Algebraic Geometry

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ALGEBRAIC GEOMETRY

Papers presented at the Bombay Colloquium 1968, by

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INTERNATIONAL COLLOQUIUM ON ALGEBRAIC GEOMETRY

Bombay, 16-23 January 1968

REPORT

An International Colloquium on Algebraic Geometry was held at the Tata Institute of Fundamental Research, Bombay on 16-23 January, 1968. The Colloquium was a closed meeting of experts and others seriously interested in Algebraic Geometry. It was attended by twenty-six members and thirty-two other participants, from France, West Germany, India, Japan, the Netherlands, the Soviet Union, the United Kingdom and the United States.

The Colloquium was jointly sponsored, and financially supported, by the International Mathematical Union, the Sir Dorabji Tata Trust and the Tata Institute of Fundamental Research. An Organizing Committee consisting of Professor K. G. Ramanathan (Chairman), Professor M. S. Narasimhan, Professor C. S. Seshadri, Professor C. P. Ramanujam, Professor M. F. Atiyah and Professor A. Grothendieck was in charge of the scientific programme. Professors Atiyah and Grothendieck represented the International Mathematical Union on the Organizing Committee. The purpose of the Colloquium was to discuss recent developments in Algebraic Geometry.

The following twenty mathematicians accepted invitations to address the Colloquium: S. S. Abhyankar, M. Artin, B. J. Birch, A. Borel, J. W. S. Cassels, B. M. Dwork, P. A. Griffiths, A Grothendieck, F. Hirzebruch, J.-I. Igusa, Yu. I. Manin, T. Matsusaka, D. Mumford, M. Nagata, M. S. Narasimhan, S. Ramanan, C. S. Seshadri, T. A. Springer, J. L. Verdier and A. Weil. Professor H. Hironaka, who was unable to attend the Colloquium, sent in a paper.

The Colloquium met in closed sessions. There were nineteen lectures in all, each lasting fifty minutes, followed by discussions. Informal lectures and discussions continued during the week, outside the official programme.

The social programme included a tea on 15 January, a dinner on 16 January, a programme of classical Indian dances on 17 January, a dinner at the Juhu Hotel on 20 January, an excursion to Elephanta on the morning of 22 January and a farewell dinner the same evening.

Contents

RESOLUTION OF SINGULARITIES OF ALGEBRAIC SURFACES

By Shreeram Shankar Abhyankar

1 Introduction. The theorem of resolution of singularities of algebraic surfaces asserts the following : 1

Surface Resolution. *Given a projective algebraic irreducible surface Y over a field k, there exists a projective algebraic irreducible nonsingular surface Y' over k together with a birational map of Y' onto Y (without fundamental points on Y').*

For the case when k is the field of complex numbers, after several geometric proofs by the Italians (see Chapter I of [15]), the first rigorous proof of Surface Resolution was given by Walker [14]. For the case when k is a field of zero characteristic, Surface Resolution was proved by Zariski ([16],[17]); and for the case when k is a perfect field of nonzero characteristic, it was proved by Abhyankar ([2],[3],[4]).

A stronger version of Surface Resolution is the following :

Embedded Surface Resolution. *Let X be a projective algebraic irreducible nonsingular three-dimensional variety over a field k, and let Y be an algebraic surface embedded in X. Then there exists a finite sequence $X \to X_1 \to X_2 \to \ldots \to X_t \to X'$ of monoidal transformations, with irreducible nonsingular centers, such that the total transform of Y in X' has only normal crossings and the proper transform of Y in X' is nonsingular.*

For the case when k is of zero characteristic and Y is irreducible, the part of Embedded Surfaces Resolution concerning the proper transform of Y was proposed by Levi [13] and proved by Zariski [18]. Again for the case when k is of zero characteristic, Hironaka [12] proved Embedded Resolution for algebraic varieties of any dimension. For the case when k is a perfect field of nonzero characteristic, Embedded Surface Resolution was proved by Abhyankar ([7],[9],[10],[11]).

2 Then in January 1967, in a seminar at Purdue University, I gave a proof of Embedded Surface Resolution (and hence *a fortiori* also of Surface Resolution) for an arbitrary field k, i.e. without assuming k to the perfect. The details of this proof will be published elsewhere in due course of time. This new proof is actually only a modification of my older proofs cited above. One difference between them is this. In the older proofs I passed to the albegraic closure of k, did resolution there, and then pulled it down to the level of k. In case k is imperfect, the pulling down causes difficulties. In the new modified proof I work directly over k. In this connection, a certain lemma about polynomials in one indeterminate (with coefficients in some field) plays a significant role.

In order that my lecture should not get reduced to talking only in terms of generalities, I would like to show you, concretely, the proof of something. The said lemma being quite simple, I shall now state and prove it.

2 The lemma.

Let k' be a field and let q be a positive integer. Assume that q is a power of the characteristic exponent of k'; recall that, by definition : (the characteristic exponent of k') = (the characteristic of k') if k' is of nonzero characteristic, and (the characteristic exponent of k') = 1 if k' is of zero characteristic. Let A be the ring of all polynomials in an indeterminate T with coefficients in k'. As usual, by A^q we denote the set $\{h^q : h \in A\}$. Note that then A^q is a subring of A; this is the only property of q which we are going to use. For any $h \in A$, by $\deg h$ we shall denote the degree of h in T; we take $deg 0 = -\infty$. Let $f \in A$ and let $d = \deg f$. Assume that $f \notin k$, i.e. $d > 0$. Let $r : A \to A/(fA)$ be the canonical epimorphism. For any nonnegative integer e let

$$W(q, d, e) = [e/(dq)]q + [\max\{q - (q/d), (e/d) - [e/(dq)]q\}]$$

where the square brackets denote the integral part, i.e. for any real number a, by $[a]$ we denote the greatest integer which is $\leqslant a$.

The Lemma. *Given any $g \in A$ with $g \notin A^q$, let $e = \deg g$. Then we can express g in the form*

$$g = g' + g^* f^u$$

where $g' \in A^q$, $g^* \in A$ *with* $g^* \notin fA$, *and* u *is a nonnegative integer such that* 3

$$u \leqslant W(q, d, e)$$

and: either $u \not\equiv 0(q)$ *or* $r(g^*) \notin (r(A))^q$.

Before proving the lemma we shall make some preliminary remarks.

Remark 1. The assumption that $g \in A$ and $g \notin A^q$ is never satisfied if $q = 1$. Thus the lemma has significance only when k' is of nonzero characteristic and q is a positive power of the characteristic of k'. The lemma could conceivably be generalized by replacing A^q by some other subset of A.

Remark 2. For any integers q, d, e with $q > 0$, $d > 0$, $e \geqslant 0$, we clearly have $W(q, d, e) \geqslant 0$.

Remark 3. The bound $W(q, d, e)$ for u can be expressed in various other forms. Namely, *we claim that for any integers q, d, e with* $q > 0$, $d > 0$, $e \geqslant 0$, *we have*

$$W(q, d, e) = W_1(q, d, e) = W_2(q, d, e) = W_3(q, d, e)$$

where

$$\begin{aligned}
W_1(q, d, e) &= [[e/d]/q]q + [\max\{q - (q/d), [e/d] - [[e/d]/q]q\}],\\
W_2(q, d, e) &= [[e/d]/q]q + \max\{[q - (q/d)], [e/d] - [[e/d]/q]q\},\\
W_3(q, d, e) &= \max\{[e/d], [[e/d]/q]q + [q - (q/d)]\}.
\end{aligned}$$

To see this, first note that by the division algorithm we have

$$e = [e/d]d + j \quad \text{with} \quad 0 \leqslant j \leqslant d - 1$$

and

$$[e/d] = [[e/d]/q]q + j' \quad \text{with} \quad 0 \leqslant j' \leqslant q - 1;$$

upon substituting the second equation in the first equation we get

$$e = [[e/d]/q]dq + (dj' + j) \quad \text{and} \quad 0 \leqslant dj' + j \leqslant dq - 1$$

and hence

$$[[e/d]/q] = [e/(dq)].$$

4 For any two real numbers a and b we clearly have

$$[\max\{a, b\}] = \max\{[a], [b]\},$$

and hence in view of the last displayed equation we see that

$$W(q, d, e) = W_2(q, d, e) = W_1(q, d, e).$$

For any real numbers a, b, c we clearly have

$$a + \max\{b, c\} = \max\{a + b, a + c\},$$

and hence we see that

$$W_2(q, d, e) = W_3(q, d, e).$$

Thus our claim is proved. Clearly $[e/d] \leqslant W_3(q, d, e)$; since $W(q, d, e) = W_3(q, d, e)$, we thus get the following

Remark 4. For any integers q, d, e with $q > 0$, $d > 0$, $e \geqslant 0$, we have $[e/d] \leqslant W(q, d, e)$.

Remark 5. Again let q, d, e be any integers with $q > 0$, $d > 0$, $e \geqslant 0$. Concerning $W(q, d, e)$ we note the following.

If $d = 1$ then clearly $W(q, d, e) = e$.

If $d > 1$ and $e \leqslant q$ then : $[e/(dq)] = 0$ and $\max\{q-(q/d), e/d\} < q$, and hence $W(q, d, e) \leqslant q - 1$.

If $d > 1$ and $e > q$ then :

$$\begin{aligned}[][e/d]/q]q + [q - (q/d)] &\leqslant (e/d) + q - (q/d) \\ &= (e/d) + (q/d)(d - 1) \\ &< (e/d) + (e/d)(d - 1) \\ &= e\end{aligned}$$

and hence

$$[[e/d]/q]q + [q - (q/d)] \leqslant e - 1;$$

also $[e/d] \leqslant e-1$, and hence $W_3(q,d,e) \leqslant e-1$ where W_3 is as in Remark 3; since $W(q,d,e) = W_3(q,d,e)$ by Remark 3, we get that $W(q,d,e) \leqslant e-1$.

Thus : *if* $d=1$ *then* $W(q,d,e)=e$*; if* $d>1$ *and* $e \leqslant q$ *then* $W(q,d,e) \leqslant q-1$*; if* $d>1$ *and* $e>q$ *then* $W(q,d,e) \leqslant e-1$.

In particular : *either* $W(q,d,e)/q \leqslant e/q$ *or* $W(q,d,e)/q < 1$*; if* 5
$d>1$ *then either* $W(q,d,e)/q < e/q$ *or* $W(q,d,e)/q < 1$.

Thus the lemma has the following

Corollary. *The same statement as that of the lemma, except that we replace the inequality* $u \leqslant W(q,d,e)$ *by the following weaker estimates: either* $u/q \leqslant e/q$ *or* $u/q<1$*; if* $d>1$ *then either* $u/q<e/q$ *or* $u/q<1$.

For applications, this corollary is rather significant.

Remark 6. For a moment suppose that $r(g) \in (r(A))^q$. Then $r(g) = h'^q$ for some $h' \in r(A)$. Now there exists a unique $h^* \in A$ such that $\deg h^* \leqslant d-1$ and $r(h^*) = h'$. Let $h = h^{*q}$. Then $h \in A^q$, $\deg h \leqslant dq-q$, and $g-h \in fA$. Since $g \notin A^q$ and $h \in A^q$, we must have $g-h \notin A^q$; hence in particular $g-h \neq 0$. Since $0 \neq g-h \in fA$, there exists $g_1 \in A$ and a positive integer v such that $g_1 \notin fA$ and $g-h = g_1 f^v$. Since $g_1 f^v = g-h \notin A^q$ we get that: either $v \not\equiv 0(q)$ or $g_1 \notin A^q$. Now $g_1 f^v = g-h$, $\deg f = d$, $\deg g = e$, and $\deg h \leqslant dq-q$; therefore upon letting $e_1 = \deg g_1$ we have

$$dv + e_1 = \deg(g_1 f^v) = \deg(g-h) \begin{cases} = e & \text{if } e > dq-q \\ \leqslant dq-q & \text{if } e \leqslant dq-q. \end{cases}$$

Thus we have proved the following

Remark 7. If $r(g) \in (r(A))^q$ then $g = h + g_1 f^v$ where $h \in A^q$, $g_1 \in A$ with $g_1 \notin fA$, v is a positive integer, and letting $e_1 = \deg g_1$ we have

$$dv + e_1 \begin{cases} = e & \text{if } e > dq-q \\ \leqslant dq-q & \text{if } e \leqslant dq-q \end{cases}$$

and: either $v \not\equiv 0(q)$ or $g_1 \notin A^q$.

Proof of the Lemma. We shall make induction on $[e/(dq)]$. First consider the case when $[e/(dq)] = 0$. If $r(g) \notin (r(A))^q$ then, in view of

Remark 2, it suffices to take $g' = 0$, $g^* = g$, $u = 0$. If $r(g) \in (r(A))^q$ then let the notation be as in Remark 7; since $[e/(dq)] = 0$ and by Re-
6 mark 7 we have $dv \leqslant \max\{dq - q, e\}$, we see that $v \leqslant W(q, d, e)$ and $v < q$; since $0 < v < q$, we see that $v \not\equiv 0(q)$; therefore it suffices to take $g' = h$, $g^* = g_1$, $u = v$.

Now let $[e/(dq)] > 0$ and assume that the assertion is true for all values of $[e/(dq)]$ smaller than the given one. If $r(g) \notin (r(A))^q$ then, in view of Remark 2, it suffices to take $g' = 0$, $g^* = g$, $u = 0$. So now suppose that $r(g) \in (r(A))^q$ and let the notation be as in Remark 7. Since $[e/(dq)] > 0$, by Remark 7 we have

$$dv + e_1 = e. \tag{*}$$

Therefore $v \leqslant [e/d]$ and hence if $v \not\equiv 0(q)$ then, in view of Remark 4, it suffices to take $g' = h$, $g^* = g_1$, $u = v$. So now also suppose that

$$v \equiv 0(q). \tag{**}$$

Then by Remark 7 we must have $g_1 \notin A^q$; since $v > 0$, by (*) and (**) we see that $[e_1/(dq)] < [e/(dq)]$; therefore by the induction hypothesis we can express g_1 in the form

$$g_1 = g_1' + g^* f^{u_1}$$

where $g_1' \in A^q$, $g^* \in A$ with $g^* \notin fA$, and u_1 is a nonnegative integer such that $u_1 \leqslant W(q, d, e_1)$ and either $u_1 \not\equiv 0(q)$ or $r(g^*) \notin (r(A))^q$. Let $u = u_1 + v$. Then u is a nonnegative integer, and in view of (**) we see that $u \equiv 0(q)$ if and only if $u_1 \equiv 0(q)$; consequently: either $u \not\equiv 0(q)$ or $r(g^*) \notin (r(A))^q$. Let $g' = h + g_1' f^v$; since h and g_1' are in A^q, by (**) we get that $g' \in A^q$. Clearly

$$g = g' + g^* f^u.$$

By (*) and (**) we get that

$$e \equiv e_1 \bmod dq \quad \text{and} \quad (e - e_1)/(dq) = v/q,$$

and hence

$$[e/(dq)] = [e_1/(dq)] + (v/q)$$

and

$$(e/(dq)) - [e/(dq)] = (e_1/(dq)) - [e_1/(dq)];$$

therefore

$$W(q, d, e) = W(q, d, e_1) + v;$$

since $u = u_1 + v$ and $u_1 \leqslant W(q, d, e_1)$, we conclude that $u \leqslant W(q, d, e)$. **7**

3 Use of the lemma. To give a slight indication of how the lemma is used, let R and R^* be two-dimensional regular local rings such that R^* is a quadratic transform of R. Let M and M^* be the maximal ideals in R and R^* respectively. Let $k' = R/M$ and let J be a coefficient set of R, i.e. J is a subset of R which gets mapped one-to-one onto k' by the canonical epimorphism $R \to R/M$. We can take a basis (x, y) of M such that $MR^* = xR^*$. Then $y/x \in R^*$. Let $s : R[y/x] \to R[y/x]/(xR[y/x])$ be the canonical epimorphism and let $T = s(y/x)$. Then $s(R)$ is naturally isomorphic to k' and, upon identifying $s(R)$ with k' and letting $A = k'[T]$, we have that T is transcendental over k', $s(R[y/x]) = A$, and there exists a unique nonconstant monic irreducible polynomial

$$f = T^d + f_1 T^{d-1} + \cdots + f_d \quad \text{with} \quad f_i \in k'$$

such that $s(R[y/x] \cap M^*) = fA$. Take $f_i' \in J$ with $s(f_i') = f_i$, and let

$$y^* = (y/x)^d + f_1'(y/x)^{d-1} + \cdots + f_d'.$$

Now R^* is the quotient ring of $R[y/x]$ with respect to the maximal ideal $R[y/x] \cap M^*$ in $R[y/x]$; consequently (x, y^*) is a basis of M^*, and, upon letting $s^* : R^* \to R^*/M^*$ be the canonical epimorphism and identifying $s^*(R)$ with k', we have that $R^*/M^* = k'(s^*(y/x))$, $s^*(y/x)$ is algebraic over k', and f is the minimal monic polynomial of $s^*(y/x)$ over k'.

Given any element G in R we can expand G as a formal power series $H(x, y)$ in (x, y) with coefficients in J; since $G \in R^*$, we can also expand G as a formal power series $H^*(x, y^*)$ in (x, y^*) with coefficients

in a suitable coefficient set J^* of R^*. In our older proofs we needed to show that if H satisfies certain structural conditions then H^* satisfies certain other structural conditions (for instance see (2.5) of [4], (1.5) of [5], and §7 of [11]); there we were dealing with the case when R/M is algebraically closed (and hence with the case when $d = 1$, the equation $y^* = (y/x) + f_1'$ expressing the quadratic transformation is linear in y/x,
8 $R^*/M^* = R/M$, and one may take $J^* = J$). The lemma enables us to do the same sort of thing in the general case, i.e. when R/M is not necessarily algebraically closed and we may have $d > 1$.

In passing, it may be remarked that if R is of nonzero characteristic and R/M is imperfect then, in general, it is not possible to extend a coefficient field of the completion of R to a coefficient field of the completion of R^*.

4 Another aspect of the new proof.

Another difference between the new modified proof and the older proofs is that the new modified proof gives a unified treatment for zero characteristic and nonzero characteristic; this is done by letting the characteristic exponent play the role previously played by the characteristic. To illustrate this very briefly, consider a hypersurface given by $F(Z) = 0$ where $F(Z)$ is a nonconstant monic polynomial in an indeterminate Z with coefficients in a regular local ring R, i.e.

$$F(Z) = Z^m + \sum_{i=1}^{m} F_i Z^{m-i} \quad \text{with} \quad F_i \in R.$$

Let M be the maximal ideal in R and suppose that $F_i \in M^i$ for all i. Now the hypersurface given by $F(Z) = 0$ has a point of multiplicity m at the "origin", and one wants to show that, by a suitable sequence of monoidal transformations, the multiplicity can be decreased.

In the previous proofs of this, dealing with zero characteristic (for instance see [16], [18], [12], and (5.5) to (5.8) of [10]), F_1 played a dominant role. In our older proofs, dealing with nonzero characteristic (for instance see [9] and [11]), the procedure was to reduce the problem to the case when m is a power of the characteristic and then to do that case by letting F_m play the dominant role.

In the new modified proof we directly do the general case by letting F_q play the dominant role where q is the greatest positive integer such that q is a power of the characteristic exponent of R/M and q divides m. Note that on the one hand, if R/M is of zero characteristic (or, more generally, if m is not divisible by the characteristic of R/M) then $q = 1$;
and on the other hand, if R/M is of nonzero characteristic and m is a 9
power of the characteristic of R/M then $q = m$.

5 More general surfaces. Previously, in ([1], [5], [6], [7], [8]), I had proved Surface Resolution also in the arithmetical case, i.e. for "surfaces" defined over the ring of integers; in fact what I had proved there was slightly more general, namely, Surface Resolution for "surfaces" defined over any pseudogeometric Dedekind domain k satisfying the condition that k/P is perfect for every maximal ideal P in k. In view of the new modified proof spoken of in §1, this last condition can now be dropped. The final result which we end up with can be stated using the language of models (alternatively, one could use the language of schemes), and is thus:

SURFACE RESOLUTION OVER EXCELLENT RINGS. *Let k be an excellent (in the sense of Grothendieck, see* (1.2) *of [10]) noetherian integral domain. Let K be a function field over k such that* $\dim_k K = 2$*; (by definition,* $\dim_k K =$ *the Krull dimension of $k +$ the transcendence degree of K over k). Let Y be any projective model of K over k. Then there exists a projective nonsingular model Y' of K over k such that Y' dominates Y.*

In ([7], [9], [10], [11]) I had proved Embedded Surface Resolution for models over any excellent noetherian integral domain k such that for every maximal ideal P in k we have that k/P has the same characteristic as k, and k/P is perfect. In view of the new modified proof spoken of in §1 the condition that k/P be perfect can now be dropped. What we end up with can be stated thus:

EMBEDDED SURFACE RESOLUTION OVER EQUICHARACTERISTIC EXCELLENT RINGS. *Let k be an excellent noetherian integral domain such that for* *every maximal ideal P in k we have that k/P has the same characteristic as k. Let K be a function field over k such that* $\dim_k K = 3$. *Let X be a projective nonsingular model of K over k, and let Y be a surface in X. Then there exists a finite sequence $X \to X_1 \to X_2 \to \ldots \to X_t \to X'$ of monoidal transformations, with irreducible nonsingular centers, such that the total transform of Y in X' has only normal crossings and the proper transform of Y in X' is nonsingular.*

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THE IMPLICIT FUNCTION THEOREM IN ALGEBRAIC GEOMETRY

By M. Artin*

Several years ago, Matsusaka introduced the concept of *Q-variety* in 13
order to study equivalence relations on algebraic varieties. *Q-varieties* are essentially quotients of algebraic varieties by algebraic equivalence relations. The theory of these structures is developed in Matsusaka [24]. In this paper, we discuss a special case of Matsusaka's notion in the context of arbitrary schemes. We call the structure *scheme for the etale topology,* or *algebraic space.* One obtains an algebraic space from affine schemes via gluing by etale algebraic functions, i.e. via an etale equivalence relation. Thus the concept is similar in spirit to that of *Nash manifold* [29], [5]. It is close to the classical concept of variety, and gives a naturally geometric object. In particular, an algebraic space over the field of complex numbers has an underlying analytic structure (1.6).

We have tried to choose the notion most nearly like that of scheme with which one can work freely without projectivity assumptions. The assertion that a given object is an algebraic space will thus contain a lot of information. Consequently, the definition given is rather restrictive, and interesting structures such as Mumford's moduli topology [26] have been excluded (for the moment, let us say), as being not scheme-like enough.

Our point of view is that a construction problem should be solved first in the context of algebraic spaces. In the best cases, one can deduce a posteriori that the solution is actually a scheme. We give some criteria for this in Section 3, but the question of whether a given algebraic space is a scheme may sometimes be very delicate. Thus a construction as algebraic space simply ignores a difficult and interesting side of the problem. On the other hand, it cannot be said that a construction as
scheme solves such a problem completely either. For one wants to prove 14

*Sloan foundation fellow.

that the result is projective, say, and where possible to give a description via explicit equations; and projectivity can presumably be shown as easily for an algebraic space as for a scheme (cf. (3.4) in this connection). The question of what constitutes a solution is thus largely a matter of fashion.

We give here an outline of a theory of algebraic spaces, of which details will be published elsewhere. The foundations of this theory, very briefly indicated in Sections 1, 2, are being developed jointly with D. Knutson.[†] Section 4 contains a fundamental result on approximation of formal sections locally for the etale topology, with some applications. In Section 5, we give the basic existence theorem (5.2) for algebraic spaces. This theorem allows one to apply deformation theory methods directly to global modular problems, in the context of algebraic spaces. Various applications are given in Section 6.

In Sections 4–6, we assume that the schemes considered are locally of finite type over a field. The techniques are actually available to treat the case of schemes of finite type over an excellent discrete valuation ring, so that the case that the base is Spec $\mathbf{Z}$ should be included. However, all details have not been written out in that case.

1 Schemes for the etale topology. We assume throughout that the base scheme S is *noetherian.*

Let

$$F : (S\text{-schemes})^0 \to (\text{Sets}) \tag{1.1}$$

be a (contravariant) functor. When $X = \operatorname{Spec} A$ is an affine S-scheme, we will often write $F(A)$ for $F(X)$. Recall that F is said to be a *sheaf for the etale topology* if the following condition holds:

1.2 Let $U_i \to V (i \in I)$ be etale maps of S-schemes such that the union of their images in V. Then the canonical sequence of maps

$$F(V) \to \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \underset{V}{\times} U_j)$$

15 is exact.

[†]Cf. D. Knutson: *Algebraic spaces,* Thesis, M.I.T. 1968 (to appear).

We assume the reader familiar with the basic properties of this notion (cf. [6] I, II, [7] IV).

A morphism $f : U \to F$ (i.e. an element $f \in F(U)$) of an S-scheme to a functor F (1.1) is said to be *representable by etale surjective maps* if for every map $V \to F$, where V is an S-scheme, the fibred product $U \underset{F}{\times} V$ (considered as a contravariant functor) is representable by a scheme, and if the projection map $U \underset{F}{\times} V \to V$ is an etale surjective map.

Definition 1.3. *An S-scheme for the etale topology, or an algebraic space over S, locally of finite type, if a functor*

$$X : (S\text{-schemes})^0 \to (\text{Sets})$$

satisfying the following conditions :

(1) *X is a sheaf for the etale topology on (S-schemes).*

(2) *X is locally representable : There exists an S-scheme U locally of finite type, and a map $U \to X$ which is representable by etale surjective maps.*

It is important not to confuse this notion of algebraic space S with that of scheme over S whose structure map to S is etale. There is scarcely any connection between the two. Thus *any* ordinary S-scheme locally of finite type is an algebraic space over S.

We will consider only algebraic spaces which are locally of finite type over a base, and so we drop that last phrase.

Most algebraic spaces X we consider will satisfy in addition some separation condition :

Separation Conditions 1.4. With the notation of (1.3), consider the functor $U \underset{X}{\times} U$. This functor is representable, by (1.3) [2]. The algebraic space X is said to be

(i) *separated* if $U \underset{X}{\times} U$ is represented by a closed subscheme of $U \underset{S}{\times} U$;

(ii) *locally separated*, if $U \underset{X}{\times} U$ is represented by a locally closed sub- **16**
scheme of $U \underset{S}{\times} U$;

(iii) *locally quasi-separated*, if the map $U \underset{X}{\times} U \to U \underset{S}{\times} U$ is of finite type.

It has of course to be shown that these notions are independent of U. Although the general case, and the case that (iii) holds, are of considerable interest, we will be concerned here primarily with cases (i) and (ii).

Note that $U \underset{X}{\times} U = R$ is the graph of an *etale* equivalence relation on U (meaning that the projection maps are etale). It follows from general sheaf-theoretic considerations [6, II.4.3] that in fact X is the quotient U/R as sheaf for the etale topology on the category (S-schemes). Conversely, any etale equivalence relation defines an algebraic space $X = U/R$. Thus we may view an algebraic space as given by an *atlas* consisting of its *chart* U (which may be taken to be a sum of affine schemes) and its *gluing data* $R \rightrightarrows U$, an etale equivalence relation. The necessary verifications for this are contained in the following theorem, which is proved by means of Grothendieck's descent theory [14, VIII].

Theorem 1.5. *Let U be an S-scheme locally of finite type and let $R \rightrightarrows U$ be an etale equivalence relation. Let $X = U/R$ be the quotient as sheaf for the etale topology. Then the map $U \to X$ is represented by etale surjective maps. Moreover, for any maps $V \to X$, $W \to X$, where V, W are schemes, the fibred product $V \underset{X}{\times} W$ is representable.*

One can of course re-define other types of structure, such as that of analytic space by introducing atlases involving etale equivalence relations $R \rightrightarrows U$. However, it is a simple exercise to check that in the separated and locally separated cases, i.e. those in which R is immersed in $U \times U$, this notion of analytic space is not more general than the usual one, so that every separated analytic etale space is an ordinary analytic space. Thus we obtain the following observation, which is important for an intuitive grasp of the notion of algebraic space.

17 **Corollary 1.6.** *Suppose $S = \operatorname{Spec} \mathbf{C}$, where $\mathbf{C}$ is the field of complex numbers. Then every (locally) separated algebraic space X over S has an underlying structure of analytic space.*

Examples 1.7. (i) If G is a finite group operating freely on an S-scheme U locally of finite type, then the resulting equivalence relation $R = G \times U \rightrightarrows U$ is obviously etale, hence U/R has the structure of an algebraic space. Thus one can take for instance the example of Hironaka [16] of a nonsingular 3-dimensional variety with a free operation of $\mathbf{Z}/2$ whose quotient is not a variety.

(ii) Let $S = \operatorname{Spec} \mathbf{R}[x]$, $U = \operatorname{Spec} \mathbf{C}[x]$. Then $U \underset{S}{\times} U \approx U \amalg U$, where say the first U is the diagonal. Now put $V = \operatorname{Spec} \mathbf{C}[x, x^{-1}]$. Then $R = U \amalg V \to U \amalg U$ is an etale equivalence relation on U. The quotient X is locally separated. It is isomorphic to S outside the origin $x = 0$. Above the origin, X has two geometric points which are conjugate over $\mathbf{R}$. Here $\mathbf{R}$ and $\mathbf{C}$ denote the fields of real and complex numbers respectively.

(iii) Let $S = \operatorname{Spec} k$, where k is a field of characteristic not 2, $U = \operatorname{Spec} k[x]$, and let $R = \Delta \amalg \Gamma \to U \times U$, where Δ is the diagonal, and where Γ is the complement of the origin in the anti-diagonal

$$\Gamma = \{(x, -x) | x \neq 0\}.$$

Then $X = U/R$ is locally quasi-separated, but not locally separated.

Proposition 1.8. *Let*

$$\begin{tikzcd} X \arrow[dr] & & Z \arrow[dl] \\ & Y & \end{tikzcd}$$

be a diagram of algebraic spaces over S. Then the fibred product $X \underset{Y}{\times} Z$ is again an algebraic space.

2 Elementary notions.

Definition 2.1. *A property P of schemes is said to be local for the etale topology if*

(i) *$U' \to U$ an etale map and $U \in P$, implies $U' \in P$;* **18**

(ii) *$U' \to U$ etale and surjective and $U' \in P$, implies $U \in P$.*

A property P of morphisms $f : U \to V$ of schemes is said to be local for the etale topology if

(i) *$V' \to V$ etale and $f \in P$, implies $f \underset{V}{\times} V' \in P$;*

(ii) *let $\phi : U' \to U$ be etale and surjective. Then $f \in P$ if and only if $f\phi \in P$.*

Clearly, any property of S-schemes which is local for the etale topology, carries over to the context of algebraic spaces. One just requires that the property considered hold for the scheme U of (1.3). Examples of this are *reduced, geometrically unibranch, normal, nonsingular,* etc...

Similarly, any property of morphisms of schemes which is local for the etale topology carries over to the case of algebraic spaces. Thus the notions of *locally quasi-finite, unramified, flat, etale, surjective,* etc... are defined. In particular, an algebraic space X comes with a natural *etale topology* [1], whose objects are etale maps $X' \to X$ with X' an algebraic space, and whose covering families are surjective families. In this language, the map $U \to X$ of (1.3) is an etale covering of X by a scheme.

We extend the notion of *structure sheaf* $\mathcal{O}_X$ to algebraic spaces with the above topology, in the obvious way. For an etale morphism $U \to X$ where U is a scheme, we put

$$\Gamma(U, \mathcal{O}_X) = \Gamma(U, \mathcal{O}_U)$$

where the term on the right hand side is understood to have its usual meaning. Then this definition is extended to all etale maps $X' \to X$ so as to give a sheaf for the etale topology (applying ([6] VII.2.c)).

Definition 2.2. *A morphism $f : Y \to X$ of algebraic spaces over S is called an immersion (resp. open immersion, resp. closed immersion) if for any map $U \to X$ where U is a scheme, the product $f \underset{X}{\times} U : Y \underset{X}{\times} U \to U$ is an immersion (resp......).*

Using descent theory ([14] VIII), one shows that it is enough to 19
check this for a single etale covering $U \to X$ of X by a scheme. The notions of open and of closed *subspaces* of X are defined in the obvious way, as equivalence classes of immersions. Thus X has, in addition to its etale topology above, also a *Zariski topology* whose objects are the open subspaces.

Definition 2.3. *A point x of an algebraic space X is an isomorphism class of S-monomorphisms $\phi : \operatorname{Spec} K \to X$, where K is a field.*

The map ϕ is said to be *isomorphic* to $\phi' : \operatorname{Spec} K' \to X$ if there is a map $\epsilon : \operatorname{Spec} K' \to \operatorname{Spec} K$ such that $\phi' = \phi\epsilon$. The map ϵ is then necessarily a uniquely determined isomorphism. We refer to the field K, unique up to unique isomorphism, as the *residue field* of x, denoted as usual by $k(x)$. It is easily seen that this definition is equivalent to the usual one when X is an ordinary scheme.

Theorem 2.4. *Let $x \in X$ be a point. There is an etale map $X' \to X$ with X' a scheme, and a point $x' \in X'$ mapping to $x \in X$, such that the induced map on the residue fields $k(x) \to k(x')$ is an isomorphism.*

Thus every $x \in X$ admits an *etale neighborhood* (X', x') (without residue field extension !) which is a scheme. The category of all such etale neighborhoods is easily seen to the filtering.

Definition 2.5. *The local ring of X at a point x (for the etale topology) is defined to be*

$$\mathcal{O}_{X,x} = \varinjlim_{(X',x')} \Gamma(X', \mathcal{O}_X)$$

where (X', x') runs over the category of etale neighborhoods of x.

This ring is henselian ([8] IV_4.18). If X is a scheme, it is the henselization of the local ring of X at x for the Zariski topology. However, if we define a local ring for the Zariski topology of an algebraic space in the obvious way, the ring $\mathcal{O}_{X,x}$ will not in general be its henselizaton.

20 A *quasi-coherent sheaf* F on X is a sheaf of $\mathcal{O}_X$-modules on the etale topology of X which induces a quasi-coherent sheaf in the usual sense on each scheme X' etale over X. The notion of *proper map* of algebraic spaces is defined exactly as with ordinary schemes. One has to develop the cohomology theory of quasi-coherent sheaves and to prove the analogues in this context of Serre's finiteness theorems and of Grothendieck's existence theorem for proper maps ([8] III_1.3.2.1, 4.1.5, 5.1.4). However, the details of this theory are still in the process of being worked out, and so we will not go into them here[‡].

3 Some cases in which an algebraic space is a scheme.

The question of whether or not a given algebraic space X over S is a scheme is often very delicate. In this section we list a few basic cases in which the answer is affirmative, but we want to emphasize that these cases are all more or less elementary, and that we have not tried to make the list complete. Some very delicate cases have been treated (cf. [25], [30]), and a lot remains to be done.

One has in complete generality the following.

Theorem 3.1. *Let X be a locally quasi-separated algebraic space over S. Then there is a dense Zariski-open subset $X' \subset X$ which is a scheme.*

Theorem 3.2. *Let $X_0 \subset X$ be a closed subspace defined by a nilpotent ideal in $\mathcal{O}_X$. Then X_0 a scheme implies that X is one. In particular, X is a scheme if X_{red} is one.*

Using descent theory ([14] VIII), one proves

Theorem 3.3. *Let $f : X \to Y$ be a morphism of algebraic spaces. If f is separated and locally quasi-finite, and if Y is a scheme, then X is a scheme.*

Corollary 3.4. *An algebraic space X which is quasi-projective (resp. quasi-affine) over a scheme Y is a scheme.*

21 Via (3.1), (3.2), and Weil's method [34] of construction of a group from birational data, we have

[‡]Cf. Knutson, op. cit.

Theorem 3.5. *Let $S = \operatorname{Spec} A$, where A is an artin ring, and let X be a group scheme over S for the etale topology. Then X is a scheme.*

4 Approximation of formal sections.

In this section we assume that the base scheme S is of finite type over a filed k. We consider primarily some questions which are *local for the etale topology,* for which there is no difference between schemes and *algebraic spaces*.

Let X be an algebraic space or a scheme locally of finite type, and let $s \in S$ be a point. By *formal section* $\overline{f}$ of X/S at s we mean an S-morphism

$$\overline{f} : \widehat{S} \to X$$

where $\widehat{S} = \operatorname{Spec} \widehat{O}_{S,s}$. By *local section* (for the etale topology) of X/S at s we mean a triple

$$(S', s', f)$$

where (S', s') is an etale neighborhood of s in S and where

$$f : S' \to X$$

is an S-morphism. A local section induces a formal section since $\widehat{O}_{S,s} \approx \widehat{O}_{S',s'}$. Finally, we introduce the schemes

$$S_n = \operatorname{Spec} O_{S,s}/\mathfrak{m}^{n+1}, \quad \mathfrak{m} = \max(O_{S,s}).$$

They map to $\widehat{S}$. We will say that two (local or formal) sections are *congruent modulo* $\mathfrak{m}^{n+1}$ if the composed maps

$$S_n \to X$$

are equal.

The basic result is the following. It answers in a special case the question raised in [3].

Theorem 4.1. *With the above notation, let $\overline{f}$ be a formal section of X/S at s. Then there exists a local section f (for the etale topology) such that*

$$f \equiv \overline{f} (\operatorname{mod} \mathfrak{m}^{n+1}).$$

It can be sharpened as follows. 22

Theorem 4.2. *Suppose X/S of finite type. Let n be an integer. There exists an integer $N \geqslant n$ such that for every S-morphism*

$$f' : S_N \to X$$

there is a local section f such that

$$f \equiv f' (\mathrm{mod}\ \mathfrak{m}^{n+1}).$$

We remark that (4.2) has been previously proved by Greenberg [10] and Raynaud in the case that S is the spectrum of an arbitrary excellent discrete valuation ring, and we conjecture that in fact the following holds.

Conjecture 4.3. *Theorems (4.1), (4.2) can be extended to the case of an arbitrary excellent scheme S.*

Theorem 4.1 allows one to approximate any algebraic structure given over $\widehat{S}$ by a structure over S locally for the etale topology, provided the algebraic structure can be described by solutions of finitely many equations. This condition is usually conveniently expressed in terms of a *functor locally of finite presentation*. We recall that (following Grothendieck) a functor F (1.1) is said to be locally of finite presentation if for every filtering inverse system of affine S-schemes $\{\mathrm{Spec}\, B_i\}$, we have

$$F(\varinjlim B_i) = \varinjlim F(B_i). \tag{4.4}$$

From (4.1) follows

Theorem 4.5. *Suppose that the functor F (1.1) is locally of finite presentation. Let n be an integer. Then for every $\overline{z} \in F(S')$, there exists an etale neighborhood (S', s') of s in S and an element $z' \in F(S')$ such that*

$$\overline{z} \equiv z' \quad (\mathrm{mod}\ \mathfrak{m}^{n+1}),$$

i.e. such that the elements of $F(S_n)$ induced by $\overline{z}$, z' are equal.

There are many applications of the above results, of which we will list a few here, without attempting to give the assertions in their greatest generality. For a more complete discussion, see [4].

Theorem 4.6. *Let X_1, X_2 be schemes of finite type over k, and let* 23
$x_i \in X_i$ be closed points. Suppose $\widehat{X}_1$, $\widehat{X}_2$ are k-isomorphic, where $\widehat{X}_i = \operatorname{Spec} \widehat{O}_{X_i,x_i}$. Then X_1, X_2 are locally isomorphic for the etale topology, i.e. there are etale neighborhoods (X_i', x_i') of x_i in X_i which are isomorphic.

One obtains the following result, a conjecture of Grauert which was previously proved in various special cases ([31], [17], [2]), by applying results of Hironaka and Rossi [18], [15].

Theorem 4.7. *Let $\overline{A}$ be a complete noetherian local k-algebra with residue field k whose spectrum is formally smooth except at the closed point. Then $\overline{A}$ is algebraic, i.e. is the completion of a local ring of an algebraic scheme over k.*

A simple proof of the base change theorem for π_1 ([6] XII) results from (4.5) and Grothendieck's theory of specialization of the fundamental group.

Theorem 4.8. *Let $f : X \to Y$ be a proper morphism of finite presentation, where Y is the spectrum of an equicharacteristic hensel ring. Let X_0 be the closed fibre of X/Y. Then*

$$\pi_1(X_0) \approx \pi_1(X).$$

5 Algebrization of formal moduli.

Let $S = \operatorname{Spec} k$, where k is a field, and let F (1.1) be a functor. Let

$$z_0 \in F(k')$$

be an element, where k' is a finite field extension of k. An *infinitesimal deformation* of z_0 is a pair (A, z) with $z \in F(A)$, where

5.1

(i) *A is a finite local k-algebra with residue field k';*

(ii) *the element z induces z_0 by functorality.*

We say that F is *pro-representable at* z_0 if the functor F_{z_0} assigning to each algebra A (5.1)(i) the set of infinitesimal deformations (A, z) is pro-representable [11], and we will consider only the case that F is pro-represented by a *noetherian* complete local k-algebra $\overline{A}$, (with residue field k'). If in addition there is an element $\overline{z} \in F_{z_0}(\overline{A})$ which is universal
24 with respect to infinitesimal deformations, then we say that F is *effectively* pro-representable at z_0. We recall that this is an extra condition; the pro-representability involves only a compatible system of elements $z_n \in F(\overline{A}/\mathfrak{m}^{n+1})$. However, one can often apply Grothendieck's existence theorem ([8] III_1.5.1.4) to deduce effective pro-representability from pro-representability. We will say that F is *effectively pro-representable* if F is pro-representable on the category of finite k-algebras, and if each local component $\overline{A}$ (cf. [11]) admits a universal element $\overline{z} \in F(\overline{A})$.

The following is the main result of the section. It allows one to construct local modular varieties in certain cases, which reminds one of the theorem of Kuranishi [20] for analytic varieties. (However, Kuranishi's theorem is non-algebraic in an essential way.)

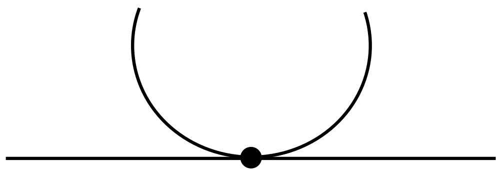
Theorem 5.2. (Algebrization of formal moduli) *With the above notation, suppose that F is locally of finite presentation and that F is effectively pro-representable at $z_0 \in F(k')$, by a pair $(\overline{A}, \overline{z})$ where $\overline{A}$ is a complete noetherian local ring. Then there is an S-scheme X of finite type, a closed point $x \in X$, a k-isomorphism $k(x) \approx k'$, and an element $z \in F(X)$ inducing $z_0 \in F(k')$, such that the triple (X, x, z) pro-represents F at z_0.*

One obtains from (5.2) in particular a canonical isomorphism

$$\overline{A} \approx \widehat{O}_{X,x}.$$

In practice, it usually happens that the element $\widehat{z} \in F(\overline{A})$ induced from z via this isomorphism is $\overline{z}$, but a slight extra condition is needed to guarantee this. For, the universal element $\overline{z} \in F(\overline{A})$ may not be uniquely determined. An example is furnished by two lines with an "infinite order contact".

Example 5.3.



By this we mean the ind-object X obtained as the limit **25**

$$X_1 \to X_2 \to \ldots \to X_n \to \ldots$$

where X_n is the locus in the plane of

$$y(y - x^n) = 0$$

and the map $X_n \to X_{n+1}$ sends

$$(x, y) \mapsto (x, xy).$$

We take X as ind-object on the category of affine S-schemes. Thus by definition

$$\operatorname{Hom}(Z, X) = \varinjlim_i \operatorname{Hom}(Z, X_i) \tag{5.4}$$

for any *affine* S-scheme Z. For arbitrary Z, a map $Z \to X$ is given by a compatible set of maps on an affine open covering.

However, if such examples are avoided, then (X, x, z) is essentially unique.

Theorem 5.5. (Uniqueness). *With the notation of (5.2), suppose that in addition the universal element $\overline{z} \in F(\overline{A})$ is uniquely determined. Then the triple (X, x, z) is unique up to unique local isomorphism, for the etale topology, at x.*

Suppose for the moment that the functor F considered in (5.2) is represented by an algebraic space over S, and that the map $z_0 : \operatorname{Spec} k' \to F$ is a monomorphism, so that z_0 represents a point, (denote it also by z_0) of F (2.3). Then it is easily seen that the map

$$z : X \to F$$

of (5.2) is etale (Section 2) at the point $x \in X$. Thus if we replace X by a suitable Zariski open neighborhood of x, we obtain an etale neighborhood of z_0 in F.

Now, taking into account the definition (1.3), the representability of F by an algebraic space will follow from the existence of a covering by etale neighborhoods. Thus it is intuitively clear that one will be able to derive criteria of representability in the category of algebraic spaces
26 from (5.2), with effective pro-representability as a starting point. The following is such a criterion. It is proved in a rather formal way from (5.2).

Theorem 5.6. *Let F be a functor* (1.1), *with $S = \operatorname{Spec} k$. Then F is represented by a separated (respectively locally separated) algebraic space if and only if the following conditions hold.*

[0] *F is a sheaf for the etale topology.*

[1] *F is locally of finite presentation.*

[2] *F is effectively pro-representable by complete noetherian local rings.*

[3] *Let X be an S-scheme of finite type, and $z_1, z_2 \in F(X)$. Then the kernel of the pair of maps*

$$z_i : X \to F$$

is represented by a closed subscheme (resp. a subscheme) of X.

[4] *Let R be a k-geometric discrete valuation ring with field of fractions K, and let A_K be a finite local K-algebra with residue field*

K. Suppose given elements $z_1 \in F(R)$, $z_2 \in F(A_K)$ which induce the same element of $F(K)$. Then there is a finite R-algebra A, an augmentation $A \to R$, an isomorphism $A_K \approx K \otimes_R A$, and an element $z \in F(A)$ which induces z_1, z_2.

[5] *Let X be a scheme of finite type over k and let $z \in F(X)$. The condition that the map $z : X \to F$ be etale is an open condition on X.*

Here a map $X \to F$ is called *etale at $x \in X$* if for every map $Y \to F$ there is an open neighborhood U of x in X such that the product $U \underset{F}{\times} Y$ is represented by a scheme etale over Y.

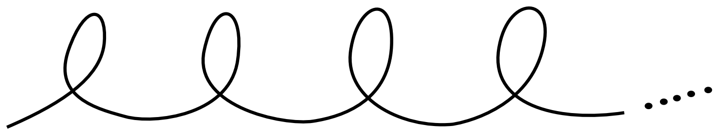
Except for [4], the conditions are modifications of familiar ones used in previous results of Murre [27] and Grothendieck [28]. Note that condition [0] is just the natural one which assures that F extend to a functor on the category of etale schemes.

The result should be taken primarily as a guide, which can be modified in many ways. This is especially true of conditions [2]-[5]. They
can be rewritten in terms of standard deformation theory. Thus for in- **27**
stance condition [2] can be revised by writing out the conditions of pro-representability of Schlessinger [32] and Levelt [21], and condition [3] can be rewritten by applying conditions [0]-[5] to the kernel functor in question. Condition [4] is usually quite easy to verify by deformation theory. The condition which is most difficult to verify as it stands is conditions [5], but this too can be interpreted by infinitesimal methods. In fact, condition [5] can sometimes be dispensed with completely. One has

Theorem 5.7. *Let F be a functor on S-schemes satisfying [0]-[4] of (5.2). Suppose that the complete local-rings $\overline{A}$ of condition [2] are all geometrically unibranch and free of embedded components (e.g. normal). Then [5] holds as well, i.e. F is representable by an algebraic space.*

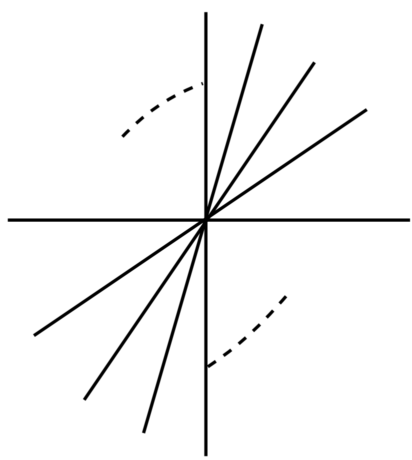
Here are some examples which illustrate the various conditions of (5.6) and the relations between them. To begin with, all conditions but [3] hold in example (5.3).

Example 5.8. The ind-object X (cf. (5.4)) obtained by introducing more and more double points into a line:



Neither condition [3] nor [5] hold.

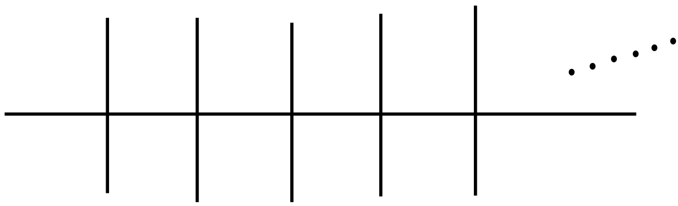
Example 5.9. The ind-object X (cf. (5.4)) obtained as union



28 of more and more lines through the origin in the plane.

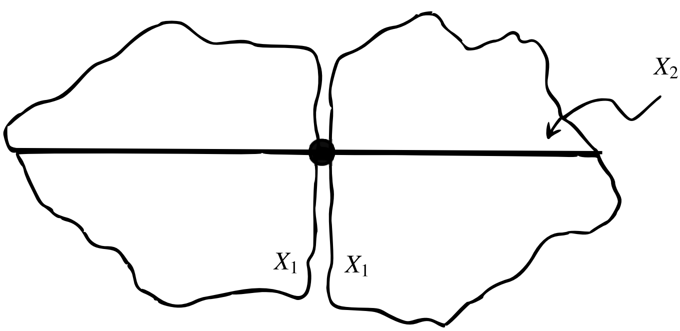
All conditions hold except that in condition [2] the functor is not effectively pro-representable at the origin. Its formal moduli there exist however; they are those of the plane.

Example 5.10. The ind-object X (cf. (5.4)) obtained by adding more and more lines crossing a given line at distinct points.



All conditions but [5] hold.

Example 5.11. The object X is the union of the two schemes $X_1 = \operatorname{Spec} k[x,y][1/x]$ and $X_2 = \operatorname{Spec} k[x,y]/(y)$ (the x-axis) in the (x,y)-plane.



It is viewed as the following sub-object of the (x,y)-plane : A map $f : Z \to \operatorname{Spec} k[x,y]$ represents a map to X if there is an open covering $Z = Z_1 \cup Z_2$ of Z such that the restriction of f to Z_i factors through X_i. In this example, all conditions but [4] hold.

29

6 Applications. Our first application is to Hilbert schemes. We refer to [12] for the definitions and elementary properties. Recall that Grothendieck [12] has proved the existence and (quasi)-projectivity of Hilbert schemes $\operatorname{Hilb} X/S$, $\operatorname{Quot} F/X/S$, etc..., when X is (quasi)-projective over S. Moreover, Douady [9] showed their existence as analytic spaces when $X \to S$ is a morphism of analytic spaces. Now if X is not projective over S, one can not expect $\operatorname{Hilb} X/S$ to be a scheme, in general. For, consider the example of Hironaka [16] of a nonsingular variety

X over a field and admitting a fixed point free action of $\mathbf{Z}/2$ whose quotient Y is not a scheme (though it has a structure of algebraic space). Clearly Y is the sub-object of $\operatorname{Hilb} X/S$ which parametrizes the pairs of points identified by the action, whence $\operatorname{Hilb} X/S$ is not a scheme, by (3.4). Thus it is natural to consider the problem in the context of etale algebraic spaces.

Theorem 6.1. *Let $f : X \to S$ be a morphism locally of finite type of noetherian schemes over a field k. Let F be a coherent sheaf on X. Then* $\operatorname{Quot} F/X/S$ *is represented by a locally separated algebraic space over S. It is separated if f is. In particular,* $\operatorname{Hilb} X/S$ *is represented by such an algebraic space.*

Assuming that the cohomology theory for algebraic spaces goes through as predicted, one will be able to replace f above by a morphism of algebraic spaces over S, and will thus obtain an assertion purely in that category[§].

Next, we consider the case of relative Picard schemes (cf. [13] for definitions) for proper maps $f : X \to S$. An example of Mumford ([13] VI) shows that $\operatorname{Pic} X/S$ is in general not a scheme if the geometric fibres of f are reducible. The following result is proved from (5.6) using the general techniques of [13]. The fact that condition [3] of (5.6) holds for $\operatorname{Pic} X/S$ had been proved previously by Raynaud [30].

Theorem 6.2. *Let $f : X \to S$ be a proper map of noetherian schemes over a field k. Suppose f cohomologically flat in dimension zero, i.e.*
30 *that $f_*\mathcal{O}_X$ commutes with base change. Then* $\operatorname{Pic} X/S$ *is represented by a locally separated algebraic space over S.*

Again, f can conjecturally be replaced by a morphism of algebraic spaces. When $S = \operatorname{Spec} k$, we obtain from (3.5) the following theorem of Murre and Grothendieck [27].

Theorem 6.3. *Let $f : X \to S$ be a proper map of schemes, where $S = \operatorname{Spec} k$. Then* $\operatorname{Pic} X/S$ *is represented by a scheme.*

This proof is completely abstract, and is strikingly simple even in the case that X is projective, when one can give a "classical" proof using

[§]These results are now available, cf. Knutson, op. cit.

Hilbert schemes. The verification can be reduced to a minimum using the following result. It is the theorem of Murre ([27], cf. also [23]). However, in Murre's formulation there is a condition of existence of a "module" for a map of a curve to the group, which is difficult to verify. Here we can just drop that condition completely, and we can remove the hypothesis that the groups be abelian.

Theorem 6.4. *Let F be a contravariant functor from (S-schemes) to (groups), where $S = \operatorname{Spec} k$. Then F is represented by a scheme locally of finite type over k if and only if*

[0] *F is a sheaf for the etale topology.*

[1] *F is locally of finite presentation.*

[2] *F is effectively pro-representable by a sum of complete noetherian local rings.*

[3] *Let X be an S-scheme of finite type, and $z_1, z_2 \in F(X)$.*

Then the kernel of the pair of maps

$$z_i : X \to F$$

is represented by a subscheme of X.

As a final application, one obtains the criterion of representability of unramified functors of Grothendieck [28] in the case that the base S is of finite type over a field. Here again, one can conclude a posteriori that the algebraic space is actually a scheme, by (3.3). Since the statement 31
[28] is rather technical, we will not repeat it here.

7 Passage to quotient.

The following result shows that the definition of algebraic space could not be generalized in an essential way by allowing flat equivalence relations. The theorem was proved independently by Raynaud [30] and me. It shows the strong similarity between algebraic spaces and Q-varieties [24]. However Mumford has pointed out to us that there is a beautiful example due to Holmann ([19] p.342) of a Q-variety which admits no underlying analytic structure.

Theorem 7.1. *Let U be an S-scheme of finite type and let $i : R \to U \underset{S}{\times} U$ be a flat equivalence relation. (By this we mean that i is a monomorphism, R is a categorical equivalence relation, and the projection maps $R \rightrightarrows U$ are flat.) Let X be the quotient U/R as sheaf for the fppf topology (cf. [7] IV). Then X is represented by an algebraic space over S. It is separated (resp. locally separated) if $i : R \to U \underset{S}{\times} U$ is a closed immersion (resp. an immersion). Moreover, X is a universal geometric quotient (cf. [25]).*

One wants of course to have more general results on quotients by pre-equivalence relations and by actions of algebraic groups. In the analytic case, there questions have been treated in detail by Holmann [19] and it seems likely that many of his results have algebraic analogues. Some algebraic results have already been obtained by Seshadri [33].

The following is an immediate corollary of (7.1).

Corollary 7.2. *Let $f : Y' \to Y$ be a faithfully flat morphism of algebraic spaces over S. Then any descent data for an algebraic space X' over Y' with respect to f is effective.*

Note that because of our definitions, the map f is locally of finite type. We have not proved the result for flat extensions of the base $S' \to S$ which are not of finite type, although in the case that S is of finite type over a field, a proof might be based on (5.6).

32 Another application is to groups in the category of algebraic spaces.

Corollary 7.3. (i) *Let $H \to G$ be a morphism of algebraic spaces of groups over S, which is a monomorphism. Assume H flat over S. Then the cokernel G/H as fppf-sheaf is represented by an algebraic space.*

(ii) *Let A, B be algebraic spaces of abelian groups flat over S, and let E be an extension of B by A, as fppf-sheaves. Then E is represented by an algebraic space.*

It follows for instance from (i) that one can define groups Ext^q on the category of algebraic spaces of abelian groups flat over S, via the definition given in MacLane ([22] p. 367), taking as distinguished exact sequences the sequences

$$0 \to A \xrightarrow{i} E \to B \to 0$$

which are exact as $fppf$-sheaves, i.e. such that i is a monomorphism and $B = E/A$. When the base S is not a field, very little is known about these Ext^q.

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DIOPHANTINE ANALYSIS AND MODULAR FUNCTIONS

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1. In 1952, Heegner published a paper [5] in which he discussed certain 35
curves parametrised by modular functions. By evaluating the modular functions at certain points, he showed that these curves had points whose coordinates were integers of certain class-fields. Unfortunately, the style of his proofs was unconvincing, so his paper has been discounted; none the less, his main assertions appear correct and interesting.

In particular, if E is an elliptic curve over the rationals $\mathbf{Q}$, parametrised by modular functions, there are well-known conjectures (see Appendix) about the group $E_{\mathbf{Q}}$ of rational points of E. At the expense of being rather special, we can be very explicit; take E in Weierstrass form

$$E : y^2 = x^3 + Ax + B, \quad A, B \text{ integers;}$$

and write E_K for the group of points of E with coordinates in a field K. If K is a number field, E_K is finitely generated; write $g(E_K)$ for the number of independent generators of infinite order. Let D be an integer, and $E^{(-D)}$ the curve $-Dy^2 = x^3 + Ax + B$; then $g(E_{\mathbf{Q}(\sqrt{-D})}) = g(E_{\mathbf{Q}}) + g(E_{\mathbf{Q}}^{(-D)})$. The conjectures assert that, for fixed A, B, the parity of $g(E_{\mathbf{Q}(\sqrt{-D})})$ and so of $g(E_{\mathbf{Q}}^{(-D)})$ depend on the sign of D and the congruence class of D modulo a power of $6(4A^3 + 27B^2)$. Heegner's paper seems at present the only hope of approaching such conjectures - at any rate, it provides infinitely many cases for which they are true.

I will give two illustrations of Heegner's argument. The best known assertion in his paper is the enumeration of the complex quadratic fields of class number 1: the complex quadratic field of discriminant D has class number 1 if and only if $D = 3, 4, 7, 8, 11, 19, 43, 67, 163$. Subsequently, the first accepted proof of this has been given by Stark [7], and
Baker [1] has given another approach. I will give a proof, essentially 36

the same as Heegner's; on the way, I will re-prove and extend classical results of Weber [9] on 'class invariants'. Afterwards, I will exhibit a family of curves related to $H/\Gamma_0(17)$, each of which have infinitely many rational points.

The theory of complex multiplication can of course be built up algebraically [3]. Though I will be using function theory, because it is traditional and probably easier, all the constructions are basically algebraic; in particular, we have an algebraic solution of the class number 1 problem.

2. From now on, p will always be a prime $p \equiv 3(4)$, $p > 3$. $j(z)$ is the modular function, defined for z in the open upper half plane H, invariant by the modular group $\Gamma(1)$, and mapping $i, \rho, i\infty$ to 1728, 0, ∞; note that $j(\frac{1}{2} + it)$ is real and negative for real $t > \sqrt{3}/2$. $\Gamma(N)$ is the subgroup of $\Gamma(1)$ consisting of maps $z \to \dfrac{az+b}{cz+d}$ with $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ (N).

If ω is quadratic over $\mathbf{Q}$, then $A\omega^2 + B\omega + C = 0$, with A, B, C integers without common factor; define $D(\omega) = |B^2 - 4AC|$. Call D a field discriminant if D is the absolute value of the discriminant of a complex quadratic field; so either $D \equiv 3(4)$ and D is square free, or $\frac{1}{4}D \equiv 1, 2(4)$ and $\frac{1}{4}D$ is square free.

