13. Applications of Bézout's Theorem

Bézout's Theorem (as in Proposition 12.16 or Corollary 12.26) is clearly one of the most powerful results in algebraic geometry that we will discuss in this class. To illustrate this, let us now take some time to study several of its applications, which are in fact of quite different flavors.

Our first application is actually not much more than a simple remark. It concerns the question whether the ideal of a given variety of pure dimension $n-k$ in affine space $\mathbb{A}^n$ or projective space $\mathbb{P}^n$ can be generated by k elements. We have seen in Exercise 7.16 (a) already that this is always the case if $k=1$. On the other hand, Example 0.11 and Exercise 2.32 show that for $k \geq 2$ one sometimes needs more than k generators. Bézout's Theorem can often be used to detect when this happens, e. g. in the following setting.

Proposition 13.1. *Let $X \subset \mathbb{P}^3$ be a curve that is not contained in any proper linear subspace of $\mathbb{P}^3$. If $\deg X$ is a prime number, then $I(X)$ cannot be generated by two elements.*

Proof. Assume for a contradiction that we have $I(X) = (f,g)$ for two homogeneous polynomials $f,g \in K[x_0,x_1,x_2,x_3]$. Clearly, g does not vanish identically on any irreducible component of $V(f)$, since otherwise the zero locus of (f,g) would have codimension 1. By Proposition 12.16 and Example 12.17 we therefore have

$$\deg X = \deg((f)+(g)) = \deg(f) \cdot \deg g = \deg f \cdot \deg g.$$

As $\deg X$ is a prime number, this is only possible if $\deg f = 1$ or $\deg g = 1$, i. e. if one of these polynomials is linear. But then X is contained in a proper linear subspace of $\mathbb{P}^3$, in contradiction to our assumption. □

Example 13.2. Let X be the degree-3 Veronese embedding of $\mathbb{P}^1$ as in Construction 7.27, i. e.

$$X = \{(s^3 : s^2t : st^2 : t^3) : (s:t) \in \mathbb{P}^1\} \quad \subset \mathbb{P}^3.$$

By Exercise 12.15 (b) we know that X is a cubic curve, i. e. $\deg X = 3$. Moreover, X is not contained in any proper linear subspace of $\mathbb{P}^3$, since otherwise the monomials s^3, s^2t, st^2, t^3 would have to satisfy a K-linear relation. Hence Proposition 13.1 implies that the ideal $I(X)$ cannot be generated by two elements.

However, one can check directly that $I(X)$ can be generated by three elements. For example, we can write

$$I(X) = (x_0x_2 - x_1^2, x_1x_3 - x_2^2, x_0x_3 - x_1x_2).$$

In the spirit of Bézout's Theorem, we can also see geometrically why none of these three generators is superfluous: if we leave out e. g. the last generator and consider $I = (x_0x_2 - x_1^2, x_1x_3 - x_2^2)$ instead, we now have $\deg I = 2 \cdot 2 = 4$. Clearly, $V(I)$ still contains the cubic X, and hence by the additivity of degrees as in Example 12.14 (b) there must be another 1-dimensional component in $V(I)$ of degree 1. In fact, this component is easy to find: we have $V(I) = X \cup L$ for the line $L = V(x_1,x_2)$.

Exercise 13.3. Let $X \subset \mathbb{P}^n$ be an irreducible curve of degree d. Show that X is contained in a linear subspace of $\mathbb{P}^n$ of dimension at most d.

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As another application of Bézout's Theorem, we are now able to prove the result already announced in Example 7.6 (a) that any isomorphism of $\mathbb{P}^n$ is linear, i. e. a projective automorphism in the sense of this example. Note that the corresponding statement would be false in the affine case, as e. g. $f : \mathbb{A}^2 \to \mathbb{A}^2,\ (x_1,x_2) \mapsto (x_1, x_2 + x_1^2)$ is an isomorphism with inverse $f^{-1} : (x_1,x_2) \mapsto (x_1, x_2 - x_1^2)$.

