On the application of voltage graphs to the degree/diameter problem

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Abstract

Following the techniques used by Loz and Širáň (Australasian Journal of Combinatorics 41 63-80), we improve some of the best available lower bounds on the order of graphs with given maximum degree and diameter. For maximum degree up to 15 and diameter up to 10, a table with the best values is maintained online by Comellas, Delorme and Loz. Our results have already been updated in that table.

1 Introduction

The problem of *degree/diameter* is the study of the maximum number of nodes $N_{\Delta,k}$ that a graph with degree at most Δ and diameter k can have. Except for the trivial cases $\Delta = 1$ or 2 and $k = 1$, the exact value, is known only for seven values of the pair (Δ, k) [8]. Therefore most studies focus on finding upper and lower bounds, as well as in the asymptotic behavior of these bounds when either Δ or k tends to infinity.

The topology of a network (telecommunications, microprocessors or computers) can clearly be modeled by a graph. The construction of large networks is usually bounded by the degree and diameter of such network. In these contexts those approximations or solutions for the *degree/diameter* problem are relevant.

It is easy to see that $N_{\Delta,k}$, has as upper bound the following expression:

$$N_{\Delta,k} \leq \frac{\Delta(\Delta-1)^k - 2}{\Delta - 2},$$

known as the Moore bound. Except for the trivial cases $\Delta = 1$ or 2 and $k = 1$, the Moore bound could only be achieved for $k = 2$ and $\Delta \in \{2, 3, 7, 57\}$, see [7, 1, 4]. The existence and uniqueness of graphs that reach the bound for $\Delta \in \{2, 3, 7\}$ is proved in [7]. The case $\Delta = 57$, is one of the outstanding open problems in the area. Further work has just led to slight improvements in Moore's bound [11].

Regarding lower bounds, these are often found through the explicit construction of graphs with maximum degree Δ and diameter k. The best lower bounds for $\Delta \leq 16$ and $k \leq 10$ are maintained by Charles Delorme, Francesc Comellas and more recently by Eyal Loz [3, 5, 8]. Currently, most of the values in that table were obtained by using a technique known as *voltage graphs*. From now on we will call this table *the Table*.

1.1 Voltage graphs

The voltage graph technique was first used by Ringel and Young in the proof of the Headwood conjecture [12]. In the context of the *degree/diameter* problem, the technique has been successfully applied by Loz and Širáň improving most entries in the Table, particularly for large values of Δ and k. Next we present a brief description of the technique; refer to [10, 9] for further details.

Let $\Gamma(V, E)$ be a graph with loops, multi-edges and *semi-edges*, i.e., edges with a free extreme. Given a group G we will call *voltages assignment* on the graph Γ, to any function $\alpha : E \to G$ such that if $e \in E$ is a semi-edge, then $\alpha(e) = \alpha(e)^{-1}$ and if e is a loop, then $\alpha(e) \neq \alpha(e)^{-1}$. Finally let us define the *lift of G by α* as the graph $\Gamma^\alpha = (V^\alpha, E^\alpha)$ where:

$$\begin{aligned} V^\alpha &= V \times G = \{v_g : v \in V, g \in G\}, \\ E^\alpha &= \{\{u_g, v_{g\alpha(e)}\} : e = \{u, v\} \in E, g \in G\} \cup \\ &\quad \{\{u_g, u_{g\alpha(e)}\} : e \text{ } \textit{semi-edge} \text{ of } u, g \in G\}. \end{aligned}$$

The graph Γ is called *quotient graph of the lift*. Note that:

1. The degree of the a vertex v_g in the lift Γ^α is equal to the degree of v in Γ, where the degree of a loop is two and the degree of a semi-edge is one.

2. The group of automorphism of the lift act transitively on the *fibre* $\{v\} \times G$ of G for each $v \in V$.

3. The edges of the lift are simple (not multi-edges) if and only if given two different parallel edges e and f, then $\alpha(e) \notin \{\alpha(f), \alpha(f)^{-1}\}$.