I quote the standard theorems about the value of a modular function $f(\omega)$ at a complex quadratic value ω. For proofs of Theorems 1 and 2, see [3].

Theorem 1. *If $D(\omega)$ is a filed discriminant, then $j(\omega)$ is an algebraic integer and generates the class field K_1 of $\mathbf{Q}(\omega)$ over $\mathbf{Q}(\omega)$.*

Theorem 2. *If $D(\omega) = M^2 D_1$ with M integral and D_1 a field discriminant, then $j(\omega)$ generates the ring class field K_M modulo M of $\mathbf{Q}(\omega)$ over $\mathbf{Q}(\omega)$.*

Theorem 3 (Söhngen [6]). *Suppose that $f(z)$ is invariant by $\Gamma(N)$, and*
37 *the Fourier expansions of f at every cusp of $H/\Gamma(N)$ have coefficients in $\mathbf{Q}(\sqrt[N]{1})$. If $D(\omega) = M^2 D_1$ as above, then $f(\omega) \in K'_{MN}$, the ray class field modulo MN of $\mathbf{Q}(\omega)$.*

Corollary. $\mathbf{Q}(\omega, f(\omega))$ *is an abelian extension of* $\mathbf{Q}(\omega)$.

Now we set $\omega = \frac{1}{2}(1 + \sqrt{-p})$, where p is still a prime $\equiv 3(4)$, so $D(\omega) = p$. Then $K_1 = \mathbf{Q}(\omega, j(\omega))$ is the class filed of $\mathbf{Q}(\omega)$, and $j(\omega)$ is a negative real algebraic integer.

We know that $[K_1 : \mathbf{Q}(\omega)]$ is odd. The $K_M/\mathbf{Q}(\omega)$ norms of *integral* ideals of K_M prime to M are precisely the principal ideals (α) where α is an integer of $\mathbf{Q}(\omega)$ congruent modulo M to a rational integer prime to M. Accordingly, we see easily that

$$[K_2 : K_1] = \begin{cases} 1 & \text{for} \quad p \equiv 7(8) \\ 3 & \text{for} \quad p \equiv 3(8), \end{cases}$$

and

$$[K_3 : K_1] = \begin{cases} 2 & \text{for} \quad p \equiv 2(3) \\ 4 & \text{for} \quad p \equiv 1(3). \end{cases}$$

Further, $K_2' = K_2$ and $K_3' = K_3$.

All this is very classical, see [9], [6] or [3].

3. Now let us look at some particular functions.

Example 1 (Wiber [9] §125). There is a function $\gamma(z)$ invariant by $\Gamma(3)$ with $\gamma^3(z) = j(z)$ and $\gamma(z)$ real for $\mathrm{Re}(z) = \frac{1}{2}$. By Theorem 3, $\gamma(\omega) \in K_3$; but $[K_3 : K_1]$ is a power of 2, and obviously $[K_1(\gamma(\omega)) : K_1]$ is odd; so

$$\gamma(\omega) \in K_1. \tag{1}$$

Example 2. There is a function $\sigma(z)$ invariant by $\Gamma(48)$ related to $j(z)$ by

$$(\sigma^{24}(z) - 16)^3 = \sigma^{24}(z) j(z). \tag{2}$$

If j is real and negative, $(U - 16)^3 = Uj$ has a unique real root which is positive; if $z = \frac{1}{2} + it$ with $t > \frac{1}{2}\sqrt{3}$, then $\sigma(z)$ is the unique positive real root of (2). $\sigma^{24}(z)$ is invariant by $\Gamma(2)$, and in fact $\sigma^{24}(\omega)$ generates K_2 over K_1. So far, all is well known and in [9], though the normalisation is different.

Now we restrict $p \equiv 3(8)$, so that $[K_2 : K_1] = 3$, $\sigma^{24}(\omega)$ is cu- **38**
bic over K_1. By Theorem 3, $\sigma(\omega)$ is abelian over K_1, and by (2) and Theorem 1, $\sigma(\omega)$ is an algebraic integer.

I assert that

$$\sigma^{12}(\omega) \in K_2. \tag{3}$$

For σ^{12} is quadratic over K_2, and abelian of degree 6 over K_1, so $\sigma^{12} \in K_2(a)$ with $a^2 \in K_1$. Suppose $\sigma^{12} \notin K_2$. Then $\sigma^{24} \in K_2$, so $\sigma^{12} = ab$ with $b \in K_2$. So $\sigma^{24} = a^2b^2$. So $2^{12} = N_{K_2/K_1}(\sigma^{24}) = (N_{K_2/K_1}b)^2a^6$; so $a = \pm 2^6(a^2Nb)^{-1} \in K_1$. So $\sigma^{12} \in K_2$, as required. Also, $N_{K_2/K_1}(\sigma^{12}) > 0$, so $N_{K_2/K_1}(\sigma^{12}) = 2^6$.

Repeat the argument :

$$\sigma^6(\omega) \in K_2. \tag{4}$$

Repeat it again : $\sqrt{2}\sigma^3(\omega) \in K_2$.

This is an old conjecture of Weber (see [9] §127); however, for our applications we will use no more than (4), which was already proved by Weber.

Using our first example, we have $\sigma^8(\omega) = (\sigma^{24}(\omega) - 16)/\gamma(\omega) \in K_2$, so by (4)

$$\sigma^2(\omega) \in K_2 \quad \text{for} \quad p \equiv 3(8). \tag{5}$$

Example 3 ([9] §134). There is a function $g(z)$ invariant by $\Gamma(2)$ with $g^2 = j - 1728$; $g(\frac{1}{2} + it)$ must be pure imaginary when t is real, we may take it to have positive imaginary part.

$g(\omega) \in K_2$, but K_2 is an odd extension of K_1, so $g(\omega) \in K_1$ and

$$\sqrt{(-p)}g(\omega) \in K_1 \cap \mathbf{R}. \tag{6}$$

4. Heegner applies these examples to enumerate the complex quadratic fields with class number 1, and to exhibit infinite families of elliptic curves with non-trivial rational points. In this paragraph, we will restrict $p \equiv 3(8)$; this is enough for the class number 1 problem, as the other cases are easy.

For the moment, suppose $\mathbf{Q}(\sqrt{-p})$ has class number 1, so

$$K_1 = \mathbf{Q}(\sqrt{-p}). \tag{7}$$

Write $V = \sigma^2(\omega)$. Then V satisfies a cubic equation over K_1, say 39

$$V^3 - \alpha V^2 + \beta V - 2 = 0; \tag{8}$$

since V is a real algebraic integer, α, β are real algebraic integers in K_1; so α, β are rational integers.

Since the roots of (8) are three of the roots of $V^{12} - \gamma V^4 - 16 = 0$, the left hand side of (8) divides $(V^{12} - \gamma V^4 - 16)$, so there must be a relation between α and β; it turns out to be

$$\beta^2 - 4\beta\alpha^2 + 2\alpha^4 - 2\alpha = 0.$$

So we have reduced (7) to the problem of solving the Diophantine equation $(\beta - 2\alpha^2)^2 = 2\alpha(1 + \alpha^3)$ in integers. This is easy, see [5]; the complete solution is

	α	=	0	1	1	-1	2	2
	β	=	0	0	4	2	2	14
corresponding to	p	=	3	11	67	19	43	163.

This is a complete enumeration of complex quadratic fields $\mathbf{Q}(\sqrt{-p})$ with class number 1 and $p \equiv 3(8)$.

Now we exhibit some curves with points. By (2),

$$j - 1728 = (\sigma^{24} - 64)(\sigma^{24} + 8)^2\sigma^{-24},$$

so

$$\sigma^{24} - 64 = \left(\frac{\sigma^{12}g}{\sigma^{24} + 8}\right)^2.$$

By (4) and (6), we deduce that

$$-pu^2 = v^4 - 64 \tag{9}$$

is soluble in $K_2 \cap \mathbf{R}$. But $[K_2 \cap \mathbf{R} : \mathbf{Q}]$ is odd; so the curve (9) has an odd divisor defined over $\mathbf{Q}$: it clearly has a divisor of order 2; so (9) is soluble in $\mathbf{Q}$. But (9) is a non-trivial 2-covering of the elliptic curve

$$Y^2 = X(X^2 + p^2) \tag{10}$$

which accordingly has infinitely many rational points. (These points have rather large coordinates - indeed, they account for several gaps in the tables of [2]).

A similar argument works, with appropriate modifications, when $p \equiv 7(8)$; see [5]. We have thus confirmed the main conjecture of [2], for the particular curves (10) with p a prime congruent to 3 modulo 4; for that conjecture predicts that the group of rational points of (10) should have structure $\mathbf{Z}_2 \times \mathbf{Z}$.

5. Finally we give another example, using a different method. We consider the subgroup $\Gamma_0(17)$ of $\Gamma(1)$ consisting of maps $z \to \dfrac{az+b}{cz+d}$ with $ad - bc = 1$, $c \equiv 0(17)$; w_{17} is the map $z \to -\dfrac{1}{17z}$, and Γ_0^* is the group generated by $\Gamma_0(17)$ and w_{17}. Then $H/\Gamma_0(17)$ has genus 1, with function field generated over $\mathbf{Q}$ by $j(\omega)$, $j(17\omega)$; w_{17} is an involution on $H/\Gamma_0(17)$, and H/Γ_0^* has genus zero. Suppose H/Γ_0^* is uniformised by $\tau(z)$. Then $\tau(z) \in \mathbf{Q}(j(z).j(17z), j(z) + j(17z))$, and if we specialise $z \to \omega$, $\tau(\omega) \in \mathbf{Q}(j(\omega), j(17\omega))$. [A priori, there may be finitely many exceptions, corresponding to specialisations which make both the numerator and the denominator vanish.] Now write $k(z) = j(z) - j(17z)$, then k is invariant by Γ_0 but not Γ_0^*, k^2 is invariant by Γ_0^*, and $H/\Gamma_0(17)$ has equation $k^2 = G(\tau)$, with $G(X) \in \mathbf{Q}(X)$. By a suitable birational transformation (making a bilinear transformation on τ, and replacing k by $\sigma f(\tau)$ with $f(X) \in \mathbf{Q}(X)$) we may obtain Fricke's equation $\sigma^2 = \tau^4 - 6\tau^3 - 27\tau^2 - 28\tau - 16$; see [4].

As usual, write $E_{\mathbf{Q}(\sqrt{-D})}$ for the group of points of the elliptic curve $E : Y^2 = G(X)$ which have coordinates in $\mathbf{Q}(\sqrt{-D})$; the conjectures predict that $E_{\mathbf{Q}(\sqrt{-D})}$ should have an odd number of generators of infinite order, and so be infinite, if D is a positive integer congruent to 3 modulo 4 which is a quadratic residue modulo 17.

Suppose then that p is prime, $p \equiv 3(4)$, $\left(\frac{p}{17}\right) = +1$. Then we can find ω with $\Delta(\omega) = p$, $17\omega = 1/\overline{\omega}$; for, take ω as a root of $17A\omega^2 + B\omega + A = 0$ where $B^2 - 68A^2 = -p$. But now $j(17\omega) = j(1/\overline{\omega}) = j(-\overline{\omega}) = \overline{j(\omega)}$, and $j(\omega) \in K_1$ the class field of $\mathbf{Q}(\sqrt{-p})$. Hence $j(\omega) \cdot j(17\omega)$, $j(\omega) + j(17\omega)$ and so $\tau(\omega)$ are real, and $k(\omega) = j(\omega) - j(17\omega)$ is pure imaginary. We deduce that

$$\sqrt{(-p)}k(\omega), \tau(\omega) \in K_1 \cap \mathbf{R},$$

an extension of $\mathbf{Q}$ of odd degree. As in the previous paragraph, $-pY^2 =$ **41**
$G(X)$ has a point in $K_1 \cap \mathbf{R}$ and so in $\mathbf{Q}$, and this implies that $E_{\mathbf{Q}(\sqrt{-p})}$ is infinite (except possibly in finitely many cases).

Appendix. The so-called Birch-Swinnerton-Dyer conjectures were officially stated in [2]; since then, they have been extended and generalised in various ways, and nowadays the standard account is [8]. However, the particular case we are quoting is not made quite explicit; though it was remarked by Shimura some years ago.

We suppose that $E : y^2 = x^3 + Ax + B$ is a 'good' elliptic curve with conductor N. This means (for motivation, see Weil [10]), *inter alia*, that E is parametrised by functions on $H/\Gamma_0(N)$, and corresponds to a differential $f(z)dz = \Sigma a_n e^{2\pi i n z} dz$ on $H/\Gamma_0(N)$; the essential part of the zeta function of E is $L_E(s) = \Sigma a_n n^{-s} = (2\pi)^s(\Gamma(s))^{-1} \int_0^\infty f(iz)z^{s-1}dz$; and E has a good reduction modulo p precisely when p does not divide N, so N divides a power of $6(4A^3 + 27B^2)$. The involution $\omega_N : z \to -1/Nz$ of $H/\Gamma_0(N)$ will take $f(z)dz$ to $\pm f(z)dz$; so $L_E(s)$ has a functional equation $\Lambda_E(s) = \epsilon N^{1-s}\Lambda_E(2-s)$, where $\Lambda_E(s) = \Gamma(s)(2\pi)^{-s}L_E(s)$ and $\epsilon^2 = 1$. Let $\chi(n)$ be a real character with conductor M, with $(M, N) = 1$, and $L_E(s,\chi) = \Sigma a_n \chi(n) n^{-s}$; then $L_E(s,\chi)$ has functional equation $\Gamma_E(s,\chi) = \epsilon\chi(-N)N^{1-s}\Lambda_E(2-s,\chi)$, where $\Lambda_E(s,\chi) = (M/2\pi)^s\Gamma(s)$. $\Lambda_E(2-s,\chi)$. So $L_E(s,\chi)$ has a zero at $s = 1$ of odd or even order according to the sign of $\chi(-N)$.

Say $D > 0$, $D \equiv 3(4)$, $\chi(n) = \left(\frac{n}{D}\right)$; then $L_E(s,\chi)$ is the essential part of the zeta function of $E^{(-D)} : -Dy^2 = x^3 + Ax + B$. The main conjecture of [2] asserts that $L_E(s,\chi)$ should have a zero of order

$g(E_{\mathbf{Q}}^{(-D)})$ at $s = 1$; so the parity of $g(E_{\mathbf{Q}}^{(-D)})$ should be determined by the Legendre symbol $\left(\frac{-N}{D}\right)$.

Finally, we remark that if $\mathfrak{U}$ is a point of $E_{\mathbf{Q}(\sqrt{-D})}$ and $\overline{\mathfrak{U}}$ its conju-
42 gate over $\mathbf{Q}$, then $\mathfrak{U} + \overline{\mathfrak{U}} \in E_{\mathbf{Q}}$ and $\mathfrak{U} - \overline{\mathfrak{U}}$ is a point of $E_{\mathbf{Q}(\sqrt{-D})}$ with x real and y pure imaginary, so $\mathfrak{U} - \overline{\mathfrak{U}}$ gives a point of $E_{\mathbf{Q}}^{(-D)}$. Hence $g(E_{\mathbf{Q}(\sqrt{-D})}) = g(E_{\mathbf{Q}}) + g(E_{\mathbf{Q}}^{(-D)})$, and so forth.

Added in proof. Since this talk was given, I have heard that Deuring, and Stark, too, have independently decided that Heegner was right after all. Deuring's paper was published in *Inventiones Mathematicae* 5 (1968); Stark's has yet to appear.

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ON THE AUTOMORPHISMS OF CERTAIN SUBGROUPS OF SEMI-SIMPLE LIE GROUPS

By Armand Borel

43 Let L be a group. We denote by $E(L)$ the quotient group $\mathrm{Aut}\, L/\,\mathrm{Int}\, L$ of the group $\mathrm{Aut}\, L$ of automorphisms of L by the group $\mathrm{Int}\, L$ of inner automorphisms $\mathrm{Int}\, a : x \to a.x.a^{-1} (a, x \in L)$ of L. Our first aim is to show that $E(L)$ is finite if L is arithmetic, S-arithmetic (see 3.2) or uniform in a semi-simple Lie group (with some exceptions, see Theorem 1.5, Theorem 3.6 and Theorem 5.2 for the precise statements). A slight variant of the proof also shows that in these cases L is not isomorphic to a proper subgroup of finite index. As a consequence, a Riemannian symmetric space with negative curvature, and no flat component, has infinitely many non-homeomorphic compact Clifford-Klein forms, Theorem 6.2.

Further information on $E(L)$ is obtained when L is an S-arithmetic group of a semi-simple k-group G (with some conditions on G and S). If L contains the center of G_k, and G is simply connected, then $E(L)$ is essentially generated by four kinds of automorphisms: exterior automorphisms of G, automorphisms deduced from certain automorphisms of k, automorphisms of the form $x \to f(x) \cdot x$ where f is a suitable homomorphism of L into the center of G_k, and automorphisms induced by the normalizer $N(L)$ of L in G (see Lemma 1.8, Remark in 1.9). In the case where G is split and L is the group of $\mathfrak{o}(S)$-points of G for its canonical integral structure, there results are made more precise (Theorem 2.2, Theorem 4.3), and $N(L)/L$ is put into relation with the S-ideal class group of k and S-units (see Lemma 2.3, Lemma 4.5; these results overlap with those of Allan [1], [2]). As an illustration, we discuss $\mathrm{Aut}\, L$ for some classical groups (Examples 2.6, 4.6). The results are related to those of O'Meara [24] if $G = \mathbf{SL}_n$, and of Hua-Reiner [12], [13] and Reiner [27] if $G = \mathbf{SL}_n, \mathbf{Sp}_{2n}$, and $k = \mathbf{Q}, \mathbf{Q}(i)$.

The finiteness of $E(L)$ follows here rather directly from rigidity theorems [25], [26], [32], [33]. The connection between the two is established by Lemma 1.1. **44**

Notation. In this paper, all algebraic groups are linear, and we follow in general the notations and conventions of [9]. In particular, we make no notational distinction between an algebraic group G over a field k and its set of points in an algebraic closure $\overline{k}$ of the field of definition (usually $\mathbf{C}$ here). The Lie algebra of a real Lie group or of an algebraic group is denoted by the corresponding German letter.

If L is a group and V a L-module, then $H^1(L, V)$ is the 1st cohomology group of L with coefficients in V. In particular, if L is a subgroup of the Lie group G, then $H^1(L, \mathfrak{g})$ is the 1st cohomology group of L with coefficients in the Lie algebra $\mathfrak{g}$ of G, on which L operates by the adjoint representation.

A closed subgroup L of a topological group G is *uniform* if G/L is compact.

1 Uniform or arithmetic subgroups.

Lemma 1.1. *Let G be an algebraic group over $\mathbf{R}$, L a finitely generated subgroup of $G_{\mathbf{R}}$ and N the normalizer of L in $G_{\mathbf{R}}$. Assume that $H^1(L, \mathfrak{g}) = 0$. Then the group of automorphisms of L induced by elements of N has finite index in* Aut *L.*

Let L_0 be a group isomorphic to L and ι an isomorphism of L_0 onto L. For $M = G_{\mathbf{R}}$, G, let $R(L_0, M)$ be the set of homomorphisms of L_0 into M. Let $(x_i)(1 \leqslant i \leqslant q)$ be a generating set of elements of L. Then $R(L_0, M)$ may be identified with a subset of M^q, namely, the set of m-uples (y_i) which satisfy a set of defining relations for L_0 in the x_i, x_i^{-1}. In particular $R(L_0, G)$ is an affine algebraic set over $\mathbf{R}$, whose set of real points is $R(L_0, G_{\mathbf{R}})$. The group M operates on $R(L_0, M)$, by composition with inner automorphisms, and G is an algebraic transformation group of $R(L_0, G)$, with action defined over $\mathbf{R}$.

To $\alpha \in \operatorname{Aut} L$, let us associate the element $j(\alpha) = \alpha \circ \iota$ of $R(L_0, G)$. **45**
The map j is then a bijection of $\operatorname{Aut} L$ onto the set $I(L_0, L) \subset R(L_0, G_{\mathbf{R}})$

of isomorphisms of L_0 onto L. If $a, b \in I(L_0, L)$, then $b \in G_{\mathbf{R}}(a)$ if and only if there exists $n \in N$ such that $b = (\operatorname{Int} n) \circ a$. Our assertion is therefore equivalent to: "$I(L_0, L)$ is contained in finitely many orbits of $G_{\mathbf{R}}$," which we now prove.

Let $b \in I(L_0, L)$. Since

$$H^1(b(L^0), \mathfrak{g}) = H^1(L, \mathfrak{g}) = H^1(L, \mathfrak{g}_{\mathbf{R}}) \otimes \mathbf{C},$$

we also have $H^1(b(L_0), \mathfrak{g}) = 0$. By the lemma of [33], it follows that the orbit $G(b)$ contains a Zariski-open subset of $R(L_0, G)$. Since the latter is the union of finitely many irreducible components, this shows that $I(L_0, L)$ is contained in finitely many orbits of G. But an orbit of G containing a real point can be identified to a homogeneous space G/H where H is an algebraic subgroup of G, defined over $\mathbf{R}$. Therefore its set of real points is the union of finitely many orbits of $G_{\mathbf{R}}$([8], §6.4), whence our contention.

Remark 1.2. (i) The lemma and its proof remain valid if $\mathbf{R}$ and $\mathbf{C}$ are replaced by a locally compact field of characteristic zero K and an algebraically closed extension of K.

(ii) The group $\mathbf{SL}(2, \mathbf{Z})$ has a subgroup of finite index L isomorphic to the free group on m generators, where $m \geqslant 2$ (and in fact may be taken arbitrarily large). The group $E(L)$ has the group $\mathbf{GL}(m, \mathbf{Z}) = \operatorname{Aut}(L/(L, L))$ as a quotient, hence is infinite. On the other hand, L has finite index in its normalizer in $\mathbf{SL}(2, \mathbf{C})$, as is easily checked (and follows from Proposition 3.3(d)). Thus, 1.1 implies that $H^1(L, \mathfrak{g}) \neq 0$, as is well known. Similarly, taking (i) into account, we see that the free uniform subgroups of $\mathbf{PSL}(2, \mathbf{Q}_p)$ constructed by Ihara [15] have non-zero first cohomology group with coefficients in $\mathfrak{g}$.

Lemma 1.3. *Let G be an algebraic group over $\mathbf{R}$, L a finitely generated discrete subgroup of $G_{\mathbf{R}}$ such that $G_{\mathbf{R}}/L$ has finite invariant measure. Assume that $H^1(L', \mathfrak{g}) = 0$ for all subgroups of finite index L' of L. Then L is not isomorphic to a proper subgroup of finite index.*

46 Define L_0, M, ι, $R(L_0, M)$ and the action of M on $R(L_0, M)$ as in the proof of Lemma 1.1. Let C be the set of monomorphisms of L_0 onto subgroups of finite index of L. Then, $j : \alpha \mapsto \alpha \circ \iota$ is a bijection of C onto a subset J of $R(L_0, G_{\mathbf{R}})$, and the argument of Lemma 1.1 shows that J is contained in the union of finitely many orbits of $G_{\mathbf{R}}$. Fix a Haar measure on $G_{\mathbf{R}}$, and hence on all quotients of $G_{\mathbf{R}}$ by discrete subgroups. The total measure $m(G_{\mathbf{R}}/L')$ is finite for every subgroup of finite index of L, since $m(G_{\mathbf{R}}/L)$ is finite. If $b, c \in C$, and $b = \operatorname{Int} g \circ c$, $(g \in G_{\mathbf{R}})$, then $m(G_{\mathbf{R}}/b(L)) = m(G_{\mathbf{R}}/c(L))$. Consequently, $m(G_{\mathbf{R}}/L')$ takes only finitely many values, as L' runs through the subgroups of finite index of L, isomorphic to L. But, if there is one such group $L' \neq L$, then there is one of arbitrary high index in L, a contradiction.

Lemma 1.4. *Let L be a finitely generated group, M a normal subgroup of finite index, whose center is finitely generated, N a characteristic finite subgroup of L.*

(a) *If $E(M)$ is finite, then $E(L)$ is finite.*

(b) *If $E(L/N)$ is finite, then $E(L)$ is finite.*

(a) It is well known and elementary that a finitely generated group contains only finitely many subgroups of a given finite index (see e.g. [11]). Therefore, the group $\operatorname{Aut}(L, M)$ of automorphisms of L leaving M stable has finite index in $\operatorname{Aut}(L)$. Since L/M is finite, the subgroup Q of elements of $\operatorname{Aut}(L, M)$ inducing the identity on L/M has also finite index. Let $r : Q \to \operatorname{Aut} M$ be the restriction map. Our assumption implies that $r^{-1}(\operatorname{Int} M)$ has finite index in Q, hence that $\operatorname{Int} L . \ker r$ is a subgroup of finite index of $Q . \operatorname{Int} L$. It suffices therefore to show that $\ker r \cap \operatorname{Int} L$ has finite index in $\ker r$. Let $b \in \ker r$. Write

$$b(x) = u_x \cdot x (x \in L).$$

Then $u_x \in M$ and routine checking shows: the map $u : x \mapsto u_x$ is a 1-cocycle on L, with coefficients in the center C of M, which is constant on the cosets mod M, and may consequently be viewed

as a 1-cocycle of L/M with coefficients in C; furthermore, two cocycles thus associated to elements b, $c \in Q$ are cohomologous
47 if and only if there exists $n \in C$ such that $b = \operatorname{Int} n \circ c$, and any such cocycle is associated to an automorphism. Therefore

$$\ker r/\operatorname{Int}_L C \cong H^1(L/M, C).$$

By assumption, L/M is finite, and C is finitely generated. Hence the right hand side is finite, which implies our assertion.

(b) The group N being characteristic, we have a natural homomorphism $\pi : \operatorname{Aut} L \to \operatorname{Aut} L/N$. The finiteness of $E(L/N)$ implies that $\operatorname{Int} L.\ker\pi$ has finite index in $\operatorname{Aut} L$. Moreover $\ker\pi$ consists of automorphisms of the form $x \mapsto x \cdot v_x$, $(x \in L, v_x \in N)$, and is finite, since N is finite and L is finitely generated.

Theorem 1.5. *Let G be a semi-simple Lie group, with finitely many connected components, whose identity component G^0 has a finite center, and L a discrete subgroup of G. Then $E(L)$ is finite if one of the two following conditions is fulfilled:*

(a) *G/L is compact, G^0 has no non-compact three-dimensional factor;*

(b) $\operatorname{Aut}(\mathfrak{g} \otimes \mathbf{C})$ *may be identified with an algebraic group G' over $\mathbf{Q}$, such that the image L' of $L \cap G$ in G' by the natural projection is an arithmetic subgroup of G', and $G'_{\mathbf{R}}$ has no factor locally isomorphic to* $\mathbf{SL}(2, \mathbf{R})$ *on which the projection of L' is discrete.*

(a) Let A be the greatest compact normal subgroup of G^0 and $\pi : G^0 \to G^0/A$ the canonical projection. Since the center of G^0 is finite, it is contained in A, and G^0/A is the direct product of non-compact simple groups with center reduced to $\{e\}$. The group $\pi(L \cap G^0)$ is discrete and uniform in G^0/A. By density [4], its center is contained in the center of G^0/A, hence is reduced to $\{e\}$. Consequently, the center of $L \cap G^0$ is contained in $A \cap L$, hence

is finite. We may then apply Lemma 1.4(a), which reduces us to the case where G is connected. Moreover, by [4], any finite normal subgroup of $\pi(L)$ is central in G/A, hence reduced to $\{e\}$. Therefore $A \cap L$ is the greatest finite normal subgroup of L, and is characteristic. By Lemma 1.4(b), it suffices to show that $E(\pi(L))$
is finite. We are thus reduced to the case where G has no center, 48
and is a direct product of non-compact simple groups of dimension > 3. In particular, G is of finite index in the group of real points of an algebraic group defined over $\mathbf{R}$, namely $\mathrm{Aut}(\mathfrak{g} \otimes \mathbf{C})$. By a theorem of Weil [33], $H^1(L, \mathfrak{g}) = 0$, hence by Lemma 1.1, it is enough to show that L has finite index in its normalizer $N(L)$. By [4], $N(L)$ is discrete. Since G/L is fibered by $N(L)/L$, and is compact, $N(L)/L$ is finite.

(b) Let A be the greatest normal $\mathbf{Q}$-subgroup of ${G'}^0$, whose group of real points is compact, and let π be the composition of the natural homomorphisms

$$G^0 \to \mathbf{Ad}\mathfrak{g} \to ({G'}^0)_{\mathbf{R}}/A_{\mathbf{R}}.$$

G'/A is a $\mathbf{Q}$-group without center, which is a product of $\mathbf{Q}$-simple groups, each of which has dimension > 3 and a non-compact group of real points. $\pi(L)$ is arithmetic in G'/A([6], Theorem 6) and $L \cap \ker\pi$ is finite. By Zariski-density ([6], Theorem 1), any finite normal subgroup of $\pi(L)$ is central in G'/A, hence reduced to $\{e\}$. Thus $L \cap \ker\pi$ is the greatest finite normal subgroup of L, and is characteristic. Also the center of $\pi(L \cap G^0)$ is central in $\mathbf{Ad}G'$, hence reduced to $\{e\}$, and the center of $L \cap G^0$ is compact, and therefore finite. By Lemma 1.4, we are thus reduced to the case where G, G' are connected, $A = \{e\}$, and L is arithmetic in G'. Let $G_1, \ldots, G_q$ be the simple $\mathbf{Q}$-factors of G'. The group L is commensurable with the product of the intersections $L_i = L \cap G_i$, which are arithmetic ([7], 6.3). If $G_i \cap L$ is uniform in $G_{i\mathbf{R}}$, then $H^1(L_i, \mathfrak{g}_{\mathbf{R}}) = 0$ by [33]. If not, then $rk_Q G_i \geqslant 1$, and $H^1(L_i, \mathfrak{g}_i) = 0$ by theorems of Raghunathan [25], [26]. Consequently, $H^1(L \cap G', \mathfrak{g}'_{\mathbf{R}}) = 0$. Moreover, $L \cap G'$ is of finite index in its normalizer

N. In fact, N is closed, belongs to $G'_{\mathbf{Q}}$ ([6], Theorem 2), hence to $G'_{\mathbf{R}}$, and $N/(L \cap G')$ is compact since $G'_{\mathbf{R}}/(L \cap G')$ has finite invariant measure [7]. The conclusion now follows from Lemma 1.1 and Lemma 1.4.

Lemma 1.6. *Let L be a finitely generated group, N a normal subgroup of finite index. Assume L is isomorphic to a proper subgroup of finite index. Then N has a subgroup of finite index which is isomorphic to a proper subgroup of finite index.*

49 The assumption implies the existence of a strictly decreasing sequence $(L_i)(i = 1, 2, \ldots)$ of subgroups of finite index of L and of isomorphisms $f_i : L \xrightarrow{\sim} L_i (i = 1, 2, \ldots)$. Let a be the index of N in L. Then $L_i \cap N$ has index $\leqslant a$ in L_i, hence $M_i = f_i^{-1}(L_i \cap N)$ has index $\leqslant a$. Passing to a subsequence if necessary, we may assume that M_i is independent of i. Then, $L_i \cap N$ is isomorphic to a proper subgroup of finite index.

Proposition 1.7. *Let G and L be as in Theorem 1.5. Assume one of the conditions (a), (b) of Theorem 1.5 to be fulfilled. Then L is not isomorphic to a proper subgroup of finite index.*

By use of Lemma 1.6, the proof is first reduced to the case where G is connected. Let π be as in the proof of (a) or (b) in Theorem 1.5. Then $L \cap \ker \pi$ is the greatest finite normal subgroup of L. Similarly $L' \cap \ker \pi$ is the greatest finite normal subgroup of L', if L' has finite index in L. Therefore, if L' is isomorphic to L, the groups $\ker \pi \cap L$ and $\ker \pi \cap L'$ are equal, and are mapped onto each other by any isomorphism of L onto L'; hence $\pi(L) \cong \pi(L')$, and $\pi(L) \neq \pi(L')$ if $L \neq L'$. We are thus reduced to the case where the group A of (a) or (b) in Lemma 1.4 is $= \{e\}$. Moreover, in case (b), it suffices to consider $L \cap G'$ in view of Lemma 1.6. Our assertion then follows from the rigidity theorems of Weil and Raghunathan and from Lemma 1.3.

We shall need the following consequence of a theorem of Raghunathan:

Lemma 1.8. *Let G, G' be connected semi-simple $\mathbf{Q}$-groups, which are almost simple over $\mathbf{Q}$. Let L be a subgroup of $G_{\mathbf{Q}}$ containing an arith-*

metic subgroup L_0 of G, and s an isomorphism of L onto a subgroup of $G'_{\mathbf{Q}}$ which maps L_0 onto a Zariski-dense subgroup of G'. Assume that $rk_{\mathbf{Q}}(G) \geqslant 2$.

(i) *G and G' are isogeneous over* $\mathbf{Q}$.

(ii) *If G is simply connected, or G' is centerless, there exists a* $\mathbf{Q}$-*isogeny s' of G onto G', and a homomorphism g of L into the center of $G'_{\mathbf{Q}}$ such that $s(x) = s'(x) \cdot g(x)(x \in L)$.*

Let $\widehat{G}$ be the universal covering group of $G, \pi : \widetilde{G} \to G$ the canonical **50**
projection and $\widetilde{L} = \pi^{-1}(L) \cap G_{\mathbf{Q}}$. The group $\widetilde{L}_0 = \pi^{-1}(L_0) \cap \widetilde{L}$ is arithmetic, as follows e.g. from ([7], §6.11); in particular, $\pi(\widetilde{L}_0)$ has finite index in L_0, and $L'_0 = s \circ \pi(\widetilde{L}_0)$ is Zariski-dense in G'.

We identify G' with a $\mathbf{Q}$-subgroup of $\mathbf{GL}(n, \mathbf{C})$, for some n. The map $r = s \circ \pi$ may be viewed as a linear representation of $\widetilde{L}$ into $\mathbf{GL}(n, \mathbf{Q})$. By Theorem 1 of [25], there exists a normal subgroup $\widetilde{N}$ of $\widetilde{L}_0$, Zariski-dense in $\widetilde{G}$, and a morphism $t : \widetilde{G} \to \mathbf{GL}(n, \mathbf{C})$ which coincides with r on $\widetilde{N}$. Let C be the Zariski-closure of $t(\widetilde{N})$. It is an algebraic subgroup contained in $t(\widetilde{G}) \cap G'$. Since $t(\widetilde{N})$ is normal in L'_0, and L'_0 is Zariski-dense, the group C is normal in G'. However, ([9], §6.21(ii)), the group G' is isogeneous to a group $R_{k/\mathbf{Q}}H$, where k is a number field, H an absolutely simple k-group, and $R_{k/\mathbf{Q}}$ denotes restriction of the scalars ([31], Chap. I). Consequently, an infinite subgroup of $G'_{\mathbf{Q}}$ is not contained in a proper direct factor of G', whence $C = G' = t(\widetilde{G})$. If f is a regular function defined over $\mathbf{Q}$ on G', then $f \circ t$ is a regular function on $\widetilde{G}$, which takes rational values on the dense set $\widetilde{N}$. It follows immediately that $f \circ t$ is defined over $\mathbf{Q}$, hence t is defined over $\mathbf{Q}$. Its kernel is a proper normal $\mathbf{Q}$-subgroup of G', hence is finite, and t is a $\mathbf{Q}$-isogeny. This implies (i).

If G' is centerless, then $Z(\widetilde{G})$ belongs to the kernel of t. Thus, if G is simply connected, or G' centerless, t defines a $\mathbf{Q}$-isogeny s' of G onto G', which coincides with s on the Zariski-dense subgroup $N = \pi(\widetilde{N})$. The group $s(N)$ is then Zariski-dense in G'. Let $x \in L$, $y \in N$. Then $x \cdot y \cdot x^{-1} \in N$, hence $s(x) \cdot s'(x)^{-1}$ centralizes $s(N)$, and therefore also

G'. Consequently, $s(x) \cdot S'(x)^{-1}$ belongs to $G'_{\mathbf{Q}} \cap Z(G')$, and $f : x \mapsto s(x) \cdot s'(x)^{-1}$, and s' fulfil our conditions.

Theorem 1.9. *Let G and L be as in Lemma 1.8. Assume G to be centerless and L to be equal to its normalizer in $G_{\mathbf{Q}}$. Then $E(L)$ may be identified with a subgroup of* $(\operatorname{Aut} G)_{\mathbf{Q}}/(\operatorname{Int} G)_{\mathbf{Q}}$.

If G is centerless, and M is a Zariski-dense subgroup of $G_{\mathbf{Q}}$, then the
51 normalizer of M in G belongs to $G_{\mathbf{Q}}$. This follows from Theorem 2 in [6] if M is arithmetic, but the proof yields this more general statement, as well as Proposition 3.3(b) below. In view of this, Theorem 1.9 is a consequence of Lemma 1.8,

Remark. It is no great loss in generality to assume that L contains the center of $G_{\mathbf{Q}}$, and this assumption will in fact be fulfilled in the cases to be considered below. In this case, $\operatorname{Aut} L$ is generated by three kinds of automorphisms: (a) exterior automorphisms of G leaving L stable, (b) automorphisms $x \mapsto f(x) \cdot x$, where f is a homomorphism of L into its center, (c) automorphisms of the form $x \mapsto y \cdot x \cdot y^{-1}$, where y belongs to the normalizer of L in G.

Using some information on these three items, we shall in the following paragraph give a more precise description of $\operatorname{Aut} L$, when G is a split group.

2 Arithmetic subgroups of split groups over Q.

In this paragraph G is a connected semi-simple and almost simple $\mathbf{Q}$-group, which is split, of $\mathbf{Q}$-rank $\geqslant 2$; L is the group of integral points of G for the canonical $\mathbf{Z}$-structure associated to a splitting of G[10], [18], and $N(L)$ the normalizer of L in $G_{\mathbf{C}}$.

2.1 The group L is equal to its normalizer in $G_{\mathbf{Q}}$, and also to its normalizer in G if G has no center. To see this, we first notice that L has finite index in its normalizer in G. In fact, since the image of L in $\operatorname{Int} G$ is arithmetic ([7], §6.11), it suffices to show that if G is centerless, any arithmetic subgroup of G is of finite index in its normalizer, which follows from the end argument of Theorem 1.5(b). Our assertion is then a consequence of ([6], Theorem 7). Another proof will be given below (Theorem 4.3).

Theorem 2.2. *If G is centerless, then* $\operatorname{Aut} L$ *is a split extension of* $E(G) = \operatorname{Aut} G/\operatorname{Int} G$ *by L. If G is simply connected, then* $\operatorname{Aut} L$ *is a split extension of $E(G)$ by the subgroup A of automorphisms of the form $x \mapsto f(x) \cdot y \cdot x \cdot y^{-1}$ where $y \in N(L)$ and f is a homomorphism of L into its center.*

If G is centerless or simply connected, the standard construction of automorphisms of G leaving stable the splitting of G yields a subgroup $E'(G)$ of $(\operatorname{Aut} G)_{\mathbf{Q}}$, isomorphic to $E(G)$ under the canonical pro- 52
jection, and leaving the $\mathbf{Z}$-structure of G invariant. Thus $E(G)$ may be identified with a subgroup of $\operatorname{Aut} L$. If G is centerless, the theorem follows then from Theorem 1.9, and 2.1. Let now G be simple connected. Let $s \in \operatorname{Aut} L$. By Lemma 1.8, we can find $s' \in (\operatorname{Aut} G)_{\mathbf{Q}}$ and $f \in \operatorname{Hom}(L, Z(G)_{\mathbf{Q}})$ such that $s(x) = f(x) \cdot s'(x) (x \in L)$. However, L contains $Z(G)_{\mathbf{Q}}$ by 2.1. Therefore, f maps L into $Z(G) \cap L$. But $Z(G) \cap L$ is equal to the center of L, since L is Zariski-dense in G, whence our assertion.

If G has a non-trivial center the image of $N(L)$ in $\operatorname{Aut} L$ is in general different from $\operatorname{Int} L$. The quotient $N(L)/L$ has been studied in various cases, including those of Examples 2.6(1), (2), notably by Maass, Ramanathan, Allan (see [1], where references to earlier work are also given). We shall discuss it here and in §4 from a somewhat different point of view. In the following statement, the group $G' = \operatorname{Int} G$ is endowed with the $\mathbf{Z}$-structure associated to the splitting defined by the given splitting of G.

Lemma 2.3. *Let $\pi : G \to G' = \operatorname{Int} G$ be the canonical projection, T the maximal torus given by the splitting of G, and $T' = \pi(T)$. Then*

$$\pi(N(L)) = G'_{\mathbf{Z}}, \tag{1}$$

$$\pi(N(L))/\operatorname{Int} L \cong T'_{\mathbf{Z}}/\pi(T_{\mathbf{Z}}) \cong Z(L). \tag{2}$$

The group L is the normalizer in $G_{\mathbf{Q}}$ of a Chevalley lattice $\mathfrak{g}_{\mathbf{Z}}$ in $\mathfrak{g}$ as follows from 2.17 in [16]. Moreover, a Chevalley lattice is spanned by the logarithms of the unipotent elements in L. Consequently $N(L)$ is the normalizer in G of $\mathfrak{g}_{\mathbf{Z}}$. From this (1) follows.

Let B be the maximal solvable subgroup of G corresponding to the positive roots in the given splitting of G and U its unipotent radical. Then $B = T \cdot U$ (semi-direct). Let $x \in N(L)$. Since $\operatorname{Int} x$ preserves the $\mathbf{Q}$-structure of G, the group $x \cdot B \cdot x^{-1}$ is a maximal connected solvable subgroup defined over $\mathbf{Q}$, hence ([9], §4.13) there exists $z \in G_{\mathbf{Q}}$ such that $z \cdot B \cdot z^{-1} = x \cdot B \cdot x^{-1}$. But we have $G_{\mathbf{Q}} = L \cdot B_{\mathbf{Q}}$([6], Lemma 1). Since B is equal to its normalizer, it follows that $N(L) = L \cdot (N(L) \cap B)$. Let now $x \in N(L) \cap B$. Write $x = t \cdot v (t \in T, v \in U)$. We have $\pi(x) \in L$
53 (see 2.1). But, with respect to a suitable basis of a Chevalley lattice in $\mathfrak{g}_{\mathbf{Q}}$, $\pi(t)$ is diagonal, and $\pi(u)$ upper triangular, unipotent, therefore $\pi(t)$, $\pi(u) \in L$. However [10], π defines a $\mathbf{Z}$-isomorphism of U onto $\pi(U)$, hence $u \in L$, which shows that

$$N(L) = L \cdot (N(L) \cap T). \tag{3}$$

The kernel of π is contained in T, therefore (1) implies that $N(L) \cap T$ is the full inverse image of $T'_{\mathbf{Z}}$, which yields the first equality of (2). The groups $T_{\mathbf{Z}}$ and $T'_{\mathbf{Z}}$ consist of the elements of order 2 of T and T' respectively and are both isomorphic to $(\mathbf{Z}/2\mathbf{Z})^l$, where l is the rank of G. Consequently $T'_{\mathbf{Z}}/\pi(T_{\mathbf{Z}})$ is isomorphic to the kernel of $\pi : T_{\mathbf{Z}} \to T'_{\mathbf{Z}}$, i.e. to $Z(L)$, which ends the proof of (2).

The determination of $\operatorname{Aut} L / \operatorname{Int} L$ is thus to a large extent reduced to that of the center $Z(L)$ of L, and of the quotient of L by its commutator subgroup (L, L). We now make some remarks on these two groups.

2.4 The center $Z(L)$ of L is of order two if G is simply connected of type $\mathbf{A}_n$ (n odd), $\mathbf{B}_n$, $\mathbf{C}_n (n \geqslant 1)$, $\mathbf{D}_n (n \geqslant 3, n$ odd), $\mathbf{E}_7$, of type (2.2) if $G = \mathbf{Spin}\ 4m$ (m positive integer), of order one in the other cases.

In fact the $\mathbf{Z}$-structure on G may be defined by means of an admissible lattice in the representation space of a faithful representation defined over $\mathbf{Q}$. If we assume $G \subset \mathbf{GL}(n, \mathbf{C})$ and $\mathbf{Z}^n$ to be an admissible lattice, then $Z(L)$ is represented by diagonal matrices with integral coefficients, which shows first that $Z(L)$ is an elementary abelian 2-group. All almost simple simply connected groups have faithful irreducible representations, except for the type $\mathbf{D}_{2m}$. Thus, except in that case, $Z(L)$ is of order 2 (resp. 1) if $Z(G)$ has even (resp. odd) order, whence our contention. The case of $\mathbf{D}_{2m}$ is settled by considering the sum of the two half-spinor representations.

2.5 It is well known that $\mathbf{SL}(n,\mathbf{Z})$ is equal to its commutator subgroup if $n \geqslant 3$ (see [3] e.g.). Also the commutator subgroup of $\mathbf{Sp}(2n,\mathbf{Z})$ is equal to $\mathbf{Sp}(2n,\mathbf{Z})$ if $n \geqslant 3$, has index two if $n = 2$ [3], [28]. More **54**
generally, if the congruence subgroup theorem holds, which is the case if $rkG \geqslant 2$ and G is simply connected, according to [22], then $L/(L,L)$ is the product of the corresponding local groups $G_{0_p}/(G_{0_p}, G_{0_p})$. Serre has pointed out to me that, using this, one can show that $L = (L,L)$ if G has rank $\geqslant 3$, and is simply connected. Another more direct proof was mentioned to me by R. Steinberg, who also showed that $L/(L,L)$ is of order two if $G = \mathbf{G}_2$. He uses known commutation rules among unipotent elements of L, and the fact that they generate L.

Examples 2.6. (1) $G = \mathbf{SL}(n,\mathbf{C})$, $L = \mathbf{SL}(n,\mathbf{Z})$, $(n \geqslant 3)$. In this case, $E(G)$ is of order two, generated by the automorphism $\sigma : x \mapsto {}^t x^{-1}$. By Lemma 2.3 and 2.4, $\operatorname{Int} L$ has index one (resp. two) in the image of $N(L)$ if n is odd (resp. even). Furthermore, it is easily seen, and will follow from Lemma 4.5, that, in the even dimensional case, the non-interior automorphisms defined by $N(L)$ are of the form $x \mapsto y \cdot x \cdot y^{-1} (y \in \mathbf{GL}(n,\mathbf{Z}), \det y = -1)$. Thus, taking 2.5 into account, we see that $\operatorname{Aut} L$ is generated by $\operatorname{Int} L$, σ, and, for n even, by one further automorphism induced by an element of $\mathbf{GL}(n,\mathbf{Z})$ of determinant -1. This is closely related to results of Hua-Reiner [12], [13].

(2) $G = \mathbf{Sp}(2n,\mathbf{C})$, $L = \mathbf{Sp}(2n,\mathbf{Z})$, $(n \geqslant 2)$. Here, $E(G)$ is reduced to the identity. Thus, by the above, $\operatorname{Int} L$ has index two in $\operatorname{Aut} L$ if $n \geqslant 3$, index four if $n = 2$. The non-trivial element of $N(L)/L$

is represented by an automorphism of the form $x \mapsto y \cdot x \cdot y^{-1}$ where y is an element of $\mathbf{GL}(2n, \mathbf{Z})$ which transforms the bilinear form underlying the definition of $\mathbf{Sp}(2n, \mathbf{C})$ into its opposite (see Examples 4.6). For $n = 2$, one has to add the automorphism $x \to \chi(x) \cdot x$, where χ is the non-trivial character of L. This result is due to Reiner [27].

(3) G is simply connected, of type $\mathbf{D}_{2m}$. Then we have a composition series

$$\operatorname{Aut} L \supset A \supset \operatorname{Int} L,$$

where $\operatorname{Aut} L/A$ is of order two, and $A/\operatorname{Int} L$ has order two if m is odd, is of type (2.2) if m is even. This follows from (2.2), (2.3), (2.4), (2.5). The other simple groups of rank $\geqslant 3$ are discussed similarly.

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3 S-arithmetic groups over number fields.

3.1 Throughout the rest of this paper, k is an algebraic number field of finite degree over $\mathbf{Q}$, $\mathfrak{o}$ its ring of integers, V the set of primes of k, V_∞ the set of infinite primes of k, S a finite subset of V containing V_∞, and $\mathfrak{o}(S)$ the subring of $x \in k$ which are integral outside S. We let $I(k, S)$ be the S-ideal class group of k, i.e. the quotient of the group of fractional $\mathfrak{o}(S)$-ideals by the group of principal $\mathfrak{o}(S)$-ideals. We follow the notation of [5]. In particular k_v is the completion of k at $v \in S$, $\mathfrak{o}_v$ the ring of integers of k_v. If G is a k-group, then G^0 is its identity component, and

$$G_v = G_{k_v} (v \in S), G_S = \prod_{v \in S} G_v, G_\infty = \prod_{v \in V_\infty} G_v.$$

Moreover $G' = R_{k/\mathbf{Q}} G$ is the group obtained from G by restriction of the groundfiled from k to $\mathbf{Q}$([31], Chap. I), and we let μ denote the canonical isomorphism of G_k onto $G'_{\mathbf{Q}}$.

If A is an abelian group, and q a positive integer, we let ${}_qA$ and $A^{(q)}$ denote the kernel and the image of the homomorphism $x \mapsto x^q$.

3.2 Let G be a k-group. A subgroup L of G_k is S-arithmetic if there is a faithful k-morphism $r : G \to \mathbf{GL}_n$ such that $r(L)$ is commensurable with $r(G)_{\mathfrak{o}(S)}$.

If S' is a finite set of primes of $\mathbf{Q}$, including ∞, and S is the set of primes dividing some element of S', then $v : G_k \xrightarrow{\sim} G'_{\mathbf{Q}}$ induces a bijection between S-arithmetic subgroups of G and S'-arithmetic subgroups of G'. This follows directly from the remarks made in ([5], §1).

In the following proposition, we collect some obvious generalizations of known facts.

Proposition 3.3. *Let G be a semi-simple k-group, L a S-arithmetic subgroup of G, N the greatest normal k-subgroup of G^0 such that N_∞ is compact, and $\pi : G \to G/N$ the natural projection.*

(a) *If N is finite and G is connected, L is Zariski-dense in G.*

(b) *If G is connected, the commensurability group $C(L)$ of L in G is equal to $\pi^{-1}((G/N)_k)$.*

(c) *If $\sigma : G \to H$ is a surjective k-morphism, $\sigma(L)$ is S-arithmetic in H.* 56

(d) *If N is finite, L has finite index in its normalizer in G.*

(a) follows from ([6], Theorem 3), and from the fact that L contains an arithmetic subgroup of G.

(b) We recall that $C(L)$ is the group of elements $x \in G$ such that $x \cdot L \cdot x^{-1}$ is commensurable with L. The proof of (b) is the same as that the Theorem 2 in [6]. In fact, this argument shows that if G is centerless, then $C(M) \subset G_k$ whenever M is a subgroup of G_k Zariski-dense in G.

(c) If σ is an isomorphism, the argument is the same as that of ([7], §6.3). If σ is an isogeny, this has been proved in ([5], §8.12). From there, the extension to the general case proceeds exactly in the same way as in the case $S = V_\infty$([6], Theorem 6).

(d) We may assume G to be connected and $N = \{e\}$. Then by (c), $N(L) \subset G_k$. We view G_k and L as diagonally embedded in G_S. Then L is discrete in G_S.

This group L has a finite system of generators [17], say $(x_i)_{1 \leqslant i \leqslant q}$. Since L is discrete, there exists a neighbourhood U of e in G_S such that if $x \in N(L) \cap U$, then x centralizes the x_i's, hence L. The latter being Zariski-dense in G, this implies that the component x_v of x in $G_v (v \in S)$ is central in G_v, whence $x_v = e$, which shows that $N(L)$ is discrete in G_S. But G_S/L has finite invariant volume ([5], §5.6) and is fibered by $N(L)/L$, hence $N(L)/L$ is finite.

Proposition 3.4. *Let G be a semi-simple k-group, L a subgroup of G_k which is Zariski-dense, and is equal to its normalizer in G_k, and $N(L)$ the normalizer of L in G. Then $N(L)/L$ is a commutative group whose exponent divides the order m of the center $Z(G)$ of G.*

We show first that if $x \in N(L)$, then $x^m \in L$. In view of the assumption, it suffices to prove that $x^m \in G_k$. Let $\pi : G \to G/Z(G)$ be the canonical projection. The fiber $F_x = \pi^{-1}(\pi(x))$ of x consists of the elements $x \cdot z_i (1 \leqslant i \leqslant m)$, where z_i runs through $Z(G)$, and belongs to
57 $N(L)$. By the remark made in Proposition 3.3(b), $\pi(x)$ is rational over k, hence F_x is defined over k, and its points are permuted by the Galois group of $\bar{k}$ over k. Since the z_i's are central, the product of the xz_i is equal to $x^m \cdot z_1 \ldots z_m$ and is rational over k. Similarly the product of the z_i's is rational over k, whence our assertion.

It is possible to embed $Z(G)$ as a k-subgroup in a k-torus T' whose first Galois cohomology group is zero (see Ono, Annals of Math. (2), 82 (1965), p. 96). Let $H = (G \times T')/Z(G)$ where $Z(G)$ is embedded diagonally in $G \times T'$. Then $G/Z(G)$ may be identified with H/T'. Let $x \in N(L)$. We have already seen that $\pi(x)$ is rational over k. But, since T' has trivial first Galois-cohomology group, the map $H_k \to (H/T')_k$ is surjective. There exists therefore $d \in T'$ such that $d \cdot x \in H_k$. Thus, if $x, y \in N(L)$, we can find two elements $x', y' \in H_k$, which normalize L, whose commutator (x', y') is equal to (x, y). But, obviously, $G = (H, H)$, therefore $(x, y) \in N(L)_k$ hence $(x, y) \in L$, and $(N(L), N(L)) \subset$

L. This argument also proves that

$$N(L) = G \cap (N_H(L)_k \cdot T'), \tag{1}$$

where $N_H(L)$ is the normalizer of L in H, and $N_H(L)_k = N_H(L) \cap H_k$.

For the sake of reference, we state as a lemma a remark made by Ihara in ([14], p. 269).

Lemma 3.5. *Let A be a group, B a subgroup, and V a A-module. Assume that for any $a \in A$, there is no non-zero element of V fixed under $a \cdot B \cdot a^{-1} \cap B$. Then the restriction map $r : H^1(A, V) \to H^1(B, V)$ is injective.*

It suffices to show that if z is a 1-cocycle of A which is zero on B, then z is zero. Let $a \in A$, $b \in B$ be such that $a \cdot b \cdot a^{-1} = b' \in B$. We have then

$$z(a \cdot b) = z(a) = z(b' \cdot a) = b' \cdot z(a),$$

which shows that $z(a)$ is fixed under $a \cdot B \cdot a^{-1} \cap B$, hence is zero.

Theorem 3.6. *Let G be a semi-simple k-group and L a S-arithmetic subgroup. Then $E(L)$ is finite if one of the following conditions is fulfilled :*

(a) *G has no normal k-subgroup N such that N_∞ has a non-compact factor of type $\mathbf{SL}(2, \mathbf{R})$, or also of type $\mathbf{SL}(2, \mathbf{C})$ if G_S/L is not compact;*

(b) *G is of type $\mathbf{SL}_2$ over k, and S has at least two elements.* **58**

By Lemma 1.4(b), we may assume G to be connected. Let N be the greatest normal k-subgroup of G such that N_∞ is compact and $\pi : G \to G/N$ the natural projection.

(a) Arguing as in Lemma 1.4, we see that it suffices to show that $E(\pi(L))$ is finite, which reduces us to the case where G is a direct product of simple k-groups G_i. The group $L_i = G_i \cap L$ is S-arithmetic in G_i and the product of the L_i is normal of finite index in L. By Lemma 1.4, we may therefore assume L to be the product of its intersection with the G_i's. The group L has finite index in its normalizer (Proposition 3.3) and is finitely generated [17], so that, in order to deduce our assertion from Lemma 1.1, applied to L and G_∞, it suffices to show that $H^1(L, \mathfrak{g}_\infty) = 0$. Since this group is isomorphic to the product of the groups $H^1(L_i, \mathfrak{g}_{i\infty})$, we may assume G to be simple over k. Let $L_0 = L \cap G_\mathfrak{o}$.

The group L_0 is arithmetic, and therefore so is $x \cdot L_0 \cdot x^{-1} \cap L_0 = L_{0,x} (x \in L)$. Consequently, $L_{0,x}$ is Zariski-dense in G ([6], Theorem 1), and has no non-zero fixed vector in $\mathfrak{g}_\infty$. By Lemma 3.5, the restriction map : $H^1(L, \mathfrak{g}_\infty) \to H^1(L_0, \mathfrak{g}_\infty)$ is injective. But $H^1(L_0, \mathfrak{g}_\infty) = 0$: if $\mathrm{rk}_k G \geqslant 1$, this follows from [25], [26]. Let now $\mathrm{rk}_k G = 0$. Then G_∞ / L_0 is compact ([4], §11.6). In view of 3.2, we may further assume G to be almost absolutely simple over k. Let J be the set of $v \in V_\infty$ such that G_v is not compact and H the subgroup of G generated by the G_v's $(v \in J)$. Then, by Weil's theorem ([32], [33]),

$$H^1(\Gamma_0, \mathfrak{h}) = 0. \tag{1}$$

But we have

$$H^1(\Gamma_0, \mathfrak{g}_v) = H^1(\Gamma_0, {}^v\mathfrak{g}_k) \otimes_{v(k)} k_v, \quad (v \in V_\infty) \tag{2}$$

$$H^1(\Gamma_0, \mathfrak{g}) = \prod_{v \in V_\infty} H^1(\Gamma_0, \mathfrak{g}_v), \tag{3}$$

$$H^1(\Gamma_0, \mathfrak{h}) = \prod_{v \in J} H^1(\Gamma_0, \mathfrak{g}_v), \tag{4}$$

whence $H^1(\Gamma_0, \mathfrak{g}_\infty) = 0$.

59 (b) N is finite, and therefore, L has finite index in its normalizer (Proposition 3.3). Again, there remains to show that $H^1(L, \mathfrak{g}_\infty) = 0$. Let $\mathbf{SL}_2 \to G$ be the covering map, and L' the inverse image of L in $\mathbf{SL}(2, k)$. The homomorphism $H^1(L, \mathfrak{g}_\infty) \to H^1(L', \mathfrak{g}_\infty)$ is injective, hence we may assume $G = \mathbf{SL}_2$. But then the vanishing of H^1 follows from [29].

Remark. It is also true that in both cases of Theorem 3.6, the group L is not isomorphic to a proper subgroup of finite index. This is seen by modifying the proof of Theorem 3.6, in the same way as Proposition 1.7 was obtained from Theorem 1.5.

If G is almost simple over k, of k-rank $\geqslant 2$, then we may apply Theorem 1.9 to G'. Thus, in that case, we see that, if L contains $Z(G)_k$, the determination of $\operatorname{Aut} L$ is essentially reduced to that of the normalizer of L in G, of the homomorphisms of L into its center, and of the exterior automorphisms of G' leaving stable. We shall use this in §4 to get more explicit information when G is a split group. Here, we mention another consequence of Theorem 1.9.

Proposition 3.7. *Let G be an almost absolutely simple k-group, of k-rank $\geqslant 2$, k' a number field, and G' an almost absolutely simple k'-group. Let L be an arithmetic subgroup of G_k, and s an isomorphism of L onto an arithmetic subgroup of G'. Then there is an isomorphism ϕ of k' onto k and the k-group ${}^{\phi}G'$ obtained from G' by change of the groundfield ϕ is k-isogeneous to G.*

Let $H = R_{k/\mathbf{Q}}G$, $H' = R_{k'/\mathbf{Q}}G'$, and M, M' the images of L and $L' = s(L)$ under the canonical isomorphisms $G_k \xrightarrow{\sim} H_{\mathbf{Q}}$ and $G'_{k'} \xrightarrow{\sim} H'_{\mathbf{Q}}$.

