- **Schur positivity**

- **k-Schur functions**

- **Catalania**
  (joint with Blasiak, Pun, and Summers)

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Schur functions are...

Symmetric functions

Multivariate polynomials invariant under S_n-action: $\sigma : x_i \mapsto x_{\sigma(i)}$

symmetric: $3x_1^2+3x_2^2+3x_3^2-7x_1x_2-7x_1x_3-7x_2x_3$

$x_1 + x_2 + x_3$

$x_1x_2 + x_1x_3 + x_2x_3$

$x_1x_2x_3$

not symmetric: $5x_1^2+5x_2^2+8x_3^2 \neq 5x_3^2+5x_2^2+8x_1^2$

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symmetric: $3x_1^2+3x_2^2+3x_3^2-7x_1x_2-7x_1x_3-7x_2x_3$

$$\begin{aligned} e_1 &= x_1+x_2+x_3 \\ e_2 &= x_1x_2+x_1x_3+x_2x_3 \\ e_3 &= x_1x_2x_3 \end{aligned}$$

$$e_r = \sum_{i_1<i_2<\cdots<i_r} x_{i_1}x_{i_2}\cdots x_{i_r} \quad \text{or} \quad h_r = \sum_{i_1\leq i_2\leq\cdots\leq i_r} x_{i_1}x_{i_2}\cdots x_{i_r}$$

Polynomials in $e_1, e_2, \ldots$ or in $h_1, h_2, \ldots,$

$$3h_1^2h_2 - h_2^2 + 6h_3h_1 = 3h_{(112)} - h_{(22)} + 6h_{(31)}$$

Schur functions are...

defined for $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ by:

$$s_\alpha = \prod_{i<j}(1 - R_{ij})h_\alpha \, ,$$

Raising operator: $R_{ij}h_\alpha = h_{\alpha+\epsilon_i-\epsilon_j}$

$$R_{12}h_{(2,2)} = h_{(2+1,2-1)} = h_{(3,1)}$$

$$R_{24}h_{(1,6,2,7,5,1)} = h_{(1,6+1,2,7-1,5,1)}$$

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$$s_{211} = (1-R_{12})(1-R_{23})(1-R_{13}) h_{211}$$

$$= h_{211} - h_{301} - h_{220} - h_{310} + h_{310} + h_{32-1} + h_{400} - h_{41-1}$$

some terms cancel

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$$s_{23} = (1-R_{23})h_{23} = h_{23} - h_{32} = 0$$

not linearly independent

Schur function basis

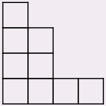
Schur function straightening

$$s_\alpha = \prod_{i<j}(1 - R_{ij})h_\alpha = \begin{cases} \pm s_\lambda & \text{for a partition } \lambda \\ 0 & \end{cases}$$

Partitions $\lambda = (\lambda_1 \geq \cdots \geq \lambda_\ell > 0)$

$$\lambda = (4, 2, 2, 1) =$$

1
2
2
4



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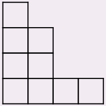


- orthonormal basis for Λ
- irreducible S_n-modules
- representatives for Schubert classes in $H^*(Gr)$

Schur positivity example

Harmonic polynomials

$\mathcal{M}$ = polynomials killed by all symmetric differential operators
= linear span of all partial derivatives of Vandermonde

$$\Delta = \det \begin{vmatrix} x_1^2 & x_1^1 & 1 \\ x_2^2 & x_2^1 & 1 \\ x_3^2 & x_3^1 & 1 \end{vmatrix} = x_1^2(x_2 - x_3) - x_2^2(x_1 - x_3) + x_3^2(x_1 - x_2)$$

$\mathcal{M}$ is an S_n-module

$$\mathcal{M} = \mathrm{sp}\{\Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2,\ 2x_2(x_3 - x_1) - x_3^2 + x_1^2,\ x_3 - x_1,\ x_2 - x_3,\ 1\}$$

$1 \mapsto 2 \quad 2 \mapsto 3 \quad 3 \mapsto 1$

Harmonic polynomials

$\mathcal{M}$ = linear span of all partial derivatives of Vandermonde

decompose into irreducible submodules (indexed by partitions)

$$\underbrace{\mathrm{sp}\{\Delta\}} \oplus \underbrace{\mathrm{sp}\{2x_1(x_2-x_3)-x_2^2+x_3^2,\, x_1^2-2x_2(x_1-x_3)-x_3^2\}} \oplus \underbrace{\mathrm{sp}\{x_3-x_1,\, x_2-x_3\}} \oplus \underbrace{\mathrm{sp}\{1\}}$$

- How many times does a particular submodule occur?

Harmonic polynomials

$\mathcal{M}$ = linear span of all partial derivatives of Vandermonde

decompose into irreducible submodules (indexed by partitions)

$$\underbrace{\mathrm{sp}\{\Delta\}}_{\tiny\yng(1,1,1)} \oplus \underbrace{\mathrm{sp}\{2x_1(x_2-x_3)-x_2^2+x_3^2,\, x_1^2-2x_2(x_1-x_3)-x_3^2\}}_{\tiny\yng(1,2)} \oplus \underbrace{\mathrm{sp}\{x_3-x_1,\, x_2-x_3\}}_{\tiny\yng(1,2)} \oplus \underbrace{\mathrm{sp}\{1\}}_{\tiny\yng(3)}$$

- How many times does a particular submodule occur?

