathrm{char}(\mathrm{F}) = 0$, so $\mathrm{Y}_{\mathbf{Q}}$ is geometrically irreducible of dimension $d-1$ over the algebraic closure K of $\mathbf{Q}$ in F. Replacing Y with a dense open subscheme brings us to the case that Y is an $\mathscr{O}_{\mathrm{K}}$-scheme with geometrically irreducible fiber Y_v of dimension $d-1$ over the closed points v in a dense open subset $\mathrm{U} \subset \mathrm{Spec}(\mathscr{O}_{\mathrm{K}})$ [**EGA**, IV_3, 9.7.7(i)]. Since $\zeta_{\mathrm{Y}}(s) = \prod_{v \in \mathrm{U}} \zeta_{\mathrm{Y}_v}(s)$ with

Y_v geometrically irreducible of dimension $d-1$ over $\kappa(v)$, the analysis just given in positive characteristic exhibits $\zeta_{\mathrm{Y}}(s)$ as the product of $\zeta_{\mathrm{U}}(s-(d-1))$ and a holomorphic function in the half-plane $\mathrm{Re}(s) > 1 + (2(d-1)-1)/2 = d - 1/2$. By meromorphicity of ζ_{U} on $\mathbf{C}$ with a simple pole at $s = 1$, we are done.

Step 3. Now we may assume k is finite, so $\mathrm{C}_0(\mathrm{V}, q)$ is a matrix algebra over its center. Let $\rho : \mathrm{Spin}(q) \to \mathrm{GL}(\mathrm{W})$ be the associated k-descent of the spin representation for odd $n \geqslant 3$ and of the direct sum of the two half-spin representations for even $n \geqslant 2$. There is a general identity (brought to my attention by Z. Yun) on the *entirety* of $\mathrm{Spin}(q)$ without any intervention of squaring ambiguity or finiteness hypotheses on k:

Lemma C.5.11. — *Let* (V, q) *be a non-degenerate quadratic space of dimension* $n \geqslant 2$ *over an arbitrary field* k *with* $\mathrm{char}(k) \neq 2$. *For any* $\widetilde{h} \in \mathrm{Spin}(q)$ *with image* $h \in \mathrm{SO}(q)$,

$$(1/2)\det(1+h) = \mathrm{Tr}(\rho(\widetilde{h}))^2 \textit{ for odd } n, \quad \det(1+h) = \mathrm{Tr}(\rho(\widetilde{h}))^2 \textit{ for even } n.$$

Granting this, we shall prove $\mathrm{sp}_q(g) = \det((1+g)/2) \bmod (k^\times)^2$ for any $g \in \mathrm{SO}(q)(k)$ without -1 as an eigenvalue, where k is finite (as above). By finiteness of k, it is equivalent to prove $\mathrm{sp}_q(g) = 1$ if and only if $\det((1+g)/2)$ is a square in $k^\times$. If $\mathrm{sp}_q(g) = 1$ then g lifts to some $\widetilde{g} \in \mathrm{Spin}(q)(k)$ and Lemma C.5.11 gives that $\det((1+g)/2)$ is a square in k. Conversely, if $\det((1+g)/2)$ is a square in k then we want to show that the degree-2 finite étale g-fiber in $\mathrm{Spin}(q)$ splits.

Suppose not, so for the quadratic extension k'/k there is a point $\widetilde{g} \in \mathrm{Spin}(q)(k')$ over g and its k'/k-conjugate is $\widetilde{g}z$ for the nontrivial central element z in $\mathrm{Spin}(q)(k)$. But $\mathrm{Tr}(\widetilde{g})^2/\det((1+g)/2) \in (k^\times)^2$ by Lemma C.5.11, and by hypothesis $\det((1+g)/2) \in (k^\times)^2$, so $\mathrm{Tr}(\rho(\widetilde{g})) \in k^\times$. Galois-equivariance of ρ then implies $\mathrm{Tr}(\rho(\widetilde{g}z)) = \mathrm{Tr}(\rho(\widetilde{g}))$. By construction of the spin representation for odd n and both half-spin representations for even n, the central involution z satisfies $\rho(z) = -1$ (as we may check over k_s) and hence

$$\mathrm{Tr}(\rho(\widetilde{g})) = \mathrm{Tr}(\rho(\widetilde{g}z)) = -\mathrm{Tr}(\rho(\widetilde{g})),$$

forcing $\mathrm{Tr}(\rho(\widetilde{g})) = 0$. But the square of this trace is $\det(1+g) \neq 0$ by Lemma C.5.11, so we have a contradiction and hence g lifts into $\mathrm{Spin}(q)(k)$ as desired.

Step 4. It remains to prove Lemma C.5.11. We may assume k is *algebraically closed*, so (V, q) is split. We treat the case of even and odd n separately, based on how the structure of the Clifford algebra depends on the parity of n.

First consider odd n, so the center Z_q of $\mathrm{C}(\mathrm{V}, q)$ is finite étale of degree 2 over k (i.e., $\mathrm{Z}_q = k \times k$ as k-algebras) and the natural map $\mathrm{Z}_q \otimes_k \mathrm{C}_0(\mathrm{V}, q) \to \mathrm{C}(\mathrm{V}, q)$ is an isomorphism. The conjugation action on the $\mathbf{Z}/2\mathbf{Z}$-graded algebra $\mathrm{C}(\mathrm{V}, q)$ by the subgroup $\mathrm{Spin}(q) \subset \mathrm{C}_0(\mathrm{V}, q)^\times$ is trivial on Z_q, so $\mathrm{C}(\mathrm{V}, q)$ as a representation of $\mathrm{Spin}(q)$ is a direct sum of two copies of $\mathrm{C}_0(\mathrm{V},q) = \mathrm{End}_k(\rho) = \rho \otimes \rho^*$ where ρ is the spin representation.

Consider the conjugation action on $\mathrm{C}(\mathrm{V},q)$ by any $\widetilde{h} \in \mathrm{Spin}(q)$. We compute the trace of this action in two different ways. On the one hand, this is $2\chi_{\rho\otimes\rho^*}(\widetilde{h}) = 2\chi_\rho(\widetilde{h})\chi_{\rho^*}(\widetilde{h})$, yet $\chi_{\rho^*} = \chi_\rho$ in characteristic 0 (by highest-weight theory) and hence in general by specialization from characteristic 0 (consider the Clifford algebra of the standard split quadratic space of rank n over $\mathbf{Z}$), so the trace of the action is $2\chi_\rho(\widetilde{h})^2$.

On the other hand, by definition of $\mathrm{Spin}(q)$ inside the Clifford algebra, this action preserves V with resulting representation on V that is the composition of the standard quotient map $\mathrm{Spin}(q) \to \mathrm{SO}(q)$ and the inclusion $\mathrm{SO}(q) \hookrightarrow \mathrm{GL}(\mathrm{V})$, so this action preserves the *filtration* of $\mathrm{C}(\mathrm{V},q)$ defined by degree of (possibly mixed) tensors. The trace of the $\widetilde{h}$-conjugation on $\mathrm{C}(\mathrm{V},q)$ is therefore the same as that of its effect on the associated graded space for this filtration, which is the exterior algebra $\wedge^\bullet(\mathrm{V})$. In other words, the trace of $\widetilde{h}$-conjugation is the trace of the action of $h \in \mathrm{SO}(q)$ on $\wedge^\bullet(\mathrm{V})$, which is $\det(1+h)$.

Now consider even n. Exactly as for odd n, the trace of $\widetilde{h}$-conjugation on $\mathrm{C}(\mathrm{V},q)$ is $\det(1+h)$. On the other hand, for a Lagrangian (i.e., maximal isotropic) subspace $\mathrm{W} \subset \mathrm{V}$ of dimension $n/2$ and the associated graded components $\mathrm{A}_+ = \oplus \wedge^{2j}(\mathrm{W})$ and $\mathrm{A}_- = \oplus \wedge^{2j+1}(\mathrm{W})$ of the exterior algebra of W, there is an isomorphism of algebras

$$\mathrm{C}(\mathrm{V},q) \simeq \mathrm{End}(\mathrm{A}) = (\mathrm{End}(\mathrm{A}_+)\times\mathrm{End}(\mathrm{A}_-))\oplus(\mathrm{Hom}(\mathrm{A}_+,\mathrm{A}_-)\oplus\mathrm{Hom}(\mathrm{A}_-,\mathrm{A}_+))$$

(depending on a choice of Lagrangian complement of W in V) in which $\mathrm{C}_0(\mathrm{V},q)$ is identified with the subalgebra $\mathrm{End}(\mathrm{A}_+)\times\mathrm{End}(\mathrm{A}_-)$ of linear endomorphisms of A that respect its $\mathbf{Z}/2\mathbf{Z}$-grading. This identifies A_+ and A_- as underlying spaces of the two half-spin representations $\rho_\pm$ of $\mathrm{Spin}(q) \subset \mathrm{C}_0(\mathrm{V},q)^\times$.

We conclude that $\mathrm{C}(\mathrm{V},q)$ as a representation space for $\mathrm{Spin}(q)$ via conjugation is isomorphic to

$$(\rho_+\otimes\rho_+^*)\oplus(\rho_-\otimes\rho_-^*)\oplus(\rho_-\otimes\rho_+^*)\oplus(\rho_+\otimes\rho_-^*).$$

Since the half-spin representations each have a self-dual character (by specialization from characteristic 0, as in the treatment of odd n), it follows that the character of this representation is therefore equal to

$$\chi_{\rho_+}^2 + \chi_{\rho_-}^2 + 2\chi_{\rho_+}\chi_{\rho_-} = \chi_{\rho_+\oplus\rho_-}^2.$$

But $\rho_+\oplus\rho_- = \rho$ by definition, so we are done.

Remark C.5.12. — Consider a non-degenerate (V,q) over a field k with characteristic 2. If $n = \dim \mathrm{V}$ is even then $\mathrm{Pin}(q)$ is a central extension of $\mathrm{O}(q)$ by μ_2 yielding a spinor norm $\mathrm{sp}_q : \mathrm{O}(q)(k) \to k^\times/(k^\times)^2$ characterized by the property $\mathrm{sp}_q(r_v) = q(v) \bmod (k^\times)^2$ for non-isotropic $v \in \mathrm{V}$. A formula for

$\mathrm{sp}_q(g)$ in the spirit of Zassenhaus' theorem is given in [**Ha**, Cor. 2.7], building on earlier work of Wall [**Wa59**], [**Wa63**]. However, this formula involves an "anisotropic" factor that is difficult to use in theoretical arguments. (Note that [**Ha**] uses sp_q^-, as shown by the Example after [**Ha**, Thm. 1.4], but this sign aspect is invisible in characteristic 2.)

If n is odd then the disconnected $\mathrm{Pin}(q)$ is *not* a central extension of the connected $\mathrm{O}(q) = \mu_2 \times \mathrm{SO}(q)$ by μ_2. However, $\mathrm{O}(q)(k) = \mathrm{SO}(q)(k)$ in such cases, so the central extension $\mathrm{Spin}(q)$ of $\mathrm{SO}(q)$ by μ_2 (for the fppf topology) defines a connecting homomorphism

$$\mathrm{O}(q)(k) = \mathrm{SO}(q)(k) \to \mathrm{H}^1(k, \mu_2) = k^\times/(k^\times)^2$$

that we call the "spinor norm" sp_q; it encodes the obstruction to lifting $g \in \mathrm{O}(q)(k) = \mathrm{SO}(q)(k)$ to $\mathrm{Spin}(q)(k)$. This homomorphism is determined by its values on reflections r_v in non-isotropic v, and we may restrict attention to $v \notin \mathrm{V}^\perp$ since $r_v = 1$ when $v \in \mathrm{V}^\perp - \{0\}$. For $v \notin \mathrm{V}^\perp$ we claim that $\mathrm{sp}_q(r_v) = q(v) \bmod (k^\times)^2$, exactly as for even n. Indeed, since $v \notin \mathrm{V}^\perp$ there exists $w \in \mathrm{V}$ such that $\mathrm{B}_q(v, w) \neq 0$, and w is linearly independent from v since $\mathrm{B}_q(v, v) = 0$ (as $\mathrm{char}(k) = 2$). Thus, for the span P of v and w we see that $(\mathrm{P}, q|_\mathrm{P})$ is a quadratic space containing v on which B_q has discriminant $\mathrm{B}_q(v, w)^2 \neq 0$, so P is non-degenerate and $\mathrm{V} = \mathrm{P} \oplus \mathrm{P}^\perp$. Consequently, $\mathrm{sp}_q(r_v)$ coincides with the analogue for P. But $\dim \mathrm{P}$ is even, so the claim follows.

The restriction $q|_{\mathrm{V}^\perp}$ to the defect line has the form cx^2 for $c \in k^\times$ well-defined up to $(k^\times)^2$-multiple, so for any nonzero $v \in \mathrm{V}^\perp$ we have $q(v) \in c(k^\times)^2$ yet $r_v = 1$. Hence, if $\mathrm{V}^\perp$ does not contain a unit vector (i.e., if c is not a square) there is *no* well-defined homomorphism $\mathrm{O}(q)(k) \to k^\times/(k^\times)^2$ carrying r_v to the class of $q(v)$ for *all* non-isotropic v, but sp_q achieves this for all such $v \notin \mathrm{V}^\perp$. The composition of this spinor norm with the quotient map

$$k^\times/(k^\times)^2 \to k^\times/\langle q(\mathrm{V}^\perp - \{0\})\rangle = k^\times/\langle c, (k^\times)^2\rangle$$

carries r_v to the residue class of $q(v)$ for *all* non-isotropic $v \in \mathrm{V}$, and it is this composite map that is called the "spinor norm" in [**Ha**], where a formula in the spirit of Zassenhaus' theorem is given in [**Ha**, Cor. 2.7] (involving a non-explicit "anisotropic" factor as for even n).

C.6. Accidental isomorphisms. — The study of (special) orthogonal groups provides many *accidental isomorphisms* between low-dimensional members of distinct "infinite families" of algebraic groups. This is analogous to isomorphisms between small members of distinct "infinite families" of finite groups. Using the hyperplane $\mathrm{H} = \{\sum x_i = 0\} \subset \mathbf{A}^n$ over $\mathbf{Z}$, quadratic form $q = \sum x_i^2$ on H over $\mathbf{Z}[1/2]$, and line $\mathrm{L} = \{x_1 = \cdots = x_n\} \subset \mathrm{H}$ over $\mathbf{F}_p$ with $p|n$, the natural action on $\mathbf{A}^n$ by the symmetric group $\mathfrak{S}_n$ yields isomorphisms:

— $\mathfrak{S}_3 \simeq \mathrm{SL}_2(\mathbf{F}_2)$ (use $\mathrm{H} \subset \mathbf{A}^3$),

— $\mathfrak{S}_4 \simeq \mathrm{PGL}_2(\mathbf{F}_3)$ (identify $\mathbf{P}^1_{\mathbf{F}_3}$ with the smooth conic $\{q = 0\}$ in $\mathbf{P}(\mathrm{H}^*)$ for $\mathrm{H} \subset \mathbf{A}^4_{\mathbf{F}_3}$),

— $\mathfrak{S}_5 \simeq \mathrm{PGL}_2(\mathbf{F}_5)$ (identify $\mathbf{P}^1_{\mathbf{F}_5}$ with the smooth conic $\{\overline{q} = 0\}$ in the plane $\mathbf{P}((\mathrm{H}/\mathrm{L})^*)$, where $\mathrm{H} \subset \mathbf{A}^5_{\mathbf{F}_5}$ and L is the defect line for q on H),

— $\mathfrak{A}_6 \simeq \mathrm{SL}_2(\mathbf{F}_9)/\langle -1\rangle$ (for $\mathrm{H} \subset \mathbf{A}^6_{\mathbf{F}_3}$ and the defect line L of q on H, $(\mathrm{H}/\mathrm{L}, \overline{q}) \simeq x^2 + y^2 + z^2 - t^2$ and the $\mathfrak{S}_6$-action on the projective quadric $\{\overline{q} = 0\} \subset \mathbf{P}((\mathrm{H}/\mathrm{L})^*)$ defines a map $\mathfrak{S}_6 \to \mathrm{O}(\overline{q})(\mathbf{F}_3)/\langle -1\rangle \subset (\mathrm{O}(\overline{q})/\mu_2)(\mathbf{F}_3)$ that must be injective and carry the simple $\mathfrak{A}_6$ into the normal subgroup $\mathrm{Spin}(\overline{q})(\mathbf{F}_3)/\mathrm{Z}_{\mathrm{Spin}(\overline{q})}(\mathbf{F}_3)$ that is identified with $\mathrm{SL}_2(\mathbf{F}_9)/\langle -1\rangle$ in Remark C.6.4),

— $\mathfrak{S}_6 \simeq \mathrm{Sp}_4(\mathbf{F}_2)$ (use $(\mathrm{H}/\mathrm{L}, \psi)$ where $\mathrm{H} \subset \mathbf{A}^6_{\mathbf{F}_2}$, L is the defect line for $\mathrm{Q} = \sum_{i<j} x_i x_j$ on H, and ψ is induced by the symplectic form $\mathrm{B}_{\mathrm{Q}}(x, y) = \sum_{i \neq j} x_i y_j$),

— $\mathfrak{S}_8 \simeq \mathrm{O}_6(\mathbf{F}_2)$ (use the non-degenerate quadratic space $(\mathrm{H}/\mathrm{L}, \overline{\mathrm{Q}})$ where $\mathrm{H} \subset \mathbf{A}^8_{\mathbf{F}_2}$, $\mathrm{Q} = \sum_{i<j} x_i x_j$, and L is the defect line for $\mathrm{Q}|_{\mathrm{H}}$).

