ON LINEAR POLYGON TRANSFORMATIONS[1]

JESSE DOUGLAS

1. **Introduction, definitions.** In a recent paper[2] the author has developed a general theory of linear transformations of polygons, regarded as lying in the complex plane. By a *polygon* we understand a system of points, or complex numbers: $(z_1, z_2, \cdots, z_n)$, called *vertices*, which are taken in a definite *cyclic* order. This is to say that two n-gons, $(z_1, z_2, \cdots, z_n)$ and $(w_1, w_2, \cdots, w_n)$, are the same when and only when $z_i = w_{i+k}$ for $i = 1, 2, \cdots, n$, where k has any fixed one of the values $0, 1, 2, \cdots, n-1$ and all indices are taken modulo n. To be distinguished from a polygon is a *multipoint*, where the definition of equivalence is the identity of corresponding points in the order given: $z_i = w_i$ for $i = 1, 2, \cdots, n$.

A linear multipoint transformation is simply the general linear transformation of n complex variables:

$$z_i' = \sum_{j=1}^{n} a_{ij} z_j, \qquad i = 1, 2, \cdots, n, \tag{1}$$

where the coefficients a_{ij} may be any complex numbers. For a linear polygon transformation, on the other hand, a certain cyclicity is required: a cyclic permutation of the z's must produce the same cyclic permutation of the z''s. Thus we may write down arbitrarily the first line of an L.P.T.:[3]

$$z_1' = \alpha_0 z_1 + \alpha_1 z_2 + \cdots + \alpha_{r-1} z_r,$$

[1] Presented to the Society, October 29, 1938, under the title *Geometry of polygons in the complex plane.*

[2] *Geometry of polygons in the complex plane,* Journal of Mathematics and Physics, vol. 19 (1940), pp. 93–130, and this Bulletin, abstract 44-9-390. See, as a preliminary to the present theory, papers by E. Kasner and his students in Scripta Mathematica, vol. 2 (1934), pp. 131–138, and vol. 4 (1936), pp. 37–49. Kasner considers the polygon derived from a given one by taking the midpoint of each side, and, more generally, by taking the centroid of r consecutive vertices. These are special linear polygon transformations (2), where $\alpha_0 = \alpha_1 = \cdots = \alpha_{r-1} = 1/r$. Kasner uses real cartesian coordinates, and his polygons may lie in euclidean space of any number of dimensions.

The basic ideas of the present paper are (i) to regard the polygons as lying in the *complex plane,* (ii) to consider general linear polygon transformations (2) with any *complex* coefficients. As pointed out in §1, this is equivalent to taking a "center of gravity" with complex "weights."

The author delivered a series of lectures on the present topic at Columbia University in July, 1939.

[3] L.P.T. denotes "linear polygon transformation" throughout this paper.

but then the whole transformation is determined by cyclic permutation, that is:

(2) $$z_k' = \alpha_0 z_k + \alpha_1 z_{k+1} + \cdots + \alpha_{r-1} z_{k+r-1}, \qquad k = 1, 2, \cdots, n,$$

where the indices of z are to be taken modulo n. Here r may have any value from 1 to n inclusive, and we may suppose $\alpha_0 \neq 0$,[4] $\alpha_{r-1} \neq 0$, the transformation being then called *r-ary*. If $r < n$, we may fill out formula (2) to the general form (1) by using zero coefficients in each line of (2) for those z's which do not effectively appear.

Formula (2) represents the general *linear polygon transformation L.*[5] Its characteristic property is to be transformed into itself by a cyclic permutation $C = (z_1, z_2, \cdots, z_n)$ of the z's: $C^{-1}LC = L$, or to be commutative with such a permutation: $CL = LC$. Since $C^{-1}LC = L$ and $C^{-1}L'C = L'$ imply by multiplication $C^{-1}LL'C = LL'$, it follows that *the product of two L.P.T.'s is again an L.P.T.* Thus, all L.P.T.'s for a given value of n form a group,[6] a subgroup G_n of n complex parameters in the total linear group G_{n^2} of n^2 complex parameters represented by (1). In §2 another proof of the same fact will appear (Theorem II).

