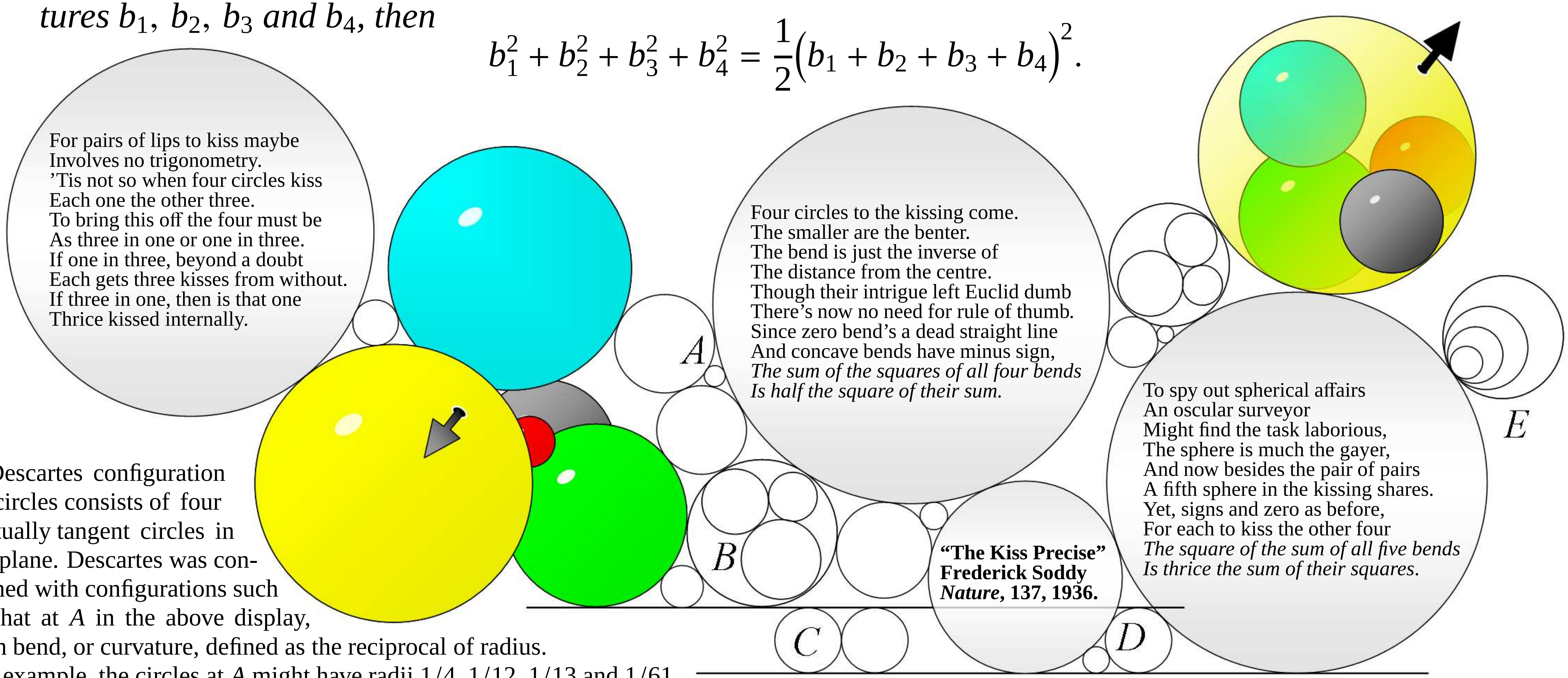
THEOREM OF THE DAY

The Descartes Circle Theorem *If four circles forming a Descartes configuration have respective curvatures b_1, b_2, b_3 and b_4, then*

$$b_1^2 + b_2^2 + b_3^2 + b_4^2 = \frac{1}{2}\left(b_1 + b_2 + b_3 + b_4\right)^2.$$



A Descartes configuration of circles consists of four mutually tangent circles in the plane. Descartes was concerned with configurations such as that at A in the above display, with bend, or curvature, defined as the reciprocal of radius. For example, the circles at A might have radii 1/4, 1/12, 1/13 and 1/61. Then $(b_1 + b_2 + b_3 + b_4)^2/2 = (4 + 12 + 13 + 61)^2/2 = 8100/2 = 4050$ and $16 + 144 + 169 + 3721 = 4050$ also, conforming to Descartes' theorem. If we adopt the convention that *negative* curvature corresponds to the interior of a circle being *outside* it, then configuration B is equally valid. For example the circles at B might have radii $-1/7$ (the outside circle) and 1/12, 1/17 and 1/24 (the inside circles). Then $(-7 + 12 + 17 + 24)^2/2 = 1058 = 49 + 144 + 289 + 576$ as required. If horizontal lines are taken as circles of infinite radius—"zero bend's a dead straight line"—then the configurations at C and D are again valid. But configuration E is not valid.

As the third stanza of Soddy's poem celebrates, configurations of five spheres in $\mathbb{R}^3$ can also be made to work, the fraction in the theorem becoming 1/3; with the fraction $1/n$, we can go to n dimensions; and adding curvature of space into the equation, even non-Euclidean Descartes Circle Theorems obtain!

Although he did not prove it completely, this result appears originally in a letter sent by René Descartes to Princess Elisabeth of Bohemia in 1643. There have been many rediscoveries with complete proofs, Jakob Steiner's, in 1826, being perhaps the first. The $\mathbb{R}^n$ version is due to Thorold Gosset in 1937, with $n = 3$ being given by Robert Lachlan in 1886.

Web link: arxiv.org/abs/math/0101066. Additions by Gosset and Fred Lunnon to Soddy's poem, which is reproduced courtesy of Nature Publishing Group, can be found here: pballew.blogspot.com/2014/10/the-kiss-precise-soddys-circle-theorem.html.

Further reading: *Introduction to Circle Packing: The Theory of Discrete Analytic Functions,* by Kenneth Stephenson, CUP, 2005.