Maximizing the Edelman-Greene Statistic

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Introduction

The *Edelman-Greene statistic* of S. Billey-B. Pawlowski measures the "shortness" of the Schur expansion of a Stanley symmetric function. We show that the maximum value of this statistic on permutations of Coxeter length n is the number of involutions in the symmetric group S_n, and explicitly describe the permutations that attain this maximum. Our proof confirms a recent conjecture of C. Monical, B. Pankow, and A. Yong: we give an explicit combinatorial injection between a certain collections of Edelman-Greene tableaux and standard Young tableaux.

Background

Reduced Words

S_n is the set of permutations on $\{1, 2, \ldots, n\}$, and $S_\infty = \cup_{n=1}^{\infty} S_n$. For $n \in \mathbb{N}$, the *simple transposition* $s_n :\in S_{n+1}$ is the permutation expressed in cycle notation as $(n, n+1)$.

Any permutation can be expressed as the product of simple transpositions. The minimum length of such an expression for $w \in S_\infty$ is the *Coxeter length* $\ell(w)$. Any expression of w using $\ell(w)$ simple transpositions is called a *reduced word* for w. The set of all reduced words for w is denoted $\mathrm{Red}(w)$.

Example 1 $s_1 s_2 s_3 \in \mathrm{Red}(2341)$, *but* $s_1 s_2 s_1 s_2 \notin \mathrm{Red}(312)$*, since* $s_2 s_1$ *is an equivalent expression for* 312 *of shorter length.*

Tits Lemma gives all the ways that reduced words can be transformed:

Lemma 1 *One can go from any element of* $\mathrm{Red}(w)$ *to any other using the following two relations:*

$$s_i s_j = s_j s_i \text{ for all } |i-j| \geq 2$$

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$$

Additionally, define $\mathrm{Inv}(n)$ to be the number of *involutions* (permutations that are their own inverses) of S_n.

The Edelman-Greene Statistic

For a partition λ and $w \in S_\infty$, an *Edelman-Greene tableau* (or *EG tableau*) of type (λ, w) is a filling of the cells of a Young diagram λ such that the cells are strictly increasing on rows and columns, and that if $i_1, i_2, \ldots, i_{|\lambda|}$ is the sequence that results from reading the tableau top-to-bottom and right-to-left, then $s_{i_1} s_{i_2} \ldots s_{i_{|\lambda|}} \in \mathrm{Red}(w)$. Let $\mathrm{EG}(\lambda, w)$ be the set of these tableaux.

Example 2 *Because* $s_5 s_7 s_1 s_3, s_3 s_7 s_1 s_5 \in \mathrm{Red}(21436587)$,

$$\left\{ \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 5 & 7 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 1 & 5 \\ \hline 3 & 7 \\ \hline \end{array} \right\} \subseteq \mathrm{EG}\left(21436587, \begin{array}{|c|c|}\hline \ & \ \\ \hline \ & \ \\ \hline \end{array}\right)$$

The *Edelman-Greene statistic* is defined by

$$\mathrm{EG}(w) = \sum_{\lambda} a_{w,\lambda} \text{, where } a_{w,\lambda} := |\mathrm{EG}(\lambda, w)|$$

Standardization

For a fixed λ, recall that a *semi-standard Young tableau* is a labeling of the cells of λ that is non-decreasing on rows and strictly increasing on columns, and a *standard Young tableau* is a labeling of the cells of λ with $1, \ldots, |\lambda|$ that is strictly increasing on rows and columns.

Let $\mathrm{SSYT}(\lambda)$ and $\mathrm{SYT}(\lambda)$ be the sets of all semi-standard Young tableaux and standard Young tableaux of shape λ respectively.

Suppose $T \in \mathrm{SSYT}(\lambda)$ and k_i is the number of i's appearing in T. Now replace all 1's in T from left to right by $1, 2, \ldots, k_1$. Then replace all of the (original) 2's in T by $k_1+1, k_1+2, \ldots, k_1+k_2$, *etc.* The result of this procedure is $\mathrm{std}(T)$, the *standardization* map applied to T.

Example 3

$$\mathrm{std}\left(\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array} \right) = \begin{array}{|c|c|c|c|}\hline 1 & 3 & 5 & 6 \\ \hline 2 & 7 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array}$$

$$\mathrm{std}\left(\begin{array}{|c|c|c|c|}\hline 1 & 3 & 3 & 6 \\ \hline \end{array} \right) = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline \end{array}$$

Since $\mathrm{EG}(w, \lambda) \subseteq \mathrm{SSYT}(\lambda)$, one can talk about the standardization map restricted to $\mathrm{EG}(w, \lambda)$. This map is in fact an injection, as conjectured by C. Monical, B. Pankow, and A. Yong.