Proposition 13.4. *Every isomorphism $f : \mathbb{P}^n \to \mathbb{P}^n$ is linear, i. e. of the form $f(x) = Ax$ for an invertible matrix $A \in \mathrm{GL}(n+1,K)$.*

Proof. Let $H \subset \mathbb{P}^n$ be a hyperplane (given as the zero locus of one homogeneous linear polynomial), and let $L \subset \mathbb{P}^n$ be a line not contained in H. Clearly, the intersection $L \cap H$ consists of one point with multiplicity 1 (i. e. $I(L) + I(H)$ has multiplicity 1 in the sense of Definition 12.23 (a)). As f is an isomorphism, $f(L)$ and $f(H)$ must again be a curve resp. a hypersurface that intersect in one point with multiplicity 1. By the local version of Bézout's Theorem as in Corollary 12.26 (a), this means that $\deg f(L) \cdot \deg f(H) = 1$. This is only possible if $\deg f(H) = 1$. In other words, f must map hyperplanes to hyperplanes.

By composing f with a suitable projective automorphism (i. e. a linear map as in Example 7.6 (a)) we can therefore assume that f maps the affine part $\mathbb{A}^n = \mathbb{P}^n \backslash V(x_0)$ isomorphically to itself. Passing to this affine part with coordinates $x_1, \dots, x_n$, the above argument shows that $f^{-1}(V(x_i)) = V(f^* x_i)$ is an affine linear space for all i, so that $f^* x_i$ must be a power of a linear polynomial. But $f^* : K[x_1, \dots, x_n] \to K[x_1, \dots, x_n]$ is an isomorphism by Corollary 4.8 and thus maps irreducible polynomials to irreducible polynomials again. Hence $f^* x_i$ is itself linear for all i, which means that f is an affine linear map on $\mathbb{A}^n$, i. e. a linear map on $\mathbb{P}^n$. □

For the rest of this chapter we will now restrict to plane curves. One consequence of Bézout's Theorem in this case is that it gives an upper bound on the number of singular points that such a curve can have, in terms of its degree.

Proposition 13.5. *Let $X \subset \mathbb{P}^2$ be an irreducible curve of degree d. Then X has at most $\binom{d-1}{2}$ singular points.*

Example 13.6.

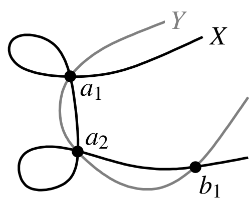
(a) A plane curve of degree 1 is a line, which is isomorphic to $\mathbb{P}^1$. A curve of degree 2, i. e. a conic, is always isomorphic to $\mathbb{P}^1$ as well, as we have seen in Example 7.6 (d). So in both these cases the curve does not have any singular points, in accordance with the statement of Proposition 13.5.

(b) By Proposition 13.5, a cubic curve in $\mathbb{P}^2$ can have at most one singular point. In fact, we have already seen both a cubic with no singular points (e. g. the Fermat cubic in Example 10.20) and a cubic with one singular point (e. g. $V_p(x_0 x_2^2 - x_0 x_1^2 - x_1^3)$ or $V_p(x_0 x_2^2 - x_1^3)$, whose affine parts we have studied in Example 10.3).

(c) Without the assumption of X being irreducible the statement of Proposition 13.5 is false: a union of two intersecting lines in $\mathbb{P}^2$ has degree 2, but a singular point (namely the intersection point). However, we will see in Exercise 13.7 that there is a similar statement with a slightly higher bound also for not necessarily irreducible curves.

Proof of Proposition 13.5. By Example 13.6 (a) it suffices to prove the theorem for $d \geq 3$. Assume for a contradiction that there are distinct singular points $a_1, \dots, a_{\binom{d-1}{2}+1}$ of X. Moreover, pick $d-3$ arbitrary further distinct points $b_1, \dots, b_{d-3}$ on X, so that the total number of points is

$$\binom{d-1}{2} + 1 + d - 3 = \binom{d}{2} - 1.$$



We claim that there is a curve Y of degree at most $d-2$ that passes through all these points. The argument is essentially the same as in Exercise 7.31 (c): the space $K[x_0, x_1, x_2]_{d-2}$ of all homogeneous polynomials of degree $d-2$ in three variables is a vector space of dimension $\binom{d}{2}$, with the coefficients of the polynomials as coordinates. Moreover, the condition that such a polynomial vanishes at a given point is clearly a homogeneous linear equation in these coordinates. As $\binom{d}{2} - 1$ homogeneous linear equations in a vector space of dimension $\binom{d}{2}$ must have a non-trivial common solution, we conclude that there is a non-zero polynomial $f \in K[x_0, x_1, x_2]_{d-2}$ vanishing at all points a_i and b_j. The corresponding curve $Y = V_p(f)$ then has degree at most $d-2$ (strictly less if f contains repeated factors) and passes through all these points.