Then s may be viewed as an isomorphism of M onto M'. The group M' is infinite, hence $H'_{\mathbf{R}}$ is not compact, and M' is Zariski-dense in H' ([6], Theorem 1). By Lemma 1.8, H and H' are $\mathbf{Q}$-isogeneous. There exists therefore an isomorphism α of $\mathfrak{h}_{\mathbf{Q}}$ onto $\mathfrak{h}'_{\mathbf{Q}}$. But the commuting algebra of $\operatorname{ad}\mathfrak{h}_{\mathbf{Q}}$ (resp. $\operatorname{ad}\mathfrak{h}'_{\mathbf{Q}}$) in the ring of linear transformations of $\mathfrak{h}_{\mathbf{Q}}$ (resp. $\mathfrak{h}'_{\mathbf{Q}}$) into itself is isomorphic to k (resp. k'). Hence α induces an isomorphism $\phi : k' \xrightarrow{\sim} k$. Let $\mathfrak{g}''_k = {}^{\phi}\mathfrak{g}_{k'}$ be the Lie algebra over k obtained from $\mathfrak{g}'$ by the change of ground-field ϕ. Then, it is clear from **60**
the definition of ϕ that $\alpha = R_{k/\mathbf{Q}}\beta$, where β is a k-isomorphism of $\mathfrak{g}$ onto $\mathfrak{g}''$. This isomorphism is then the differential of a k-isogeny of the universal covering of G onto the k-group ${}^{\phi}G'$.

3.8 We need some relations between $(\operatorname{Aut} G)_k$ and $(\operatorname{Aut} G')_{\mathbf{Q}}$. For simplicity, we establish them in the context of Lie algebras, and assume G

to be almost simple over $\overline{k}$. The Lie algebra $\mathfrak{g}'_{\mathbf{Q}}$ is just $\mathfrak{g}_k$, viewed as a Lie algebra over $\mathbf{Q}$. Since $\mathfrak{g}_k$ is absolutely simple, the commuting algebra of $\operatorname{ad}\mathfrak{g}'_{\mathbf{Q}}$ in $\mathfrak{gl}(\mathfrak{g}'_{\mathbf{Q}})$ may be identified to k. Let $a \in \operatorname{Aut}\mathfrak{g}'_{\mathbf{Q}}$. Then a defines an automorphism of $\mathfrak{gl}(\mathfrak{g}'_{\mathbf{Q}})$ leaving $\operatorname{ad}\mathfrak{g}'_{\mathbf{Q}}$ stable, and therefore an automorphism $\beta(a)$ of k. If β is the identity, this means that a is a k-linear map of $\mathfrak{g}'_{\mathbf{Q}}$, hence comes from an automorphism of $\mathfrak{g}_k$. We have therefore an exact sequence

$$1 \to \operatorname{Aut}\mathfrak{g}_k \to \operatorname{Aut}\mathfrak{g}'_{\mathbf{Q}} \to \operatorname{Aut}k. \tag{1}$$

Let k_0 be the fixed field of $\operatorname{Aut}k$ in k. Assume that $\mathfrak{g}_k = \mathfrak{g}_0 \otimes_{k_0} k$, where $\mathfrak{g}_0$ is a Lie algebra over k_0. Then, for $s \in \operatorname{Aut}k$, ${}^s\mathfrak{g}_k = \mathfrak{g}_k$, and s, acting by conjugation with respect to $\mathfrak{g}_0$, defines a s-linear automorphism of $\mathfrak{g}_k$, and therefore an automorphism a of $\operatorname{Aut}\mathfrak{g}_{\mathbf{Q}}$ such that $\beta(a) = s$. Thus, in this case, the sequence

$$1 \to \operatorname{Aut}\mathfrak{g}_k \to \operatorname{Aut}\mathfrak{g}'_{\mathbf{Q}} \to \operatorname{Aut}k \to 1 \tag{2}$$

is exact and split. Translated into group terms, this yields the following lemma:

Lemma 3.9. *Let G be absolutely almost simple over k. Then we have an exact sequence*

$$1 \to (\operatorname{Aut}G)_k \to (\operatorname{Aut}G')_{\mathbf{Q}} \to \operatorname{Aut}k. \tag{1}$$

Let k_0 be the fixed field of Aut k *and assume that G is obtained by extension of the field of definition from a k_0-group G_0. Then the sequence*

$$1 \to (\operatorname{Aut}G)_k \to (\operatorname{Aut}G')_{\mathbf{Q}} \to \operatorname{Aut}k \to 1 \tag{2}$$

is exact and split. On G_k, identified with $G_{0,k}$, the group Aut k *acts by conjugation.*

61 Strictly speaking, the sequences (3.8) (1), (2) give Lemma 3.9 (1), (2) if G is centerless or simply connected (the only cases of interest below). But in the general case, we may argue in the same way as above, replacing $\operatorname{Aut}\mathfrak{g}_k$ and $\operatorname{Aut}\mathfrak{g}'_{\mathbf{Q}}$ by the images of $(\operatorname{Aut}G)_k$ and $(\operatorname{Aut}G')_{\mathbf{Q}}$

in those groups. The proof can also be carried out directly in G and G', using the structure of $R_{k/\mathbf{Q}}G$, and is then valid of $\mathbf{Q}$, k and $\mathrm{Aut}\,k$ are replaced by a field K, a finite separable extension K' of K, and $\mathrm{Aut}(K'/K)$.

Remark. The above lemma was obtained with the help of Serre, who has also given examples where G has no k_0-form and (2) is not exact.

4 Split groups over number fields.

In this paragraph, G is a connected almost simple k-split group. G is viewed as obtained by extension of the groundfield from a $\mathbf{Q}$-split group G_0, endowed with the $\mathbf{Z}$-structure associated to a splitting over $\mathbf{Q}$. G is then endowed with an $\mathfrak{o}$-structure associated to its given splitting, and G_B is well defined for any $\mathfrak{o}$-algebra B. We shall be interested mainly in the canonical S-arithmetic subgroup $G_{\mathfrak{o}(S)}$.

Lemma 4.1. *Let G be split over k, almost simple over k, and $L = G_{\mathfrak{o}(S)}$.*

(i) *L is equal to its normalizer in G_k. The image in $G/Z(G)$ of the normalizer $N(L)$ of L in G is equal to $(G/Z(G))_{\mathfrak{o}(S)}$. In particular, $L = N(L)$ if G is centerless.*

(ii) *The group $N(L)/L$ is a finite commutative group whose exponent divides the order m of $Z(G)$.*

(i) Let Γ be a Chevalley lattice in $\mathfrak{g}_{0,\mathbf{Q}}$. Then ([16], 2.17) shows that $G_{\mathfrak{o}(S)}$ is the stabilizer of $\mathfrak{o}(S)$. Γ in G_k, operating on $\mathfrak{g}$ by the adjoint representation. The lattice Γ is spanned by the logarithms of the unipotent elements in $G_{0,\mathbf{Z}}$, hence $\mathfrak{o}(S)$. Γ is spanned by the logarithms of unipotent elements in $G_{\mathfrak{o}(S)}$. It is then clear that if $x \in G$ normalizes $G_{\mathfrak{o}(S)}$, then $\mathbf{Ad}x$ normalizes $\mathfrak{o}(S)\cdot\Gamma$. If moreover $x \in G_k$, then $x \in G_{\mathfrak{o}(S)}$, which proves the first assertion. Together with Proposition 3.3, this proves (i).

(ii) The group $N(L)/L$ is finite by Proposition 3.3. The other asser- **62**
tions of (ii) follow from (i) and Proposition 3.4.

Lemma 4.2. *Let $G = \mathbf{SL}_2$, $\mathbf{PSL}_2$ and L a S-arithmetic subgroup of G. Assume that S has at least two elements. Let s be an automorphism of L. There exists an automorphism s' of G', defined over $\mathbf{Q}$, and a homomorphism f of L into $Z(G')_{\mathbf{Q}}$ such that $s(x) = f(x) \cdot s'(x) (x \in L)$.*

Let $\widetilde{G} = \mathbf{SL}_2$, $\pi : \widetilde{G} \to G$ the natural homomorphism and $\widetilde{L} = \pi^{-1}(L) \cap G_k$. Then $\widetilde{L}$ is S-arithmetic in $\widetilde{G}$. The map $s \circ \pi$ defines a homomorphism of $\widetilde{L}$ into $G'_{\mathbf{Q}}$. It follows from [29] that there exists a $\mathbf{Q}$-morphism $t : R_{k/\mathbf{Q}}\widetilde{G} \to G'$, which coincides with $s \circ \pi$ on a normal subgroup of finite index of $\widetilde{L}$. The end of the argument is then the same as in Lemma 1.8.

Theorem 4.3. *Let* $\mathrm{Aut}(k, S)$ *be the subgroup of* $\mathrm{Aut}\, k$ *leaving S stable. Assume either* $\mathrm{rk}_k\, G \geqslant 2$ *or* $\mathrm{rk}_k\, G = 1$ *and* $\mathrm{Card}\, S \geqslant 2$. *Let $L = G_{\mathfrak{o}(S)}$.*

(i) *If G is centerless,* $\mathrm{Aut}\, L$ *is generated by $E(G)$, the group* $\mathrm{Aut}(k, S)$ *acting by conjugation, and* $\mathrm{Int}\, L$.

(ii) *If G is simply connected,* $\mathrm{Aut}\, L$ *is generated by* $E(G)$, $\mathrm{Aut}(k, S)$, *and automorphisms of the form $x \mapsto f(x) \cdot y \cdot x \cdot y^{-1}$ where f is a homomorphism of L into its center, and y belongs to the normalizer of L in G.*

By Lemma 4.1, L contains $Z(G)_k$. Let $s \in \mathrm{Aut}\, L$. By Lemma 1.8 and Lemma 4.2 we may write $s(x) = f(x) \cdot s'(x)$ where s' is a $\mathbf{Q}$-automorphism of G' and f a homomorphism of L into $Z(G')_{\mathbf{Q}} \cong Z(G)_k$, hence of L into its center.

The group G comes by extension of the groundfield from a split $\mathbf{Q}$-group G_0. Therefore Lemma 3.9 obtains. After having modified s by a field automorphism J, we may consequently assume s' to belong to $(\mathrm{Aut}\, G)_k$. In both cases (i), (ii) $(\mathrm{Aut}\, G)_k$ is a split extension of $E(G)$ by $(\mathrm{Int}\, G)_k$; moreover, the representative $E'(G)$ of $E(G)$ alluded to in
63 Theorem 2.2 leaves L stable. Thus, after having multiplied s' by an element of $E(G)$, we may assume $s' \in (\mathrm{Int}\, G)_k$, hence $s' = \mathrm{Int}\, y$, $(y \in N(L))$.

4.4 Let G have a non-trivial center. We assume that the underlying $\mathbf{Q}$-split group G_0 may be (and is) identified with a $\mathbf{Q}$-subgroup of $\mathbf{GL}_n$ by means of an irreducible representation all of whose weights are extremal, i.e. form one orbit under the Weyl group, in such a way that $\mathbf{Z}^n$ is an admissible lattice, in the sence of [10]. (This assumption is fulfilled in all cases, except for the one of the spinor group in a number of variables multiple of four.)

Let D be the group of scalar multiples of the identity in $\mathbf{GL}_n$, and $H = D \cdot G$. The group H is the identity component of the normalizer of G in $\mathbf{GL}_n$. The group D is a one-dimensional split torus. In particular, its first Galois cohomology group is zero. We have $G \cap D = Z(G)$, and $G \subset \mathbf{SL}_n$, therefore the order m of $Z(G)$ divides n, and Proposition 3.4 (i) yeilds

$$N(L) = G \cap N_H(L)_k \cdot D. \tag{1}$$

Lemma 4.5. *We keep the assumptions of 4.4. Let m be the order of $Z(G)$. Let A and B be the images of $N(L)$ and $H_{\mathfrak{o}(S)}$ in* Aut L.

(i) *The enveloping algebra M of L over $\mathfrak{o}(S)$ is $\mathbf{M}(n, \mathfrak{o}(S))$.*

(ii) *A/B is isomorphic to a subgroup of*

$${}_mI(k, S) \quad \textit{and} \quad B \quad \textit{to} \quad \mathfrak{o}(S)^*/\mathfrak{o}(S)^{*(m)}.$$

(i) In view of the definition of admissible lattices [10], the maximal k-split torus T of the given splitting G may be assumed to be diagonal and the $\mathfrak{o}(S)$-lattice $\Gamma_0 = \mathfrak{o}(S)^n$ is the direct sum of its intersections with the eigenspaces of T. Out assumption on the weights implies further that these eigenspaces are one-dimensional, permuted transitively by the normalizer $N(T)$ of T.

Given a prime ideal $v \in V = V_\infty$, we denote by F_v the residue field $\mathfrak{o}/v$ and by $\overline{F}_v$ an algebraic closure of F_v. By [10], reduction mod v of G, (endowed with its canonical $\mathfrak{o}$-structure), yields a F_v-subgroup $G_{(v)}$ of $\mathbf{GL}(n, \overline{F}_v)$ which is connected, almost simple, has the same Dynkin diagram as G, and is simply connected if G is. The reduction mod v also defines an isomorphism of the **64** character group $X^*(T)$ of T onto the character group $X^*(T_{(v)})$ of the reduction mod v of T, which induces a bijection of the weights of the identity representation of G onto those of the identity representation of $G_{(v)}$. Thus the eigenspaces of $T_{(v)}$ are one-dimensional, and permuted transitively by the normalizer of $T_{(v)}$. Consequently, the identity representation of $G_{(v)}$ is irreducible.

The given splitting of G defines one of the universal covering $\widetilde{G}$ of G, hence an $\mathfrak{o}$-structure on $\widetilde{G}$. The reduction mod $v\widetilde{G}_{(v)}$ of G is the universal covering group of $G_{(v)}$ and the identity representation may be viewed as a irreducible representation of $\widetilde{G}_{(v)}$, say $f_{(v)}$. But $f_{(v)}$ has only extremal weights, therefore is a fundamental representation. It follows then from results of Steinberg ([30]; 1.3, 7.4) that the representation $f_{(v)}$ of the finite group $\widetilde{G}_{(v),F_v}$ is absolutely irreducible. Now, since reduction mod v is good, $\widetilde{G}_{(v),F_v}$ is the reduction of $\widetilde{G}_{\mathfrak{o}_v}$. Moreover, $\widetilde{G}$ being split and simply connected, strong approximation is valid in $\widetilde{G}$, hence $\widetilde{G}_{\mathfrak{o}}$ is dense in $\widetilde{G}_{\mathfrak{o}_v}$, which implies that reduction mod v maps $\widetilde{G}_{\mathfrak{o}}$ onto $\widetilde{G}_{(v),F_v}$. But the canonical projection of $\widetilde{G}$ onto G maps $\widetilde{G}_{\mathfrak{o}}$ into $\widetilde{G}_{\mathfrak{o}}$. Consequently, the image of $G_{\mathfrak{o}}$ in $G_{(v)}$ by reduction is a subgroup which contains $f_{(v)}(\widetilde{G}_{(v),F_v})$, hence is irreducible. Therefore

$$M \otimes F_v = \mathbf{M}(n, F_v), (v \in V - S).$$

This shows that the index of M in $\mathbf{M}(n, \mathfrak{o}(S))$ is prime to all elements in $V - S$, whence (i).

(ii) By 4.4 (1), the image of $N(L)$ in Aut L is the same as that of $N' = N_H(L)_k$. Let $x \in N'$ and $\Gamma = x \cdot \Gamma_0$ be the transform under x of the standard lattice $\Gamma_0 = \mathfrak{o}(S)^n$. This is a $\mathfrak{o}(S)$-lattice stable under L hence, by (i), also stable under $\mathbf{GL}(n, \mathfrak{o}(S))$. For $v \in V - S$, the local lattice $\mathfrak{o}_v \cdot \Gamma$ in k_v^n is then stable under $\mathbf{GL}(n, \mathfrak{o}_v)$. There exists therefore a power $v^{a(v)} (a(v) \in Z)$ of v such that $\mathfrak{o}_v \cdot \Gamma = v^{a(v)} \cdot \mathfrak{o}_v^n$.
65 We have then also $\mathfrak{o}_v \cdot (\det x) = v^{n \cdot a(v)}$. In view of the relation between a lattice and its localizations, we have then $\Gamma = \mathfrak{a} \cdot \Gamma_0$ with $\mathfrak{a} = \Pi \mathfrak{v}^{a(v)}$, and moreover $\mathfrak{a}^n \cdot \mathfrak{o}(S) = \mathfrak{o}(S) \cdot (\det x)$. By

assigning to x the image of $\mathfrak{a} \cdot \mathfrak{o}(S)$ in $I(k, S)$, we define therefore a map α of N' into ${}_nI(k, S)$, which is obviously a homomorphism. If $d \in D$, then $\alpha(d \cdot x) = \alpha(x)$, whence a homomorphism of A into ${}_nI(k, S)$, to be denoted also by α. Clearly, $H_{\mathfrak{o}(S)} \subset \ker \alpha$. Conversely, assume that $x \in \ker \alpha$. Then $x \cdot \Gamma_0$ is homothetic to Γ_0, and there exists $d \in k^*$ such that $d \cdot x$ leaves Γ_0 stable. But then $d \cdot x \in H_{\mathfrak{o}(S)}$, so that the image of x in $\operatorname{Aut} L$ belongs to B. Thus, A/B is isomorphic to a subgroup of ${}_nI(k, S)$. But $N(L)/L$ is of exponent m by Lemma 4.1, and m divides n, therefore α maps A/B into a subgroup of ${}_mI(k, S)$.

Let $\sigma : H \to H/G$ be the canonical projection. Its restriction to D is the projection $D \to D/Z(G)$, and $H/G = D/Z(G)$. If an element $x \in H_{\mathfrak{o}(S)}$ defines an inner automorphism of L, then $x \in D_{\mathfrak{o}(S)} \cdot L$, and $\sigma(x) \in \sigma(D_{\mathfrak{o}(S)})$. Since elements of $D_{\mathfrak{o}(S)}$ define trivial automorphisms of L, we see that

$$B/\operatorname{Int} L \cong \sigma(H_{\mathfrak{o}(S)})/\sigma(D_{\mathfrak{o}(S)}). \tag{1}$$

Identify $D/Z(G)$ to $\mathbf{GL}_1$. Then $\sigma(H_{\mathfrak{o}(S)})$ is an S-arithmetic subgroup of $\mathbf{GL}_1$ hence a subgroup of finite index of $\mathfrak{o}(S)^*$. The group $Z(G)$ is cyclic of order m, therefore the projection $D = \mathbf{GL}_1 \to D'$ is either $x \mapsto x^m$ or $x \mapsto x^{-m}$, hence $\sigma(D_{\mathfrak{o}(S)}) \cong \mathfrak{o}(S)^{*(m)}$, so that $B/\operatorname{Int} L$ may be identified to a subgroup of $\mathfrak{o}(S)^*/\mathfrak{o}(S)^{*(m)}$. Thus (1) yields an injective homomorphism $\tau : B/\operatorname{Int} L \to \mathfrak{o}(S^*)/\mathfrak{o}(S)^{*(m)}$. There remains to show that τ is surjective.

Let $\pi : H \to H/D = G/Z(G) = \operatorname{Int} G$ be the canonical projection, T the maximal torus given by the splitting of G and $T' = \pi(T)$. We have already remarked that $x \in H_{\mathfrak{o}(S)}$ defines an inner automorphism of L if and only if $x \in D_{\mathfrak{o}(S)} \cdot L$, so $B/\operatorname{Int} L \cong \pi(H_{\mathfrak{o}(S)})/\pi(L)$. By Lemma 4.1, $\pi(N(L)) \cong (G/Z(G))_{\mathfrak{o}(S)}$. On the other hand, since TD is split, D is a direct factor over k; this implies immediately that $\pi : (TD)_{\mathfrak{o}(S)} \to T'_{\mathfrak{o}(S)}$ is surjective, hence $\pi(H_{\mathfrak{o}(S)}) \cap T' = T'_{\mathfrak{o}(S)}$. We have $\pi(D_{\mathfrak{o}(S)} \cdot L) = \pi(L)$, and consequently, since $\ker \pi \cap G \subset T$,

$$\pi(D_{\mathfrak{o}(S)} \cdot L) \cap T' = \pi(L \cap T)\pi = (T_{\mathfrak{o}(S)});$$

66 hence $B/\operatorname{Int} L$ contains a subgroup isomorphic to $T'_{\mathfrak{o}(S)}/\pi(T_{\mathfrak{o}(S)})$. However, the kernel of $\pi : T \to T'$ is a cyclic group of order m. It is then elementary that we can write $T = T_1 \times T_2$, over k, with T_1 containing $Z(G)$ of dimension one. This implies

$$T'_{\mathfrak{o}(S)}/\pi(T_{\mathfrak{o}(S)}) \cong \pi(T_1)_{\mathfrak{o}(S)}/\pi(T_{1,\mathfrak{o}(S)}) = \mathfrak{o}(S)^*/\mathfrak{o}(S)^{*(m)};$$

this shows that the order of $B/\operatorname{Int} L$ exceeds that of $\mathfrak{o}(S)^*/\mathfrak{o}(S)^{*(m)}$. Therefore T is surjective.

Examples 4.6. (1) $G = \mathbf{SL}_n \cdot H = \mathbf{GL}_n$, $(n \geqslant 3)$. The group $L = \mathbf{SL}(n, \mathfrak{o}(S))$ is equal to its derived group [3], Corollary 4.3. By Lemma 4.1 $\operatorname{Aut} L$ is generated by $\operatorname{Aut}(k, S)$, acting by conjugation on the coefficients, by the automorphism $x \mapsto {}^t x^{-1}$, and by the image A in $\operatorname{Aut} L$ of $N(L)$.

If $\mathfrak{a}$ is an $\mathfrak{o}(S)$-ideal, then $\mathfrak{a} \cdot \Gamma_0$ is isomorphic to $\mathfrak{a}^n \oplus \mathfrak{o}(S)^{n-1}$ by standard facts on lattices. Therefore, if $\mathfrak{a}^n$ is principal, then $\mathfrak{a} \cdot \Gamma_0$ is isomorphic to Γ_0 and there exists $g \in \mathbf{GL}(n, k)$ such that $g \cdot \Gamma_0 = \mathfrak{a} \cdot \Gamma_0$. But the stabilizer of Γ_0 in G is the same as that of $\mathfrak{a} \cdot \Gamma_0$, hence $g \in N_H(L)_k$, which shows that, in this case, the monomorphism $A/B \to {}_nI(k, S)$ is an isomorphism. We have therefore a composition series

$$\operatorname{Aut} L \supset A' \supset A \supset B \supset \operatorname{Int} L,$$

whose successive quotients are isomorphic to $\operatorname{Aut}(k, S)$, $\mathbf{Z}/2\mathbf{Z}$, ${}_nI(k, S)$ and $\mathfrak{o}(S)^*/\mathfrak{o}(S)^{*(n)}$.

This result is contained in [24], where $\operatorname{Aut} \mathbf{SL}(n, Q)$ is determined for any commutative integral domain Q, except for the fact that the structure of the subgroup corresponding to $A/\operatorname{Int} L$ is not discussed there. For $\mathfrak{o}(S) = \mathfrak{o}$, it is related to those of [19] if k has class number one, and of [20] if $k = \mathbf{Q}(i)$.

(2) $G = \mathbf{SL}_2$, $\operatorname{card} S \geqslant 2$. The above discussion of $A/\operatorname{Int} L$ is still valid, (without restriction on S, in fact). Furthermore, the contragredient mapping $x \mapsto {}^t x^{-1}$ is an inner automorphism for $n = 2$.

However, in general, L is not equal to its commutator subgroup, and $L/(L, L)$ has a non-trivial 2-primary component. Therefore there may be non-trivial automorphisms of the form $x \to f(x) \cdot x$ where f is a character of order two of L. Clearly, such a homo- **67**
morphism of L into itself is bijective if and only if $\chi(-1) = 1$. It follows from Lemma 4.2 that $\operatorname{Aut} L$ is generated by automorphisms of the previous type, field automorphisms, and elements of A.

We note that this conclusion does not hold true without some restriction on k, S. For instance, there is one further automorphism if $k = \mathbf{Q}(i)$, $\mathfrak{o}(S) = \mathbf{Z}(i)$, (see [20], and also [21] for a further discussion of the case $n = 2$).

(3) $G = \mathbf{Sp}_{2n}$, $L = \mathbf{Sp}(2n, \mathfrak{o}(S))$. The commutator subgroup of L is equal to L if $n \geqslant 3$, and has index a power of two if $n = 2$ ([3], Remark to 12.5). The group G has no outer automorphisms, therefore, if $n \geqslant 3$, Theorem 4.3, and Lemma 4.5 show that we have a composition series

$$\operatorname{Aut} L \supset A \supset B \supset \operatorname{Int} L,$$

with

$$\operatorname{Aut} L/A \cong \operatorname{Aut}(k, S), \quad B/\operatorname{Int} L \cong \mathfrak{o}(S)^*/\mathfrak{o}(S)^{*(2)},$$

and A/B isomorphic to a subgroup of ${}_2I(k, S)$. We claim that in fact

$$A/B \cong {}_2I(k, S).$$

We write the elements of $\mathbf{GL}_{2n}$ as 2×2 matrices whose entries are $n \times n$ matrices. $\mathbf{Sp}_{2n}$ is the group of elements in $\mathbf{GL}_{2n}$ leaving $J = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ invariant, and its normalizer H in $\mathbf{GL}_{2n}$ is the group of similitudes of J. Let $\mathfrak{a}$ be an $\mathfrak{o}(S)$-ideal such that $\mathfrak{a}^2$ is principal. As remarked above, there exists $x \in \mathbf{GL}(2, k)$ such that $x \cdot \mathfrak{o}(S)^2 = \mathfrak{a} \cdot \mathfrak{o}(S)^2$. Let y be the element of $\mathbf{GL}(2n, k)$ which acts via x on the space spanned by the i-th and $(n+i)$-th canonical basis vectors

$(i = 1, \ldots, n)$. Then $y \in H_k$, and $y \cdot \mathfrak{o}(S)^{2n} = \mathfrak{a} \cdot \mathfrak{o}(S)^{2n}$. Thus, y is an element of $N_H(L)_k$ which is mapped onto the image of $\mathfrak{a}$ in ${}_2I(k, S)$ by the homomorphism $\alpha : A/B \to {}_2I(k, S)$ of Lemma 4.5. Hence, α is also surjective.

If $n = 2$, $\operatorname{Aut} L$ is obtained by combining automorphisms of the above types with those of the form $x \mapsto f(x) \cdot x$, where f is a homomorphism of L into ± 1 whose kernel contains -1.

68 **Remark 4.7.** It was noticed in Lemma 4.1 that L is equal to its normalizer in G_k. Since L has finite index in its normalizer in G, (Proposition 3.3 (d)), this means that L is not a proper normal subgroup of an arithmetic subgroup of G. More generally, we claim that L is maximal among arithmetic subgroups, i.e. that no subgroup M of G_k contains L as a proper subgroup of finite index. This was proved by Matsumoto [23] when $S = V_\infty$, and his proof extends immediately to the present case. In fact, the argument in the proof of Theorem 1 of [23] shows that if L has finite index in $M \subset G_k$, then the closure of M in $G_v (v \in V, v \notin S)$ is contained in $G_{\mathfrak{o}_v}$, whence $M \subset L$.

5 Uniform subgroups in G_S.

In §1, we proved the finiteness of $E(L)$ for subgroups which are either uniform or arithmetic. In §3 the arithmetic case was extended to S-arithmetic groups. Now a S-arithmetic group may be viewed as a discrete subgroup of G_S, which is irreducible in the sense that its intersection with any proper partial product of the G_v's $(v \in S)$ reduces to the identity. We wish to point out here that there is also a generalization to G_S of the uniform subgroup case. We assume that $S \neq V_\infty$. Such groups have been considered by Ihara [14] for $G = \mathbf{SL}_2$, and Lemma 5.1 is an easy extension of results of his.

Lemma 5.1. *Let G be a connected semi-simple, almost simple k-group, L a uniform irreducible subgroup of G_S, and L' its projection on G_∞.*

(i) *L is finitely generated.*

(ii) *If $\operatorname{rank} G \geqslant 2$ and G_∞ has no compact or three-dimensional factor or $k = \mathbf{Q}$, $G = \mathbf{SL}_2$, then $H^1(L', \mathfrak{g}_\infty) = 0$.*

(i) Let $S' = S - V_\infty$, $G_{S'} = \prod_{v \in S'} G_v$, and $K = G_\infty \times \prod_{v \in S'} G_{0v}$. The latter is an open subgroup of G_S. The orbits of K in G_S/L are open, hence closed, hence compact. Therefore $L_0 = L \cap K$ is uniform in K. Since K is the product of G_∞ by a compact group, the projection L_0' of L_0 in G_∞ is a discrete uniform subgroup of G_∞. But G_∞ is a real Lie group with a finite number of connected components. Therefore the standard topological argument shows **69**
that L_0 is finitely generated. Let L'' be the projection of L on $G_{S'}$. Since L is uniform in G_S, there exists a compact subset C of $G_{S'}$ such that $G_S = L'' \cdot C$. On the other hand, it follows from ([9], 13.4) that $G_{S'}$ has a compact set of generators, say D. Then the standard Schreier-Reidemeister procedure to find generators for a subgroup shows that L is generated by $L \cap (G_\infty \times D \cdot C \cdot D^{-1})$ and consequently by L_0 and finitely many elements. (This argument is quite similar to the one used by Kneser [17] to prove the finite generation of G_S.)

(ii) We first notice that the restriction map

$$r : H^1(L', \mathfrak{g}_\infty) \to H^1(L_0', \mathfrak{g}_\infty)$$

is injective. The argument is the same as one of Ihara's ([14], p.269) in the case $G = \mathbf{SL}_2$: if $x \in G_S$, then $x \cdot K \cdot x^{-1}$ is commensurable with K, hence, if $x \in L$, the group $L_{0,x} = x \cdot L_0 \cdot x^{-1} \cap L_0$ has finite index in L_0. In particular, $L_{0,x}'$ is uniform in G_∞, hence, by density [4], has no fixed vector $\neq 0$ in $\mathfrak{g}_\infty$. This implies by Lemma 3.5 that $\ker r = 0$. If $\mathrm{rk}(G) \geqslant 2$, then $H^1(L_0', \mathfrak{g}_\infty) = 0$ by [32] and [33], whence our assertion in this case. If $G = \mathbf{SL}_2$, $k = \mathbf{Q}$, the vanishing of $H^1(L', \mathfrak{g}_\infty)$ has been proved by Ihara, loc. cit. (it is stated there only in the case where S consists of ∞ and one prime, but the proof is a *fortiori* valid in the more general case).

Theorem 5.2. *Let G and L be as in Lemma 5.1 (ii). Then $E(L)$ is finite, and L is not isomorphic to a proper subgroup of finite index.*

Identify L to its projection L' in G_∞. Then the theorem follows from Lemma 1.1 and Lemma 1.3 in the same way as in the case $S = V_\infty$.

APPENDIX

6 On compact Clifford-Klein forms of symmetric spaces with negative curvature.

6.1 Let M be a simply connected and connected Riemannian symmetric space of negative curvature, without flat component. A Clifford-Klein form of M is the quotient M/L of M by a properly discontinuous group of isometries acting freely, endowed with the metric induced from the given metric on M. In an earlier paper (Topology 2 (1963), 111-122),
70 it was proved that M always has at least one compact Clifford-Klein form. In answer to a question of H. Hopf, we point out here that M always has infinitely many different compact forms. More precisely:

Theorem 6.2. *Let M be as in 6.1. Then M has infinitely many compact Clifford-Klein forms with non-isomorphic fundamental groups.*

M is the direct product of irreducible symmetric spaces. We may therefore assume M to be irreducible. Then $M = G/K$, where G is a connected simple non-compact Lie group, with center reduced to $\{e\}$, and K is a maximal compact subgroup of G. Moreover, G is the identity component of the group of isometries of M. Let L be a discrete uniform subgroup of G, without elements of finite order $\neq e$. Then L operates freely, in a properly discontinuous manner, on M, and M/L is compact. Moreover, by a known result of Selberg (see e.g. loc. cit., Theorem B), L has subgroups of arbitrary high finite index. Since M is homeomorphic to euclidean space, L is isomorphic to the fundamental group of M/L; it suffices therefore to show that L is not isomorphic to any proper subgroup L' of finite index. If $\dim G = 3$, then M is the upper half-plane, and this is well known. It follows for instance from the relations

$$\chi(M/L') = [L : L'] \cdot \chi(M/L) \neq 0, \tag{1}$$

where $[L : L']$ is the index of L' in L, and $\chi(X)$ denotes the Euler-Poincaré-characteristic of the space X. If $\dim G > 3$, our assertion is a consequence of Proposition 1.7. If $\chi(M/L) \neq 0$, which is the case if and only if G and K have the same rank, one can of course also use (1).

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ON “ABSTRACT” HOMOMORPHISMS OF SIMPLE ALGEBRAIC GROUPS

By A. Borel *and* J. Tits

75 This Note describes some results pertaining chiefly to homomorphisms of groups of rational points of semi-simple algebraic groups, and gives an application to a conjecture of Steinberg’s [9] on irreducible projective representations. Some proofs are sketched. Full details will be given elsewhere.

Notation. The notation and conventions of [1] are used. In particular, all algebraic groups are affine, k is a commutative field, $\bar{k}$ an algebraic closure of k, p its characteristic, and G is a k-group. In this Note, G is moreover assumed to be *connected*. k' also denotes a commutative field.

Let $\phi : k \to k'$ be a (non-zero) homomorphism. We let ${}^{\phi}G$ be the k'-groups $G \underset{k}{\otimes} k'$ obtained from G by the change of basis ϕ, and ϕ_0 be the canonical homomorphism $G_k \to {}^{\phi}G_{k'}$ associated to ϕ.

If $p \neq 0$, then Fr^i denotes the p^i-th power homomorphism $\lambda - \lambda^{p^i}$ of a field of characteristic $p (i = 0, 1, 2, \ldots)$. If $p = 0$, Fr^i is the identity.

A connected semi-simple k-group H is *adjoint* if it is isomorphic to its image under the adjoint representation, *almost simple* (resp. *simple*) over k if it has no proper normal k-subgroup of strictly positive dimension (resp. $\neq \{e\}$).

1 Homomorphisms.

1.1 Let G be semi-simple. G^+ will denote the subgroup of G_k generated by the groups U_k, where U runs through the unipotent radicals of the parabolic k-subgroups of G. The group G^+ is normal in G_k; it is $\neq \{e\}$ if and only if $\mathrm{rk}_k(G) > 0$. If, moreover, G is almost simple over k, then G^+ is Zariski-dense in G, and the quotient of G^+ by its center is

simple except in finitely many cases where k has two or thee elements [10]. If $f : G \to H$ is a central k-isogeny, then $f(G^+) = H^+$. The group G^+ is equal to G_k if $k = \overline{k}$, or if G is k-split and simple connected; it is conjectured to be equal to G_k if G is simply connected and **76**
$\mathrm{rk}_k(G) > 0$[10]. It is always equal to its commutator subgroup.

Theorem 1.2. *Assume k to be infinite, and G to be almost absolutely simple, of strictly positive k-rank. Let H be a subgroup of G_k containing G^+. Let k' be a commutative field, G' a connected almost absolutely simple k'-group, and $\alpha : H \to G'_{k'}$ a homomorphism whose kernel does not contain G^+, and whose image contains ${G'}^+$. Assume finally that either G is simply connected or G' is adjoint. Then there exists an isomorphism $\phi : k \xrightarrow{\sim} k'$, a$k'$-isogeny $\beta : {}^{\phi}G \to G'$, and a homomorphism γ of H into the center of $G'_{k'}$ such that $\alpha(x) = \beta(\phi_0(x)) \cdot \gamma(x)(x \in H)$. Moreover, β is central, except possibly in the cases : $p = 3$, G, G' split of type $\mathbf{G}_2$; $p = 2$, G, G' split of type $\mathbf{F}_4$; $p = 2$, G, G' split of type $\mathbf{B}_n$, $\mathbf{C}_n$, where β may be special.*

(The special isogenies are those discussed in [3, Exp. 21-24].) In the following corollary, G and G' need not satisfy the last assumption of the theorem.

Corollary 1.3. *Assume G_k is isomorphic to $G'_{k'}$. Then k is isomorphic to k', and G, G' are of the same isogeny class.*

Let $\overline{G}$ and $\overline{G}'$ be the adjoint groups of G and G'. The assumption implies the existence of an isomorphism $\alpha : \overline{G}^+ \xrightarrow{\sim} \overline{G'}^+$. By the theorem there is an isomorphism ϕ of k onto k' and an isogeny μ of ${}^{\phi}\overline{G}$ onto $\overline{G'}$, whence our assertion.

Remarks 1.4. (i) It may be that the homomorphism γ in (1.2) is always trivial. It is obviously so if G' is adjoint, or if H is equal to its commutator subgroup. Since G^+ is equal to its commutator group, this condition will be fulfilled if G is simply connected and the conjecture $G_k = G^+$ of [10] is true, thus in particular if G splits over k. Moreover, in that case the assumption $G^+ \subset \ker\alpha$ would be superfluous.

(ii) The theorem has been known in many special cases, starting with the determination of the automorphism group of the projective linear group [7]. We refer to Dieudonné's survey [4] for the automorphisms of the classical groups. For split groups over infinite fields, see also [6].

77 (iii) Assume $k = k'$, $G = G'$, G adjoint, and k not to have any automorphism $\neq$ id. Theorem 1.2 implies then that every automorphism of G_k is the restriction of an automorphism of G, which is then necessarily defined over k. In particular, if k is the field of real numbers $\mathbf{R}$, every automorphism of G_k is continuous in the ordinary topology, as was proved first by Freudenthal [5].

(iv) The assumption $\mathrm{rk}_k\, G > 0$ is essential for our proof, but it seems rather likely that similar results are valid for anisotropic groups. This is the case for many classical groups [4]. Also, Freudenthal's proof is valid for compact groups. In fact, the continuity of any abstract-group automorphism of a compact semi-simple Lie group had been proved earlier, independently, by E. Cartan [2] and van der Waerden [11]. We note also that van der Waerden's proof remains valid in the p-adic case.

(v) The group $\mathrm{Aut}\, G_k$ has also been studied when k is finite. See [4] for the classical groups, and [8] for the general case.

Theorem 1.5. *Assume k to be infinite, and G to be almost simple, split over k. Let G' be a semi-simple split k'-group, $G'_i (1 \leqslant i \leqslant s)$ the almost simple normal subgroups of G', and $\alpha : G_k \to G'_{k'}$ a homomorphism whose image is Zariski-dense. If $G_k = G^+$, then G' is connected. Assume G' to be connected and either G simply connected or G' adjoint. Then there exist homomorphisms $\phi_i : k \to k'$ and k'-isogenies $\beta_i : \phi_i G \to G'_i (1 \leqslant i \leqslant s)$, which are either central or special, such that*

$$\alpha(x) = \prod_i (\beta_i \circ \phi_{i,0})(x), \quad (x \in G_k).$$

Moreover, $Fr^a \circ \phi_i \neq Fr^b \circ \phi_j$ *if* $p = 0$ *and* $i \neq j$*, or if* $p \neq 0$ *and* $(a, i) \neq (b, j)(1 \leqslant i, j \leqslant s; a, b = 0, 1, 2, \ldots)$.

The proof of Theorem 1.5 goes more or less along the same lines as that of Theorem 1.2. In fact, it seems not unlikely that Theorem 1.5 can be generalized so as to contain Theorem 1.2. We hope to come back to this question on another occasion.

Example 1.6. The following example, which admits obvious generalizations, shows that the assumption of semi-simplicity made on G' in Theorem 1.5 cannot be dropped. **78**

Let $G = \mathbf{SL}_2$, and N be the additive group of 2×2 matrices over $\overline{k}$, of trace zero. Let d be a non-trivial derivation of k. Extend it to a derivation of N_k by letting it operate on the coefficients, and define $h : G_k \to N_k$ by $h(g) = g^{-1} \cdot dg$. Let $G' = G \cdot N$ be the semi-direct product of G and N, where G acts on N by the adjoint representation. Then $g \to (g, h(g))$ is easily checked to be a homomorphism of G_k into G'_k with dense image; clearly, it defines an "abstract" Levi section of G'_k.

2 Projective representations.

2.1 Assume $p \neq 0$. For G semi-simple, let $\mathscr{R}$ or $\mathscr{R}(G)$ be the set of $p^l (l = \operatorname{rank} G)$ irreducible projective representations whose highest weight is a linear combination of the fundamental highest weights with coefficients between 0 and $p - 1$. The following theorem, in a slightly different formulation, was conjectured by R. Steinberg [9], for $k = \overline{k}$. We show below how it follows from Theorem 1.5 and [9], (Theorem 1.1).

Theorem 2.2. *Assume* k *to be infinite,* $p \neq 0$*, and* G *k-split, simple, adjoint. Let* $\pi : G^+ \to \mathbf{PGL}(n, \overline{k})$ *be an irreducible (not necessarily rational) projective representation of* G^+*. Then there exist distinct homomorphisms* $\phi_j : k \to \overline{k}$*, and elements* $\pi_j \in \mathscr{R}({}^{\phi_j}G)(1 \leqslant j \leqslant t)$*, such that* $\pi : \prod_j \pi_j \circ \phi_{j,0}$.

Proof. Let G' be the Zariski-closure of $\pi(G_k)$ in $\mathbf{PGL}_n$. It is also an irreducible projective linear group, hence its center and also its centralizer in $\mathbf{PGL}_n$, or in the Lie algebra of $\mathbf{PGL}_n$, are reduced to $\{e\}$. Thus G' is semi-simple, and its identity component is adjoint. Moreover, by Theorem 1.5, G' is connected. By Theorem 1.1 of [9], there exist elements $\pi_a \in \mathscr{R}(G')$, $(1 \leqslant a \leqslant q)$ such that the identity representation of G' is equal to $\prod\limits_a \pi_a \circ (Fr^a)$. Let $G'_i (1 \leqslant i \leqslant s)$ be the simple factors of G'.

The tensor product defines a bijection of $\mathscr{R}(G'_1) \times \cdots \times \mathscr{R}(G'_s)$ onto $\mathscr{R}(G')$. We may therefore write

$$\pi_a = \prod_{1 \leqslant i \leqslant s} \pi_{a,i}, \quad (\pi_{ai} \in \mathscr{R}(G'_i); 1 \leqslant a \leqslant q).$$

79 Let now $\phi_i : k \to \overline{k}$ and $\beta_i : {}^{\phi_i}G \to G_i$ be as in Theorem 1.5 (with $\overline{k} = k'$). We have then

$$\pi = \prod_{a,i} \pi_{a,i} \circ (Fr^a)_0 \circ \beta_i \circ \phi_{i,0}. \tag{1}$$

But $(Fr^a)_0 \circ \beta_i = \beta_{a,i} \circ (Fr^a)_0$, where $\beta_{a,i}$ is the transform of β_i under Fr^a. Let $\phi_{a,i} = Fr^a \circ \phi_i$. Since G, G_i are adjoint, the morphisms $\beta_{a,i}$ are either isomorphisms or special isogenies. Therefore, taking ([9], §11) into account, we see that

$$\pi'_{a,i} = \pi_{a,i} \circ \beta_{a,i} \in \mathscr{R}({}^{\phi_{a,i}}(G)), \quad (1 \leqslant i \leqslant s; 1 \leqslant a \leqslant q),$$

and (1) yields

$$\pi = \prod_{a,i} \pi'_{a,i} \circ (\phi_{a,i})_0, \tag{2}$$

which proves the theorem, in view of the fact that the $\phi_{a,i}$ are distinct by Theorem 1.5. □

3 Sketch of the proof of Theorem 1.2.

In this paragraph, k is infinite and G is semi-simple, of strictly positive k-rank.

The two following propositions are the starting point of the proofs of Theorem 1.2 and Theorem 1.5.

Proposition 3.1. *Let G' be a k'-group, and $\alpha : G^+ \to G'_{k'}$ a non-trivial homomorphism. Let P be a minimal parabolic k-subgroup of G and U its unipotent radical. Then $\alpha(U_k)$ is a unipotent subgroup contained in the identity component of G' and $\alpha(G^+) \subset G'^0$. The field k' is also of characteristic p.*

Let S be a maximal k-split torus of P. It is easily seen that $S^+ = S \cap G^+$ is dense in S. It follows then from [1, §11.1] that any subgroup of finite index of $S^+ \cdot U_k$ contains elements $s \in S^+$ such that $(s, U_k) = U_k$. From this we deduce first that U_k is contained in any normal subgroup of finite index of $S^+ \cdot U_k$, and then, that it is also contained in the commutator subgroup of any such subgroup. It follows that $\alpha(U_k)$ is contained in the derived group of the identity component of the 80
Zariski closure of $\alpha(S^+ \cdot U_k)$. The latter being solvable, this implies that $\alpha(U_k)$ is unipotent.

Let $p' = \text{char.}\, k'$. If $p \neq 0$, then U_k is a p-group. Its image is a p-group and is $\neq \{e\}$ since α is non-trivial, and G^+ is generated by the conjugates of U_k; hence $p = p'$. If $p = 0$ and $p' \neq 0$, then $\ker \alpha \cap U_k$ has finite index in U, whence easily a contradiction with the main theorem of [10].

Proposition 3.2. *Let G' be a connected semi-simple k'-group. Let P, S, U be as above, P^- the parabolic k-subgroup opposed to P and containing $\mathscr{Z}(S)$, and $U^- = R_u(P^-)$. Let H be a subgroup of G_k containing G^+ and $\alpha : H \to G'_{k'}$ be a homomorphism with dense image. Then the Zariski-closures Q, Q^- of $\alpha(P \cap H)$ and $\alpha(P^- \cap H)$ are two opposed parabolic k'-subgroups, and $Q \cap Q^-$, $R_u(Q)$, $R_u(Q^-)$ are the Zariski-closures of $\alpha(Z(S) \cap H), \alpha(U_k)$ and $\alpha(U^-_k)$ respectively.*

Let M, V, V^- be the Zariski-closures of $\alpha(\mathscr{Z}(S) \cap H)$, $\alpha(U_k)$ and $\alpha(U^-_k)$ respectively. The groups V, V^- are unipotent, by Proposition 3.1. The group G is the union of finitely many left translates of $U^- \cdot P$. Since $\alpha(H)$ is dense, this implies that $V^- \cdot M \cdot V$ contains a non-empty open subset of G'. Let T be a maximal torus of M and Y, Y^- be two maximal unipotent subgroups of M^0 normalized by T such that $Y^- \cdot T \cdot Y$ is open in M^0 (see [1], §2.3, Remarque). Then $V^- \cdot Y^-$ and $Y \cdot V$ are unipotent subgroups of G' normalized by T and $V^- \cdot Y^- \cdot T \cdot Y \cdot V$

contains a non-empty open set of G'. Consequently ([1], §2.3), T is a maximal torus of G', and $V^- \cdot Y^-$, $Y \cdot V$ are two opposed maximal unipotent subgroups. This shows that Q, Q^- are parabolic subgroups, M is reductive, connected, and $V = R_u(Q)$, (resp. $V^- = R_u(Q^-)$). The groups Q, Q^- are obviously k'-closed. Arguing as in Proposition 3.1, we may find $s \in S \cap H$ such that $(s, U_k) = U_k$, $(s, U_k^-) = U_k^-$. It follows then from ([1], §11.1) that $\mathscr{Z}(\alpha(s))^0 = M$. Hence M is defined over k'([1], §10.3). By Grothendieck's theorem ([1], §2.14), it contains a maximal torus defined over k'. Hence ([1], §3.13), Q, Q^-, V, V^- are defined over k'.

3.3 We now sketch the proof of Theorem 1.2, assuming for simplicity that G, G' are adjoint and $H = G_k$. Then α is injective. Propo-
81 sition 3.2, applied to α and α^{-1}, shows that Q, Q^- are two opposed minimal parabolic k'-subgroups of G'. Consequently, α induces an isomorphism of $\mathscr{N}(S)/\mathscr{Z}(S)$ onto $\mathscr{N}(M)/M$, i.e. of ${}_kW(G) = {}_kW$ onto ${}_{k'}W' = {}_{k'}W(G')$. For $a \in {}_k\Phi(G)$, let $U_a = U_{(a)}/U_{(2a)}$, where we put $U_{(2a)} = \{e\}$ if $2a \notin {}_k\Phi$. It may be shown that $U_{(2a)}$ is the center of $U_{(a)}$. The groups $U^-_{(a)}$ may be characterized as minimal among the intersections $U \cap w(P)(w \in {}_kW)$ not reduced to $\{e\}$. It then follows that α induces a bijection $\alpha_* : {}_k\Phi(G) \to {}_{k'}\Phi(G')$ preserving the angles, and isomorphisms $U_{a,k} \xrightarrow{\sim} V_{\alpha*(a),k'}$. The group U_a (resp. $V_{\alpha*(a)}$) may be endowed canonically with a vector space structure such that S (resp. a maximal k'-split torus S' of M) acts on it by dilatations. The next step is to show that $\alpha : U_{a,k} \xrightarrow{\sim} V_{\alpha*(a),k'}$ induces a bijection ϕ_a between the algebras of dilatations. Let L_a be the subgroup of G generated by $U_{(a)}$ and $U_{(-a)}$. The assumption that G is almost absolutely simple is equivalent to the existence of one element $a \in {}_k\Phi$ such that the intersection X_a of L_a with the center C of $\mathscr{Z}(S)$ is one-dimensional, hence such that $X_a^0 \subset S$. This is the main tool used in showing that $\alpha(S_k) \subset S'_{k'}$, hence that α maps dilatations by elements of $(k^*)^2$ into dilatations. If $p \neq 2$, this suffices to yield the existence of $\phi_a : k \xrightarrow{\sim} k'$. In characteristic two, some further argument, based on properties of groups of rank one, is needed. It is clear that $\phi_a = \phi_b$ if $b \in {}_kW(a)$. Using further some facts

about commutators, it is then easily proved that $\phi_a = \phi_b (a, b \in {}_k\Phi)$ if α_* preserves the lengths. If not, we show that we are in one of the exceptional cases listed in the theorem, and we reduce it to the preceding one by use of a special isogeny. Write then ϕ instead of ϕ_a. Replacing G by ${}^{\phi}G$, we may assume $k = k'$, $\phi = \mathrm{id}$. It is then shown that $\alpha : U_k \xrightarrow{\sim} V_k$ is the restriction of a k-isomorphism of varieties. On the other hand, since G' is adjoint, $\mathscr{Z}(S')$ is isomorphic to its image in $GL(\mathbf{b})$ under the adjoint representation, where $\mathbf{b}$ is the sum of Lie algebras of the $V_{a'} (a' \in {}_k\Phi(G'))$. This implies readily that the restriction of α to $U_k^- \cdot P_k$ is the restriction of a k-isomorphism of varieties of $U^- \cdot P$ onto $V^- \cdot Q$. The conclusion then follows readily from the fact that G is a finite union of translates $x \cdot U^- \cdot P (x \in G_k)$.

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RATIONAL POINTS ON CURVES OF HIGHER GENUS

By J. W. S. Cassels

The old conjecture of Mordell [3] that a curve of genus greater than 1 defined over the rationals has at most one rational point still defies attack. Recently Dem'janenko [2] has given a quite general theorem which enables one to prove the existence of only finitely many rational points in a wide variety of cases. In this lecture I show how his theorem is an immediate consequence of the basic properties of heights of points on curves. The details will be published in the Mordell issue of the Journal of the London Mathematical Society [1]. **83**

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A DEFORMATION THEORY FOR SINGULAR HYPERSURFACES

By B. Dwork

85 In previous articles [1], [2] we have given a theory for the zeta function of non-singular hypersurfaces defined over a finite field. Here we shall discuss the zeta function of the complement, $\overline{V'}$, in projective n-space of the algebraic set

$$X_1 X_2 \dots X_{n+1} \overline{f}(X) = 0$$

defined over $GF[q]$, where $\overline{f}$ is homogeneous of degree d. Let f be a lifting of $\overline{f}$. We shall make no hypothesis that f be non-singular in general position and shall study the variation of the zeta function of $\overline{V}$ as f varies. This involves a generalization of the non-singular case and we will review the situation for that case.

1 Notation.

In the following the field of coefficients will be a suitably chosen field of characteristic zero. The precise choice of field will usually be clear from the context.

$\underline{L}^*$ = all infinite sums of the form $\Sigma_{dw_0 = w_1 + \cdots + w_{n+1}} A_w X^{-w}$, $w_i \geqslant 0$, $\forall i$;

$\underline{L}$ = all finite sums of the form $\Sigma A_w X^w$, the range of w being as above;

$\underline{K} = \{\xi^* \in L^* | D_i^* \xi^* = 0, i = 1, 2, \dots, n+1\}$;

$D_i^* = \gamma_- \circ (E_i + {}_\pi X_0 E_i f)$, $D_i = E_i + {}_\pi X_0 E_i f$;

$$E_i = X_i \frac{\partial}{\partial X_i};$$

$$\pi^{p-1} = -p;$$

$$\gamma_- X^w = \begin{cases} X^w & \text{if each } w_i \leqslant 0, \\ 0 & \text{otherwise;} \end{cases}$$

$\Phi X^w = X^{qw}$;

$\alpha^* = \gamma_- \circ \exp\{\pi X_0 f(X) = {}_\pi X_0^q f(X^q)\} \circ \phi$ **86**

$\underline{K}^\infty = \{\xi^* \in \underline{L}^* | D_i^{*v} \xi^* = 0,\ \forall i, \forall v \text{ large enough}\}$;

$\underline{L}_g^*$ = elements of $\underline{L}^*$ with suitable growth conditions;

$\underline{K}_g^\infty$ = elements of $\underline{K}^\infty$ with suitable growth conditions;

(suitable growth simply means that α^* operates.)

R = resultant of $(f, E_1 f, \ldots, E_n f)$;

K = algebraic number field;

$\underline{O_K}$ = ring of integers of K;

$f \in \underline{O_K}[X_1, \ldots, X_{n+1}]$;

$b(\lambda)$ is a suitably chosen element of $\underline{O_k}[\lambda]$.

For each prime p we assume an extension of p to K has been chosen and that q is the cardinality of the residue class field.

2 Non-singular case. In this paragraph we suppose that f defines a non-singular hypersurface in general position (hence $R \neq 0$). In this case $\dim \underline{K} = d^n$ and if $\overline{f}$ is non-singular and in general position (i.e. $\overline{R} \neq 0$) then all elements of $\underline{K}$ satisfy growth conditions

$$\operatorname{ord} A_w = -0(\log w_0). \tag{1}$$

The zeta function of $\overline{V'}$ is given by the characteristic polynomial of $\alpha^* | \underline{K}$. The Koszul complex of $D_1^*, D_2^*, \ldots, D_{n+1}^*$ acting on $\underline{L^*}$ and L_g^* is acyclic.

If $f_\lambda \in \underline{O}_K[\lambda, X]$ where $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(\mu)})$ is a set of independent parameters and if $R(\lambda)$ is not identically zero then as above we may define $\underline{K_\lambda}$ as a $K(\lambda)$ space and any basis has the form

$$\xi_{i,\lambda}^* = b^{-1} \Sigma_w G_w^{(i)}(\lambda) X^{-w} (\pi R(\lambda))^{-w_0}, i = 1, 2, \ldots, d^n, \tag{2}$$

with $G_w^{(i)} \in \underline{O}_K[\lambda]$, $i = 1, \ldots, d^n$ and $\deg G_w^{(i)} \leqslant k w_0$ for suitable constant k. For λ p-adically close to λ_0 we have the map of $\underline{K_{\lambda_0}}$ onto $\underline{K_\lambda}$ (with suitable extension of field of coefficients)

$$T_{\lambda_0, \lambda} = \gamma_- \circ \exp \pi X_0 (f_{\lambda_0} - f_\lambda)$$

87 and we have the commutative diagram

$$\begin{tikzcd}
K_{\lambda_0^q} \arrow[r, "T_{\lambda_0^q,\lambda^q}"] \arrow[d, "\alpha^*_{\lambda_0}"'] & K_{\lambda^q} \arrow[d, "\alpha^*_\lambda"] \\
K_{\lambda_0} \arrow[r, "T_{\lambda_0,\lambda}"'] & K_\lambda
\end{tikzcd}$$

where α^*_λ is defined by modifying the formula for α^*, replacing $f(X)$ by $f_\lambda(X)$ and $f(X^q)$ by $f_{\lambda^q}(X^q)$. The matrix C_λ of $T_{\lambda_0,\lambda}$ (relative to the bases (2)) is the solution matrix of a system of linear differential equations (with coefficients in $K(\lambda)$)

$$\frac{\partial}{\partial\lambda^{(t)}}\underline{X} = \underline{X}B^{(t)}, t = 1, 2, \ldots, \mu, \tag{3}$$

which is independent of p and λ_0. The matrix of α^*_λ relative to our basis is holomorphic (as function of λ) in a region

$$W = \{\lambda|(R(\lambda))| > 1 - \epsilon, |\lambda| < 1 + \epsilon\} \tag{4}$$

for some $\epsilon > 0$. It follows from Krasner that for $\lambda \in W$, $|\lambda| = 1$, the zeta function of $\overline{V}'_\lambda$ is determined by (3) and the matrix of $\alpha^*_{\lambda_0}$ for one specialization of λ_0 in W. This has seemed remarkable and it is this situation which we wish to extend to the singular case.

3 Explanation of equation (3) **(Katz [5]).** $\underline{K}$ is dual to $\underline{L}/\Sigma D_i\underline{L}$. Again let w be in Z^{n+2} such that $dw_0 = w_1 + \cdots + w_{n+1}$. Let $\underline{\widetilde{L}}^0$ be the span of all X^w such that $w_0 > 0$ (but $w_1, \ldots, w_{n+1}$ may be negative). It is shown by Katz that

$$X^w \to \frac{(w_0-1)!}{(-\pi)^{w_0-1}}\frac{d(X_1/X_{n+1})}{(X_1/X_{n+2})} \wedge \ldots \wedge \frac{d(X_n/X_{n+1})}{(X_n/X_{n+1})}\frac{X^w}{(X_0 f)^w},$$

modulo exact $n-1$ forms gives an isomorphism

$$\underline{\widetilde{L}}^0 \Sigma_{i=1}^{n+1} \quad D_i\underline{\widetilde{L}^0} \simeq H^n(V')$$

and if f is replaced by f_λ then the endomorphisms $\sigma_t = \dfrac{\partial}{\partial\lambda^{(t)}} + \pi X_0 \dfrac{\partial f_\lambda}{\partial\lambda^{(t)}}$ **88**
on the left is transformed into differentaition of the n^{th} de-Rham space on the right with respect to $\lambda^{(t)} (t = 1, 2, \ldots, \mu)$. This is valid without restriction on f but if f is non-singular in general position then by comparison of dimensions, he showed that $\underline{L}'/\Sigma D_i \underline{L}$ is identified with the factor space of $H^n(V')$ modulo an $n+1$ dimensional subspace consisting of invariant classes. (L' is the set of all elements of $\underline{L}$ with zero constant term.) This identifies Equation (3) with the "dual" of the Fuchs-Picard equation of $H^n(V')$ if we use the fact that (3) is equivalent to the fact that $\sigma_t \circ T_{\lambda_0,\lambda}$ annihilates $\underline{K}_{\lambda_0}$.

4 Singular case [3].

Here we know finiteness of the Koszul complex for $\underline{L}^*$ and for $\underline{K}^\infty$ but not in general for $\underline{L}_g^*$ or for $\underline{K}_g^\infty$. However $\underline{K}^\infty = \underline{K}_g^\infty$ for almost all primes p and in this way the theory has been developed only for a generic prime.

