Minkowski weights and the Grothendieck Group

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Abstract

For a fan Δ, we introduce a ring of Grothendieck weights $\mathrm{GW}(\Delta)$. When Δ is proper, we show that $\mathrm{GW}(\Delta)$ is isomorphic to the operational K-theory of the associated toric variety. We demonstrate that $\mathrm{GW}(\Delta)$ admits a Riemann-Roch morphism from the ring of Minkowski weights on Δ, and a forgetful map from the ring of piecewise exponential functions $\mathrm{PExp}(\Delta)$. We provide applications to lattice point counting and the study of vector bundles on toric surfaces.

Motivation

Let X be an algebraic variety. Recall $A_*(X)$ which generated by classes $[V]$ for each k-dimensional subvariety of X, modulo relations given by rational equivalences

- for X singular, $A_*(X)$ is no longer a ring - instead we use the operational Chow ring $A^*(X)$ originally introduced by [FM]
- $A^*(X) \cong A_*(X)$ when X is smooth

Now, suppose X is complete and toric:

- then $A^*(X)$ is isomorphic to the ring $MW^*(\Delta)$ of Minkowski weights on Δ - these are functions from Δ to $\mathbb{Z}$ that satisfy a **balancing condition**
- the balancing condition for Minkowski weights also appears in tropical intersection theory.

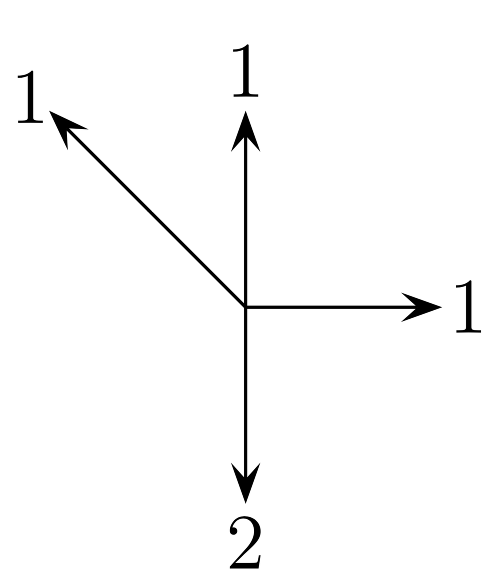

Figure 1: A Minkowski weight

A zoo of K-Theories

The **Grothendieck group** of coherent sheaves $K_\circ(X)$ is generated by $[\mathcal{F}]$ for each coherent sheaf $\mathcal{F}$, modulo relations from short exact sequences. This is a ring via $[\mathcal{F}][\mathcal{G}] = [\mathcal{F} \otimes^L \mathcal{G}]$ when X is smooth but not in general (analogous to the Chow groups). There are other theories which are always ring-valued:

- $K^\circ(X)$: this functor is simple to define (replace "coherent sheaf" with "vector bundle," in the previous paragraph), but is badly-behaved
- $\mathrm{opK}^\circ(X)$: this functor is harder to define and coarser than algebraic K-theory, but rectifies some of these issues (see [AP])
- By [AP]: When X is a complete toric variety, $\mathrm{opK}^\circ(X) \cong Hom_{\mathbb{Z}}(K_\circ(X), \mathbb{Z})$

Toric varieties and Grothendieck weights

Recall that a **toric variety** is a normal algebraic variety with the action of $(\mathbb{C}^*)^n$, such that there is a dense orbit.

- a toric variety $X \leftrightarrow$ a polyhedral fan Δ

Let X be a toric variety, and let Δ be its fan. Then:

- $K_\circ(X)$ is generated by the classes $[\mathcal{O}_{V(\sigma)}]$
- ensures there is an inclusion $Hom_{\mathbb{Z}}(K_\circ(X), \mathbb{Z}) \hookrightarrow \{c : \Delta \to \mathbb{Z}\}$ sending c to the function sending σ to $c([\mathcal{F}_\sigma])$
- we define **Grothendieck weights** on Δ – denoted $\mathrm{GW}(\Delta)$ – as the image of $Hom_{\mathbb{Z}}(K_\circ(X), \mathbb{Z})$ via this map

Theorem

Let Δ be a fan. Then we give an explicit list of relations on $c : \Delta \to \mathbb{Z}$ that is equivalent to c being a Grothendieck weight. When Δ is complete, this gives a presentation of $\mathrm{opK}^\circ(X)$.