When accidental isomorphisms among algebraic groups are applied to rational points over *finite* fields one obtains some of the accidental isomorphisms among small finite simple groups. This is seen in the discussion of $\mathfrak{A}_6$ above. As another illustration, in Example C.6.6 we will see that $\mathrm{SL}_4/\mu_2 \simeq \mathrm{SO}_6$ as $\mathbf{Z}$-groups, so by passing to $\mathbf{F}_2$-points and using the vanishing of $\mathrm{H}^1(\mathbf{F}_2, \mu_2)$ (fppf Kummer theory) we see that $\mathrm{SL}_4(\mathbf{F}_2) = \mathrm{SO}_6(\mathbf{F}_2)$, an index-2 subgroup of $\mathrm{O}_6(\mathbf{F}_2) = \mathfrak{S}_8$. The only such subgroup is $\mathfrak{A}_8$, so $\mathrm{SL}_4(\mathbf{F}_2) = \mathfrak{A}_8$.

The case of $\mathfrak{A}_6$ above can be pushed a bit further. First, we note that $\mathfrak{S}_6 \not\simeq \mathrm{PGL}_2(\mathbf{F}_9)$, despite the isomorphism between their index-2 perfect commutator subgroups, since $\mathfrak{S}_6$ has no element of order 8. More interesting is that we can interpret the nontrivial outer automorphism of $\mathfrak{S}_6$. (For $n > 2$, $\mathrm{Out}(\mathfrak{S}_n)$ is trivial for $n \neq 6$ and has order 2 for $n = 6$.) This rests on an isomorphism associated to any non-degenerate quadratic space (W, Q) of rank 4 over a finite field k: by Example C.6.3 below, for some degree-2 étale k-algebra k' we have

$$\mathrm{O}(\mathrm{Q})/\mu_2 \simeq \mathrm{Aut}_{\mathrm{R}_{k'/k}(\mathbf{P}^1_{k'})/k} = \mathrm{R}_{k'/k}(\mathrm{PGL}_2) \rtimes \langle -1\rangle$$

where the nontrivial k-automorphism of k' defines the action of $\langle -1\rangle$. On identity components this gives $\mathrm{SO}(\mathrm{Q})/\mu_2 \simeq \mathrm{R}_{k'/k}(\mathrm{PGL}_2)$, so if Q is not k-split (hence $\mathrm{SO}(\mathrm{Q})/\mu_2$ is not k-split, by Proposition C.3.14) then k' is a field. We conclude that $\mathfrak{S}_6$ is a subgroup of $\mathrm{PGL}_2(\mathbf{F}_9) \rtimes \langle -1\rangle$ making $\mathfrak{A}_6$ the index-4 image of $\mathrm{SL}_2(\mathbf{F}_9)$, so $\mathfrak{S}_6$ is an index-2 subgroup of $\mathrm{PGL}_2(\mathbf{F}_9) \rtimes \langle -1\rangle$ (the unique one distinct from $\mathrm{PGL}_2(\mathbf{F}_9)$ that does not contain the Galois involution). This index-2 inclusion defines the nontrivial outer automorphism of $\mathfrak{S}_6$.

Just as isomorphisms among small finite groups are due to the limited possibilities for finite groups of small size, accidental isomorphisms between low-dimensional semisimple groups are due to a limitation in the possibilities for a "small" case of the root datum that governs the (geometric) isomorphism class of a connected semisimple group. We now work out the accidental isomorphisms for SO_n with $2 \leqslant n \leqslant 6$; for $3 \leqslant n \leqslant 6$ these correspond to equalities between distinct classical infinite families of root systems, as we will explain. (See [**KMRT**, IV, § 15] for further discussion over fields, using algebras with involution, and [**Knus**, V] for further discussion over rings.) In what follows, (V, q) is a quadratic space (so q is $\mathscr{O}_{\mathrm{S}}$-valued and non-degenerate); some aspects go through with little change for line bundle-valued q, especially when the odd part of the Clifford algebra does not intervene (see [**Au**, § 5] for $n = 2, 4, 6$).

Example C.6.1. — Suppose $n = 2$. In this case, Lemma C.2.1 and calculations with $q_2 = xy$ show that $\mathrm{SO}(q)$ is a rank-1 torus and $\mathrm{E}_q := \{q = 0\} = \mathrm{Proj}(\mathrm{Sym}(\mathrm{V}^*)/(q)) \subset \mathbf{P}(\mathrm{V}^*)$ is a degree-2 finite étale S-scheme. By Proposition C.3.15(1) (and its proof), the coordinate ring of E_q over $\mathscr{O}_{\mathrm{S}}$ naturally coincides with the quadratic étale commutative $\mathscr{O}_{\mathrm{S}}$-algebra $\mathrm{Z}_q = \mathrm{C}_0(\mathrm{V}, q)$ and there is a natural isomorphism between $\mathrm{GSO}(q) = \mathbf{G}_{\mathrm{m}} \times^{\mu_2} \mathrm{SO}(q)$ and the Weil restriction torus $\mathrm{R}_{\mathrm{E}_q/\mathrm{S}}(\mathbf{G}_{\mathrm{m}})$. This isomorphism respects the natural inclusion of $\mathbf{G}_{\mathrm{m}}$ into each torus and identifies $\mathrm{SO}(q)$ with the norm-1 subtorus of $\mathrm{R}_{\mathrm{E}_q/\mathrm{S}}(\mathbf{G}_{\mathrm{m}})$. The discussion preceding Lemma C.4.1 identifies the S-group $\mathrm{GSpin}(q) = \mathbf{G}_{\mathrm{m}} \times^{\mu_2} \mathrm{Spin}(q)$ with the torus $\mathrm{R}_{\mathrm{Z}_q/\mathrm{S}}(\mathbf{G}_{\mathrm{m}})$ (essentially by definition), respecting the natural inclusion of $\mathbf{G}_{\mathrm{m}}$ into each and (as we saw in the discussion following Remark C.4.2) identifying $\mathrm{Spin}(q)$ with the norm-1 subtorus of $\mathrm{R}_{\mathrm{Z}_q/\mathrm{S}}(\mathbf{G}_{\mathrm{m}})$. In this way we obtain a natural isomorphism between the tori $\mathrm{SO}(q)$ and $\mathrm{Spin}(q)$ via their identifications with the norm-1 subtori in the respective tori $\mathrm{R}_{\mathrm{E}_q/\mathrm{S}}(\mathbf{G}_{\mathrm{m}})$ and $\mathrm{R}_{\mathrm{Z}_q/\mathrm{S}}(\mathbf{G}_{\mathrm{m}})$ that we have seen are naturally isomorphic. (These isomorphisms are also established in [**Knus**, V, 2.5.2].)

This natural identification of $\mathrm{SO}(q)$ and its double cover $\mathrm{Spin}(q)$ is a special case of the general fact that for any $d \geqslant 1$ and degree-d isogeny $f : \mathrm{T}' \to \mathrm{T}$ between rank-1 tori (such as the degree-2 isogeny $\mathrm{Spin}(q) \to \mathrm{SO}(q)$) there is a unique isomorphism $\mathrm{T}' \simeq \mathrm{T}$ that identifies f with $t \mapsto t^d$ on T. To see this agreement of isomorphisms, we have to show that the composition of the isomorphism $\mathrm{SO}(q) \simeq \mathrm{Spin}(q)$ between norm-1 subtori of $\mathrm{R}_{\mathrm{E}_q/\mathrm{S}}(\mathbf{G}_{\mathrm{m}})$ and $\mathrm{R}_{\mathrm{Z}_q/\mathrm{S}}(\mathbf{G}_{\mathrm{m}})$ with the natural isogeny $\mathrm{Spin}(q) \to \mathrm{SO}(q)$ is the squaring endomorphism of $\mathrm{SO}(q)$. It suffices to check this comparison of tori on geometric fibers over S, so we may assume $\mathrm{S} = \mathrm{Spec}\, k$ for an algebraically closed field k with $\mathrm{V} = ke_1 \oplus ke_2$ and $q = xy$. For $a \in k^\times$, the element $\mathrm{diag}(a, 1/a) \in \mathrm{SO}_2(k)$ is associated to the unit

$$u(a) = ae_1e_2 + (1/a)(1 - e_1e_2) = 1/a + (a - 1/a)e_1e_2 \in \mathrm{Spin}(q_2)(k) \subset \mathrm{Z}_{q_2}^\times$$

due to the computations in the proof of Proposition C.3.15(1). Since $e_1e_2e_1 = e_1$ and $e_2e_1e_2 = e_2$ inside $\mathrm{C}(k^2, q_2)$ and $u(a)^{-1} = u(1/a)$, the image of $u(a)$ in

$\mathrm{GL}_2(k)$ under the natural isogeny $\mathrm{Spin}(q_2) \to \mathrm{SO}_2$ is the linear automorphism of k^2 defined by

$$e_1 \mapsto u(a)e_1u(a)^{-1} = a^2e_1, \;\; e_2 \mapsto u(a)e_2u(a)^{-1} = (1/a)^2e_2.$$

Thus, we obtain the point $\mathrm{diag}(a, 1/a)^2 \in \mathrm{SO}_2(k)$ as desired.

Via the action of $\mathrm{PGL(V)}$ on $\mathbf{P}(\mathrm{V}^*)$ we can interpret the torus $\mathrm{SO}(q)/\mu_2 = \mathrm{GSO}(q)/\mathbf{G}_{\mathrm{m}}$ in an interesting way, as follows. Consideration of the relative homogenous coordinate ring shows that the $\mathrm{PGL(V)}$-stabilizer G_q of E_q is $\mathrm{GO}(q)/\mathbf{G}_{\mathrm{m}}$. Thus, we get a homomorphism

$$\alpha_q : \mathrm{GO}(q) \twoheadrightarrow \mathrm{GO}(q)/\mathbf{G}_{\mathrm{m}} = \mathrm{G}_q \to \underline{\mathrm{Aut}}_{\mathrm{S}}(\mathrm{E}_q) = (\mathbf{Z}/2\mathbf{Z})_{\mathrm{S}}.$$

We claim that α_q is the enhanced Dickson invariant GD_q from (C.3.1) and Remark C.3.11 (for $n = 2$), so $\mathrm{GSO}(q)/\mathbf{G}_{\mathrm{m}}$ is the subgroup of points of G_q whose action on E_q is trivial. The map α_q must kill the fiberwise connected open and closed subgroup $\mathrm{GSO}(q)$, so it factors through the quotient $\mathrm{GO}(q)/\mathrm{GSO}(q)$ that GD_q identifies with $(\mathbf{Z}/2\mathbf{Z})_{\mathrm{S}}$. Hence, α_q is the composition of GD_q with a uniquely determined endomorphism of $(\mathbf{Z}/2\mathbf{Z})_{\mathrm{S}}$, and we have to show that this endomorphism is the identity map. The identity map is the unique automorphism of $(\mathbf{Z}/2\mathbf{Z})_{\mathrm{S}}$, so our problem is reduced to the geometric fibers, where it amounts to the assertion that the G_q-action on E_q is nontrivial over an algebraically closed field k. This nontriviality is easily verified by using a point of $\mathrm{O}_2(k)$ not in $\mathrm{SO}_2(k)$ (e.g., the point that swaps the standard basis vectors).

Example C.6.2. — Suppose $n = 3$. This corresponds to the equality $\mathrm{A}_1 = \mathrm{B}_1$ for adjoint groups. Consider the zero scheme $\mathrm{C}_q = \{q = 0\} \subset \mathbf{P}(\mathrm{V}^*)$. By the definition of non-degeneracy, this is a smooth conic in a $\mathbf{P}^2$-bundle, so it is a $\mathbf{P}^1$-bundle. The automorphism scheme $\mathrm{Aut}_{\mathrm{C}_q/\mathrm{S}}$ is therefore a form of $\mathrm{Aut}_{\mathbf{P}^1_{\mathrm{S}}/\mathrm{S}} = \mathrm{PGL}_2$. Under the left action of $\mathrm{PGL(V)}$ on $\mathbf{P}(\mathrm{V}^*)$, the stabilizer G_q of C_q is $\mathrm{GO}(q)/\mathbf{G}_{\mathrm{m}} = \mathrm{SO}(q)$ (see Lemma C.3.12 with $n = 3$), and we claim that the action map

$$\alpha_q : \mathrm{SO}(q) \to \mathrm{Aut}_{\mathrm{C}_q/\mathrm{S}}$$

to a form of PGL_2 is an isomorphism. (For $q = q_3$ this is a map $\mathrm{SO}_3 \to \mathrm{PGL}_2$.)

The source and target of α_q are smooth with fibers that are connected of the same dimension, and it is easy to check that α_q is injective on geometric points (since a smooth plane conic over an algebraically closed field contains many triples of points in general position). Thus, α_q is a purely inseparable isogeny on fibers over each $s \in \mathrm{S}$, so by Proposition 6.1.10 (and Definition 3.3.9) it is finite flat as an S-morphism. To prove that α_q has constant degree 1, we may first pass to $q = q_3$ over $\mathbf{Z}$ (by Lemma C.2.1) and then check over $\mathbf{Q}$. A purely inseparable isogeny in characteristic 0 is an isomorphism, so we are done.

In the special case $q = q_3$, an isomorphism in the opposite direction can be described by the following alternative procedure. Consider the linear "conjugation" action of $\mathrm{PGL}_2 = \mathrm{GL}_2/\mathbf{G}_\mathrm{m}$ on the rank-3 affine space $\mathfrak{sl}_2$. This action preserves the non-degenerate quadratic form Q on $\mathfrak{sl}_2$ given by the determinant. Explicitly, $\mathrm{Q}(\begin{smallmatrix} x & y \\ z & -x \end{smallmatrix}) = -(x^2 + yz)$ is, up to sign, the quadratic form q_3. Preservation of q_3 is the same as that of $-q_3$, so the sign does not affect the orthogonal group. Thus, we obtain a homomorphism $\mathrm{PGL}_2 \to \mathrm{O}_3 = \mu_2 \times \mathrm{SO}_3$ over $\mathbf{Z}$ with trivial kernel. By computing over $\mathbf{Q}$, the map to the μ_2-factor must be trivial. Thus, the map $\mathrm{PGL}_2 \to \mathrm{O}_3$ factors through SO_3. Since PGL_2 is smooth and fiberwise connected of dimension 3, it follows that the monic map $\mathrm{PGL}_2 \to \mathrm{SO}_3$ is an isomorphism on fibers and hence is an isomorphism (Lemma B.3.1). We leave it to the interested reader to relate this isomorphism to $\alpha_{q_3}^{-1}$ (relative to a suitable isomorphism $\mathrm{C}_{q_3} \simeq \mathbf{P}^1$).