We mention that this restriction could be removed if we could show:

Conjecture. A linear differential operator in one variable with polynomial coefficients operating on functions holomorphic in an "open" disk has finite cokernel. (This is known in the complex case. In the p-adic case it is known only for disks which are either small enough or large enough. It is true without restriction if the coefficients are constants. The conjecture is false for "closed" disks.)

In any case the zeta function of $\overline{V}'$ is given by the action of α^* on the factor spaces of the Koszul complex of $\underline{K}_g^\infty$. The first term, i.e. the characteristic polynomial of $\alpha^*|\underline{K}_g$ dominates the zeta function in that up to a factor of power of q, the zeros and poles of the zeta function occur in this factor. In the following we consider the variation of $\underline{K}$ (and of the corresponding factor of the zeta function) as f varies.

If we again consider $f_\lambda \in K[\lambda, X]$, (but now $R(\lambda)$ may be identically zero) then we may again construct $\underline{K}_\lambda$; its dimension N over $K(\lambda)$ is not **89**
less than $N_{\lambda_0} = \dim_{K(\lambda_0)} \underline{K}_{\lambda_0}$ for each specialization, λ_0, of λ.

Theorem. *Each basis $\{\xi^*_{i,\lambda}\}_{i=1}^N$ of $\underline{K}_\lambda$ is of the form*

$$\xi^*_{i,\lambda} = b(\lambda)^{-1}\Sigma M_w^{(i)}(\lambda)X^{-w}/(\pi G(\lambda))^{w_0} \tag{5}$$

where $G(\lambda_0) = 0$ if and only if $N_{\lambda_0} < N$. If $N_{\lambda_0} = N$ then the basis may be chosen such that $b(\lambda_0) \neq 0$. For each pair (i, w), $M_w^{(i)}$ is a polynomial whose degree is bounded by a constant multiple of w_0.

For almost all p

$$\mathrm{ord}'_p(M_w^{(i)}/(\pi G)^{w_0}) = 0(\log w_0) \tag{6}$$

where the left side refers to the p-adic ordinal extended to $K(\lambda)$ in a formal way (generic value on circumference of unit poly disk). We again have the mapping $T_{\lambda_0,\lambda}$ of $\underline{K}_{\lambda_0}$ into $\underline{K}_\lambda$ for λ close enough to λ_0 and if $N_{\lambda_0} = N$ then the matrix of this mapping is again a solution matrix of Equation (3). Also the matrix of α^*_λ is holomorphic in a region of the same type as before,

$$|G(\lambda)| > 1 - \epsilon, |\lambda| < 1 + \epsilon,$$

for some $\epsilon > 0$ (this region may be empty for a finite set of p) and the theory of Krasner may again be applied. This completes our statement of results.

We now discuss equation (5). If f_λ is generically singular, choose a new family, $f_{\lambda,\Gamma}$, which is generically non-singular and which coincides with f_λ when $\Gamma = 0$. We have the mapping $T^{(\lambda,\Gamma)}$ of $\underline{K}_\lambda$ into $\underline{K}_{\lambda,\Gamma}$ given by $\gamma_- \circ \exp(\pi X_0(f_\lambda - f_{\lambda,\Gamma}))$ and for ξ^* in $\underline{K}_\lambda$ we may write

$$T^{(\lambda,\Gamma)}\xi^* = \Sigma_j \underline{X}_j \xi^*_{j,\lambda,\Gamma} \tag{7}$$

where $\{\xi^*_{j,\lambda,\Gamma}\}_{j=1}^{d^n}$ is a basis of $\underline{K}_{\lambda,\Gamma}$ given by (2). The left side of (6) lies in $K(\lambda)[[\Gamma, X^{-1}]]$, $\underline{X}_j$ lies in $K(\lambda)[[\Gamma]]$ and ξ^* may be recovered by setting $\Gamma = 0$ (i.e. by determining the coefficient of Γ^0 on the right side).

Our only information about the vector $\underline{X} = (\ldots, \underline{X}_j, \ldots)$ is that it satisfies a differential equation

$$\frac{\partial \underline{X}}{\partial \Gamma} = \underline{X}B \tag{8}$$

90 when B is rational in λ, Γ. If this equation has (for generic λ) a regular singular point at $\Gamma = 0$ (which is not clear since the theorem of Griffiths need not apply to $V'_{\lambda,\Gamma}$, even though $f_{\lambda,\Gamma}$ is generically non-singular) and if further $\Gamma B = B_0(\lambda) + \Gamma B_1(\lambda) + \cdots$ then each formal power series solution $\sum_{s=0}^{\infty} A_s \Gamma^s$ must have the form (for some $h \in K[\lambda]$, $v \in Z_+$)

$$A_s = \text{ polynomial } / \left\{ h(\lambda)^s \prod_{t=v}^{s} \det(tI - B_0) \right\}.$$

Since B_0 is a function of λ this leads to the possibility that the singular locus (in λ) of the formal solution is an infinite union of varieties,

$$\{\det(tI - B_0) = 0\}_{t=v}^{\infty},$$

and this would leave the same possibility for the singular locus of the coefficients of ξ^*. We indicate two methods by which this difficulty may be overcome.

Method 1. In the above analysis, the hypothesis of regularity of singularity of (8) at $\Gamma = 0$ was not essential but now we use this hypothesis to conclude (with the aid of §3) that for fixed λ, the zeros of the polynomial $\det(tI - B_0)$ (i.e. the roots of the indicial polynomial of (7)) are related to the eigenvalues of the monodromy matrix for $H^n(V'_{\lambda,\Gamma})$ for a circuit about $\Gamma = 0$. Since this matrix can be represented by a matrix with integral coefficients which is continuous as function of λ for λ near a generic point, the conclusion is that the polynomial $\det(tI - B_0)$ is independent of λ. With this conclusion the method of the previous paragraph easily leads to equation (5). However as noted a (probably not serious) gap remains in this treatment since the question of regularity of singularity of (7) is not settled.

Method 2. By means of Equations (6) and (7) together with crude estimates for growth conditions of formal power series solutions of ordinary differential equations we show for each prime p, a constant c_p and an **91**
element a_0 of $K[\lambda]$ such that each $\Sigma A_w X^{-w}$ in $\underline{K}_{\lambda_0}$ satisfies growth conditions

$$\operatorname{ord} A_w \geqslant -c_p w_0 + 0(1) \tag{9}$$

provided

(i) λ_0 lies in a certain Zariski open set defined over K,

(ii) $|\lambda_0| \leqslant 1$,

(iii) $a_0(\lambda_0)$ is a unit,

(iv) p is not one of a certain finite set of primes.

Now let λ_0 be algebraic over K, in the Zariski open set of (i) and such that $a_0(\lambda_0) \neq 0$. We may choose p such that conditions (ii), (iii), (iv) are satisfied and then for λ_1 close enough to λ_0 the conditions (i)-(iv) remain satisfied. We may put upon λ_1 the further conditions that $\operatorname{ord}(\lambda_1 - \lambda_0) > c_p$ and that λ_1 be of maximal transcendence degree over K. We conclude that the dimension of $\underline{K}_{\lambda_1}$ (over $K(\lambda_1)$) is N, that the elements of $\underline{K}_{\lambda_1}$ satisfy (9) and hence that T_{λ_1,λ_0} is defined. We conclude (since T_{λ_1,λ_0} is injective) that the dimension of K_{λ_0} is N. From this we conclude that

$$N_{\lambda_0} = N$$

for all λ_0 in a Zariski open set. This is the central point (which one might expect to follow from general principles); from this and equations (6) and (7) the remainder of the results may be deduced. The details are explained in [4].

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SOME RESULTS ON ALGEBRAIC CYCLES ON ALGEBRAIC MANIFOLDS

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0 Introduction. The basic problem we have in mind is the classification of the algebraic cycles on an algebraic manifold V. The first invariant is the homology class $[Z]$ of a cycle Z on V; if Z has codimension q, then $[Z] \in H_{2n-2q}(V, \mathbf{Z})(n = \dim V)$. By analogy with divisors (c.f. [18]), and following Weil [22], if $[Z] = 0$, then we want to associate to Z a point $\phi_q(Z)$ in a complex torus $T_q(V)$ naturally associated with V. The classification question then becomes two problems :

(a) Find the image of ϕ_q (inversion theorem);

(b) Find the equivalence relation given by ϕ_q (Abel's theorem).

We are unable to make substantial progress on either of these. On the positive side, our results do cover the foundational aspects of the problem and give some new methods for studying subvarieties of general codimension. In particular, the issue is hopefully clarified to the extent that we can make a guess as to what the answers to (a) and (b) should be. This supposed solution is a consequence of the (rational) Hodge conjecture; conversely, if we know (a) and (b) in suitable form, then we can construct algebraic cycles.

We now give an outline of our results and methods.

For the study of q-codimensional cycles on V, Weil introduced certain complex tori $J_q(V)$; as a real torus,

$$J_q(V) = H^{2q-1}(V, \mathbf{R})/H^{2q-1}(V, \mathbf{Z}).$$

These tori are abelian varieties. We use the same real torus, but with a different complex structure (c.f. §§1,2); these tori $T_q(V)$ vary holomorphically with V (the $J_q(V)$ don't) and have the necessary functorial

properties. In general, they are not abelian varieties, but have an r-convex polarization [9]. However, the polarizing line bundle is positive on the "essential part" of $T_q(V)$. Also, $T_1(V) = J_1(V)$ (= Picard variety 94
of V) and $T_n(V) = J_n(V)$ (= Albanese variety of V).

Let Σ_q be the cycles of codimension q algebraically equivalent to zero on V. There is defined a homomorphism $\phi_q : \Sigma_q \to T_q(V)$ by $\phi_q(Z) = \begin{bmatrix} \vdots \\ \int_\Gamma \omega^\alpha \\ \vdots \end{bmatrix}$/(periods), where Γ is a $2n-2q+1$ chain with $\partial\Gamma = Z$ and $\omega^1, \dots, \omega^m \in H^{2n-2q+1}(V, \mathbf{C})$ are a basis for the holomorphic one-forms on $T_q(V)$. Using the torus $T_q(V)$, this mapping is holomorhic and depends only on the complex structure of V (c.f. §3); this latter result follows from a somewhat interesting theorem on the cohomology of Kähler manifolds given in the Appendix following §10. In §3, we also give the infinitesimal calculation of ϕ; the transposed differential ϕ^* is essetially the Poincaré residue operator (c.f. (3.8)). For hypersurfaces $(q = 1)$, the Poincaré residue and geometric residue operators coincide, and the (well-known) solutions to (a) and (b) follow easily.

In §4, we relate the functorial properties of the tori $T_q(V)$ to geometric operations on cycles. The expected theorems turn up, but the proofs require some effort. We use the calculus of differential forms with singularities. In particular, the notion of a residue operator associated to an irreducible subvariety $Z \subset V$ appears. Such a residue operator is given by a C^∞ form ψ on $V - Z$ such that: (1) ψ is of type $(2q-1, 0) + \cdots + (q, q-1)$; (2) $\partial\psi = 0$ and $\bar{\partial}\psi$ is a C^∞ (q, q) form on V which gives the Poincaré dual $\mathscr{D}[Z] \in H^{q,q}(V)$ of $[Z]$; and (3) for Γ a $2n-k$ chain on V meeting Z transversely and η a smooth $2q-k$ form on V, we have the residue formula: $\lim\limits_{\epsilon \to 0} \int_{\Gamma\cdot(\partial T_\epsilon)} \psi \wedge \eta = \int_{\Gamma \cdot Z} \eta$, where T_ϵ is the ϵ-neighborhood of Z in V. The construction of residue operators is done using Hermitian differential geometry; the techniques involved give a different method of approaching the theorem of Bott-Chern [4]. One use of the residue operators is the explicit construction, on the form level, of the Gysin homomorphism $i_* : H^k(Z) \to H^{2q+k}(V)$ where we can keep close track of the complex structure (c.f. the Appendix to §4, 95
section (e)). This is useful in proving the functorial properties.

In §5 we give one of our basic constructions. If $[Z] = 0$ in $H_{2n-2q}(V, \mathbf{Z})$, and if ψ is a residue operator for Z, we may assume that $d\psi = 0$. Then ψ is the general codimensional analogue of a logarithmic integral of the third kind ([17]). The trouble is that ψ has degree $2q - 1$ and so cannot directly be integrated on V to give a function. However, ψ can be integrated on the set of algebraic cycles of dimension $q - 1$ on V. We show then that Z defines a divisor $D(Z)$ on a suitable Chow variety associated to V, and that ψ induces an integral of the third kind on this Chow variety. The generalization of Abel's theorem we give is then : $D(Z)$ is linearly equivalent to zero if $\phi_q(Z) = 0$ in $T_q(V)$. As in the classical case, the proof involves a bilinear relation between ψ and the holomorphic differentials on $T_q(V)$. Also, as mentioned above, the "only if" part of this statement (which is trivial when $q = 1$) depends upon the Hodge problem. Our conclusion from this, as regards problem (b) is: The equivalence relation defined by ϕ should be linear equivalence on a suitable Chow variety. In particular, we don't see that this equivalence should necessarily be rational equivalence on V.

In §6 we give our main result trying to determine the image of ϕ. To explain this formula (given by (6.8) in §6) we let $\{\mathbf{E}_\lambda\}$ be a holomorphic family of holomorphic vector bundles over V. We denote by $Z_q(\mathbf{E}_\lambda)$ the q^{th} Chern class in the rational equivalence ring, so that $\{Z_q(\mathbf{E}_\lambda)\}$ gives a family of codimension q cycles on V. Our formula gives a method for calculating the infinitesimal variation of $Z_q(\mathbf{E}_\lambda)$ in $T_q(V)$; it involves the curvature matrix Φ in $\mathbf{E}_\lambda$ and the Kodaira-Spencer class giving the variation of $\mathbf{E}_\lambda$.

The crux of this formula is that it relates the Poincaré and geometric residues in higher codimension. The proof involves a somewhat delicate computation using forms with singularities and the curvature in $\mathbf{E}_\lambda$. In §8 we give the argument for the highest Chern class of an ample bundle. In §7 it is shown that we need only check the theorem for ample bundles; however, in general the Chern classes, given by Schubert cy-
96 cles, will be singular, except of course for the highest one. So, to prove our formula in general we give in §9 an argument, which is basically differential-geometric, but which requires that we examine the singularities of $Z_q(\mathbf{E}_\lambda)$.

The reason for proving such a formula is that the Chern classes $Z_q(\mathbf{E}_\lambda)$ generate the rational equivalence ring on V. So, if we could effectively use the main result, we could settle problem (a). For example, for line bundles $(q = 1)$, the mapping in question is the identity; this gives once more the structure theorems of the Picard variety. However, we are unable to make effective use of the formula, except in rather trivial cases, so that our result has more of an intrinsic interest and illuminating proof than the applications we would like.

In the last part of §9 we give an integral-geometric argument, using the transformation properties of the tori $T_q(V)$ and the relation of these properties to cycles, of the main formula (6.8).

Finally, in §10 we attempt to put the problem in perspective. We formulate possible answers to (a) and (b) and show how these would follow if we knew the Hodge problem. The construction of algebraic cycles, assuming the answer to (a) and (b), is based on a generalization of the Poincaré normal functions (c.f. [19]) and will be given later.

To close this introduction, I would like to call attention to the paper of David Lieberman [20] on the same subject and which contains several of the results given below. Lieberman uses the Weil Jacobians [22] to study intermediate cycles; however, his results are equally valid for the complex tori we consider. His methods are somewhat different from the ones used below; many of our arguments are computational whereas Lieberman uses functorial properties of the Weil mapping and his proofs have an algebro-geometric flavor.

More specifically, Lieberman proves the functorial properties of the Weil mapping in somewhat more precise form than given below. Thus his results include the functorial properties (4.2) (the hard one arising from the Gysin map) and (4.14) (the easy one using restriction of cohomology), as well as (4.12) which we only state conjecturally. From **97**
the functorial properties and the fact that the Weil mapping is holomorphic for codimension one, Lieberman concludes the analyticity of this mapping (given by (3.2)) in general. (It is interesting to contrast his conceptual argument with the computational one given in [9].) In summary, Lieberman's results include the important general properties of the intermediate Jacobians given in §1-4 below. Also, the conjectured

Abelian variety for which the inversion theorem ((a) above) holds was found by Lieberman using his Poincaré divisor, and the proof of (10.4) is due to Lieberman.

The reason for this overlap is because this manuscript was done in Berkeley, independently but at a later time than Lieberman (most of his results are in his M. I. T. thesis). By the time we talked in Princeton, this paper was more or less in the present form and, because of the deadline for these proceedings, could not be rewritten so as to avoid duplication.

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7. *Properties of the Mapping ζ in (6.8).*
   (Here we discuss the formula (6.8) and prove it for $q = 1$. It is also shown that it suffices to verify it for ample bundles.)
8. *Proof of* (6.4) *for the Highest Chern Class.*
   (This is the basic integral-differential-geometric argument relating the Poincaré and geometric residues via the Chern forms.)
9. *Proof of* (6.8) *for the General Chern Classes.*
   (Here we discuss the singularities of the Chern classes and show how to extend Poincaré residues and the argument of §8 to the general case.)
10. *Concluding Remarks.*
    (We formulate what we feel are reasonable solutions to problems (a) and (b) above, and discuss what is needed to prove these.)

Appendix: A Theorem on the Cohomology of Algebraic Manifolds.

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1 Complex Tori associated to Algebraic Manifolds.

Let V be an n-dimensional algebraic manifold and $\mathbf{L} \to V$ the *positive line bundle* giving the polarization on V. The *characteristic class* $\omega \in H^{1,1}(V) \cap H^2(V, \mathbf{Z})$ may be locally written as $\omega = \dfrac{i}{2}\{\Sigma g_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta\}$ where $\sum\limits_{\alpha,\beta} g_{\alpha\bar{\beta}} dz^\alpha d\bar{z}^\beta$ gives a Kähler metric on V. **99**

According to Hodge, the cohomology group $H^s(V, \mathbf{C})$ decomposes as a sum:

$$H^s(V, \mathbf{C}) = \sum_{p+q=s} H^{p,q}(V),$$

where $H^{p,q}(V)$ are the cohomology classes represented by differential forms of type (p, q). Under complex conjugation, $\overline{H^{p,q}(V)} = H^{q,p}(V)$.

Consider now the cohomology group

$$H^{2n-2q+1}(V,\mathbf{C}) = \sum_{r=0}^{2n-2q+1} H^{2n-2q+1-r,r}(V) \tag{1.1}$$

and choose a complex subspace $S \subset H^{2n-2q+1}(V,\mathbf{C})$ such that

$$S \cap \overline{S} = 0 \quad \text{and} \quad S + \overline{S} = H^{2n-2q+1}(V,\mathbf{C}); \tag{1.2}$$

$$S = \sum_{r=0}^{2n-2q+1} S \cap H^{2n-2q+1-r,r}(V) \tag{1.3}$$

(i.e. S is compatible with the *Hodge decomposition* (1.1));

$$H^{n-q+1,n-q}(V) \subset S. \tag{1.4}$$

Under these conditions we shall define a complex torus $T_q(S)$ such that the space of holomorphic 1-forms on $T_q(S)$ is just S. There are three equivalent definitions of $T_q(S)$.

Definition 1. Choose a basis $\omega^1, \ldots, \omega^m$ for S and define the lattice $\Gamma(S) \subset \mathbf{C}^m$ of all column vectors

$$\pi_\gamma = \begin{bmatrix} \int_\gamma \omega^1 \\ \vdots \\ \int_\gamma \omega^m \end{bmatrix}$$

where $\gamma \in H_{2m-2q+1}(V,\mathbf{Z})$. To see that $\Gamma(S)$ is in fact a lattice, we observe tht rank $(H_{2n-2q+1}(V,\mathbf{Z})) = 2m$ and so we must show :

If $\gamma_1, \ldots, \gamma_k \in H_{2n-2q+1}(V,\mathbf{Z})$ are linearly independent over $\mathbf{R}$, then $\pi_{\gamma_1}, \ldots, \pi_{\gamma_k}$ are also linearly independent over $\mathbf{R}$. But if

$$\sum_{j=1}^{k} \alpha_j \int_{\gamma_j} \omega^\alpha = \int_{\sum_{j=1}^{k} \alpha_j\gamma_j} \omega^\alpha = 0$$

100 then also

$$\int_{\sum_{j=1}^{k} \alpha_j\gamma_j} \overline{\omega}^{\alpha} = 0, \quad \text{since} \quad \overline{\alpha}_j = \alpha_j.$$

This says that $\sum_{j=1}^{k} \alpha_j\gamma_j$ is orthogonal to $S + \overline{S} = H^{2n-2q+1}(V, \mathbf{C})$ and so $\sum_{j=1}^{k} \alpha_j\gamma_j = 0$.

If then $T_q(S) = \mathbf{C}^m/\Gamma(S)$, then $T_q(S)$ is a complex torus associated to $S \subset H^{2n-2q+1}(V, \mathbf{C})$.

Definition 2. Let $H_{2n-2q+1}(V, \mathbf{C}) = H^{2n-2q+1}(V, \mathbf{C})^*$ be the dual space of $H^{2n-2q+1}(V, \mathbf{C})$ so that $0 \to S \to H^{2n-2q+1}(V, \mathbf{C})$ dualizes to

$$0 \leftarrow S^* \leftarrow H_{2n-2q+1}(V, \mathbf{C}). \tag{1.5}$$

Then $H_{2n-2q+1}(V, \mathbf{Z}) \subset H_{2n-2q+1}(V, \mathbf{C})$ projects onto a lattice $\Gamma(S) \subset S^*$ and $T_q(S) = S^*/\Gamma(S)$. (Proof that Definition 1 = Definition 2: choosing a basis $\omega^1, \ldots, \omega^m$ for S makes $S^* \cong \mathbf{C}^m$ by $l(\omega^{\alpha}) = l_{\alpha}$ where $= \begin{bmatrix} l_1 \\ \vdots \\ l_m \end{bmatrix}$. Then $\pi_{\gamma}(\omega^{\alpha}) = \int_{\gamma} \omega^{\alpha} = \langle \omega^{\alpha}, \gamma \rangle$ so that $\Gamma(S)$ is the same lattice in both cases.)

Definition 3. Let $\mathscr{D} : H_{2n-2q+1}(V, \mathbf{C}) \to H^{2q-1}(V, \mathbf{C})$ be the *Poincaré duality isomorphism* and $0 \leftarrow S^* \leftarrow H^{2q-1}(V, \mathbf{C})$ the sequence corresponding to (1.5), $\Gamma(S) \subset S^*$ the lattice corresponding to $\Gamma(S)$. Then $T_q(S) = S^*/\Gamma(S)$.

Observe that if $H^{r,s}(V) \subset S$, then $H^{n-r,n-s}(V) \subset S^*$ and vice-versa. In particular, S is the dual space of S^* by:

$$\langle \omega, \phi \rangle = \int_V \omega \wedge \phi \qquad (\omega \in S, \ \phi \in S^*). \tag{1.6}$$

Thus $S \cong H^{1,0}(T_q(S))$, the space of holomorphic 1-forms on $T_q(S)$. 101

2 Special Complex Tori. The choice of $S \subset H^{2n-2q+1}(V, \mathbf{C})$ depends on the properties we want $T_q(S)$ to have; the results on algebraic cycles will be essentially independent of S because of condition (1.4).

Example 1. We let $S = \sum_{r=0}^{n-q} H^{n-q+r+1, n-q-r}$ and set $T_q(S) = T_q(V)$. These tori have been studied in [9], where it is proved that $T_q(V)$ varies holomorphically with V.

The trouble with $T_q(V)$ is that it is not polarized in the usual sense; however, for our purposes we can do almost as well as follows. Recall [23] that there is defined on $H^{2q-1}(V, \mathbf{C})$ a quadratic form Q with the following properties:

$$\left.\begin{array}{ll} \text{(a)} & Q \text{ is skew-symmetric and integral on} \\ & H^{2q-1}(V, \mathbf{Z}) \subset H^{2q-1}(V, \mathbf{C}); \\ \text{(b)} & Q(H^{r,s}(V), \overline{H^{r',s'}(V)}) = 0 \text{ if } r \neq r', s \neq s'; \\ \text{(c)} & Q(H^{r,s}(V), \overline{H^{r,s}(V)}) \text{ is nonsingular; and} \\ \text{(d)} & iQ(H^{q-1,q}(V), \overline{H^{q-1,q}(V)}) > 0. \end{array}\right\} \qquad (2.1)$$

It follows that $Q(S^*, S^*) = 0$ and that, choosing a basis $\omega^1, \ldots, \omega^m$ for S, there is a complex line bundle $\mathbf{L} \to T_q(S)$ whose characteristic class $\omega(\mathbf{L}) \in H^2(T_q(S), \mathbf{Z}) \cap H^{1,1}(T_q(S))$ is given by

$$\omega(\mathbf{L}) = \frac{i}{2\pi} \left\{ \sum_{\alpha,\beta=1}^{m} h_{\alpha\bar{\beta}} \omega^\alpha \wedge \overline{\omega}^\beta \right\},$$

where the matrix $H = (h_{\alpha\bar{\beta}}) = \{iQ(e_\alpha, \bar{e}_\beta)\}^{-1}$ and $\int_V \omega^\alpha \wedge e_\beta = \delta^\alpha_\beta$. Diagonalizing H, we may write

$$\omega(\mathbf{L}) = \frac{i}{2\pi} \left\{ \sum_{\alpha=1}^{m} \epsilon_\alpha \omega^\alpha \wedge \overline{\omega}^\alpha \right\}, \qquad (2.2)$$

where $\epsilon_\alpha = \pm 1$ and $\epsilon_\alpha = +1$ if $\omega^\alpha \in H^{n-q+1,n-q}(V)$. Thus we may say that :

There is a natural *r-convex polarization* [10] $\mathbf{L} \to T_q(V)$ (2.3)

(r = number of α such that $\epsilon_\alpha = -1$) and the characteristic class of $\mathbf{L}$ **102**
is positive on the translates of $H^{q-1,q}(V)$.

Example 2. We let $S = \sum_r H^{n-q+2r+1,n-q-2r}$ and set $J_q(V) = T_q(S)$. This torus is Weil's *intermediate Jacobian* [22] and from (2.1) we find :

There is a natural 0-*convex polarization* (= *positive line bundle*)

$$\mathbf{K} \to J_q(V). \tag{2.4}$$

Referring to (2.2), we let $\phi^\alpha = \omega^\alpha$ if $\epsilon_\alpha = +1$, $\phi^\alpha = \overline{\omega}^\alpha$ if $\epsilon_\alpha = -1$. Then the ϕ^α give a basis for $H^{1,0}(J_q(V))$ and

$$\omega(\mathbf{K}) = \frac{i}{2\pi}\left\{\sum_{\alpha=1}^{m} \phi^\alpha \wedge \overline{\phi}^\alpha\right\}. \tag{2.5}$$

We recall [23] that $H^s(J_q(V), \mathscr{O}(\mathbf{K}^\mu)) = 0$ for $\mu > 0$, $s > 0$ and that $H^0(J_q(V), \mathscr{O}(\mathbf{K}^\mu))$ has a basis $\theta_0, \ldots, \theta_N$ given by *theta functions of weight* μ.

Comparison of $T_q(V)$ and $J_q(V)$. In [9] it is proved that there is a *real* linear isomorphism $\xi : T_q(V) \to J_q(V)$ such that

$$\left.\begin{array}{rl}
\text{(i)} & \xi^*\phi^\alpha = \omega^\alpha \ \text{ if } \ \epsilon_\alpha = +1 \ \text{ and } \ \xi^*\phi^\alpha = \overline{\omega}^\alpha \ \text{ if } \ \epsilon_\alpha = -1; \\
\text{(ii)} & \xi^*(\mathbf{K}) = \mathbf{L}; \text{ and} \\
\text{(iii)} & \text{if } \ \Omega_p = \xi^*(\theta_p)\left\{\prod_{\epsilon_\alpha=-1} \overline{\omega}^\alpha\right\}, \ \text{ then the } \Omega_p \ \text{ give a basis of} \\
& H^r(T_q(V), \mathscr{O}(\mathbf{L}^\mu)), \quad \text{and} \quad H^s(T_q(V), \mathscr{O}(\mathbf{L}^\mu)) = 0 \quad \text{for} \\
& \mu > 0, s \neq r.
\end{array}\right\} \tag{2.6}$$

Some Special Cases. For $q = 1$, $T_1(V) = J_1(V) = H^{0,1}(V)/H^1(V, \mathbf{Z})$ is the *Picard variety* of V [22]. For $q = n$, $T_n(V) = J_n(V) = H^{n-1,n}(V)$ $/H^{2n-1}(V, \mathbf{Z})$ is the *Albanese variety* [3] of V. For $q = 2$, $T_2(V) = H^{1,2}(V) + H^{0,3}(V)/H^3(V, \mathbf{Z})$ and $J_2(V) = H^{1,2}(V) + H^{3,0}(V)/H^3(V, \mathbf{Z})$; this is the simplest case where $T_q(V) \neq J_q(V)$.

Some Isogeny Properties. We let $S_q \subset H^{2n-2q+1}(V, \mathbf{C})$ be the subspace corresponding to either $T_q(V)$ or $J_q(V)$ constructed above, and we let $S_q^* \subset H^{2q-1}(V, \mathbf{C})$ be the dual space. Then we have

$$\left.\begin{tikzcd}
H^{2q-1}(V, \mathbf{C}) \arrow[r] & S_q^* \arrow[r] & 0 \\
H^{2q-1}(V, \mathbf{Z}) \arrow[u] & &
\end{tikzcd}\right\}$$

103 and $T_q(V)$ or $J_q(V)$ is given as S_q^*/Γ_q^* where Γ_q^* is the projection of $H^{2q-1}(V, \mathbf{Z})$ on S_q^*.

Suppose now that $\psi \in H^{p,p}(V) \cap H^{2p}(V, \mathbf{Z})$. Then, by cup-product, we have induced :

$$\begin{tikzcd}
S_q^* \arrow[r, "\psi"] & S_{p+q}^* \\
\Gamma_q^* \arrow[u] \arrow[r, "\psi"] & \Gamma_{p+q}^* \arrow[u]
\end{tikzcd} \tag{2.7}$$

which gives $\psi : T_q(V) \to T_{p+q}(V)$ or $\psi : J_q(V) \to J_{p+q}(V)$. We want to give this mapping in terms of the coordinates given in the first definition of paragraph 1.

Let $\omega^1, \ldots, \omega^m = \{\omega^\alpha\}$ be a basis for $S_q \subset H^{2n-2q+1}(V, \mathbf{C})$ and $\phi^1, \ldots, \phi^k = \{\phi^\rho\}$ be a basis for $S_{p+q} \subset H^{2n-2p-2q+1}(V, \mathbf{C})$. Then $\psi \wedge \phi^\rho = \sum\limits_\alpha m_{\rho\alpha}\omega^\alpha$ and

$$\int\limits_\gamma \psi \wedge \phi^\rho = \int\limits_{\gamma \cdot \mathscr{D}(\psi)} \phi^\rho, \tag{2.8}$$

where $\mathscr{D}(\psi) \in H_{2n-2p}(V, \mathbf{Z})$ and $\gamma \in H_{2n-2q+1}(V, \mathbf{Z})$. Now $M = (m_{\rho\alpha})$

is a $k \times m$ matrix giving $\psi : \mathbf{C}^m \to \mathbf{C}^k$ by $\psi \begin{pmatrix} \vdots \\ \lambda^\alpha \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ \sum_{\alpha=1}^{m} m_{\rho\alpha}\lambda^\alpha \\ \vdots \end{pmatrix}$ and

$$\psi \begin{pmatrix} \vdots \\ \int_\gamma \omega^\alpha \\ \end{pmatrix} = \begin{pmatrix} \vdots \\ \Sigma m_{\rho\alpha} \int_\gamma \omega^\alpha \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ \int_\gamma \psi \wedge \phi^\rho \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ \int_{\gamma \cdot \mathscr{D}(\psi)} \phi^\rho \\ \vdots \end{pmatrix},$$

so that $\psi(\Gamma_q) \subset \Gamma_{p+q}$. It follows that, in terms of the coordinates in Definition 1, ψ is given by the matrix M.

Now suppose that $\psi : H^{2q-1}(V, \mathbf{C}) \to H^{2p+2q-1}(V, \mathbf{C})$ is an isomorphism. Then $\psi : S_q^* \cong S_{p+q}^*$ and $\psi(\Gamma_q^*)$ is of finite index in Γ_{p+q}^*. Thus $\psi : T_q(V) \to T_{p+q}(V)$ is an *isogeny*, as is also $\psi : J_q(V) \to J_{p+q}(V)$. Taking $\psi = \omega^{n-2q+1}$, where ω is the polarizing class, and using [23], page 75, we have : **104**

$$\left.\begin{aligned} \omega^{n-2q+1} &: T_q(V) \to T_{n-q+1}(V), \quad \text{and} \\ \omega^{n-2q+1} &: J_q(V) \to J_{n-q+1}(V) \end{aligned}\right\} \tag{2.9}$$

are both isogenies for $q \leqslant \left[\dfrac{n+1}{2}\right]$.

Finally, using [23], Chapter IV, we have :

For $p \leqslant n - 2q + 1$, the mappings

$$\left.\begin{aligned} \omega^p &: T_q(V) \to T_{p+q}(V), \quad \text{and} \\ \omega^p &: J_q(V) \to J_{p+q}(V) \end{aligned}\right\} \tag{2.10}$$

make $T_q(V)$ isogenous to a sub-torus of $T_{p+q}(V)$, and similarly for $J_q(V)$ and $J_{p+q}(V)$.

Some Functionality Properties. Given a holomorphic mapping $f : V' \to V$, there is induced a cohomology mapping $f^* : H^{2q-1}(V, \mathbf{C}) \to H^{2q-1}(V', \mathbf{C})$ with $f^*(S_q^*(V)) \subset S_q^*(V'), f^*(\Gamma_q^*(V)) \subset \Gamma_q^*(V')$ (using the obvious notation).

This gives

$$\left.\begin{aligned} f^* &: T_q(V) \to T_q(V'), \quad \text{and} \\ f^* &: J_q(V) \to J_q(V'). \end{aligned}\right\} \tag{2.11}$$

On the other hand, if $\dim V = n$ and $\dim V' = n'$, we set $k = n - n'$ and from $f_* : H_{2n'-2q+1}(V', \mathbf{C}) \to H_{2n-2(k+q)+1}(V, \mathbf{C})$ we find a mapping

$$\left.\begin{aligned} f_* : T_q(V') \to T_{q+k}(V), \quad \text{and} \\ f_* : J_q(V') \to J_{q+k}(V). \end{aligned}\right\} \tag{2.12}$$

Suppose now that $f : V' \to V$ is an embedding so that V' is an
105 algebraic submanifold of V. Then V' defines a class $[V'] \in H_{2n-2k}(V, \mathbf{Z})$ and $\mathscr{D}[V'] = \Psi \in H^{2k}(V, \mathbf{Z}) \cap H^{k,k}(V)$. We assert that :

In (2.11) and (2.12), the composite mapping

$$f_* f^* : T_q(V) \to T_{q+k}(V) \text{ is just } \Psi : T_q(V) \to T_{q+k}(V) \text{ as given by (2.7) (and similarly for } J_q(V)) \tag{2.13}$$

Proof. We have to show that the composite

$$H^{2q-1}(V, \mathbf{C}) \xrightarrow{f^*} H^{2q-1}(V', \mathbf{C}) \xrightarrow{f^*} H^{2q+2k-1}(V, \mathbf{C}) \tag{2.14}$$

is cup product with Ψ. In homology (2.14) dualizes to

$$H_{2q-1}(V, \mathbf{C}) \xleftarrow{f_*} H_{2q-1}(V', \mathbf{C}) \xleftarrow{f^*} H_{2q+2k-1}(V, \mathbf{C}) \tag{2.15}$$

where f^* is defined by

$$\begin{tikzcd} H_{2q+2k-1}(V, \mathbf{C}) \arrow[r, "f^*"] \arrow[d, "\mathscr{D}"] & H_{2q-1}(V', \mathbf{C}) \\ H^{2n-2q-2k+1}(V, \mathbf{C}) \arrow[r, "f^*"] & H^{2n-2k-2q+1}(V', \mathbf{C}) \arrow[u, "\mathscr{D}^{-1}"'] \end{tikzcd} \tag{2.16}$$

If we can show that $f_* f^*(\gamma) = [V'] \cdot \gamma$ for $\gamma \in H_{2q+2k-1}(V, \mathbf{C})$, then $\int\limits_{\gamma} f_* f^* \phi = \int\limits_{f_* f^* \gamma} \phi = \int\limits_{[V'] \cdot \gamma} \phi = \int\limits_{\gamma} \Psi \wedge \phi (\phi \in H^{2q-1}(V, \mathbf{C}))$, and we are done. So we must show that, in (2.15), f^* is intersection with V', and this a standard result on the *Gysin homomorphism* (2.16) (c.f. (4.11) and the accompanying Remark). □

3 Algebraic Cycles and Complex Tori. Let $V = V_n$ be an algebraic manifold, $S \subset H^{2n-2q+1}(V, \mathbf{C})$ a subspace satisfying (1.2)-(1.4), and $T_q(S)$ the resulting complex torus. We choose a suitable basis $\omega^1, \dots, \omega^m$ for $S \cong H^{1,0}(T_q(S))$ and let Σ_q = {set of algebraic cycles $Z \subset V$ which are of codimension q in V and are homologous to zero}. Following Weil [22], we define

$$\phi_q : \Sigma_q \to T_q(S) \tag{3.1}$$

as follows: if $Z \in \Sigma_q$, then $Z = \partial C_{2n-2q+1}$ for some $2n - 2q + 1$ chain C, and we set

$$\phi_q(Z) = \begin{bmatrix} \vdots \\ \int_C \omega^\alpha \\ \vdots \end{bmatrix}. \tag{3.2}$$

Since C is determined up to cycles, $\phi_q(Z)$ is determined up to vectors 106 $\begin{bmatrix} \vdots \\ \int_\gamma \omega^\alpha \\ \vdots \end{bmatrix}$ $(\gamma \in H_{2n-2q+1}(V, \mathbf{Z}))$, and so ϕ_q *is defined and depends on the subspace of the closed C^∞ forms spanned by* $\omega^1, \dots, \omega^m$; this restriction will be removed in the Appendix to §3.

Now, while it should be the case that ϕ_q is holomorphic, we shall be content with recalling from [9] a special result along these lines. Consider on V an analytic family $\{Z_\lambda\}_{\lambda\in\Delta}(\Delta =$ disc in λ-plane) of q-codimensional algebraic subvarieties $Z_\lambda \subset V$. Locally on V, $\{Z_\lambda\}_{\lambda\in\Delta}$ is given by the vanishing of analytic functions

$$f_1(z^1, \dots, z^n; \lambda), \dots, f_l(z^1, \dots, z^n; \lambda).$$

We define $\phi : \Delta \to T_q(S)$ by $\phi(\lambda) = \phi_q(Z_\lambda - Z_0)$. Using (1.4), we have proved in [9] that

$$\left.\begin{array}{l} \phi : \Delta \to T_q(S) \quad \text{is holomorphic and} \\ \phi_*\{\mathbf{T}_\lambda(\Delta)\} \subset H^{q-1,q}(V). \end{array}\right\} \tag{3.3}$$

We may rephrase (3.3) by saying that $\phi^* : S_q \to \mathbf{T}_\lambda(\Delta)^*$ is determined by $\phi^*|H^{n-q+1,n-q}(V)$ (c.f. (1.4)).

Continuous Systems and The Infinitesimal Calculation of $\boldsymbol{\phi_q}$. Suppose that the $Z_\lambda \subset V$ are all nonsingular and $Z = Z_0$. We let $\mathbf{N} \to Z$ be the *normal bundle* of $Z \subset V$, so that we have the exact sheaf sequence

$$0 \to \mathcal{O}_Z(\mathbf{N}^*) \to \Omega^1_{V|Z} \to \Omega^1_Z \to 0. \tag{3.4}$$

Since $\dim Z = n - q$, from (3.4) we have induced the *Poincaré residue operator*

$$\Omega^{n-q+1}_{V|Z} \to \Omega^{n-q}_Z(\mathbf{N}^*) \to 0 \tag{3.5}$$

as follows : Let $\phi \in \Omega^{n-q+1}_{V|Z}$; $\tau_1, \ldots, \tau_{n-q}$ be tangent vectors to Z; η a
107 normal vector to Z. Lift η to a tangent vector $\widehat{\eta}$ on V along Z. Then $\langle \phi, \tau_1 \wedge \ldots \wedge \tau_{n-q} \otimes \eta \rangle = \langle \phi, \tau_1 \wedge \ldots \wedge \tau_{n-q} \wedge \widehat{\eta} \rangle$, where $\phi \in \Omega^{n-q+1}_{V|Z}$.

From (3.5) and $\Omega^{n-q+1}_V \to \Omega^{n-q+1}_{V|Z}$, we have

$$H^{n-q}(V, \Omega^{n-q+1}_V) \xrightarrow{\xi^*} H^{n-q}(Z, \Omega^{n-q}_Z(\mathbf{N}^*)). \tag{3.6}$$

On the other hand, in [16] Kodaira has defined the *infinitesimal displacement mapping*

$$\rho : \mathbf{T}_0(\Delta) \to H^0(Z, \mathcal{O}_Z(\mathbf{N})). \tag{3.7}$$

To calculate ϕ^*, we have shown in [9] that the following diagram commutes:

$$\begin{tikzcd}
& H^{n-q+1,n-q}(V) = H^{n-q}(V, \Omega^{n-q+1}_V) \arrow[dl, "\phi^*"'] \arrow[dd, "\xi^*"] \\
\mathbf{T}_0(\Delta)^* & \\
& H^{n-q}(Z, \Omega^{n-q}_Z(\mathbf{N}^*)) = H^0(Z, \mathcal{O}_Z(\mathbf{N}))^*. \arrow[ul, "\rho^*"]
\end{tikzcd} \tag{3.8}$$

In other words, infinitesimally ϕ is eseentially given by by ξ^* in (3.6).

Some Special Cases.

(i) In case $q = n$, Z is a finite set of points $z_1, \ldots, z_r$ (Z is a zero-cycle) and (3.6) becomes:

$$H^{1,0}(V) \xrightarrow{\xi^*} \sum_{j=1}^{r} \mathbf{T}_{z_j}(V)^* \tag{3.9}$$

where $\xi^*(\omega) = \sum_{j=1}^{r} \omega(z_j)$, $\omega \in H^{1,0}(V)$ being a holomorphic 1-form on V. In particular, ϕ^* is onto if ξ^* is injective.

(ii) In case $q = 1$, $Z \subset V$ is a nonsingular hypersurface. Then there is a holomorphic line bundle $\mathbf{E} \to V$ and a section $\sigma \in H^0(V, \mathscr{O}_V(\mathbf{E}))$ such that $Z = \{z \in V : \sigma(z) = 0\}$. From the exact sheaf sequence $0 \to \mathscr{O}_V \xrightarrow{\sigma} \mathscr{O}_V(\mathbf{E}) \to \mathscr{O}_Z(\mathbf{N}) \to 0$, we find

$$H^0(Z, \mathscr{O}_Z(\mathbf{N})) \xrightarrow{\xi} H^1(V, \mathscr{O}_V), \tag{3.10}$$

where we claim that ξ in (3.10) is (up to a constant) the dual of ξ^* in (3.6) (using $H^{0,1}(V) = H^{n,n-1}(V)^*$).

Proof. We may choose a covering $\{U_\alpha\}$ of V by polycylinders such that $Z \cap U_\alpha$ is given by $\sigma_\alpha = 0$ where σ_α is a coordinate function if $U_\alpha \cap Z \neq \emptyset$ and $\sigma_\alpha \equiv 1$ if $U_\alpha \cap Z = \emptyset$. Then $\sigma_\alpha/\sigma_\beta = f_{\alpha\beta}$ where $\{f_{\alpha\beta}\} \in H^1(V, \mathscr{O}_V^*)$ and gives the transition functions of $\mathbf{E} \to V$. Let $\theta = \{\theta_\alpha\} \in H^0(Z, \mathscr{O}_Z(\mathbf{N}))$ and $\omega \in H^{n,n-1}(V)$. We want to show that, for a suitable constant c, we have 108

$$\int_V \xi(\theta) \wedge \omega = c \int_Z \langle \theta, \xi^* \omega \rangle. \tag{3.11}$$

If $Z \cap U_\alpha \neq \emptyset$, we may write $\omega = \omega_\alpha \wedge d\sigma_\alpha$ where ω_α is a $C^\infty(n-1, n-1)$ form in U_α such that $\omega_\alpha | Z \cap U_\alpha$ is well-defined. In $U_\alpha \cap U_\beta$, $\omega = \omega_\alpha \wedge d\sigma_\alpha = \omega_\alpha \wedge d(f_{\alpha\beta}\sigma_\beta) = \omega_\alpha \sigma_\beta \wedge df_{\alpha\beta} + f_{\alpha\beta}\omega_\alpha \wedge d\sigma_\beta$ so that $\omega_\alpha | Z \cap U_\alpha \cap U_\beta = f_{\alpha\beta}^{-1} \omega_\beta | Z \cap U_\alpha \cap U_\beta$. This means that

$\{\omega_\alpha | Z \cap U_\alpha\}$ gives an $(n-1, n-1)$ form on Z with values in $\mathbf{N}^*$, and so $\{\theta_\alpha \omega_\alpha | Z \cap U_\alpha\}$ gives a global $C^\infty(n-1, n-1)$ form on Z (since $\theta_\alpha = f_{\alpha\beta}\theta_\beta$ on $Z \cap U_\alpha \cap U_\beta$). It is clear that $\langle \theta, \xi^* \omega \rangle | Z = \{\theta_\alpha \omega_\alpha\}$ so that the right hand side of (3.11) is

$$\int\limits_Z \{\theta_\alpha \omega_\alpha\}. \tag{3.12}$$

On the other hand, choose a C^∞ section $\Theta = \{\Theta_\alpha\}$ of $\mathbf{E} \to V$ with $\Theta | Z = \theta$. Then $\overline{\partial \Theta} = \sigma \xi(\theta)$ where $\xi(\theta)$ is a $C^\infty(0, 1)$ form giving a Dolbeault representative of $\xi(\theta) \in H^1(V, \mathscr{O}_V)$ in (3.10). Let T_ϵ be an ϵ-tube aroung Z and $\psi = \dfrac{\Theta}{\sigma}$. Then the left hand side of (3.11) is $\int\limits_V \xi(\theta) \wedge \omega = \lim\limits_{\epsilon \to 0} \int\limits_{V - T_\epsilon} \xi(\theta) \wedge \omega = -\lim\limits_{\epsilon \to 0} \int\limits_{\partial T_\epsilon} \psi \wedge \omega$ (since $d(\psi \wedge \omega) = \overline{\partial}(\psi \wedge \omega) = \xi(\theta) \wedge \omega$). Locally $\psi \wedge \omega = \Theta_\alpha \omega_\alpha \wedge \dfrac{d\sigma_\alpha}{\sigma_\alpha}$ so that $\lim\limits_{\epsilon \to 0} \int\limits_{\partial T_\epsilon} \psi \wedge \omega = \lim\limits_{\epsilon \to 0} \int\limits_{\partial T_\epsilon} \{\Theta_\alpha \omega_\alpha\} \wedge \dfrac{d\sigma_\alpha}{\sigma_\alpha} = 2\pi i \int\limits_Z \{\Theta_\alpha \omega_\alpha | Z \cap U_\alpha\} = 2\pi i \int\limits_Z \{\theta_\alpha \omega_\alpha\}$, which, by (3.12), proves (3.11). □

Appendix to §3 : Some Remarks on the Definition of ϕ_q. At the beginning of Paragraph 3 where $\phi_q : \Sigma_q \to T_q(S)$ was defined, it was stated that ϕ_q depended on the vector space spanned by $\omega^1, \ldots, \omega^m$ and not just on S. This is because, if we replace ω^α by $\omega^\alpha + d\eta^\alpha$, then $\int\limits_{C_{2n-2q+1}} \omega^\alpha + d\eta^\alpha = \int\limits_C \omega^\alpha + \int\limits_Z \eta^\alpha$ (Stokes' Theorem).

109 One way around this is to use the Kähler metric on V and choose $\omega^1, \ldots, \omega^m$ to be *harmonic*. This has the disadvantage that harmonic forms are *not* generally preserved under holomorphic mappings. However, if we agree to use the torus $T_q(V)(S = \sum\limits_r H^{n-q+1+r, n-q-r}(V))$ constructed in Example 1 of §2, it is possible to given $\phi_q : \Sigma_q \to T_q(V)$ *purely in terms of cohomology*, and so remove this problem in defining ϕ_q.

To do this, we shall use a theorem on the cohomology of algebraic manifolds which is given in the Appendix below. Let then Ω^q be the sheaf of holomorphic q-forms on V and $\Omega_c^q \subset \Omega^q$ the subsheaf of closed forms. There is an exact sequence:

$$0 \to \Omega_c^q \to \Omega^q \xrightarrow{d} \Omega_c^{q+1} \to 0 \tag{A3.1}$$

(*Poincaré lemma*), which gives in cohomology (c.f. (A.7)):

$$0 \to H^{p-1}(V, \Omega_c^{q+1}) \xrightarrow{\delta} H^p(V, \Omega_c^q) \to H^p(V, \Omega^q) \to 0. \tag{A3.2}$$

From (A3.2), we see that there is a diagram (c.f.(A.16) in the Appendix):

$$\begin{array}{ccc}
H^r(V, \mathbf{C}) & = & H^r(V, \mathbf{C}) \\
\uparrow{\scriptstyle \delta} & & \| \\
H^{r-1}(V, \Omega_c^1) & \subset & H^r(V, \mathbf{C}) \\
\uparrow & & \| \\
\vdots & & \vdots \\
\uparrow & & \| \\
H^{r-q}(V, \Omega_c^q) & \subset & H^r(V, \mathbf{C}) \\
\uparrow{\scriptstyle \delta} & & \| \\
\vdots & & \vdots \\
\uparrow & & \| \\
H^0(V, \Omega_c^r) & \subset & H^r(V, \mathbf{C}) \\
\uparrow & & \\
0 & &
\end{array} \tag{A3.3}$$

Thus $\{H^{r-q}(V, \Omega_c^q)\}$ gives a *filtration* $\{F_q^r(V)\}$ of $H^r(V, \mathbf{C})$; and **110**
$F_q^r(V)/F_{q+1}^r(V) \cong H^{r-q}(V, \Omega^q)$. It is also true that $F_q^r(V)$ depends holomorphically on V [9].

To calculate $F_q^r(V)$ using differential forms, we let $A^{s,r-s}$ be the C^∞ forms of type $(s, r-s)$ on V, $B^{r,q} = \sum_{s \geqslant q} A^{s,r-s}$, and $B_c^{r,q}$ the d-closed forms in $B^{r,q}$. Then $dB^{r,q} \subset B_c^{r+1,q}$, and it is shown in the Appendix (c.f. (A.18)) that

$$F_q^r(V) \cong B_c^{r,q}/dB^{r-1,q} \subset H^r(V, \mathbf{C}). \tag{A3.4}$$

We conclude then from (A3.4) that:

$$\left.\begin{array}{l}\text{A class } \phi \in F_q^r(V) \subset H^r(V, \mathbf{C}) \text{ is represented by a closed } C^\infty \\ \text{form } \phi = \sum\limits_{s \geqslant q} \phi_{s,r-s} (\phi_{s,r-s} \in A^{s,r-s}, \text{ defined up to forms} \\ d\eta = \sum\limits_{s \geqslant q} d\eta_{s,r-1-s}.\end{array}\right\} \tag{A3.5}$$

In particular, look at

$$F_{n-q+1}^{2n-2q+1}(V) \cong \sum_{r \geqslant 0} H^{n-q+1+r,n-q-r}(V).$$

$A\phi \in B_c^{2n-2q+1,n-q+1}$ is defined up to

$$\sum_{s \geqslant 0} d\eta_{n-q+1+s,n-q-1-s}$$

and

$$\int_Z \eta_{n-q+1+s,n-q-1-s} = 0$$

for an algebraic cycle Z of codimension q (Z is of type $(n-q, n-q)$). Thus $\int\limits_{C_{2n-2q+1}} \phi (\partial C = Z)$ depends *only* on the class of

$$\phi \in F_{n-q+1}^{2n-2q+1}(V) \subset H^{2n-2q+1}(V, \mathbf{C}).$$

This proves that:

$$\left.\begin{array}{l}\text{For the torus } T_q(V) \text{ constructed in §3, the mapping} \\ \phi_q : \Sigma_q \to T_q(V) \text{ depends only on the complex structure of } V.\end{array}\right\} \tag{A3.6}$$

For the general tori $T_q(S)$ we may prove the analogue of (A3.6) as follows. First, we may make the forms $\omega^1, \ldots, \omega^m$ subject to $\partial\omega^\alpha = 0$,

$\bar{\partial}\omega^\alpha = 0$, because $S = \sum_r S \cap H^{2n-2q+1-r,r}(V)$ and so $\omega^\alpha = \mathscr{H}(\omega^\alpha) + d\xi^\alpha$ ($\mathscr{H}$ = harmonic part of ω^α) and $\mathscr{H}(\omega^\alpha) = \sum_r \mathscr{H}(\omega^\alpha_{n-q+1+r,n-q-r})$ with $\partial\mathscr{H}(\omega^\alpha_{n-q+1+r,n-q-r}) = 0 = \bar{\partial}\mathscr{H}(\omega^\alpha_{n-q+1+r,n-q-r})$. Thus we may choose a basis $\omega^1, \dots, \omega^m$ for S with $\partial\omega^\alpha = 0 = \bar{\partial}\omega^\alpha$.

Second, let η be a C^∞ form on V with $\partial d\eta = 0 = \bar{\partial} d\eta$. We claim **111**
that $d\eta = \partial\bar{\partial}\xi$ for some ξ. Since $d\eta = \partial\eta + \bar{\partial}\eta$, it will suffice to do this for $\partial\eta$. Now write $\eta = \mathscr{H}_\partial\eta + \partial\partial^* G_\partial\eta + \partial^*\partial G_\partial\eta$, where $\mathscr{H}_\partial$ is the *harmonic projector* for $\Box_\partial = \partial\partial^* + \partial^*\partial$ and G_∂ is the corresponding *Green's operator*. Then $\partial_\eta = \partial\partial^*\partial G_\partial\eta$. On the other hand, since $\bar{\partial}\partial = 0$, $\partial\eta = \mathscr{H}_\partial(\partial\eta) + \bar{\partial}\bar{\partial}^* G_\partial\partial\eta$. But $\mathscr{H}_\partial = \mathscr{H}_\partial$ and $G_\partial = G_\partial$ so that $\partial\eta = \partial\bar{\partial}(\bar{\partial}^* G_\partial\eta)$ as desired.

Finally, let $\omega \in S$ satisfy $\partial\omega = 0 = \bar{\partial}\omega$ and change ω to $\omega + d\eta$ with $\partial(\omega + d\eta) = 0 = \bar{\partial}(\omega + d\eta)$. Then $\omega + d\eta = \omega + \partial\bar{\partial}\xi$ for some ξ. We claim that $\int_C \omega = \int_C \omega + \partial\bar{\partial}\xi$, where C is a $2n - 2q + 1$ chain with $\partial C = Z$. If $\xi = \sum_r \xi_{n-q+r,n-q-1-r}$, then

$$\int_C \partial\bar{\partial}\xi = \int_C d(\bar{\partial}\xi) = \int_Z (\bar{\partial}\xi)_{n-q,n-q} = \int_Z \bar{\partial}\xi_{n-q,n-q-1} = 0$$

since $Z \subset V$ is a complex submanifold. This proves that:

$$\left.\begin{array}{l}\text{If, in the definition of } \phi_q : \Sigma_q \to T_q(V) \text{ in (3.1), we make}\\ \text{the } \omega^\alpha \text{ subject to } \partial\omega^\alpha = 0 = \bar{\partial}\omega^\alpha\text{, then } \phi_q \text{ is well-defined}\\ \text{and depends only on the complex structure of } V.\end{array}\right\} \tag{A3.7}$$

This is the procedure followed by Weil [22].

Remark. A(3.8). Let $D = \partial\bar{\partial}$; then $D : A_c^{p,q} \to A_c^{p+1,q+1}$ and $D^2 = 0$. If $H_D^r(V)$ are the cohomology groups constructed from $D = \partial\bar{\partial}$ and $H_d^r(V)$ the deRham groups, there is a natural mapping:

$$H_D^r(V) \xrightarrow{\alpha} H_d^r(V). \tag{A3.9}$$

4 Some Functorial Properties. (a) Let $W_{n-k} \subset V_n$ be an algebraic submanifold of codimension k. We shall assume for the moment that there is a holomorphic vector bundle $\mathbf{E} \to V$, with fibre $\mathbf{C}^s$, and holomorphic sections $\sigma_1, \ldots, \sigma_{s-k+1}$ of $\mathbf{E}$ such that W is given by $\sigma_1 \wedge \ldots \wedge \sigma_{s-k+1} = 0$. Thus, the homology class carried by W is the *k-th Chern class* of $\mathbf{E} \to V$ (c.f. [5]). Following (2.11) there is a mapping

$$T_q(V) \xrightarrow{i^*} T_q(W) \tag{4.1}$$

112 induced from $H^{2q-1}(V, \mathbf{C}) \to H^{2q-1}(W, \mathbf{C})$. We want to interpret this mapping geometrically.

For this, let $\{Z_\lambda\}_{\lambda \in \Delta}$ be a *continuous system* as in paragraph 3. Assume that each intersection $Y_\lambda = Z_\lambda \cdot W$ is transverse so that $\{Y_\lambda\}_{\lambda \in D}$ gives a continuous system of W. Letting $\phi_q(V)(\lambda) = \phi_q(Z_\lambda - Z_0) \in T_q(V)$ and $\phi_q(W)(\lambda) = \phi_q(Y_\lambda - Y_0) \in T_q(W)$, we would like to show that the following diagram commutes:

$$\begin{tikzcd} & T_q(V) \arrow[dd, "i^*"] \\ \Delta \arrow[ur, "\phi_q(V)"] \arrow[dr, "\phi_q(W)"'] & \\ & T_q(W) \end{tikzcd} \tag{4.2}$$

This would interpret (4.1)geometrically as "*intersection with W*".

Proof. Let $S_V = \sum\limits_{r \geqslant 0} H^{n-q+1+r, n-q-r}(V) \subset H^{2n-2q+1}(V, \mathbf{C})$ be the space of holomorphic 1-forms on $T_q(V)$ and $\omega^1, \ldots, \omega^m$ a basis for S_V. If C_λ is a $2n - 2q + 1$ chain on V with $\partial C_\lambda = Z_\lambda - Z_0$ then $\phi_q(V)(\lambda) = \begin{bmatrix} \vdots \\ \int_{C_\lambda} \omega^\alpha \\ \vdots \end{bmatrix}$. Similarly, let $S_W \subset H^{2n-2q-2k+1}(W, \mathbf{C})$ be the holomorphic 1-forms on $T_q(W)$ and $\phi^1, \ldots, \phi^r$ a basis for S_W. Letting $D_\lambda = C_\lambda \cdot W$, $\partial D_\lambda = Y_\lambda - Y_0$ and $\phi_q(W)(\lambda) = \begin{bmatrix} \vdots \\ \int_{D_\lambda} \phi^\rho \\ \vdots \end{bmatrix}$.

Actually, in line with the Appendix to §3, we should use the isomorphisms $F^{2n-2q+1}_{n-q+1}(V) \cong S_V$, $F^{2n-2k-2q+1}_{n-k-q+1}(W) \cong S_W$ (c.f. (A3.5)), and choose $\omega^1, \ldots, \omega^m$ and $\phi^1, \ldots, \phi^r$ as bases of

$$F^{2n-2q+1}_{n-q+1}(V) \text{ and } F^{2n-2k-2q+1}_{n-k-q+1}(W)$$

respectively. We assume this is done.