Another presentation

A fan Δ is *simplicial* if each $\sigma \in \Delta$ is generated by some subset of a $\mathbb{Q}$-basis, and it is *projective* if it is the dual fan of a lattice polytope P. In this case, the fan Δ is indexed by faces F of P.

- If Δ is projective, then an element $c \in \mathrm{GW}(\Delta)$ is *polytopal* if there exists a lattice polytope P with normal fan Δ, such $c(\sigma_F)$ is the number of lattice points in F.

Theorem

If Δ is projective and simplicial, then polytopal Grothendieck weights generate $\mathrm{GW}(\Delta)_{\mathbb{Q}}$ as a $\mathbb{Q}$-vector space.

Example

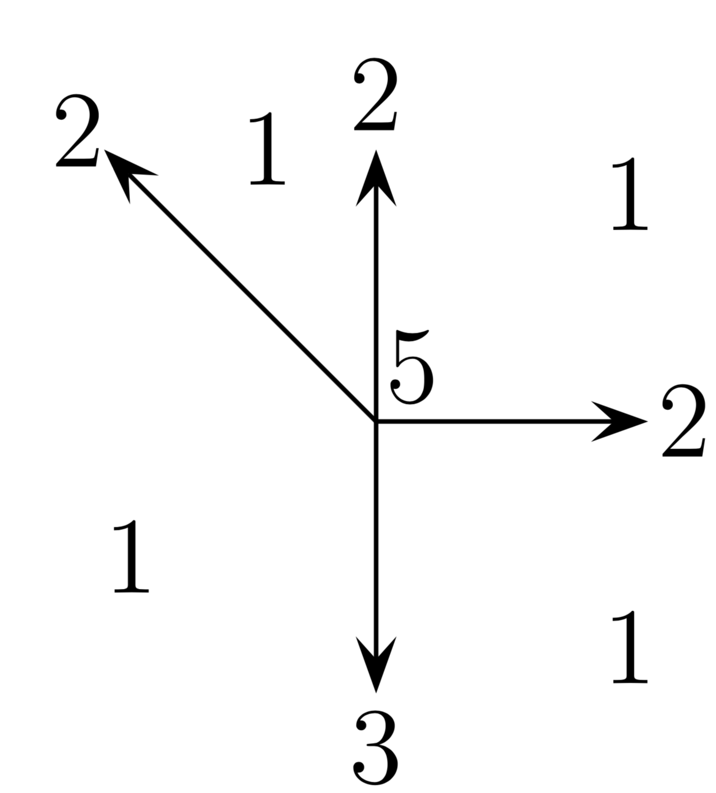

Figure 2: A Grothendieck weight

Product formula

For Minkowski weights, the product is given by the "displacement rule". For Grothendieck weights:

Theorem

Let Δ be a fan. There is an explicit formula for the product on $GW(\Delta)$. When X is complete, this rule is compatible with the product on $\mathrm{opK}^\circ(X)$.

Applications

Though $\mathrm{opK}^\circ(X)$ is coarser than $K^\circ(X)$, there are still relationships between the two invariants (see [AP],[AGP]) and their equivariant analogues:

Theorem

Let Δ be a fan. There exists a complete (simplicial) toric surface X on which $\mathrm{opK}^\circ_T(X)$ does not surject onto $\mathrm{opK}^\circ(X)$. This implies that this toric surface X has a vector bundle with no resolution by T-equivariant vector bundles.

A Square

Let X be complete toric. Then, there is a Riemann-Roch square – vertical maps are forgetful and horizontal maps are the operational Riemann-Roch of [AGP]:

$$\begin{array}{ccc} \mathrm{PExp}(\Delta) & \longrightarrow & \mathrm{PP}^*(\Delta)_{\mathbb{Q}} \\ \downarrow & & \downarrow \\ \mathrm{GW}(\Delta) & \longrightarrow & \mathrm{MW}^*(\Delta)_{\mathbb{Q}} \end{array}$$

References

A. Shah. Minkowski weights and the Grothendieck group of a toric variety. *In preparation.*