Note that X and Y cannot have a common irreducible component, since X is irreducible and of bigger degree than Y. Hence Corollary 12.26 (b) shows that the curves X and Y can intersect in at most $\deg X \cdot \deg Y = d(d-2)$ points, counted with multiplicities. But the intersection multiplicity at all a_i is at least 2 by Exercise 12.27 since X is singular there. Hence the number of intersection points that we know already, counted with their respective multiplicities, is at least

$$2 \cdot \left(\binom{d-1}{2} + 1 \right) + (d-3) = d(d-2) + 1 > d(d-2),$$

which is a contradiction. □

Exercise 13.7. Show that a (not necessarily irreducible) curve of degree d in $\mathbb{P}^2$ has at most $\binom{d}{2}$ singular points. Can you find an example for each d in which this maximal number of singular points is actually reached?

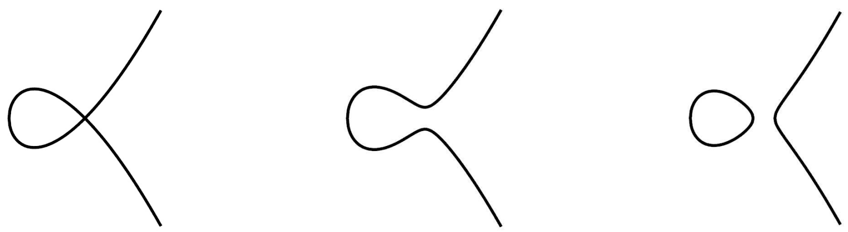
Let us now study smooth plane curves in more detail. An interesting topic that we have neglected entirely so far is the *classical* topology of such curves when we consider them over the real or complex numbers, e. g. their number of connected components in the standard topology. We will now see that Bézout's Theorem is able to answer such questions.

Of course, for these results we will need some techniques and statements from topology that have not been discussed in this class. The following proofs in this chapter should therefore rather be considered as sketch proofs, which can be made into exact arguments with the necessary topological background. However, all topological results that we will need should be intuitively clear — although their exact proofs are often quite technical. Let us start with the real case, as real curves are topologically simpler than complex ones.

Remark 13.8 (Real curves)**.** Note that we have developed most of our theory only for algebraically closed ground fields, so that our constructions and results are not directly applicable to real curves. However, as we will not go very deep into the theory of real algebraic geometry it suffices to note that the definition of projective curves and their ideals works over $\mathbb{R}$ in the same way as over $\mathbb{C}$. To apply other concepts and theorems to a real curve X with ideal $I(X) = (f)$ for a real homogeneous polynomial f, we will simply pass to the corresponding complex curve $X_{\mathbb{C}} = V_p(f) \subset \mathbb{P}^2_{\mathbb{C}}$ first. For example, we will say that X is smooth or irreducible if $X_{\mathbb{C}}$ is.

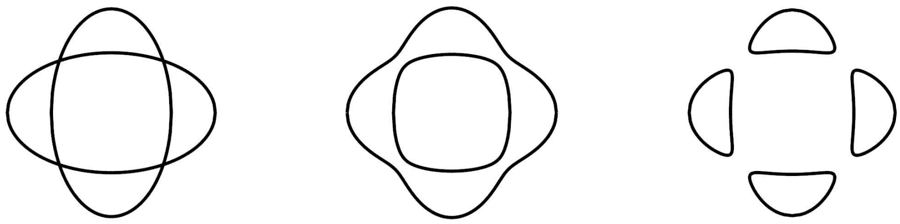
Remark 13.9 (Loops of real projective curves)**.** Let $X \subset \mathbb{P}^2_{\mathbb{R}}$ be a smooth projective curve over $\mathbb{R}$. In the classical topology, X is then a compact 1-dimensional manifold (see Remark 10.14). This means that X is a disjoint union of finitely many connected components, each of which is homeomorphic to a circle. We will refer to these components as *loops* of X.