Particularly interesting are the special L.P.T.'s, to be denoted by M, for which

(3) $$\alpha_0 + \alpha_1 + \cdots + \alpha_{r-1} = 1.$$

As is immediately verifiable, (3) is the necessary and sufficient condition that a given L.P.T. (2) be permutable with an arbitrary similitude transformation $S: z' = Az + B$ (A, B complex); that is, $MS = SM$, or $S^{-1}MS = M$, for every S and every M. It follows, as in the preceding paragraph, that *the transformations M form a group*, a subgroup G_{n-1} of the group G_n of all L.P.T.'s.

If the polygon P' is the image of the polygon P by any transformation M, the remarks just made state that the relation between P and P' is invariant under an arbitrary similitude transformation; we may say that P' is a "similitude concomitant" of P. Accordingly, we shall term any L.P.T., M, which obeys the condition (3) a *similitude*

[4] Evidently we can always bring about $\alpha_0 \neq 0$ by cyclic renumbering of the vertices z.

[5] We dispense with the consideration of nonhomogeneous L.P.T.'s, since these have the same constant term in each line and therefore differ from a homogeneous L.P.T. only by a translation.

[6] It should be emphasized that here, and throughout this paper, we use the term "group" only in the sense of possession of the specific group property: closure with respect to composition of elements. Since the determinant of an L.P.T. may be zero, an inverse transformation does not always exist.

construction on the polygon $P=(z_1, z_2, \cdots, z_n)$ and a *binary*, *ternary*, $\cdots$, *r-ary* S.C.[7] according as two, three, $\cdots$, r consecutive vertices of P are effectively involved in the formula (2) of the transformation ("effectively" meaning $\alpha_0 \neq 0$, $\alpha_{r-1} \neq 0$).

If furthermore the α's are *real*, then, as is easily verified, we have for every affine transformation a, that is, $z'=Az+B\bar{z}+C$ (A, B, C complex, $\bar{z}$ the conjugate of z), the relation $a^{-1}Ma=M$ or $Ma=aM$. The polygon P' is then an "affine concomitant" of the polygon P; accordingly, for *real* values of the α's obeying (3), we term M an *affine construction.*

Evidently, under the condition (3), we can consider that (2) defines z_k' as a kind of "center of gravity" of $z_k, z_{k+1}, \cdots, z_{k+r-1}$ with *complex* "weights" proportional to $\alpha_0, \alpha_1, \cdots, \alpha_{r-1}$. In the case of an affine construction, where the weights α are real, we have an actual center of gravity in the ordinary sense.

We may observe here, as in our cited paper,[8] that the centroid of any polygon is invariant under any S.C., M. For by addition of all the equations (2),

$$\text{(3a)} \quad \begin{aligned} &z_1' + z_2' + \cdots + z_n' \\ &\qquad = (\alpha_0 + \alpha_1 + \cdots + \alpha_{r-1})(z_1 + z_2 + \cdots + z_n); \end{aligned}$$

then take account of (3) and divide by n.

Of special importance is the *binary* S.C., M_2:

$$\text{(4)} \qquad z_k' = \alpha_0 z_k + \alpha_1 z_{k+1}, \qquad k = 1, 2, \cdots, n,$$

$$\text{(4')} \qquad \alpha_0 + \alpha_1 = 1.$$

The point z_k' is here determined by the simple geometric condition that *the triangle* (z_k, z_{k+1}, z_k') *is directly*[9] *similar to the fixed triangle* $(-\alpha_1, \alpha_0, 0)$. This is seen by considering the equations in A, B

$$\text{(5)} \qquad z_k = -\alpha_1 A + B, \quad z_{k+1} = \alpha_0 A + B, \quad z_k' = B,$$

whose solvability is the condition for the existence of a similitude transformation $z'=Az+B$ converting the triangle $(-\alpha_1, \alpha_0, 0)$ into (z_k, z_{k+1}, z_k'). By use of (4'), the solution of the first two of these equations is

$$A = z_{k+1} - z_k, \; B = \alpha_0 z_k + \alpha_1 z_{k+1};$$

[7] S.C. is "similitude construction" throughout this paper.