Example C.6.3. — Suppose $n = 4$. This corresponds to the equality of root systems $\mathrm{D}_2 = \mathrm{A}_1 \times \mathrm{A}_1$. More specifically, in this case $\mathrm{SO}(q)$ is not "absolutely simple"; i.e., on geometric fibers it contains nontrivial smooth connected proper normal subgroups. (This is the only $n \geqslant 3$ for which that happens.) In more concrete terms, we claim that

$$(\mathrm{SL}_2 \times \mathrm{SL}_2)/\mathrm{M} \simeq \mathrm{SO}_4 \tag{C.6.1}$$

with $\mathrm{M} = \mu_2$ diagonally embedded in the evident central manner. As in Example C.6.2, we will also show that there is a geometrically-defined isomorphism in the opposite direction from $\mathrm{SO}(q)/\mu_2$ onto an adjoint semisimple group of type $\mathrm{A}_1 \times \mathrm{A}_1$ for *any* q (not just q_4).

First we explain the concrete isomorphism (C.6.1) for $q = q_4$, since the relative geometry underlying the isomorphism in the opposite direction for general q is more complicated than in Example C.6.2. Apply a sign to the third standard coordinate to convert q_4 into $\mathrm{Q} = x_1x_2 - x_3x_4$, which we recognize as the determinant of a 2×2 matrix. The group SL_2 acts on the rank-4 space of such matrices in two evident commutating ways, via $(g, g').x = gxg'^{-1}$, and these actions preserve the determinant by the definition of SL_2. This defines a homomorphism $\mathrm{SL}_2 \times \mathrm{SL}_2 \to \mathrm{SO}'(\mathrm{Q}) \simeq \mathrm{SO}'_4$ whose kernel is easily seen to be M. This map visibly lands in SO_4 since SL_2 is fiberwise connected and $\mathrm{O}_4/\mathrm{SO}_4 = \mathbf{Z}/2\mathbf{Z}$. Hence, we obtain a monomorphism $(\mathrm{SL}_2 \times \mathrm{SL}_2)/\mathrm{M} \to \mathrm{SO}_4$ that must be an isomorphism on fibers (as both sides have smooth connected fibers of the same dimension), and thus is an isomorphism.

For general q with $n = 4$, we shall build an isomorphism $\varphi : \mathrm{SO}(q)/\mu_2 \simeq \mathrm{R}_{\mathrm{S}'/\mathrm{S}}(\mathrm{G}')$ for a canonically associated degree-2 finite étale cover $\mathrm{S}' \to \mathrm{S}$ and S'-form G' of PGL_2. (For $q = q_4$ we will have $\mathrm{S}' = \mathrm{S} \coprod \mathrm{S}$ and $\mathrm{G}' = \mathrm{PGL}_2$ over S', yielding a canonical isomorphism

$$\mathrm{SO}_4/\mu_2 \simeq \mathrm{PGL}_2 \times \mathrm{PGL}_2$$

that the interested reader can relate to the isomorphism of adjoint groups induced by (C.6.1).) There are two methods to construct $(\mathrm{S}' \to \mathrm{S}, \mathrm{G}', \varphi)$: an algebraic method via Clifford algebras and a geometric method via automorphism schemes. The algebraic method is simpler, so we explain that one first (see [**Knus**, V, 4.4] for a related discussion).

By Proposition C.2.2 (and its proof), the even part $\mathrm{C}_0(\mathrm{V}, q)$ of the Clifford algebra $\mathrm{C}(\mathrm{V}, q)$ is a quaternion algebra (i.e., rank-4 Azumaya algebra) over a rank-2 finite étale $\mathscr{O}_\mathrm{S}$-algebra Z_q. Denote by $\mathrm{C}_0(\mathrm{V}, q)^\times$ the associated unit group over Z_q. This is an inner form of GL_2 over Z_q, and $\mathrm{GSpin}(q)$ is a reductive closed subgroup of $\mathrm{R}_{\mathrm{Z}_q/\mathrm{S}}(\mathrm{C}_0(\mathrm{V}, q)^\times)$. The derived group $\mathrm{Spin}(q)$ has fibers of dimension 6, yet the reductive $\mathrm{R}_{\mathrm{Z}_q/\mathrm{S}}(\mathrm{C}_0(\mathrm{V}, q)^\times)$ also has derived group with fibers of dimension 6, so the containment

$$\mathrm{Spin}(q) \subset \mathscr{D}(\mathrm{R}_{\mathrm{Z}_q/\mathrm{S}}(\mathrm{C}_0(\mathrm{V}, q)^\times))$$

is an equality. Passing to adjoint quotients, we get

$$\mathrm{SO}(q)/\mu_2 = \mathrm{Spin}(q)/\mathrm{Z}_{\mathrm{Spin}(q)} \simeq \mathrm{R}_{\mathrm{Z}_q/\mathrm{S}}(\mathrm{C}_0(\mathrm{V}, q)^\times/\mathbf{G}_\mathrm{m}).$$

Thus, we take $\mathrm{S}' = \mathrm{Z}_q$ and $\mathrm{G}' = \mathrm{C}_0(\mathrm{V}, q)^\times/\mathbf{G}_\mathrm{m}$.

The geometric method rests on the zero scheme $\Sigma_q = \{q = 0\} \subset \mathbf{P}(\mathrm{V}^*)$, a smooth proper S-scheme with geometric fibers over S given by a ruled quadric $\mathbf{P}^1 \times \mathbf{P}^1$ in $\mathbf{P}^3$. The PGL(V)-stabilizer of Σ_q is $\mathrm{GO}(q)/\mathbf{G}_\mathrm{m} = \mathrm{O}(q)/\mu_2$, so we get an action map

$$f_q : \mathrm{O}(q)/\mu_2 \to \mathrm{Aut}_{\Sigma_q/\mathrm{S}}\,.$$

We claim that for any smooth proper map $\mathrm{X} \to \mathrm{S}$ whose geometric fibers are $\mathbf{P}^1 \times \mathbf{P}^1$ (e.g., Σ_q), there is a unique triple $(\mathrm{S}' \to \mathrm{S}, \mathrm{C}', \varphi)$ consisting of a degree-2 finite étale cover $\mathrm{S}' \to \mathrm{S}$, a $\mathbf{P}^1$-bundle $\mathrm{C}' \to \mathrm{S}'$, and an S-isomorphism $\varphi : \mathrm{X} \simeq \mathrm{R}_{\mathrm{S}'/\mathrm{S}}(\mathrm{C}')$. Here, "unique" means that if $(\mathrm{S}'', \mathrm{C}'', \psi)$ is second such triple then there is a unique pair consisting of an S-isomorphism $\alpha : \mathrm{S}' \simeq \mathrm{S}''$ and an isomorphism $\mathrm{C}' \simeq \mathrm{C}''$ over α such that the induced S-isomorphism $\mathrm{R}_{\mathrm{S}'/\mathrm{S}}(\mathrm{C}') \simeq \mathrm{R}_{\mathrm{S}''/\mathrm{S}}(\mathrm{C}'')$ coincides with $\psi \circ \varphi^{-1}$.

To prove the existence and uniqueness of $(\mathrm{S}', \mathrm{C}', \varphi)$, by limit arguments and the uniqueness assertion we may assume $\mathrm{S} = \mathrm{Spec}\,\mathrm{A}$ for a noetherian local ring A. Uniqueness allows us to work fppf-locally, so we may increase the residue field k by a finite amount to make the special fiber isomorphic to $\mathbf{P}^1_k \times \mathbf{P}^1_k$. Since $\mathrm{H}^1(\mathbf{P}^1_k \times \mathbf{P}^1_k, \mathscr{O}) = 0$, by standard cohomological and deformation theory arguments the Isom-functor $\underline{\mathrm{Isom}}(\mathrm{X}, \mathbf{P}^1 \times \mathbf{P}^1)$ is formally smooth. Hence, by formal GAGA this functor has an $\widehat{\mathrm{A}}$-point. By the uniqueness assertion and fpqc descent, it suffices to solve existence and uniqueness over $\widehat{\mathrm{A}}$, so we may assume $\mathrm{X} = \mathbf{P}^1 \times \mathbf{P}^1$. It now suffices to prove that the $\mathbf{Z}$-homomorphism

$$(\mathrm{PGL}_2 \times \mathrm{PGL}_2) \rtimes (\mathbf{Z}/2\mathbf{Z}) \to \mathrm{Aut}_{\mathbf{P}^1 \times \mathbf{P}^1}$$

is an isomorphism, and this is part of Exercise 1.6.3(iv).

Since $(\mathrm{S}', \mathrm{C}', \varphi)$ has been built for $\mathrm{X} = \Sigma_q$, we have an S-homomorphism

$$\mathrm{O}(q)/\mu_2 \to \mathrm{Aut}_{\mathrm{R}_{\mathrm{S}'/\mathrm{S}}(\mathrm{C}')/\mathrm{S}}. \tag{C.6.2}$$

This automorphism scheme is an étale form of $(\mathrm{PGL}_2 \times \mathrm{PGL}_2) \rtimes (\mathbf{Z}/2\mathbf{Z})_{\mathrm{S}}$, so it is a smooth S-affine S-group and the natural map $\mathrm{R}_{\mathrm{S}'/\mathrm{S}}(\mathrm{Aut}_{\mathrm{C}'/\mathrm{S}'}) \to \mathrm{Aut}_{\mathrm{R}_{\mathrm{S}'/\mathrm{S}}(\mathrm{C}')/\mathrm{S}}$ is an open and closed immersion onto the open relative identity component. Thus, we get an induced S-homomorphism

$$\mathrm{SO}(q)/\mu_2 \to \mathrm{R}_{\mathrm{S}'/\mathrm{S}}(\mathrm{Aut}_{\mathrm{C}'/\mathrm{S}'}) \tag{C.6.3}$$

between adjoint semisimple S-groups with the same constant fiber dimension. Arguing similarly to the treatment of α_q in Example C.6.2 (by passing to q_4 over $\mathbf{Z}$), it follows that (C.6.2) is an isomorphism.

To link the algebraic and geometric methods, we claim that the composite S-group isomorphism

$$\mathrm{R}_{\mathrm{S}'/\mathrm{S}}(\mathrm{Aut}_{\mathrm{C}'/\mathrm{S}'}) \simeq \mathrm{SO}(q)/\mu_2 \simeq \mathrm{R}_{\mathrm{Z}_q/\mathrm{S}}(\mathrm{C}_0(\mathrm{V}, q)^{\times}/\mathbf{G}_{\mathrm{m}})$$

arises from a unique pair (β_q, h_q) consisting of an S-isomorphism $\beta_q : \mathrm{S}' \simeq \mathrm{Z}_q$ and isomorphism $h_q : \mathrm{Aut}_{\mathrm{C}'/\mathrm{S}'} \simeq \mathrm{C}_0(\mathrm{V}, q)^{\times}/\mathbf{G}_{\mathrm{m}}$ over β_q. Since $\mathrm{PGL}_2 \times \mathrm{PGL}_2$ is the open relative identity component in its own automorphism scheme and $\mathrm{SO}(q)$ has connected fibers, the claim follows from a descent argument similar to the general construction of $(\mathrm{S}' \to \mathrm{S}, \mathrm{C}', \varphi)$ above. Thus, $\mathrm{S}' \to \mathrm{S}$ corresponds to the center Z_q of $\mathrm{C}_0(\mathrm{V}, q)$, and the map $\delta : \mathrm{H}^1(\mathrm{Z}_q, \mathrm{PGL}_2) \to \mathrm{Br}(\mathrm{Z}_q)[2]$ carries the class of the $\mathbf{P}^1$-bundle (more precisely, the associated PGL_2-torsor) C' over $\mathrm{S}' = \mathrm{Z}_q$ to the class of the quaternion algebra $\mathrm{C}_0(\mathrm{V}, q)$ over Z_q. However, $\ker \delta$ can be nontrivial (e.g., when S is Dedekind and $\mathrm{Pic}(\mathrm{Z}_q) \neq 1$), and working with cohomology only keeps track of torsors and Azumaya algebras up to an equivalence. Hence, it is better to express the link between geometry and algebra via the canonical β_q and h_q.

Remark C.6.4. — Example C.6.3 can be made concrete for $\mathrm{S} = \mathrm{Spec}\, k$ with a finite field k, as follows. Since central simple algebras over finite fields are matrix algebras, and $\mathbf{P}^1$-bundles over finite fields are trivial, we get an isomorphism $\mathrm{SO}(q)/\mu_2 \simeq \mathrm{R}_{k'/k}(\mathrm{PGL}_2)$ for some quadratic étale k-algebra k'. Assume that q is non-split, so $\mathrm{SO}(q)$ is not k-split (by Proposition C.3.14) and hence k' must be a field. The isomorphism $\mathrm{SO}(q)/\mu_2 \simeq \mathrm{R}_{k'/k}(\mathrm{PGL}_2)$ lifts uniquely to an isomorphism $\mathrm{Spin}(q) \simeq \mathrm{R}_{k'/k}(\mathrm{SL}_2)$ between the simply connected central covers, so for non-split q of rank 4 over a finite field k,

$$\mathrm{Spin}(q)(k)/\mathrm{Z}_{\mathrm{Spin}(q)}(k) \simeq \mathrm{R}_{k'/k}(\mathrm{SL}_2)(k)/\mathrm{R}_{k'/k}(\mu_2)(k) = \mathrm{SL}_2(k')/\mu_2(k')$$

with k' a quadratic extension field.

Example C.6.5. — Suppose $n = 5$. This corresponds to the isomorphism $\mathrm{B}_2 = \mathrm{C}_2$ for adjoint groups, which says that SO_5 is isomorphic to the quotient of Sp_4 by its center μ_2. We shall work over a base scheme S that is $\mathbf{Z}_{(2)}$-flat at all points of residue characteristic 2. This includes the base scheme $\mathrm{Spec}\,\mathbf{Z}$, so we will get an isomorphism $\mathrm{Sp}_4 \simeq \mathrm{Spin}_5$ over $\mathbf{Z}$, hence over any scheme (including $\mathbf{F}_2$-schemes) by base change.

To that end, consider a rank-4 symplectic space (V, ω_0) over such an S, with $\omega_0 \in \wedge^2(\mathrm{V})^* = \wedge^2(\mathrm{V}^*)$ the given symplectic form. The rank-6 vector bundle $\wedge^2(\mathrm{V})$ contains the rank-5 subbundle W of sections killed by ω_0, and on $\wedge^2(\mathrm{V})$ there exists a natural non-degenerate quadratic form q valued in $\mathrm{L} = \det(\mathrm{V})$ defined by $\omega \mapsto \omega \wedge \omega$ that is actually valued in the subsheaf $2\mathrm{L} \subset \mathrm{L}$. Thus, by the $\mathbf{Z}_{(2)}$-flatness hypothesis on S we can define the L-valued quadratic form $q(\omega) = (1/2)(\omega \wedge \omega)$ that is readily checked to be fiberwise non-degenerate. The action of $\mathrm{SL}(\mathrm{V})$ clearly preserves q, the restriction $q|_{\mathrm{W}}$ is non-degenerate (by calculation), and $\mathrm{Sp}(\omega_0)$ preserves W (due to the definition of W). Thus, the $\mathrm{Sp}(\omega_0)$-action on W defines a homomorphism $\mathrm{Sp}(\omega_0) \to \mathrm{O}(q|_{\mathrm{W}})$ that obviously kills the center μ_2. We claim that the resulting homomorphism $f : \mathrm{Sp}(\omega_0)/\mu_2 \to \mathrm{O}(q|_{\mathrm{W}})$ is an isomorphism onto $\mathrm{SO}(q|_{\mathrm{W}})$. For this purpose we may work Zariski-locally on S so that the symplectic space (V, ω_0) is standard. Now our problem is a base change from $\mathbf{Z}$, so we may assume $\mathrm{S} = \mathrm{Spec}\,\mathbf{Z}$.