[Frobenius] **replace irreducible** λ **by** s_λ

$$s_{\tiny\yng(1,1,1)} + s_{\tiny\yng(1,2)} + s_{\tiny\yng(1,2)} + s_{\tiny\yng(3)} = (x_1+x_2+x_3)^3$$

- Expand $(x_1+\cdots+x_n)^n$ into Schur functions

Combinatorial Bonanza

$$s_{\ydiagram{1,2}} = h_{21} - h_{30}$$

$$= 2\,x_1x_2x_3 + x_1x_1x_2 + x_1x_2x_2 + x_1x_1x_3 + x_1x_3x_3 + x_2x_3x_3 + x_2x_2x_3$$

Combinatorial Bonanza

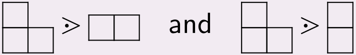
$$s_{21} = h_{21} - h_{30}$$

$$= 2\,x_1x_2x_3 + x_1x_1x_2 + x_1x_2x_2 + x_1x_1x_3 + x_1x_3x_3 + x_2x_3x_3 + x_2x_2x_3$$

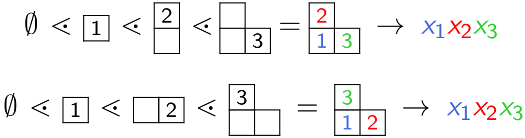
Young's Partition Poset

order: containment of shapes

cover: $\lambda \gtrdot \mu$ when $\mu = \lambda$ - box



Schur function = generating function of chains in Young's poset (tableaux)

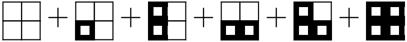


The quantum craze

- Symmetric functions over $\mathbb{Q}(q)$

$$\frac{2}{(1-q)}x_1^2 + \frac{2}{(1-q)}x_2^2 + \frac{17q}{2}x_1x_2 = \frac{2}{(1-q)}s_{(1,1)} + \frac{17q}{2}s_{(2)}$$

- Representation theory: graded modules
- Geometry: quantum cohomology, string theory
- Combinatorics: q-counting

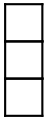
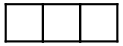


$$\rightarrow (1 + q + 2q^2 + q^3 + q^4)$$

Harmonic polynomials

$\mathcal{M}$ = linear span of all partial derivatives of Vandermonde

$$\underbrace{\mathrm{sp}\{\Delta\}}\oplus\underbrace{\mathrm{sp}\{2x_1(x_2-x_3)-x_2^2+x_3^2,\, x_1^2-2x_2(x_1-x_3)-x_3^2\}}\oplus\underbrace{\mathrm{sp}\{x_3-x_1,\, x_2-x_3\}}\oplus\underbrace{\mathrm{sp}\{1\}}$$

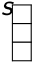
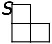
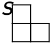
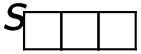
degree 2 polynomials

degree 1 polynomials

How many times does a particular submodule occur?

[Frobenius] irreducible $\mapsto$ s_λ

$$(x_1+x_2+x_3)^3 = s \;+\; s \;+\; s \;+\; s$$

Harmonic polynomials

$\mathcal{M}$ = linear span of all partial derivatives of Vandermonde

$$\underbrace{\mathrm{sp}\{\Delta\}}_{\ydiagram{1,1,1}} \oplus \underbrace{\mathrm{sp}\{2x_1(x_2-x_3)-x_2^2+x_3^2,\, x_1^2-2x_2(x_1-x_3)-x_3^2\}}_{\ydiagram{1,2}} \oplus \underbrace{\mathrm{sp}\{x_3-x_1,\, x_2-x_3\}}_{\ydiagram{1,2}} \oplus \underbrace{\mathrm{sp}\{1\}}_{\ydiagram{3}}$$

degree 2 polynomials   degree 1 polynomials

How many times does a particular submodule occur?

[Frobenius]   irreducible of degree $d \mapsto \; q^d\, s_\lambda$

$$?????\; = \; q^3\, s_{\ydiagram{1,1,1}} + q^2\, s_{\ydiagram{1,2}} + q\, s_{\ydiagram{1,2}} + s_{\ydiagram{3}}$$

Macdonald goes wild (1980's)

Eigenfunctions of: $$\sum_{\substack{I\subset[1,n]\\|I|=1}} \prod_{\substack{i\in I\\ j\notin I}} \frac{tx_i - x_j}{x_i - x_j} \prod_{i\in I} T_{q,x_i} - \sum_{i=1}^{n} t^{-1}$$

where $T_{q,x_i}(f(x_1,\ldots,x_n)) = f(x_1,\ldots,qx_i,\ldots,x_n)$

Macdonald goes wild (1980's)

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where $T_{q,x_i}(f(x_1,\ldots,x_n)) = f(x_1,\ldots,qx_i,\ldots,x_n)$

$$\frac{-4q}{t-1}x_1^2x_2 + \frac{-4q}{t-1}x_2^2x_3 + \frac{-4q}{t-1}x_1^2x_3 + (t^2-7q)x_1x_2x_3$$

[Macdonald:1981] symmetric function basis over $\mathbb{Q}(q,t)$

Macdonald goes wild (1980's)

Eigenfunctions of: $$\sum_{\substack{I\subset[1,n]\\|I|=1}} \prod_{\substack{i\in I\\ j\notin I}} \frac{tx_i - x_j}{x_i - x_j} \prod_{i\in I} T_{q,x_i} - \sum_{i=1}^{n} t^{-1}$$

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[Macdonald:1981] symmetric function basis over $\mathbb{Q}(q,t)$

Conjecture: q,t-sum of funny functions

$$J_{2,2} = t^2 f_{(4)} + (qt^2+qt+t)\,f_{(3,1)} + (q^2t^2+1)\,f_{(2,2)} + (q^2t+qt+q)\,f_{(2,1,1)} + q^2 f_{(1,1,1,1)}$$

Macdonald polynomials

basis defined (obscurely) as eigenfunctions

Open problem: q, t-enumeration

of monomial terms = # of tableaux

$$J_{2,2} = t^2 f_{(4)} + (qt^2 + qt + t) f_{(3,1)} + \color{red}{(q^2 t^2 + 1)} f_{(2,2)} + (q^2 t + qt + q) f_{(2,1,1)} + q^2 f_{(1,1,1,1)}$$

$$\begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline\end{array} \qquad \begin{array}{c}4\\1\;2\;3\end{array} \quad \begin{array}{c}3\\1\;2\;4\end{array} \quad \begin{array}{c}2\\1\;3\;4\end{array} \qquad \color{red}{\begin{array}{cc}3&4\\1&2\end{array} \quad \begin{array}{cc}2&4\\1&3\end{array}} \qquad \begin{array}{c}4\\3\\1\;2\end{array} \quad \begin{array}{c}4\\2\\1\;3\end{array} \quad \begin{array}{c}3\\2\\1\;4\end{array} \qquad \begin{array}{c}4\\3\\2\\1\end{array}$$