[8] In this way we shall always refer to the paper whose title is given in the first footnote.

[9] That is, with preservation of sense.

and then equation (4) expresses exactly that the third equation (5) is satisfied.

If α_0, α_1 are real, the point z_k' lies upon the side $z_k z_{k+1}$ and divides it in the fixed ratio $z_k z_k' : z_k' z_{k+1} = \alpha_1 : \alpha_0$.

2. L.P.T.'s **and cyclic matrices.** The matrix of the general L.P.T. is a *cyclic matrix*

$$L = \left\| \begin{matrix} \alpha_0 & \alpha_1 & \cdots & \alpha_{n-2} & \alpha_{n-1} \\ \alpha_{n-1} & \alpha_0 & \alpha_1 \cdots & \cdot & \alpha_{n-2} \\ \alpha_{n-2} & \alpha_{n-1} & \alpha_0 \cdots & \cdot & \alpha_{n-3} \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} & \alpha_0 \end{matrix} \right\|, \tag{6}$$

where the circular derivation of each row from the preceding is the characteristic feature. Any number of the α's may be zero, so that the L.P.T. can be r-ary with $r=1, 2, \cdots, n$. We shall speak interchangeably of an L.P.T., L, and its corresponding cyclic matrix L.

The simplest L.P.T. is a cyclic permutation of vertices, C:

$$z_1' = z_2, \quad z_2' = z_3, \quad \cdots, \quad z_{n-1}' = z_n, \quad z_n' = z_1.$$

As a matrix, this is

$$C = \left\| \begin{matrix} 0 & 1 & 0 \cdots 0 \\ 0 & 0 & 1 \cdots 0 \\ \cdot & \cdot & \cdots\cdot \\ 0 & 0 & \cdots 1 \\ 1 & 0 & \cdots 0 \end{matrix} \right\|.$$

We may say that C has 1's in its second "cyclic diagonal" and 0's everywhere else. In general, as is seen at once, C^k has 1's in its $(k+1)$th cyclic diagonal and 0's elsewhere. Obviously, $C^n = I$, the identity matrix, which has 1's in its first (cyclic) diagonal and 0's elsewhere.

From these remarks we have directly by comparison with (6):

$$L = \alpha_0 I + \alpha_1 C + \alpha_2 C^2 + \cdots + \alpha_{n-1} C^{n-1}. \tag{7}$$

Conversely, every polynomial in C of this form represents a cyclic matrix, namely the one whose first, or generating, row consists of the coefficients of the polynomial in order. Hence we have the following theorem.

THEOREM I. *The necessary and sufficient condition that an n-rowed square matrix L be cyclic is that it be expressible as a polynomial of degree $n-1$ at most in the matrix C of a cyclic permutation. This mode of representation of L is unique.*

If in (7) we replace C by a (complex) numerical variable x, and I by 1, we obtain a polynomial $\phi(x)=\alpha_0+\alpha_1 x+\cdots+\alpha_{n-1}x^{n-1}$, that we shall term the *auxiliary polynomial*[10] of L.

If any two polynomials of the form (7) are multiplied together, the product can be reduced to the degree $n-1$ at most by means of the relation $C^n=I$. From this remark and Theorem I the following three theorems result immediately.

THEOREM II. *The product of two cyclic matrices is again a cyclic matrix.*

THEOREM III. *The multiplication of cyclic matrices is isomorphic with the multiplication of their auxiliary polynomials, modulo x^n-1.*

THEOREM IV. *The multiplication of cyclic matrices is commutative.*

Thus cyclic matrices form a commutative subalgebra of order n in the non-commutative algebra of order n^2 formed by general n-rowed square matrices.

