International Journal of Computer Discovered Mathematics (IJCDM)
ISSN 2367-7775 ©IJCDM
Volume 3, 2018, pp.49-54
Received 8 April 2018. Published on-line 12 April 2018
web: http://www.journal-1.eu/

A New Proof and Some Generalizations of the Bottema Theorem

Nguyen Ngoc Giang
Banking University of Ho Chi Minh City
36 Ton That Dam street, district 1,
Ho Chi Minh City, Vietnam
nguyenngocgiang.net@gmail.com

Abstract. The article gives a new proof and some new generalizations of the Bottema theorem.

Keywords. Bottema theorem, similar rectangulars.

1. Introduction

The Bottema theorem is a famous one which has many applications in solving problems. they find out many proofs of the Bottema theorem such as the synthetic solution, complex solution and analytic solution nowadays. This article gives some new proofs of generalizations of the Bottema theorem.

Theorem 1.1. *(The Bottema theorem)*
Given a triangle ABC. Two vertices A, B are fixed and the vertex C is movable. Two squares ACB_cB_a and BCA_cA_b are constructed on two sides CA, CB having the same orientation. Prove that the midpoint M of A_bB_a does no depend on the position of point C.

2. Theorems

Let W, U, V be the projections from M, B_a, A_b to AB, respectively. We have:

$$\overrightarrow{WM} = \frac{1}{2}\left(\overrightarrow{UB_a} + \overrightarrow{VA_b}\right) \tag{1}$$

We have:

$$\overrightarrow{UB_a} = \overrightarrow{UA} + \overrightarrow{AB_a} \tag{2}$$

$$\overrightarrow{VA_b} = \overrightarrow{VB} + \overrightarrow{BA_b} \tag{3}$$

Considering the rotation of vectors with angle -90^0, we have:

$$\overrightarrow{UA} \rightarrow CZ \ (since\ \Delta AB_aU = \Delta CAZ) \tag{4}$$

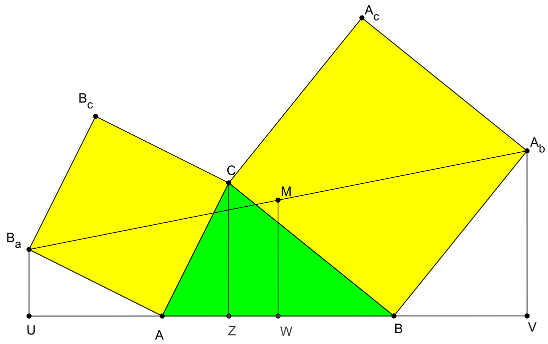


Fig1. The Bottema's theorem

$$AB_a \rightarrow AC \tag{5}$$

$$\overrightarrow{VB} \rightarrow \overrightarrow{ZC} \ (since\ \Delta BA_bV = \Delta CBZ) \tag{6}$$

$$\overrightarrow{BA_b} \rightarrow \overrightarrow{CB} \tag{7}$$

Since (1), (2), (3), (4), (5), (6), (7), we have:

$$\overrightarrow{WM} = \frac{1}{2}\left(\overrightarrow{UB_a} + \overrightarrow{VA_b}\right) = \frac{1}{2}\left(\overrightarrow{UA} + \overrightarrow{AB_a} + \overrightarrow{VB} + \overrightarrow{BA_b}\right)$$

$$\rightarrow \frac{1}{2}\left(\overrightarrow{CZ} + \overrightarrow{AC} + \overrightarrow{ZC} + \overrightarrow{CB}\right) = \frac{1}{2}\overrightarrow{AB}.$$

From the above we have $AU = CZ = BV$. It follows that W is the midpoint of UV then W is the midpoint of AB. Since $WM = \frac{1}{2}AB$; $WM \perp AB$, W is the midpoint of fixed segment AB, it follows M is fixed (Q. E. D).

The Bottema theorem only considers two squares rotated about the vertex C (as the above figure).We know that square is a special rectangular. If two arbitrary rectangulars are rotated about the vertex C then the above thing is true? We go to the following problem:

Theorem 2.1. *(The generalization 1) (Source: The Cut-the-knot)*
Given a triangle ABC. Two vertices A, B are fixed and the vertex C is movable. Two similar rectangulars ACB_cB_a and BCA_cA_b are constructed on two sides CA, CB having the same orientation such that CA, CB . Prove that the midpoint M of A_bB_a does not depend on the position of point C.