4. If $|V| = 1$ and $S = \alpha(E)$ is the set of assigned voltages, then the lift Γ^α is the Cayley graph over G generated by $S \cup S^{-1}$.

As in [9], we use semidirect products of cyclic groups. These products have the form $\mathbb{Z}_m \rtimes_f \mathbb{Z}_n$, where $f_r : \mathbb{Z}_m \to \mathcal{A}ut(\mathbb{Z}_n)$ must have the form $(f(x))(y) = xr^y$ for certain natural r such that $r^n \equiv 1 \bmod m$. Specifically, the product of two elements (a, b) and (c, d) is given by

$$(a, b)(c, d) = (a + r^b c, b + d).$$

This family of groups was successfully applied, before the introduction of the voltage graphs technique, to generate entries in the Table; see Dineen-Hafner [6], Comellas-Mitjana and Sampels [13].

Following [9], we use quotients graphs with $|V|$ between one and four.

2 Algorithm and Results

Instead of maximize the order of a graph with maximum degree Δ and diameter k, our algorithm minimizes the diameter of a Δ-regular graph with given order N. In fact, since we know that the minimum possible diameter is k, the algorithm simply searches for a Δ-regular graph with N vertices and diameter k.

The implementation of the search, which is available online [2], can be split into two stages. In the first stage, given a Δ-regular quotient Γ, the target diameter k and a range of possible values for the group parameters (m, n), the algorithm performs a preliminary analysis, shown in pseudocode in Figure 1. This analysis determines the best candidates for r, which are those r such that $\langle r\rangle = \{r^i \bmod m : 1 \leq i \leq n\}$ has a maximum cardinality for that m and n. Next, the tuples (m, n, r) are evaluated as shown in pseudocode in Figure 2. In order to increase the probability of success, we assess the tuple by computing the proportion of graphs with diameter equal to $k+1$, as it serves as an indicator of the quality the tuple is. Finally the preliminary search sorts the list of tuples by their evaluation value. Of course, if a graph with diameter k is found, the search exits there.

The second stage corresponds to the pseudocode in Figure 3, which is a sequential search on tuples resulting from the previous stage. Notice that the maximum degree Δ is implicitly determined by the quotient graph Γ.

Most of the time spent for this search is consumed by the evaluation of the lift diameter. This operation has order $O(\Delta|\Gamma|^2|mn|)$. Also search for values of r that maximizes $|\langle r\rangle|$, is an expensive operation. In this case, the evaluation of each value has order $O(n)$ whenever the operation consists of evaluating $r^i \bmod m$; since it is made with integer arithmetics, it is not as heavy as evaluating the lift diameter.

Notice that since $r^n \equiv 1 \bmod m$, i.e. r is a n-th root of unity in Z_m, then r is coprime with m. Besides, n divides the Carmicheal function $\lambda(m)$ of m. Thus, the greatest possible $| < r > |$ is $gcd(n, \lambda(m))$. We can find r either by checking those numbers coprime with m that have order equal to $gcd(n, \lambda(m))$ or by means of the decomposition of $(\mathbb{Z}_m^\times, *)$ as a direct product of cyclic groups. Indeed, given the prime factorization $\prod p_i^{k_i}$ of m, we known that there is an isomorphism f from $(\mathbb{Z}_m^\times, *)$ to $\bigotimes(\mathbb{Z}_{p_i^{k_i}}^\times, *)$. Since the group $G_i = (\mathbb{Z}_{p_i^{k_i}}^\times, *)$ is cyclic of order $|G_i| = \varphi(p_i^{k_i})$, where φ is the Euler's totient function, we can take a generator g_i of G_i, i.e. an element g_i such that $< g_i > = G_i$. Then it is enough to take $r = \prod g_i^{r_i}$ with $gcd(n, \lambda(m)) = \prod |G_i|^{r_i}$.