Note that X can consist of several loops in the classical topology even if f is irreducible (so that X is irreducible in the Zariski topology). A convenient way to construct such curves is by deformations of singular curves. For example, consider the singular cubic curve X in $\mathbb{P}^2_{\mathbb{R}}$ whose affine part in $\mathbb{A}^2_{\mathbb{R}}$ is the zero locus of $f_3 := x_2^2 - x_1^2 - x_1^3$ as in Example 10.3. It has a double point at the origin, as shown in the picture below on the left. In $\mathbb{P}^2_{\mathbb{R}}$, the curve contains one additional point at infinity that connects the two unbounded branches, so that X is homeomorphic to two circles glued together at a point.



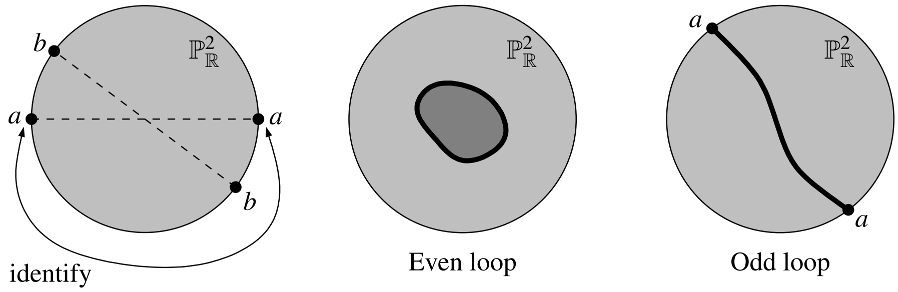
$f_3(x_1,x_2) = 0$ $\qquad$ $f_3(x_1,x_2) - \varepsilon = 0$ $\qquad$ $f_3(x_1,x_2) + \varepsilon = 0$

Let us now perturb f_3 and consider the curves $f_3(x_1,x_2) \pm \varepsilon = 0$ for a small number $\varepsilon \in \mathbb{R}_{>0}$ instead. This deforms X into a smooth cubic with one or two loops depending on the sign of the perturbation, as shown in the picture above. The same technique applied to a singular quartic curve, e. g. the union of two ellipses given by $f_4 = (x_1^2 + 2x_2^2 - 1)(x_2^2 + 2x_1^2 - 1)$, yields two or four loops as in the following picture.



$f_4(x_1,x_2) = 0$ $\qquad$ $f_4(x_1,x_2) - \varepsilon = 0$ $\qquad$ $f_4(x_1,x_2) + \varepsilon = 0$

Remark 13.10 (Even and odd loops)**.** Although all loops of smooth curves in $\mathbb{P}^2_{\mathbb{R}}$ are homeomorphic to a circle, there are two different kinds of them when we consider their embeddings in projective space. To understand this, recall from Remark 6.3 that $\mathbb{P}^2_{\mathbb{R}}$ is obtained from $\mathbb{A}^2_{\mathbb{R}}$ (which we will draw topologically as an open disc below) by adding a point at infinity for each direction in $\mathbb{A}^2_{\mathbb{R}}$. This has the effect of adding a boundary to the disc, with the boundary points corresponding to the points at infinity. But note that opposite points of the boundary of the disc belong to the same direction in $\mathbb{A}^2_{\mathbb{R}}$ and hence are the same point in $\mathbb{P}^2_{\mathbb{R}}$. In other words, $\mathbb{P}^2_{\mathbb{R}}$ is topologically equivalent to a closed disc with opposite boundary points identified, as in the picture below on the left.