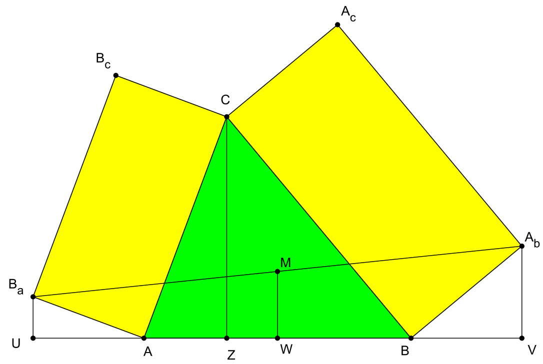


Fig2. The generalization 1 of Bottema's theorem

Solution 1

Drop the perpendicular lines B_aU, MW, AV, CZ to AB. We see that MW is the midline of trapezium B_aUVA_b which satisfies that:

$$MW = \frac{B_aU + A_bV}{2} \tag{8}$$

Since the right triangles UB_aU, AZC and BA_bV, BCZ are similar, we have:

$$\frac{UB_a}{AZ} = \frac{AB_a}{AC} = \frac{BA_b}{BC} = \frac{VA_b}{BZ} = k(k \; is \; the \; similar \; ratio \; of \; two \; rectangulars).$$

Since $\frac{UB_a}{AZ} = \frac{VA_b}{BZ} = \frac{UB_a + VA_b}{AZ + BZ} = \frac{2MW}{AB} = k \Rightarrow MW = \frac{kAB}{2} = const.$

Thus MW is constant, W is fixed, it follows M is fixed.

The above problem is proved by the synthetic method. We can use the roation of vectors to prove this theorem with new and interesting aspects.

Solution 2

Let W, U, V be the projections from W, U, V to AB. We have:

$$\overrightarrow{WM} = \frac{1}{2}\left(\overrightarrow{UB_a} + \overrightarrow{VA_b}\right) \tag{9}$$

We also have:

$$\overrightarrow{UB_a} = \overrightarrow{UA} + \overrightarrow{AB_a} \tag{10}$$

$$\overrightarrow{VA_b} = \overrightarrow{VB} + \overrightarrow{BA_b} \tag{11}$$

Let k be the ratio of two sides of the rectangulars: $k = \frac{AB_a}{AC} = \frac{BA_b}{BC}$. Considering the rotation of vectors with angle -90^0, we have:

$$\overrightarrow{UA} \to kCZ(since \; \Delta AB_aU \sim \Delta CAZ) \tag{12}$$

$$\Delta AB_aU \sim \Delta CAZ \tag{13}$$

$$\overrightarrow{VB} \to k\overrightarrow{ZC}(since \; \Delta BA_bV \sim \Delta CBZ) \tag{14}$$

$$\overrightarrow{BA_b} \to k\overrightarrow{CB} \tag{15}$$

Since (9), (10), (11), (12), (13), (14), (15), we have:

$$\overrightarrow{WM} = \frac{1}{2}\left(\overrightarrow{UB_a} + \overrightarrow{VA_b}\right) = \frac{1}{2}\left(\overrightarrow{UA} + \overrightarrow{AB_a} + \overrightarrow{VB} + \overrightarrow{BA_b}\right)$$

$$\rightarrow \frac{1}{2}\left(k\overrightarrow{CZ} + k\overrightarrow{AC} + k\overrightarrow{ZC} + k\overrightarrow{CB}\right) = \frac{1}{2}k\overrightarrow{AB}.$$

From the above we have $AU = kCZ = BV$. It follows that W is the midpoint of UV then W is the midpoint of AB. Since $WM = \frac{1}{2}AB$; $WM \perp AB$, W is the midpoint of fixed segment AB. Thus M is fixed (Q. E. D).

We see that two right angles $\widehat{CAB_a}$ and $\widehat{CBA_b}$ are supplemental and $\frac{UB_a}{AZ} = \frac{VA_b}{BZ} = k$. If we consider two arbitrary angles $\widehat{CAB_a}$ and $\widehat{CBA_b}$ are supplemental and $\frac{UB_a}{AZ} = \frac{VA_b}{BZ} = k$ then we obtain the generalization of theorem 2 as follows:

Theorem 2.2. *(The generalization 2) (Source: The Cut-the-knot)*
Given a triangle ABC. Two vertices A, B are fixed and the vertex C is movable. Two triangles CAB_a and CBA_b are constructed on two sides CA, CB having the same orientation such that $\widehat{CAB_a} = \alpha$ is supplemental with $\widehat{A_bBC} = \pi - \alpha = \beta$ and $\frac{AB_a}{CA} = \frac{BA_b}{CB} = k$. Prove that the midpoint M of segment A_bB_a does not depend on the position of point C.