By using connectedness of symplectic groups over $\mathbf{Q}$ we see that f factors through $\mathrm{SO}(q|_{\mathrm{W}})$, so we obtain a map $h : \mathrm{Sp}_4/\mu_2 \to \mathrm{SO}_5$ over $\mathbf{Z}$ that we need to prove is an isomorphism. A computation shows that $\ker h$ has trivial intersection with the "diagonal" maximal torus, so $\ker h$ is quasi-finite (as it is normal in a reductive group) and hence h is surjective for fibral dimension reasons. This forces h to be finite flat (Proposition 6.1.10), so h has locally constant fiber degree that we claim is 1. It suffices to compute the fiber degree over $\mathbf{Q}$. Isogenies between connected semisimple groups of adjoint type are isomorphisms in characteristic 0 (false in characteristic > 0, via Frobenius).

Example C.6.6. — Suppose $n = 6$. This case corresponds to the equality $\mathrm{D}_3 = \mathrm{A}_3$ for groups that are neither adjoint nor simply connected: we claim that the $\mathbf{Z}$-group SO_6 is the quotient of SL_4 by the subgroup μ_2 in the central μ_4 (and hence likewise over any scheme by base change). To construct the isomorphism, we work in the setup of Example C.6.5 (over $\mathrm{Spec}\,\mathbf{Z}$) and again use the natural action of $\mathrm{SL}(\mathrm{V})$ on the rank-6 bundle $\wedge^2(\mathrm{V})$ equipped with the non-degenerate quadratic form $q(\omega) = (1/2)(\omega \wedge \omega)$ valued in the line bundle $\det \mathrm{V}$. The homomorphism $\mathrm{SL}(\mathrm{V}) \to \mathrm{O}(q)$ defined in this way clearly kills the central μ_2, and it factors through $\mathrm{SO}(q)$ (as $\mathrm{O}(q)/\mathrm{SO}(q) = \mathbf{Z}/2\mathbf{Z}$).

To prove that the resulting map $h : \mathrm{SL}(\mathrm{V})/\mu_2 \to \mathrm{SO}(q)$ is a $\mathbf{Z}$-isomorphism, as in Example C.6.5 we reduce to the isomorphism problem over $\mathbf{Q}$. Isogenies between smooth connected groups in characteristic 0 are always central, and μ_4/μ_2 is not killed by h, so we are done.

Appendix D

Proof of Existence Theorem over $\mathbf{C}$

In this appendix, we prove the Existence Theorem over $\mathbf{C}$. The main difficulty is the construction of connected semisimple groups whose semisimple root datum is simply connected and has irreducible underlying root system. (See Definition 1.3.11 and Lemma 6.3.1.) Over any algebraically closed field k of characteristic 0, algebraic methods can easily settle the case of semisimple reduced root data that are adjoint (see Proposition D.1.1). From that case, one can bootstrap to simply connected semisimple reduced root data via an existence theorem for *projective* representations (modeled on the Borel–Weil construction for semisimple complex Lie groups). This method is explained in [**BIBLE**, 15.3, 23.1], and is cited in [**SGA3**, XXV, 1.4]. In this appendix, we present an alternative analytic argument via covering space methods over the ground field $\mathbf{C}$, linking up the analytic viewpoint with algebraic techniques. The references we shall use are [**BtD**] for the "algebraicity" of compact Lie groups and [**Ho65**] for general facts related to complex Lie groups and maximal compact subgroups of Lie groups with a finite component group. An elegant summary (without proofs) of much of this analytic background is given in [**Ser01**, VIII].

But first, we sketch an algebraic proof based on enveloping algebras; this can be extracted from [**Ho70**] and was pointed out by P. Polo. Let $\mathfrak{g}$ be a semisimple Lie algebra over k having the desired root system (this exists, by [**Hum72**, 18.4(a)]). Consider the universal enveloping algebra $\mathrm{U}(\mathfrak{g})$, an associative k-algebra whose representation theory coincides with that of $\mathfrak{g}$. As J varies through the 2-sided ideals of finite codimension in $\mathrm{U}(\mathfrak{g})$, the dual spaces $(\mathrm{U}(\mathfrak{g})/\mathrm{J})^*$ exhaust the k-algebra generated by the "matrix coefficients" (on $\mathrm{U}(\mathfrak{g})$) of the finite-dimensional representations of $\mathfrak{g}$. This equips $\mathrm{H}(\mathfrak{g}) := \varinjlim(\mathrm{U}(\mathfrak{g})/\mathrm{J})^*$ with a natural commutative k-algebra structure and a compatible Hopf algebra structure that defines an affine k-group scheme $\mathrm{G}(\mathfrak{g}) = \operatorname{Spec} \mathrm{H}(\mathfrak{g})$ (see [**Ho70**, § 3]). Every dominant integral weight for $\mathfrak{g}$ is a $\mathbf{Z}_{\geqslant 0}$-linear combination of the finitely many fundamental weights (dual basis to the coroots), so by highest-weight theory for $\mathfrak{g}$ the k-algebra $\mathrm{H}(\mathfrak{g})$ is finitely generated (see the end of [**Ho59**, § 5]). The linear dual of $\mathrm{U}(\mathfrak{g})$ is a k-algebra that contains $\mathrm{H}(\mathfrak{g})$ as a subalgebra, so $\mathrm{H}(\mathfrak{g})$ is a domain since $\mathrm{U}(\mathfrak{g})^*$ is a formal power series ring (see [**Ho59**, § 2]). Hence, $\mathrm{G}(\mathfrak{g})$ is connected and smooth.

By [**Ho70**, 3.1] (or [**Ho70**, 6.1]), $\mathfrak{g}$ is identified with $\mathrm{Lie}(\mathrm{G}(\mathfrak{g}))$ (so $\mathrm{G}(\mathfrak{g})$ is semisimple; see Exercise 6.5.5). Every finite-dimensional $\mathfrak{g}$-representation is naturally an $\mathrm{H}(\mathfrak{g})$-comodule, which is to say a $\mathrm{G}(\mathfrak{g})$-representation, and this is inverse to applying the Lie functor to a $\mathrm{G}(\mathfrak{g})$-representation. Since every representation $\mathfrak{g} \to \mathfrak{gl}(\mathrm{V})$ has just been "integrated" to a representation

$G(\mathfrak{g}) \to GL(V)$, an argument with highest-weight theory for $\mathfrak{g}$ (as in the proof of "$\Rightarrow$" in Proposition D.4.1) shows $G(\mathfrak{g})$ has a simply connected root datum.

D.1. Preliminary steps. — By the self-contained (and rather formal) Lemma 6.3.1, to prove the Existence Theorem over $\mathbf{C}$ it suffices to consider semisimple root data R that are simply connected. We will approach such R through preliminary consideration of the adjoint case, beginning with a basic existence result over any algebraically closed field k with $\mathrm{char}(k) = 0$.

Proposition D.1.1. — *Let* $R = (X, \Phi, X^\vee, \Phi^\vee)$ *be an adjoint semisimple reduced root datum. There exists a connected semisimple k-group $(\mathscr{G}, \mathscr{T})$ such that* $R(\mathscr{G}, \mathscr{T}) \simeq R$*.*

Proof. — By [**Hum72**, 18.4(a)], there exists a semisimple Lie algebra $\mathfrak{g}$ over k and a Cartan subalgebra $\mathfrak{t}$ such that the root system $\Phi(\mathfrak{g}, \mathfrak{t})$ (see [**Hum72**, 8.5, 15.3]) coincides with $(X_{\mathbf{Q}}, \Phi)$. By Ado's theorem [**Bou1**, I, § 7.3, Thm. 2], there exists an injective map of Lie algebras $\mathfrak{g} \hookrightarrow \mathfrak{gl}_n$ over k. Since $\mathfrak{g}$ is its own derived subalgebra (due to semisimplicity) and $\mathrm{char}(k) = 0$, it is an "algebraic" subalgebra of $\mathfrak{gl}_n$ in the sense that there exists a (unique) connected linear algebraic k-subgroup $G \subset GL_n$ satisfying $\mathrm{Lie}(G) = \mathfrak{g}$ inside $\mathfrak{gl}_n$ [**Bo91**, 7.9]. The k-group G is necessarily semisimple, since the Lie algebra $\mathrm{Lie}(\mathscr{R}(G))$ of the radical is a solvable ideal in the semisimple $\mathfrak{g}$ and hence vanishes (so $\mathscr{R}(G) = 1$).

Let $T \subset G$ be a maximal torus, so the abelian subalgebra $\mathrm{Lie}(T) \subset \mathfrak{g}$ is its own Lie-theoretic centralizer (as $T = Z_G(T)$ and $\mathrm{char}(k) = 0$). Hence, $\mathrm{Lie}(T)$ is a Cartan subalgebra of the semisimple Lie algebra $\mathfrak{g}$ (i.e., it is maximal among commutative subalgebras whose elements have semisimple adjoint action on $\mathfrak{g}$). All Cartan subalgebras of $\mathfrak{g}$ are in the same $\mathrm{Aut}(\mathfrak{g})$-orbit [**Hum72**, 16.2], so we get an isomorphism of root systems

$$(X(T)_{\mathbf{Q}}, \Phi(G, T)) = \Phi(\mathfrak{g}, \mathrm{Lie}(T)) \simeq \Phi(\mathfrak{g}, \mathfrak{t}) = (X_{\mathbf{Q}}, \Phi).$$

This identifies the Weyl groups $W(\Phi(G, T))$ and $W(\Phi)$ since the root system determines the Weyl group (without reference to a Euclidean structure). The roots determine the coroots since the reflection $s_a : v \mapsto v - \langle v, a^\vee \rangle a$ in a is uniquely determined by preservation of the set of roots, so the isomorphism $X(T)_{\mathbf{Q}} \simeq X_{\mathbf{Q}}$ identifies $\Phi(G, T)^\vee$ with $\Phi^\vee$ as well.

Since $X(T/Z_G) = \mathbf{Z}\Phi(G, T)$ by Corollary 3.3.6(1), and $\mathbf{Z}\Phi = X$ by our hypothesis that R is adjoint, the equality $\Phi(G, T) = \Phi$ via $X(T)_{\mathbf{Q}} \simeq X_{\mathbf{Q}}$ forces the compatible identification $X(T/Z_G) \simeq X$. This yields an isomorphism of root data $R(G/Z_G, T/Z_G) \simeq (X, \Phi, X^\vee, \Phi^\vee) = R$. Set $(\mathscr{G}, \mathscr{T}) = (G/Z_G, T/Z_G)$. $\square$

Fix a semisimple reduced root datum $\mathrm{R} = (\mathrm{X}, \Phi, \mathrm{X}^\vee, \Phi^\vee)$ that is simply connected. Let $\mathrm{R}^{\mathrm{ad}} = (\mathbf{Z}\Phi, \Phi, (\mathbf{Z}\Phi)^*, \Phi^\vee)$ denote the associated adjoint semisimple root datum. Let k be an algebraically closed field of characteristic 0. By Proposition D.1.1, there exists a connected semisimple k-group (G, T) such that $\mathrm{R}(\mathrm{G}, \mathrm{T}) \simeq \mathrm{R}^{\mathrm{ad}}$. We seek a connected semisimple k-group G' and an isogeny $h : \mathrm{G}' \to \mathrm{G}$ such that for the maximal torus $\mathrm{T}' = h^{-1}(\mathrm{T})$ in G' there exists a (necessarily unique) isomorphism $\mathrm{R}(\mathrm{G}', \mathrm{T}') \simeq \mathrm{R}$ over the isomorphism $\mathrm{R}(\mathrm{G}, \mathrm{T}) \simeq \mathrm{R}^{\mathrm{ad}}$ via the central isogenies $\mathrm{R}(\mathrm{G}', \mathrm{T}') \to \mathrm{R}(\mathrm{G}, \mathrm{T})$ and $\mathrm{R} \to \mathrm{R}^{\mathrm{ad}}$. Over $\mathbf{C}$ we can build covers via topology, so now we set $k = \mathbf{C}$ and pass to the analytic theory.

D.2. Compact and complex Lie groups. — For a connected semisimple $\mathbf{C}$-group G, the group $\mathrm{G}(\mathbf{C})$ is connected for the analytic topology (since for an open cell $\Omega \subset \mathrm{G}$ around 1, $\Omega(\mathbf{C})$ is visibly connected and $g\Omega(\mathbf{C})$ meets $\Omega(\mathbf{C})$ for any $g \in \mathrm{G}(\mathbf{C})$, due to the irreducibility of G). Thus, $\mathrm{G}(\mathbf{C})$ is a connected Lie group. To build isogenous covers of G, we will use maximal compact subgroups of $\mathrm{G}(\mathbf{C})$ and topological facts from the theory of Lie groups.

Proposition D.2.1. — *The functor* $\mathrm{G} \rightsquigarrow \mathrm{G}(\mathbf{C})$ *from connected reductive* $\mathbf{C}$*-groups to complex Lie groups is fully faithful. More generally, if* G *is a connected reductive* $\mathbf{C}$*-group and* G' *is a linear algebraic* $\mathbf{C}$*-group then every analytic homomorphism* $f : \mathrm{G}(\mathbf{C}) \to \mathrm{G}'(\mathbf{C})$ *arises from a unique* $\mathbf{C}$*-homomorphism* $\mathrm{G} \to \mathrm{G}'$.

The essential image of the functor in Proposition D.2.1 is identified in Example D.3.3.

Proof. — First we treat semisimple G, and then we allow a central torus. Assume G is semisimple, so the Lie algebra $\mathfrak{g}$ is semisimple (Exercise 6.5.5). The graph Γ_f is a connected closed complex Lie subgroup of $\mathrm{G}(\mathbf{C}) \times \mathrm{G}'(\mathbf{C})$, so its Lie algebra is a subalgebra of $\mathfrak{g} \bigoplus \mathfrak{g}'$ that projects isomorphically onto the semisimple $\mathfrak{g}$ and hence is its own derived subalgebra. But in characteristic 0, a Lie subalgebra $\mathfrak{h}$ of the Lie algebra of a linear algebraic group arises from a connected linear algebraic subgroup provided that $\mathfrak{h}$ is its own derived subalgebra [**Bo91**, 7.9]. We thereby get a connected linear algebraic subgroup $\mathrm{H} \subset \mathrm{G} \times \mathrm{G}'$ such that $\mathrm{Lie}(\mathrm{H}) = \mathrm{Lie}(\Gamma_f)$ inside $\mathfrak{g} \bigoplus \mathfrak{g}'$, and H is semisimple since $\mathrm{Lie}(\mathrm{H}) \simeq \mathfrak{g}$ is semisimple. The connected closed Lie subgroups $\mathrm{H}(\mathbf{C})$ and Γ_f in $\mathrm{G}(\mathbf{C}) \times \mathrm{G}'(\mathbf{C})$ must therefore be equal. Thus, the algebraic projection $\mathrm{H} \to \mathrm{G}$ induces a holomorphic isomorphism on $\mathbf{C}$-points, so it is étale and bijective on $\mathbf{C}$-points, hence an isomorphism (by Zariski's Main Theorem). It follows that the $\mathbf{C}$-subgroup H in $\mathrm{G} \times \mathrm{G}'$ is the graph of a $\mathbf{C}$-homomorphism $\mathrm{G} \to \mathrm{G}'$; this clearly analytifies to f.