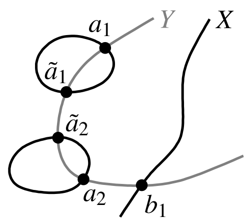
The consequence of this is that we have two different types of loops. An *even loop* is a loop such that its complement has two connected components, which we might call its "interior" (shown in dark in the picture above, homeomorphic to a disc) and "exterior" (homeomorphic to a Möbius strip), respectively. In contrast, an *odd loop* does not divide $\mathbb{P}^2_{\mathbb{R}}$ into two regions; its complement is a single component homeomorphic to a disc. Note that the distinction between even and odd is *not* whether the affine part of the curve is bounded: whereas an odd loop always has to be unbounded, an even loop may well be unbounded, too. Instead, if you know some topology you will probably recognize that the statement being made here is just that the fundamental group $\pi_1(\mathbb{P}^2_{\mathbb{R}})$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$; the two types of loops simply correspond to the two elements of this group.

In principle, a real curve can have even as well as odd loops. There is one restriction however: as the complement of an odd loop is simply a disc, all other loops in this complement will have an interior and exterior, so that they are even. In other words, a smooth curve in $\mathbb{P}^2_{\mathbb{R}}$ can have at most one odd loop.

We are now ready to find a bound on the number of loops in a smooth curve in $\mathbb{P}^2_{\mathbb{R}}$ of a given degree. Interestingly, the idea in its proof is almost identical to that in Proposition 13.5, although the resulting statement is quite different.

Proposition 13.11 (**Harnack's Theorem**)**.** *A smooth real projective curve of degree d in $\mathbb{P}^2_{\mathbb{R}}$ has at most $\binom{d-1}{2} + 1$ loops.*

Example 13.12. A line ($d = 1$) has always exactly one loop. A conic ($d = 2$) is a hyperbola, parabola, or ellipse, so in every case the number of loops is again 1 (after adding the points at infinity). For $d = 3$ Harnack's Theorem gives a maximum number of 2 loops, and for $d = 4$ we get at most 4 loops. We have just seen examples of these numbers of loops in Remark 13.9. In fact, one can show that the bound given in Harnack's theorem is sharp, i. e. that for every d one can find smooth real curves of degree d with exactly $\binom{d-1}{2}+1$ loops.



Proof sketch of Proposition 13.11. By Example 13.12 it suffices to consider the case $d \geq 3$. Assume that the statement of the proposition is false, i. e. that there are at least $\binom{d-1}{2}+2$ loops. We have seen in Remark 13.10 that at least $\binom{d-1}{2}+1$ of these loops must be even. Hence we can pick points $a_1, \dots, a_{\binom{d-1}{2}+1}$ on distinct even loops of X, and $d-3$ more points $b_1, \dots, b_{d-3}$ on another loop (which might be even or odd). So as in the proof of Proposition 13.5, we have a total of $\binom{d}{2}-1$ points.

Again as in the proof of Proposition 13.5, it now follows that there is a real curve Y of degree at most $d-2$ passing through all these points. Note that the corresponding complex curves $X_\mathbb{C}$ and $Y_\mathbb{C}$ as in Remark 13.8 cannot have a common irreducible component since $X_\mathbb{C}$ is assumed to be smooth, hence irreducible by Exercise 10.22 (a), and has bigger degree than $Y_\mathbb{C}$. So Bézout's Theorem as in Corollary 12.26 (b) implies that $X_\mathbb{C}$ and $Y_\mathbb{C}$ intersect in at most $d(d-2)$ points, counted with multiplicities. But recall that the even loops of X containing the points a_i divide the real projective plane into two regions, hence if Y enters the interior of such a loop it has to exit it again at another point $\tilde{a}_i$ of the same loop as in the picture above (it may also happen that Y is tangent to X at a_i, in which case their intersection multiplicity is at least 2 there by Exercise 12.27). So in any case the total number of intersection points, counted with their respective multiplicities, is at least

$$2 \cdot \left(\binom{d-1}{2} + 1 \right) + (d-3) = d(d-2) + 1 > d(d-2),$$

which is a contradiction. □

Let us now turn to the case of complex curves. Of course, their topology is entirely different, as they are 2-dimensional spaces in the classical topology. In fact, we have seen such an example already in Example 0.1 of the introduction.