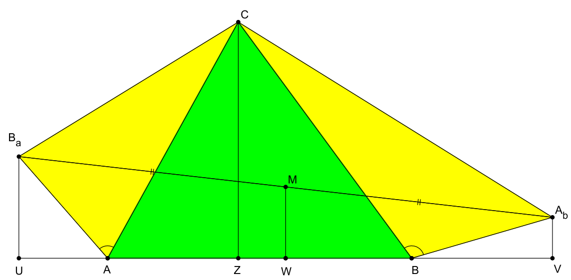
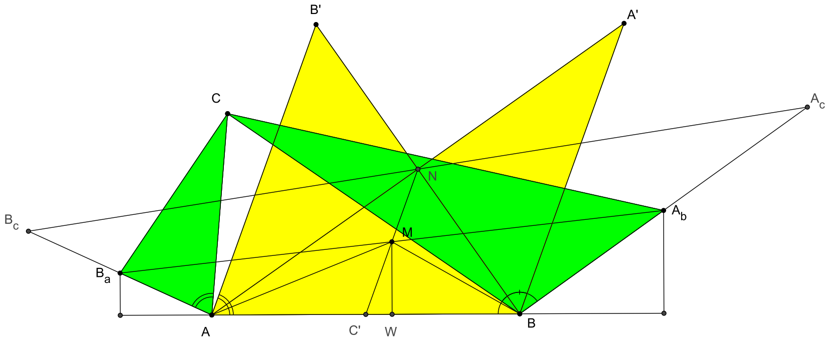


Fig 3. The generalization 2 of Bottema's theorem

Let B', B_c be the images of points B, C under the rotation A with rotated angle $(\overrightarrow{AC}, \overrightarrow{AB_a}) = \alpha$.

Similarly, A', A_c are the images of points A, C under the rotation B with rotated angle $(\overrightarrow{BC}, \overrightarrow{BA_b}) = \alpha - \pi = -\beta$.

Since ABB', $BB'A'$ are isosceless, it follows $AB' = AB = A'B$. We ea sily prove that $AB' \,//\, BA'$. It follows the quadrilateral $AB'A'B$ is paralleogram having two equal adjacent sides. It follows that $AB'A'B$ is a lozenge. It follows $AA' \perp BB'$ at N. According to the property of rotation, N is the intersection point of two lines connecting the image A' with preimage A and the image B' with preimage B under two above rotations, so the rotation $Q_{(N,\, \alpha+\beta)}$ is the product of two rotations $Q_{(B,\, \beta)} \circ Q_{(A,\, \alpha)}$.



We have that C is the image of A_c under the rotation $Q_{(B,\,\beta)}$; B_c is the image of C under the rotation $Q_{(A,\,\alpha)}$.

Considering the rotation $Q_{(N,\,\alpha+\beta)}$, we have $A_c \mapsto B_c$. Since $\alpha + \beta = \pi$, B_c, N, A_c are collinear and N is the midpoint of BC.

Since the lozenge $ABA'B'$ have fixed vertices, it follows N is a fixed point.

Since C' is the midpoint of AB, we go to prove that C', M, N are collinear. We have:

$$\overrightarrow{C'M} = \frac{1}{2}\left(\overrightarrow{AB_a} + \overrightarrow{BA_b}\right) \tag{16}$$

$$\overrightarrow{C'N} = \frac{1}{2}\left(\overrightarrow{AB_c} + \overrightarrow{BA_c}\right) \tag{17}$$

$$\frac{AB_a}{AC} = \frac{AB_a}{AB_c} = k \Rightarrow \overrightarrow{AB_a} = k\overrightarrow{AB_c} \tag{18}$$

$$\frac{BA_b}{BC} = \frac{BA_b}{BA_c} = k \Rightarrow \overrightarrow{BA_b} = k\overrightarrow{BA_c} \tag{19}$$

Since (16), (17), (18), (19) we have: $\overrightarrow{C'M} = k\overrightarrow{C'N}$, it follows that C', M, N are collinear. Since C', N is fixed, M is fixed.

We have some small discoveries about the Bottema theorem. The different solutions, generalized problems make us interesting. The following is a problem for the readers.

Problem 4

Given a triangle ABC. Two vertices A, B are fixed and the vertex C is moveable. Two triangles CAB_a and CBA_b are constructed on two sides CA, CB of triangle ABC having the same orientation such that $\widehat{CAB_a} + \widehat{CBA_b} = 180^0$.

a) *Prove that the circle (ACB_a) meets the circle (BCA_b) at the second point P lying on the line A_bB_a.*
b) *Find the locus of point P when C is movable.*

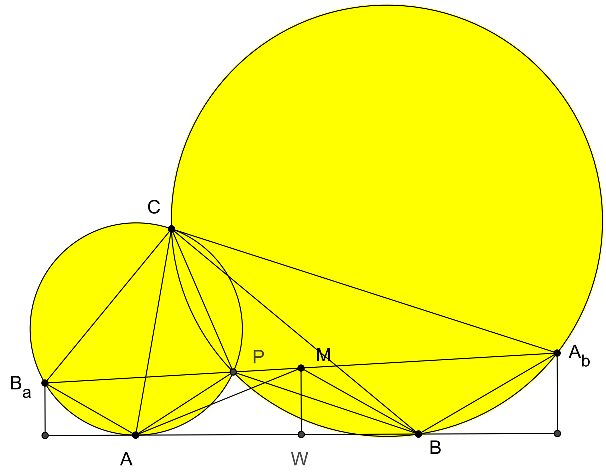


Fig 5

REFERENCES

[1] https://www.cut-the-knot.org/m/Geometry/GeneralBottema.shtml