Now consider the general case, so $G = (Z \times \mathscr{D}(G))/\mu$ for the maximal central torus Z, connected semisimple derived group $\mathscr{D}(G)$, and a finite central subgroup $\mu \subset Z \times \mathscr{D}(G)$. The algebraic theory of quotients by finite subgroups over $\mathbf{C}$ analytifies to the analogous theory on the analytic side, so it suffices to treat the cases of T and $\mathscr{D}(G)$. We have already settled $\mathscr{D}(G)$, so we may now assume $G = T$, and more specifically $G = \mathbf{G}_m$. Our problem is to prove the algebraicity of any holomorphic homomorphism $\mathbf{C}^\times \to G'(\mathbf{C})$. Using a faithful representation $G' \hookrightarrow GL_n$, we are reduced to checking the algebraicity of any holomorphic action of $\mathbf{C}^\times$ on a finite-dimensional $\mathbf{C}$-vector space V. The action of each finite subgroup $\mathbf{C}^\times[n] = \mu_n$ diagonalizes, and the $\mathbf{C}^\times$-action preserves each μ_n-isotypic subspace of V. Thus, by induction on $\dim V$ we can assume that *every* μ_n acts through a character $\chi_n : \mu_n \to \mathbf{C}^\times$. This says that the given holomorphic map $\rho : \mathbf{C}^\times \to GL(V)$ carries each μ_n into the central $Z = \mathbf{C}^\times$, so ρ does as well (since the complex-analytic subgroup $\rho^{-1}(Z) \subset \mathbf{C}^\times$ contains all μ_n, so it is infinite and thus exhausts the connected 1-dimensional $\mathbf{C}^\times$). In other words, we are reduced to showing that a holomorphic homomorphism $\chi : \mathbf{C}^\times \to \mathbf{C}^\times$ must be $\chi(z) = z^m$ for some $m \in \mathbf{Z}$. Since $\mathbf{C}^\times = \mathbf{C}/2\pi i\mathbf{Z}$ with $\mathbf{C}$ simply connected, this clearly holds. □

Remark D.2.2. — The analogue of Proposition D.2.1 over $\mathbf{R}$ fails, even for semisimple groups whose set of $\mathbf{R}$-points is connected. For example, the natural isogeny $SL_{2n+1} \to PGL_{2n+1}$ of degree $2n+1$ induces an isomorphism on $\mathbf{R}$-points, the inverse of which is non-algebraic.

Corollary D.2.3. — *The finite-dimensional holomorphic representations of* $G(\mathbf{C})$ *are completely reducible for any connected reductive* $\mathbf{C}$*-group* G.

Proof. — By Proposition D.2.1, it suffices to prove the analogous result for the finite-dimensional algebraic representations of G. As in the proof of Proposition D.2.1, we immediately reduce to the separate cases of tori and semisimple G. The case of tori is clear (weight spaces), so we can assume that G is semisimple. Let $\rho : G \to GL(V)$ be a finite-dimensional representation, and let $\mathfrak{g} \to End(V)$ be the associated Lie algebra representation. Since we are in characteristic 0 and G is connected, a linear subspace of V is G-stable if and only if it is $\mathfrak{g}$-stable. Hence, it is sufficient to prove complete reducibility of the representation theory of $\mathfrak{g}$. This in turn follows from the fact that $\mathfrak{g}$ is semisimple (by Exercise 6.5.5). □

The mechanism by which we will keep track of algebraicity when working on the analytic side is to use maximal compact subgroups of complex Lie groups and the "algebraicity" of the theory of compact Lie groups:

Theorem D.2.4 **(Chevalley–Tannaka).** — *The functor* $\mathrm{H} \rightsquigarrow \mathrm{H}(\mathbf{R})$ *is an equivalence from the category of* $\mathbf{R}$*-anisotropic reductive* $\mathbf{R}$*-groups whose connected components contain* $\mathbf{R}$*-points to the category of compact Lie groups.*

The $\mathbf{R}$*-group* H *is connected if and only if* $\mathrm{H}(\mathbf{R})$ *is connected, and* H *is semisimple if and only if the Lie group* $\mathrm{H}(\mathbf{R})$ *has semisimple Lie algebra.*

Proof. — The semisimplicity criterion for H follows from Exercise 6.5.5 since $\mathrm{Lie}(\mathrm{H}(\mathbf{R})) = \mathrm{Lie}(\mathrm{H})$ (which is semisimple over $\mathbf{R}$ if and only if the complex Lie algebra $\mathrm{Lie}(\mathrm{H})_{\mathbf{C}} = \mathrm{Lie}(\mathrm{H}_{\mathbf{C}})$ is semisimple). It is not at all obvious that $\mathrm{H}(\mathbf{R})$ is compact when H is $\mathbf{R}$-anisotropic, nor that $\mathrm{H}(\mathbf{R})$ is connected when H is moreover connected. We begin by proving the compactness of $\mathrm{H}(\mathbf{R})$ via a variant of the method used by G. Prasad in his elementary proof of the analogous result over non-archimedean local fields in [**Pr**].

Fix a faithful finite-dimensional representation $\rho : \mathrm{H} \hookrightarrow \mathrm{GL}(\mathrm{V})$ over $\mathbf{R}$ (i.e., $\ker\rho = 1$). By Exercise 1.6.11, $\mathrm{V}_{\mathbf{C}}$ is completely reducible as an $\mathrm{H}_{\mathbf{C}}$-representation. Let $\bigoplus \mathrm{W}_j$ be a decomposition of $\mathrm{V}_{\mathbf{C}}$ into a direct sum of irreducible subrepresentations of $\mathrm{H}(\mathbf{C})$. The group $\mathrm{H}^0(\mathbf{R})$ is Zariski-dense in H^0 by the unirationality of connected reductive groups over fields [**Bo91**, 18.2(ii)]. By hypothesis every connected component of H contains an $\mathbf{R}$-point and hence is an $\mathrm{H}(\mathbf{R})$-translate of H^0, so $\mathrm{H}(\mathbf{R})$ is Zariski-dense in H. Thus, $\mathrm{H}(\mathbf{R})$ is Zariski-dense in $\mathrm{H}_{\mathbf{C}}$, so each W_j is irreducible as a representation of $\mathrm{H}(\mathbf{R})$. If $\mathrm{H}(\mathbf{R})$ has compact image in each $\mathrm{GL}(\mathrm{W}_j)$ then $\mathrm{H}(\mathbf{R})$ is compact. The image H_j of H in $\mathrm{R}_{\mathbf{C}/\mathbf{R}}(\mathrm{GL}(\mathrm{W}_j))$ is $\mathbf{R}$-anisotropic, and $\mathrm{H}_j(\mathbf{R})$ must be Zariski-dense in H_j since even the image of $\mathrm{H}(\mathbf{R})$ in H_j is Zariski-dense. We may therefore replace H with each H_j to reduce to the case that H is a closed subgroup of $\mathrm{R}_{\mathbf{C}/\mathbf{R}}(\mathrm{GL}(\mathrm{W}))$ with W irreducible as a $\mathbf{C}$-linear representation of $\mathrm{H}(\mathbf{R})$.

For $h \in \mathrm{H}(\mathbf{R})$, its eigenvalues λ on W satisfy $|\lambda| = 1$. Indeed, by Jordan decomposition in $\mathrm{H}(\mathbf{R})$ we may assume h is semisimple, and it is harmless to replace h with h^n for $n > 0$. Thus, we can assume that h lies in the identity component of the commutative semisimple Zariski closure of $h^{\mathbf{Z}}$. That identity component is an $\mathbf{R}$-torus T. All $\mathbf{R}$-tori in H are $\mathbf{R}$-anisotropic and hence have a compact group of $\mathbf{R}$-points, so the eigenvalues of the integral powers of h are bounded in $\mathbf{C}$. This forces the eigenvalues to lie on the unit circle.

Since $\mathrm{H}(\mathbf{R})$ acts irreducibly on W, its image in $\mathrm{GL}(\mathrm{W})$ must generate the $\mathbf{C}$-algebra $\mathrm{End}(\mathrm{W})$ (Burnside), so $\mathrm{H}(\mathbf{R})$ spans $\mathrm{End}(\mathrm{W})$ over $\mathbf{C}$. Choose $\{h_j\}$ in $\mathrm{H}(\mathbf{R})$ that is a $\mathbf{C}$-basis of $\mathrm{End}(\mathrm{W})$, and consider the dual basis $\{\mathrm{L}_j\}$ under the perfect $\mathbf{C}$-bilinear trace pairing $(\mathrm{T}, \mathrm{T}') = \mathrm{Tr}(\mathrm{T} \circ \mathrm{T}')$ on $\mathrm{End}(\mathrm{W})$. The eigenvalue condition forces $|\mathrm{Tr}(hh_j)| \leqslant n := \dim \mathrm{W}$ for all $h \in \mathrm{H}(\mathbf{R})$, so (exactly as in the proof of finite generation of integer rings of number fields) the coefficients of $\rho(h)$ relative to $\{\mathrm{L}_j\}$ are bounded in $\mathbf{C}$ independently of h. Hence, the closed set $\rho(\mathrm{H}(\mathbf{R}))$ in $\mathrm{GL}(\mathrm{W})$ is bounded in $\mathrm{End}(\mathrm{W})$. It is also closed in $\mathrm{End(W)}$ because $|\det(\rho(h))| = 1$ for all $h \in \mathrm{H}(\mathbf{R})$ (due to the eigenvalues of $\rho(h)$ lying on the unit circle), so $\rho(\mathrm{H}(\mathbf{R}))$ is compact.

To establish the desired equivalence of categories between anisotropic reductive $\mathbf{R}$-groups and compact Lie groups (including the connectedness aspect), we shall now proceed in reverse by constructing a functor from compact Lie groups to reductive $\mathbf{R}$-groups. This rests on the *real representation algebra* $\mathrm{R(K)}$ of a compact Lie group K: the $\mathbf{R}$-algebra generated inside the $\mathbf{R}$-algebra of continuous $\mathbf{R}$-valued functions on K by the matrix entries of the finite-dimensional continuous representations of K over $\mathbf{R}$. Let $\mathrm{R}(\mathrm{K}, \mathbf{C})$ denote the analogue defined using continuous linear representations over $\mathbf{C}$. The operation of scalar extension of $\mathbf{R}$-linear representations to $\mathbf{C}$-linear representations defines an $\mathbf{R}$-algebra injection $\mathrm{R(K)} \to \mathrm{R}(\mathrm{K}, \mathbf{C})$. There is a natural involution ι of $\mathrm{R}(\mathrm{K}, \mathbf{C})$ over complex conjugation on $\mathbf{C}$ (using scalar extension through complex conjugation on $\mathbf{C}$-linear representations of K), and this acts trivially on $\mathrm{R(K)}$. Galois descent gives that $\mathrm{R}(\mathrm{K}, \mathbf{C}) = \mathbf{C} \otimes_{\mathbf{R}} \mathrm{R}(\mathrm{K}, \mathbf{C})^{\iota=1}$, so the natural map of $\mathbf{C}$-algebras

$$\mathbf{C} \otimes_{\mathbf{R}} \mathrm{R(K)} \to \mathrm{R}(\mathrm{K}, \mathbf{C}) \tag{D.2.1}$$

is injective. But every $\mathbf{C}$-linear representation can be viewed as an $\mathbf{R}$-linear representation, so (via a choice of $i = \sqrt{-1} \in \mathbf{C}$) if $f = u + iv \in \mathrm{R}(\mathrm{K}, \mathbf{C})$ with $\mathbf{R}$-valued u and v then $u, v \in \mathrm{R(K)}$. In other words, (D.2.1) is an isomorphism.

There exists a faithful representation $\rho : \mathrm{K} \hookrightarrow \mathrm{GL}_n(\mathbf{R})$ (see [**BtD**, III, 4.1], which rests on the Peter–Weyl theorem). Upon choosing such a ρ, we get an $\mathbf{R}$-algebra map $\mathbf{R}[x_{ij}][1/\det] \to \mathrm{R(K)}$. The representation algebra is generated by the matrix functions from any single faithful representation [**BtD**, III, 1.4(iii), 2.7(i)], so this map is surjective. In particular, $\mathrm{R(K)}$ is finitely generated over $\mathbf{R}$, and likewise $\mathrm{R}(\mathrm{K}, \mathbf{C})$ is finitely generated over $\mathbf{C}$. There is a natural Hopf algebra structure on $\mathrm{R(K)}$ that is contravariant in K (see [**BtD**, III, § 7]), and by Tannaka duality [**BtD**, III, 7.15] the natural map $\mathrm{K} \to (\mathrm{Spec}\,\mathrm{R(K)})(\mathbf{R})$ defined via evaluation is an isomorphism of Lie groups. The linear algebraic $\mathbf{R}$-group $\mathrm{K}^{\mathrm{alg}} := \mathrm{Spec}\,\mathrm{R(K)}$ may be disconnected, but we can detect its connected components using $\mathbf{R}$-points:

Lemma D.2.5. — *The locus* $\mathrm{K} = \mathrm{K}^{\mathrm{alg}}(\mathbf{R})$ *is Zariski-dense in* $\mathrm{K}^{\mathrm{alg}}$*, and* $(\mathrm{K}^{\mathrm{alg}})^0$ *is* $\mathbf{R}$*-anisotropic reductive. In particular, each connected component of* $\mathrm{K}^{\mathrm{alg}}$ *contains an* $\mathbf{R}$*-point and hence is geometrically connected over* $\mathbf{R}$*.*

Proof. — By definition $\mathrm{R(K)}$ is a subalgebra of the $\mathbf{R}$-algebra of continuous $\mathbf{R}$-valued functions on K, so every element of the finite type $\mathbf{R}$-algebra $\mathrm{R(K)}$ is uniquely determined by its restriction to the set K of $\mathbf{R}$-points. This exactly encodes the Zariski-density property in the affine algebraic $\mathbf{R}$-scheme $\mathrm{K}^{\mathrm{alg}}$.

To prove reductivity, assume to the contrary. Any nontrivial connected unipotent group over a perfect field contains the additive group as a closed subgroup, so K^{alg} contains $\mathbf{G}_a$ as a closed $\mathbf{R}$-subgroup. Hence, $\mathbf{R} = \mathbf{G}_a(\mathbf{R})$ is a closed subgroup of the group $K^{alg}(\mathbf{R}) = K$ that is compact, a contradiction. The same reasoning shows that the connected reductive group $(K^{alg})^0$ is $\mathbf{R}$-anisotropic. $\square$

Every continuous homomorphism $f : K \to K'$ between compact Lie groups yields a map of Hopf algebras $R(K') \to R(K)$ by composing K'-representations with f. The corresponding $\mathbf{R}$-group homomorphism $f^{alg} : K^{alg} \to K'^{alg}$ recovers f on $\mathbf{R}$-points, and it is uniquely determined by this condition since K is Zariski-dense in K^{alg}. In particular, for *any* $\mathbf{R}$-group map $\varphi : K^{alg} \to K'^{alg}$, necessarily $\varphi = f^{alg}$ where f is the restriction of φ to $\mathbf{R}$-points. Also, if K is a closed subgroup of K' then $R(K') \to R(K)$ is surjective [**BtD**, III, 4.3], so f^{alg} is a closed immersion when f is injective.

Lemma D.2.6. — $K^0 = (K^{alg})^0(\mathbf{R})$ *and* $(K^{alg})^0 = (K^0)^{alg}$.

Proof. — Let $j : K^0 \to K$ be the natural map, so j^{alg} is a closed immersion since j is injective. The dimensions of $(K^0)^{alg}$ and K^{alg} coincide because the Lie algebra of a linear algebraic $\mathbf{R}$-group can be naturally computed using the Lie group of $\mathbf{R}$-points. Thus, j^{alg} is an open and closed immersion, so the identity component of K^{alg} is the same as that of $(K^0)^{alg}$.

We claim that $(K^0)^{alg}$ is the identity component of K^{alg}; i.e., $(K^0)^{alg}$ is connected. Since $(K^{alg})^0(\mathbf{R})$ is an open and closed subgroup of $K^{alg}(\mathbf{R}) = K$, it contains K^0. But we just saw that $(K^{alg})^0 \subset (K^0)^{alg}$, so

$$K^0 \subset (K^{alg})^0(\mathbf{R}) \subset (K^0)^{alg}(\mathbf{R}) = K^0,$$

forcing the containment $(K^{alg})^0 \subset (K^0)^{alg}$ to be an equality on $\mathbf{R}$-points and hence an equality of $\mathbf{R}$-groups (due to the Zariski-density of $\mathbf{R}$-points in K^{alg} and $(K^0)^{alg}$). $\square$

To summarize, we have defined a fully faithful functor $K \rightsquigarrow K^{alg}$ from the category of compact Lie groups into the category of anisotropic reductive $\mathbf{R}$-groups all of whose connected components contain $\mathbf{R}$-points, and this functor is compatible with identity components (i.e., K is connected if and only if K^{alg} is connected).