Garsia-Haiman modules, $\mathcal{M}_\mu$

$\mathcal{M}_\mu$ = span of partial derivatives of Δ_μ

$$\Delta_{(2,1)} = \det \begin{vmatrix} 1 & y_1 & x_1 \\ 1 & y_2 & x_2 \\ 1 & y_3 & x_3 \end{vmatrix} = x_3y_2 - y_3x_2 - y_1x_3 + y_1x_2 + y_3x_1 - y_2x_1$$

S_n-module in $\mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ under $\sigma : x_iy_j \mapsto x_{\sigma(i)}y_{\sigma(j)}$

$$\mathcal{M}_{2,1} = \underbrace{\mathrm{sp}\{1\}}_{(3),\ (0,0)} \oplus \underbrace{\mathrm{sp}\{x_3 - x_1, x_1 - x_2\}}_{(2,1)} \oplus \underbrace{\mathrm{sp}\{y_3 - y_1, y_1 - y_2\}}_{(2,1),\ (1,0)} \oplus \underbrace{\mathrm{sp}\{\Delta_{(2,1)}\}}_{(1,1,1),\ (1,1)}$$

degree 0 in y   degree 1 in x

$$H_{2,1} = t^0 q^0 s_{(3)} + t^0 q^1 s_{(2,1)} + t^1 q^0 s_{(2,1)} + t^1 q^1 s_{(1,1,1)}$$

- **Schur positivity**

- **k-Schur functions**

- **Catalania**

k-Schur Catalan functions

by Jennifer Morse

- **Schur positivity**
- **k-Schur functions**
- **Catalania**

q, t-Kostka coefficients

```
|1|2|3|4| |2|       |3|       |4|       |2|4| |3|4| |3|     |4|     |4|     |4|
          |1|3|4|   |1|2|4|   |1|2|3|   |1|3| |1|2| |2|     |2|     |3|     |3|
                                                    |1|4|   |1|3|   |1|2|   |2|
                                                                             |1|
```

$$H_{(1,1,1,1)} = t^6 s_{(4)} + \left(t^3 + t^4 + t^5\right) s_{(3,1)} + \left(t^2 + t^4\right) s_{(2,2)} + \left(t + t^2 + t^3\right) s_{(2,1,1)} + s_{(1,1,1,1)}$$

$$H_{(2,1,1)} = t^3 s_{(4)} + \left(t + t^2 + qt^3\right) s_{(3,1)} + \left(t + qt^2\right) s_{(2,2)} + \left(1 + qt + qt^2\right) s_{(2,1,1)} + q s_{(1,1,1,1)}$$

$$H_{(2,2)} = t^2 s_{(4)} + \left(qt^2 + qt + t\right) s_{(3,1)} + \left(q^2 t^2 + 1\right) s_{(2,2)} + \left(q^2 t + qt + q\right) s_{(2,1,1)} + q^2 s_{(1,1,1,1)}$$

$$H_{(3,1)} = t s_{(4)} + \left(q^2 t + qt + 1\right) s_{(3,1)} + \left(q^2 t + q\right) s_{(2,2)} + \left(q^3 t + q^2 + q\right) s_{(2,1,1)} + q^3 s_{(1,1,1,1)}$$

$$H_{(4)} = s_{(4)} + \left(q^3 + q^2 + q\right) s_{(3,1)} + \left(q^2 + q^4\right) s_{(2,2)} + \left(q^5 + q^4 + q^3\right) s_{(2,1,1)} + q^6 s_{(1,1,1,1)}$$

q, t-Kostka coefficients

```
|1|2|3|4|  |2|        |3|        |4|        |2|4|  |3|4|  |3|    |4|    |4|    |4|
           |1|3|4|    |1|2|4|    |1|2|3|    |1|3|  |1|2|  |2|    |2|    |3|    |3|
                                                           |1|4|  |1|3|  |1|2|  |2|
                                                                                |1|
```

$$H_{(1,1,1,1)} = t^6 s_{(4)} + (t^3 + t^4 + t^5)\, s_{(3,1)} + (t^2 + t^4)\, s_{(2,2)} + (t + t^2 + t^3)\, s_{(2,1,1)} + s_{(1,1,1,1)}$$

$$H_{(2,1,1)} = t^3 s_{(4)} + (t + t^2 + qt^3)\, s_{(3,1)} + (t + qt^2)\, s_{(2,2)} + (1 + qt + qt^2)\, s_{(2,1,1)} + q s_{(1,1,1,1)}$$

$$H_{(2,2)} = t^2 s_{(4)} + (qt^2 + qt + t)\, s_{(3,1)} + (q^2 t^2 + 1)\, s_{(2,2)} + (q^2 t + qt + q)\, s_{(2,1,1)} + q^2 s_{(1,1,1,1)}$$

$$H_{(3,1)} = t s_{(4)} + (q^2 t + qt + 1)\, s_{(3,1)} + (q^2 t + q)\, s_{(2,2)} + (q^3 t + q^2 + q)\, s_{(2,1,1)} + q^3 s_{(1,1,1,1)}$$

$$H_{(4)} = s_{(4)} + (q^3 + q^2 + q)\, s_{(3,1)} + (q^2 + q^4)\, s_{(2,2)} + (q^5 + q^4 + q^3)\, s_{(2,1,1)} + q^6 s_{(1,1,1,1)}$$

q, t-Kostka coefficients

2-bounded shapes

$(2,2)\quad (2,1,1)\quad (1,1,1,1)$

$$H_{(1,1,1,1)} = t^4\left(s_{(2,2)} + t s_{(3,1)} + t^2 s_{(4)}\right) + \underbrace{(t^2+t^3)}_{\text{positive sum of } q,\,t\text{-monomials}} \underbrace{\left(s_{(2,1,1)} + t s_{(3,1)}\right)}_{t\text{-positive sum of Schur functions}} + \left(s_{(1,1,1,1)} + t s_{(2,1,1)} + t^2 s_{(2,2)}\right)$$

$$H_{(2,1,1)} = t\left(s_{(2,2)} + t s_{(3,1)} + t^2 s_{(4)}\right) + (1+qt^2)\left(s_{(2,1,1)} + t s_{(3,1)}\right) + q\left(s_{(1,1,1,1)} + t s_{(2,1,1)} + t^2 s_{(2,2)}\right)$$