The following alternative form of proof of these theorems, directly in terms of the explicit representation (6) of a cyclic matrix, may be of interest. Let α_m, $(m=0, 1, \cdots, n-1)$, denote any n complex numbers. Then a cyclic matrix of order n, $\|a_{ij}\|$, is one where $a_{ij}=\alpha_{j-i}$, *the index $j-i$ being taken modulo n*; that is, in case $j-i$ is negative, $j-i+n$ is to be used instead. If $\|b_{ij}\|$ is another cyclic matrix: $b_{ij}=\beta_{j-i}$, $j-i$ taken modulo n, then $\|a_{ij}\|\times\|b_{ij}\|=\|c_{ij}\|$, where

$$c_{ij}=\sum_{k=1}^{n} a_{ik}b_{kj}=\sum_{k=1}^{n}\alpha_{k-i}\beta_{j-k}=\sum \alpha_s\beta_t$$

(in the last summation s, t evidently take once and only once every combination of values such that $s+t\equiv j-i$, modulo n). In other words, defining

$$\gamma_m=\sum \alpha_s\beta_t, \qquad s+t\equiv m \pmod{n}, \tag{8}$$

we have $c_{ij}=\gamma_{j-i}$, $j-i$ taken modulo n.

This proves Theorem II. Theorem III follows because (8) repre-

[10] This is only of degree $r-1$ if L is r-ary (see (9)). In our cited paper we defined the auxiliary polynomial as $\alpha_0 x^{r-1}+\alpha_1 x^{r-2}+\cdots+\alpha_{r-1}$. With this difference of notation in mind, no ambiguity should arise in comparing the two papers.

sents the law of multiplication of the auxiliary polynomials modulo x^n-1:

$$\sum_{m=0}^{n-1} \gamma_m x^m \equiv \sum_{s=0}^{n-1} \alpha_s x^s \cdot \sum_{t=0}^{n-1} \beta_t x^t \pmod{(x^n - 1)}.$$

Theorem IV expresses the symmetry of (8) in the indices s, t of α, β.

As has been pointed out, all these theorems can be stated in terms of L.P.T.'s[11] instead of cyclic matrices. The auxiliary polynomial of an r-ary L.P.T. is of degree $r-1$:

$$\phi(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_{r-1} x^{r-1}. \tag{9}$$

We term $\phi(x)=0$ the *auxiliary equation* of the L.P.T.

By factorization of its auxiliary polynomial, or solution of its auxiliary equation, *every r-ary L.P.T. can be factored into $r-1$ binary L.P.T.'s*. This results from Theorem III; in fact, if

$$\phi(x) = (\alpha_0^{(1)} + \alpha_1^{(1)} x)(\alpha_0^{(2)} + \alpha_1^{(2)} x) \cdots (\alpha_0^{(r-1)} + \alpha_1^{(r-1)} x), \tag{10}$$

then L is equal to the product of the $r-1$ binary L.P.T.'s

$$L_2^{(j)} = \alpha_0^{(j)} I + \alpha_1^{(j)} C, \qquad j = 1, 2, \cdots, r-1.$$

The binary components $L_2^{(j)}$ are evidently not unique; rather, as is evident, each is subject to multiplication by a (complex) scalar $\lambda^{(j)}$ provided that the product of all the $\lambda^{(j)}$'s is unity.