Theorem 1 *For any partition* λ *and* $w \in S_\infty$,

$$\mathrm{std} : \mathrm{EG}(w, \lambda) \to \mathrm{SYT}(\lambda)$$

is an injection.

Total Commutativity and λ-Maximality

We define $w \in S_\infty$ to be *totally commutative* if

$$\exists s_{i_1} \ldots s_{i_{\ell(w)}} \in \mathrm{Red}(w) \ \textit{with} \ |i_j - i_k| \geq 2 \ \forall \ j \neq k$$

Example 4 $s_3 s_1 s_5 \in \mathrm{Red}(214365)$*, so* 214365 *is totally commutative. However,* 32154 *is not totally commutative, as*

$$\mathrm{Red}(32154) = \{s_1 s_2 s_4, s_4 s_1 s_2\}$$

Because $\mathrm{std} : \mathrm{EG}(w, \lambda) \to \mathrm{SYT}(\lambda)$ is an injection,

$$a_{w,\lambda} = |\mathrm{EG}(w, \lambda)| \leq |\mathrm{SYT}(\lambda)| := f^\lambda$$

For a Young diagram λ, define $w \in S_\infty$ to be *λ-maximal* if $a_{w,\lambda} = f^\lambda$, or equivalently $\mathrm{std} : \mathrm{EG}(\lambda) \to \mathrm{SYT}(\lambda)$ is a bijection.

It is in fact possible to exactly classify which w are λ-maximal for any given λ or for any given w.

Theorem 2 *All totally commutative permutations* w *are* λ*-maximal whenever* $|\lambda| = \ell(w)$.

Theorem 3 *If any element of* $\mathrm{Red}(w)$ *repeats a simple transposition, then* w *is not* λ*-maximal for any* λ.

Theorem 4 *If* $\lambda = (n)$ *for some* $n \in \mathbb{N}$*, then* w *is* λ*-maximal if and only if* $\exists \ i_1 > i_2 > \cdots > i_n$ *such that* $s_{i_1} s_{i_2} \ldots s_{i_n} \in \mathrm{Red}(w)$.

Theorem 5 *If* $\lambda = (1)^n$ *for some* $n \in \mathbb{N}$*, then* w *is* λ*-maximal if and only if* $\exists \ i_1 < i_2 < \cdots < i_n$ *such that* $s_{i_1} s_{i_2} \ldots s_{i_n} \in \mathrm{Red}(w)$.

In particular, the conditions in both of the previous 2 theorems are satisfied by all totally commutative permutations $w \in S_\infty$ with $\ell(w) = n$.

Theorem 6 *If* λ *has more than one row and more than one column,* w *is* λ*-maximal if and only if* λ *is totally commutative and* $\ell(w) = |\lambda|$.

The Main Theorem

The above theorems, combined with the fact that $\sum_{|\lambda|=n} f^\lambda = \mathrm{Inv}(n)$, provide an upper bound for the Edelman-Green statistic, and exactly characterize the permutations that maximize the statistic.

Theorem 7

$$\max\{\mathrm{EG}(w) : w \in S_\infty, \ell(w) = n\} = \mathrm{inv}(n)$$

And the maximum is attained by $w \in S_\infty$ *if and only* w *is totally commutative.*

Outside Connections

First, B. Pawlowski has proved that $\mathbb{E}[\mathrm{EG}] \geq (0.072)(1.299)^m$ where the expectation is taken over $w \in S_m$ [6, Theorem 3.2.7]). More recently, C. Monical, B. Pankow, and A. Yong show that $EG(w)$ is "typically" exponentially large on S_m [4, Theorem 1.1]. In comparison, the above theorem combined with a using a standard estimate (using Stirling's formula) for $\mathrm{inv}(n)$ gives

$$\max\{\mathrm{EG}(w) : w \in S_\infty, \ell(w) = n\} \sim \left(\frac{n}{e}\right)^{\frac{n}{2}} \frac{e^{\sqrt{n}}}{(4e)^{\frac{1}{4}}}$$

Second, in [5], maximums for the Littlewood-Richardson coefficients and their generalization, the Kronecker coefficients, were determined. We remark that the $a_{w,\lambda}$'s are also generalizations of the Littlewood-Richardson coefficients; this follows from [1, Corollary 2.4].

Finally, the results of V. Reiner-M. Shimozono (specifically [7, Theorem 33]) appear related to ours. Our work does not depend on their paper and is combinatorial and self-contained.

References

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