Remark 13.13 (Classical topology of complex curves)**.** Let $X \subset \mathbb{P}^2_\mathbb{C}$ be a smooth curve. Then X is a compact 2-dimensional manifold in the classical topology (see Remark 10.14). Moreover, one can show:

(a) X is always an *oriented manifold* in the classical topology, i. e. a "two-sided surface", as opposed to e. g. a Möbius strip. To see this, note that all tangent spaces T_aX of X for $a \in X$ are isomorphic to $\mathbb{C}$, and hence admit a well-defined multiplication with the imaginary unit i. Geometrically, this means that all tangent planes have a well-defined notion of a *clockwise* rotation by 90 degrees, varying continuously with a — which defines an orientation of X. In fact, this statement holds for all smooth complex curves, not just for curves in $\mathbb{P}^2_\mathbb{C}$.

(b) In contrast to the real case that we have just studied, X is always *connected.* In short, the reason for this is that the notion of degree as well as Bézout's Theorem can be extended to compact oriented 2-dimensional submanifolds of $\mathbb{P}^2_\mathbb{C}$. Hence, if X had (at least) two connected components X_1 and X_2 in the classical topology, each of these components would be a compact oriented 2-dimensional manifold itself, and there would thus be well-defined degrees $\deg X_1, \deg X_2 \in \mathbb{N}_{>0}$. But then X_1 and X_2 would have to intersect in $\deg X_1 \cdot \deg X_2$ points (counted with multiplicities), which is obviously a contradiction.

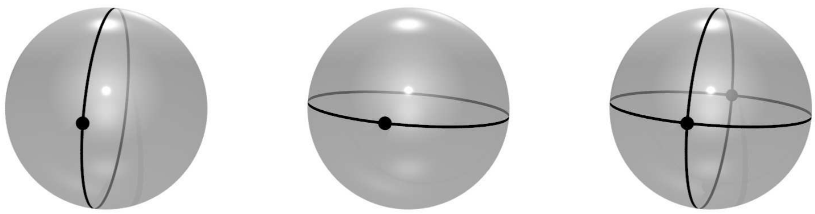
Of course, the methods needed to prove Bézout's Theorem in the topological setting are entirely different from ours in Chapter 12. If you know some algebraic topology, the statement here is that the 2-dimensional homology group $H_2(\mathbb{P}^2_\mathbb{C}, \mathbb{Z})$ is isomorphic to $\mathbb{Z}$. With this isomorphism, the class of a compact oriented 2-dimensional submanifold in $H_2(\mathbb{P}^2_\mathbb{C}, \mathbb{Z})$

is a positive number, and the intersection product $H_2(\mathbb{P}^2_\mathbb{C},\mathbb{Z}) \times H_2(\mathbb{P}^2_\mathbb{C},\mathbb{Z}) \to H_0(\mathbb{P}^2_\mathbb{C},\mathbb{Z}) \cong \mathbb{Z}$ (using Poincaré duality) is just the product of these numbers.

It is now a (non-trivial but intuitive) topological result that a connected compact orientable 2-dimensional manifold X is always homeomorphic to a sphere with some finite number of "handles". This number of handles is called the **genus** of X. Hence every curve in $\mathbb{P}^2_\mathbb{C}$ can be assigned a genus that describes its topological type in the classical topology. The picture on the right shows a complex
22 curve of genus 2.

Our goal for the rest of this chapter will be to compute the genus of a smooth curve in $\mathbb{P}^2_\mathbb{C}$ in terms of its degree, as already announced in Example 0.3. To do this, we will need the following technique from topology.

Construction 13.14 (Cell decompositions)**.** Let X be a compact 2-dimensional manifold. A *cell decomposition* of X is given by writing X topologically as a finite disjoint union of points, (open) lines, and (open) discs. This decomposition should be "nice" in a certain sense, e. g. the boundary points of every line in the decomposition must be points of the decomposition. We do not want to give a precise definition here (which would necessarily be technical), but only remark that every "reasonable" decomposition that one could think of will be allowed. For example, the following picture shows three valid decompositions of the complex curve $\mathbb{P}^1_\mathbb{C}$, which is topologically a sphere.



In the left two pictures, we have 1 point, 1 line, and 2 discs (the two halves of the sphere), whereas in the picture on the right we have 2 points, 4 lines, and 4 discs.