If L is an S.C., M, so that (3) is verified, we have $\phi(1)=1$, or by (10),

$$1 = (\alpha_0^{(1)} + \alpha_1^{(1)})(\alpha_0^{(2)} + \alpha_1^{(2)}) \cdots (\alpha_0^{(r-1)} + \alpha_1^{(r-1)}). \tag{11}$$

Dividing (10) by (11), we obtain

$$\phi(x) = (\beta_0^{(1)} + \beta_1^{(1)} x)(\beta_0^{(2)} + \beta_1^{(2)} x) \cdots (\beta_0^{(r-1)} + \beta_1^{(r-1)} x)$$

where $\beta_0^{(j)} = \alpha_0^{(j)}/(\alpha_0^{(j)}+\alpha_1^{(j)})$, $\beta_1^{(j)} = \alpha_1^{(j)}/(\alpha_0^{(j)}+\alpha_1^{(j)})$; hence

$$\beta_0^{(j)} + \beta_1^{(j)} = 1. \tag{12}$$

Accordingly $M_2^{(j)} = \beta_0^{(j)} I + \beta_1^{(j)} C$ is a binary S.C. for $j=1, 2, \cdots, r-1$, and we have the following theorem.

[11] The commutativity of L.P.T.'s seems the more noteworthy because it is certainly not evident geometrically that two binary S.C.'s are commutative. See the geometric interpretation of a binary S.C. at the end of §1.

THEOREM V. *Every r-ary S.C., M_r, is representable uniquely as the product of $r-1$ binary S.C.'s*: $M_r=\prod_{j=1}^{r-1}M_2^{(j)}$.

The uniqueness is a consequence of the fact that the components $M_2^{(j)}$ are not subject to multiplication by a scalar, because that would disturb condition (12).

An interesting illustration of Theorem V is furnished by the ternary centroidal construction

$$z_k' = \tfrac{1}{3}z_k + \tfrac{1}{3}z_{k+1} + \tfrac{1}{3}z_{k+2}.$$

Here the auxiliary polynomial factors as follows:

$$\frac{1}{3}(1 + x + x^2) = \frac{\omega - x}{\omega - 1}\cdot\frac{\omega^2 - x}{\omega^2 - 1},$$

where $\omega=e^{2\pi i/3}$ is a complex cube root of unity. By the remarks associated with (4), the component binary S.C.'s are those which employ triangles similar respectively to $(1, \omega, 0)$, $(1, \omega^2, 0)$. Hence the centroid of any three points z_1, z_2, z_3 will be arrived at as follows. On z_1z_2, z_2z_3 construct 120° isosceles triangles to the left; let z_1', z_2' be their respective vertices. On $z_1'z_2'$ construct a 120° isosceles triangle to the right; its vertex z_1'' is the centroid of z_1, z_2, z_3.

We conclude this section by observing the following factorization of a cyclic determinant:

$$\det L = \phi(1)\cdot\phi(\omega)\cdot\phi(\omega^2)\cdot \cdots \cdot\phi(\omega^{n-1}), \tag{13}$$

where $\omega=e^{2\pi i/n}$ is a primitive nth root of unity. The proof is easily given as follows. Det L is a homogeneous polynomial of degree n in the α's. By addition of all the columns of L (refer to (6)), $\phi(1)$ is seen to be a factor of det L. By using $1, \omega, \omega^2, \cdots, \omega^{n-1}$ as multipliers of the successive columns and adding, $\phi(\omega)$ is seen to be a factor of det L. Similarly $\phi(\omega^2), \cdots, \phi(\omega^{n-1})$ are factors. Each of these n factors is a linear polynomial in the α's. Hence (13) holds to within a numerical factor, which is verified to be 1 by comparison of the terms in α_0^n.

3. **Geometrical applications.** In our cited paper, we consider the relations

$$R_p \equiv \sum_{k=1}^{n} \omega^{(k-1)p}z_k = 0, \quad p = 1, 2, \cdots, n-1, \tag{14}$$

as applied to any polygon. Each relation $R_p=0$ expresses an intrinsic geometric property of the polygon, being invariant under any similitude transformation, $z_k' = Az_k + B$; this is easily verified if account is taken of the well known equation[12] $\sum_{k=0}^{n-1}\omega^{kp} = 0$, $(p = 1, 2, \cdots, n-1)$.