It remains to show that if G is an anisotropic reductive $\mathbf{R}$-group whose connected components have $\mathbf{R}$-points then for the compact Lie group $K := G(\mathbf{R})$ (which is Zariski-dense in G) there exists an $\mathbf{R}$-group isomorphism $G \simeq K^{alg}$ extending the equality of $\mathbf{R}$-points. (Such an isomorphism is unique if it exists, due to the Zariski-density of $\mathbf{R}$-points in these $\mathbf{R}$-groups, and likewise it is necessarily functorial in G.)

Since $\mathscr{O}(G)$ is exhausted by finite-dimensional G-stable $\mathbf{R}$-subspaces [**Bo91**, I, § 1.9], we get a restriction map from $\mathscr{O}(G)$ to the real representation algebra

R(K). This map is injective, due to the Zariski-density of K in G. To complete the proof of Theorem D.2.4, we just have to check that the map $\mathscr{O}(\mathrm{G}) \hookrightarrow \mathrm{R}(\mathrm{K}) = \mathscr{O}(\mathrm{K}^{\mathrm{alg}})$ of $\mathbf{R}$-algebras is an isomorphism of Hopf algebras.

The corresponding $\mathbf{R}$-scheme map $\mathrm{K}^{\mathrm{alg}} \to \mathrm{G}$ induces a group isomorphism between $\mathbf{R}$-points, so it is an $\mathbf{R}$-group map (as K is Zariski-dense in $\mathrm{K}^{\mathrm{alg}}$). In other words, $\mathscr{O}(\mathrm{G}) \hookrightarrow \mathrm{R}(\mathrm{K})$ is a map of Hopf algebras. To check surjectivity, we note that a choice of faithful representation $\mathrm{G} \hookrightarrow \mathrm{GL}_n$ over $\mathbf{R}$ defines a faithful representation $\mathrm{K} \hookrightarrow \mathrm{GL}_n(\mathbf{R})$. The induced pullback map $\mathscr{O}(\mathrm{GL}_n) \to \mathrm{R}(\mathrm{K})$ is surjective, as we noted in the discussion preceding Lemma D.2.5, so this forces the map $\mathscr{O}(\mathrm{G}) \hookrightarrow \mathrm{R}(\mathrm{K})$ to be surjective. □

Recall that for any connected Lie group H, the universal covering space $q : \widetilde{\mathrm{H}} \to \mathrm{H}$ (equipped with a marked point over the identity of H) has a unique compatible structure of real Lie group, and the discrete closed normal subgroup $\ker q = \pi_1(\mathrm{H}, 1)$ is central (as in any connected Lie group). This recovers the fact that $\pi_1(\mathrm{H}, 1)$ is abelian, so $\ker q = \mathrm{H}_1(\mathrm{H}, \mathbf{Z})$. In the compact case this universal covering has finite degree:

***Proposition D.2.7* (Weyl).** — *If* K *is a connected compact Lie group and* Lie(K) *is semisimple then the universal cover* $\widetilde{\mathrm{K}} \to \mathrm{K}$ *is a finite-degree covering. In particular,* $\widetilde{\mathrm{K}}$ *is compact.*

The hypothesis on the Lie algebra cannot be dropped; consider $\mathrm{K} = \mathrm{S}^1$.

Proof. — The kernel of the covering map is the group $\mathrm{H}_1(\mathrm{K}, \mathbf{Z})$ that is finitely generated (as K is a compact manifold), so this group is finite if and only if its finite-order quotients have bounded cardinality. In other words, we seek an upper bound on the degree of isogenies $f : \mathrm{K}' \to \mathrm{K}$ from connected compact Lie groups K' onto the given group K. By Theorem D.2.4, f "algebraizes" to an isogeny of connected semisimple $\mathbf{R}$-groups $\mathrm{K}'^{\mathrm{alg}} \to \mathrm{K}^{\mathrm{alg}}$. By extending scalars to $\mathbf{C}$, we are reduced to checking that for any connected semisimple $\mathbf{C}$-group G there exists an upper bound on the degree of all connected isogenous covers $f : \mathrm{G}' \to \mathrm{G}$.

Such a bound is obtained from the root datum of G, as follows. Let $\mathrm{T} \subset \mathrm{G}$ be a maximal torus and $\mathrm{T}' = f^{-1}(\mathrm{T})$ the associated maximal torus in G', so the centrality of $\ker f$ forces $\ker f = \ker(\mathrm{T}' \to \mathrm{T})$. Hence, to bound the degree of f it is equivalent to bound the index of the inclusion of lattices $\mathrm{X}(\mathrm{T}) \to \mathrm{X}(\mathrm{T}')$. The centrality of $\ker f$ implies that the isomorphism $\mathrm{X}(\mathrm{T})_{\mathbf{Q}} \simeq \mathrm{X}(\mathrm{T}')_{\mathbf{Q}}$ identifies $\Phi(\mathrm{G}, \mathrm{T})$ with $\Phi(\mathrm{G}', \mathrm{T}')$ and that the dual isomorphism identifies $\Phi(\mathrm{G}, \mathrm{T})^\vee$ with $\Phi(\mathrm{G}', \mathrm{T}')^\vee$. (This is seen most concretely by using that f even identifies $\mathfrak{g}'$ with $\mathfrak{g}$, as we are in characteristic 0.) Thus, by (1.3.2) we have

$$\mathbf{Z}\Phi(\mathrm{G}, \mathrm{T}) \subset \mathrm{X}(\mathrm{T}) \subset \mathrm{X}(\mathrm{T}') \subset (\mathbf{Z}\Phi(\mathrm{G}, \mathrm{T})^\vee)^*.$$

Hence, an upper bound on $\deg f$ is the index of $\mathbf{Z}\Phi(\mathrm{G},\mathrm{T})$ in $(\mathbf{Z}\Phi(\mathrm{G},\mathrm{T})^\vee)^*$ (i.e., the absolute determinant of the non-degenerate pairing between $\mathbf{Z}\Phi(\mathrm{G},\mathrm{T})$ and $\mathbf{Z}\Phi(\mathrm{G},\mathrm{T})^\vee$). $\square$

The following hard result concerning maximal compact subgroups is fundamental:

Theorem D.2.8. — *Let $\mathscr{H}$ be a Lie group with finite component group. Every compact subgroup of $\mathscr{H}$ is contained in a maximal one, and all maximal compact subgroups* K *of $\mathscr{H}$ are conjugate to each other. Moreover, for any* K *there are closed vector subgroups $\mathrm{V}_1,\dots,\mathrm{V}_n \subset \mathscr{H}$ such that the multiplication map $\mathrm{K}\times\mathrm{V}_1\times\cdots\times\mathrm{V}_n \to \mathscr{H}$ is a C^∞ isomorphism. In particular,* K *is a deformation retract of $\mathscr{H}$, so $\pi_0(\mathrm{K}) = \pi_0(\mathscr{H})$ and $\pi_1(\mathrm{K},1) = \pi_1(\mathscr{H},1)$.*

Proof. — This is [**Ho65**, XV, 3.1]. (We only need the case of connected $\mathscr{H}$ in the proof of the Existence Theorem over **C**. An application in the disconnected case is given in Example D.4.2. The connected case does not seem to formally imply the general case.) $\square$

As an elementary illustration of Theorem D.2.8 in the disconnected case, consider $\mathscr{H} = \mathrm{GL}(\mathrm{W})$ for a finite-dimensional nonzero **R**-vector space W. We can take $\mathrm{K} = \mathrm{O}(q)$ for a positive-definitive quadratic form q on W. Let V be the vector space of endomorphisms of W that are self-adjoint with respect to the symmetric auto-duality $\mathrm{W} \simeq \mathrm{W}^*$ arising from B_q. Exponentiation identifies V with the closed subgroup $\mathrm{P} \subset \mathscr{H}$ of self-adjoint automorphisms of W whose eigenvalues (all in $\mathbf{R}^\times$, by the spectral theorem) are positive. The isomorphism $\mathrm{K}\times\mathrm{V} \simeq \mathscr{H}$ is defined by $(k,\mathrm{T}) \mapsto ke^{\mathrm{T}}$, recovering the classical "polar decomposition" when $\mathrm{W} = \mathbf{R}^n$ and $q = \sum x_i^2$.

D.3. Complexification. — To go further, we need to link up compact Lie groups and complex Lie groups. For any Lie group H, the *complexification* of H is an initial object among C^∞ homomorphisms from H to complex Lie groups; i.e., it is a C^∞ homomorphism $j_{\mathrm{H}} : \mathrm{H} \to \mathrm{H}_{\mathbf{C}}$ to a complex Lie group such that any C^∞ homomorphism $f : \mathrm{H} \to \mathscr{H}$ to a complex Lie group has the form $\mathrm{F}\circ j_{\mathrm{H}}$ for a unique holomorphic homomorphism $\mathrm{F} : \mathrm{H}_{\mathbf{C}} \to \mathscr{H}$. We emphasize that this definition is intrinsic to the C^∞ and holomorphic theories, having *no reliance on the algebraic theory.* In particular, if $\mathrm{H} = \mathrm{G}(\mathbf{R})$ for a linear algebraic **R**-group G then it is not at all clear if the canonical map $\mathrm{H} \to \mathrm{G}(\mathbf{C})$ is a complexification, nor is it clear if $\ker j_{\mathrm{H}} = 1$ (or if $\dim\ker j_{\mathrm{H}} = 0$).

Here is a basic example of complexifications (whose verification is easy): if we denote by $\exp_{\mathbf{R}}(\mathfrak{h})$ the unique connected and simply connected real Lie group H such that Lie(H) is equal to a given real Lie algebra $\mathfrak{h}$, and similarly for complex Lie algebras, then $\exp_{\mathbf{R}}(\mathfrak{h}) \to \exp_{\mathbf{C}}(\mathfrak{h}_{\mathbf{C}})$ is a complexification.

In the commutative case, this says that if V is a real vector space then the inclusion $V \hookrightarrow \mathbf{C} \otimes_{\mathbf{R}} V$ is a complexification.

If a Lie group H admits a complexification $H \to H_{\mathbf{C}}$ and Λ is a discrete central subgroup of H whose image Λ' in $H_{\mathbf{C}}$ is discrete then $H/\Lambda \to H_{\mathbf{C}}/\Lambda'$ is a complexification. (For example, the inclusion $S^1 = \mathbf{R}/\mathbf{Z} \hookrightarrow \mathbf{C}/\mathbf{Z} \simeq \mathbf{C}/2\pi i\mathbf{Z} \simeq \mathbf{C}^\times$ given by $\theta \mapsto \exp(2\pi i\theta)$, where $i^2 = -1$, is a complexification.) A formal argument using the settled simply connected case (and a pushout construction when H is disconnected) shows that a complexification exists for any H [**Bou1**, III, 6.10, Prop. 20] (also see [**Ho65**, XVII, § 5]). The construction gives that $H_{\mathbf{C}}$ is connected when H is connected (though this is also clear by applying the universal property of $H_{\mathbf{C}}$ to $H \to (H_{\mathbf{C}})^0 \hookrightarrow H_{\mathbf{C}}$) and that the $\mathbf{C}$-linearization $\mathbf{C} \otimes_{\mathbf{R}} \mathrm{Lie}(H) \to \mathrm{Lie}(H_{\mathbf{C}})$ is surjective, but in general $\ker j_H$ can have positive dimension, as the following example from [**OV**, Ch. 1, § 4.1, Thm. 7.2] shows.

Example D.3.1. — A maximal compact subgroup of $\mathrm{SL}_2(\mathbf{R})$ is $\mathrm{SO}_2(\mathbf{R}) = S^1$, so the universal cover G of $\mathrm{SL}_2(\mathbf{R})$ fits into an exact sequence

$$1 \to \mathbf{Z} \to G \to \mathrm{SL}_2(\mathbf{R}) \to 1$$

and the induced natural map $G \to \mathrm{SL}_2(\mathbf{C})$ is the complexification of G (since $\mathrm{SL}_2(\mathbf{C})$ is simply connected; i.e., $\mathrm{SL}_2(\mathbf{C}) = \exp_{\mathbf{C}}(\mathfrak{sl}_2(\mathbf{C}))$). Consider the central pushout H of G along an injection $j : \mathbf{Z} \hookrightarrow S^1$, so $H = (S^1 \times G)/\mathbf{Z}$. Then $H_{\mathbf{C}}$ is the quotient of $\mathbf{C}^\times \times \mathrm{SL}_2(\mathbf{C})$ modulo the unique minimal complex Lie subgroup containing the central subgroup $j(\mathbf{Z})$ in the first factor. In other words, $H_{\mathbf{C}} = \mathrm{SL}_2(\mathbf{C})$, so $\ker(H \to H_{\mathbf{C}}) = S^1$.

Fortunately, in the compact case the pathological situation $\dim \ker j_H > 0$ does not arise:

Proposition D.3.2. — *If* K *is a compact Lie group then* $K = K^{\mathrm{alg}}(\mathbf{R}) \to K^{\mathrm{alg}}(\mathbf{C}) =: K^{\mathrm{an}}$ *is a complexification and* K *is a maximal compact subgroup of* K^{an}. *In particular, the complexification* $j_K : K \to K_{\mathbf{C}}$ *is a closed embedding and the* $\mathbf{C}$*-linearization of* $\mathrm{Lie}(j_K)$ *is an isomorphism* (*so* $\mathrm{Lie}(K_{\mathbf{C}})$ *is semisimple when* K *has finite center*).

See Example D.4.2 for a discussion of examples and counterexamples related to complexification beyond the compact case. The main difficulty in the proof of Proposition D.3.2 is that we are demanding the universal property for maps into arbitrary complex Lie groups; the weaker version of Proposition D.3.2 that only asserts the universal property relative to maps to complex Lie groups arising from linear algebraic groups over $\mathbf{C}$ (which is entirely sufficient for our needs) is much easier to prove (see [**BtD**, III, 8.6]).

Proof. — Every connected component of K^{an} meets $K = K^{\mathrm{alg}}(\mathbf{R})$ (as $(K^{\mathrm{an}})^0 = (K^{\mathrm{alg}})^0(\mathbf{C})$ and the connected components of K^{alg} have $\mathbf{R}$-points), and $(\mathrm{K}^{\mathrm{alg}})^0 = (\mathrm{K}^0)^{\mathrm{alg}}$. Granting the case with connected K, we deduce the general case as follows. Let $f : \mathrm{K} \to \mathscr{H}$ be a Lie group homomorphism to a complex-analytic Lie group, so the restriction $f^0 : \mathrm{K}^0 \to \mathscr{H}$ uniquely extends to a holomorphic homomorphism $\mathrm{F} : \mathrm{K}^{\mathrm{alg}}(\mathbf{C})^0 = (\mathrm{K}^0)^{\mathrm{alg}}(\mathbf{C}) \to \mathscr{H}$. We just need to check that F uniquely extends to a holomorphic homomorphism $\mathrm{K}^{\mathrm{alg}}(\mathbf{C}) \to \mathscr{H}$ satisfying $\mathrm{F}|_{\mathrm{K}} = f$. Such an F is unique if it exists since $\mathrm{K} \cdot \mathrm{K}^{\mathrm{alg}}(\mathbf{C})^0 = \mathrm{K}^{\mathrm{alg}}(\mathbf{C})$ (as K meets every connected component of $\mathrm{K}^{\mathrm{alg}}(\mathbf{C})$, since all connected components of $\mathrm{K}^{\mathrm{alg}}$ have an $\mathbf{R}$-point and so are geometrically connected), and the existence of F is a group-theoretic problem: we just have to check that for $k \in \mathrm{K}$, k-conjugation on $\mathrm{K}^{\mathrm{alg}}(\mathbf{C})^0$ is compatible via F with $f(k)$-conjugation on $\mathscr{H}$. This is a comparison of two holomorphic homomorphisms $\mathrm{K}^{\mathrm{alg}}(\mathbf{C})^0 \rightrightarrows \mathscr{H}$, so by the assumed universal property in the connected case it suffices to check equality of their restrictions to K^0. But $\mathrm{F}|_{\mathrm{K}^0} = f^0$ by design.