The implementation of the introduced algorithm allowed us to reproduce some results from [9] as well as to obtain new ones. In particular, for maximum degree 9 and diameter 5, we found a graph with order 8 268 where the maximum known graph had 8 200 vertices. The quotient used was a node with four loops and one semi-edge. The parameters of the group were $m = 159$, $n = 52$ and $r = 2$, the voltages were $(41, 14), (112, 47), (82, 37), (113, 10)$ for the loops and $(147, 26)$ for the semi-edge. We also found a graph with maximum degree 6, diameter 9 and order 331 387, for the group $m = 6\,763, n = 49, r = 41$. In that case we used the same quotient used in [9], consisting of a vertex with three loops.

Input: (m_a, m_b), (n_a, n_b), k target diameter, Γ quotient graph, i_{max} evaluations count
Output: L group parameter list (m, n, r)
$E \leftarrow \emptyset$;
for $m = m_a$ **to** m_b **do**
    for $n = n_a$ **to** n_b **do**
        $RList \leftarrow \{r : \langle r \rangle \text{ is maximal}\}$;
        foreach $r \in RList$ **do**
            $E \leftarrow E \cup evaluate\_tuple((m, n, r), \Gamma, i_{max}, k)$;
        end
    end
end
return $sort(E)$

Figure 1: Pseudocode of the preliminary search.

Input: $G = (m, n, r)$ group parameters, Γ quotient graph, i_{max} evaluations count, k expected diameter
Output: dga amount of graphs with diameter $= k + 1$
$I \leftarrow involutory\_elements(G)$;
for $i = 1$ **to** i_{max} **do**
    $\alpha \leftarrow random\_voltage(\Gamma, G, I)$;
    $\Gamma^\alpha \leftarrow compute\_lift(\Gamma, \alpha)$;
    $k' \leftarrow diameter(\Gamma^\alpha)$;
    if $k' = k + 1$ **then**
        $dga \leftarrow dga + 1$;
    end
end

Figure 2: Pseudocode of the *evaluate_tuple* routine.

The voltage values in this case were: $(1\,254, 25), (541, 18)$ and $(4\,642, 47)$. Finally, we found a graph with diameter 9, order $1\,697\,688$ and diameter 8 using the group $m = 23\,579, n = 72, r = 1\,413$. The quotient was a node with four loops and one semi-edge, the voltages were: $(5\,958, 8), (6\,086, 27), (22\,093, 37), (22\,621, 33)$ and $(2\,717, 36)$ as involutive value for the semi-edge.

All graphs presented are lifts on a quotient graph with a single node, so they are Cayley graphs as well. Thus, all graphs improve the largest known lower bound for the degree/diameter problem in the Cayley context, as well as in the vertex-transitive context.

3 Remarks

At the beginning of our search for graphs that meet the criteria of the problem, several experiments with identical quotient graph were performed, changing only the value of r in the group $\mathbb{Z}_m \rtimes_r \mathbb{Z}_n$. Those experiments showed a particular effect. For certain r, a relatively large set of samples have the same proportion of graphs which satisfy the constraints of the problem. This leads to the following

Input: L tuple list (m, n, r), group parameters, Γ quotient graph, i_{max} evaluations count, k expected diameter
Output: A generated tuples set *lifts* graphs with expected diameter

```
A ← ∅;
foreach G = (m, n, r) ∈ L do
    I ← involutory_elements(G);
    for i = 1 to i_max do
        α ← random_voltage(Γ, G, I);
        Γ^α ← compute_lift(Γ, α);
        k' ← diameter(Γ^α);
        if k' = k then
            A ← A ∪ (m, n, r);
        end
    end
end
```

Figure 3: Sequential search pseudocode.

conjecture: the search space generated by each r hosts different probabilities of finding viable graphs. Moreover, some of these search spaces seemed to be equivalent. We next attempt to explain part of the observations by proving a proposition. First, let us fix some notation.