We now need to give $i^* : \mathbf{C}^m \to \mathbf{C}^r$ explicitly using the above bases. Let $e_1, \ldots, e_m \in S^*_V \subset H^{2q-1}(V, \mathbf{C})$ be dual to $\omega^1, \ldots, \omega^m$ and $f_1, \ldots, f_r \in S^*_W \subset H^{2q-1}(W, \mathbf{C})$ dual to $\phi^1, \ldots, \phi^r$. Then $\int\limits_V \omega^\alpha \wedge e_\beta = \delta^\alpha_\beta$, $\int\limits_W \phi^\rho \wedge f_\sigma = \delta^\rho_0$. Now $i^*(e_\alpha) = \sum\limits_{\rho=1}^{r} m_{\rho\alpha} f_\rho$ for some $r \times m$ matrix M, and 113
$i^* : T_q(V) \to T_q(W)$ is given by $M : \mathbf{C}^m \to \mathbf{C}^r$ (c.f. just below (2.7)).

To calculate $M \begin{bmatrix} \vdots \\ \int_\gamma \omega^\alpha \\ \vdots \end{bmatrix}$, we let

$$i_* : H^{2n-2k-2q+1}(W, \mathbf{C}) \to H^{2n-2q+1}(V, \mathbf{C})$$

be the *Gysin homomorphism* defined by:

$$\begin{tikzcd} H^{2n-2k-2q+1}(W, \mathbf{C}) \arrow[r, "i_*"] & H^{2n-2q+1}(V, \mathbf{C}) \\ H_{2q-1}(W, \mathbf{C}) \arrow[u, "\mathscr{D}_W"'] \arrow[r, "i_*"] & H_{2q-1}(V, \mathbf{C}). \arrow[u, "\mathscr{D}_V"'] \end{tikzcd} \tag{4.3}$$

Then, $i_*(\phi^p) = \sum\limits_{\alpha=1}^{m} m_{p\alpha}\omega^\alpha$, and $M \begin{bmatrix} \vdots \\ \int_\gamma \omega^\alpha \\ \vdots \end{bmatrix} = \begin{bmatrix} \sum\limits_{\alpha=1}^{m} m_{\rho\alpha} \int\limits_\gamma \omega^\alpha \end{bmatrix} = \begin{bmatrix} \vdots \\ \int_\gamma i_*(\phi^\rho) \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ \int_{W\cdot\gamma} \phi^\rho \\ \vdots \end{bmatrix}$ (c.f. (2.16)) where $\gamma \in H_{2n-2q+1}(V, \mathbf{Z})$ is a cycle on V. This gives the equation

$$M \begin{bmatrix} \vdots \\ \int\limits_\gamma \omega^\alpha \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ \int\limits_\gamma i_*\phi^\rho \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ \int\limits_{W\cdot\gamma} \phi^\rho \\ \vdots \end{bmatrix}, \tag{4.4}$$

for $\gamma \in H_{2n-2q+1}(V, \mathbf{Z})$. To prove (4.2), we must prove (4.4) for the chain C_λ with $\partial C_\lambda = Z_\lambda - Z_0$; this is because, in (4.2),

$$i^*\phi_q(V)(\lambda) = M \begin{bmatrix} \vdots \\ \int\limits_{C_\lambda} \omega^\alpha \\ \vdots \end{bmatrix} \quad \text{and} \quad \phi_q(W)(\lambda) = \begin{bmatrix} \vdots \\ \int\limits_{W\cdot C_\lambda} \phi^\rho \\ \vdots \end{bmatrix}.$$

Thus, to prove the formula (4.2), we must show :

The *Gysin homomorphism*

$$i_* : H^{2n-2k-2q+1}(W, \mathbf{C}) \to H^{2n-2q+1}(V, \mathbf{C}) \tag{4.5}$$

114 (given in (4.3)) has the properties :

$$i_* : F^{2n-2q-2k+1}_{n-q-k+1}(W) \to F^{2n-2q+1}_{n-q+1}(V); \tag{4.6}$$

and

$$\int\limits_C i_*\phi = \int\limits_{W\cdot C} \phi, \tag{4.7}$$

where C is a $2n - 2q + 1$ chain on V with $\partial C = Z$, Z being an algebraic cycle on V meeting W transversely.

This is where we use the bundle $\mathbf{E} \to V$. Namely, it will be proved in the Appendix to §4 below that there is a $C^\infty(k, k-1)$ form ψ defined on $V - W$ having the properties :

$\partial\psi = 0$ and $\overline{\partial}\psi = \Psi$ is a C^∞ form on V which represents the

$$\text{Poincaré dual } \mathscr{D}(W) \in H^{k,k}(V, \mathbf{C}) \cap H^{2k}(V, \mathbf{Z}); \tag{4.8}$$

to give i_* in (4.6), we let $\phi \in B_c^{2n-2k-2q+1, n-k-q+1}(W)$ represent a class in $F^{2n-2k-2q+1}_{n-k-q+1}(W)$ and choose $\widehat{\phi} \in B^{2n-2k-2q+1, n-k-q+1}(V)$ with $\widehat{\phi}|W = \phi$. Then $d(\psi \wedge \phi)$ is a *current* on V and

$$i_*(\phi) = d(\psi \wedge \widehat{\phi}); \quad \text{and} \tag{4.9}$$

$$\lim_{\epsilon \to 0} \int\limits_{c\cdot\partial T_\epsilon} \psi \wedge \eta = \int\limits_{C\cdot W} \eta, \tag{4.10}$$

where T_ϵ is the ϵ-tube around W and $\eta \in B^{2n-2k-2q+1, n-k-q+1}(V)$. □

Remark. The composite

$$F^{2n-2k-2q+1}_{n-k-q+1}(V) \xrightarrow{i^*} F^{2n-2k-2q+1}_{n-k-q+1}(W) \xrightarrow{i_*} F^{2n-2q+1}_{n-q+1}(V) \tag{4.11}$$

is given by $i_* i^* \eta = d(\psi \wedge \eta) = \Psi \wedge \eta (\eta \in B_c^{2n-2k-2q+1, n-k-q+1}(V))$; this should be compared with (2.16) above.

Proof of (4.6) **and** (4.7) **from** (4.8)–(4.10)**.** Since $i_*(\phi) = d(\psi \wedge \widehat{\phi})$ and $\psi \wedge \widehat{\phi} \in B^{2n-2q, n-q+1}$, $i_*(\phi) \in B_c^{2n-2q+1, n-q+1}(V)$ which proves (4.6) (c.f. (e) in the Appendix to Paragraph 4).

To prove (4.7), we will have

$$\int\limits_C i_* \phi = \lim_{e \to 0} \int\limits_{C - C\cdot T_\epsilon} i_*(\phi) = \text{(by Stokes' theorem)} \int\limits_{\partial(C - C\cdot T_\epsilon)} \psi \wedge \widehat{\phi}.$$

But $\partial(C - C\cdot T_\epsilon) = (Z - Z\cdot T_\epsilon) - C\cdot \partial T_\epsilon$ and so $\int\limits_C i_*\phi = -\lim\limits_{\epsilon \to 0} \int\limits_{C\cdot \partial T_\epsilon} \psi \wedge \widehat{\phi}$ **115**
(since $\int\limits_{Z - Z\cdot T_\epsilon} \psi \wedge \widehat{\phi} = 0) = \int\limits_{C\cdot W} \phi$ by (4.10).

Remark 4.12. Actually (4.2) will hold in the following generality. Let V, V' be algebraic manifolds and $f : V' \to V$ a holomorphic mapping. Let $\Sigma_q(V)$ be the algebraic cycles $Z \subset V$ of codimension q which are homologous to zero and similarly for $\Sigma_q(V')$. Then there is a commutative diagram :

$$\begin{tikzcd}
\Sigma_q(V)/\text{S.E.R.} \arrow[r, "\phi_q(V)"] \arrow[d, "f^*"] & T_q(V) \arrow[d, "F^*"] \\
\Sigma_q(V')/\text{S.E.R.} \arrow[r, "\phi_q(V')"] & T_q(V')
\end{tikzcd}$$

where S.E.R. = suitable equivalence relation (including rational equivalence), and where $f^*(Z) = f^{-1}(Z) = \{z' \in V' : f(z) \in V\}$ in case Z is transverse to $f(V')$.

(b) Keeping the notation and assumptions of (4a) above, following (2.12) we have:

$$i_* : T_q(W) \to T_{q+k}(V), \tag{4.13}$$

and we want also these maps geometrically. For this, let $\{Y_\lambda\}_{\lambda\in\Delta}$ be a continuous system of subvarieties $Y_\lambda \subset W$ of codimension q. Then $Y_\lambda \subset V$ has codimension $k+q$ and so we may set

$$\phi_{q+k}(V)(\lambda) = \phi_q(Y_\lambda - Y_0) \in T_{q+k}(V), \phi_q(W)(\lambda) = \phi_q(Y_\lambda - Y_0) \in T_q(W).$$

We assert that the following diagram commutes:

$$\begin{tikzcd} & T_q(W) \arrow[dd, "i_*"] \\ \Delta \arrow[ur, "\phi_q(W)"] \arrow[dr, "\phi_{q+k}(V)"'] & \\ & T_{q+k}(V) \end{tikzcd} \tag{4.14}$$

116 This interprets i_* in (4.13) as "*inclusion of cycles lying on W into V*".

Proof. As in the proof of (4.2), we choose bases $\omega^1, \ldots, \omega^m$ for $S_V \subset H^{2n-2k-2q+1}(V, \mathbf{C})$ and $\phi^1, \ldots, \phi^r$ for $S_W \subset H^{2n-2k-2q+1}(W, \mathbf{C})$. Then

$$\phi_q(W)(\lambda) = \begin{bmatrix} \vdots \\ \int\limits_{C_\lambda} \phi^\rho \\ \vdots \end{bmatrix} \quad \text{and} \quad \phi_{q+k}(V)(\lambda) = \begin{bmatrix} \vdots \\ \int\limits_{C_\lambda} \omega^\alpha \\ \vdots \end{bmatrix}$$

where $\partial C_\lambda = Y_\lambda - Y_0$.

We now need i_* explicitly. Let $e_1, \ldots, e_m$ be a dual basis in $S_V^* \subset H^{2q+2k-1}(V, \mathbf{C})$ to $\omega^1, \ldots, \omega^m$ and $f_1, \ldots, f_r$ in $S_W^* \subset H^{2q-1}(W, \mathbf{C})$ be a dual basis to $\phi^1, \ldots, \phi^r$. Then i_* in (4.14) is induced by the Gysin homomorphism (4.6) $i_* : H^{2q-1}(W, \mathbf{C}) \to H^{2q+2k-1}(V, \mathbf{C})$. Write $i_*(f_\rho) = \sum\limits_{\alpha=1} m_{\alpha\rho} e_\alpha$ so that $M = (m_{\alpha\rho})$ is an $m \times r$ matrix $M : \mathbf{C}^r \to \mathbf{C}^m$ which gives $i_* T_q(W) \to T_{q+k}(V)$.

Now $M\phi_q(W)(\lambda) = \begin{bmatrix} \vdots \\ \sum\limits_{\rho=1}^{r} m_{\alpha\rho} \int\limits_{C} \phi^\rho \\ \vdots \end{bmatrix}$ so that, to prove (4.14), we

$$\int\limits_\gamma \omega^\alpha = \sum_{\rho=1}^{r} m_{\alpha\rho} \int\limits_\gamma \phi^\rho \tag{4.15}$$

for γ a suitable $2n-2k-2q+1$ chain on W. Since

$$i^* : H^{2n-2k-2q+1}(V, \mathbf{C}) \to H^{2n-2k-2q+1}(W, \mathbf{C})$$

satisfies $\int\limits_W i^*(\omega) \wedge \phi = \int\limits_V \omega \wedge i_* \phi$, we have $i^* \omega^\alpha = \sum\limits_{\rho=1}^{r} m_{\alpha\rho} \phi^\rho$ in $H^{2n-2k-2q+1}(W, \mathbf{C})$. On the other hand, since i^* satisfies $i^* \{F_q^r(V)\} \subset F_q^r(W)$, we have, as forms

$$i^* \omega^\alpha = \sum_{\rho=1}^{r} m_{\alpha\rho} \phi^\rho + d\mu^\alpha (\mu^\alpha \in B^{2n-2k-2q+1, n-k-q+1}(W))$$

so that $\int\limits_{C_\lambda} \omega^\alpha = \sum\limits_{\rho=1}^{r} m_{\alpha\rho} \int\limits_{C_\lambda} \phi^\rho$ as needed. □

Remark. To prove (4.14) for $J_q(W)$ and $J_{q+k}(V)$, we use that $i^* \partial = \partial i^*$ **117**
and $i^* \bar{\partial} = \bar{\partial} i^*$ on the form level, so that $i^* \omega^\alpha = \sum\limits_{\rho=1}^{r} m_{\alpha\rho} \phi^\rho + \partial\bar{\partial} \xi^\alpha$ and then, as before,

$$\int_{C_\lambda} \omega^\alpha = \sum_{\rho=1}^{r} m_{\alpha\rho} \int_{C_\lambda} \phi^\rho.$$

(c) We now combine (a) and (b) above. Thus let $W \subset V$ be a submanifold of codimension k and $\{Z_\lambda\}_{\lambda \in \Delta}$ be a continuous system of codimension q on V such that $Z_\lambda \cdot W = Y_\lambda$ is a proper intersection. Then $\{Z_\lambda\}_{\lambda \in \Delta}$ defines $\phi_q : \Delta \to T_q(V)$ and $\{Y_\lambda\}_{\lambda \in \Delta}$ defines $\phi_{q+k} : \Delta \to T_{q+k}(V)$. Combining (4.2) and (4.14), we find that the following is a commutative diagram:

$$\begin{tikzcd}
 & T_q(V) \arrow[d, "i^*"] \\
\Delta \arrow[ur, "\phi_q(V)"] \arrow[r, "\phi_q(W)"] \arrow[dr, "\phi_{q+k}(V)"'] & T_q(W) \arrow[d, "i_*"] \\
 & T_{q+k}(V)
\end{tikzcd} \tag{4.16}$$

Combining (2.13) with (4.16), we have the following commutative diagram:

$$\begin{tikzcd} & T_q(V) \arrow[dd, "\Psi"] \\ \Delta \arrow[ur, "\phi_q"] \arrow[dr, "\phi_{q+k}"'] & \\ & T_{q+k}(V) \end{tikzcd} \tag{4.17}$$

where $\Psi \in H^{k,k}(V) \cap H^{2k}(V, \mathbf{Z})$ is the Poincaré dual of $W \in H_{2n-2k}(V, \mathbf{Z})$ (c.f. (2.7)).

Remark. Actually, we see that (4.17) holds for *all* algebraic cycles $W_{n-k} \subset V_n$, provided we assume a foundational point concerning the *suitable equivalence relation* (= S.E.R.) in Remark 4.12. Let $\Sigma_q(V)$ be
118 the algebraic cycles of codimension q which are homologous to zero, and assume that S.E.R. has the property that, for any $W_{n-k} \subset V_n$, the mapping $\Sigma_q(V)/\text{S.E.R.} \xrightarrow{W} \Sigma_{q+k}(V)/\text{S.E.R.}$ is defined and $W(Z) = W \cdot Z$ if the intersection is proper $(Z \in \Sigma_q(V))$. Then we have that: The following diagram commutes:

$$\begin{tikzcd} \Sigma_q(V)/\text{S.E.R.} \arrow[r, "\phi_q"] \arrow[d, "W"] & T_q(V) \arrow[d, "\Psi"] \\ \Sigma_{q+k}(V)/\text{S.E.R.} \arrow[r, "\phi_{q+k}"] & T_{q+k}(V) \end{tikzcd} \qquad (\Psi = \mathscr{D}(W)). \tag{4.18}$$

Proof. The proof of (4.17) will show that (4.18) commutes when W is a Chern class of an *ample bundle* [11]. However, by [12] the Chern classes of ample bundles generate the *rational equivalence ring* on V, so that (4.18) holds in general. □

Appendix to Paragraph 4. Let $\mathbf{E} \to V$ be a holomorphic vector bundle with fibre $\mathbf{C}^k$, and $\sigma_1, \ldots, \sigma_{k-q+1}$ holomorphic cross-sections of $\mathbf{E} \to V$ such that the subvariety $W = \{\sigma_1 \wedge \ldots \wedge \sigma_{k-q+1} = 0\}$ is a generally singular subvariety $W_{n-q} \subset V_n$ of codimension q. (Note the shift in indices from §4.) Then the homology class $W \in H_{2n-2k}(V, \mathbf{Z})$ is the *Poincaré dual* of the q^{th} *Chern class* $c_q \in H^{2q}(V, \mathbf{Z})$ (c.f. [11]). We shall prove: There exists a differential form ψ on V such that

$$\psi \text{ is of type } (q, q-1), \text{ is } C^\infty \text{ in } V - W, \text{ and } \partial\psi = 0; \tag{A4.1}$$

$$\bar{\partial}\psi = d\psi \text{ is } C^\infty \text{ on } V \text{ and represents } c_q \text{ (via deRham)}; \tag{A4.2}$$

ψ has a pole of order $2q-1$ along W and, if ω is any closed $2n-2q$ form on V, $\displaystyle\int_V c_a \wedge \omega = \lim_{\epsilon \to 0} \int_{\partial T_\epsilon} \psi \wedge \omega$ where $T_\epsilon \subset V$ is the ϵ-tubular neighbourhood around W. (A4.3)

Proof. For a $k \times k$ matrix A, define $P_q(A)$ by:

$$\det\left(\frac{i}{2\pi}A + tI\right) = \sum_{q=0}^{k} P_q(A)t^{k-q}. \tag{A4.4}$$

Let $P_q(A_1, \ldots, A_q)$ be the invariant, symmetric multilinear form ob- **119**
tained by polarizing $P_q(A)$ (for example,

$$P_k(A_1, \ldots, A_k) = 1/k! \sum_{\pi_1, \ldots, \pi_k} \det(A^1_{\pi_1}, \ldots, A^k_{\pi_k})$$

where $A^\alpha_{\pi_\alpha}$ is the α^{th} column of A_{π_α}; cf. (6.5) below). Choose an Hermitian metric in $\mathbf{E} \to V$ and let $\Theta \in A^{1,1}(V, \operatorname{Hom}(\mathbf{E}, \mathbf{E}))$ be the curvature of the metric connection. Then (c.f. [11]):

$c_q \in H^{2q}(V, \mathbf{C})$ is represented by the differential form

$$P_q(\Theta) = P_q(\Theta, \ldots, \Theta). \tag{A4.5}$$

□

What we want to do is to construct ψ, depending on $\sigma_1, \ldots, \sigma_{k-q+1}$ and the metric, such that (A4.1)–(A4.3) are satisfied. The proof proceeds in four steps.

(a) Some Formulae in Local Hermitian Geometry. Suppose that Y is a complex manifold (Y will be $V - W$ in applications) and that $\mathbf{E} \to Y$ is a holomorphic vector bundle such that we have an exact sequence :

$$0 \to \mathbf{S} \to \mathbf{E} \to \mathbf{Q} \to 0 \tag{A4.6}$$

(in applications, $\mathbf{S}$ will be the trivial sub-bundle generated by $\sigma_1, \ldots,$ σ_{k-q+1}). We assume that there is an Hermitian metric in $\mathbf{E}$ and let D be the metric connection [11]. Let $e_1, \ldots, e_k$ be a unitary frame for $\mathbf{E}$ such that $e_1, \ldots, e_s$ is a frame for $\mathbf{S}$. Then $De_\rho = \sum\limits_{\sigma=1}^{k} \theta_\rho^\alpha e_\sigma$ where $\theta_\rho^\sigma + \overline{\theta}_\sigma^\rho = 0$. By the formula $D_{\mathbf{S}} e_\alpha = \sum\limits_{\beta=1}^{s} \theta_\alpha^\beta e_\beta (\alpha = 1, \ldots, s)$, there is defined a connection $D_{\mathbf{S}}$ in $\mathbf{S}$, and we claim that $D_{\mathbf{S}}$ is the connection for the induced metric in $\mathbf{S}$ (c.f. [11], §1.d).

Proof. Choose a *holomorphic* section $e(z)$ of $\mathbf{S}$ such that $e(0) = e_\alpha(0)$ (this is over a small coordinate neighborhood on Y). Then $D''e = 0$ since $D'' = \overline{\partial}$. Thus, writing $e(z) = \sum\limits_{\alpha=1}^{s} \xi^\alpha e_\alpha$, $0 = D''e = \sum\limits_{\alpha=1}^{s} \overline{\partial} \xi^\alpha e_\alpha +$ $\sum\limits_{\alpha,\beta=1}^{s} \xi^\alpha \theta_\alpha^{\beta''} e_\beta + \sum\limits_{\mu=s+1}^{k} \sum\limits_{\alpha=1}^{s} \xi^\alpha \theta_\alpha^{\mu''} e_\mu$. At $z = 0$, this gives $\sum\limits_{\beta=1}^{s} (\overline{\partial} \xi^\beta(0) +$ $\theta_\alpha^{\beta''}) e_\beta + \sum\limits_{\mu=s+1}^{k} \theta_\alpha^{\mu''} e_\mu = 0$. Thus $\theta_\alpha^{\mu''} = 0$ and, since $(D'' - D''_{\mathbf{S}}) e_\alpha =$
120 $\sum\limits_{\mu=s+1} \theta_\alpha^{\mu''} e_\mu$, $D''_{\mathbf{S}} = \overline{\partial}$. By uniqueness, $D_{\mathbf{S}}$ is the connection of the induced metric in $\mathbf{S}$. □

Remark. **(A4.7)** Suppose that $\mathbf{S}$ has a *global holomorphic frame* $\sigma_1, \ldots, \sigma_s$. Write $\sigma_\alpha = \sum\limits_{\beta=1}^{s} \xi_\alpha^\beta e_\beta$. From $0 = \overline{\partial} \sigma_\alpha = D'' \sigma_\alpha = \sum\limits_{\beta=1}^{s} (\overline{\partial} \xi_\alpha^\beta +$

$\sum_{\gamma=1}^{s} \xi_\alpha^\gamma \theta_\gamma^{\beta''})e_\beta$, we get $\overline{\partial}\xi + \theta_{\mathbf{S}}''\xi = 0$ or $\theta_{\mathbf{S}}'' = -\overline{\partial}\xi\xi^{-1}$. This gives

$$\theta_{\mathbf{S}} = {}^t\overline{\xi}^{-1}\partial\overline{\xi} - \overline{\partial}\xi\xi^{-1}. \tag{A4.8}$$

Now write $\theta = \begin{pmatrix} \theta_1^1 & \theta_2^1 \\ \theta_1^2 & \theta_2^2 \end{pmatrix}$ where $\theta_1^1 = (\theta_\beta^\alpha)$, $\theta_1^2 = (\theta_\alpha^\mu)$, etc. Then $\theta_1^{2''} = 0 = \theta_2^{1'}$ (since $\theta_1^2 + {}^t\overline{\theta}_2^1 = 0$). Let $\phi = \begin{pmatrix} 0 & -\theta_2^1 \\ -\theta_1^2 & 0 \end{pmatrix}$ and $\widehat{\theta} = \theta + \phi = \begin{pmatrix} \theta_1^1 & 0 \\ 0 & \theta_2^2 \end{pmatrix}$. Then θ and $\widehat{\theta}$ give connections D and $\widehat{D}$ in $\mathbf{E}$ with curvatures Θ and $\widehat{\Theta}$. Setting $\theta_t = \theta + t\phi$, we have a homotopy from θ to $\widehat{\theta}$ with $\dot{\theta}_t = \phi \left(\dot{\theta}_t = \dfrac{\partial \theta_t}{\partial t}\right)$.

Now let $P(A)$ be an invariant polynomial of degree q (c.f. §6 below) and $P(A_1, \ldots, A_q)$ the corresponding invariant, symmetric, multilinear from (c.f. (6.5) for an example). Thus $P(A) = P(\underbrace{A, \ldots, A}_{q})$. Set

$$Q_t = 2\sum_{j=1}^{q} P(\theta_t, \ldots, \phi_j', \ldots, \Theta_t), \tag{A4.9}$$

and define Ψ_1 by:

$$\Psi_1 = \int_0^1 Q_t dt. \tag{A4.10}$$

What we want to prove is (c.f. [11], §4):

$$\left.\begin{array}{l} \Psi_1 \text{ is a } C^\infty \text{ form of type } (q, q-1) \text{ on } Y \text{ satisfying} \\ \partial\Psi_1 = 0,\ \overline{\partial}\Psi_1 = P(\Theta) - P(\widehat{\Theta}). \end{array}\right\} \tag{A4.11}$$

Proof. It will suffice to show that

$$\partial Q_t = 0, \quad \text{and} \tag{A4.12}$$

$$\dot{P}(\Theta_t) = \overline{\partial} Q_t. \tag{A4.13}$$

By the *Cartan structure equation,* $\Theta_t = d\theta_t + \theta_t \wedge \theta_t = d(\theta + t\phi) +$ **121**
$(\theta + t\phi) \wedge (\theta + t\phi) = d\theta + \theta \wedge \theta + t(d\phi + \phi \wedge \theta + \theta \wedge \phi) + t^2\phi \wedge \phi =$
$\Theta + tD\phi + t^2\phi \wedge \phi$. Now

$$\phi \wedge \phi = \begin{pmatrix} \theta_2^1\theta_1^2 & 0 \\ 0 & \theta_1^2\theta_2^1 \end{pmatrix} \text{ and } D\phi = \begin{pmatrix} 0 & -\Theta_2^1 \\ -\Theta_1^2 & 0^2 \end{pmatrix} + 2\begin{pmatrix} -\theta_2^1\theta_1^2 & 0 \\ 0 & \theta_1^2\theta_2^1 \end{pmatrix}.$$

This gives

$$\Theta_t = \Theta + t\begin{pmatrix} 0 & -\Theta_2^1 \\ -\Theta_1^2 & 0 \end{pmatrix} + (t^2 - 2t)\begin{pmatrix} \theta_2^1\theta_1^2 & 0 \\ 0 & \theta_1^2\theta_2^1 \end{pmatrix}; \tag{A4.14}$$

$$D'\phi' = 0; \quad \text{and} \tag{A4.15}$$

$$D''\phi' = \begin{pmatrix} 0 & 0 \\ -\Theta_1^2 & 0 \end{pmatrix} + \begin{pmatrix} -\theta_2^1\theta_1^2 & 0 \\ 0 & -\theta_1^2\theta_2^1 \end{pmatrix} \tag{A4.16}$$

It follows that Θ_t is of type $(1,1)$ and so Q_t is of type $(q, q-1)$, as is Ψ_1.

By symmetry, to prove (A4.12) it will suffice to have

$$\partial P(\Theta_t, \ldots, \Theta_t, \phi') = 0.$$

Let $D_t = D_t' + D_t''$ be the connection corresponding to θ_t. Then $D_t'\Theta_t = 0$ (*Bianchi identity*) and $\partial P(\Theta_t, \ldots, \Theta_t, \phi') = \Sigma P(\Theta_t, \ldots, D_t'\Theta_t, \ldots, \Theta_t, \phi')$ $+ P(\Theta_t, \ldots, \Theta_t, D_t'\phi') = P(\Theta_t, \ldots, \Theta_t, D_t'\phi')$. But $D_t'\phi' = D'\phi' + t[\phi, \phi']'$ $= 0 + t[\phi', \phi'] = 0$ by (A4.15). This proves (A4.12).

We now calculate

$$\begin{aligned} \overline{\partial} P(\Theta_t, \ldots, \phi', \ldots, \Theta_t) &= \Sigma P(., D_t''\Theta_t, \ldots, \phi', \ldots, \Theta_t) + \\ + P(\Theta_t, \ldots, D_t''\phi', \ldots, \Theta_t) &+ \Sigma P(\Theta_t, \ldots, \phi', \ldots, D_t''\Theta_t, .) \\ &= P(\Theta_T, \ldots, D_t''\phi', \ldots, \Theta_t) \end{aligned}$$

(since $D_t''\Theta_T = 0$ by Bianchi). Then we have $D_t''\phi' = D''\phi' + t[\phi, \phi']'' =$

(by (A4.16)) $\begin{pmatrix} 0 & 0 \\ -\Theta_1^2 & 0 \end{pmatrix} + (t-1)\begin{pmatrix} \theta_2^1\theta_1^2 & 0 \\ 0 & \theta_1^2\theta_2^1 \end{pmatrix}$. But, by (A4.14),

$$\dot{\Theta}_t = \begin{pmatrix} 0 & -\Theta_2^1 \\ -\Theta_1^2 & 0 \end{pmatrix} + 2(t-1)\begin{pmatrix} \theta_2^1\theta_1^2 & 0 \\ 0 & \theta_1^2\theta_2^1 \end{pmatrix},$$

so that

$$2D_t''\phi' - \dot{\Theta}_T = \begin{pmatrix} 0 & \Theta_2^1 \\ -\Theta_1^2 & 0 \end{pmatrix} = [\pi, \Theta], \tag{A4.17}$$

where $\pi = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Using (A4.17), $\bar{\partial} Q_t - \dot{P}(\Theta_t) = \Sigma\{2P(\Theta_t, \ldots, D_t''\phi', \ldots, \Theta_t) - P(\Theta_t, \ldots, \dot{\Theta}_t, \ldots, \Theta_t)\} = \Sigma P(\Theta_t, \ldots, [\pi, \Theta_t], \ldots, \Theta_t) = 0$. This proves (A4.13) and completes the proof of (A4.11). □

Return now to the form Q_t defined by (A4.9). Since $\Theta_T = \Theta + tD\phi + t^2\phi \wedge \phi$, we see that $\underline{Q_t}$ is a polynomial in the differential forms $\Theta_\sigma^\rho, \theta_\alpha^\mu, \theta_\mu^\alpha (1 \leqslant \rho, \sigma \leqslant k; 1 \leqslant \alpha \leqslant s; s+1 \leqslant \mu \leqslant k)$. Write $Q_t \equiv 0(l)$ to symbolize that each term in Q_t contains *no more* than $l - t$ terms involving the θ_α^μ and θ_μ^α. We claim that **122**

$$Q_t \equiv 0(2q-1). \tag{A4.18}$$

Proof. The term of highest order (i.e. containing the most θ_α^μ and θ_μ^α) in Q_t is $(t^2/2)^{q-1}\Sigma P([\phi,\phi], \ldots, \phi', \ldots, [\phi,\phi])$. Now, by invariance,

$$\begin{aligned} P([\phi,\phi], \ldots, \phi', \ldots, [\phi,\phi]) &= -P(\phi, [\phi,\phi], \ldots, [\phi,\phi'], \ldots, [\phi,\phi]) \\ &= -\frac{1}{2}P(\phi, [\phi,\phi], \ldots, [\phi,\phi]) \end{aligned}$$

since $[\phi, [\phi,\phi]] = 0$ and $[\phi,\phi'] = \frac{1}{2}[\phi,\phi]$. But, by invariance again, $P(\phi, [\phi,\phi], \ldots, [\phi,\phi]) = 0$. Since all other terms in Q_t are of order $2q-2$ or less, we obtain (A4.18).

It follows from (A4.10) that

$$\Psi_1 \equiv 0(2q-1). \tag{A4.19}$$

(b) Some further formulae in Hermitian geometry. Retaining the situation (A4.6), we have from (A4.11) and (A4.19) that:

$$P(\Theta) - P(\widehat{\Theta}) = \bar{\partial}\Psi_1 \quad \text{where} \quad \partial\Psi_1 = 0, \Psi_1 \equiv 0(2q-1). \tag{A4.20}$$

Now suppose that $\mathbf{S}$ has fibre dimension $k - q + 1$ and that:

$$\mathbf{S} \quad \text{has a global holomorphic frame} \quad \sigma_1, \dots, \sigma_{k-q+1}. \tag{A4.21}$$

Let $\mathbf{L}_1 \subset \mathbf{S}$ be line bundle generated by σ_1 and $\mathbf{S}_1 = \mathbf{S}/\mathbf{L}_1$. Then σ_2 gives a non-vanishing section of $\mathbf{S}_1$ and so generates a line bundle $\mathbf{L}_2 \subset \mathbf{S}_1$. Continuing, we get a diagram:

$$\left.\begin{array}{llllll} 0 \longrightarrow \mathbf{L}_1 & \longrightarrow \mathbf{S} & \longrightarrow \mathbf{S}_1 & \longrightarrow 0 \\ 0 \longrightarrow \mathbf{L}_2 & \longrightarrow \mathbf{S}_1 & \longrightarrow \mathbf{S}_2 & \longrightarrow 0 \\ & \quad\vdots & & \\ 0 \longrightarrow \mathbf{L}_{k-q} & \longrightarrow \mathbf{S}_{k-q-1} & \longrightarrow \mathbf{S}_{k-q} & \longrightarrow 0 \\ 0 \longrightarrow \mathbf{L}_{k-q+1} & \longrightarrow \mathbf{S}_{k-q} & \longrightarrow 0 & \end{array}\right\} \tag{A4.22}$$

123 All the bundles in (A4.22) have metrics induced from $\mathbf{S}$; as a C^∞ bundle, $\mathbf{S} \cong \mathbf{L}_1 \oplus \cdots \oplus \mathbf{L}_{k-q+1}$ (this is actually true as holomorphic bundles, but the splitting will *not* be this orthonormal one).

Now suppose that we use unitary frames $(e_1, \dots, e_{k-q+1})$ for $\mathbf{S}$ where e_α is a unit vector in $\mathbf{L}_\alpha$. If $\theta_{\mathbf{S}} = (\theta^\alpha_\beta)$ is the metric connection in $\mathbf{S}$, then θ^α_α gives the connection of the induced metric in $\mathbf{L}_\alpha$ (c.f. (a) above). This in turn gives a connection

$$\gamma_{\mathbf{S}} = \begin{bmatrix} \gamma^1_1 & & 0 \\ & \ddots & \\ 0 & & \gamma^{k-q+1}_{k-q+1} \end{bmatrix} \quad (\gamma^\alpha_\alpha = \theta^\alpha_\alpha) \text{ with curvature}$$

$$\Gamma_{\mathbf{S}} = \begin{bmatrix} \Gamma^1_1 & & 0 \\ & \ddots & \\ 0 & & \Gamma^{k-q+1}_{k-q+1} \end{bmatrix}$$

in $\mathbf{S}$. Now the connection $\widehat{\theta} = \theta_{\mathbf{S}} \oplus \theta_{\mathbf{Q}}$ in $\mathbf{E}$ has curvature $\widehat{\Theta} = \left[\begin{smallmatrix} \Theta_{\mathbf{S}} & 0 \\ 0 & \Theta_{\mathbf{Q}} \end{smallmatrix}\right]$. We let $\Gamma = \left[\begin{smallmatrix} \Gamma_{\mathbf{S}} & 0 \\ 0 & \Theta_{\mathbf{Q}} \end{smallmatrix}\right]$ be the curvature of the connection $\left[\begin{smallmatrix} \gamma_{\mathbf{S}} & 0 \\ 0 & \theta_{\mathbf{Q}} \end{smallmatrix}\right]$ in $\mathbf{E}$. Then the same argument as used in (a) to prove (A4.20), when iterated, gives

$$P(\widehat{\Theta}) - P(\Gamma) = \overline{\partial}\Psi_2 \quad \text{where} \quad \partial\Psi_2 = 0 \text{ and } \Psi_2 \equiv 0(2q). \tag{A4.23}$$

The congruence $\Psi_2 \equiv 0(2q)$ is trivial in this case since $\deg \Psi_2 = 2q - 1$. Adding (A4.20) and (A4.23) gives:

$$P(\Theta) - P(\Gamma) = \bar{\partial}(\Psi_1 + \Psi_2). \tag{A4.24}$$

The polynomial $P(A)$ is of degree q, and we assume now that:

$$P(A) = 0 \text{ if } A = \begin{pmatrix} 0 & 0 \\ 0 & A' \end{pmatrix} \text{ where } A' \text{ is a } (q-1) \times (q-1) \text{ matrix.} \tag{A4.25}$$

□

We claim then that

$$P(\Gamma) = \bar{\partial}\Psi_3 \text{ where } \partial\Psi_3 = 0 \text{ and } \Psi_3 \equiv 0(2q). \tag{A4.26}$$

Proof. Each line bundle $\mathbf{L}_\alpha$ has a holomorphic section $\sigma_\alpha = |\sigma_\alpha| e_\alpha$. From $0 = \bar{\partial}\sigma_\alpha = \bar{\partial}|\sigma_\alpha| e_\alpha + |\sigma_\alpha| \theta_\alpha^{\alpha''} e_\alpha$, we find $\theta_\alpha^{\alpha''} = -\bar{\partial} \log |\sigma_\alpha|$ and

$$\theta_\alpha^\alpha = (\partial - \bar{\partial}) \log |\sigma_\alpha|, \quad \text{and} \tag{A4.27}$$

$$\Gamma_\alpha^\alpha = 2\bar{\partial}\partial \log |\sigma_\alpha|. \tag{A4.28}$$

Now **124**

$$P(\Gamma) = P\underbrace{(\Gamma_S + \Theta_{\mathbf{Q}}, \ldots, \Gamma_{\mathbf{S}} + \Theta_{\mathbf{Q}})}_{q} = \sum_{\substack{r+s=q \\ r>0}} \binom{q}{r} P(\underbrace{\Gamma_S}_{r}; \underbrace{\Theta_{\mathbf{Q}}}_{s})$$

(since $P(\underbrace{\Theta_{\mathbf{Q}}, \ldots, \Theta_{\mathbf{Q}}}_{q}) = 0$ by (A4.25)). Let

$$\xi = 2 \begin{bmatrix} \theta_1^{1'} & & & \\ & \ddots & 0 & 0 \\ 0 & & \theta_{k-q+1}^{k-q+1'} & \\ & & 0 & \theta_\nu^\mu \end{bmatrix}.$$

Then $D'_\gamma\xi = 0$ where γ is the connection

$$\begin{bmatrix} \theta_1^1 & & & \\ & \ddots & & 0 \\ & & \theta_{k-q+1}^{k-q+1} & \\ & & 0 & \theta_\nu^\mu \end{bmatrix}$$

in $\mathbf{L}_1 \oplus \cdots \oplus \mathbf{L}_{k-q+1}\Theta\mathbf{Q}$. Also $D''_\gamma\xi = \left(\begin{smallmatrix}\Gamma_\mathbf{S} & 0 \\ 0 & 0\end{smallmatrix}\right)$. Set

$$\Psi_3 = \sum_{\substack{r+s=q \\ r>0}} \binom{q}{r} P(\xi, \underbrace{\Gamma_\mathbf{S}}_{r-1}; \underbrace{\Theta_\mathbf{Q}}_{s}). \tag{A4.29}$$

Then $\partial\Psi_3 = 0$ since $D'_\gamma\xi = 0 = D'_\gamma\Gamma_\mathbf{S} = D'_\gamma\Theta_\mathbf{Q}$, and

$$\bar\partial\Psi_3 = \sum_{\substack{r+s=q \\ r>0}} \binom{q}{r} P(\underbrace{\Gamma_\mathbf{S}}_{r}; \underbrace{\Theta_\mathbf{Q}}_{s}) = P(\Gamma)$$

since $D''_\gamma\xi = \Gamma_\mathbf{S}$, $D''_\gamma\Gamma_\mathbf{S} = 0 = D''_\gamma\Theta_\mathbf{Q}$. This shows that Ψ_3 defined by (A4.29) satisfies (A4.26).

Combining (A4.24) and (A4.26) gives:

$$P(\Theta) = \bar\partial\Psi \text{ where } \Psi = \Psi_1 + \Psi_2 + \Psi_3, \partial\Psi = 0, \Psi \equiv 0(2q). \tag{A4.30}$$

Let Ψ be given as just above by (A4.30); Ψ is a form of type $(q, q-1)$ on Y. Suppose we refine the congruence symbol $\equiv$ so that $\eta \equiv 0(l)$ means that η contains at most $l = 1$ terms involving $\theta_1^{1'}, \theta_1^{k-2+q}, \ldots, \theta_1^k$, $\theta_{k-q+2}^1, \ldots, \theta_k^1$. Then, for some constant c,

$$\Psi \equiv c\theta_1^{1'}\theta_1^{k-2+q} \ldots \theta_1^k\theta_{k-q+2}^1 \ldots \theta_k^1(2q-1). \tag{A4.31}$$

We want to calculate c when $P(A) = P_q(A)$ corresponds to the q^{th} Chern class (c.f. (A4.4)). By (A4.19), $\Psi_1 \equiv 0(2q-1)$ and an inspection of (A4.9) shows that $\Psi_2 \equiv 0(2q-1)$. Thus $\Psi \equiv \Psi_3(2q-1)$.

To calculate $\Gamma_{\mathbf{S}}$, we have $\Gamma_\alpha^\alpha = d\theta_\alpha^\alpha = d\theta_\alpha^\alpha + \sum_{\rho=1}^{k} \theta_\rho^\alpha \wedge \theta_\alpha^\rho - \sum_{\rho=1}^{k} \theta_\rho^\alpha \wedge \theta_\alpha^\rho = \Theta_\alpha^\alpha - \sum_{\rho=1}^{k} \theta_\rho^\alpha \wedge \theta_\alpha^\rho$. Thus $\Gamma_\alpha^\alpha \equiv 0(0)$ for $\alpha > 1$ and

$$\Gamma_1^1 \equiv - \sum_{\mu=k-q+2}^{k} \theta_\mu^1 \wedge \theta_1^\mu (0).$$

It follows that $P_q(\xi, \underbrace{\Gamma_{\mathbf{S}}}_{r-1} \underbrace{\Theta_{\mathbf{Q}}}_{s}) \equiv 0(2q-1)$ if $r-1>0$, and so 125
$\Psi_3 \equiv P_q(\xi, \underbrace{\Theta_{\mathbf{Q}}}_{q-1})(2q-1)$.

Now, by the definition of

$$P_q, P_q(\xi, \underbrace{\Theta_{\mathbf{Q}}}_{q-1}) = (i/2\pi)^q (1/q) \theta_1^{1'} \det(\Theta_{\mathbf{Q}}).$$

But $(\Theta_{\mathbf{Q}})_\nu^\mu = \Theta_\nu^\mu + \sum_{\alpha=1}^{k-q+1} \theta_\alpha^\mu \wedge \theta_\nu^\alpha$, so that $(\Theta_{\mathbf{Q}})_\nu^\mu \equiv \theta_1^\mu \wedge \theta_\nu^1(0)$. Combining these relations gives $\Psi_3 \equiv (i/2\pi)^q (1/q) \theta_1^{1'} \det(\theta_1^\mu \theta_\nu^1)(2q-1)$ or

$$\Psi_3 \equiv \left(\frac{1}{2\pi i}\right)^q (-1)^{q(q-1)/2} \theta_1^{1'} \theta_1^{k-q+2} \ldots \theta_1^k \theta_{k-q+2}^1 \ldots \theta_k^1 (2q-1). \tag{A4.32}$$

Combining (A4.30) and (A4.32) gives

$$P_q(\Theta) = \bar{\partial}\Psi \quad \text{where} \quad \partial\psi = 0 \tag{A4.33}$$

and

$$\left.\begin{array}{c} \Psi \equiv -\Gamma(q) \theta_1^{1'} \theta_1^{k-q+2} \ldots \theta_1^k \theta_{k-q+2}^1 \ldots \theta_k^1 (2q-1) \\ (\Gamma(q) = (1/2\pi i)(-1)^{q(q-1)/2}). \end{array}\right\} \tag{A4.34}$$

□

(c) Reduction to a local problem. Return now to the notation and assumptions at the beginning of the Appendix to §4. Taking into account (A4.6), (A4.21), (A4.30), and letting $Y = V - W$, we have constructed a $(q, q-1)$ form ψ on $V - W$ such that $\partial\psi = 0$, $\bar{\partial}\psi = P_q(\Theta) = c_q$, and such that $\psi \equiv 0(2q)$. This proves (A4.1), (A4.2), and, in the following section, we will interpret $\psi \equiv 0(2q)$ to mean that ψ has a pole of order $2q - 1$ along W.

Let ω be a closed $2n - 2q$ form on V and T_ϵ the ϵ-tube around W. Then $\int_V c_q \wedge \omega = \lim_{\epsilon \to 0} \int_{V - T_\epsilon} c_q \wedge \omega = \lim_{\epsilon \to 0} - \int_{\partial T_\epsilon} \psi \wedge \omega$ since $d(\psi \wedge \omega) = P_q(\Theta) \wedge \omega$. This proves (A4.3).

For the purposes of the proof of (4.2), we need a stronger version of (A4.3); namely, we need that

$$\lim_{\epsilon \to 0} - \int_{C \cdot \partial T_\epsilon} \psi \wedge \eta = \int_{C \cdot W} \eta, \tag{A4.35}$$

126 where $\eta \in B^{2n-2k-2q+1, n-k-q+1}(V)$ and C is the $2n - 2q + 1$ chain on V used in the proof of (4.2). In other words, we need to show that *integration with respect to ψ is a residue operator along W*. Because both sides of (A4.35) are linear in η, we may assume that η has support in a coordinate neighborhood. Also, because ψ will have a pole only of order $2q - 1$ along W, it will be seen that both sides of (A4.35) will remain unchanged if we take out of W_{n-q} an algebraic hypersurface H_{n-q-1} which is in general position with respect to C. Thus, to prove (A4.35), we may assume that:

$$\left.\begin{array}{l} \eta \text{ has support in a coordinate neighborhood on } V \\ \text{where } \sigma_2 \wedge \ldots \wedge \sigma_{k-q+2} \neq 0, \ \ \sigma_1 \neq 0. \end{array}\right] \tag{A4.36}$$

This is a local question which will be resolved in the next section.

We note in passing that (4.9) follows from (4.10) when C is a cycle on V, so that (4.2) will be completely proved when (A4.35) is proved in the local form (A4.35) above.

(d) COMPLETION OF THE PROOF. Over $\mathbf{C}^n$ consider the trivial bundle $\mathbf{E} = \mathbf{C}^n \times \mathbf{C}^k$ in which we have a Hermitian metric $(h_{\rho\sigma}(z))$ ($z = \begin{bmatrix} z^1 \\ \vdots \\ z^n \end{bmatrix}$ are coordinates in $\mathbf{C}^n$; $1 \leqslant \rho, \sigma \leqslant k$). We suppose that there are holomorphic sections $\sigma_2, \ldots, \sigma_{k-q+1}$ generating the sub-bundle $\mathbf{S}' = \mathbf{C}^n \times \{\mathbf{O}^q \times \mathbf{C}^{k-q}\}$ of $\mathbf{E}$, and let σ_1 be a holomorphic section of the form $\sigma_1(z) = \begin{bmatrix} z^1 \\ \vdots \\ z^q \\ 0 \\ \vdots \\ 0 \end{bmatrix}$. Then the locus $\sigma_1 \wedge \ldots \wedge \sigma_{k-q+1} = 0$ is given by $z^1 = \ldots = z^q = 0$, so that we have the local situation of (A4.6) ($\mathbf{S}$ is generated by $\mathbf{S}'$ and σ_1 on $\mathbf{C}^n - \mathbf{C}^{n-q}$), (A4.21) and (A4.36). We consider unitary frames $e_1, \ldots, e_k$ for $\mathbf{E}$ where $e_1 = \dfrac{\sigma_1}{|\sigma_1|}$, and $e_2, \ldots, e_{k-q+1}$ is a frame for $\mathbf{S}'$. Thus $e_1, \ldots, e_{k-q+1}$ is a frame for $\mathbf{S}|\mathbf{C}^n - \mathbf{C}^{n-q}$.

Write $De_1 = \sum_{\rho=1}^{k} \theta_1^\rho e_\rho$ (θ_1^ρ is of type $(1,0)$ for $\rho > 1$) and set $\Omega =$ **127**
$-\Gamma(q)\theta_1^{1'}\theta_1^{k-q+2} \ldots \theta_1^k \theta_{k-q+2}^1 \ldots \theta_k^1$. If η is a compactly supported $2n - 2q$ form on $\mathbf{C}^n$, we want to show:

$$\int\limits_{\mathbf{C}^{n-q}} \eta = -\lim_{\epsilon \to 0} \int\limits_{\partial T_\epsilon} \Omega \wedge \eta, \tag{A4.37}$$

where T_ϵ is an ϵ-ball around $\mathbf{C}^{n-q} \subset \mathbf{C}^n$. Having done this, we will, by almost exactly the same argument, prove (A4.35).

Using the metric connection, we write $De_\rho = \sum_{\sigma=1}^{k} \theta_\rho^\sigma e_\sigma$. Then the 1-forms θ_α^ρ are smooth on $\mathbf{C}^n$ for $\rho \neq 1, \sigma \neq 1$. If we can show that the θ_1^ρ have a first order pole along $\mathbf{C}^{n-q} \subset \mathbf{C}^n$, then it will follow that Ω has a pole of order $2q - 1$ along $\mathbf{C}^{n-q}$ and that our congruence symbol "$\equiv$" (c.f. just above (A4.31)) refers to the order of pole along $\mathbf{C}^{n-q}$. We consider each vector $e_\rho = \begin{bmatrix} e_\rho^1(z) \\ \vdots \\ e_\rho^k(z) \end{bmatrix}$ as a vector field $f_\rho = \sum_{\sigma=1}^{k} e_\rho^\sigma(z) \dfrac{\partial}{\partial z^\sigma}$

on $\mathbf{C}^n$ and, letting $f_a = \dfrac{\partial}{\partial z^a}(a = k+1, \ldots, n)$, we have a *tangent vector frame* $f_1, \ldots, f_n$ on $\mathbf{C}^n$ such that $f_2, \ldots, f_{k-q+1}, f_{k+1}, \ldots, f_n$ are tangent to $\mathbf{C}^{n-q} \subset \mathbf{C}^n$ along $\mathbf{C}^{n-q}$. Let $\omega^1, \ldots, \omega^n$ be the co-frame of $(1,0)$ forms; then if $z = \sum_{j=1}^{n} z^j \dfrac{\partial}{\partial z^j}$, $dz = \sum_{j=1}^{n} f_j \omega^j$. But $z = \sigma_1 + \sum_{\alpha=2}^{k-q+2} \lambda_\alpha e_\alpha + \sum_{a=k+1}^{n} z^a f_a$, so that $dz = D'\sigma_1 \sum_{\alpha=2}^{k-q+2} (\partial\lambda_\alpha e_\alpha + \lambda_\alpha D' e_\sigma) + \sum_{a=k+1}^{n} f_a \omega^a$. This gives:

$$\left.\begin{aligned} \omega^1 &= 2|\sigma_1|\theta_1^{1'}; \\ \omega^\mu &= |\sigma_1|\theta_1^\mu + \sum_{\alpha=2}^{k-q+1} \lambda_\alpha \theta_\alpha^\mu \quad && (\mu = k-q+2, \ldots, k); \\ \omega^\alpha &= \partial\lambda_\alpha + \sum_{\beta=2}^{k-q+1} \lambda_\beta \theta_\beta^{\alpha'} && (\alpha = 2, \ldots, k-q+1); \text{ and} \\ \omega^a &= dz^a && (a = k+1, \ldots, n). \end{aligned}\right\} \tag{A4.38}$$

128 It follows that θ_1^1, θ_1^μ have a first order pole along $\mathbf{C}^{n-q}$ and that

$$\Omega \equiv \left(\frac{i}{2}\right)^q (-1)^{q(q-1)/2} \frac{1}{|\sigma_1|^{2q-1}} \omega^1 \omega^{k-q+2} \ldots \omega^k \overline{\omega}^{k-q+2} \ldots \overline{\omega}^k (2q-1). \tag{A4.39}$$

The situation is now this: On $\mathbf{C}^n$, let $f_1, \ldots, f_n$ be a tangent frame such that $f_{q+1}, \ldots, f_n$ is a frame for $\{\mathbf{Q}^q \times \mathbf{C}^{n-q}\} \subset \mathbf{C}^n$ (thus $f_1, \ldots, f_q$ is a normal frame for $\mathbf{C}^{n-q} \subset \mathbf{C}^n$). Let $\omega^1, \ldots, \omega^n$ be the dual co-frame and η be a compactly supported $2n - 2q$ form. Then we need

$$-\lim_{\epsilon \to 0} \int_{\partial T\epsilon} \eta \wedge \Lambda = \int_{\mathbf{C}^{n-q}} \eta \tag{A4.40}$$

where $\Lambda = \left(\dfrac{i}{2}\right)^q (-1)^{q(q-1)/2} \dfrac{1}{|\sigma|^{2q-1}} \omega^1 \omega^2 \ldots \omega^q \overline{\omega}^1 \ldots \overline{\omega}^q$, T_ϵ is the

ϵ-tube around $\mathbf{C}^{n-q} \subset \mathbf{C}^n$, and $\sigma = \begin{bmatrix} z^1 \\ \vdots \\ z^q \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $f_1 = \dfrac{\sigma}{|\sigma|}$.

If now the metric in the tangent frame is the flat Euclidean one and T_ϵ the normal neighborhood of radius ϵ, then Λ is minus the volume element on the normal sphere of radius ϵ. Writing $\eta = (\eta(0,z) + |f_1|\widehat{\eta})\omega^{q+1} \wedge \ldots \wedge \omega^n \wedge \overline{\omega}^{q+1} \wedge \ldots \wedge \overline{\omega}^n + \eta'$ where

$$\eta' \equiv 0(\omega^1, \ldots, \omega^q, \overline{\omega}^1, \ldots, \overline{\omega}^q),$$

it follows that

$$-\lim_{\epsilon \to 0} \int_{\partial T_\epsilon} \eta \wedge \Lambda = \int_{\mathbf{C}^{n-q}} \eta(0,z)\omega^{q+1} \ldots \omega^n \overline{\omega}^{q+1} \ldots \overline{\omega}^n = \int_{\mathbf{C}^{n-q}} \eta.$$

On the other hand, if $\widehat{T}_\epsilon$ is another ϵ-tube aroung $\mathbf{C}^{n-q}$, by Stokes' theorem

$$\Big| \int_{\partial T_\epsilon} \eta \wedge \Lambda - \int_{\partial \widehat{T}_\epsilon} \eta \wedge \Lambda \Big| \leqslant \Big| \int_{T_\epsilon \cup \widehat{T}_\epsilon} d(\eta \wedge \Lambda) \Big|.$$

Since η is smooth and $d\Lambda$ has a pole of order $\leqslant 2q-1$ along $\mathbf{C}^{n-q}$ (in fact, we may assume $d\Lambda = 0$), $\lim_{\epsilon \to 0} \big| \int_{T_\epsilon \cup \widehat{T}_\epsilon} d(\eta \wedge \Lambda) \big| = 0$. Thus, the limit on the left hand side of (A4.40) is independent of the T_ϵ (as should be the case).

Now $z = \sum_{\alpha=1}^{q} \lambda_\alpha f_\alpha(z) + \sum_{\mu=q+1}^{m} \lambda_\mu f_\mu(z)$, and we set $z_\eta = \sum_{\alpha=1}^{q} \lambda_\alpha f_\alpha$; **129**
then the left hand side of (A4.40) is $-\lim_{\epsilon \to 0 |z_\eta| = \epsilon} \eta \wedge \Lambda$. But $z_\eta = |z_\eta| f_1$, and by iterating the integral, we have

$$-\lim_{\epsilon \to 0} \int_{T_\epsilon} \eta \wedge \Lambda = \int_{\mathbf{C}^{n-q}} \left\{ -\lim_{\epsilon \to 0} \int_{\substack{|z_\eta| = \epsilon \\ z - z_\eta = \text{ constant}}} \eta \wedge \Lambda \right\} =$$

$$\int\limits_{\mathbf{C}^{n-q}} \eta(0,z) \left\{ -\lim_{\epsilon\to 0} \int\limits_{\substack{|z_\eta|=\epsilon \\ z-z_\eta=\text{ constant}}} \Lambda \right\} \omega^{q+1}\dots\omega^n\overline{\omega}^{q+1}\dots\overline{\omega}^n = \int\limits_{\mathbf{C}^{n-q}} \eta.$$

To prove (A4.35), we refer to the proof of (4.2) (c.f. the proof of (3.3) in [9]) and see that we may assume that $\mathbf{C}$ is a (real) manifold with boundary $\partial C = Z$. In this case the argument is substantially the same as that just given.

(e) Concluding Remarks on Residues, Currents, and the Gysin Homomorphism. Let V be an algebraic manifold and $W \subset V$ an irreducible subvariety which is the q^{th} Chern class of an ample bundle $\mathbf{E} \to V$. Given an Hermitian metric in $\mathbf{E}$, the differential form $P_q(\Theta)$ (Θ = curvature form in $\mathbf{E}$) represents the Poincaré dual $\mathscr{D}(W) \in H^{2n-2q}(V, \mathbf{Z})$ of $W \in H_{2n-2q}(V, \mathbf{Z})$. The differential form ψ (having properties (4.8)-(4.10) which we constructed is a *residue operator for* W; that is to say:

$$\begin{array}{l}\psi \ \text{ is a } \ C^\infty(q, q-1) \ \text{ form on } \ V - W \\ \text{which has a pole of order } \ 2q-1 \ \text{ along } \ W;\end{array} \tag{A4.41}$$

$$\partial\psi = 0 \ \text{ and } \ d\psi = \overline{\partial}\psi = P_q(\Theta) \ \text{ is the Poincaré dual of } \ W; \tag{A4.42}$$

and for any $2n-k$ chain Γ meeting W transversely and any smooth $2n-2q-k$ form η,

$$\lim_{\epsilon\to 0} - \int\limits_{\Gamma\cdot\partial T_\epsilon} \psi \wedge \eta = \int\limits_{\Gamma\cdot W} \eta \quad (\textit{Residue formula}). \tag{A4.43}$$

This formalism is perhaps best understood in the language of *currents* [14]. Let then $C^m(V)$ be the currents of degree m on V; by defini-
130 tion, $\theta \in C^m(V)$ is a linear form on $A^{2n-m}(V)$ (the C^∞ forms of degree $2n-m$) which is continuous in the distribution topology (c.f. Serre [21]). The derivative $d\theta \in C^{m+1}(V)$ is defined by

$$\langle d\theta, \lambda\rangle = \langle \theta, d\lambda\rangle \quad \text{for all} \quad \lambda \in A^{2n-m-1}(V). \tag{A4.44}$$

Of course we may define $\partial\theta$, $\bar{\partial}\theta$, and speak of currents of type (r, s), etc. If $Z^m(V) \subset C^m(V)$ are the closed currents $(d\theta = 0)$, then we may set $\mathscr{H}^m(V) = Z^m(V)/dC^{m-1}(V)$ (*cohomology computed from currents*), and it is known that (c.f. [14])

$$H^m(V) \cong \mathscr{H}^m(V). \tag{A4.45}$$

Now $P_q(\Theta)$ gives a current in $C^{q,q}(V)$ by $\langle P_q(\Theta), \lambda\rangle = \int\limits_V P_q(\Theta) \wedge \lambda (\lambda \in A^{2n-2q}(V))$. By Stokes' theorem, $dP_q(\Theta)$ in the sense of currents is the same as the usual exterior derivative. Thus $dP_q(\Theta) = 0$ and $P_q(\Theta) \in H^{q,q}(V)$.

Also, W gives a current in $C^{q,q}(V)$ by $\langle W, \lambda\rangle = \int\limits_W \lambda (\lambda \in A^{2n-2q}(V))$. By Stokes' theorem again, $dW = 0$ (if W were a manifold with boundary, then dW would be just ∂W).