Now we may assume K is connected, so $\mathrm{K}^{\mathrm{alg}}$ is connected. Let $\mathrm{K}' = \mathscr{D}(\mathrm{K}^{\mathrm{alg}})(\mathbf{R})$, and let $\mathrm{T} = \mathscr{Z}(\mathbf{R})$ for the maximal central $\mathbf{R}$-torus $\mathscr{Z}$ in $\mathrm{K}^{\mathrm{alg}}$, so K' and T are connected and $\mathrm{K}' \times \mathrm{T} \to \mathrm{K}$ is *surjective* (due to connectedness of K) with finite central kernel μ. Clearly μ^{alg} is a central constant finite $\mathbf{R}$-subgroup of $\mathrm{K}'^{\mathrm{alg}} \times \mathrm{T}^{\mathrm{alg}}$, and upon passing to $\mathbf{R}$-points the natural map $\mathrm{K}'^{\mathrm{alg}}(\mathbf{R}) \times \mathrm{T}^{\mathrm{alg}}(\mathbf{R}) \to \mathrm{K}^{\mathrm{alg}}(\mathbf{R})$ with kernel $\mu^{\mathrm{alg}}(\mathbf{R})$ is exactly the surjection $\mathrm{K}' \times \mathrm{T} \to \mathrm{K}$ with kernel μ.

The finite complex Lie group associated to the finite abelian group μ is a complexification of μ, and a complexification of $\mathrm{K} = (\mathrm{K}' \times \mathrm{T})/\mu$ is given by $(\mathrm{K}'_{\mathbf{C}} \times \mathrm{T}_{\mathbf{C}})/\mu_{\mathbf{C}}$ where $\mu_{\mathbf{C}}$ is the image of μ in $\mathrm{K}'_{\mathbf{C}} \times \mathrm{T}_{\mathbf{C}}$. Provided that $\mathrm{T}_{\mathbf{C}} = \mathrm{T}^{\mathrm{alg}}(\mathbf{C})$ and $\mathrm{K}'_{\mathbf{C}} = \mathrm{K}'^{\mathrm{alg}}(\mathbf{C})$ via the natural maps, it follows that $\mu_{\mathbf{C}} = \mu^{\mathrm{alg}}(\mathbf{C})$ and $\mathrm{K}_{\mathbf{C}} = \mathrm{K}^{\mathrm{alg}}(\mathbf{C})$ with the desired canonical map from $\mathrm{K} = (\mathrm{K}' \times \mathrm{T})/\mu$. Hence, it suffices to treat K' and T rather than K. The case of $\mathrm{T} = (\mathrm{S}^1)^r$ is elementary, so now we may assume that $\mathrm{K}^{\mathrm{alg}}$ is semisimple. Equivalently, Lie(K) is semisimple.

Let $j_{\mathrm{K}} : \mathrm{K} \to \mathrm{K}_{\mathbf{C}}$ be the complexification. The natural map $j : \mathrm{K} \to \mathrm{K}^{\mathrm{an}}$ uniquely factors as $f \circ j_{\mathrm{K}}$ for a holomorphic map $f : \mathrm{K}_{\mathbf{C}} \to \mathrm{K}^{\mathrm{an}}$ between connected complex Lie groups, and f is an isomorphism on Lie algebras since $\mathbf{C} \otimes_{\mathbf{R}} \mathrm{Lie}(\mathrm{K}) \to \mathrm{Lie}(\mathrm{K}^{\mathrm{an}})$ is an isomorphism and the map $\mathbf{C} \otimes_{\mathbf{R}} \mathrm{Lie}(\mathrm{K}) \to \mathrm{Lie}(\mathrm{K}_{\mathbf{C}})$ via j_{K} is surjective. Hence, the $\mathbf{C}$-linearization of $\mathrm{Lie}(j_{\mathrm{K}})$ is an isomorphism and f is a quotient map modulo a discrete subgroup. Thus, the real Lie group $\mathrm{K}_{\mathbf{C}}$ has dimension $2 \dim \mathrm{K}$ and to show that f is an isomorphism it suffices to show that f is injective. Note that j_{K} is injective (and so is a closed embedding) since we have factored it through the injective map $\mathrm{K} \to \mathrm{K}^{\mathrm{an}}$.

By Proposition D.2.7, the universal covering $\pi : \widetilde{\mathrm{K}} \to \mathrm{K}$ has finite degree. Let $q : \widetilde{\mathrm{K}}^{\mathrm{alg}} \to \mathrm{K}^{\mathrm{alg}}$ be the corresponding map between $\mathbf{R}$-anisotropic connected reductive $\mathbf{R}$-groups, and let $\mu = \ker q$ be its finite central kernel, so

$\mathrm{K}^{\mathrm{an}} = \widetilde{\mathrm{K}}^{\mathrm{an}}/\mu(\mathbf{C})$. The induced map $\pi_{\mathbf{C}} : \widetilde{\mathrm{K}}_{\mathbf{C}} \to \mathrm{K}_{\mathbf{C}}$ is an isomorphism on Lie algebras, so it is surjective between the connected complexifications and has discrete central kernel. If we can handle the simply connected case (i.e., if $\widetilde{\mathrm{K}}_{\mathbf{C}} \to \widetilde{\mathrm{K}}^{\mathrm{an}}$ is an isomorphism) then $\pi_{\mathbf{C}}$ has degree dividing $\deg \pi^{\mathrm{an}} = \#\mu(\mathbf{C})$ with equality if and only if $\mathrm{K}_{\mathbf{C}} \to \mathrm{K}^{\mathrm{an}}$ is an isomorphism.

Thus, to reduce to the case that K is simply connected we just need to show that $\ker \pi_{\mathbf{C}}$ has size at least $\#\mu(\mathbf{C})$. Let $\mathrm{T} \subset \mathrm{K}$ be a maximal (compact) torus, so its preimage $\widetilde{\mathrm{T}} = \pi^{-1}(\mathrm{T})$ in $\widetilde{\mathrm{K}}$ is also a maximal torus (due to the conjugacy and self-centralizing properties of maximal tori in connected compact Lie groups), and clearly $\ker \pi = \ker(\widetilde{\mathrm{T}} \to \mathrm{T})$. Since the case of tori is settled and we are granting the simply connected case as also being settled, the natural map of complexifications $\widetilde{\mathrm{T}}_{\mathbf{C}} \to \widetilde{\mathrm{K}}_{\mathbf{C}}$ is identified with $\widetilde{\mathrm{T}}^{\mathrm{an}} \to \widetilde{\mathrm{K}}^{\mathrm{an}}$ and hence is *injective*, so $\ker \pi_{\mathbf{C}}$ has size at least that of $\ker(\widetilde{\mathrm{T}}_{\mathbf{C}} \to \mathrm{T}_{\mathbf{C}})$. But $\widetilde{\mathrm{T}}_{\mathbf{C}} \to \mathrm{T}_{\mathbf{C}}$ is the map on $\mathbf{C}$-points arising from the map $\widetilde{\mathrm{T}}^{\mathrm{alg}} \to \mathrm{T}^{\mathrm{alg}}$ induced by $q : \widetilde{\mathrm{K}}^{\mathrm{alg}} \to \mathrm{K}^{\mathrm{alg}}$ between maximal $\mathbf{R}$-tori, so this map between maximal $\mathbf{R}$-tori has kernel μ as well and hence $\mu(\mathbf{C})$ is identified with the kernel of $\widetilde{\mathrm{T}}_{\mathbf{C}} \to \mathrm{T}_{\mathbf{C}}$. Now we can assume K is simply connected.

By the construction of complexification for simply connected Lie groups, to show $f : \mathrm{K}_{\mathbf{C}} \to \mathrm{K}^{\mathrm{an}}$ is an isomorphism it suffices to show that K^{an} is simply connected if K. The group $\mathrm{K}^{\mathrm{alg}}(\mathbf{R}) = \mathrm{K}$ is a compact subgroup of K^{an}, and any connected Lie group has a deformation retract onto its maximal compact subgroups (and so has their homotopy type) by Theorem D.2.8.

It now suffices to prove that if K is *any* connected compact Lie group (not necessarily simply connected) then it is maximal as a compact subgroup of K^{an}. Since the maximal compact subgroups of K^{an} are connected, it suffices to show that the common dimension of the maximal compact subgroups of K^{an} is $\dim \mathrm{K}$.

To control the possibilities for the maximal compact subgroups of K^{an}, we consider its finite-dimensional holomorphic representations. The group K^{an} has a faithful finite-dimensional holomorphic representation (using a faithful representation of the $\mathbf{R}$-group $\mathrm{K}^{\mathrm{alg}}$) and the finite-dimensional holomorphic representations of K^{an} are completely reducible (Corollary D.2.3). Thus, the connected complex Lie group K^{an} is "reductive" in the sense of [**Ho65**, XVII, § 5]. It then follows from [**Ho65**, XVII, 5.3] that K^{an} is the complexification of its maximal compact subgroups. That is, if $\mathscr{K}$ is a maximal compact subgroup of K^{an} then the map $\mathscr{K}_{\mathbf{C}} \to \mathrm{K}^{\mathrm{an}}$ induced by the inclusion $\mathscr{K} \hookrightarrow \mathrm{K}^{\mathrm{an}}$ is an isomorphism. But we showed earlier in the proof that the complexification of a compact Lie group has twice the real dimension in general, so applying that to $\mathscr{K}$ and using the equality $\mathscr{K}_{\mathbf{C}} = \mathrm{K}^{\mathrm{an}} = \mathrm{K}^{\mathrm{alg}}(\mathbf{C})$ gives that $\dim \mathscr{K} = (1/2) \dim_{\mathbf{R}} \mathrm{K}^{\mathrm{alg}}(\mathbf{C}) = \dim \mathrm{K}^{\mathrm{alg}} = \dim \mathrm{K}$. $\square$

Example D.3.3. — By Proposition D.3.2, every compact Lie group is a maximal compact subgroup of its complexification, whose component group is finite. Turning this around, let K be the maximal compact subgroup of a complex Lie group $\mathscr{G}$ with finite component group, and assume $\mathrm{Lie}(\mathscr{G})$ is semisimple (over $\mathbf{R}$ or $\mathbf{C}$; these are equivalent). The inclusion $j : \mathrm{K} \hookrightarrow \mathscr{G}$ factors through a unique holomorphic homomorphism $\mathrm{K}_{\mathbf{C}} \to \mathscr{G}$, and we claim that this latter map is an isomorphism.

In other words, we claim that every complex Lie group with semisimple Lie algebra and finite component group is the complexification of its maximal compact subgroups. (In particular, using Proposition D.2.1, $\mathrm{G} \rightsquigarrow \mathrm{G}(\mathbf{C})$ is an equivalence from connected semisimple $\mathbf{C}$-groups to connected complex Lie groups with semisimple Lie algebra. The analogous equivalence for connected reductive $\mathbf{C}$-groups is onto the category of connected complex Lie groups $\mathscr{H}$ such that $\mathrm{Lie}(\mathscr{H})$ is reductive and $\mathrm{Z}^0_{\mathscr{H}} \simeq (\mathbf{C}^\times)^r$ for some $r \geqslant 0$.)

To prove that $\mathscr{G}$ is the complexification of K, we use the criterion in [**Ho65**, XVII, 5.3]: it suffices to prove that $\mathscr{G}$ has a faithful finite-dimensional representation (i.e., an injective holomorphic homomorphism $\mathscr{G} \to \mathrm{GL}(\mathrm{V})$ for a finite-dimensional $\mathbf{C}$-vector space V) and that the holomorphic finite-dimensional representations of $\mathscr{G}$ are completely reducible. Complete reducibility is inherited from semisimplicity of $\mathrm{Lie}(\mathscr{G})$ (and finiteness of $\pi_0(\mathscr{G})$), as in the proof of Corollary D.2.3. The existence of a faithful finite-dimensional representation of $\mathscr{G}$ lies deeper (and its analogue in the $\mathbf{R}$-theory is false; e.g., the universal cover of $\mathrm{SL}_2(\mathbf{R})$). We only need the case when $\mathscr{G}$ is the Lie group of $\mathbf{C}$-points of a connected semisimple $\mathbf{C}$-group, for which the existence of a faithful finite-dimensional representation is obvious. The general case is [**Ho65**, XVII, 3.2].

D.4. Construction of covers and applications. — Returning to the root datum R and the connected semisimple $\mathbf{C}$-group G of adjoint type at the end of § D.1, we now use § D.2–§ D.3 to build an isogeny $\mathrm{G}' \to \mathrm{G}$ from a connected semisimple $\mathbf{C}$-group G' such that the semisimple root datum of G' is simply connected. Let H be the connected Lie group $\mathrm{G}(\mathbf{C})$ whose Lie algebra is semisimple, and let K be a maximal compact subgroup of H, so $\mathrm{K}^{\mathrm{alg}}$ is an $\mathbf{R}$-anisotropic connected reductive $\mathbf{R}$-group. By Proposition D.3.2 and Example D.3.3, $\mathrm{K}^{\mathrm{alg}}(\mathbf{C}) \simeq \mathrm{H} = \mathrm{G}(\mathbf{C})$. It follows that $\mathrm{K}^{\mathrm{alg}}(\mathbf{C})$ has finite center, so the connected reductive $\mathbf{R}$-group $\mathrm{K}^{\mathrm{alg}}$ is semisimple. By Proposition D.2.1, the analytic isomorphism $\mathrm{K}^{\mathrm{alg}}(\mathbf{C}) \simeq \mathrm{G}(\mathbf{C})$ arises from a $\mathbf{C}$-group isomorphism $\mathrm{K}^{\mathrm{alg}}_{\mathbf{C}} \simeq \mathrm{G}$; i.e., $\mathrm{K}^{\mathrm{alg}}$ is an $\mathbf{R}$-descent of G (the "compact form" of G).

Since $\mathrm{K}^{\mathrm{alg}}$ is an $\mathbf{R}$-descent of G, we now convert our problem for isogenous covers of G over $\mathbf{C}$ into an analogous problem for isogenous covers of $\mathrm{K}^{\mathrm{alg}}$ over $\mathbf{R}$. Inspired by Theorem D.2.4, we will make constructions in the category of compact connected Lie groups, and then pass back to the algebraic theory over $\mathbf{R}$ (and finally extend scalars to $\mathbf{C}$). To this end, we use the connected compact universal covering space $\widetilde{\mathrm{K}} \to \mathrm{K}$ (see Proposition D.2.7).

Passing to the corresponding $\mathbf{R}$-anisotropic connected semisimple $\mathbf{R}$-groups, we get an isogeny $\widetilde{\mathrm{K}}^{\mathrm{alg}} \to \mathrm{K}^{\mathrm{alg}}$. Extending scalars to $\mathbf{C}$ defines an isogeny

$$f : \mathrm{G}' := \widetilde{\mathrm{K}}^{\mathrm{alg}}_{\mathbf{C}} \to \mathrm{K}^{\mathrm{alg}}_{\mathbf{C}} = \mathrm{G}$$

between connected semisimple $\mathbf{C}$-groups. Let $\mathrm{T}' = f^{-1}(\mathrm{T})$ be the maximal torus of G' corresponding to a choice of maximal torus T of G, so we get an isogeny of root data

$$\mathrm{R}(\mathrm{G}', \mathrm{T}') \to \mathrm{R}(\mathrm{G}, \mathrm{T}) = \mathrm{R}^{\mathrm{ad}}.$$

We will prove that $\mathrm{R}(\mathrm{G}', \mathrm{T}')$ is simply connected (i.e., the coroots span the cocharacter group of T'), so $\mathrm{R}(\mathrm{G}', \mathrm{T}') = \mathrm{R}$ compatibly with the initial identification $\mathrm{R}(\mathrm{G}, \mathrm{T}) = \mathrm{R}^{\mathrm{ad}}$. This will complete the proof of the Existence Theorem over $\mathbf{C}$.