A polygon is called *regular of ω^q-type*[13] if it is similar to the polygon $(1, \omega^q, \omega^{2q}, \cdots, \omega^{(n-1)q})$. A regular polygon of ω-type is simply a (convex) regular polygon in the ordinary sense. A regular polygon of ω^q-type may be star-shaped (for example, $n=5$, $q=2$: a star-shaped regular pentagon), or it may resolve into an s-gon described t times, where $st=n$ (for example, $n=6$, $q=2$: an equilateral triangle described twice, considered as a form of regular hexagon).

It was shown (loc. cit., §5) that *the criterion for a polygon to be regular of ω^q-type is that it obey the following $n-2$ conditions*:

$$R_p = 0 \qquad \text{for } p = 1, 2, \cdots, n-1, \text{ except } n-q. \tag{15}$$

Each condition $R_p = 0$ may therefore be regarded as expressing a "degree of regularity," and any number $k < n-2$ of these conditions as expressing "partial regularity" of degree k.

By multiplying equation (2) by $\omega^{(k-1)p}$ and summing for $k=1, 2, \cdots, n$, we find

$$R_p' = \phi(\omega^{-p})R_p = \phi(\omega^{n-p})R_p, \tag{16}$$

or, interchanging p and $n-p$,

$$R_{n-p}' = \phi(\omega^p)R_{n-p}. \tag{16'}$$

We infer from (16) or (16′) the following two facts:

(i) *If a polygon obeys the relation $R_p = 0$, this relation persists after any L.P.T.* (That is, R_p is a *relative invariant* of the group G_n of all L.P.T.'s.)

(ii) *If ω^p is a root of the auxiliary equation $\phi(x)=0$ of an L.P.T., then this L.P.T. converts every polygon P into a polygon P' obeying the relation $R_{n-p}=0$.*

A binary S.C., $M_2 = \alpha_0 I + \alpha_1 C$, $(\alpha_0 + \alpha_1 = 1)$, has the auxiliary linear polynomial $\phi(x) = \alpha_0 + \alpha_1 x$, and hence obeys the hypothesis of (ii): $\phi(\omega^p) = 0$, when and only when $-\alpha_1 : \alpha_0 = 1 : \omega^p$. By the statement following (4′), this means that the triangle (z_k, z_{k+1}, z_k') is similar to $(1, \omega^p, 0)$. Then z_k' is at the vertex of an isosceles triangle with vertex angle $2p\pi/n$ based on $z_k z_{k+1}$ and to the left[14] of this base. We shall denote this construction whereby the polygon (z_k') is derived from the polygon (z_k) by $\Delta(2p\pi/n)$.

[12] ω^p is a root, not 1, of $x^n - 1 \equiv (x-1)(1+x+x^2+\cdots+x^{n-1}) = 0$.

[13] As defined, loc. cit., §2, Definition (iii).

[14] With this interpretation: an isosceles triangle to the left with vertex angle $\pi+\theta$ is an isosceles triangle to the right with vertex angle $\pi-\theta$.

We may now put (i), (ii), and the criterion (15) together so as to give the next theorem:

THEOREM VI. *If, on an* arbitrary *polygon* P_0, *the* $n-2$ *constructions* $\Delta(2p\pi/n)$ *for* $p=1, 2, \cdots, n-1$ *except* q, *are performed successively in any order, giving the series of polygons* $P_1, P_2, \cdots, P_{n-2}$, *then* P_{n-2} *is* regular *of* ω^q-*type. Further,* P_{n-2} *is independent of the order of these constructions. Also, the centroid of* P_{n-2} *and of all the intermediate polygons* $P_1, P_2, \cdots$ *is the same as that of* P_0.[15]

(The last statement refers to the remark associated with (3a).)

PROOF. Each operation $\Delta(2p\pi/n)$ confers the property $R_{n-p}=0$ on the new polygon, according to (ii); and the subsequent operations allow the polygon to keep this property, according to (i); therefore the final polygon P_{n-2} has all the properties (15) requisite for regularity of ω^q-type.