Now ψ gives a current in $C^{q,q-1}(V)$ by $\langle \psi, \lambda\rangle = \int\limits_W \psi \wedge \lambda$ (this is because ψ has a pole of order $2q-1$). To compute $d\psi \in C^{q,q}(V)$, we have, for any $\lambda \in A^{2n-2q}(V)$,

$$\int\limits_V \psi \wedge d\lambda = \lim_{\epsilon\to 0} \int\limits_{V-T_\epsilon} \psi \wedge d\lambda = \lim_{\epsilon\to 0} \left\{ \int\limits_{V-T_\epsilon} -d(\psi \wedge \lambda) + d\psi \wedge \lambda \right\}$$

$$= \lim_{\epsilon\to 0} \int\limits_{V-T_\epsilon} d(\psi \wedge \lambda) + \lim_{\epsilon\to 0} \int\limits_{V-T_\epsilon} P_q(\Theta) \wedge \lambda = -\int\limits_W \lambda + \int\limits_V P_q(\Theta) \wedge \lambda,$$

which says that, *in the sense of currents*,

$$d\psi = P_q(\Theta) - W. \tag{A4.45}$$

Thus, among other things, the residue operator ψ expresses the fact that, in the cohomology group $\mathscr{H}^{q,q}(V)$, $P_q(\Theta) = W$ (which proves also that $P_q(\Theta) = \mathscr{D}(W)$). The point in the above calculation is that $d\psi$ in the sense of currents is *not* just the exterior derivative of ψ; the singularities force us to be careful in Stokes' theorem, so that we get (A4.45).

Suppose that W is non-singular and consider the Gysin homomor- **131**
phism $H^k(W) \to H^{k+2q}(V)$. Given a smooth form $\phi \in A^k(W)$ which is closed, we choose $\widehat{\phi} \in A^k(V)$ with $\widehat{\phi}|W = \phi$. Then the differential form $i_*(\phi) = d(\psi \wedge \widehat{\phi}) = P_q(\Theta) \wedge \widehat{\phi} - \psi \wedge d\widehat{\phi}$ will have only a pole of order $2q-2$ along W (since $d\widehat{\phi}|W = 0$, the term of highest order in ψ involves only *normal* differentials along W, as does $d\widehat{\phi}$), and so $i_*(\phi)$ is a current in $C^{k+2q}(V)$. We claim that, in the sense of currents, $di_*(\phi) = 0$.

Proof. $\int\limits_V d(\psi \wedge \widehat{\phi}) \wedge d\lambda = \lim\limits_{\epsilon \to 0} \int\limits_{V - T_\epsilon} d(\psi \wedge \widehat{\phi}) \wedge d\lambda = -\lim\limits_{\epsilon \to 0} \int\limits_{\partial T_\epsilon} d(\psi \wedge \widehat{\phi}) \wedge \lambda = \lim\limits_{\epsilon \to 0} \int\limits_{\partial T_\epsilon} \psi \wedge d\widehat{\phi}\lambda$

(since $d\psi \wedge \widehat{\phi} = P_q(\Theta) \wedge \widehat{\phi}$ is smooth). But $\psi \wedge d\widehat{\phi}$ has a pole of order $2q-2$ along W so that $\lim\limits_{\epsilon \to 0} \int\limits_{\partial T_\epsilon} \psi \wedge d\widehat{\phi} \wedge \lambda = 0$).

Thus $i_*(\phi)$ is a *closed current* and so defines a class in $\mathscr{H}^{k+2q}(V) \cong H^{k+2q}(V)$; because of the residue formula (A4.43), $i_*(\phi)$ is the Gysin homomorphism on $\phi \in H^k(V)$.

Of course, if we are interested only in the de Rham groups $H^k(W)$, we may choose $\widehat{\phi}$ so that $d\widehat{\phi} = 0$ in T_ϵ for small ϵ (since W is a C^∞ *retraction* of T_ϵ). Then $d(\psi \wedge \widehat{\phi})$ is smooth and currents are unnecessary. However, if we want to keep track of the complex structure, we must use currents because W is generally *not a holomorphic retraction* of T_ϵ. Thus, if $\phi \in F_l^k(W)$ (so that $\phi = \phi_{k,0} + \cdots + \phi_{l,k-l}$), we may choose $\widehat{\phi} \in F_l^k(V)$ with $\widehat{\phi}|W = \phi$, but we *cannot* assume that $d\widehat{\phi} = 0$ in $T_\in$. The point then is that, if we let $\mathscr{F}_l^k(W)$ and $\mathscr{F}_{l+q}^{k+2q}(V)$ be the cohomology groups computed from the *Hodge filtration* using currents, then we have

$$\mathscr{F}_{l+q}^{k+2q}(V) \cong F_{l+q}^{k+2q}(V); \tag{A4.46}$$

and the Gysin homomorphism $i_* : H^k(W) \to H^{k+2q}(V)$ satisfies $i_* : F_l^k(W) \to F_{l+q}^{k+2q}(V)$ and is given, as explained above, by

$$i_*(\widehat{\phi}) = d(\psi \wedge \phi) \in \mathscr{F}_{l+q}^{k+2q}(V). \tag{A4.47}$$

In other words, by using residues and currents, we have proved that the Gysin homomorphism is compatible with the complex structure and can be computed using the residue form. □ 132

5 Generalizations of the Theorems of Abel and Lefschetz.

Let $V = V_n$ be an algebraic manifold and $\mathbf{Z} = \mathbf{Z}_{n-q}$ an *effective algebraic cycle* of codimension q; thus $\mathbf{Z} = \sum_{\alpha=1}^{l} n_\alpha \mathbf{Z}_\alpha$ where $\mathbf{Z}_\alpha$ is irreducible and $n_\alpha > 0$. We denote by $\Phi = \Phi(\mathbf{Z})$ an irreducible component containing $\mathbf{Z}$ of the *Chow variety* [13] of effective cycles Z on V which are algebraically equivalent to Z. If $Z \in \Phi$, then $Z - \mathbf{Z}$ is homologous to zero and so, as in §3, we may define $\phi_q : \Phi \to T_q(V)$. Letting $\mathrm{Alb}(\Phi)$ be the *Albanese variety* of Φ, we in fact have a diagram of mappings :

$$\begin{tikzcd} & \mathrm{Alb}(\Phi) \arrow[d, "\alpha_\Phi"] \\ \Phi \arrow[ur, "\delta_\Phi"] \arrow[r, "\phi_q"'] & T_q(\Phi, V). \end{tikzcd} \tag{5.1}$$

Here $T_q(\Phi, V)$ is the torus generated by $\phi_q(\Phi)$ and δ_Φ is the usual mapping of an irreducible variety to its Albanese. Thus, if $\psi^1, \ldots, \phi^m$ are a basis for the holomorphic 1-forms on Φ, then $\delta_\Phi(Z) = \begin{bmatrix} \vdots \\ \int_Z^Z \phi^\rho \\ \vdots \end{bmatrix}$, where $\int_Z^Z \phi^\rho$ means that we take a path on Φ from $\mathbf{Z}$ to Z and integrate ψ^ρ. We may assume that $\psi^1 = \phi_q^*(\omega^1), \ldots, \psi^k = \phi_q^*(\omega^k)$ where $\omega^1, \ldots, \omega^k$ give a basis for the holomorphic 1-forms on $T_q(\Phi, V)$ ($\omega^\alpha \in H^{n-q+1,n-q}(V)$), and then $\alpha_\Phi \delta_\Phi(Z) = \alpha_\Phi \begin{bmatrix} \int_{\mathbf{Z}}^Z \psi^1 \\ \vdots \\ \int_{\mathbf{Z}}^Z \psi^m \end{bmatrix} = \begin{bmatrix} \int_{\mathbf{Z}}^Z \omega^1 \\ \vdots \\ \int_{\mathbf{Z}}^Z \omega^k \end{bmatrix}$, where $\int_{\mathbf{Z}}^Z \omega^\alpha$ means $\int_\Gamma \omega^\alpha$ if Γ is a $2n - 2q + 1$ chain on V with $\partial\Gamma = Z - \mathbf{Z}$.

Let now $\mathbf{W} = \mathbf{W}_{q-1}$ be a sufficiently general irreducible subvariety of dimension $q-1$ (codimension $n-q+1$) and $\Sigma = \Sigma(\mathbf{W})$ an irreducible component of the Chow variety of $\mathbf{W}$. Each $Z \in \Phi$ defines a
133 divisor $D(Z)$ on Σ by letting $D(Z) = \{\text{all } W \in \Sigma \text{ such that } W \text{ meets } Z\}$. Thus, if $Z = \sum_{\alpha=1}^{l} n_\alpha Z_\alpha$, $D(Z) = \sum_{\alpha=1}^{l} n_\alpha D(Z_\alpha)$. Letting "$\equiv$" denote *linear equivalence of divisors*, we will prove as a generalization of *Abel's theorem* that:

$$\begin{array}{l} D(Z) \text{ is algebraically equivalent to } D(\mathbf{Z}), \\ \text{even if we only assume that } Z \text{ is homologous to } \mathbf{Z}; \end{array} \tag{5.2}$$

and

$$D(Z) \equiv D(\mathbf{Z}) \quad \text{if} \quad \phi_q(Z) = 0 \ \text{ in } \ T_q(\Phi, V). \tag{5.3}$$

Example 1. Suppose that $\mathbf{Z}$ is a divisor on V; then Φ is a projective fibre space over (part of) $\mathrm{Pic}(V)$ (= Picard variety of V) and the fibre through $Z \in \Phi$ is the *complete linear system* $|Z|$. Now $\mathbf{W}$ is a point on V and $\Sigma = V$, and $D(Z) = Z$ as divisor on Σ. In this case, (5.3) is just the classical Abel's theorem for divisors [17]; (5.2) is the statement (well known, of course) that homology implies algebraic equivalence. The converse to (5.3), which reads :

$$\phi_q(Z) = 0 \quad \text{if} \quad D(Z) \equiv D(\mathbf{Z}), \tag{5.4}$$

is the trivial part of Abel's theorem in this case.

Remark. We may give (5.3) as a functorial statement as follows. The mapping $\Phi \to \mathrm{Div}(\Sigma)$ (given by $Z \to D(Z)$) induces $\Phi \to \mathrm{Pic}(\Sigma)$. From this we get $\mathrm{Alb}(\Phi) \to \mathrm{Alb}(\mathrm{Pic}(\Sigma)) = \mathrm{Pic}(\Sigma)$, which combines with (5.1) to give

$$\begin{tikzcd} \mathrm{Alb}(\Phi) \arrow[r, "\xi_\Phi"] \arrow[d, "\alpha_\Phi"] & \mathrm{Pic}(\Sigma) \\ T_q(\Phi, V) \arrow[ur, dashed, "\zeta_\Phi"'] & \end{tikzcd} \tag{5.5}$$

Then (5.3) is equivalent to saying that ξ_Φ factors in (5.5).

Proof. For $z_0 \in \mathrm{Alb}(\Phi)$, there exists a zero-cycle $Z_1 + \cdots + Z_N$ on Φ such that $z_0 = \delta_\Phi(Z_1 + \cdots + Z_N)$. Let $Z = Z_1 + \cdots + Z_N$ be the corresponding subvariety of V. Then $\alpha_\Phi(Z) = \phi_q(Z - N\mathbf{Z})$ and, assuming (5.3), if $\phi_q(Z - N\mathbf{Z}) = 0$, then $\xi_\Phi(Z) = 0$ in $\mathrm{Pic}(\Sigma)$. Thus, if (5.3) holds, $\ker \alpha_\Phi \supset \ker \xi_\Phi$ and so ξ_Φ factors in (5.5). □

Example 2. Let $\mathbf{Z}$ = point on V so that $\Phi = V$, $\mathrm{Alb}(\Pi) = \mathrm{Alb}(V)$. **134**
Choose $\mathbf{W}$ to be a very ample divisor on V; then Σ is a *projective fibre bundle* over $\mathrm{Pic}(V)$ with $|W|$ as fibre through W (c.f. [18]). Now $D(Z)$ consists of all divisors $W \in \Sigma$ which pass through Z. In this case, (5.3) reads:

$$\begin{array}{l}\text{Albanese equivalence of points on } V \text{ implies linear} \\ \text{equivalence of divisors on } \Sigma.\end{array} \tag{5.6}$$

Remark. There is a reciprocity between Φ and Σ; each $W \in \Sigma$ defines a divisor $D(W)$ on Φ so that we have $\mathrm{Alb}(\Sigma) \xrightarrow{\xi_\Sigma} \mathrm{Pic}(\Phi)$. Then (5.5) dualizes to give :

$$\begin{tikzcd} \mathrm{Alb}(\Sigma) \arrow[r, "\xi_\Sigma"] \arrow[d, "\alpha_\Sigma"] & \mathrm{Pic}(\Phi) \\ T_{n-q+1}(\Sigma, V) \arrow[ur, dashed, "\zeta_\Sigma"'] & \end{tikzcd} \tag{5.7}$$

For example, suppose that $\dim V = 2m + 1$ and $q = m + 1$. We may take $\mathbf{W} = \mathbf{Z}$, $\Sigma = \Phi$, and then (5.5) and (5.7) coincide to give :

$$\begin{tikzcd} \mathrm{Alb}(\Phi) \arrow[r, "\xi_\Phi"] \arrow[d, "\alpha_\Phi"] & \mathrm{Pic}(\Phi) \\ T_q(\Phi, V) \arrow[ur, dashed, "\zeta_\Phi"'] & \end{tikzcd} \tag{5.8}$$

Given $\mathbf{Z}$, Φ as above, there is a mapping

$$H_r(\Phi, \mathbf{Z}) \xrightarrow{\tau} H_{2n-2q+r}(V, \mathbf{Z}) \tag{5.9}$$

as follows. Given an r-cycle Γ on Φ, $\tau(\Gamma)$ is the cycle traced out by the varieties Z_γ for $\gamma \in \Gamma$. Suppose that Φ is nonsingular. Then the adjoint $\tau^* : H^{2n-2q+r}(V) \to H^r(\Phi)$ is given as follows. On $\Phi \times V$, there is a
135 cycle T with $\mathrm{pr}_V T \cdot \{Z \times V\} = Z (Z \in \Phi)$. We then have

$$\begin{tikzcd} T \arrow[r, "\widetilde{\omega}"] \arrow[d, "\pi"] & V \\ \Phi & \end{tikzcd}$$

and :

$$\tau^* = \pi_* \widetilde{\omega}^* : H^{2n-2q+r}(V) \to H^r(\Phi) \tag{5.10}$$

(here π_* is *integration over the fibre*). Since Φ is nonsingular $D(\mathbf{W})$ (= divisor on Φ) gives a class in $H^{1,1}(\Phi)$. In fact, we will show, as a generalization of the *Lefschetz theorem* [19], that

$$\tau^* : H^{n-q+s,n-q+t}(V) \to H^{s,t}(\Phi); \tag{5.11}$$

and, if $\omega \in H^{n-q+1,n-q+1}(V)$ is the dual of $\mathbf{W} \in H_{q-1,q-1}(V) \cap H_{2q-2}(V, \mathbf{Z})$, then :

$$\text{The dual of } D(\mathbf{W}) \text{ is } \tau^*\omega \in H^{1,1}(\Phi). \tag{5.12}$$

In other words, an *integral cohomology class* ω of type $(n-q+1, n-q+1)$ on V defines a *divisor* on Φ.

Remark. In (5.11), we have

$$\tau^* : H^{n-q+1,n-q}(V) \to H^{1,0}(\Phi); \tag{5.13}$$

this τ^* is just $\phi_q^* : H^{1,0}(T_q(V)) \to H^{1,0}(\mathrm{Alb}(\Phi))$ where ϕ_q is given by (5.1).

Remark 5.14. The gist of (5.2), (5.3) and (5.11), (5.12) may be summarized by saying: The cohomology of type (p, p) gives algebraic cycles, and the equivalence relation defined by the tori $T_q(V)$ implies rational equivalence, both on suitable Chow varieties attached to the original algebraic manifold V.

The problem of dropping back down to V still remains of course.

(a) A GENERALIZATION OF INTERALS OF THE 3RD KIND TO HIGHER CODIMENSION. We want to prove (5.2) and (5.3) above. Since changing $\mathbf{Z}$ or Z by rational equivalence will change $D(\mathbf{Z})$ or $D(Z)$ by linear equivalence and will not alter $\phi_q(Z)$, and since we may add to $\mathbf{Z}$ and Z a common cycle, we will assume that $\mathbf{Z} = \sum\limits_{\alpha=1}^{l} n_\alpha \mathbf{Z}_\alpha$, $Z = \sum\limits_{\rho=1}^{k} m_\rho Z_\rho$ where the $\mathbf{Z}_\alpha$, Z_ρ are *Chern classes of ample bundles* (c.f. §4 above) and that all inter- **136**
sections are transversal. To simplify notation then, we write $Y = Z - \mathbf{Z}$ and $Y = \sum\limits_{j=1}^{l} n_j Y_j$ where the Y_j are nonsingular Chern classes which meet transversely. We also set $|Y| = \bigcup\limits_{j=1}^{l} Y_j$, $V - Y = V - |Y|$.

A *residue operator for* Y (c.f. Appendix to §4, section (e) above) is given by a C^∞ differential form ψ on $V - Y$ such that :

(i) ψ is of degree $2q-1$ and $\psi = \psi_{2q-1,0} + \cdots + \psi_{q,q-1}$ ($\psi_{s,t}$ is the part of ψ of type (s,t));

(ii) $\partial\psi = 0$ and $\overline{\partial}\psi = \Phi$ where Φ is a $C^\infty(q,q)$ form on V giving the Poincaré dual of $Y \in H_{2n-2q}(V, \mathbf{Z})$;

(iii) $\psi - \psi_{q,q-1}$ is C^∞ on V and ψ has a pole of order $2q-1$ along Y; and

(iv) for any $k+2q$ chain Γ on V which meets Y transversely and smooth k-form ω on V

$$\int\limits_{\Gamma\cdot Y} \omega = -\lim_{\epsilon\to 0} \int\limits_{\Gamma\cdot\partial T_\epsilon} \psi \wedge \omega (\text{Residue formula}) \tag{5.15}$$

where T_ϵ is an ϵ-tube around Y.

From (A4.1)-(A4.3) and (A4.35), we see that a residue operator ψ_j for each Y_j exists. Then $\psi = \sum\limits_{j=1}^{l} n_j\psi_j$ is a residue operator for Y (the formula (5.15) has to be interpreted suitably).

If $Y = 0$ in $H_{2n-2q}(V, \mathbf{Z})$, then we may assume that $\overline{\partial}\psi = 0$. (5.16)

Proof. $\overline{\partial}\psi = \Phi$ is a C^∞ form and $\Phi = 0$ in $H^{q,q}_{\overline{\partial}}(V)$. Then $\Phi = \overline{\partial}\eta$ where $\eta = \overline{\partial}^* G_{\overline{\partial}}\Phi$ and $\partial\eta = 0$ since $\partial\overline{\partial}^* = -\overline{\partial}^*\partial$, $\partial G_{\overline{\partial}} = G_{\overline{\partial}}\partial$, $\partial\Phi = 0$. Since η is of type $(q, q-1)$, we may take $\psi - \eta$ as our residue operator. □

Remark 5.17. If Y is a divisor which is zero in $H_{2n-2}(V, \mathbf{Z})$, then ψ is a holomorphic differential on $V - Y$ having Y as its *logarithmic residue locus* (c.f. [18]).

137 **Remark 5.18.** Let Y be homologous to zero and ψ be a residue operator for Y with $d\psi = 0$ (c.f. (5.16)). Then ψ gives a class in $H^{2q-1}(V - Y)$, and ψ is determined up to $H^{2q-1,0}(V) + \cdots + H^{q,q-1}(V)$. We claim that $H^{2q-1}(V - Y)$ is generated by $H^{2q-1}(V)$ and the ψ_j.

Proof. Let δ_j be a normal sphere to Y_j at some simple point not on any of the other Y_k's. We map $\mathbf{Z}^{(l)} = \underbrace{\mathbf{Z} \oplus \cdots \oplus \mathbf{Z}}_{l}$ into $H_{2q-1}(V - Y)$ by $(\alpha_1, \ldots, \alpha_l) \to \sum_{j=1}^{l} \alpha_j \delta_j$. Since $\int_{\delta_j} \psi_j = +1$, we must show that the sequence

$$\mathbf{Z}^{(l)} \to H_{2q-1}(V - Y) \xrightarrow{i_*} H_{2q-1}(V) \to 0 \tag{5.19}$$

is exact. By dimension, $H_{2q-1}(V - Y)$ maps onto $H_{2q-1}(V)$. If $\sigma \in H_{2q-1}(V - Y)$ is an integral cycle which bounds in V, then $\sigma = \delta\gamma$ for some $2q$-chain γ where γ meets Y transversely in nonsingular points $p_\rho \in Y$. If $p_\rho \in Y_{j(\rho)}$, then clearly $\sigma \sim \sum_\rho \delta_{j(\rho)}$ so that $Z^{(l)}$ generates the kernel of i_* in (5.19).

Consider now our subvariety $\mathbf{W} = \mathbf{W}_{q-1}$ with Chow variety Σ. We may assume that $\mathbf{W}$ lies in $V - Y$ and, for $W \in \Sigma$, W lying in $V - Y$, we may write $W - \mathbf{W} = \partial\Gamma$ where Γ is a $2q - 1$ chain not meeting Y. Clearly Γ is determined up to $H_{2q-1}(V - Y)$. We will show :

There exists an integral of the 3rd kind θ on Σ whose logarithmic residue locus is $D(Y)$, provided that $Y = 0$ in $H_{2n-2q}(V, \mathbf{Z})$. (5.20)

□

Proof. Let ψ be a residue operator for Y with $d\psi = 0$. Define a 1-form θ on $\Sigma - D(Y)$ by :

$$\theta = d\left\{\int_{\mathbf{W}}^{W} \psi\right\} = d\left\{\int_{\Gamma} \psi\right\}. \tag{5.21}$$

This makes sense since $d\psi = 0$. We claim that

$$\theta \textit{ is holomorphic on } \Sigma - D(Y).$$

□

Proof. Let $\Sigma^* = \Sigma - D(Y)$ and $T^* \subset \Sigma^* \times V$ the graph of the correspondence $(W, z)(z \in W)$ (i.e. $W \in \Sigma^*$ is a subvariety of V and $z \in V$ lies on W). Then we have

$$\begin{array}{ccc} T^* & \xrightarrow{\widetilde{\omega}} & V. \\ \big\downarrow{\scriptstyle \pi} & & \\ \Sigma & & \end{array}$$

Now $\widetilde{\omega}^* : A^{r,s}(V) \to A^{r,s}(T^*)(A^{r,s}(*) = C^\infty$ forms of type (r, s) 138
on*); since $\widetilde{\omega}$ is holomorphic, $\widetilde{\omega}^*\overline{\partial} = \overline{\partial}\widetilde{\omega}^*$. On the other hand, the *integration over the fibre* $\pi_* : A^{r+q-1,s+q-1}(T^*) \to A^{r,s}(\Sigma^*)$ is defined and is determined by the equation :

$$\int_{\Sigma^*} \pi_*\phi \wedge \eta = \int_{T^*} \phi \wedge \pi^*\eta, \tag{5.22}$$

where η is a compactly supported form on T^*. Since $\int_{\Sigma^*} \overline{\partial}\pi_*\phi \wedge \eta = (-1)^{r+s} \int_{T^*} \phi \wedge \pi^*\overline{\partial}\eta = \int_{T^*} \partial\phi \wedge \pi^*\eta = \int_{\Sigma^*} \pi_*(\overline{\partial}\phi) \wedge \eta$ for all η, $\overline{\partial}\pi_* = \pi_*\overline{\partial}$. Let $\tau^* : A^{r+q-1,s+q-1}(V) \to A^{r,s}(\Sigma^*)$ be the composite $\pi_*\widetilde{\omega}^*$. Then $\overline{\partial}\tau^* = \tau^*\overline{\partial}$ (this proves (5.11)).

Now let $\psi \in A^{q,q-1}(V-Y)$ be a residue operator for Y. Then by the definition (5.21), $\theta = \tau^*\psi \in A^{1,0}(\Sigma^*)$ and $d\theta = \tau^* d\psi = 0$. This proves that θ is holomorphic on Σ^*.

Now $Y = \sum\limits_{j=1} n_j Y_j$ where the Y_j are subvarieties of codimension q on V. We have that $D(Y) = \sum\limits_{j=1}^{l} n_j D(Y_j)$ and $\psi = \sum\limits_{j=1} n_j \psi_j$. We will prove that ψ has a pole of order one on Y_j with logarithmic residue n_j there. □

Proof. We give the argument when Y is irreducible and $q = n$. From this it will be clear how the general case goes.

Let Δ be the unit disc in the complex t-plane and $\{W_t\}_{t\in\Delta}$ a holomorphic curve on Σ meeting $D(Y)$ simply at the point $t = 0$. Then W_0 meets Y simply at a point $z_0 \in V$. We may choose local coordinates $z^1, \ldots, z^n$ on V such that $z_0 = Y$ is the origin. Now $\psi =$
139 $\dfrac{1}{|z|^{2n-1}} \left\{ \sum\limits_{\alpha=1}^{n} \psi_\alpha dz^1 \ldots dz^n d\bar{z}^1 \ldots \widehat{d\bar{z}^\alpha} \ldots d\bar{z}^n \right\}$ where $|z|^2 = \sum\limits_{\alpha=1}^{n} |z^\alpha|^2$ and ψ_α is smooth. We may assume that W_t is given by $z^1 = t$, and, to prove that θ has a pole of order one at $t = 0$, it will suffice to show that $\iint\limits_{\Delta} |\theta \wedge d\bar{t}|$ is finite. It is clear, however, that $\iint\limits_{\Delta} |\theta \wedge d\bar{t}|$ will be finite if $\int\limits_{|z^\alpha|<1} |\psi \wedge d\bar{z}^1|$ is finite. But

$$|\psi \wedge d\bar{z}^1| \leqslant c \left\{ \frac{|dz^1 \ldots dz^n \, d\bar{z}^1 \ldots d\bar{z}^n|}{|z|^{2n-1}} \right\} \qquad (c = \text{constant}),$$

so that $\int\limits_{|z^\alpha|<1} |\psi \wedge d\bar{z}^1|$ is finite.

We now want to show that $\int\limits_{|t|=1} \theta = +1$ (i.e. θ has logarithmic residue $+1$ on $D(Y)$). Let $\delta = \bigcup\limits_{|t|=1} W_t$. Then $\int\limits_{|t|=1} \theta = \int\limits_{\delta} \psi$. If $T_\epsilon =$

$\{z : |z| < \epsilon\}$, then setting $\Gamma = \left\{ \bigcup_{|t| \leqslant 1} W_t \right\} - T_\epsilon, \partial\Gamma = \delta + \partial T_\epsilon$. Thus $\int_\delta \psi = -\int_{\partial T_\epsilon} \psi = +1$ as required. □

Remark. Let $Y \subset V$ be as above but without assuming that $Y = 0$ in $H_{2n-2q}(V, \mathbf{Z})$. Let ψ be a residue operator for Y and $\theta = \tau^*(\psi) = \widetilde{\omega}_* \pi^* \psi$. The above argument generalizes to prove :

$$\theta = \tau^*(\psi) \ \textit{ is a residue operator for } \ D(Y). \tag{5.23}$$

We have now proved (5.20), and with it have proved (5.2), since $D(Y)$ will be algebraically equivalent to zero on Σ because of the existence of an integral of the 3rd kind associated to $D(Y)$.

Proof of (5.12)**.** Let $Y \subset V$ be as above and interchange the roles of Y and $\mathbf{W}$ in the statement of (5.12). Let $\omega \in H^{q,q}(V)$ be the dual of $Y \in H_{2n-2q}(V, \mathbf{Z})$ and let ψ be a residue operator for Y. Then (c.f. (5.23) above) $\tau^*\psi = \theta$ is a residue operator for $D(Y) \subset \Sigma$, and so (c.f. Appendix to §4, section (e)) $\overline{\partial}\theta$ is the dual of $D(Y) \in H_{2N-2}(\Sigma, \mathbf{Z})(N = \dim \Sigma)$. But $\overline{\partial}\theta = \tau^*\overline{\partial}\psi = \tau^*\omega$, and so (5.12) is proved. □

(b) Reciprocity Relations in Higher Codimension. Let $Y = Z - \mathbf{Z}$ be as in beginning of §5, section (a) above. We assume that $Y = 0$ in $H_{2n-2q}(V, \mathbf{Z})$ so that $D(Y)$ is algebraically equivalent to zero on $\Sigma = \Sigma(\mathbf{W})$. Let ψ be a residue operator for Y and $\theta = \tau^*\psi$ be defined by **140**
(5.21). Then (c.f. (5.20)) θ is an integral of the 3rd kind on Σ whose logarithmic residue locus in $D(Y)$.

Now ψ is determined up to $S = H^{2p-1,0}(V) + \cdots + H^{p,p-1}(V)$. Since $\tau^*(H^{p+r,p-1-r}(V)) = 0$ for $r > 0$, θ is determined up to $\tau^*(S)$ where only $\tau^*(H^{p,p-1}(V)) \subset H^{1,0}(\Sigma)$ (c.f. (5.11)) counts. Let us prove now :

$$\begin{aligned} &D(Y) \equiv 0 \textit{ on } \Sigma \textit{ if, and only, if, there exists } \omega \in H^{1,0}(\Sigma) \\ &\textit{such that } \textstyle\int_\delta \theta + \omega \equiv 0(1) \textit{ for all } \delta \in H_1(\Sigma - D(Y), \mathbf{Z}). \end{aligned} \tag{5.24}$$

Proof. If ω exists satisfying $\int_\delta \theta+\omega \equiv 0(1)$ for all $\delta \in H_1(\Sigma-D(Y),\mathbf{Z})$, then we may set :

$$f(W) = \exp\left(\int_{\mathbf{W}}^{W} \theta + \omega\right), \quad (\exp\xi = e^{2\pi i\xi}). \tag{5.25}$$

This $f(W)$ is a single-valued meromorphic function and, by (5.20), $(f) = D(Y)$.

Conversely, assume that $D(Y) = (f)$. Then $\theta - \dfrac{1}{2\pi i}\dfrac{df}{f} = -\omega$ will be a holomorphic 1-form in $H^{1,0}(\Sigma)$, and for $\delta \in H_1(\Sigma - D(Y),\mathbf{Z})$, $\int\limits_\gamma \theta + \omega = \dfrac{1}{2\pi i}\int\limits_\gamma \dfrac{df}{f} = \dfrac{1}{2\pi i}\int\limits_\gamma d\log f \equiv 0(1)$. This proves (5.24).

Suppose we can prove :

There exists $\eta \in S$ such that $\int_\Gamma \psi + \eta \equiv 0(1)$ for all $\Gamma \in H_{2q-1}(V-Y,\mathbf{Z})$ if and only if, $\phi_q(Y) = 0$ in $T_q(\Phi, V) \subset T_q(V)$. (5.26)

□

Then we can prove the Abel's theorem (5.3) as follows.

Proof. If $\phi_q(Y) = 0$ in $T_q(\Phi, V)$, then by (5.26) we may find $\eta \in S$ such that $\int_\Gamma \psi+\eta \equiv 0(1)$ for all $\Gamma \in H_{2q-1}(V-Y,\mathbf{Z})$. Set $\omega = \tau^*\eta \in H^{1,0}(\Sigma)$. Then, for $\delta \in H_1(\Sigma-D(Y),\mathbf{Z})$, $\int_\delta \theta+\omega = \int_\delta \tau^*(\psi+\eta) = \int_{\tau(\delta)} \psi+\eta \equiv 0(1)$, where τ is given by (5.9). Using (5.24), we have proved (5.2). □

Remark 5.27. The converse to Abel's theorem (5.2), which reads :

$$\phi_q(Y) = 0 \text{ in } T_q(Y) \text{ if } D(Y) \equiv 0 \text{ in } \Sigma, \tag{5.28}$$

141 will be true, up to isogeny, if we have :

$$\textit{The mapping } \tau^* : H^{q,q-1}(V) \to H^{1,0}(\Sigma) \textit{ is into.} \tag{5.29}$$

Proof. Referring to (5.5), we see that τ^* is

$$(\zeta_\Phi)_* : T_0(T_q(\Phi, V)) \to T_0(\mathrm{Pic}(\Sigma)),$$

so that ζ_Φ is an isogeny of $T_q(\Phi, V)$ onto an abelian subvariety of $\mathrm{Pic}(\Sigma)$. $\square$

Proof of (5.26)**.** Let $\Gamma_1, \ldots, \Gamma_{2m}$ be a set of free generators of $H_{2q-1}(V, \mathbf{Z})$ (mod torsion). We may assume that Γ_ρ lies in $H_{2q-1}(V - Y, \mathbf{Z})$, since $\int_\delta \psi \equiv 0(1)$ for all δ in $H_{2q-1}(V - Y, \mathbf{Z})$ which are zero in $H_{2q-1}(V, \mathbf{Z})$ (c.f. (5.19)). Choose a basis $\eta^1, \ldots, \eta^m$ for S and set $\pi_\alpha : \left\{\begin{array}{c} \vdots \\ \int_{\Gamma_\rho} \eta^\alpha \\ \vdots \end{array}\right\}$.

Then $\pi_\alpha \in \mathbf{C}^{2m}$ and we let $\mathbf{S}$ be the subspace generated by $\pi_1, \ldots, \pi_m$. The lattice generated by integral vectors $\left\{\begin{array}{c} k^1 \\ \vdots \\ k^{2m} \end{array}\right\}$ projects onto a lattice in $\mathbf{C}^{2m}/\mathbf{S}$, and the resulting torus is $T_q(V)$.

Proof. We may identify $\mathbf{C}^{2m}$ with $H^{2q-1}(V, \mathbf{C}) = H_{2q-1}(V, \mathbf{C})^*$; $\mathbf{S}$ is the subspace $H^{2q-1,0}(V) + \cdots + H^{q,q-1}(V)$, and the integral vectors are just $H^{2q-1}(V, \mathbf{Z})$. Thus the torus above is $H^{q-1,q}(V) + \cdots + H^{0,2q-1}(V)/H^{2q-1}(V, \mathbf{Z})$. Let $\pi(\psi) = \left\{\begin{array}{c} \vdots \\ \int_{\Gamma_\rho} \psi \\ \vdots \end{array}\right\}$; $\pi(\psi)$ projects onto a point $\pi(\psi) \in T_q(V)$, and we see that:

The congruence $\int_\Gamma \psi + \eta \equiv 0(1)$ $(\Gamma \in H_{2q-1}(V - Y, \mathbf{Z}))$ can be solved for some $\eta \in S$ if, and only if, $\pi(\psi) = 0$ in $T_q(V)$. (5.30)

Thus, to prove (5.26), we need to prove the following *reciprocity relation:*

$$\pi(\psi) = \phi_q(Y) \quad \text{in} \quad T_q(V). \tag{5.31}$$

142 Let $e_\rho \in H^{2n-2q+1}(V, \mathbf{Z})$ be the harmonic form dual to $\Gamma_\rho \in H_{2q-1}(V, \mathbf{Z})$. We claim that, if we can find $\eta \in S$ such that we have

$$\int_{\Gamma_\rho} \psi - \int_C e_\rho = (\Gamma_\rho, C) + \int_{\Gamma_\rho} \eta (\rho = 1, \ldots, 2m), \tag{5.32}$$

then (5.31) holds. □

Proof. By normalizing ψ, we may assume that $\eta = 0$ in (5.32). Let $e_\rho^* \in H^{2q-1}(V)$ be the harmonic form defined by $\int V_\rho^e \wedge e_\sigma^* = \delta_\sigma^\rho$. Choose a harmonic basis $\omega^1, \ldots, \omega^m$ for $H^{2n-2q+1,0} + \cdots + H^{n-q+1,n-q}$ and let $\phi^1, \ldots, \phi^m$ be a dual basis for $H^{q-1,q} + \cdots + H^{0,2q-1}$. Then $\omega^\alpha = \sum_{\rho=1}^{2m} \mu_{\rho\alpha} e_\rho$ and $e_\rho^* = \sum_{\alpha=1}^{m} (\mu_{\rho\alpha}\phi^\alpha + \overline{\mu}_{\rho\alpha}\overline{\phi}^\alpha)$. It follows that $\pi(\psi)$ is given by the column vector $\left\{ \begin{array}{c} \vdots \\ \sum_{\rho=1}^{2m} \mu_{\rho\alpha} \int_{\Gamma_\rho} \psi \\ \vdots \end{array} \right\}$. From (5.32), we have $\sum_{\rho=1}^{2m} \mu_{\rho\alpha} \int_{\Gamma_\rho} \psi - \int_C \sum_{\rho=1}^{2m} \mu_{\rho\alpha} e_\rho = \sum_{\rho=1}^{2m} \mu_{\rho\alpha} (\Gamma_\rho \cdot C)$, which says that

$$\left\{ \begin{array}{c} \vdots \\ \sum_{\rho=1}^{2m} \mu_{\rho\alpha} \int_{\Gamma_\rho} \psi \\ \vdots \end{array} \right\} - \left\{ \begin{array}{c} \vdots \\ \int_C \omega^\alpha \\ \vdots \end{array} \right\} = \sum_{\rho=1}^{2m} (\Gamma_\rho \cdot C) \left\{ \begin{array}{c} \mu_{\rho^1} \\ \vdots \\ \mu_{\rho^m} \end{array} \right\},$$

which lies in the lattice defining $T_q(V)$. Thus $\pi(\psi) = \phi_q(Y)$ in $T_q(V)$. Q.E.D. □

Thus we must prove (5.32), which is a generalization of the *bilinear relations involving integrals of the third kind* on a curve (c.f. [24]). We observe that, because of the term involving η, (5.32) is *independent of which residue operator we choose*. We shall use the method of Kodaira [17] to find one such ψ; in this, we follow the notations of [17].

Let then $\gamma^{2n-2q}(z,\xi)$ on $V \times V$ be the double *Green's form* associated to the $2n-2q$ forms on V and the Kähler metric. This is the unique form satisfying **143**

(a) $$\Delta_z \gamma^{2n-2q}(z,\xi) = \sum_{j=1}^{l} \theta^j(z) \wedge \theta^j(\xi),$$

where the θ^j are a basis for the harmonic $2n-2q$ forms;

(b) $\gamma^{2n-2q}(z,\xi)$ is smooth for $z \neq \xi$ and has on the diagonal $z = \xi$ the singularity of a fundamental solution of the Laplace equation;

(c) $\gamma^{2n-2q}(z,\xi) = \gamma^{2n-2q}(\xi,z)$ and is orthogonal to all harmonic $2n-2q$ forms (i.e. $\int_V \gamma^{2n-2q}(z,\xi) \wedge_* \theta^j(\xi) = 0$ for all z and $j = 1,\ldots,l$);

(d) $\delta_z \gamma^{2n-2q}(z,\xi) = d_\xi \gamma^{2n-2q-1}(z,\xi)$,
and $*_z *_\xi \gamma^{2n-2q}(z,\xi) = \gamma^{2q}(z,\xi)$.

Define now a $2n-2q$ form ϕ by the formula :

$$\phi(z) = \int\limits_{\xi \in Y} \gamma^{2n-2q}(z,\xi) d\xi. \tag{5.33}$$

Then ϕ is smooth in $V - Y$ and, by (b) above, can be shown to have a pole or order $2q-2$ along Y. We let

$$\psi = * \, d\phi. \tag{5.34}$$

Then ψ is a real $2q-1$ form. Since Y is an algebraic cycle, ϕ will have type $(n-q,n-q)$ and so $\psi = \psi' + \psi''$ where ψ' has type $(q,q-1)$ and $\psi'' = \overline{\psi}'$. We will show that $2\psi'$ is a residue operator for Y and satisfies (5.32).

We recall from [17], the formula :

$$\int\limits_{\Gamma_\rho} \psi - \int\limits_{C} e_\rho = (\Gamma_\rho \cdot C), \tag{5.35}$$

which clearly will be used to give (5.32).

First, ψ has singularities only on Y and $d\psi = d * d\phi = - * \delta d\phi = 0$ (c.f. Theorem 4 in [17]), and $\delta\psi = \delta * d\phi = \pm * d^2\phi = 0$, so that ψ is harmonic in $V - Y$. Thus ψ' and ψ'' are harmonic in $V - Y$.

Let J be the operator on forms induced by the complex structure. Then $J^* = *J$ and $J\phi = \phi$ (since $J\mathbf{T}_\xi(Y) = \mathbf{T}_\xi(Y)$). Thus $\psi = *dJ\phi = *JJ^{-1}dJ\phi = J * (L\delta - \delta L)\phi = -J^{-1} * \delta L\phi$ since $\delta\phi = 0$. This gives that $J\psi = -d * L\phi$, so that, using $J\psi = i(\psi' - \psi')$, we find :

$$2\psi' = \psi - J\psi = \psi + d(* L\phi). \tag{5.36}$$

144 Now $2\psi'$ is a form of type $(q, q-1)$ satisfying $\partial\psi' = 0 = \overline{\partial}\psi'$ and combining (5.35) and (5.36),

$$\int\limits_{\Gamma_\rho} 2\psi' - \int\limits_{C} e_\rho = (\Gamma_\rho \cdot C). \tag{5.37}$$

Finally, the same argument as used in [17], pp. 121-123, shows that $2\psi'$ has a pole of order $2q-1$ along Y and gives a residue operator for Y. This completes the proof of (5.32) and hence of (5.3).

6 Chern Classes and Complex Tori.

Let V be an algebraic manifold and $\mathbf{E}_\infty \to V$ a C^∞ vector bundle with fibre $\mathbf{C}^k$. We let $\Sigma(\mathbf{E}_\infty)$ be the set of *complex structures* on $\mathbf{E}_\infty \to V$ (i.e. the set of holomorphic bundles $\mathbf{E} \to V$ with $\mathbf{E} \underset{C^\infty}{\cong} \mathbf{E}_\infty$). For such a holomorphic bundle $\mathbf{E} \to V(\mathbf{E} \in \Sigma(\mathbf{E}_\infty))$, the *Chern cycles* $Z_q(\mathbf{E})(q = 1, \ldots, k)$ (c.f. [11], [12], [13]) are virtual subvarieties of codimension q, defined up to *retional equivalence*. Fixing $\mathbf{E}_0 \in \Sigma(\mathbf{E}_\infty)$, $Z_q(\mathbf{E}) - Z_q(\mathbf{E}_0) \in \Sigma_q$ and we define

$$\phi_q : \Sigma(\mathbf{E}_\infty) \to T_q(S), \tag{6.1}$$

by $\phi_q(\mathbf{E}) = \phi_q(Z_q(\mathbf{E}) - Z_q(\mathbf{E}_0))(\mathbf{E} \in \Sigma(\mathbf{E}_\infty))$. We may think of $\phi_q(\mathbf{E})$ as giving the *periods of the holomorphic bundle* $\mathbf{E}$. In addition to asking for the image $\phi_q(\Sigma(\mathbf{E}_\infty)) \subset T_q(S)$, we may also ask to what extent *do the periods of* $\{\mathbf{E}\} \in \Sigma(\mathbf{E}_\infty)$ *give the moduli of* $\mathbf{E}$? By putting things into the context of deformation theory, we shall infinitesimalize these questions.

Let then $\{\mathbf{E}_\gamma\}_{\lambda\in\Delta}$ be a family of holomorphic bundles over V ($\Delta =$ disc in λ-plane). Relative to a suitable covering $\{U_\alpha\}$ of V, we may give this family by holomorphic transition functions $g_{\alpha\beta}(\lambda) : U_\alpha \cap U_\beta \to GL(k)$ which satisfy the cocycle rule :

$$g_{\alpha\beta}(\lambda)g_{\beta\gamma}(\lambda) = g_{\alpha\gamma}(\lambda) \quad \text{in} \quad U_\alpha \cap U_\beta \cap U_\gamma. \tag{6.2}$$

We recall that Kodaira and Spencer [15] have defined the *infinitesimal deformation mapping :*

$$\delta : \mathbf{T}_\lambda(\Delta) \to H^1(V, \mathcal{O}(\mathrm{Hom}(\mathbf{E}_\lambda, \mathbf{E}_\lambda))). \tag{6.3}$$

Explicitly, $\delta\left(\dfrac{\delta}{\delta\lambda}\right)$ is given by the Čech cocycle $\xi_{\alpha\beta} = \dot{g}_{\alpha\beta}(\lambda)g_{\alpha\beta}(\lambda)^{-1}$ **145**
($\dot{g}_{\alpha\beta} = \partial g_{\alpha\beta}/\partial\lambda$); the cocycle rule here follows by differentiating (6.2).

Now define $\phi_q : \Delta \to T_q(S)$ by $\phi_q(\lambda) = \phi_q(\mathbf{E}_\lambda)$ ($\mathbf{E}_0$ being the base point). Recall (c.f. (3.3)) that $(\phi_q)_* : (\mathbf{T}_0(\Delta)) \subset H^{q-1,q}(V)$, so that we have a diagram ($\phi_* = (\phi_q)_*$) :

$$\begin{tikzcd} & H^1(V, \mathcal{O}(\mathrm{Hom}(\mathbf{E}, \mathbf{E}))) \arrow[dd, dashed, "\zeta"] \\ \mathbf{T}_0(\Delta) \arrow[ur, "\delta"] \arrow[dr, "\phi_*"'] & \\ & H^q(V, \Omega^{q-1}) \end{tikzcd} \tag{6.4}$$

What we want is $\zeta : H^1(V, \mathcal{O}(\mathrm{Hom}(\mathbf{E}, \mathbf{E}))) \to H^q(V, \Omega^{q-1})$ which will always complete (6.4) to a commutative diagram.

We have a formula for ζ (c.f. (6.8)) which we shall give after some preliminary explanation.

First we consider *symmetric, multilinear, invariant* forms

$$P(A_1, \ldots, A_q)$$

where the A_α are $k \times k$ matrices. Invariance means that

$$P(gA_1g^{-1}, \ldots, gA_qg^{-1}) = P(A_1, \ldots, A_q)(g \in GL(k)).$$

Such a symmetric, invariant form gives an *invariant polynomial* $P(A) = P(A, \ldots, A)$. Conversely, an invariant polynomial gives, by *polarization*, a symmetric invariant form. For example, if $P(A) = \det(A)$, then

$$P(A_1, \ldots, A_k) = \frac{1}{k!} \sum_{\pi=(\pi_1,\ldots,\pi_k)} \det(A^1_{\pi_1} \ldots A^k_{\pi_k}), \tag{6.5}$$

where $\pi = (\pi_1, \ldots, \pi_k)$ is a permutation of $(1, \ldots, k)$ and $A^\alpha_{\pi_\alpha}$ is the α^{th} column of A_{π_α}.

The invariant polynomials form a graded ring $I_* = \sum_{q \geqslant 0} I_q$, which is discussed in [11], §4(b). In particular, I_* is generated by $P_0, P_1, \ldots, P_k$ where $P_q \in I_q$ is defined by

$$\det\left(\frac{iA}{2\pi} + \lambda I\right) = \sum_{q=0}^{k} P_q(A)\lambda^{k-q}. \tag{6.6}$$

146 Let now $P \in I_r$ be an invariant polynomial. If

$$A_\alpha \in A^{p_\alpha, q_\alpha}(V, \mathrm{Hom}(\mathbf{E}, \mathbf{E}))$$

(= space of C^∞, $\mathrm{Hom}(\mathbf{E}, \mathbf{E})$-valued, (p_α, q_α) forms on V), then

$$P(A_1, \ldots, A_r) \in A^{p,q}(V) \left(p = \sum_{\alpha=1}^{r} p_\alpha, q = \sum_{\alpha=1}^{r} q_\alpha\right)$$

is a global form and $\overline{\partial} P(A_1, \ldots, A_q) = \sum_{\alpha=1}^{r} \pm P(\ldots, \overline{\partial} A_\alpha, \ldots)$. We conclude that *$P$ gives a mapping on cohomology :*

$$P : H^{q_1}(V, \Omega^{p_1}(\mathrm{Hom}(\mathbf{E}, \mathbf{E}))) \otimes \cdots \otimes H^{q_r}(V, \Omega^{p_r}(\mathrm{Hom}(\mathbf{E}, \mathbf{E}))) \\ \to H^q(V, \Omega^p). \tag{6.7}$$

Secondly, $\mathbf{E} \to V$ defines a cohomology class

$$\Theta \in H^1(V, \Omega^1(\mathrm{Hom}(\mathbf{E}, \mathbf{E})))(\Theta \text{ is the } \textit{curvature} \text{ in } \mathbf{E}; \text{ c.f. [1]}),$$

which is constructed as follows: Let $\theta = \{\theta_\alpha\}$ be a connection of type $(1, 0)$ for $\mathbf{E} \to V$ Thus θ_α is a $k \times k$-matrix-valued $(1, 0)$ form in U_α with $\theta_\alpha - g_{\alpha\beta}\theta_\beta g_{\alpha\beta}^{-1} = g_{\alpha\beta}^{-1} dg_{\alpha\beta}$ in $U_\alpha \cap U_\beta$. Letting $\Theta_\alpha = \overline{\partial}\theta_\alpha$, $\Theta_\alpha = g_{\alpha\beta}\Theta_\beta g_{\alpha\beta}^{-1}$ in $U_\alpha \cap U_\beta$ and so defines $\Theta \in H^1(V, \Omega^1(\mathrm{Hom}(\mathbf{E}, \mathbf{E})))$ (Θ is the $(1, 1)$ component of the curvature of θ).

Our formula is that, if we set

$$\zeta(\eta) = qP_q(\underbrace{\Theta, \ldots, \Theta}_{q-1}, \eta) \quad (\eta \in H^1(V, \mathcal{O}(\mathrm{Hom}(\mathbf{E}, \mathbf{E})))), \tag{6.8}$$

then (6.4) will be commutative. Note that, according to (6.7), $\zeta(\eta) \in H^q(V, \Omega^{q-1})$, so that the formula makes sense.

We shall give two proofs of the fact that ζ defined by (6.8) gives the infinitesimal variation in the periods of $\mathbf{E}$. The first will be by explicit computation relating the *Chern polynomials* $P_q(\underbrace{\Theta, \ldots, \Theta}_{q-1}; \eta)$ to the *Poincaré residue operator* alogn $Z_q(\mathbf{E})$; both the Chern polynomials and Poincaré residues will be related to geometric residues in a manner somewhat similar to §4 (especially the Appendix there). After preliminaries in §7, this first proof (which we give completely only for the top Chern class) will be carried out in §8. The general argument is complicated by the singularities of the Chern classes.

The second proof is based on the transformation formulae developed **147**
in §4; it uses an integral-geometric argument and requires that the family of bundles be globally parametrized.

Some Examples. Let $\mathbf{E} \to V$ be a holomorphic vector bundle and $\theta \in H^0(V, \Theta)$ a holomorphic vector field. Then θ exponentiates to a one-parameter group $f(\lambda) : V \to V$ of holomorphic automorphisms, and we may set $\mathbf{E}_\lambda = f(\lambda)^*\mathbf{E}$ (i.e., $(\mathbf{E}_\lambda)_z = \mathbf{E}_{f(\lambda), z}$). Let $\omega = P_q(\Theta, \ldots, \Theta)$ be a (q, q) form representing the q^{th} Chern class; we claim that the infinitesimal variation in the periods of $\mathbf{E}$ is given by

$$\langle \theta, \omega \rangle \in H^{q, q-1}(V). \tag{6.9}$$

Proof. Since $\langle \theta, \omega \rangle = \langle \theta, P_q(\Theta, \dots, \Theta) \rangle$

$$= \Sigma P_q(\Theta, \dots, \langle \theta, \Theta \rangle, \dots, \Theta) = q P_q(\underbrace{\Theta, \dots, \Theta}_{q-1}; \langle \theta, \Theta \rangle),$$

using (6.8) it will suffice to show that $\langle \theta, \Theta \rangle \in H^1(V, \mathrm{Hom}(\mathbf{E}, \mathbf{E}))$ is the infinitesimal deformation class for the family $\{\mathbf{E}_\lambda\} = \{f(\lambda)^* \mathbf{E}\}$.

Let $\mathbf{P} \to V$ be the principal bundle of $\mathbf{E} \to V$ and $0 \to \mathrm{Hom}(\mathbf{E}, \mathbf{E}) \to \mathbf{T}(\mathbf{P})/G \to \mathbf{T}(V) \to 0$ the *Atiyah sequence* [1]. The cohomology sequence goes $H^0(V, \mathcal{O}(\mathbf{T}(\mathbf{P})/G)) \to H^0(V, \Theta) \xrightarrow{\delta} H^1(V, \mathrm{Hom}(\mathbf{E}, \mathbf{E}))$, and in [8] it is proved that $\delta(\theta) = \langle \theta, \Theta \rangle$ and is the *Kodaira-Spencer class* for the family $\{\mathbf{E}_\lambda\}$. (This is easy to see directly;

$$\Theta \in H^1(V, \mathrm{Hom}(\mathbf{T}(V), \mathrm{Hom}(\mathbf{E}, \mathbf{E})))$$

is the obstruction to splitting the Atiyah sequence holomorphically, and the coboundary δ is contraction with Θ. But $\delta(\theta)$ is the obstruction to lifting θ to a bundle automorphism of $\mathbf{E}$, and so gives the infinitesimal variation of $f(\lambda)^* \mathbf{E}$). □

Remark. The formula (6.9) is easy to use on abelian varieties (ω and θ have constant coefficients) but, in the absence of knowledge about the algebraic cycles on V, fails to yield much new.

Example 2. Suppose that $\{\mathbf{E}_\lambda\}_{\lambda \in \Delta}$ is a family of *flat bundles* (i.e. having *constant transition funcitons*). Then, by (6.8), we see that:

$$\text{The periods } \phi_q(\mathbf{E}_\lambda) \text{ are constant for } q > 1. \qquad (6.10)$$

148 **Remark.** This *should* be the case because $\mathbf{E}_\lambda$ is given by a repersentation $\rho_\lambda : \pi_1(V) \to GL(k)$. If we choose a general curve $C \subset V$, then $\pi_1(C)$ maps *onto* $\pi_1(V)$, and so $\{\mathbf{E}_\lambda\}$ is given by $\rho_\lambda : \pi_1(C) \to GL(k)$. Thus $\mathbf{E}_\lambda$ is determined by $\mathbf{E}_\lambda | C$, and here the period $\phi_1(\mathbf{E}_\lambda)$ is only one which is non-zero (recall that we have $0 \to T_1(V) \to T_1(C)$).

Example 3. From (6.8), it might seem possible that the periods of $\mathbf{E}_\lambda$ are constant if all of the Chern classes of $\mathbf{T}_\lambda$ are topologically zero and

$\det \mathbf{E}_\lambda = \mathbf{L}$ is constant. This is not the case. Let C be an elliptic curve and $V = P_1 \times C$. Take the bundle $\mathbf{H} \to P_1$ degree 1 and let $\mathbf{J}_\lambda \to C$ be a family of bundles of degree zero parametrized by C. Set $\mathbf{E}_\lambda = (\mathbf{H} \otimes \mathbf{J}_\lambda) \oplus (\mathbf{H} \otimes \mathbf{J}_\lambda)^*$. Then $\det \mathbf{E}_\lambda = 1$, $c_2(\mathbf{E}_\lambda) = -c_1(\mathbf{H})^2 = 0$. If $\theta \in H^{0,1}(C)$ is the tangent to $\{\mathbf{J}_\lambda\} \to C$, then the tangent η to $\{\mathbf{E}_\lambda\}$ is $\begin{pmatrix} \theta & 0 \\ 0 & -\theta \end{pmatrix}$, and, if Θ is the curvature in $\mathbf{H}$, then the curvature in $\mathbf{E}_\lambda$ is $\Theta_{\mathbf{E}} = \begin{pmatrix} \Theta & 0 \\ 0 & -\Theta \end{pmatrix}$. Then $P_2(\Theta_{\mathbf{E}}; \eta) = -(\Theta\theta) \neq 0$ in $H^{1,2}(V)$.

Example 4. Perhaps the easiest construction of $\mathrm{Pic}(V)$ (c.f. [18]) is by using a *very positive line bundle* $\mathbf{L} \to V$, and so we may wonder what the effect of making vector bundles very positive is. For this, we let $A_q(V) = \phi_q(\Sigma_q(V)) \subset T_q(V)$ ($A_q(V)$ is the part cut out by algebraic cycles algebraically equivalent to zero); $A_q(V)$ is an abelian subvariety of $T_q(V)$ which is the range of the Weil mapping. Let Φ^p be the algebraic cycles, modulo rational equivalence, of codimension p. Then we have (c.f. §4)

$$\Phi^p \otimes A_q(V) \to A_{p+q}(V) \tag{6.11}$$

(obtained by intersection of cycles). We set $I_r(V) = \sum\limits_{\substack{p+q=r \\ p>0}} \Phi^p \otimes A_q(V)$ (this is the stuff of codimension r obtained by intersection with cycles of higher dimension) and let

$$N_r(V) = A_r(V)/I_r(V) \tag{6.12}$$

(here $N_r(V)$ stands for the *new cycles* not coming by operations in lower codimension). Then (c.f. §7 below):

Let $\{\mathbf{E}_\lambda\}$ be a family of bundles and $\mathbf{L} \to V$ any line bundle. Then **149**

$$\phi_r(\mathbf{E}_\lambda) = \phi_r(\mathbf{E}_\lambda \otimes \mathbf{L}) \text{ in } N_r(V). \tag{6.13}$$

In other words, as expected, the essential part of the problem is n't changed by making the $\mathbf{E}_\lambda$ very positive.

Example 5. Here is a point we don't quite understand. Let $\{\mathbf{E}_\lambda\}$ be a family of bundles on $V = V_n (n \geqslant 4)$ and let $S \subset V$ be a very positive *two-dimension subvariety*. Then $\mathbf{E}_\lambda \to V$ is *uniquely determined* by $\mathbf{E}_\lambda \to S$ (c.f. [8]). From this it might be expected that, if the periods $\phi_1(\mathbf{E}_\lambda)$ and $\phi_2(\mathbf{E}_\lambda)$ are constant, then all of the periods $\phi_q(\mathbf{E}_\lambda)$ are constant. However, let A be an abelian variety and $\{\mathbf{J}_\lambda\}_{\lambda \in H^1(A,0)}$ a family of topologically trivial line bundles parametrized by $\lambda \in H^1(A, \mathcal{O})$. We let $\mathbf{L}_1$, $\mathbf{L}_2$, $\mathbf{L}_3$ be fixed line bundles with characteristic classes ω_1, ω_2, ω_3 and set

$$\mathbf{E}_\lambda = (\mathbf{J}_{\lambda_1}\mathbf{L}_1) \oplus (\mathbf{J}_{\lambda_2}\mathbf{L}_2) \oplus (\mathbf{J}_{\lambda_3}\mathbf{L}_3).$$

Then the tangent η to the family $\{\mathbf{E}_\lambda\}$ is $\eta = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$ and the curvature $\Theta = \begin{bmatrix} \omega_1 & 0 & 0 \\ 0 & \omega_2 & 0 \\ 0 & 0 & \omega_3 \end{bmatrix}$. Then $P_1(\Theta; \eta) = \text{Trace}\, \eta = \lambda_1 + \lambda_2 + \lambda_3$. Setting $\lambda_3 = -\lambda_1 - \lambda_2$ we have $P_1(\Theta; \eta) = 0$. Now $P_2(\Theta; \eta) = \lambda_1\omega_2 + \lambda_2\omega_1 + \lambda_1\omega_3 + \lambda_3\omega_1 + \lambda_2\omega_3 + \lambda_3\omega_2 = \lambda_1(\omega_3 - \omega_1) + \lambda_2(\omega_3 - \omega_2)$, and $P_3(\Theta; \eta) = \lambda_1\omega_2\omega_3 + \lambda_2\omega_1\omega_3 + \lambda_3\omega_1\omega_2 = \lambda_1\omega_2(\omega_3 - \omega_1) + \lambda_2\omega_1(\omega_3 - \omega_2)$. Clearly we can have $P_2(\Theta; \eta) = 0$, $P_3(\Theta; \eta) = \lambda_1(\omega_3 - \omega_1)(\omega_2 - \omega_1) \neq 0$.