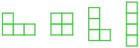
$$H_{(2,2)} = \left(s_{(2,2)} + t s_{(3,1)} + t^2 s_{(4)}\right) + (tq+q)\left(s_{(2,1,1)} + t s_{(3,1)}\right) + q^2\left(s_{(1,1,1,1)} + t s_{(2,1,1)} + t^2 s_{(2,2)}\right)$$

$$H_{(3,1)} = \left(\boxed{q}\, s_{(2,2)} + s_{(3,1)} + t s_{(4)}\right) + (q+q^2)\left(s_{(2,1,1)} + t s_{(3,1)}\right) + q^2\left(q s_{(1,1,1,1)} + qt s_{(2,1,1)} + t s_{(2,2)}\right)$$

$$H_{(4)} = \left(q^2 s_{(2,2)} + q s_{(3,1)} + s_{(4)}\right) + (q^2+q^3)\left(q s_{(2,1,1)} + s_{(3,1)}\right) + q^4\left(q^2 s_{(1,1,1,1)} + q s_{(2,1,1)} + s_{(2,2)}\right)$$

Bounded Macdonald polynomials

3-bounded shapes



positive sum of q, t-monomials — t-positive sum of Schurs — t-positive sum of Schurs

$$H_{(1,1,1,1)} = t^4\big((s_{(2,2)}) + t(s_{(3,1)} + t s_{(4)})\big) + \underbrace{(t^2 + t^3)}_{\text{positive sum of } q,\,t\text{-monomials}}\underbrace{(s_{(2,1,1)} + t s_{(3,1)})}_{t\text{-positive sum of Schurs}} + \big(\underbrace{(s_{(1,1,1,1)} + t s_{(2,1,1)})}_{t\text{-positive sum of Schurs}} + t^2 (s_{(2,2)})\big)$$

$$H_{(2,1,1)} = t\big((s_{(2,2)}) + t(s_{(3,1)} + t s_{(4)})\big) + (1 + qt^2)\left(s_{(2,1,1)} + t s_{(3,1)}\right) + q\left(\left(s_{(1,1,1,1)} + t s_{(2,1,1)}\right) + t^2 s_{(2,2)}\right)$$

$$H_{(2,2)} = \big((s_{(2,2)}) + t(s_{(3,1)} + t s_{(4)})\big) + (tq + q)\left(s_{(2,1,1)} + t s_{(3,1)}\right) + q^2\left(\left(s_{(1,1,1,1)} + t s_{(2,1,1)}\right) + t^2 s_{(2,2)}\right)$$

$$H_{(3,1)} = \big(q(s_{(2,2)}) + (s_{(3,1)} + t s_{(4)})\big) + (q + q^2)\left(s_{(2,1,1)} + t s_{(3,1)}\right) + q^2\left(q\left(s_{(1,1,1,1)} + t s_{(2,1,1)}\right) + t s_{(2,2)}\right)$$

$$H_{(4)} = (q^2 s_{(2,2)} + q s_{(3,1)} + s_{(4)}) + (q^2 + q^3)\left(q s_{(2,1,1)} + s_{(3,1)}\right) + q^4\left(q^2 s_{(1,1,1,1)} + q s_{(2,1,1)} + s_{(2,2)}\right)$$

Conjecture [Lapointe,Lascoux,M:1998]

$$H_{(1,1,1,1)} = \overbrace{t^4\Big(\overbrace{s_{(2,2)}}^{s^{(3)}_{(2,2)}} + t\,\overbrace{\big(s_{(3,1)} + t s_{(4)}\big)}^{s^{(3)}_{(3,1)}}\Big)}^{s^{(2)}_{(2,2)}} + (t^2+t^3)\overbrace{\big(s_{(2,1,1)} + t s_{(3,1)}\big)}^{s^{(2)}_{(2,1,1)} \,=\, s^{(3)}_{(2,1,1)}} + \overbrace{\Big(\overbrace{\big(s_{(1,1,1,1)} + t s_{(2,1,1)}\big)}^{s^{(3)}_{(1,1,1,1)}} + t^2\,\overbrace{\big(s_{(2,2)}\big)}^{s^{(3)}_{(2,2)}}\Big)}^{s^{(2)}_{(1,1,1,1)}}$$

$$H_{(2,1,1)} = t\big(s_{(2,2)} + t\,(s_{(3,1)} + t s_{(4)})\big) + (1+qt^2)\big(s_{(2,1,1)} + t s_{(3,1)}\big) + q\Big(\big(s_{(1,1,1,1)} + t s_{(2,1,1)}\big) + t^2 s_{(2,2)}\Big)$$

$$H_{(2,2)} = \big(s_{(2,2)} + t\,(s_{(3,1)} + t s_{(4)})\big) + (tq+q)\big(s_{(2,1,1)} + t s_{(3,1)}\big) + q^2\Big(\big(s_{(1,1,1,1)} + t s_{(2,1,1)}\big) + t^2 s_{(2,2)}\Big)$$

For each $k > 0$, there are symmetric functions $s^{(k)}_\lambda(x;t)$ which

- form a basis for the span of k-bounded Macdonald polynomials
- are a t-positive sum of $\{s^{(k+1)}_\mu\}$ (and of Schur functions)

2 decades pass...