By Proposition D.3.2, the Lie group $\mathrm{G}'(\mathbf{C}) = \widetilde{\mathrm{K}}^{\mathrm{alg}}(\mathbf{C})$ contains $\widetilde{\mathrm{K}} = \widetilde{\mathrm{K}}^{\mathrm{alg}}(\mathbf{R})$ as a maximal compact subgroup. Thus, $\mathrm{G}'(\mathbf{C})$ inherits the simply connectedness property from $\widetilde{\mathrm{K}}$ (Theorem D.2.8). Our problem is therefore reduced to verifying a relationship between combinatorial and topological notions of being "simply connected":

Proposition D.4.1. — *If* $(\mathrm{G}', \mathrm{T}')$ *is a connected semisimple* $\mathbf{C}$*-group then* $\mathrm{G}'(\mathbf{C})$ *is topologically simply connected if and only if* $\mathrm{R}(\mathrm{G}', \mathrm{T}')$ *is simply connected in the sense of root data.*

Proof. — We will use an "existence theorem" (for highest-weight representations) in the representation theory of semisimple Lie algebras over $\mathbf{C}$ and exponentiation of Lie algebra representations to Lie group representations in the topologically simply connected case.

Note that the $\mathbf{Q}$-vector space $(\mathbf{Z}\Phi'^{\vee})_{\mathbf{Q}} = \mathrm{X}_*(\mathrm{T}')_{\mathbf{Q}}$ has complexification that is naturally identified with $\mathfrak{t}' = \mathrm{Lie}(\mathrm{T}')$ (by identifying $\lambda \in \mathrm{Hom}(\mathbf{G}_{\mathrm{m}}, \mathrm{T}')$ with $\mathrm{d}\lambda := \mathrm{Lie}(\lambda)(z\partial_z) \in \mathfrak{t}'$). The set $\Phi'^{\vee}$ of coroots spans $\mathrm{X}_*(\mathrm{T}')$ over $\mathbf{Z}$ provided that every $\mathbf{Z}$-linear form $\ell : \mathbf{Z}\Phi'^{\vee} \to \mathbf{Z}$ is $\mathbf{Z}$-valued on $\mathrm{X}_*(\mathrm{T}')$ (i.e., arises from $\mathrm{X}(\mathrm{T}')$) when ℓ is viewed as a $\mathbf{Q}$-linear form on $\mathrm{X}_*(\mathrm{T}')_{\mathbf{Q}}$, or equivalently when ℓ is viewed as a $\mathbf{C}$-linear form on $\mathrm{X}_*(\mathrm{T}')_{\mathbf{C}} = \mathfrak{t}'$. This integrality property of ℓ is invariant with respect to the action of $\mathrm{W}_{\mathrm{G}'}(\mathrm{T}')$, so to check whether or not it holds for a particular $(\mathrm{G}', \mathrm{T}')$ there is no loss of generality in picking a positive system of roots Φ'^{+} and imposing the additional requirement on the element $\ell \in \mathrm{X}(\mathrm{T})_{\mathbf{Q}}$ that it lies in the associated Weyl chamber of $\mathrm{X}(\mathrm{T})_{\mathbf{R}}$ (i.e., $\langle \ell, a^{\vee} \rangle \geqslant 0$ for all $a \in \Phi'^{+}$).

Let $\mathrm{d}\ell = \mathrm{Lie}(\ell) : \mathfrak{t}' \to \mathrm{Lie}(\mathbf{G}_{\mathrm{m}}) = \mathbf{C}$ be the linear form associated to $\ell \in (\mathbf{Z}\Phi'^{\vee})^*$ satisfying $\langle \ell, a^{\vee} \rangle \geqslant 0$ for all $a \in \Phi'^{+}$. We have $\langle \mathrm{d}\ell, \mathrm{d}(a^{\vee}) \rangle =$

$\langle \ell, a^\vee \rangle \in \mathbf{Z}_{\geqslant 0}$ for all $a \in \Phi' = \Phi(\mathfrak{g}', \mathfrak{t}')$. Thus, by the existence theorem for highest-weight representations of semisimple Lie algebras over $\mathbf{C}$ (Theorem 1.5.7), there exists a (unique) irreducible representation

$$\mathfrak{g}' \to \mathfrak{gl}(\mathrm{V}) = \mathrm{Lie}(\mathrm{GL}(\mathrm{V})) \tag{D.4.1}$$

having $\mathrm{d}\ell$ as its highest weight (relative to the Cartan subalgebra $\mathfrak{t}'$ and $\Phi'^{+} \subset \Phi(\mathfrak{g}', \mathfrak{t}')$).

Now assume that $\mathrm{G}'(\mathbf{C})$ is simply connected, so every finite-dimensional representation of $\mathfrak{g}'$ over $\mathbf{C}$ exponentiates to a holomorphic representation of $\mathrm{G}'(\mathbf{C})$ on the same vector space. By Proposition D.2.1, any finite-dimensional holomorphic representation of $\mathrm{G}'(\mathbf{C})$ arises from a unique algebraic representation of G' on the same vector space, due to the semisimplicity of G'. Thus, (D.4.1) arises from a $\mathbf{C}$-group representation $\rho : \mathrm{G}' \to \mathrm{GL}(\mathrm{V})$, and the highest weight vector $v \in \mathrm{V}$ for $(\mathfrak{g}', \mathfrak{t}')$ (which is unique up to $\mathbf{C}^\times$-scaling) is a T'-eigenvector since it is a $\mathfrak{t}'$-eigenvector. The corresponding weight homomorphism $w : \mathrm{T}' \to \mathbf{G}_{\mathrm{m}}$ via the T'-action on $\mathbf{C}v$ induces $\mathrm{d}\ell$ on Lie algebras, so $w = \ell$ in $\mathrm{X}(\mathrm{T}')_{\mathbf{C}}$. In particular, $\ell \in \mathrm{X}(\mathrm{T}')$. That is, $\mathrm{X}(\mathrm{T}') = (\mathbf{Z}\Phi'^\vee)^*$, or equivalently $\mathrm{X}_*(\mathrm{T}') = \mathbf{Z}\Phi'^\vee$, so $\mathrm{R}(\mathrm{G}', \mathrm{T}')$ is simply connected.

For the converse, assume that $\mathrm{R}(\mathrm{G}', \mathrm{T}')$ is simply connected. Let K be a maximal compact subgroup of $\mathrm{G}'(\mathbf{C})$, so $\mathrm{K}^{\mathrm{alg}}(\mathbf{C}) \simeq \mathrm{G}'(\mathbf{C})$ by Proposition D.3.2 and Example D.3.3. Hence, $\mathrm{K}^{\mathrm{alg}}$ is an $\mathbf{R}$-descent of G', by Proposition D.2.1. Consider the finite-degree universal cover $f : \widetilde{\mathrm{K}} \to \mathrm{K}$. By Propositions D.2.8 and Proposition D.3.2, $\widetilde{\mathrm{K}}^{\mathrm{alg}}(\mathbf{C})$ is simply connected in the topological sense. Since f "algebraizes" to an isogeny $f^{\mathrm{alg}} : \widetilde{\mathrm{K}}^{\mathrm{alg}} \to \mathrm{K}^{\mathrm{alg}}$ between connected semisimple $\mathbf{R}$-groups, $\widetilde{\mathrm{K}}^{\mathrm{alg}}(\mathbf{C})$ is the universal cover of $\mathrm{K}^{\mathrm{alg}}(\mathbf{C}) = \mathrm{G}'(\mathbf{C})$. (This covering map is the complexification $f_{\mathbf{C}}$ of f, and it arises from f^{alg} on $\mathbf{C}$-points.) The description of the effect of central isogenies at the level of root data (see Example 6.1.9) shows that any isogeny onto G' from a connected semisimple $\mathbf{C}$-group is an isomorphism because $\mathrm{R}(\mathrm{G}', \mathrm{T}')$ is simply connected. Thus, the $\mathbf{C}$-homomorphism $(f^{\mathrm{alg}})_{\mathbf{C}}$ is an isomorphism, so its analytification $f_{\mathbf{C}}$ is an isomorphism and therefore $\mathrm{G}'(\mathbf{C})$ is topologically simply connected. □

Example D.4.2. — As an application of our work with complex Lie groups and the Existence Theorem over $\mathbf{C}$, we now relate the Lie group notion of complexification to the algebraic notion of scalar extension from $\mathbf{R}$ to $\mathbf{C}$ for Lie groups arising from semisimple $\mathbf{R}$-groups, going beyond the $\mathbf{R}$-anisotropic case. This is not used elsewhere in these notes.

Let H be a connected semisimple $\mathbf{R}$-group (so $\mathrm{H}_{\mathbf{C}}$ denotes the associated semisimple $\mathbf{C}$-group, not to be confused with the complexification $\mathrm{H}(\mathbf{R})_{\mathbf{C}}$ of the Lie group $\mathrm{H}(\mathbf{R})$). Even when $\mathrm{H}(\mathbf{R})$ is connected, it can happen that the natural map $j : \mathrm{H}(\mathbf{R}) \to \mathrm{H}(\mathbf{C})$ is not the complexification. For example, the isomorphism $\mathrm{SL}_3(\mathbf{R}) \simeq \mathrm{PGL}_3(\mathbf{R})$ provides a homomorphism $\mathrm{PGL}_3(\mathbf{R}) \to \mathrm{SL}_3(\mathbf{C})$ to a degree-3 connected cover of $\mathrm{PGL}_3(\mathbf{C})$. This problem disappears if the root datum for $\mathrm{H}_{\mathbf{C}}$ is simply connected, as we now explain.

Consider a homomorphism $f : \mathrm{H}(\mathbf{R}) \to \mathscr{G}$ to a complex Lie group $\mathscr{G}$. The map $\mathrm{Lie}(f)$ is a homomorphism from $\mathfrak{h} = \mathrm{Lie}(\mathrm{H})$ into the underlying real Lie algebra of $\mathrm{Lie}(\mathscr{G})$, so it linearizes to a map of complex Lie algebras $\widetilde{f} : \mathfrak{h}_{\mathbf{C}} \to \mathrm{Lie}(\mathscr{G})$. But $\mathfrak{h}_{\mathbf{C}} = \mathrm{Lie}(\mathrm{H})_{\mathbf{C}} = \mathrm{Lie}(\mathrm{H}_{\mathbf{C}}) = \mathrm{Lie}(\mathrm{H}(\mathbf{C}))$, and $\mathrm{H}(\mathbf{C})$ is simply connected in the topological sense by Proposition D.4.1 (whose proof relied on our proof of the Existence Theorem over $\mathbf{C}$). Hence, $\widetilde{f}$ exponentiates to a holomorphic homomorphism $\mathrm{F} : \mathrm{H}(\mathbf{C}) \to \mathscr{G}$. The maps $\mathrm{F} \circ j, f : \mathrm{H}(\mathbf{R}) \rightrightarrows \mathscr{G}$ agree on Lie algebras by construction of F, so they coincide on $\mathrm{H}(\mathbf{R})^0$. Likewise, by Lie algebra considerations, F is uniquely determined on $\mathrm{H}(\mathbf{C})$ by the equality $\mathrm{F} \circ j = f$ on $\mathrm{H}(\mathbf{R})^0$.

It remains to prove that $\mathrm{H}(\mathbf{R})$ is connected when $\mathrm{H}_{\mathbf{C}}$ has a simply connected root datum. The connectedness of $\mathrm{H}(\mathbf{R})$ in such cases is a deep result of E. Cartan, originally proved by Riemannian geometry (going through the theory of compact groups, for which the main connectedness ingredient is proved in [**He**, VII, Thm. 8.2]).

Here is a sketch of a proof of Cartan's connectedness theorem via an algebraic connectedness result of Steinberg. The real Lie group underlying $\mathrm{H}(\mathbf{C})$ is connected with an involution θ (complex conjugation) having fixed-point locus $\mathrm{H}(\mathbf{R})$. Note that $\mathrm{H}(\mathbf{R})$ has *finite* component group, as follows either from a general result of Whitney on $\mathbf{R}$-points of affine algebraic varieties (see [**Mil68**, App. A], which rests on [**AF**, § 1, Lemma]) or from a result of Matsumoto for $\mathbf{R}$-groups (see [**BoTi**, 14.4, 14.5]). Thus, Theorem D.2.8 is applicable to $\mathrm{H}(\mathbf{R})$ (so $\mathrm{H}(\mathbf{R})$ admits a good theory of maximal compact subgroups).

By a result of Mostow [**Mos**, § 6], there is a θ-stable maximal compact subgroup K' of $\mathrm{H}(\mathbf{C})$ such that $\mathrm{K} := \mathrm{K}' \bigcap \mathrm{H}(\mathbf{R})$ is a maximal compact subgroup of $\mathrm{H}(\mathbf{R})$. (Mostow's proof gives that $\mathrm{K}' \mapsto \mathrm{K}' \bigcap \mathrm{H}(\mathbf{R})$ is a bijection from the set of θ-stable maximal compact subgroups of $\mathrm{H}(\mathbf{C})$ to the set of maximal compact subgroups of $\mathrm{H}(\mathbf{R})$, with H any connected reductive $\mathbf{R}$-group.) Fix such a K', so K' is connected (as $\mathrm{H}(\mathbf{C})$ is connected) and $\pi_0(\mathrm{H}(\mathbf{R})) = \pi_0(\mathrm{K})$ for the maximal compact subgroup $\mathrm{K} = \mathrm{K}' \bigcap \mathrm{H}(\mathbf{R}) = \mathrm{K}'^{\theta}$ in $\mathrm{H}(\mathbf{R})$ (applying Theorem D.2.8 to $\mathrm{H}(\mathbf{R})$). Connectedness of $\mathrm{H}(\mathbf{R})$ is reduced to connectedness of K, so it suffices to show that the fixed-point locus of *any* involution θ of K' is connected. Note that the $\mathbf{C}$-group $(\mathrm{K}'^{\mathrm{alg}})_{\mathbf{C}} = \mathrm{H}_{\mathbf{C}}$ has a simply connected root datum.

The involution θ of the connected compact Lie group K' arises from an involution θ^{alg} of $\mathrm{K}'^{\mathrm{alg}}$, so $(\mathrm{K}'^{\mathrm{alg}})^{\theta^{\mathrm{alg}}}$ is a closed $\mathbf{R}$-subgroup of $\mathrm{K}'^{\mathrm{alg}}$ whose group of $\mathbf{R}$-points is K'^{θ}. The fixed-point subgroup for an involution of a connected reductive $\mathbf{R}$-group has reductive identity component (see [**PY02**, 2.2], or [**PY02**, 2.4] for an algebraic proof), so $(\mathrm{K}'^{\mathrm{alg}})^{\theta^{\mathrm{alg}}}$ has $\mathbf{R}$-anisotropic reductive identity component. (This identity component may not be semisimple; e.g., for $\mathrm{K}' = \mathrm{SU}(2)$ we have $\mathrm{K} = \mathrm{S}^1$.) Thus, by Theorem D.2.4, if $(\mathrm{K}'^{\mathrm{alg}})^{\theta^{\mathrm{alg}}}$ is connected for the Zariski topology then its group K of $\mathbf{R}$-points is connected for the analytic topology.

We are reduced to an algebraic assertion over $\mathbf{R}$: for any $\mathbf{R}$-anisotropic connected semisimple $\mathbf{R}$-group G such that the semisimple $\mathbf{C}$-group $\mathrm{G}_{\mathbf{C}}$ has a simply connected root datum (e.g, $\mathrm{G} = \mathrm{K}'^{\mathrm{alg}}$) and any involution ι of G (e.g., θ^{alg}), the linear algebraic $\mathbf{R}$-subgroup $\mathrm{G}^{\iota=1}$ of G is connected for the Zariski topology. We can reformulate this assertion over any field k: if G is a connected semisimple k-group such that the root datum for $\mathrm{G}_{\overline{k}}$ is simply connected and if θ is an automorphism of G that is semisimple (i.e., the induced automorphism of the coordinate ring is semisimple, such as an involution when $\mathrm{char}(k) \neq 2$), then the closed fixed-point scheme G^{θ} is connected for the Zariski topology. It suffices to consider algebraically closed k and to work with k-valued points, in which case the connectedness assertion is a theorem of Steinberg [**St68**, 8.1] (which was partially motivated by the desire for an algebraic version of Cartan's connectedness theorem).

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July 10, 2014

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