Of course, there are many possibilities for cell decompositions of X. But there is an important number that does not depend on the chosen decomposition:

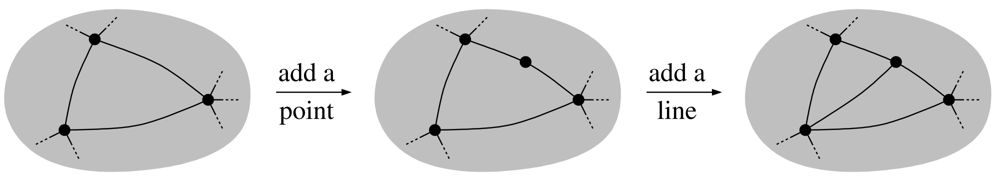
Lemma and Definition 13.15 (Euler characteristic)**.** *Let X be a compact 2-dimensional manifold. Consider a cell decomposition of X, consisting of σ_0 points, σ_1 lines, and σ_2 discs. Then the number*

$$\chi := \sigma_0 - \sigma_1 + \sigma_2$$

depends only on X, and not on the chosen decomposition. We call it the (topological) ***Euler characteristic*** *of X.*

Proof sketch. Let us first consider the case when we move from one decomposition to a finer one, i. e. if we add points or lines to the decomposition. Such a process is always obtained by performing the following steps a finite number of times:

- Adding another point on a line: in this case we raise σ_0 and σ_1 by 1 as in the picture below, hence the alternating sum $\sigma_0 - \sigma_1 + \sigma_2$ does not change.
- Adding another line in a disc: in this case we raise σ_1 and σ_2 by 1, so again $\sigma_0 - \sigma_1 + \sigma_2$ remains invariant.



We conclude that the alternating sum $\sigma_0 - \sigma_1 + \sigma_2$ does not change under refinements. But any two decompositions have a common refinement — which is essentially given by taking all the points and lines in both decompositions, and maybe adding more points where two such lines intersect. For example, in Construction 13.14 the decomposition in the picture on the right is a common refinement of the other two. Hence the Euler characteristic is independent of the chosen decomposition. □

Example 13.16 (Euler characteristic $\leftrightarrow$ genus)**.** Let X be a connected compact orientable 2-dimensional manifold of genus g, and consider the cell decomposition of X as shown on the right. It has $\sigma_0 = 2g+2$ points, $\sigma_1 = 4g+4$ lines, and 4 discs, and hence we conclude that the Euler characteristic of X is

$$\chi = \sigma_0 - \sigma_1 + \sigma_2 = 2 - 2g.$$

In other words, the genus is given in terms of the Euler characteristic as $g = 1 - \frac{\chi}{2}$.

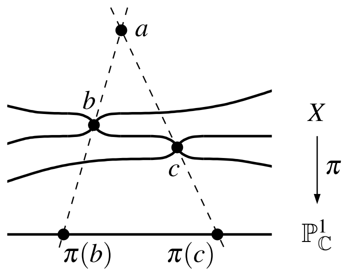
We are now ready to compute the genus of a smooth curve in $\mathbb{P}^2_{\mathbb{C}}$.

Proposition 13.17 (**Degree-genus formula**)**.** *A smooth curve of degree d in $\mathbb{P}^2_{\mathbb{C}}$ has genus $\binom{d-1}{2}$.*

Proof sketch. Let $I(X) = (f)$ for a homogeneous polynomial f of degree d. By a linear change of coordinates we can assume that $a := (1:0:0) \notin X$. Then the projection

$$\pi : X \to \mathbb{P}^1_{\mathbb{C}}, \ (x_0:x_1:x_2) \mapsto (x_1:x_2)$$

from a as in the picture on the right is a well-defined morphism on X. Let us study its inverse images of a fixed point $(x_1:x_2) \in \mathbb{P}^1_{\mathbb{C}}$. Of course, they are given by the values of x_0 such that $f(x_0,x_1,x_2) = 0$, so that there are exactly d such points — unless the polynomial $f(\cdot,x_1,x_2)$ has a multiple zero in x_0 at a point in the inverse image, which happens if and only if f and $\frac{\partial f}{\partial x_0}$ are simultaneously zero.