Proposed Definition	basis	symmetric	Schur positivity	branching
[1998:Lapointe,Lascoux,M] Tableaux and katabolism		✓	✓	
[2003:Lapointe,M] Jing vertex operators	✓	✓		
[2008:Lam,Lapointe,M,Shimozono] Bruhat order on type-A affine Weyl group				
[2010:Chen,Haiman] $GL_\ell(\mathbb{C})$-equivariant Euler characteristics (Demazure operator)		✓		
[2012:Assaf,Billey] Quasisymmetric functions				
[2015:Dalal,M] Inverting affine Kostka matrix	✓	✓		

Special case when $t = 1$

Definition [2004:Lapointe,M]

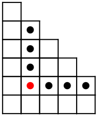
k-Schur functions are functions satisfying Pieri rule (weak order on type-A affine Weyl group)

k-**Schur functions**	basis	symmetric	positive products	branching
[2004:Lapointe,M] Quantum cohomology of Grassmannian	✓	✓	✓/2	
[2006:Lam] Schubert representatives for the affine Grassmannian	✓	✓	✓	
[2008:Lam,Lapointe,M,Shimozono]	✓	✓	✓	
[2010:Lam,Lapointe,M,Shimozono] Generating functions for marked chains in Bruhat order on type-A affine Weyl group				✓

Underlying combinatorics

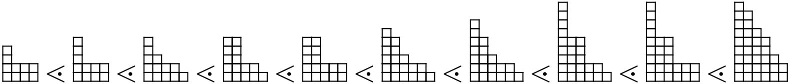
Bruhat order on Grassmannian elements of affine symmetric group

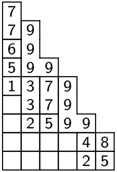
- shapes have no cell with hook-length $k+1$



● cell has hook-length 7

- ordered by containment

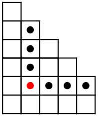




Underlying combinatorics

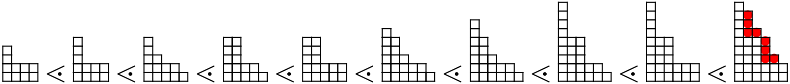
Bruhat order on Grassmannian elements of affine symmetric group

- shapes have no cell with hook-length $k+1$

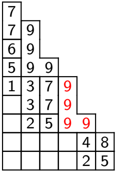


● cell has hook-length 7

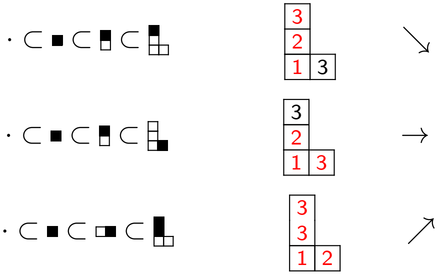
- ordered by containment



- mark one ribbon in skew of $\lambda \lessdot \mu$
  skew of adjacent shapes = copies of a ribbon



k-Schur functions



$3\,x_1x_2x_3$

[Lapointe,Lam,M,Shimozono:2008]

strong marked tableaux generating functions

$$s_\lambda^{(k)} = \sum_T x^{weight(T)}$$

Theorem at $t = 1$

For each $k > 0$, k-Schur functions are symmetric functions which

- form a basis for $\Lambda^k = \mathbb{Z}[h_1, h_2, \ldots, h_k]$
- are a positive sum of $k+1$-Schur functions
- are a positive sum of Schur functions

structure constants are positive

- generalized Young (Specht) modules
- Gromov-Witten invariants
- Stanley symmetric functions
- WZW conformal field theories
- knot invariants
- affine Stanley functions
- intersections in flag variety
- stable Schubert polynomials
- Hecke algebras at roots of unity
- positroids
- quantum cohomology
- affine Schubert calculus

without direct combinatorial interpretations

Wanted: Schur positive bases...

$$
H_{1111} = \overbrace{t^4\Big(\overbrace{s_{22}}^{s^{(3)}_{22}} + t\,\overbrace{\left(s_{31} + t s_{4}\right)}^{s^{(3)}_{31}}\Big)}^{s^{(2)}_{22}} + \overbrace{(t^2+t^3)\overbrace{\left(s_{211} + t s_{31}\right)}^{s^{(3)}_{211}}}^{s^{(2)}_{211}} + \overbrace{\Big(\overbrace{\left(s_{1111} + t s_{211}\right)}^{s^{(3)}_{1111}} + t^2\,\overbrace{\left(s_{22}\right)}^{s^{(3)}_{22}}\Big)}^{s^{(2)}_{1111}}
$$

$$
H_{211} = t\left(s_{22} + t\left(s_{31} + t s_{4}\right)\right) + (1+qt^2)\left(s_{211} + t s_{31}\right) + q\left(\left(s_{1111} + t s_{211}\right) + t^2 s_{22}\right)
$$

$$
H_{22} = \left(s_{22} + t\left(s_{31} + t s_{4}\right)\right) + (tq+q)\left(s_{211} + t s_{31}\right) + q^2\left(\left(s_{1111} + t s_{211}\right) + t^2 s_{22}\right)
$$

Mysterious branching

$$
s^{(k)}_{\lambda}(x;t) = \sum_{\mu} (\text{positive } t \text{ polynomial})\, s^{(k+1)}_{\mu}\,.
$$

k-Schur Catalan functions

- **Schur positivity**
- k-**Schur functions**
- **Catalania**

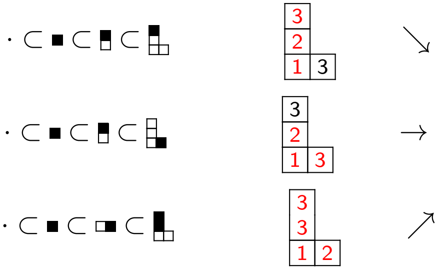
Attempts with the graded case

	basis	symmetric	Schur positivity	branching
[1998:Lapointe,Lascoux,M] Tableaux and katabolism		✓	✓	
[2003:Lapointe,M] Jing vertex operators	✓	✓		
[2008:Lam,Lapointe,M,Shimozono] Bruhat order on type-A affine Weyl group				
[2010:Chen,Haiman] $GL_\ell(\mathbb{C})$-equivariant Euler characteristics (Demazure operator)		✓		
[2012:Assaf,Billey] Quasisymmetric functions				
[2015:Dalal,M] Inverting affine Kostka matrix	✓	✓		

A proposed k-Schur candidate

$$s_\lambda^k(x;t) = \sum_{\text{strong marked tableaux T}} t^{\mathrm{spin}(T)}\, x^T$$

spin encodes # and location of the marked ribbons and their height.

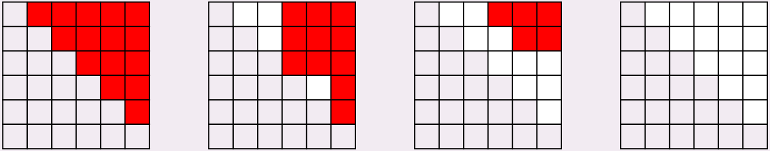


$$(1+t+t)\, x_1 x_2 x_3$$

- symmetric?
- basis for bounded Macdonalds?
- Schur and $k+1$-Schur positive?