The case $n=3$ of Theorem VI is the following well known theorem of elementary geometry: If on each side of an *arbitrary* triangle as base, a 120° isosceles triangle is constructed, always outward or always inward, then the vertices of these isosceles triangles form an *equilateral* triangle.

In conclusion, we consider the effect on a polygon of any r-ary S.C., M. Since for an S.C., $\phi(1)=1$ by (3), the determinant of M is by (13)

$$\det M = \phi(\omega)\cdot\phi(\omega^2)\cdot \cdots \cdot\phi(\omega^{n-1}).$$

Hence, if the auxiliary polynomial $\phi(x)$ is prime to x^n-1, then $\det M\neq 0$, and the transformation M is (uniquely) reversible. This means that for every polygon P' there is a (unique) polygon P such that $P'=MP$; accordingly, P' can have no special properties if P is general.

On the other hand, if the greatest common divisor of $\phi(x)$ and x^n-1 is

$$\psi(x) = (x-\omega^{p_1})(x-\omega^{p_2})\cdots(x-\omega^{p_k}),$$

so that

$$\phi(\omega^{p_1}) = 0, \quad \phi(\omega^{p_2}) = 0, \cdots, \phi(\omega^{p_k}) = 0, \qquad \phi(\omega^{p}) \neq 0 \quad \text{for } p \neq p_1 \text{ or } p_2 \cdots \text{ or } p_k, \tag{17}$$

then by (16′) the transformed polygon P' has the special properties (partial regularity of degree k)

[15] Theorem VI is the main one of our cited paper (Theorem A or A′, §7).

$$(18) \qquad R'_{n-p_1} = 0, \qquad R'_{n-p_2} = 0, \qquad \cdots, \qquad R'_{n-p_k} = 0$$

for an arbitrary original polygon P. Further, no other relations $R'_{n-p}=0$ ($p \neq p_1$ or $p_2 \cdots$ or p_k) are satisfied by P' if P remains general (P' has no higher than the kth degree of regularity). This is also seen from (16′), where $\phi(\omega^p) \neq 0$, $R_{n-p} \neq 0$ (since P is general); therefore $R'_{n-p} \neq 0$.

In fact, no relations of any kind besides (18) are satisfied by $P' = MP$ if P remains general. This is because, by the general theory of systems of linear equations, it can be readily shown that if the conditions (17) are satisfied by the coefficients α in (2), then the conditions (18) are sufficient as well as necessary in order that (2) be solvable for the z's in terms of the z''s. This is to say that for *any* polygon P' obeying (18) a polygon P can be found such that $P' = MP$; indeed, the class of such polygons P depends linearly on k complex parameters.

BROOKLYN, N. Y.

AXIOMS FOR MOORE SPACES AND METRIC SPACES[1]

C. W. VICKERY

We shall consider a set of five axioms in terms of the undefined notions of *point* and *region*. It will be shown that these axioms are independent and that they constitute a set of conditions necessary and sufficient for a space to be a complete metric space. It will also be shown that certain subsets of this set of axioms constitute necessary and sufficient conditions for a space to be (1) a metric space, (2) a Moore space, (3) a complete Moore space. Axiom 2 and a more general form of Axiom 1 have been stated by the author in an earlier paper [**1**]. Following terminology of F. B. Jones [**2**], a space is said to be a *Moore space* provided conditions (1), (2), and (3) of Axiom 1 (that is, Axiom 1_0) of R. L. Moore's *Foundations of Point Set Theory* [**3**] are satisfied. A space is said to be a *complete Moore space* provided it satisfies all the conditions of that axiom. Wherever the notion of region is employed, whether as a defined or an undefined notion, it is understood that a necessary and sufficient condition that a point P be a limit point of a point set M is that every region containing P contain a point of M distinct from P. The letter S is used to denote the set of all points.

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[1] Presented to the Society, April 20, 1935, under the title *Sets of independent axioms for complete Moore space and complete metric space.*