Example 6. Examples such as Example 5 above show that the periods fail quite badly in determining the bundle. In fact, it is clear that, if $K(V)$ is the *Grothendieck ring constructed from locally free sheaves* ([12]), the best we can hope for is that the periods determine the image of the bundle in $K(V)$.

Let us prove this for curves:

If V is an algebraic curve and $\mathbf{E} \to V$ a holomorphic vector bundle, then the image of $\mathbf{E}$ in $K(V)$ is determined by the periods of $\mathbf{E}$.
(6.14)

150 *Proof.* Let $\mathbf{I}_k$ be the trivial bundle of rank k; we have to show that $\mathbf{E} = \det \mathbf{E} \otimes \mathbf{I}_k$ in $K(V)$ (where k is the fibre dimension of $\mathbf{E}$). The assertion is trivially true for $k = 1$; we assume it for $k - 1$. Since the structure group of $\mathbf{E}$ may be reduced to the triangular group [2], in $K(V)$ we see that $\mathbf{E} = \mathbf{L}_1 \otimes \cdots \oplus \mathbf{L}_k$ where the $\mathbf{L}_\alpha$ are line bundles. We choose a very positive line bundle $\mathbf{H}$ and sections $\vartheta_\alpha \in H^0(V, \mathcal{O}(\mathbf{H} \otimes \mathbf{L}_\alpha^*))$

which have no common zeroes (since $k > 1$). Then the mapping $f \to (f\vartheta_1, \ldots, f\vartheta_k)(f \in \mathcal{O})$ gives an exact bundle sequence $0 \to \mathbf{H} \to \mathbf{L}_1 \oplus \cdots \oplus \mathbf{L}_k \to \mathbf{Q} \to 0$ where $\mathbf{Q}$ has rank $k-1$ and $\det \mathbf{Q} = \mathbf{H}^* \det \mathbf{E}$. By induction, $\mathbf{Q} = \mathbf{H}^* \det \mathbf{E} \otimes \mathbf{I}_{k-1}$ in $K(V)$, and so $\mathbf{E} = \det \mathbf{E} \otimes \mathbf{I}_k$ in $K(V)$ as required. □

7 Properties of the Mapping ζ in (6.8).

(a) Behavior under direct sums. Let $\{E_\lambda\}$, $\{\mathbf{F}_\mu\}$ be families of holomorphic bundles over V. What we claim is:

If (6.4) *holds for each of the families* $\{\mathbf{E}_\lambda\}$ *and* $\{\mathbf{F}_\mu\}$,*then it holds for* $\{\mathbf{E}_\lambda \oplus \mathbf{F}_\mu\}$. (7.1)

Proof. By linearity, we may suppose that the $\{\mathbf{F}_\mu\}$ is a constant family; thus all $\mathbf{F}_\mu = \mathbf{F}$. Letting $\mathbf{E} = \mathbf{E}_0$, the Kodaira-Spencer class $\delta(\partial/\partial\lambda)$ for $\{\mathbf{E}_\lambda \oplus \mathbf{F}\}$ lies then in $H^1(V, \mathcal{O}(\mathrm{Hom}(\mathbf{E}, \mathbf{E}))) \subset H^1(V, \mathcal{O}(\mathrm{Hom}(\mathbf{E} \oplus \mathbf{F}, \mathbf{E} \oplus \mathbf{F})))$. If $\theta_{\mathbf{E}}$ is a $(1,0)$ connection in $\mathbf{E}$ and $\theta_{\mathbf{F}}$ a $(1,0)$ connection in $\mathbf{F}$, then $\theta_{\mathbf{E}\oplus\mathbf{F}} = \theta_{\mathbf{E}} \oplus \theta_{\mathbf{F}} \left(= \begin{pmatrix} \theta_{\mathbf{E}} & 0 \\ 0 & \theta_{\mathbf{F}} \end{pmatrix} \right)$ is a $(1,0)$ connection in $\mathbf{E} \oplus \mathbf{F}$ and $\Theta_{\mathbf{E}\oplus\mathbf{F}} = \Theta_{\mathbf{F}} \oplus \Theta_{\mathbf{F}}$. From

$$\det\left(\frac{i}{2\pi}\Theta_{\mathbf{E}\oplus\mathbf{F}} + \lambda I\right) = \det\left(\frac{i}{2\pi}\Theta_{\mathbf{E}} + \lambda I\right) \det\left(\frac{i}{2\pi}\Theta_{\mathbf{F}} + \lambda I\right),$$

we get that

$$P_q(\Theta_{\mathbf{E}\oplus\mathbf{F}}) = \sum_{r+s=q} P_r(\Theta_{\mathbf{E}}) P_s(\Theta_{\mathbf{F}}). \tag{7.2}$$

Now (7.2) is the *duality theorem*; in the rational equivalence ring, we have

$$Z_q(\mathbf{E} \oplus \mathbf{F}) = \sum_{r+s=q} Z_r(\mathbf{E}) \cdot Z_s(\mathbf{F}). \tag{7.3}$$

Then **151**

$$\begin{aligned} \phi_q(\mathbf{E}_\lambda \oplus \mathbf{F}) &= \phi_q(Z_q(\mathbf{E}_\lambda \oplus \mathbf{F}) - Z_q(\mathbf{E} \oplus \mathbf{F})) \\ &= \phi_q\left(\sum_{r+s=q} \{Z_r(\mathbf{E}_\lambda) \cdot Z_s(\mathbf{F}) - Z_r(\mathbf{E}) \cdot Z_s(\mathbf{F})\}\right) \end{aligned}$$

$$= \phi_q \left(\sum_{r+s=q} [Z_r(\mathbf{E}_\lambda) - Z_r(\mathbf{E})] \cdot Z_s(\mathbf{F}) \right)$$

$$= \sum_{r+s=q} \phi_q([Z_r(\mathbf{E}_\lambda) - Z_r(\mathbf{E})] \cdot Z_s(\mathbf{F}))$$

$$= (\text{by } (4.18) \sum_{r+s=q} \Psi_s(\mathbf{F})\phi_r(Z_r(\mathbf{E}_\lambda) - Z_r(\mathbf{E}))$$

where $\Psi_s(\mathbf{F}) = P_s(\Theta_\mathbf{F}) \in H^{s,s}(V) \cap H^{2s}(V, \mathbf{Z})$ is the Poincaré dual to $Z_s(\mathbf{F})$ and $\Psi_s(\mathbf{F}) : T_r(V) \to T_{r+s}(V)$ is the mapping given by (2.7). It folows that:

$$\left\{\phi_q(\mathbf{E}_\lambda \oplus \mathbf{F})\right\}_* = \sum_{s+r=q} P_s(\Theta_\mathbf{F}) \left\{\phi_r(\mathbf{E}_\lambda)\right\}_* . \tag{7.4}$$

Assuming (6.4) for the family $\{\mathbf{E}_\lambda\}$, the right hand side of (7.4) is $\sum_{r+s=q-1} rP_{r+1}(\underbrace{\Theta_\mathbf{E}}_{r}; \eta)P_s(\Theta_\mathbf{F})$. Since we want this to equal

$$qP_q(\underbrace{\Theta_\mathbf{E} \oplus \Theta_\mathbf{F}}_{q-1}; \eta \oplus 0),$$

to prove (7.1) we must prove the algebraic identity:

$$qP_q(\underbrace{A \oplus B}_{q-1}; \xi \oplus 0) = \sum_{r+s=q-1} rP_{r+1}(\underbrace{A}_{r}; \xi)P_s(B), \tag{7.5}$$

where A, B, ξ are matrices.

Expanding $P_q(A \oplus B; \xi \oplus 0)$ gives

$$qP_q(\underbrace{A \oplus B}_{q-1}; \xi \oplus 0) = \Sigma_q \binom{q-1}{r} P_q(\underbrace{A \oplus 0}_{r}; \xi \oplus 0; \underbrace{0 \oplus B}_{s})$$

$(s = q - r - 1)$. Thus, to prove (7.5), we need to show:

$$rP_{r+1}(\underbrace{A}_{r}; \xi)P_s(B) = q\binom{q-1}{r} P_q(\underbrace{A \oplus 0}_{r}; \xi \oplus 0; \underbrace{0 \oplus B}_{s}). \tag{7.6}$$

Clearly the only question is the numerical factors; for these may take A, B, ξ to be diagonal. Now in general if $A_1, \ldots, A_t$ are diagonal matrices, say $A_\alpha = \begin{bmatrix} A_\alpha^1 & & 0 \\ & \vdots & \\ 0 & & A_\alpha^k \end{bmatrix}$, then $P_t(A_1, \ldots, A_t) = \dfrac{1}{t!} \sum_\pi A_1^{\pi_1} \ldots A_t^{\pi t}$ where **152**
the summation is over all subsets $\pi = (\pi_1, \ldots, \pi_t)$ of $(1, \ldots, k)$. Thus

$$
\begin{aligned}
q\tbinom{q-1}{r} & P_q(\underbrace{A \oplus 0}_{r}; \xi \oplus 0; \underbrace{0 \oplus B}_{s}) \\
&= \frac{1}{(q-1)!} \binom{q-1}{r} \sum_\pi A^{\pi_1} \ldots A_r^\pi \xi^{\pi_{r+1}} B^{\pi_{r+2}} \ldots B^{\pi_q} \\
&= \left(\frac{1}{r!} \sum_\pi A^{\pi_1} \ldots A^{\pi_r} \xi^{\pi_{r+1}} \right) \left(\frac{1}{s!} \sum_r B^{\tau_1} \ldots B^{\tau_s} \right) \\
&= r P_r(\underbrace{\quad A \quad}_{r}; \xi) P_s(B).
\end{aligned}
$$

This proves (7.6). □

(b) Behavior under tensor products. With the notations and assumptions of 7(a) above, we want to prove :

$$\text{If (6.4) holds for each of the families } \{\mathbf{E}_\lambda\} \text{ and } \{\mathbf{F}_\mu\}, \text{ then (6.4) holds for } \mathbf{E}_\lambda \oplus \mathbf{F}_\mu. \tag{7.7}$$

Proof. As in the proof of (7.1), we assume that all $\mathbf{F}_\mu = \mathbf{F}$, $\mathbf{E}_0 = \mathbf{E}$, and then $\delta\left(\frac{\partial}{\partial \lambda}\right) = \eta \oplus 1$ in $H^1(V, \mathcal{O}(\mathrm{Hom}(\mathbf{E}, \mathbf{E}))) \otimes H^0(V, \mathcal{O}(\mathrm{Hom}(\mathbf{F}, \mathbf{F}))) \subset H^1(V, \mathcal{O}(\mathrm{Hom}(\mathbf{E} \otimes \mathbf{F}, \mathbf{E} \otimes \mathbf{F})))$ where $\eta \in H^1(V, \mathcal{O}(\mathrm{Hom}(\mathbf{E}, \mathbf{E})))$ is $\delta\left(\dfrac{\partial}{\partial \lambda}\right)$ for the family $\{\mathbf{E}_\lambda\}$. Also, to simplify the algebra, we assume that F is a line bundle and set $\omega = \dfrac{i}{2\pi} \Theta_{\mathbf{F}} (= c_1(\mathbf{F}))$.

Now $\theta_{\mathbf{E} \otimes \mathbf{F}} = \theta_{\mathbf{E}} \otimes 1 + 1 \otimes \theta_{\mathbf{F}}$ is a $(1, 0)$ connection in $\mathbf{E} \otimes \mathbf{F}$ with curvature $\Theta_{\mathbf{E} \otimes \mathbf{F}} = \Theta_E \otimes 1 + 1 \otimes \Theta_{\mathbf{F}}$. We claim that

$$P_q(\Theta_{\mathbf{E} \otimes \mathbf{F}}) = \sum_{r+s=q} \binom{k-r}{s} \omega^s P_s(\Theta_E). \tag{7.8}$$

□

Proof. $P_q(A\otimes 1+1\otimes B) = \sum_{r+s=q} \binom{q}{r} P_q(\underbrace{A\otimes 1}_{r}, \underbrace{1\otimes B}_{s})$. Assuming that A is a $k\times k$ matrix and $B=(b)$ is 1×1, we have $P_q(A\otimes 1, 1\otimes B) =$
153 $\Sigma c_{q,r}P_r(A)b^s (s = q-r)$ and we need to determine the $c_{q,r}$. Letting $A = \begin{bmatrix} A^1 & 0 \\ & \vdots & \\ 0 & A^k \end{bmatrix}$, $P_q(A\otimes 1, 1\otimes B) = \frac{1}{q!}\sum_\pi A^{\pi_1}\ldots A^{\pi_r} b^{\pi_{r+1}}\ldots b^{\pi_q} = (s!)\frac{1}{q!}\binom{k-r}{s}\sum_\pi A^{\pi_1}\ldots A^{\pi_r}b^s$, so that $P_q(A\otimes 1 + 1\otimes B) = \sum_{\substack{r+s=q\\ \pi}} \binom{q}{r}\frac{(q-r)!}{q!}\binom{k-r}{s}A^{\pi_1}\ldots A^{\pi_r}b^s = \sum_{r+s=q}\binom{k-r}{s}P_r(A)b^s$. This proves (7.8).

In the rational equivalence ring, we have

$$Z_q(\mathbf{E}_\lambda\otimes\mathbf{F}) = \sum_{r+s=q}\binom{k-r}{s} Z_1(\mathbf{F})^s Z_r(\mathbf{E}_\lambda). \tag{7.9}$$

As in proof of (7.4) from (7.3), we have

$$\{\phi_q(\mathbf{E}_\lambda\otimes\mathbf{F})\}_* = \sum_{r+s=q}\binom{k-r}{s}\omega^s\{\phi_r(\mathbf{E}_\lambda)\}_*. \tag{7.10}$$

Using that (6.4) holds for $\{\mathbf{E}_\lambda\}$, the right hand side of (7.10) becomes $\sum_{r+s=q-1}\binom{k-r-1}{s}\omega^s r P_{r+1}(\underbrace{\Theta_{\mathbf{E}}}_{r};\eta)$; to prove (7.7) we must prove the algebraic identity :

$$qP_q(\underbrace{A\otimes 1 + 1\otimes B}_{q-1};\eta\otimes 1) = \sum_{r+s=q-1}\binom{k-r-1}{s} b^s r P_{r+1}(\underbrace{A}_{r};\eta). \tag{7.11}$$

□

Proof of (7.11). $qP_q(A\otimes 1 + 1\otimes B;\eta\otimes 1) =$

$$= \sum_{\substack{r+s=q-1\\ \pi}} q\binom{q-1}{r} P_r(\underbrace{A\otimes 1}_{r};\eta\otimes 1;\underbrace{1\otimes B}_{s}) =$$

$$= \sum_{\substack{r+s=q-1\\ \pi}} \frac{q}{q!}\binom{q-1}{r} A^{\pi_1}\dots A^{\pi_r}\eta^{\pi_{r+1}}b^{\pi_{r+2}}\dots b^{\pi_q} =$$

$$= \sum_{\substack{r+s=q-1\\ \pi}} \frac{1}{(q-1)!}\binom{q-1}{r}\cdot\binom{k-r-1}{s}(q-r-1)!$$

$$b^s A^{\pi_1}\dots A^{\pi_r}\eta^{\pi_{r+1}} =$$

$$= \sum_{\substack{r+s=q-1\\ \pi}} \binom{k-r-1}{s}\cdot\frac{1}{r!}A^{\pi_1}\dots A^{\pi_r}\eta^{\pi_{r+1}}b^s =$$

$$= \sum_{r+s=q-1} \binom{k-r-1}{s} b^s r P_{r+1}(\underbrace{A}_{r};\eta).$$

(c) Ample Bundles. If $\mathbf{E} \to V$ is a holomorphic bundle, we let $\Gamma(\mathbf{E})$ **154**
be the trivial bundle $V \times H^0(V, \mathcal{O}(\mathbf{E}))$. Then we say that $\mathbf{E}$ *is generated by its sections* if we have :

$$0 \to F \to \Gamma(\mathbf{E}) \to \mathbf{E} \to 0. \tag{7.12}$$

Now $\sigma \in \mathbf{F}_z$ is a section σ of $\mathbf{E}$ with $\sigma(z) = 0$; sending $\sigma \to d\sigma(z) \in \mathbf{E}_z \otimes \mathbf{T}^*_z(V)$ gives

$$\mathbf{F} \xrightarrow{d} \mathbf{E} \otimes \mathbf{T}^*(V). \tag{7.13}$$

In [11], $\mathbf{E}$ was said to be *ample* if (7.12) holds and if d is onto in (7.13). In this case, to describe the Chern cycles $Z_q(\mathbf{E})$, we choose k general sections $\sigma_1, \dots, \sigma_k$ of $\mathbf{E} \to V$. Then $Z_q(\mathbf{E}) \subset V$ is given by $\sigma_1 \wedge \dots \wedge \sigma_{k-q+1} = 0$. (Note that $Z_1(\mathbf{E})$ is given by $\sigma_1 \wedge \dots \wedge \sigma_k = 0$ and $Z_k(\mathbf{E})$ by $\sigma_1 = 0$.) The cycles $Z_q(\mathbf{E})$ are irreducible subvarieties defined up to rational equivalence.

If now $\mathbf{E} \to V$ is a general holomorphic bundle, we can choose an ample line bundle $\mathbf{L} \to V$ such that $\mathbf{E} \otimes \mathbf{L}$ is ample ([11]). Suppose we know (6.4) for ample bundles. Then (6.4) holds for $\mathbf{E} \otimes \mathbf{L}$ and $\mathbf{L}$. On the other hand, if (6.4) is true for a bundle, then it is also true for the dual bundle. Since $\mathbf{E} = (\mathbf{E} \otimes \mathbf{L}) \otimes \mathbf{L}^*$, using (7.6) we conclude:

$$\text{If (6.4) if true for ample bundles, then it is true for all holomorphic vector bundles.} \tag{7.14}$$

Let then $\{\mathbf{E}_\lambda\}$ be a family of ample holomorphic vector bundles and $Z_\lambda = Z_q(\mathbf{E}_\lambda)$. We may form a *continuous system* (c.f. §3); we let $Z = Z_0$ and $\mathbf{N} \to Z$ be the normal bundle and $\phi : \Delta \to T_q(V)$ the mapping (3.1) on $Z_\lambda - Z_0$. Then, combining (6.4) with the dual diagram to (3.8), we have :

$$\begin{tikzcd} T_0(\Delta) \arrow[r, "\delta"] \arrow[d, "\rho"] \arrow[dr, "\phi_*"] & H^1(V, \mathcal{O}(\mathrm{Hom}(\mathbf{E}, \mathbf{E}))) \arrow[d, "\zeta"] \\ H^0(Z, \mathcal{O}(\mathbf{N})) \arrow[r, "\xi"] & H^{q-1,q}(V) \end{tikzcd} \tag{7.15}$$

155 Actually this diagram is not quite accurate; $\mathbf{E}_\lambda$ determines $Z_q(\mathbf{E}_\lambda)$ only up to rational equivalence, and we shall see below that there is a subspace $L_q(\mathbf{E}) \subset H^0(Z, \mathcal{O}(\mathbf{N}))$ such that we have :

$$\begin{tikzcd} T_0(\Delta) \arrow[r, "\delta"] \arrow[d, "\rho"] & H^1(V, \mathcal{O}(\mathrm{Hom}(\mathbf{E}, \mathbf{E}))) \arrow[d, "\zeta"] \\ H^0(Z, \mathcal{O}(\mathbf{N}))/L_q(\mathbf{E}) \arrow[r, "\xi"] & H^{q-1,q}(V) \end{tikzcd} \tag{7.16}$$

Now in §9 below, we shall, under the assumption $H^1(V, \mathcal{O}(\mathbf{E})) = 0$, construct

$$\theta : H^1(V, \mathcal{O}(\mathrm{Hom}(\mathbf{E}, \mathbf{E}))) \to H^0(Z, \mathcal{O}(\mathbf{N}))/L_q(\mathbf{E}) \tag{7.17}$$

such that

$$\begin{tikzcd} T_0(\Delta) \arrow[rr, "\delta"] \arrow[dr, "\rho"'] & & H^1(V, \mathcal{O}(\mathrm{Hom}(\mathbf{E}, \mathbf{E}))) \arrow[dl, "\theta"] \\ & H^0(Z, \mathcal{O}(\mathbf{N}))/L_q(\mathbf{E}) & \end{tikzcd}$$

commutes. Putting this in (7.16), we have :

> In order to prove (6.4), it will suffice to
> assume that $\mathbf{E} \to V$ is ample, $H^1(V, \mathcal{O}(\mathbf{E})) = 0$, (7.18)
> and then prove that the following diagram commutes.

$$\begin{tikzcd}
H^1(V, \mathcal{O}(\mathrm{Hom}(\mathbf{E}, \mathbf{E}))) \arrow[dr, "\zeta"] \arrow[dd, "\theta"'] & \\
& H^{q-1,q}(V) \\
H^0(Z, \mathcal{O}(\mathbf{N}))/L_q(\mathbf{E}) \arrow[ur, "\xi"'] &
\end{tikzcd} \tag{7.19}$$

where ξ is given by (3.6), ζ by (6.8) and θ by the §9 below.

Remark. In case $q = k =$ fibre dimension of $\mathbf{E}$, $Z \subset V$ is the zero locus of $\sigma \in H^0(V, \mathcal{O}(\mathbf{E}))$. Then $L_k(\mathbf{E}) = rH^0(V, \mathcal{O}(\mathbf{E}))$, where $r : \mathcal{O}_V(\mathbf{E}) \to \mathcal{O}_Z(\mathbf{N})$ is the restriction mapping, and θ in (7.17) is constructed as fol- 156
lows. Let $\eta \in H^1(V, \mathcal{O}(\mathrm{Hom}(\mathbf{E}, \mathbf{E})))$. Then $\eta \cdot \sigma \in H^1(V, \mathcal{O}(\mathbf{E})) = 0$ and so $\eta \cdot \sigma = \overline{\partial}\tau$ for some $\tau \in \Gamma_\infty(V, \mathbf{E})$ ($= C^\infty$ sections of $\mathbf{E} \to V$). We set $\theta(\eta) = \tau|Z$. If also $\eta \cdot \sigma = \overline{\partial}\hat{\tau}$, then $\partial(\tau - \hat{\tau}) = 0$ so that $\theta(\eta)$ is determined up to $rH^0(V, \mathcal{O}(\mathbf{E}))$.

(d) Behavior in exact sequences. Let $\{\mathbf{E}_\lambda\}$, $\{\mathbf{S}_\lambda\}$, $\{\mathbf{Q}_\lambda\}$ be families of holomorphic vector bundles over V such that we have

$$0 \to \mathbf{S}_\lambda \to \mathbf{E}_\lambda \to \mathbf{Q}_\lambda \to 0. \tag{7.20}$$

We shall prove:

> *If* (6.4) *holds for each of the families* $\{\mathbf{S}_\lambda\}$, $\{\mathbf{Q}_\lambda\}$, *then it is true for* $\{\mathbf{E}_\lambda\}$. (7.21)

Proof. The exact sequences (7.20) are classified by classes

$$e \in H^1(V, \mathcal{O}(\mathrm{Hom}(\mathbf{Q}_\lambda, \mathbf{S}_\lambda))),$$

with e giving the same class as e' if, and only if, $e = \lambda e'(\lambda \neq 0)$. If we show that the periods of $\mathbf{E}_\lambda$ are independent of this *extension class* e, then we will have $\phi_q(\mathbf{E}_\lambda) = \phi_q(\mathbf{S}_\lambda \oplus \mathbf{Q}_\lambda)$ and so we can use (7.1). But $\phi_q(e)$ (extension class e) $= \phi_q(te)$ for all $t \neq 0$, and since $\phi_q(te)$ is continuous at $t = 0$, $\phi_q(\mathbf{E}_\lambda) = \phi_q(\mathbf{S}_\lambda \oplus \mathbf{Q}_\lambda)$. Thus, in order to prove (7.21), we must show :

Suppose the $\{\mathbf{E}_\lambda\}$ *is a family with* $0 \to \mathbf{S} \to \mathbf{E}_\lambda \to \mathbf{Q} \to 0$ *for all* λ. *Then* $\zeta(\eta) = 0$ *in* (6.8) *where*

$$\eta = \delta\left(\frac{\partial}{\partial\lambda}\right) \in H^1(V, \mathcal{O}(\operatorname{Hom}\mathbf{E}, \mathbf{E})). \tag{7.22}$$

□

Proof of (7.22)**.** Assuming that $\mathbf{E} = \mathbf{E}_0$ with

$$0 \to \mathbf{S} \to E \xrightarrow{\pi} \mathbf{Q} \to 0, \tag{7.23}$$

we clearly have $\eta \in H^1(V, \mathcal{O}(\operatorname{Hom}\mathbf{Q}, \mathbf{S})) \subset H^1(V, \mathcal{O}(\operatorname{Hom}(\mathbf{E}, \mathbf{E})))$. Let η be a $C^\infty(0, 1)$ form with values in $\operatorname{Hom}(\mathbf{Q}, \mathbf{S})$, and let $e_1, \ldots, e_k$ be a local holomorphic frame for $\mathbf{E}$ such that $e_1, \ldots, e_l$ is a frame for $\mathbf{S}$. Then $e_{l+1}, \ldots, e_k$ projects to a frame for $\mathbf{Q}$, and locally $\eta = \left(\begin{smallmatrix}\eta_{11} & \eta_{12}\\ \eta_{21} & \eta_{22}\end{smallmatrix}\right)$. Since $\eta|\mathbf{S} = 0$ and $\eta(\mathbf{E}) \subset \mathbf{S}$, $\eta_{11} = \eta_{21} = \eta_{22} = 0$ and $\eta = \left(\begin{smallmatrix}0 & \eta_{12}\\ 0 & 0\end{smallmatrix}\right)$.

157 Suppose now that we can find a $(1, 0)$ connection in $\mathbf{E}$ whose local connection matrix (using the above frame) has the form $\widehat{\theta} = \begin{pmatrix}\widehat{\theta}_{11} & \widehat{\theta}_{12}\\ 0 & \widehat{\theta}_{22}\end{pmatrix}$. Then the curvature $\overline{\partial}\widehat{\theta} = \widehat{\Theta}_{\mathbf{E}} = \begin{pmatrix}\widehat{\Theta}_{11} & \widehat{\Theta}_{12}\\ 0 & \widehat{\Theta}_{22}\end{pmatrix}$, and it follows that

$$P_q(\underbrace{\widehat{\Theta}_E}_{q-1}; \eta) \equiv 0.$$

Then let θ be an arbitrary $(1, 0)$ connection in $\mathbf{E}$. Locally $\theta = \left(\begin{smallmatrix}\theta_{11} & \theta_{12}\\ \theta_{21} & \theta_{22}\end{smallmatrix}\right)$, and we check easily that θ_{21} is a global $(1, 0)$ form with values in $\operatorname{Hom}(\mathbf{S}, \mathbf{Q})$; let $\xi = \left(\begin{smallmatrix}0 & 0\\ \theta_{21} & 0\end{smallmatrix}\right) \in A^{1,0}(V, \operatorname{Hom}(\mathbf{S}, \mathbf{Q}))$ and let $\phi : \mathbf{Q} \to \mathbf{E}$ be a C^∞ splitting of (7.23). Then $\psi = I - \phi\pi : \mathbf{E} \to \mathbf{S}$ and satisfies $\psi(v) = v$ for $v \in \mathbf{S}$. We let

$$\widehat{\theta} = \theta - \phi\xi\psi \tag{7.24}$$

be the $(1,0)$ connection for $\mathbf{E}$. Then $\pi\widehat{\theta}|\mathbf{S} = 0$ and so $\widehat{\theta} = \begin{pmatrix} \widehat{\theta}_{11} & \widehat{\theta}_{12} \\ 0 & \widehat{\theta}_{22} \end{pmatrix}$ as required.

(e) Proof of (6.4) for line bundles. Let $\mathbf{E} \to V$ be a line bundle; we want to prove (6.4) for any family $\{\mathbf{E}_\lambda\}_{\lambda \in \Delta}$ with $\mathbf{E}_0 = \mathbf{E}$. By (7.14), we may assume that $\mathbf{E} \to V$ is ample and $Z \subset V$ is the zero locus of a holomorphic section $\sigma \in H^0(V, \mathcal{O}(\mathbf{E}))$. Using (7.18), to prove (6.4) we need to show that the following diagram commutes:

$$\begin{tikzcd}
H^1(V,\mathcal{O}) \arrow[dr, "\zeta"] \arrow[dd, "\theta"] & \\
& H^{0,1}(V) \\
H^0(Z,\mathcal{O}(\mathbf{N}))/H^0(V,\mathcal{O}(\mathbf{E})) \arrow[ur, "\xi"] &
\end{tikzcd} \tag{7.25}$$

where ζ is now $\dfrac{i}{2\pi}$ (identity). Let $\omega \in H^{n,n-1}(V)$ and $\eta \in H^1(V,\mathcal{O}) \cong H^{0,1}(V)$. To prove the commutativity of (7.25), we must show:

$$\frac{i}{2\pi} \int_V \eta \wedge \omega = \int_Z \theta(\eta) \wedge \xi^* \omega. \tag{7.26}$$

The argument is now similar to the proof (3.10). Letting T_ϵ be an ϵ- 158
tubular neighborhood of Z in V, $\int_V \eta \wedge \omega = \lim_{\epsilon \to 0} \int_{V - T_\epsilon} \eta \wedge \omega$. On the other hand, $\eta\sigma = \overline{\partial}\tau$ for some $\tau \in \Gamma_\infty(V, \mathbf{E})$, and $\theta(\eta) = \tau|Z \in H^0(Z, \mathcal{O}(\mathbf{E}))$. On $V - T_\epsilon$, $\eta \wedge \omega = \eta\sigma \wedge \dfrac{\omega}{\sigma} = \overline{\partial}\left(\dfrac{\tau}{\sigma} \wedge \omega\right) = d\left(\dfrac{\tau}{\sigma} \wedge \omega\right)$, and so $\lim_{\epsilon \to 0} \int_{V - T_\epsilon} \eta \wedge \omega = \lim_{\epsilon \to 0} - \int_{\partial T_\epsilon} \dfrac{\tau\omega}{\sigma} = \dfrac{2\pi}{i} \int_Z \tau\xi^*\omega$ by the same argument as used to prove (3.10).

Corollary 7.27. (6.4) *holds whenever* $\mathbf{E} \to V$ *is restricted to have the triangular group of matrices as structure groups.*

Proof. Use (7.21) and what we have just proved about line bundles. $\square$

8 Proof of (6.4) for the Highest Chern Class. Let $\mathbf{E} \to \mathbf{V}$ be an ample holomorphic vector bundle (c.f. (7.14)) with fibre $\mathbf{C}^k$ and such that $H^1(V, \mathcal{O}(\mathbf{E})) = 0$. The diagram (7.19) then becomes, for $q = k$,

$$\begin{tikzcd} H^1(V, \mathcal{O}(\mathrm{Hom}(\mathbf{E}, \mathbf{E}))) \arrow[d, "\theta"] \arrow[dr, "\zeta"] & \\ H^0(Z, \mathcal{O}(\mathbf{N}))/rH^0(V, \mathcal{O}(\mathbf{E})) \arrow[r, "\xi^-"'] & H^{k-1,k}(V). \end{tikzcd} \tag{8.1}$$

Let $\eta \in H^1(V, \mathcal{O}(\mathrm{Hom}(\mathbf{E}, \mathbf{E})))$ be given by a global $\mathrm{Hom}(\mathbf{E}, \mathbf{E})$-valued $(0, 1)$ form η and suppose $\sigma \in H^0(V, \mathcal{O}(\mathbf{E}))$ is such that $Z = \{\sigma = 0\}$ and $\eta \cdot \sigma = 0$ in $H^1(V, \mathcal{O}(\mathbf{E}))$. Then $\eta \cdot \sigma = \overline{\partial}\tau$ where τ is a C^∞ section of $\mathbf{E} \to V$, and $\tau | Z = \theta(\eta)$. If $\omega \in H^{n-k+1,n-k}(V)$ and Θ is a curvature in $\mathbf{E}$, then we need to show that

$$\int_Z \xi^*\omega \cdot \tau = \int_V kP_k(\underbrace{\Theta}_{k-1}; \eta) \wedge \omega, \tag{8.2}$$

where $\xi^*\omega \in H^{n-k}(Z, \Omega^{n-k}(\mathbf{N}^*))$ is the *Poincaré residue operator* (3.6).

159 What we will do is write, on $V - Z$, $kP_k(\underbrace{\Theta}_{k-1}; \eta) = \overline{\partial}\psi_k$ where ψ_k is a $C^\infty(k-1, k-1)$ form. Then, if T_ϵ is the tubular ϵ-neighborhood of Z in V, $\int_V kP_k(\underbrace{\Theta}_{k-1}; \eta) \wedge \omega = \lim_{\epsilon \to 0} \int_{V - T_\epsilon} d(\psi_k \wedge \omega) = -\lim_{\epsilon \to 0} \int_{\partial T_\epsilon} \psi_k \wedge \omega$. We will then show, by a residue argument, that $-\lim_{\epsilon \to 0} \int_{\partial T_\epsilon} \psi_k \wedge \omega = \int_Z \xi^*\omega \cdot \tau$.

Suppose now that we have an Hermitian metric in $\mathbf{E} \to V$. This metric determines a $(1, 0)$ connection θ in $\mathbf{E}$ with curvature $\Theta = \overline{\partial}\theta$. Let σ^* on $V - Z$ be the C^∞ section of $\mathbf{E}^* | V - Z$ which is dual to σ (using the metric). Setting $\lambda = \tau \otimes \sigma^*$, $\widehat{\eta} = \eta - \overline{\partial}\lambda$ is $C^\infty(0, 1)$ form with values in $\mathrm{Hom}(\mathbf{E}, \mathbf{E}) | V - Z$ and $\widehat{\eta} \cdot \sigma = \eta \cdot \sigma - \overline{\partial}(\tau \otimes \sigma^* \cdot \sigma) = \eta \cdot \sigma - \overline{\partial}\tau \equiv 0$.

On the other hand, we will find a $C^\infty(1, 0)$ form γ on $V - Z$, which has values in $\mathrm{Hom}(\mathbf{E}, \mathbf{E})$, and is such that $D\sigma = \gamma \cdot \sigma$. Then $\widehat{\theta} = \theta - \gamma$

gives a C^∞ connection in $\mathbf{E}|V-Z$ whose curvature $\widehat{\Theta} = \Theta - \overline{\partial}\gamma$ satisfies $\widehat{\Theta}\cdot\sigma \equiv 0$. Since $kP_k(\underbrace{\widehat{\Theta}}_{k-1};\widehat{\eta}) \equiv 0$ (because $\widehat{\Theta}\cdot\sigma \equiv 0 \equiv \widehat{\eta}\cdot\sigma$), it is clear that, on $V-Z$,

$$kP_k(\underbrace{\Theta}_{k-1};\eta) = kP_k(\underbrace{\widehat{\Theta}+\overline{\partial}\gamma}_{k-1};\widehat{\eta}+\overline{\partial}\lambda) = \overline{\partial}\psi_k,$$

and this will be our desired form ψ_k. Having found ψ_k explicitly, we will carry out the integrations necessary to prove (8.2).

(a) An integral formula in unitary Geometry. On $\widehat{\mathbf{C}}^k = \mathbf{C}^k - \{0\}$, we consider frames $(z; e_1, \ldots, e_k)$ where $z \in \widehat{\mathbf{C}}^k$ and $e_1, \ldots, e_k$ is a unitary frame with $e_1 = \dfrac{z}{|z|}$. Using the calculus of frames as in [5], we have : $De_\rho = \sum_{\sigma=1}^{k} \theta_\rho^\sigma e_\sigma (\theta_\sigma^\rho + \overline{\theta}_\rho^\sigma = 0)$. In particular, the differential forms $\theta_1^\rho = (De_1, e_\rho)$ are *horizontal forms* in the frame bundle over $\widehat{\mathbf{C}}^k$. Since $0 = \overline{\partial} z = D^n(|z|e_1) = \overline{\partial}|z|e_1 + |z|\left(\sum_{\rho=1}^{k} \theta_1^{\rho''} e_\rho\right)$, we find that **160**
$\theta_1^{\alpha''} = 0 (\alpha = 2, \ldots, k)$ and $\theta_1^{1''} = -\dfrac{\overline{\partial}|z|}{|z|} = -\overline{\partial}\log|z|$. It follows that $\theta_\alpha^{1'} = 0 (\alpha = 2, \ldots, k)$ and $\theta_1^{1'} = \partial \log |z|$, so that

$$\theta_1^1 = (\partial - \overline{\partial})\cdot \log|z|.$$

Given a frame $(z; e_1, \ldots, e_k)$, the e_ρ give a basis for the $(1,0)$ tangent space to $\widehat{\mathbf{C}}^k$ at z. Thus there are $(1,0)$ forms $\omega^1, \ldots, \omega^k$ dual to $e_1, \ldots, e_k$, and we claim that

$$\left.\begin{aligned} \omega^1 &= 2|z|\theta_1^{1'} \\ \omega^\alpha &= |z|\theta_1^\alpha (\alpha = 2, \ldots, k) \end{aligned}\right\} \tag{8.3}$$

Proof. By definition $dz = \sum_{\rho=1}^{k} \omega^\rho e_\rho$. But $z = |z|e_1$ and so $dz = (\partial|z| + |z|\theta_1^{1'})e_1 + \sum_{\alpha=2}^{k} |z|\theta_1^\alpha e_\alpha$. Since $\theta_1^{1'} = \dfrac{\partial|z|}{|z|}$, we get (8.3) by comparing both sides of the equation :

$$\sum_{\rho=1}^{k} \omega^\rho e_\rho = 2|z|\theta_1^{1'} e_1 + \sum_{\alpha=2}^{k} |z|\theta_1^\alpha e_\alpha.$$

Now let $\tau = \sum\limits_{\rho=1}^{k} \tau^\rho e_\rho$ and $\omega = \sum\limits_{\rho=1}^{k} \xi_\rho \omega^\rho$ be respectively a smooth vector field and $(1,0)$ form on $\mathbf{C}^k$. Then $\langle \omega, \tau \rangle = \sum\limits_{\rho=1}^{k} \xi_\rho \tau^\rho$ is a C^∞ function on $\mathbf{C}^k$. We want to construct a $(k-1,k-1)$ form $\widehat{\psi}_k(\tau)$ on $\widehat{\mathbf{C}}^k$ such that

$$\langle \omega, \tau \rangle_0 = \lim_{\epsilon\to 0} \int\limits_{\partial B_\epsilon} \widehat{\psi}_k(\tau) \wedge \omega, \tag{8.4}$$

where $B_\epsilon \subset \mathbf{C}^k$ is the ball of radius ϵ. Let $\Gamma(k)$ be the reciprocal of the area of the unit $2k-1$ sphere in $\mathbf{C}^k$ and set

$$\widehat{\psi}_k(\tau) = \frac{\Gamma(k)}{|z|} \left\{ \frac{\tau^1}{2} \prod_{\alpha=1}^{k} \theta_1^\alpha \theta_\alpha^1 + \sum_{\beta=2}^{k} \tau^\beta / \theta_1^{1'} \prod_{\alpha\neq\beta} \theta_1^\alpha \theta_\alpha^1 \right\}. \tag{8.5}$$

What we claim is that $\widehat{\psi}_k(\tau)$, *as defined by* (8.5), *is a* $(k-1,k-1)$ *form on* $\widehat{\mathbf{C}}^k$ *satisfying* (8.4). □

161 *Proof.* It is easy to check that $\widehat{\psi}_k(\tau)$ is a scalar C^∞ form on $\widehat{\mathbf{C}}^k$ of type $(k-1,k-1)$. Now $\omega = \sum\limits_{\rho=1}^{k} \xi_\rho \omega^\rho = |z|(2\xi_1\theta_1^{1'} + \sum\limits_{\alpha=2}^{k} \xi_\alpha \theta_1^\alpha)$, and so

$$\omega \wedge \widehat{\psi}_k(\tau) = \Gamma(k) \left\{ \sum_{\rho=1}^{k} \tau^\rho \xi_\rho \right\} \left\{ \theta_1^{1'} \prod_{\alpha=2}^{k} \theta_1^\alpha \theta_\alpha^1 \right\}. \tag{8.6}$$

Using (8.6), we must show : If f is a C^∞ function of $\mathbf{C}^k$, then

$$\lim_{\epsilon\to 0} \Gamma(k) \int\limits_{\delta B_\epsilon} f(z) \theta_1^{1'} \prod_{\alpha=2}^{k} \theta_1^\alpha \theta_\alpha^1 = f(0). \tag{8.7}$$

But, by (8.3), $\omega^1\omega^2\overline{\omega}^2\dots\omega^k\overline{\omega}^k = 2|z|^{2k-1}\theta_1^{1'}\theta_1^2\theta_2^1\dots\theta_1^k\theta_k^1$, and so $\Gamma(k)\theta_1^{1'}\prod_{\alpha=2}^{k}\theta_1^\alpha\theta_\alpha^1$ is a $2k-1$ form on $\widehat{\mathbf{C}}^k$ having constant surface integral one over all spheres ∂B_ϵ for all ϵ. From this we get (8.7). □

Remarks. (i) Using coordinates $z = \begin{bmatrix} z^1 \\ \vdots \\ z^k \end{bmatrix}$,

$$\theta_1^{1'}\theta_1^2\theta_2^1\dots\theta_1^k\theta_k^1 = \sum_{\rho=1}^{k}\frac{(-1)^{\rho-1+k}\overline{z}^\rho dz^1\dots dz^k d\overline{z}^1\dots \widehat{d\overline{z}^\rho}\dots d\overline{z}^k}{|z|^{2k}}. \tag{8.8}$$

(ii) If μ is a C^∞ differential form on $\widehat{\mathbf{C}}^k$ which becomes infinite at zero at a slower rate than $\widehat{\psi}_k(\tau)$, then

$$\lim_{\epsilon\to 0}\int_{\partial B_\epsilon}\omega\wedge\widehat{\psi}_k(\tau) = \lim_{\epsilon\to 0}\int_{\partial B_\epsilon}\omega\wedge(\widehat{\psi}_k(\tau)+\mu). \tag{8.9}$$

(iii) On $\mathbf{C}^l\times\mathbf{C}^k$, let ω be a C^∞ form of type $(l+1,l)$. Then, if $\tau = \sum_{\rho=1}^{k}\tau^\rho e_\rho$ is a C^∞ vector on $\mathbf{C}^k$, we may write $\omega = \sum_{\rho=1}^{k}\gamma_\rho\wedge\omega^\rho$ where the C^∞ form $\xi^*\omega\cdot\tau = \sum_{\rho=1}^{k}\tau^\rho\gamma_\rho$ is of type (l,l) on $\mathbf{C}^l$ and is uniquely determined by ω and τ.

Suppose now that ω has compact support in $\mathbf{C}^l$ (i.e. is supported in $\Delta^l\times\mathbf{C}^k$ for some polycylinder $\Delta^l\subset\mathbf{C}^k$). Then, as a generalization of (8.4), we have

$$\int_{\mathbf{C}^l}\xi^*\omega\cdot\tau = \lim_{\epsilon\to 0}\int_{\mathbf{C}^l\times\partial B_\epsilon^k}\widehat{\psi}_k(\tau)\wedge\omega. \tag{8.10}$$

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Note that $\xi^*\omega = \begin{bmatrix}\gamma_1\\ \vdots \\ \gamma_k\end{bmatrix}$ is here the *Poincaré residue* of ω on $\mathbf{C}^l\times\{0\}\subset\mathbf{C}^l\times\mathbf{C}^k$.

(iv) Combining remarks (ii) and (iii) above, we have :

$$\text{Let } \omega \text{ be a } C^\infty \text{ form of type } (l+1,l) \text{ with support in } \Delta^l \times \mathbf{C}^k \text{ and let } \psi_k(\tau) \text{ be a } C^\infty \text{ form on } \mathbf{C}^l \times \widehat{\mathbf{C}}^k \text{ whose } \textit{principal part} \text{ (i.e. term with the highest order pole on } \mathbf{C}^l \times \{0\}) \text{ is } \widehat{\psi}_k(\tau) \text{ given by (8.5). Then} \tag{8.11}$$

$$\int\limits_{\mathbf{C}^l} \xi^* \omega \cdot \tau = \lim_{\epsilon \to 0} \int\limits_{\mathbf{C}^l \times \delta B_\epsilon} \omega \wedge \psi_k(\tau). \tag{8.12}$$

(b) Some Formulae in Hermitian Geometry. Let W be a complex manifold and $\mathbf{E} \to W$ a holomorphic, Hermitian vector bundle with fibre $\mathbf{C}^k$. We suppose that $\mathbf{E}$ has a *non-vanishing* holomorphic section σ and we let $\mathbf{S}$ be the trivial line sub-bundle of $\mathbf{E}$ generated by σ. Thus we have over W

$$0 \to \mathbf{S} \to \mathbf{E} \to \mathbf{Q} \to 0. \tag{8.13}$$

We consider unitary frames $e_1, \ldots, e_k$ where $e_1 = \dfrac{\sigma}{|\sigma|}$ is the unit vector in $\mathbf{S}$. The metric connection in $\mathbf{E}$ gives a covariant differentiation $De_\rho = \sum\limits_\sigma \theta^\sigma_\rho e_\sigma (\theta^\sigma_\rho + \overline{\theta}^\rho_\sigma = 0)$ with $D'' = \overline{\partial}$. From $0 = \overline{\partial}\sigma = D''(|\sigma| e_1) = (\overline{\partial}|\sigma| + |\sigma|\theta^{1''}_1)e_1 + |\sigma| \left(\sum\limits_{\alpha=2}^k \theta^{\alpha''}_1 e_\alpha \right)$, we find

$$\theta^{\alpha''}_1 = 0 (\alpha = 2, \ldots, k), \theta^1_1 = (\partial - \overline{\partial}) \log |\sigma|. \tag{8.14}$$

Now then $D\sigma = D'\sigma = (\partial|\sigma| + |\sigma|\theta^{1'}_1)e_1 + |\sigma| \left(\sum\limits_{\alpha=2}^k \theta^\alpha_1 e_\alpha \right) = |\sigma| \left\{ 2\theta^{1'}_1 e_1 + \sum\limits_{\alpha=2}^k \theta^\alpha_1 e_\alpha \right\} = \gamma \cdot \sigma$ where

$$\gamma = 2\theta^{1'}_1 e_1 \otimes e^*_1 + \sum_{\alpha=2}^k \theta^\alpha_1 e_\alpha \otimes e^*_1 \tag{8.15}$$

163 is a global $(1,0)$ form with values in $\mathrm{Hom}(\mathbf{E}, \mathbf{E})$. In terms of matrices,

$$\gamma = \begin{bmatrix} 2\theta_1^{1'} & 0\dots0 \\ \theta_1^2 & 0\dots0 \\ \vdots & \\ \theta_1^k & 0\dots0 \end{bmatrix} \tag{8.16}$$

Letting $\widehat{\theta} = \theta - \gamma$, we get a $(1,0)$ connection $\widehat{D}$ in $\mathbf{E}$ with $\widehat{D}'' = \overline{\partial}$ and $\widehat{D}\sigma = 0$. This was one of the ingredients in the construction of ψ_k outlined above.

For later use, we need to compute $\overline{\partial}\gamma = D''\gamma$. Since γ is of type $(1,0)$, $D''\gamma$ is the $(1,1)$ part of $D\gamma = d\gamma + \theta \wedge \gamma + \gamma \wedge \theta$. Also, we won't need the first column of $D''\gamma$, so we only need to know $(D\gamma)^\rho_\alpha = \sum\limits_\tau \theta^\rho_\tau \gamma^\tau_\alpha + \sum\limits_\tau \gamma^\rho_\tau \theta^\tau_\alpha$ (since $\gamma^\rho_\alpha = 0$) $= \gamma^\rho_1 \theta^1_\alpha$. This gives the formula :

$$\delta\gamma = \begin{bmatrix} * & 2\theta_1^{1'} & \theta_2^1 & \dots & 2\theta_1^{1'} & \theta_k^1 \\ \vdots & \theta_1^2 & \theta_2^1 & \dots & \theta_1^2 & \theta_k^1 \\ & \vdots & & & \vdots & \\ * & \theta_1^k & \theta_2^1 & \dots & \theta_1^k & \theta_k^1 \end{bmatrix}. \tag{8.17}$$

As another part of the construction of ψ_k with $\overline{\partial}\psi_k = kP_k(\underbrace{\Theta}_{k-1};\eta)$, we let $\tau = \sum\limits_{\rho=1}^{k} \tau^\rho e_\rho$ be a C^∞ section of $\mathbf{E}$ with $\overline{\partial}\tau = \eta \cdot \sigma$ (c.f. below (8.1)). Set

$$\lambda = \tau \otimes \sigma^* = \sum_{\rho=1}^{k} \frac{\tau^\rho}{|\sigma|} e_\rho \otimes e_1^*. \tag{8.18}$$

In terms of matrices,

$$\lambda = \frac{1}{|\sigma|} \begin{bmatrix} \tau^1 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ \tau^k & 0 & \dots & 0 \end{bmatrix}. \tag{8.19}$$

We want to compute the Hom$(\mathbf{E},\mathbf{E})$-valued $(0,1)$ form $\partial\lambda$, and we claim that

$$\overline{\partial}\lambda = \sum_{\rho=1}^{k} \eta_1^{\rho} e_\mu \otimes e_1^* - \frac{1}{|\sigma|}\sum_{\rho,\alpha} \tau^{\rho}\theta_\alpha^1 e_\rho \otimes e_\alpha^*. \tag{8.20}$$

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Proof.

$$\begin{aligned}\overline{\partial}\gamma &= \overline{\partial}\tau\otimes\sigma^* + \tau\otimes\partial\overline{\sigma}^* \\ &= \sum_{\rho=1}^{k}\eta_1^{\rho} e_\rho\otimes e_1^* + \sum_{\rho=1}^{k}\tau^{\rho} e_\rho \otimes D''\left(\frac{e_1^*}{|\sigma|}\right)\end{aligned}$$

(since $\frac{e_1^*}{|\sigma|} = \sigma^*$ and $\overline{\partial}\tau = \eta\cdot\sigma$). Now $D''\left(\frac{e_1^*}{|\sigma|}\right) = \overline{\partial}\left(\frac{1}{|\sigma|}\right)e_1^* + \frac{1}{|\sigma|}D''e_1^*$ and $D''e_1^* = \sum_{\rho=1}^{k}\theta_1^{*\rho''}e_\rho^* = -\sum_{\rho=1}^{k}\theta_\rho^{1''}e_\rho^*$ (since $\theta + {}^t\theta^* = 0$). But $\overline{\partial}\left(\frac{1}{|\sigma|}\right) - \frac{1}{|\sigma|}\theta_1^{1''} = \frac{1}{|\sigma|}(-\overline{partial}\log|\sigma| - \theta_1^{1''}) = 0$ by (8.14) so that $D''\left(\frac{e_1^*}{|\sigma|}\right) = \frac{-1}{|\sigma|}\sum_{\alpha=2}^{k}\theta_\alpha^1 e_\alpha^*$ (since $\theta_\alpha^{1'} = 0$ by (8.14)). Thus $\overline{\partial}\lambda = \sum_{\rho=1}^{k}\eta_1^{\rho}e_\rho\otimes e_1^* - \frac{1}{|\sigma|}\sum_{\rho,\alpha}\tau^{\rho}\theta_\alpha^1 e_\rho\otimes e_\alpha^*$ as required.

In terms of matrices,

$$\overline{\partial}\lambda = -\begin{bmatrix} \eta_1^1 & \frac{\tau^1\theta_2^1}{|\sigma|} & \cdots & \frac{\tau^1\theta_k^1}{|\sigma|} \\ \vdots & \vdots & & \vdots \\ \eta_1^k & \frac{\tau^k\theta_2^1}{|\sigma|} & \cdots & \frac{\tau^k\theta_k^1}{|\sigma|} \end{bmatrix} = -\frac{1}{|\sigma|}\begin{bmatrix} * & \tau^1 & \theta_2^1 & \cdots & \tau^1 & \theta_k^1 \\ \vdots & & \vdots & & & \vdots \\ * & \tau^k & \theta_2^1 & \cdots & \tau^k & \theta_k^1 \end{bmatrix}. \tag{8.21}$$

□

(c) Completion of the Proof. Given $\mathbf{E}\to V$ and $\sigma\in H^0(V,\mathcal{O}(\mathbf{E}))$ with $Z = \{z\in V : \sigma(z) = 0\}$, we let $\widehat{\mathbf{E}} = \mathbf{E} - \{0\}$ and lift $\mathbf{E}$ up to lie over $\widehat{\mathbf{E}}$. Letting $W = \widehat{\mathbf{E}}$, the considerations in 8(b) above apply, as well as the various formulae obtained there. Using $\sigma : V - Z \to \widehat{\mathbf{E}}$, we may pull everything back down to $V - Z$. In particular, $|\sigma|$ may be thought of as the distance to Z, and the forms θ_α^ρ will go to infinity like $\dfrac{1}{|\sigma|}$ near Z (c.f. 8(a)).

Now, since $\Theta \cdot \sigma = \bar{\partial}\gamma \cdot \sigma$ and $\eta \cdot \sigma = \bar{\partial}\gamma \cdot \sigma$ on $V - Z$ we have

$$0 \equiv kP_k(\underbrace{\Theta - \bar{\partial}\gamma}_{k-1}; \eta - \bar{\partial}\lambda). \tag{8.22}$$

Expanding (8.22) out, we will have 165

$$kP_k(\underbrace{\Theta}_{k-1}; \eta) = \bar{\partial}\psi_k, \tag{8.23}$$

on $V - Z$. It is clear that ψ_k will be a polynomial with terms containing Θ_σ^ρ, η_σ^ρ, θ^ρ, τ^j. Furthermore, from (8.16), (8.17), and (8.21), the highest order term of ψ_k will become infinite near Z like $\dfrac{1}{|\sigma|^{2k-1}}$. From (8.5) and (8.11), the expression

$$-\lim_{\epsilon \to 0} \int\limits_{\partial T_\epsilon} \omega \wedge \psi_k \tag{8.24}$$

will depend only on this highest order part of ψ_k. Let us use the notation $\equiv$ to symbolize "ignoring terms of order $\dfrac{1}{|\sigma|^{2k-2}}$ or less." Then from (8.22), we have

$$\psi_k \equiv (-1)^k kP_k(\underbrace{\gamma}_{1}; \underbrace{\bar{\partial}\gamma}_{k-2}; \underbrace{\bar{\partial}\lambda}_{1}). \tag{8.25}$$

This is because Θ and η are smooth over Z. Note that the right hand side of (8.25) behaves as $\dfrac{1}{|\sigma|^{2k-1}}$ near Z. Using (8.5) and (8.11) from 8(a), to prove the commutativity of (8.1), we must show:

$$(-1)^k kP_k(\underbrace{\gamma}_{1}; \underbrace{\bar{\partial}\gamma}_{k-2}; \underbrace{\bar{\partial}\lambda}_{1})$$

$$\equiv -\frac{\Gamma(k)}{2|\sigma|}\left\{\tau^1 \prod_{\alpha=2}^{k} \theta_1^\alpha \theta_\alpha^1 + 2\sum_{\beta=2}^{k} \tau^\beta \theta_\beta^1 \theta_1^{1'} \prod_{\alpha\neq\beta} \theta_1^\alpha \theta_\alpha^1\right\} \tag{8.26}$$

where γ, $\bar\partial\gamma$, and $\bar\partial\lambda$ are given by (8.16), (8.17), and (8.21).

The left hand side of (8.26) is, by (A4.4)

$$\left(\frac{1}{2\pi i}\right)\frac{1}{(k-1)!}\left\{\frac{1}{|\sigma|}\sum_{\alpha=2}^{k}\det\begin{bmatrix} 2\theta_1^{1'} & 2\theta_1^{1'} & \theta_2^1 & \dots & \tau^1\theta_\alpha^1 & \dots & 2\theta_1^{1'} & \theta_k^1 \\ \theta_1^2 & \theta_1^2 & \theta_2^1 & & \tau^2\theta_\alpha^1 & & \theta_1^2 & \theta_k^1 \\ \vdots & & \vdots & & \vdots & & & \\ \theta_1^k & \theta_1^k & \theta_2^1 & \dots & \tau^k\theta_\alpha^1 & \dots & \theta_1^k & \theta_k^1 \end{bmatrix}\right\} \tag{8.27}$$

166 Fixing α, the coefficient of $\tau^1\theta_\alpha^1$ on the right hand side of (8.27) is

$$(-1)^\alpha \det\begin{bmatrix} \theta_1^2 & \theta_1^2 & \theta_2^1 & \dots & \widehat{\alpha} & \dots & \theta_1^2 & \theta_k^1 \\ \vdots & \vdots & & & & & \vdots & \\ \theta_1^k & \theta_1^k & \theta_2^1 & & \dots & & \theta_1^k & \theta_k^1 \end{bmatrix}$$

($\widehat{\alpha}$ means that the column beginning $\theta_1^2\theta_\alpha^1$ is deleted). This last determinant is evaluated as

$$\sum_\pi \operatorname{sgn}\pi\theta_1^{\pi_1}\theta_1^{\pi_2}\theta_2^1\dots(\widehat{\alpha})\theta_1^{\pi k-1}\theta_k^1, \tag{8.28}$$

where the sum is over all permutations of $2,\dots,k$. Obviously then (8.28) is equal to $(k-1)!(-1)^{\alpha-1}\theta_1^\alpha \prod_{\beta\neq\alpha}\theta_1^\beta\theta_\beta^1$. This then gives for the coefficient of τ^1 in (8.27) the term

$$-\left(\frac{1}{2\pi i}\right)^k \frac{(k-1)}{|\sigma|}\prod_{\alpha=2}^{k}\theta_1^\alpha\theta_\alpha^1. \tag{8.29}$$

In (8.27), the term containing $\tau^\alpha\theta_\beta^1$ is

$$(-1)^{\alpha+\beta}\det_{\alpha>\beta}\begin{array}{c} \begin{array}{cccccc} & \beta & & & & \end{array} \\ \begin{bmatrix} 2\theta_1^{1'} & 2\theta_1^{1'} & \theta_2^1 & \dots & 2\theta_1^{1'} & \theta_k^1 \\ \theta_1^2 & \theta_1^2 & \theta_2^1 & & \theta_1^2 & \theta_k^1 \\ & \vdots & & & \vdots & \\ \theta_1^k & \theta_1^k & \theta_2^1 & \dots & \theta_1^k & \theta_k^1 \end{bmatrix}\end{array}$$

$$= 2(k-1)!\theta_1^{\beta}\theta_1^{1'}\left(\prod_{\gamma\neq\beta,\alpha}\theta_1^{\gamma}\theta_{\gamma}^{1}\right)\theta_{\alpha}^{1}.$$

Thus, combining, (8.27) is evaluated to be

$$-\frac{\Gamma(k)}{2|\sigma|}\left\{\tau^1\prod_{\alpha=2}^{k}\theta_1^{\alpha}\theta_{\alpha}^{1} + 2\theta_1^{1'}\sum_{\beta=2}^{k}\tau^{\beta}\theta_{\beta}^{1}\prod_{\alpha\neq\beta}\theta_1^{\alpha}\theta_{\alpha}^{1}\right\}$$

where $\Gamma(k)^{-1} = \int\limits_{\alpha B_1} \omega^1\omega^2\overline{\omega}^2\ldots\overline{\omega}^k\omega^k$ in 8(a). Comparing with (8.26) we obtain our theorem.

9 Proof of (6.8) for the General Chern Classes.

The argument given in section 8 above will generalize to an arbitrary *nonsingular* Chern class $Z_q(\mathbf{E})$. The computation is similar to, but more com- **167**
plicated than, that given in §8, (a)-(c) above. However, in general $Z_q(\mathbf{E})$ will have singularities, no matter how ample $\mathbf{E}$ is. Thus the *normal bundle* $\mathbf{N} \to Z_q(\mathbf{E})$ is *not* well-defined, and so neither the infinitesimal variation formula (3.8) nor (7.16) makes sense as it now stands.