Catalan root ideals

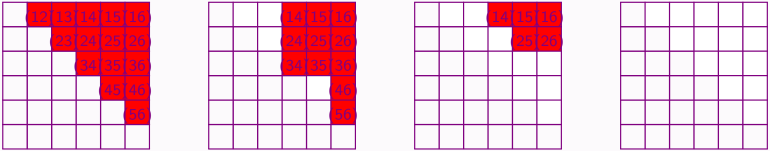
Partitions: flipped, inside a box, above the diagonal



enumerated by Catalan numbers

Catalan root ideals

Upper order ideal of roots

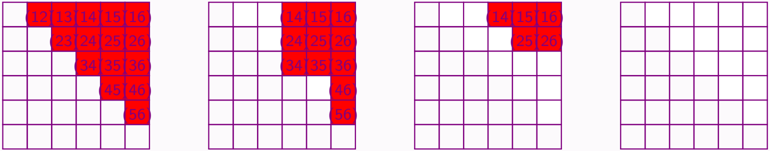


$$\downarrow$$

$$\Psi = \{(1,4),(1,5),(1,6),(2,5),(2,6)\}$$

Catalan functions

Upper order ideal of roots



$\downarrow$

$$\Psi = \{(1,4),(1,5),(1,6),(2,5),(2,6)\}$$

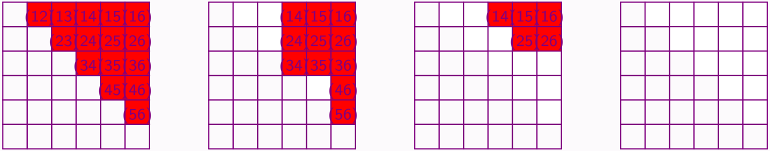
[Panyushev, Chen-Haiman (2010)]

For Ψ and $\gamma \in \mathbb{Z}^\ell$

$$H(\Psi;\gamma)(x;t) = \prod_{(i,j)\in\Psi} (1 - tR_{ij})^{-1} s_\gamma(x)$$

Catalan functions

Upper order ideal of roots



$$\downarrow$$

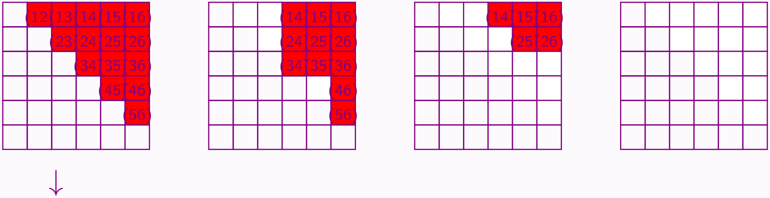
$$\Psi = \{\}$$

- Schur functions

$$H(\emptyset;\gamma)(x;t) = \prod_{\text{no roots}} (1 - tR_{ij})^{-1} s_\gamma(x) = s_\gamma(x)$$

Catalan functions

Upper order ideal of roots



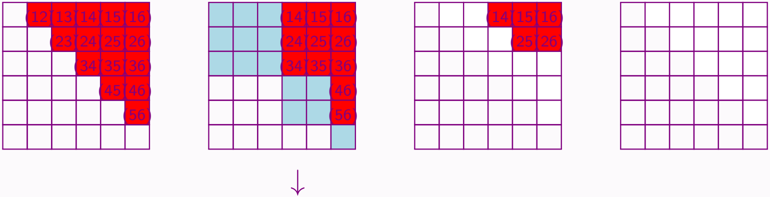
$\Psi = \{\text{all roots}\}$

- Macdonald polynomials at $q = 0$

$$H(\Delta^+; \lambda)(x; t) = \prod_{i<j} (1 - tR_{ij})^{-1} s_\lambda(x)$$

Catalan functions

Upper order ideal of roots



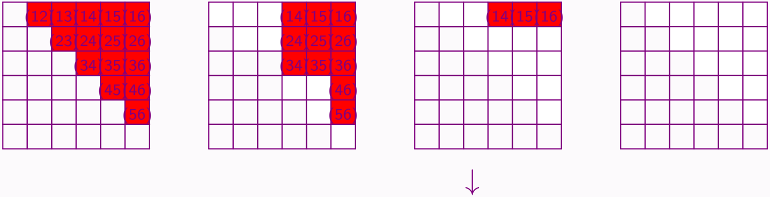
$\Psi = \{$roots lie above blocks$\}$

- Parabolic Hall-Littlewood functions
  (t-analog of Schur function products)

$$H(\emptyset;\gamma)(x;t) = \prod_{\text{parabolic ideal}} (1 - tR_{ij})^{-1} s_\gamma(x)$$

Catalan functions

Upper order ideal of roots



$$H(\emptyset;(444322))(x;t) = \prod_{\{(14),(15),(16)\}} (1-tR_{ij})^{-1} s_{444322}(x)$$

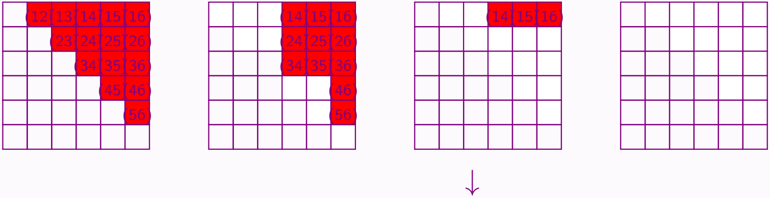
$$= (1+tR_{14}+t^2R_{14}^2+\cdots)(1+tR_{15}+t^2R_{15}^2+\cdots)(1+tR_{16}+t^2R_{16}^2+\cdots)\, s_{444322}$$

$$= (1+tR_{14}+t^2R_{14}^2+t^3R_{14}^3+\cdots+tR_{15}+t^2R_{14}R_{15}+t^3R_{14}R_{15}^2+\cdots)\, s_{444322}$$

$$= s_{444322}+ts_{544312}+t^2s_{644302}+t^3s_{7443-12}+\cdots+ts_{544321}+t^2s_{644311}+\cdots$$