- Define $\mathcal{G}_{(m,n,r)}(\Gamma)$ as the set of lifts generated by $\mathbb{Z}_m \rtimes_r \mathbb{Z}_n$ with quotient graph Γ.
- Let the set $\langle r \rangle$ be the subgroup of $(\mathbb{Z}_m^*, *)$ generated by r, i.e. $\langle r \rangle = \{r^i \bmod m : 1 \le i \le n\}$.
- If $\langle s \rangle \subseteq \langle r \rangle$, then there exists i to complete $s \equiv r^i \bmod m$. In that case, we define the function $\psi(a) = ia$, which satisfies $s^a \equiv r^{\psi(a)} \bmod m$. We also define the function $\phi : \mathbb{Z}_m \rtimes_s \mathbb{Z}_n \to \mathbb{Z}_m \rtimes_r \mathbb{Z}_n$, given by $\phi(a, b) = (a, \psi(b)) = (a, ib)$.

Proposition 1. Let m, n, s, r be such that r and s check $x^n \equiv 1 \bmod m$. If $\langle s \rangle = \langle r \rangle$, then each graph in $\mathcal{G}_{(m,n,s)}(\Gamma)$ is isomorphic to a graph in $\mathcal{G}_{(m,n,r)}(\Gamma)$.

Proof. Let $\Gamma^\alpha \in \mathcal{G}_{(m,n,s)}(\Gamma)$; we will find $\Gamma^\beta \in \mathcal{G}_{(m,n,r)}(\Gamma)$ isomorphic to Γ^α, i.e. we will find $\beta : E\Gamma \to \mathbb{Z}_m \rtimes_r \mathbb{Z}_n$ and $\Psi : V\Gamma^\alpha \to V\Gamma^\beta$ such that Ψ is a graph isomorphism from Γ^α to Γ^β.

First, we set β and Ψ and then, then we show that Ψ is actually an isomorphism. Define β as $\beta(e) = \phi(\alpha(e))$, where $\phi(x, y) = (x, iy)$ and i a natural such that $s \equiv r^i \bmod m$. The existence of i is ensured by hypotesis $\langle s \rangle = \langle r \rangle$. Finally, let us define Ψ as $\Psi(u_e) = u_{\beta(e)}$. In order to prove that Ψ is an isomorphism, we only need to see that Ψ is a bijective homomorphism of graphs.

Ψ *is graph homomorphism:* suppose that $u_{(a,b)} \sim v_{(c,d)}$ in Γ^α, $\alpha(\{u, v\}) = (e, f)$ and $(c, d) = (a, b)(e, f) = (a + s^b e, b + f)$, then

$$\begin{aligned} c &= a + s^b, \\ d &= b + f. \end{aligned}$$

We must check that $\Psi(u_{(a,b)}) = u_{(a,ib)}$ and $\Psi(v_{(c,d)}) = v_{(c,id)}$ are adjacent in Γ^β. This is subject to two conditions. First $u \sim v$ in Γ, which we know to be true. The second condition to be met is $(a, ib)\beta((u, v)) = (c, id)$, i.e. $(a, ib)(e, if) = (c, id)$. Indeed,

$$\begin{aligned}(a, ib)(e, if) &= (a + r^{ib}e, ib + if)\\ &= (a + (r^i)^b e, i(b + f))\\ &= (a + s^b e, i(b + f))\\ &= (c, id)\end{aligned}$$

Bijectivity of Ψ: just see that $x \mapsto ix$ as function of $\mathbb{Z}_n \to \mathbb{Z}_n$ is bijective. This happens if and only if i is invertible as an element of $\mathbb{Z}_n^*$. Indeed, as $\langle s\rangle = \langle r\rangle$, there exists a j such that $r \equiv s^j \bmod m$, then $r \equiv r^{ij} \bmod m$, so $1 \equiv ij \bmod n$. □

In a series of experiments, we also found that, if $\langle s\rangle \subseteq \langle r\rangle$ by using the same voltage assignment α, the diameter of Γ^{α_r} was less or equal to the diameter of Γ^{α_s} in most cases. Hence we consider r such that $|\langle r\rangle|$ were maximum. But this choice of r does not guarantee a successful search.

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