If we choose our original linear change of coordinates general enough, exactly two of the zeroes of $f(\cdot,x_1,x_2)$ will coincide at these points in the common zero locus of f and $\frac{\partial f}{\partial x_0}$, so that $\pi^{-1}(x_1:x_2)$ then consists of $d-1$ instead of d points. These points, as e. g. b and c in the picture above, are usually called the *ramification points* of π. Note that the picture might be a bit misleading since it suggests that X is singular at b and c, which is not the case. The correct topological picture of the map is impossible to draw however since it would require a real 4-dimensional space, namely an affine chart of $\mathbb{P}^2_{\mathbb{C}}$.

Let us now pick a sufficiently fine cell decomposition of $\mathbb{P}^1_{\mathbb{C}}$, containing all images of the ramification points as points of the decomposition. If $\sigma_0, \sigma_1, \sigma_2$ denote the number of points, lines, and discs in this decomposition, respectively, we have $\sigma_0 - \sigma_1 + \sigma_2 = 2$ by Example 13.16 since $\mathbb{P}^1_{\mathbb{C}}$ is topologically a sphere, i. e. of genus 0. Now lift this cell decomposition to a decomposition of X by taking all inverse images of the cells of $\mathbb{P}^1_{\mathbb{C}}$. By our above argument, all cells will have exactly d inverse images — except for the images of the ramification points, which have one inverse image less. As the number of ramification points is $|V_p(f, \frac{\partial f}{\partial x_0})| = \deg f \cdot \deg \frac{\partial f}{\partial x_0} = d(d-1)$ by Bézout's Theorem, the resulting decomposition of X has $d\sigma_0 - d(d-1)$ points, $d\sigma_1$ lines, and $d\sigma_2$ discs. Hence by Lemma 13.15 the Euler characteristic of X is

$$\chi = d\sigma_0 - d(d-1) - d\sigma_1 + d\sigma_2 = 2d - d(d-1) = 3d - d^2,$$

which means by Example 13.16 that its genus is

$$g = 1 - \frac{\chi}{2} = \frac{1}{2}(d^2 - 3d + 2) = \binom{d-1}{2}.$$ □

Example 13.18.

(a) A smooth curve of degree 1 or 2 in $\mathbb{P}^2_{\mathbb{C}}$ is isomorphic to $\mathbb{P}^1_{\mathbb{C}}$ (see Example 7.6 (d)). It is therefore topologically a sphere, i. e. of genus 0, in accordance with Proposition 13.17.

(b) By Proposition 13.17, a smooth curve of degree 3 in $\mathbb{P}^2_{\mathbb{C}}$ has genus 1, i. e. it is topologically a torus. We will study such plane cubic curves in detail in Chapter 15.

Remark 13.19. Note that every isomorphism of complex varieties is also a homeomorphism in the classical topology. In particular, two smooth connected projective curves over $\mathbb{C}$ of different genus cannot be isomorphic. Combining this with Proposition 13.17, we see that two smooth curves in $\mathbb{P}^2_{\mathbb{C}}$ of different degree are never isomorphic, unless these degrees are 1 and 2.

Exercise 13.20 (Arithmetic genus)**.** For a projective variety X the number $(-1)^{\dim X} \cdot (\chi_X(0) - 1)$ is called its *arithmetic genus*, where χ_X denotes as usual the Hilbert polynomial of X. Show that the arithmetic genus of a smooth curve in $\mathbb{P}^2_{\mathbb{C}}$ agrees with the (geometric) genus introduced above.

In fact, one can show that this is true for any smooth projective curve over $\mathbb{C}$. However, the proof of this statement is hard and goes well beyond the scope of these notes. Note that, as the definition of the arithmetic genus is completely algebraic, one can use it to extend the notion of genus to projective curves over arbitrary ground fields.

Exercise 13.21. Show that

$$\{((x_0 : x_1), (y_0 : y_1)) : (x_0^2 + x_1^2)(y_0^2 + y_1^2) = x_0 x_1 y_0 y_1\} \quad \subset \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$$

is a smooth curve of genus 1.