Catalan functions

Upper order ideal of roots



$$H(\emptyset;(444322))(x;t) = \prod_{\{(14),(15),(16)\}} (1 - tR_{ij})^{-1} s_{444322}(x)$$

$$= (1+tR_{14}+t^2R_{14}^2+\cdots)(1+tR_{15}+t^2R_{15}^2+\cdots)(1+tR_{16}+t^2R_{16}^2+\cdots)\, s_{444322}$$

$$= (1+tR_{14}+t^2R_{14}^2+t^3R_{14}^3+\cdots+tR_{15}+t^2R_{14}R_{15}+t^3R_{14}R_{15}^2+\cdots)\, s_{444322}$$

$$= s_{444322}+ts_{544312}+t^2s_{644302}+t^3s_{7443-12}+\cdots+ts_{544321}+t^2s_{644311}+\cdots$$

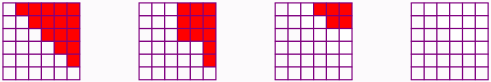
straightening introduces negatives!

$$= s_{444322} + 0 - t^2 s_{644311} - t^3 s_{744310} + \cdots + t\, s_{544321} + t^2 s_{644311} + \cdots$$

Catalan functions

Conjecture

For any upper order ideal of roots:



and partition λ,

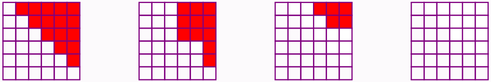
$$H(\Psi;\lambda)(x;t) = \prod_{(i,j)\in\Psi} (1 - tR_{ij})^{-1} s_\lambda(x) \quad \text{is } t\text{-Schur positive}$$

$$\begin{aligned} H(\Psi;(4322)) &= s_{4322} + 0 - t^2\, s_{6311} - t^3\, s_{7310} + \cdots + t\, s_{5321} + t^2\, s_{6311} + \cdots \\ &= s_{4322} + t\, s_{5321} \end{aligned}$$

Catalan functions

Conjecture

For any upper order ideal of roots:



and partition λ,

$$H(\Psi;\lambda)(x;t) = \prod_{(i,j)\in\Psi} (1 - tR_{ij})^{-1} s_\lambda(x) \quad \text{is } t\text{-Schur positive}$$

$$H(\Psi;(4322)) = s_{4322} + 0 - t^2\, s_{6311} - t^3\, s_{7310} + \cdots + t\, s_{5321} + t^2\, s_{6311} + \cdots$$

$$= s_{4322} + t\, s_{5321}$$

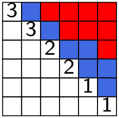
Conjecture [Chen,Haiman:2010]

k-Schur functions are some subclass of $H(\Psi;\lambda)(x;t)$

Yet another proposal for k-Schur functions

k-Schur root ideal for λ

$$\Psi^k(\lambda) = \{(i, i+j) : j > k - \lambda_i\}$$
$$= \text{root ideal with } k - \lambda_i \text{ non-roots in row } i$$

$\Psi^4(3,3,2,2,1,1) =$ 

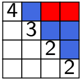
$\leftarrow$ row i has $4 - \lambda_i$ non-roots

k-Schur Catalan function

$$\mathfrak{s}_\lambda^{(k)} = H(\Psi^k(\lambda), \lambda) = \prod_{(i,j) \in \Psi^k(\lambda)} (1 - tR_{ij})^{-1} s_\lambda$$

k-Schur Catalan function

$$\mathfrak{s}^5_{4322} = \prod_{(i,j)\in\Psi^5(4322)} (1 - tR_{ij})^{-1} s_{4322}$$

$\Psi^5(4322) =$ 

$= \quad \{(1,3),(1,4)\}$

$$= (1+tR_{14}+t^2R_{14}^2+\cdots)(1+tR_{13}+t^2R_{13}^2+\cdots)\, s_{4322}$$

$$= (1+tR_{13}+t^2R_{13}^2+t^3R_{13}^3+\cdots+tR_{14}+t^2R_{14}R_{13}+\cdots)\, s_{4322}$$

$$= s_{4322} + ts_{5312} + t^2 s_{6302} + t^3 s_{73-12} + \cdots + ts_{5321} + t^2 s_{6311} + \cdots$$

straightening introduces negatives!

$$= s_{4322} + 0 - t^2 s_{6311} - t^3 s_{7310} + \cdots + t\, s_{5321} + t^2 s_{6311} + \cdots$$

Where do we stand?

Are these functions the same?

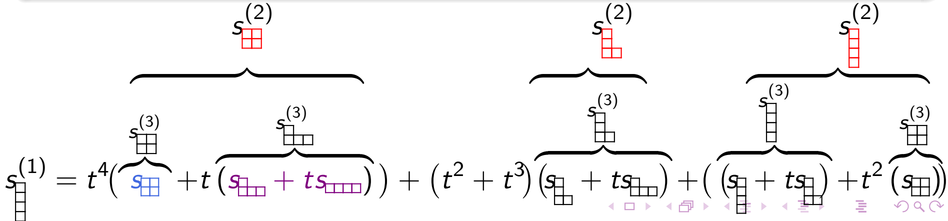
- k-Schur Catalan functions

$$H(\Psi^k(\lambda);\lambda) = \prod_{(ij)\in\Psi^k(\lambda)} (1 - tR_{ij})^{-1} s_\lambda$$

- Strong k-Schur functions

$$s_\lambda^k(x;t) = \sum_{\text{strong marked tableaux T}} t^{\mathrm{spin}(T)} x^T$$

Can we prove anything about either definition?



Are these functions the same?

- *k*-Schur Catalan functions

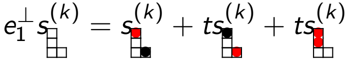
$$H(\Psi^k(\lambda);\lambda) = \prod_{(ij)\in\Psi^k(\lambda)} (1 - tR_{ij})^{-1} s_\lambda$$

- Strong *k*-Schur functions

$$s_\lambda^k(x;t) = \sum_{\text{strong marked tableaux T}} t^{\mathrm{spin}(T)}\, x^T$$

Equivalent to Pieri rule

$$e_r^\perp s_\lambda^{(k)} = \sum_{\text{marked chains:}\lambda\to\mu} t^{\mathrm{spin}}\, s_\mu^{(k)}$$



Are these functions the same?

- *k*-Schur Catalan functions

$$H(\Psi^k(\lambda);\lambda) = \prod_{(ij)\in\Psi^k(\lambda)} (1-tR_{ij})^{-1} s_\lambda$$

- Strong *k*-Schur functions

$$e_r^\perp s_\lambda^{(k)} = \sum_{\text{marked chains:}\lambda\to\mu} t^{\mathrm{spin}}\, s_\mu^{(k)}$$

Pieri rule for Catalan functions

$$e_r^\perp H(\Psi;\gamma) = \sum_{i_1<i_2<\cdots<i_r} H(\Psi;\gamma-\epsilon_{i_1}-\cdots-\epsilon_{i_r})$$

$$e_1^\perp H(\Psi^k(322),(3,2,2)) = H(\Psi^k(322),(3,1,2)) + H(\Psi^k(322),(2,2,2)) + H(\Psi^k(322),(3,2,1))$$

need straightening $\uparrow$ $\qquad$ $\uparrow$ need $\Psi^k((2,2,2))$

Shocking discovery

Shift invariance

$$e_\ell^\perp \mathfrak{s}^{(k+1)}_{\lambda+1^\ell} = \mathfrak{s}^{(k)}_\lambda$$

Shocking discovery

Shift invariance

$$e_\ell^\perp \mathfrak{s}_{\lambda+1^\ell}^{(k+1)} = \mathfrak{s}_\lambda^{(k)}$$

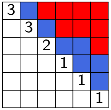
mysterious branching $\mathfrak{s}_\lambda^{(k)} = \sum ???? \, \mathfrak{s}_\nu^{(k+1)}$

follows from Pieri! $e_\ell^\perp \mathfrak{s}_{\lambda+1^\ell}^{(k+1)} = \sum_{\nu \to \lambda+1^\ell} t^{\mathrm{spin}(\mathrm{T})} \, \mathfrak{s}_\nu^{(k+1)}$

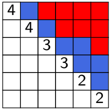
Shift invariance for $\mathfrak{s}_\lambda^{(k)} = H\left(\Psi^k(\lambda); \lambda\right)$

Property: $\Psi^k(\lambda) = \Psi^{k+1}(\lambda + 1^\ell)$

$$
\begin{aligned}
\#\text{ non-roots in row } i \text{ of } \Psi^k(\lambda) &= k - \lambda_i \\
&= (k+1) - (\lambda_i + 1) \\
&= \#\text{ non-roots in row } i \text{ of } \Psi^{k+1}(\lambda + 1^\ell)
\end{aligned}
$$

$\Psi^4(3,3,2,1,1,1) =$ 

$\leftarrow$ 4 - 2 non-roots

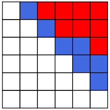
$\Psi^5(4,4,3,3,2,2) =$ 

$\leftarrow$ 5 - 3 non-roots

Shift invariance

Property: $\Psi^k(\lambda) = \Psi^{k+1}(\lambda + 1^\ell)$

$$\Psi^4(3,3,2,1,1,1) =$$



$$= \Psi^5(4,4,3,2,2,2)$$

Catalan Pieri

$$e_\ell^\perp H(\Psi; \lambda + 1^\ell) \; = H(\Psi; \lambda)$$

$\Psi^{k+1}(\lambda+1^\ell)=\Psi^k(\lambda)$

Theorem

$$e_\ell^\perp H\left(\Psi^{k+1}(\lambda + 1^\ell); \lambda + 1^\ell\right) = H\left(\Psi^k(\lambda); \lambda\right)$$

k-Schur Catalan functions

Pieri rule

$$e_\ell^\perp \mathfrak{s}_{\lambda+1^\ell}^{(k+1)}(x;t) = \sum_{\nu \lessdot \lambda+1^\ell} t^{\mathrm{spin}}\, \mathfrak{s}_\nu^{(k+1)}(x;t)$$

Shift invariance

$$e_\ell^\perp\, \mathfrak{s}_{\lambda+1^\ell}^{(k+1)} = \mathfrak{s}_\lambda^{(k)}$$

Corollaries

- Branching

$$\mathfrak{s}_\lambda^{(k)}(x;t) = \sum_\nu t^{\mathrm{spin}}\, \mathfrak{s}_\nu^{(k+1)}(x;t)$$

- k-Schur Catalans = spin strong tableaux functions

Theorem [Blasiak,M,Pun,Summers]

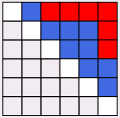
	basis	symmetric	Schur positivity	branching
[1998:Lapointe,Lascoux,M] Tableaux and katabolism		✓	✓	
[2003:Lapointe,M] Jing vertex operators	✓	✓	✓	✓
[2008:Lam,Lapointe,M,Shimozono] Bruhat order on type-A affine Weyl group	✓	✓	✓	✓
[2010:Chen,Haiman] $GL_\ell(\mathbb{C})$-equivariant Euler characteristics (Demazure operator)	✓	✓	✓	✓
[2012:Assaf,Billey] Quasisymmetric functions	✓	✓	✓	✓
[2015:Dalal,M] Inverting affine Kostka matrix	✓	✓		
[2018:Blasiak,M,Pun,Summers] Catalan functions	✓	✓	✓	✓

Future.. and THANK YOU!

- k-Schur function expansion of k-bounded Macdonald polynomials

$$H_{22} = \overbrace{\left(s_{22} + t s_{31} + t^2 s_{4}\right)}^{s^{(2)}_{22}} + (tq+q)\overbrace{\left(s_{211} + t s_{31}\right)}^{s^{(2)}_{211}} + q^2 \overbrace{\left(s_{1111} + t s_{211} + t^2 s_{22}\right)}^{s^{(2)}_{1111}}$$

- Conjecture: Catalan functions are Schur positive for dominant weights
  Refinement: k-Schur positivity when non-roots are bounded by k



maximum non-roots in row is 3

- affine Schubert calculus