We shall give two proofs of (6.8). The first and more direct argument makes use of the fact that the singularities of $Z_q(\mathbf{E})$ are not too bad; in particular, they are "rigid," and so the argument in §8 can be generalized. The second proof will use the transformation formulae of §4; it is not completely general, in that we assume the parameter space to be a compact Riemann surface and not just a disc.

First Proof of (6.4) (by direct argument). To get an understanding of the singularities of $Z_q(\mathbf{E})$, let σ_1, σ_2 be general sections of $\mathbf{E} \to V$ so that $Z_{k-1}(\mathbf{E})$ is given by $\sigma_1 \wedge \sigma_2 = 0$. If, say, $\sigma_1(z_0) \neq 0$, we may choose a local holomorphic frame $e_1, \ldots, e_k$ with $e_1 = \sigma_1$. Then $\sigma_2(z) = \sum\limits_{\alpha=1}^{k} \xi^{\alpha}(z)e_{\alpha}$, and $Z_{k-1}(\mathbf{E})$ is locally given by $\xi^2 = \ldots = \xi^k = 0$. We may thus assume that the singular points of $Z_{k-1}(\mathbf{E})$ will come where $\sigma_1 = 0 = \sigma_2$. If $n \geqslant 2k$, there will be such points; choosing a suitable holomorphic frame $e_1, \ldots, e_k$, we may assume that $\sigma_1(z) =$

$\sum_{\alpha=1}^{k} z^{\alpha} e_{\alpha}$ and $\sigma_2(z) = \sum_{\alpha=1}^{k} z^{k+\alpha} e_{\alpha}$. Then $Z_{k-1}(\mathbf{E})$ is locally given by

$$z^{\alpha} z^{k+\beta} - z^{\beta} z^{k+\alpha} = 0 \quad (1 \leqslant \alpha < \beta \leqslant k). \tag{9.1}$$

For example, when $k = 2$, (9.1) becomes $z^1 z^4 - z^2 z^3 = 0$, which is essentially an ordinary double point.

Now let $\{\mathbf{E}_\lambda\}$ be a family of ample (c.f. §7) vector bundles satisfying $H^1(V, \mathcal{O}(\mathbf{E}_\lambda)) = 0$ (c.f. (7.18)). Then we may choose general sections $\sigma_1(\lambda), \dots, \sigma_k(\lambda)$ of $\mathbf{E}_\lambda$ which depend holomorphically on λ; in this case, $Z_\lambda = Z_q(\mathbf{E}_\lambda)$ is defined by $\sigma_1(\lambda) \wedge \dots \wedge \sigma_{k-q+1}(\lambda) = 0$. Letting $Z = Z_0$, we see that, although the Z_λ are singular, the *singularities are rigid* in the following sense:

168 There are local biholomorphic mappings $f_\lambda : U \to U$ (U = open set on V) such that

$$Z_\lambda \cap U = f_\lambda(Z \cap U) \tag{9.3}$$

We now define an *infinitesimal displacement mapping:*

$$\rho : \mathbf{T}_0(\Delta) \to H^0(Z, \operatorname{Hom}(I/I^2, \mathcal{O}_Z)), \tag{9.4}$$

where $I \subset \mathcal{O}_V$ is the ideal sheaf of Z. To do this, let $z^1, \dots, z^n$ be local coordinates in U and $f(z;\lambda) = f_\lambda(z)$ the mappings given by (9.3). Let $\theta_f(z)$ be the local vector field $\sum_{i=1}^{n} \frac{\partial f^i}{\partial \lambda}(z,\lambda) \frac{\partial}{\partial z^i}$. If $\xi(z)$ is a function in I (so that $\xi(z) = 0$ on Z), then $\theta_f \cdot \xi$ gives a section of $\mathcal{O}_V / I = \mathcal{O}_Z$. Furthermore, the mapping $\xi \to \theta_f \cdot \xi | Z$ is linear over $\mathcal{O}_V$ and is zero on I^2, so that we have a section of $\operatorname{Hom}(I/I^2, \mathcal{O}_Z)$ over U.

To see that this section is globally defined on Z, we suppose that $\widehat{f}_\lambda : U \to U$ also satisfies $\widehat{f}_\lambda(Z \cap U) = Z_\lambda \cap U$. Then $\widehat{f}(z;\lambda) = f(h(z,\lambda);\lambda)$ where $h(z;\lambda) : Z \cap U \to Z \cap U$. Then

$$\sum_{i=1}^{n} \frac{\partial \widehat{f}^i}{\partial \lambda}(z,\lambda) \frac{\partial \xi}{\partial z^i}(z) = \sum_{i,j} \frac{\partial f^i}{\partial z^j} \frac{\partial h^j}{\partial \lambda} \frac{\partial \xi}{\partial z^i} + \sum \frac{\partial f^i}{\partial \lambda}(h(z,\lambda);\lambda) \frac{\partial \xi}{\partial z^i}.$$

Thus, at $\lambda = 0$ and for $z \in Z$, $\theta_{\widehat{f}} \cdot \xi - \theta_f \cdot \xi$

$$= \sum_{i,j} \frac{\partial \xi}{\partial z^i}(z) \frac{\partial f^i}{\partial z^j}(z;0) \frac{\partial h^j}{\partial \lambda}(z;0) = \left[\frac{\partial}{\partial \lambda} (\xi(f(h(z,\lambda),0))) \right]_{\substack{\lambda=0 \\ z \in Z}} = 0.$$

From this we get that $\theta_{\hat{f}}\xi = \theta_f\xi$ in $\mathcal{O}_Z$. The resulting section of $\mathrm{Hom}(I/I^2, \mathcal{O}_Z)$ is, by definition, $\rho\left(\frac{\partial}{\partial\lambda}\right)$.

Examples. (a) In case Z is nonsingular, $\mathrm{Hom}(I/I^2, \mathcal{O}_Z) = \mathcal{O}_Z(\mathbf{N})$ where $\mathbf{N} \to Z$ is the normal bundle; then $\rho\left(\frac{\partial}{\partial\lambda}\right) \in H^0(Z, \mathcal{O}_Z(\mathbf{N}))$ is just Kodaira's infinitesimal displacement mapping (3.8).

(b) In case $Z \subset V$ is a hypersurface, $\rho\left(\frac{\partial}{\partial\lambda}\right)\xi$ vanishes on the singular points of Z. This is because $\xi = \eta g$ where $g(z) = 0$ is a minimal equation for $Z \cap U$. Then, in the above notation, $\theta_f \cdot \xi|Z = \eta\theta_f \cdot g|Z$, and $\theta_f \cdot g$ vanishes on $g = 0$, $dg = 0$, which is the singular locus of Z.

Now suppose that **169**

$$\dim Z = n - q \text{ and that } \omega = \sum_{\substack{I=(i_1,\ldots,i_{n-q+1})\\ J=(j_1,\ldots,j_{n-q})}} \omega_{I\bar{J}} dz^I \bar{z} d^J$$

is a C^∞ form of type $(n-q+1, n-q)$. Then

$$\langle \theta_f, \omega\rangle = \sum_{I,J,l} \pm\omega_{I\bar{J}} \frac{\partial f^1 l}{\partial\lambda} dz^{i_1} \wedge \ldots \wedge \widehat{dz^i l} \wedge \ldots \wedge dz^{i_{n-q+1}} \wedge d\bar{z}^J$$

is a $C^\infty(n-q, n-q)$ form in U whose restriction to the manifold points $Z_{\mathrm{reg}} \subset Z$ is well-defined. Thus, there exists a $C^\infty(n-q, n-q)$ form $\Omega = \langle \xi^*\omega, \rho\left(\frac{\partial}{\partial\lambda}\right)\rangle$ on Z_{reg} such that $\int_{Z_{\mathrm{reg}}} \Omega \overset{\text{def.}}{=} \int_Z \Omega$ converges. Just as in the proof of (3.7) (c.f. [9], §4), we can now prove:

The differential $\phi_* : \mathbf{T}_0(\Delta) \to H^{q-1,q}(V)$ *of the mapping*

$$\phi(\lambda) = \phi_q(Z_\lambda - Z) \tag{9.5}$$

is given by

$$\int_V \phi_* \left(\frac{\partial}{\partial \lambda}\right) \wedge \omega = \int_Z \langle \xi^* \omega, \rho \left(\frac{\partial}{\partial \lambda}\right) \rangle, \tag{9.6}$$

where the right-hand side of (9.6) *means, as above, that we take the Poincaré residue $\xi^*\omega$ of ω on Z_{reg} and contract with*

$$\rho \left(\frac{\partial}{\partial \lambda}\right) \in H^0(Z, \mathrm{Hom}(I/I^2, \mathcal{O}_Z))$$

given by (9.4).

Example. The point of (9.5) can be illustrated by the following example. Let $Z \subset \mathbf{C}^2$ be given by $xy = 0$ and $\theta \in \mathrm{Hom}(I/I^2, \mathcal{O}_Z)$ by $\theta(xy) = 1$. Then, on the x-axis ($y = 0$), θ is the normal vector field $\dfrac{1}{x}\dfrac{\partial}{\partial y}$; on the y-axis, θ is $\dfrac{1}{y}\dfrac{\partial}{\partial x}$. If now $\omega = dxdy$, then, on the x-axis, $\langle \xi^*\omega, \theta\rangle = \dfrac{1}{x} dx$ and so $\int\limits_{Z_{\mathrm{reg}}} \langle \xi^*\omega, \theta\rangle$ becomes infinite on the singular points of Z.

More generally, if $g(x, y) = x^a - y^b$ with $(a, b) = 1$, and if $\theta \in \mathrm{Hom}(I/I^2, \mathcal{O}_Z)$ is given by $\theta(g) = 1$, then θ corresponds to the normal vector field $\dfrac{1}{\partial g/\partial y}\dfrac{\partial}{\partial y}$. Thus, if $\omega = dxdyd\overline{x}$, $\langle \xi^*\omega, \theta\rangle = \dfrac{dxd\overline{x}}{(\partial g/\partial y)}$. Letting $x = t^b$, $y = t^a$, we have

$$\langle \xi^*\omega, \theta\rangle = \left(\frac{b^2}{a}\right) \cdot \frac{|t|^{2a-2} dt d\bar{t}}{t^{a(b-1)}},$$

170 which may be highly singular at $t = 0 (= Z_{\mathrm{sing}})$.

We now reformulate (9.5) as follows.

Let $\{\mathbf{E}_\lambda\}_{\lambda \in \Delta}$ be our family of bundles and $\phi : \Delta \to T_q(V)$ the mapping (3.1) corresponding to $Z_q(\mathbf{E}_\lambda) - Z_q(\mathbf{E}_0)$. If $\sigma_1(\lambda), \ldots, \sigma_k(\lambda)$ are general sections of $\mathbf{E}_\lambda \to V$ which depend holomorphically on λ, then $Z_q(\mathbf{E}_\lambda)$ is given by $\sigma_1(\lambda) \wedge \ldots \wedge \sigma_{k-q+1}(\lambda) = 0$. We let $Y_\lambda \subset Z_\lambda$ be the Zariski open set where $\sigma_1(\lambda) \wedge \ldots \wedge \sigma_{k-q}(\lambda) \neq 0$. Then $Y_\lambda \subset V$ is a submanifold (not closed) and $\{Y_\lambda\}_{\lambda\in\Delta}$ forms a *continuous system*. We let $\rho : \mathbf{T}_0(\Delta) \to H^0(Y, \mathcal{O}(\mathbf{N}))$ (where $\mathbf{N} \to Y = Y_0$ is the normal bundle) be the infinitesimal displacement mapping. If then $\psi \in H^{n-q+1,n-q}(V)$, we have the formula:

$$\int_V \phi_* \left(\frac{\partial}{\partial\lambda}\right) \wedge \psi = \int_Y \langle \rho \left(\frac{\partial}{\partial\lambda}\right), \xi^*\psi \rangle, \tag{9.8}$$

where $\phi_* \left(\dfrac{\partial}{\partial\lambda}\right) \in H^{q-1,q}(V)$ and $\xi^*\psi \in A^{n-q,n-q}(Y, \mathbf{N}^*)$ is the Poincaré residue of ψ along Y.

With this formulation, to prove (6.4) we want to show that

$$\int_Y \langle \rho \left(\frac{\partial}{\partial\lambda}\right), \xi^*\psi \rangle = \int_V qP_q(\underbrace{\Theta, \ldots, \Theta}_{q-1}; \eta) \wedge \psi, \tag{9.9}$$

where Θ is a curvature in $\mathbf{E} \to V$ and $\eta \in H^{0,1}(V, \operatorname{Hom}(\mathbf{E}, \mathbf{E}))$ is the Kodaira-Spencer class $\delta\left(\dfrac{\partial}{\partial\lambda}\right)$ (c.f. (6.3)).

Now $Z_{q+1}(\mathbf{E})$ is defined by $\sigma_1 \wedge \ldots \wedge \sigma_{k-q} = 0$, and we let $W \subset V$ be the Zariski open set $\sigma_1 \wedge \ldots \wedge \sigma_{k-q} \neq 0$; thus $W = V - Z_{q+1}(\mathbf{E})$. Clearly we have

$$\int_V qP_q(\underbrace{\Theta, \ldots, \Theta}_{q-1}; \eta) \wedge \psi = \int_W qP_q(\underbrace{\Theta, \ldots, \Theta}_{q-1}; \eta) \wedge \psi. \tag{9.10}$$

On the other hand, over W we have an exact sequence

$$0 \to \mathbf{S} \to \mathbf{E}_W \to \mathbf{Q} \to 0, \tag{9.11}$$

where $\mathbf{S}$ is the trivial bundle generated by $\sigma_1, \ldots, \sigma_{k-q}$. Suppose that **171**
we have an Hermitian metric in $\mathbf{E} \to V$ such that Θ is the curvature of the metric connection. Using this, we want to evaluate the right hand side of (9.10).

We now parallel the argument in §8 for a while. Since $H^1(V, \mathcal{O}(\mathbf{E})) = 0$, $\eta\sigma_\alpha = \bar{\partial}\gamma_\alpha$ for some C^∞ section γ_α of $\mathbf{E} \to V (\alpha = 1, \ldots, k-q)$.

On $W = V - Z_{q+1}(\mathbf{E})$ we find a section ζ_α of $\mathbf{E}^* \to V$ such that $\langle \zeta_\alpha, \sigma_\beta \rangle = \delta^\alpha_\beta$. We claim that we can find such ζ_α having a first order pole at a general point of $Z_{q+1}(\mathbf{E})$.

Proof. On W, we look at unitary frames $e_1, \dots, e_k$ such that $e_1, \dots, e_{k-q}$ is a frame for $\mathbf{S}$. Then $\sigma_\alpha = \sum_{\beta=1}^{k-q} h_{\alpha\beta} e_\beta$ where $\det(h_{\alpha\beta})$ vanishes to first order along $Z_{q+1}(\mathbf{E})$. Set $\zeta_\alpha = \sum_{\beta=1}^{k-q} (h^{-1})_{\beta\alpha} e^*_\beta$; then $\langle \zeta_\alpha, \sigma_\beta \rangle = \sum_{\gamma,\lambda} (h^{-1})_{\gamma\alpha} h_{\beta\lambda} \langle e^*_\gamma, e_\lambda \rangle = \delta^\alpha_\beta$. □

Remark. In the case $k - q = 1$, $e_1 = \dfrac{\sigma_1}{|\sigma_1|}$ and $\zeta_1 = \dfrac{e^*_1}{|\sigma_1|}$.

On W, we define $\gamma = \left(\sum_{\alpha=1}^{k-q} \zeta_\alpha \otimes \gamma_\alpha \right)$. Then $\bar\partial \gamma \cdot \sigma_\alpha = \eta \cdot \sigma_\alpha$ and so, if $\hat\eta = \eta - \bar\partial\gamma$, $\hat\eta \cdot \sigma_\alpha \equiv 0$ and $\hat\eta$ has a pole of order one along $Z_{q+1}(\mathbf{E})$. By Stoke's theorem then,

$$\int_W q P_q(\underbrace{\Theta, \dots, \Theta}_{q-1}; \eta) = \int_W q P_q(\underbrace{\Theta, \dots, \Theta}_{q-1}; \hat\eta). \tag{9.12}$$

In terms of the natural unitary frames for $0 \to \mathbf{S} \to \mathbf{E}_W \to \mathbf{Q} \to 0$, $\hat\eta = \left(\begin{array}{c|c} 0 & * \\ \hline 0 & * \end{array} \right)$.

We now work on the curvature Θ. The curvature $\hat\Theta$ in $\mathbf{S} \oplus \mathbf{Q} \to W$ may be assumed to have the form $\hat\Theta = \begin{pmatrix} 0 & 0 \\ 0 & \Theta_{\mathbf{Q}} \end{pmatrix}$ (since $\Theta_{\mathbf{S}} = 0$), and the same techniques as used in the Appendix to §4 can be applied to show:

$$q P_q(\Theta, \dots, \Theta; \hat\eta) - q P_q(\hat\Theta, \dots, \hat\Theta; \hat\eta) = \bar\partial \lambda, \tag{9.13}$$

172 where λ has a pole of order $2q - 1$ along $Z_{q+1}(\mathbf{E})$ (c.f. (A4.24) and the accompanying calculation). By Stoke's theorem again,

$$\int_W q P_q(\underbrace{\Theta, \dots, \Theta}_{q-1}; \hat\eta) = \int_W q P_q(\underbrace{\hat\Theta, \dots, \hat\Theta}_{q-1}; \hat\eta). \tag{9.14}$$

From (9.9), (9.10), (9.12), and (9.14), we have to show

$$\int_Y \left\langle \rho\left(\frac{\partial}{\partial\lambda}\right), \xi^*\psi\right\rangle = \int_W qP_q(\underbrace{\widehat{\Theta},\ldots,\widehat{\Theta}}_{q-1};\widehat{\eta}) \wedge \psi. \tag{9.15}$$

Now write $\widehat{\eta} = \begin{pmatrix} 0 & * \\ 0 & \eta_{\mathbf{Q}} \end{pmatrix}$; clearly we have

$$P_q(\underbrace{\widehat{\Theta},\ldots,\widehat{\Theta}}_{q-1};\widehat{\eta}) = P_q(\underbrace{\Theta_{\mathbf{Q}},\ldots,\Theta_{\mathbf{Q}}}_{q-1};\eta_{\mathbf{Q}}).$$

Thus, to prove (9.9), we need by (9.15) to show that

$$\int_Y \left\langle \rho\left(\frac{\partial}{\partial\lambda}\right), \xi^*\psi\right\rangle = \int_W qP_q(\Theta_{\mathbf{Q}},\ldots,\Theta_{\mathbf{Q}};\eta_{\mathbf{Q}}). \tag{9.16}$$

The crux of the matter is this. Over W, we have a holomorphic bundle $\mathbf{Q} \to W$ and a holomorphic section $\sigma \in H^0(W, \mathcal{O}(\mathbf{Q}))$; σ is just the projection on $\mathbf{Q}$ of $\sigma_{k-q+1} \in H^0(V, \mathcal{O}(\mathbf{E}))$. The subvariety Y is given by $\sigma = 0$, and the normal bundle of Y is $\mathbf{Q} \to Y$. Thus $\rho\left(\frac{\partial}{\partial\lambda}\right)$ is a holomorphic section of $\mathbf{Q} \to Y$, and (9.16) *is essentially the exact analogue of* (8.2) *with Y replacing Z and W replacing V*. To make the analogy completely precise, we need to know that $\eta_{\mathbf{Q}}$ and $\rho\left(\frac{\partial}{\partial\lambda}\right)$ are related as in (8.2). If we know this, and if we can keep track of the singularities along $Z_{q+1}(\mathbf{E})$, then (9.16) can be proved just at (8.2) was above. Thus we need the analogues of (7.17) and (7.19); what must be proved is this:

There exists a C^∞ section τ of $\mathbf{Q} \to W$ such that $\tau|Y$ is

$$\rho\left(\frac{\partial}{\partial\lambda}\right) \quad \text{and} \quad \overline{\partial}\tau = \eta_{\mathbf{Q}}\sigma. \tag{9.17}$$

In addition, we must keep track of the singularities of τ along $Z_{q+1}(\mathbf{E})$ so as to insure that the calculations in §8 will still work.

173 For simplicity, suppose that $q = k - 1$ so that $Z_{k-1}(\mathbf{E})$ is given by $\sigma_1 \wedge \sigma_2 = 0$ and $Z_k(\mathbf{E})$ by $\sigma_1 = 0$. Let $\mathbf{E}_\lambda \to V$ be given by $\{g_{\alpha\beta}(\lambda)\}$ (c.f. §6) and $\sigma_j(\lambda)$ by holomorphic vectors $\{\sigma_{j\alpha}(\lambda)\}(j = 1, 2)$. Then $\sigma_{j\alpha}(\lambda) = g_{\alpha\beta}(\lambda)\sigma_{j\beta}(\lambda)$ and

$$\frac{\partial \sigma_{j\alpha}(\lambda)}{-\partial\lambda} = g_{\alpha\beta}(\lambda)\frac{\partial\sigma_{j\beta}(\lambda)}{\partial\lambda} + \frac{\partial g_{\alpha\beta}(\lambda)}{\partial\lambda} g_{\alpha\beta}(\lambda)^{-1}\{g_{\alpha\beta}(\lambda)\sigma_{j\beta}(\lambda)\}.$$

At $\lambda = 0$, this says that

$$\delta\left(\frac{\partial\sigma_j}{\partial\lambda}\right) = \eta \cdot \sigma_j, \tag{9.18}$$

where $\dfrac{\partial\sigma_j}{\sigma\lambda}$ is a zero cochain for the sheaf $\mathcal{O}(\mathbf{E})$ and $\eta = \{\dot{g}_{\alpha\beta} g_{\alpha\beta}^{-1}\}$ is the *Kodaira-Spencer class* (6.3).

Let $'$ denote $\left.\dfrac{\partial}{\partial\lambda}\right]_{\lambda=0}$. Then from (9.18) we have

$$(\sigma_1 \wedge \sigma_2)' = \sigma_1' \wedge \sigma_2 + \sigma_1 \wedge \sigma_2' = \eta \cdot (\sigma_1 \wedge \sigma_2). \tag{9.19}$$

Thus, over $Z_{k-1}(\mathbf{E})$, $(\sigma_1 \wedge \sigma_2)'$ is a holomorphic section of $\Lambda^2\mathbf{E} \to Z_{k-1}(\mathbf{E})$. On the other hand, over $Y = Z_{k-1}(\mathbf{E}) \to Z_k(\mathbf{E})$, σ_1 is non-zero. Since $\mathbf{S} \subset \mathbf{E}_W$ is the sub-bundle generated by σ_1, we have on W an exact sequence:

$$0 \to \mathbf{S} \to \mathbf{E}_W \to \sigma_1 \wedge \mathbf{E}_W \to 0, \tag{9.20}$$

where the last bundle is the sub-bundle of $\Lambda^2\mathbf{E}_W$ of all vectors ξ such that $\xi \wedge \sigma_1 = 0$ in $\Lambda^3\mathbf{E}_W$.

Along Y, $\sigma_1 \wedge \sigma_2 = 0$ and so $\sigma_1 \wedge (\sigma_1 \wedge \sigma_2)' = 0$; thus $(\sigma_1 \wedge \sigma_2)'$ is a section along Y of $\sigma_1 \wedge \mathbf{E}$. But $\sigma_1 \wedge \mathbf{E}$ is naturally isomorphic to $\mathbf{Q}$ and, under this isomorphism, we may see that

$$(\sigma_1 \wedge \sigma_2)' = \rho\left(\frac{\partial}{\partial\lambda}\right). \tag{9.21}$$

Thus we have identified $\rho\left(\dfrac{\partial}{\partial\lambda}\right)$.

Let now $\eta \in A^{0,1}(V, \mathrm{Hom}(\mathbf{E},\mathbf{E}))$ be a *Dolbeault class* corresponding to $\{\dot{g}_{\alpha\beta} g_{\alpha\beta}^{-1}\}$. Then $\eta\sigma_1 = \bar{\partial}\gamma_1$ and $\eta \cdot \sigma_2 = \bar{\partial}\gamma_2$ where γ_1, γ_2 are C^∞ sections of $\mathbf{E} \to V$. Clearly these equations are the global analogue of (9.18). In particular, we may assume that, along $Z_{k-1}(\mathbf{E})$,

$$\sigma_1' \wedge \sigma_2 + \sigma_1 \wedge \sigma_2' = \gamma_1 \wedge \sigma_2 + \sigma_1 \wedge \gamma_2 = \rho\left(\frac{\partial}{\partial\lambda}\right). \tag{9.22}$$

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Now $\gamma_1 \wedge \sigma_2 + \sigma_1 \wedge \gamma_2$ is a C^∞ section of $\Lambda^2\mathbf{E}_W \to W$, but will not in general lie in $\sigma_1 \wedge \mathbf{E}_W \subset \Lambda^2\mathbf{E}_W$. However, letting $\gamma = \zeta_1 \otimes \gamma_1$ be as just above (9.12) (thus ζ_1 is a C^∞ section of $\mathbf{E}_W^* \to W$ satisfying $\langle \zeta_1, \sigma_1 \rangle = 1$), we may subtract

$$\gamma \cdot (\sigma_1 \wedge \sigma_2) = \gamma_1 \wedge \sigma_2 + \sigma_1 \wedge \langle \zeta_1, \sigma_2 \rangle \gamma_1$$

from $\gamma_1 \wedge \sigma_2 + \sigma_1 \wedge \gamma_2$ without changing the value along Y. But then $\tau = \gamma_1 \wedge \sigma_2 + \sigma_1 \wedge \gamma_2 - \gamma \cdot (\sigma_1 \wedge \sigma_2) = \sigma_1 \wedge \gamma_2 - \langle \zeta_1, \sigma_2 \rangle \sigma_1 \wedge \gamma_1$ lies in $\sigma_1 \wedge \mathbf{E}_W$. This gives us that:

$$\tau \text{ is a } C^\infty \text{ section of } \mathbf{Q} \to W \text{ such that } \tau|Y = \rho\left(\frac{\partial}{\partial\lambda}\right). \tag{9.23}$$

Also, $\bar{\partial}\tau = \bar{\partial}\gamma_1 \wedge \sigma_2 + \sigma_1 \wedge \bar{\partial}\gamma_2 - \bar{\partial}\gamma \cdot (\sigma_1 \wedge \sigma_2) = \eta \cdot \sigma_1 \wedge \sigma_2 + \sigma_1 \wedge \eta\sigma_2 - \bar{\partial}\gamma \cdot (\sigma_1 \wedge \sigma_2) = (\eta - \bar{\partial}\gamma) \cdot \sigma_1 \wedge \sigma_2 = \hat{\eta} \cdot (\sigma_1 \wedge \sigma_2)$. Under the isomorphism $\sigma_1 \wedge \mathbf{E}_W \cong \mathbf{Q}$, $\hat{\eta} \cdot (\sigma_1 \wedge \sigma_2) = \sigma_1 \wedge \hat{\eta}\sigma_2$ (since $\hat{\eta} \cdot \sigma_1 = 0$) corresponds to $\eta_{\mathbf{Q}} \cdot \sigma$, i.e. we have

$$\bar{\partial}\tau = \eta_{\mathbf{Q}}\sigma. \tag{9.24}$$

Combining (9.23) and (9.24) gives (9.17).

The only possible obstacle to using the methods of §8 to prove (9.16) is the singularities along $Z_{q+1}(\mathbf{E})$. Now τ has at worst a pole of order one along $Z_{q+1}(\mathbf{E})$, $\Theta_{\mathbf{Q}}$ has a pole of order 2, and so the forms which enter into the calculation will have *at most a pole of order* $2q$ *along* $Z_{q+1}(\mathbf{E})$. But this is just right, because $Z_{q+1}(\mathbf{E})$ has (real) codimension $2q + 2$, and we can use the following general principle.

Let X be an n-dimensional compact, complex manifold and $S \subset X$ an irreducible subvariety of codimension r. If Ω is a smooth $2n$-form on $X - S$ with a pole of order $2r-1$ along S, then $\int_{X-S} \Omega$ converges. Furthermore, if Ω_1, Ω_2 are two C^∞ forms on $X - S$ such that $\deg(\Omega_1) + \deg(\Omega_2) = 2n-1$ and such that {order of pole of (Ω_1)} + {order of pole of (Ω_2)} $= 2r-2$, then $\int_{X-S} d\Omega_1 \wedge \Omega_2 = (-1)^{\deg \Omega_2} \int_{X-S} \Omega_1 \wedge d\Omega_2$.

175 *Proof.* The singularities of S will not cause trouble, so assume S is nonsingular and let T_ϵ be an ϵ-tube around S. Then clearly $\lim_{\epsilon\to 0} \int_{X-T_\epsilon} \Omega$ converges and, by definition, equals $\int_{X-S} \Omega = \int_X \Omega$. Also, $\int_{X-T_\epsilon} d\Omega_1 \wedge \Omega_2 - (-1)^{\deg \Omega_1} \int_{X-T_\epsilon} \Omega_1 \wedge d\Omega_2 = -\int_{\partial T_\epsilon} \Omega_1 \wedge \Omega_2$. But, on ∂T_ϵ, $|\Omega_1 \wedge \Omega_2| \leqslant \dfrac{c}{\epsilon^{2r-2}} d\mu$ where $d\mu$ is the volume on ∂T_ϵ. Since $\int_{\partial T_\epsilon} d\mu \leqslant c' \epsilon^{2r-1}$, $\lim_{\epsilon \to 0} \int_{\partial T_\epsilon} \Omega_1 \wedge \Omega_2 = 0$. □

Second proof of (6.4) **(by functoriality).** We shall consider over V a family of holomorphic vector bundles $\{\mathbf{E}_\lambda\}_{\lambda \in C}$ parametrized by a *nonsingular algebraic curve* C; this family is given by a holomorphic bundle $\mathcal{E} \to V \times C$ where $\mathbf{E}_\lambda \cong \mathcal{E}|V \times \{\lambda\}$. We let $X = V \times C$ and $V_\lambda = V \times \{\lambda\}$, $V = V_{\lambda_0}$ where $\lambda_0 \in C$ is the marked point. It may be assumed that $\mathcal{E} \to X$ is ample and $H^1(V, \mathcal{O}(\mathbf{E}_\lambda)) = 0 = H^1(X, \mathcal{O}(\mathcal{E})) = 0$ for all $\lambda \in C$ (c.f. §7(c)).

Let $\mathscr{Z}_q \subset X$ be the q^{th} *Chern class* of $\mathcal{E} \to X$ and $Z_q(\lambda) = \mathscr{Z}_q \cdot V_\lambda$; thus $Z_q(\lambda)$ is the q^{th} Chern class of $\mathbf{E}_\lambda \to V$. More precisely, letting $\pi : X \to V$ be the projection, $\pi(\mathscr{Z}_q \cdot V_\lambda) \to Z_q(\lambda)$ is the q^{th} Chen class of $\mathbf{E}_\lambda \to V$.

Now let

$$\mathscr{Z}_\lambda = \mathscr{Z}_q \cdot V_\lambda - \mathscr{Z}_q \cdot V_{\lambda_0} = \mathscr{Z}_q \cdot (V_\lambda - V_{\lambda_0}) \text{ and } Z_\lambda = Z_q(\lambda) - Z_q(\lambda_0).$$

Then $\mathscr{Z}_\lambda$ is a cycle of codimension $q+1$ on X which is algebraically equivalent to zero, and $Z_\lambda = \pi(\mathscr{Z}_\lambda)$ is a similar cycle of codimension q

on V. Using an easy extension of the proof of (4.14), we have :

$$\begin{tikzcd} & T_{q+1}(X) \arrow[d, "\pi_*"] \\ C \arrow[ur, "\phi^{q+1}(X)"] \arrow[r, "\phi_q(V)"'] & T_q(V), \end{tikzcd} \tag{9.26}$$

where $\pi_* : H^*(X, \mathbf{C}) \to H^*(V, \mathbf{C})$ is *integration over the fibre* and $\phi_{q+1}(X)(\lambda) = \phi_{q+1}(X)(\mathscr{Z}_\lambda)$ (similarly for $\phi_q(V)$).

In infinitesimal form, (9.26) is:

$$\begin{tikzcd} & H^{q,q+1}(X) \arrow[d, "\pi_*"] \\ T_{\lambda_0}(C) \arrow[ur, "\phi_{q+1}(X)_*"] \arrow[r, "\phi_q(V)_*"'] & H^{q-1,q}(V). \end{tikzcd} \tag{9.27}$$

We let $\omega = \phi_q(V)_* \left(\dfrac{\partial}{\partial \lambda}\right)$ and $\Omega = \phi_{q+1}(X)_* \left(\dfrac{\partial}{\partial \lambda}\right)$, so that $\pi_* \Omega = \omega$ **176**
in $H^{q-1,q}(V)$. The class $\omega \in H^{q-1,q}(V)$ is characterized by

$$\int_X \Omega \wedge \pi^* \psi = \int_V \omega \wedge \psi, \text{ for all } \psi \in H^{n-q+1,n-q}(V). \tag{9.28}$$

The family of divisors $V_\lambda \subset X$ defines $\phi_1(X) : C \to T_1(X)$, and, from the mapping

$$\phi_1(X)_* : T_{\lambda_0}(C) \to H^{0,1}(X), \tag{9.29}$$

we let $\theta = \phi_1(X)_* \left(\dfrac{\partial}{\partial \lambda}\right)$. Thus θ is the *infinitesimal variation of V_λ measured in the Picard variety of X*. Letting $\Psi \in H^{q,q}(X)$ be the Poincaré dual of $\mathscr{Z}_q$, we have by (4.17) that

$$\Omega = \theta \Psi. \tag{9.30}$$

Because $V_\lambda \subset X$ is a divisor and because of (3.10), we know how to compute $\theta \in H^{0,1}(X)$. By (9.30), $\Omega \in H^{q,q+1}(X)$ is known, and so we must find $\pi_*(\Omega)$. This calculation, when carried out explicitly, will prove (6.4).

First, let $\mathbf{L} \to X$ be the line bundle $[V_{\lambda_0}]$ and $\sigma \in H^0(X, \mathcal{O}(\mathbf{L}))$ the holomorphic section with V_{λ_0} given by $\sigma = 0$. Then $\mathbf{L}|V = \mathbf{N}$ is the *normal bundle* of V in X; in fact, $\mathbf{N} \to V$ is clearly a trivial bundle with non-vanishing section $\dfrac{\partial}{\partial\lambda}$, where λ is a local coordinate on C at λ_0. Choose a C^∞ section τ of $\mathbf{L} \to X$ with $\tau|V = \dfrac{\partial}{\partial\lambda}$ and write

$$\overline{\partial}\tau = \theta\sigma. \tag{9.31}$$

177 Then, by §7(e), $\theta \in H^{0,1}(X)$ and gives $\phi_1(X)_*\left(\dfrac{\partial}{\partial\lambda}\right)$. By the same argument as in (3.10), we have :

$$\int\limits_X (\partial\Psi) \wedge \pi^*\psi = -\lim_{\epsilon\to 0} \int\limits_{\partial T_\epsilon} \overline{\partial}\left(\frac{\tau}{\sigma}\right)\Psi \wedge \pi^*\psi = \int\limits_V \langle \frac{\partial}{\partial\lambda}, \xi^*(\Psi \wedge \pi^*\psi)\rangle$$

(ξ^* being given by (3.6)) $= \int\limits_V \langle \dfrac{\partial}{\partial\lambda}, \xi^*\Psi\rangle \wedge \psi$. Combining, we have $\int\limits_X \Omega \wedge \pi^*\psi = \int\limits_V \langle \dfrac{\partial}{\partial\lambda}, \xi^*\Psi\rangle\psi$ for all $\psi \in H^{n-q+1,n-q}(V)$; by (9.28), we see then that

$$\omega = \pi_*(\Omega) = \langle \frac{\partial}{\partial\lambda}, \xi^*\Psi\rangle. \tag{9.32}$$

This equation is the crux of the matter; in words, it says that:

The infinitesimal variation of $Z_q(\mathbf{E})$ in $T_q(V)$ is given by the Poincaré residue, relative to $\partial/\partial\lambda$ along $V \times \{\lambda_0\}$ in $V \times C$, of the form

$$P_q(\Theta, \ldots, \Theta) \text{ on } V \times C \text{ where } \Theta \text{ is a curvature in } \mathcal{E} \to V \times C. \tag{9.33}$$

Since $\Theta|V = \Theta_{\mathbf{E}}$ is a curvature in $\mathbf{E} \to V$, and since

$$\langle \frac{\partial}{\partial\lambda}, P_q(\Theta, \ldots, \Theta)\rangle = qP_q(\Theta, \ldots, \Theta, \langle \frac{\partial}{\partial\lambda}, \Theta\rangle),$$

to prove (6.4) we must show that:

$$\langle \frac{\partial}{\partial\lambda}, \Theta\rangle = \eta \in H^{0,1}(V, \mathrm{Hom}(\mathbf{E},\mathbf{E})) \text{ is the } \textit{Kodaira-Spencer class}$$
$$\delta\left(\frac{\partial}{\partial\lambda}\right) \text{ given by (6.3).} \tag{9.34}$$

Let then Δ be a neighborhood, with coordinate λ, of λ_0 on C and $\{U_\alpha\}$ an open covering for V. Then $\mathcal{E}|V \times \Delta$ is given by transition functions $\{g_{\alpha\beta}(z,\lambda)\}$, and a $(1,0)$ connection θ for $\mathcal{E} \to V \times C$ is given by matrices $\theta_\alpha = \theta_\alpha(z,\lambda; dz, d\lambda)$ of $(1,0)$ forms which satisfy

$$\theta_\alpha - g_{\alpha\beta}\theta_\beta g_{\alpha\beta}^{-1} = dg_{\alpha\beta}g_{\alpha\beta}^{-1} = \left(\sum_k \frac{\partial g_{\alpha\beta}}{\partial z^j}dz^j + \frac{\partial g_{\alpha\beta}}{\partial\lambda}d\lambda\right)g_{\alpha\beta}^{-1}. \tag{9.35}$$

The curvature $\Theta|U_\alpha \times \Delta$ is given by $\Theta|U_\alpha \times \Delta = \overline{\partial}\theta_\alpha$. Thus $\langle \frac{\partial}{\partial\lambda}, \Theta\rangle|U_\alpha \times \Delta$ is given by $\overline{\partial}\langle \frac{\partial}{\partial\lambda}, \theta_\alpha\rangle$. But, on $U_\alpha \times \{\lambda_0\}$ $(\lambda_0 = 0)$, we **178**
have $\langle \frac{\partial}{\partial\lambda}, \theta_\alpha\rangle - g_{\alpha\beta}\langle \frac{\partial}{\partial\lambda}, \theta_\beta\rangle g_{\alpha\beta}^{-1} = \dot{g}_{\alpha\beta}g_{\alpha\beta}^{-1}$, so that

$$\overline{\partial}\left\{\langle \frac{\partial}{\partial\lambda}, \theta_\alpha\rangle|U_\alpha \times \{\lambda_0\}\right\}$$

is a *Dolbeault representative* of the Cêch cocycle $\{\dot{g}_{\alpha\beta}g_{\alpha\beta}^{-1}\} = \delta\left(\frac{\partial}{\partial\lambda}\right)$ by (6.3). Thus $\delta\left(\frac{\partial}{\partial\lambda}\right)$ is given by $\langle \frac{\partial}{\partial\lambda}, \Theta\rangle|V \times \{\lambda_0\}$ which proves (9.34).

10 Concluding Remarks.

Let V be an algebraic manifold and Σ_q the group of algebraic cycles of codimension q which are algebraically equivalent to zero. Letting $T_q(V)$ be the torus constructed in §2, there is a holomorphic homomorphism

$$\phi : \Sigma_q \to T_q(V), \tag{10.1}$$

given by (3.2). Letting A_q be the image of ϕ, we have that:

$$A_q \text{ is an abelian variety (c.f. (2.6)) and } \mathbf{T}_0(A_q) \subset H^{q-1,q}(V). \quad (10.2)$$

The two main questions are: *What is the equivalence relation defined by ϕ (Abel's theorem), and what is $\mathbf{T}_0(A_q)$ (inversion theorem)?* While we have made attempts at both of these, none of our results are definitive, and we want now to discuss the difficulties.

The obvious guess about the image of ϕ is:

$$\mathbf{T}_0(A_q) \text{ is the largest } \textit{rational subspace} \text{ contained in } H^{q-1,q}(V). \quad (10.3)$$

Remark. A subspace $S \subset H^{q-1,q}(V)$ is *rational* if there exist integral cycles $\Gamma_1, \ldots, \Gamma_l \in H_{2q-1}(V, \mathbf{Z})$ such that $S = \{\omega \in H^{q-1,q}(V)$ for which $\int_{\Gamma_\rho} \omega = 0, \rho = 1, \ldots, l\}$.

We want to show that:

$$\text{(10.3) is equivalent to a special case of the (rational) } \textit{Hodge conjecture.} \quad (10.4)$$

Proof. Let $S \subset H^{q-1,q}(V)$ be a rational subspace and $S_{\mathbf{R}} \subset H^{2q-1}(V, \mathbf{R})$ the corresponding real vector space of all vectors $\omega + \overline{\omega} (\omega \in S)$. Then $S_{\mathbf{R}} \cap H^{2q-1}(V, \mathbf{Z})$ is a lattice Γ_S and $S_R/\Gamma_S = J_q(V)$ is a torus which
179 has a complex structure given by: $S \subset S_{\mathbf{R}} \otimes \mathbf{C}$ is the space of holomorphic tangent vectors of $J_q(V)$. Furthermore, $J_q(V)$ is an *abelian variety* which will vary holomorphically with V, provided that its dimension remains constant and that $S_{\mathbf{R}}(V)$ varies continuously (c.f. §2). The space of holomoprhic 1-forms on $J_q(V)$ is $S^* \subset H^{n-q+1,n-q}(V)$.

Now suppose that $Z \subset J_q \times V$ is an algebraic cycle of codimension q on $J_q \times V$ such that, for a general point $\lambda \in J_q$, $Z \cdot \{\lambda\} \times V = Z_\lambda$ is a cycle of codimension q on V. This gives a family $\{Z_\lambda\}_{\lambda \in J_q}$ of codimension q-cycles on V, and we have then a holomorphic homomorphism

$$\phi : J_q \to T_q(V). \quad (10.5)$$

At the origin, the differential is

$$\phi_* : S \to H^{q-1,q}(V), \tag{10.6}$$

and to compute ϕ_* we shall use a formula essentially proved in the last part of §9: Let $e \in S$ be a $(1,0)$ vector on J_q and Ψ on $J_q \times V$ the (q,q) form which is dual to Z. Then $\langle e, \Psi \rangle$ is a $(q-1,q)$ form on $J_q \times V$ and we have (c.f. (9.33)):

$$\phi_*(e) \text{ is } \langle e, \Psi \rangle \text{ restricted to } \{0\} \times V. \tag{10.7}$$

What we must do then is construct a rational (q,q) form Ψ on $J_q \times V$ such that, according to (10.7),

$$\langle e, \Psi \rangle \text{ is equal to } e \text{ on } \{0\} \times V. \tag{10.8}$$

Let $e_1, \ldots, e_r$ be a basis for $S \subset H^{q-1,q}$ and $\psi_1, \ldots, \psi_r$ the dual basis for $S^* \subset H^{n-q+1,n-q}$. Then the ψ_ρ can be thought of as $(1,0)$ forms on J_q, the e_ρ become $(1,0)$ vectors on J_q, and $\langle e_\rho, \psi_\sigma \rangle = \delta^\rho_\sigma$ on J_q. We let

$$\Psi = \sum_{\rho=1}^{l} (\psi_\rho \otimes e_\rho + \overline{\psi}_\rho \otimes \overline{e}_\rho). \tag{10.9}$$

Then Ψ is a real (q,q) form on $J_q \times V$ and $\langle e_\rho, \Psi \rangle = e_\rho$ is a $(q-1,q)$ form on V. Thus (10.8) is satisfied and, to prove (10.4) we need only show that Ψ is rational.

If $f_1, \ldots, f_{2r}$ is a rational basis for $S_{\mathbf{R}} \subset H^{2q-1}(V, \mathbf{R})$ and $\theta_1, \ldots, \theta_{2r}$ a dual rational basis for $S^*_{\mathbf{R}} \subset H^{2n-2q+1}(V, \mathbf{R})$, then $e_\rho = \sum_{\beta=1}^{2r} M_{\beta\rho} f_\beta$ and $f_\alpha = \sum_{\rho=1}^{r} m_{\rho\alpha} e_\rho + \overline{m}_{\rho\alpha} \overline{e}_\rho$. This gives $mM = I$ and $m\overline{M} = 0$ where m is 180
an $r \times 2r$ and M a $2r \times r$ matrix. Thus $\left(\frac{m}{\overline{m}}\right)(M\overline{M}) = \left(\begin{smallmatrix} I & 0 \\ 0 & I \end{smallmatrix}\right)$. We also see that $\psi_\rho = \sum_{\alpha=1}^{2r} m_{\rho\alpha}\theta_\alpha$ and so $\Psi = \Sigma(m_{\rho\alpha} M_{\beta\rho} + \overline{m}_{\rho\alpha}\overline{M}_{\beta\rho})\theta_\alpha \otimes f_\beta = \sum_{\alpha=1}^{2r} \theta_\alpha \otimes f_\alpha$, which is rational on $J_q \times V$. $\square$

Remark. A similar class Ψ of $J_q \times V$ has been discussed by Lieberman, who calls it a *Poincaré cycle*, from the case $q = 1$. In this case $J_1(V) = \mathrm{Pic}(V) \cong H^{0,1}(V)/H^1(V, \mathbf{Z})$, and there is a line bundle $\mathscr{L} \to J_1 \times V$ with $c_1(\mathscr{L}) = \Psi$ and such that $\mathscr{L}|\{\lambda\} \times V = \mathbf{L}_\lambda$ is the line bundle over V corresponding to $\lambda \in H^{0,1}(V)/H^1(V, \mathbf{Z}) \subset H^1(V, \mathcal{O}^*)$.

We now prove:

If (10.3) holds, then the equivalence relation defined by ϕ in (10.1) is *rational equivalence* on a suitable subvariety of a Chow variety associated to V. (10.10)

Proof. Let $\mathbf{Z} \subset V$ be an irreducible subvariety of codimension q on V, and let Φ parametrize an algebraic family of subvarieties $Z \subset V$ such that $\mathbf{Z} \in \Phi$. Then (c.f. §5) Φ is a subvariety of the Chow variety of $\mathbf{Z}$.

Now, if (10.3) holds, then in proving it we will certainly be able to find a family $\{W_\lambda\}$ of effective subvarieties $W_\lambda \subset V$ of codimension $n - q + 1$ which are parametrized by $\lambda \in J_{n-q+1}$ and such that $\phi_{n-q+1}(W_\lambda - W_0) = \lambda$. Then, as in §5, each $Z \in \Phi$ defines a divisor $D(Z)$ on J_{n-q+1} and we want to prove :

$$D(Z) \equiv D(\mathbf{Z}) \text{ if, and only if, } \phi_q(Z - \mathbf{Z}) = 0 \text{ in } T_q(V). \tag{10.11}$$

Let ψ be a *residue operator* for $Z - \mathbf{Z}$ (c.f. §5(a)) and set $\theta = d\left\{\int_{W_0}^{W_\lambda} \psi\right\}$ on J_{n-q+1} (c.f. (5.21)). Then θ is a meromorphic form of the third kind on J_{n-q+1} associated to the divisor $D(Z) - D(\mathbf{Z})$. By (5.24), we have:

$$\begin{gathered} D(Z) \equiv D(\mathbf{Z}) \text{ on } J_{n-q+1} \text{ if, and only if, there exists } \omega \in H^{1,0}(J_{n-q+1}) \\ \text{such that } \int_\delta \theta + \omega \equiv 0(1) \text{ for all } \delta \in H_1(J_{n-q+1}, \mathbf{Z}). \end{gathered} \tag{10.12}$$

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Denote by $S \subset H^{n-q,n-q+1}(V)$ the largest rational subspace; then S is the holomorphic tangent space to J_{n-q+1}. The holomorphic one forms $H^{1,0}(J_{n-q+1})$ are then $S^* \subset H^{q,q-1}(V)$. Given $\Omega \in S^*$, the corresponding form $\omega \in H^{1,0}(J_{n-q+1})$ is defined by

$$\omega = d\left\{\int_{W_0}^{W_\lambda} \Omega\right\}.$$

Given $\delta \in H_1(J_{n-q+1}, \mathbf{Z})$, there is defined a $2q-1$ cycle $T(\delta) \in H_{2q-1}(V, \mathbf{Z})$ by tracing out the W_λ for $\lambda \in \delta$. Clearly we have

$$\int_\delta \theta + \omega = \int_{T(\delta)} \psi + \Omega. \tag{10.13}$$

Combining (10.13) and (10.12), we see that:

$$\begin{aligned}&D(Z) \equiv D(\mathbf{Z}) \text{ on } J_{n-q+1}, \text{ if, and only if, } \int_\Gamma \psi + \Omega \equiv 0(1)\\ &\text{for some } \Omega \in S^* \text{ and all } \Gamma \in H_{2q-1}(V, \mathbf{Z}).\end{aligned} \tag{10.14}$$

□

Now taking into account the reciprocity relation (5.30), we find that (10.14) implies (10.10).

Remark. The mapping $T : H_1(J_{n-q+1}, \mathbf{Z}) \to H_{2q-1}(V, \mathbf{Z})$ may be divisible so that, to be precise, (10.10) holds up to isogeny.

Example 10.15. Take $q = n$, so that Φ is a family of zero-cycles on V and $\phi_n : \Phi \to T_n(V)$ is the Albanese mapping. Then $J_{n-q+1} = J_1 = \mathrm{Pic}(V)$ and we may choose $\{W_\lambda\}_{\lambda \in \mathrm{Pic}(V)}$ to be a family of ample divisors. In this case we see that:

Albanese equivalence on Φ is, up to isogeny, linear equivalence on $\mathrm{Pic}(V)$. (10.16)

The conclusion drawn from (10.4) and (10.10) is:

The generalizations to arbitrary cycles of both the inversion theorem and Abel's theorem, as formulated in (10.3) *and* (10.10)*, essentially depend on a special case of the Hodge problem.*
(10.17)

The best example I know where the inversion theorem (10.3) and Abel's theorem (10.10) hold is the case of the *cubic threefold* worked out by F. Gherardelli. Let $V \subset P_4$ be the zero locus of a nonsingular cubic polynomial. Through any point z_0 in V, there will be six lines in P_4 lying on V.

(10.18)

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Proof. Using affine coordinates x, y, z, w and taking z_0 to be the origin, V will be given by $f(x, y, z, w) = 0$ where f will have the form $f(x, y, z, w) = x + g_2(x, y, z, w) + g_3(x, y, z, w)$. Any line through z_0 will have an equation $x = \alpha_0 t$, $y = \alpha_1 t$, $z = \alpha_2 t$, $w = \alpha_3 t$. If the line is to lie on V, then we have $\alpha_0 t + g_2(\alpha_0, \alpha_1, \alpha_2, \alpha_3)t^2 + g_3(\alpha_0, \alpha_1, \alpha_2, \alpha_3)t^3 = 0$ for all t; thus $\alpha_0 = 0$ and $g_2(0, \alpha_1, \alpha_2, \alpha_3) = 0 = g_3(0, \alpha_1, \alpha_2, \alpha_3)$. Thinking z_0 are given by the points of intersection of a quadric and cubic in P_2, so there are six of them.

Let Φ be the variety of lines on V. Then it is known that Φ is a nonsingular surface and the irregularity $h^{0,1}(\Phi)$ is five. But also $h^{1,2}(V) = 5$ and $h^{0,3}(V) = 0$. Thus, in this case, $J_2(V) = T_2(V)$ is the whole torus. Fixing a base point $z_0 \in \Phi$, there is defined $\phi_2 : \Phi \to T_2(V)$ by the usual method. What Gherardelli has proved is:

$$\phi_2 : \mathrm{Alb}(\Phi) \to T_2(V) \text{ is an isogeny.} \tag{10.19}$$

Thus, in the above notation, we have:

For the cubic threefold V, $A_2 - J_2 = T_2$ and so the inversion theorem (10.3) holds. Furthermore, the equivalence relation given by the intermediate torus is, up to an isogeny, linear equivalence on Φ.

(10.20)

□

APPENDIX

A Theorem on the Cohomology of Algebraic Manifolds.

Let V be a compact, complex manifold and $A^{p,q}(V)$ the vector space of C^∞ forms of type (p,q) on V. From

$$\left.\begin{aligned}&\partial : A^{p,q}(V) \to A^{p+1,q}(V), \partial^2 = 0,\\ &\overline{\partial} : A^{p,q}(V) \to A^{p,q+1}(V), \overline{\partial}^2 = 0, \partial\overline{\partial} + \overline{\partial}\partial = 0,\end{aligned}\right\} \tag{A.1}$$

we find a *spectral sequence* (c.f. [7], section 4.5) $\{E_r^{p,q}\}$ with $E_1^{p,q} = H_{\overline{\partial}}^{p,q}(V) \cong H^q(V, \Omega^p)$ (Dolbeault). This spectral sequence was discussed by Frölicher [6], who observed that, if V was a *Kähler manifold*, **183**
then $E_1^{p,q} = E_2^{p,q} = \ldots = E_\infty^{p,q}$. This proved that:

There is a *filtration* $F_{p+q}^{p+q}(V) \subset \cdots \subset F_0^{p+q}(V) = H^{p+q}(V, \mathbf{C})$ such that

$$F_p^{p+q}(V)/F_{p+1}^{p+q}(V) \cong H_{\overline{\partial}}^{p,q}(V) \cong H^q(V, \Omega^p). \tag{A.2}$$

Thus

$$F_p^{p+q}(V) \cong \sum_{r \geqslant 0} H_{\overline{\partial}}^{p+r,q-r}(V). \tag{A.3}$$

We call the filtration (A.3) the *Hodge filtration*. Our object is to give a description of the Hodge filtration $\{F_q^r(V)\}$ *using only holomorphic functions*, from which it follows, e.g., that the Hodge filtration varies holomorphically with V. It will also prove that

$$F_p^{p+q}(V) \cong \ker d \cap \left(\sum_{r \geqslant 0} A^{p+r,q-r}(V)\right) / d \left(\sum_{r \geqslant 0} A^{p+r,q-r-1}(V)\right), \tag{A.4}$$

which is the result (A3.5) used there to prove (A3.6), the fact that the mappings $\phi_q : \Sigma_q \to T_q(V)$ depend only on the complex structure of V.

(a) Let V be a complex manifold and Ω_c^p the sheaf on V of *closed* holomorphic p-forms. There is an exact sheaf sequence:

$$0 \to \Omega_c^p \to \Omega^p \xrightarrow{\partial} \Omega_c^{p+1} \to 0. \tag{A.5}$$

Theorem A.6. (Dolbeault) *In case V is a compact Kähler manifold, we have $H^q(V,\Omega_c^p) \to H^q(V,\Omega^p) \to 0$, so that the exact cohomology sequence of* (A.5) *is*

$$0 \to H^{q-1}(V\Omega_c^{p+1}) \to H^q(V,\Omega_c^p) \to H^q(V,\Omega^p) \to 0. \tag{A.7}$$

Proof. We shall inductively define diagrams:

$$\begin{tikzcd}
 & H^{q-k-1}(V,\Omega_c^{p+k+2}) \arrow[d, "\delta"] \\
H^q(V,\Omega^p) \arrow[ur, dashed, "\alpha_{k+1}"] \arrow[r, "\alpha_k"] \arrow[dr, "\beta_k"'] & H^{q-k}(V,\Omega_c^{p+k+1}) \arrow[d] \\
 & H^{q-k}(V,\Omega^{p+k+1})
\end{tikzcd} \tag*{(A.8)$_k$}$$

$(k = 0, \ldots, q)$, where the first one is:

$$\begin{tikzcd}
 & H^{q-1}(V,\Omega_c^{p+2}) \arrow[d, "\delta"] \\
H^q(V,\Omega^p) \arrow[ur, dashed, "\alpha_1"] \arrow[r, "\alpha_0"] \arrow[dr, "\beta_0"'] & H^q(V,\Omega_c^{p+1}) \arrow[d] \\
 & H^q(V,\Omega^{p+1})
\end{tikzcd} \tag*{(A.8)$_0$}$$

184 and where (A.8)$_k$ will define α_{k+1} after we prove that $\beta_k = 0$. In (A.8)$_k$, the mapping δ is the coboundary in the exact cohomology sequence of

$$0 \to \Omega_c^{p+k+1} \to \Omega^{p+k+1} \xrightarrow{\partial} \Omega_c^{p+k+2} \to 0.$$

We want to prove that $\alpha_0 = 0$. If $\alpha_{k+1} = 0$, then $\alpha_k = 0$ so it will suffice to prove that $\alpha_q = 0$. Now (A.8)$_q$ is

$$\begin{tikzcd} & 0 \arrow[d] \\ H^q(V,\Omega^q) \arrow[r, "\alpha_q"] \arrow[dr, "\beta_q"'] & H^0(V,\Omega_c^{q+p+1}) \arrow[d] \\ & H^0(V,\Omega^{q+p+1}), \end{tikzcd} \qquad \text{(A.8)}_q$$

and so we have to show that $\beta_q = 0$. Thus, to prove Theorem A.6, we will show that:

$$\text{The maps } \beta_k \text{ in (A.8)}_k \text{ are zero for } k = 0, \dots, q. \qquad \text{(A.9)}$$

The basic fact about Kähler manifolds which we use is this:

> Let $\phi \in A^{p,q}(V)$ be a $C^\infty(p,q)$ form with $\bar\partial\phi = 0$, so that ϕ defines a class ϕ in the Dolbeault group $H^{p,q}_{\bar\partial}(V) \cong H^q(V,\Omega^p)$. Suppose that $\phi = \partial\psi$ for some $\psi \in A^{p-1,q}(V)$. Then $\phi = 0$ in $H^{p,q}_{\bar\partial}(V)$. (A.10)

□

Proof. Let $\square_{\bar\partial}$ and $\mathbf{H}_{\bar\partial}$ be the *Laplacian* and *harmonic projection* for $\bar\partial$, and similarly for $\square_\partial$ and $\mathbf{H}_\partial$. Thus $\mathbf{H}_{\bar\partial}$ is the projection of $A^{p,q}(V)$ onto 185
the kernel $\mathbf{H}^{p,q}_{\bar\partial}(V)$ of $\square_{\bar\partial}$, and likewise for $\mathbf{H}_\partial$. Since $\square_{\bar\partial}$ is self-adjoint and $\square_{\bar\partial} = \square_\partial$ (because V is Kähler), $\mathbf{H}_{\bar\partial} = \mathbf{H}_\partial$. Thus, if $\phi = \partial\psi$, $\mathbf{H}_\partial(\phi) = \mathbf{H}_{\bar\partial}(\phi) = 0$. But if $\mathbf{H}_{\bar\partial}(\phi) = 0$ and $\bar\partial\phi = 0$, $\phi = \bar\partial\bar\partial^*\mathbf{G}_{\bar\partial}\phi$ where $\bar\partial^*$ is the adjoint of $\bar\partial$ and $G_{\bar\partial}$ is the *Green's operator* for $\square_{\bar\partial}$ (recall that $\phi = \mathbf{H}_{\bar\partial}(\phi) + \square_{\bar\partial}\mathbf{G}_{\bar\partial}(\phi)$ and $\bar\partial\mathbf{G}_{\bar\partial} = \mathbf{G}_{\bar\partial}\bar\partial$). Thus $\phi = 0$ in $H^{p,q}_{\bar\partial}(V)$ if $\phi = \partial\psi$.

Now $\beta_0 : H^{p,q}_{\bar\partial}(V) \to H^{p+1,q}_{\bar\partial}(V)$ is given by $\beta_0(\phi) = \partial\phi$ so that $\beta_0 = 0$ and α_1 is defined.

Write $\partial\phi = \overline{\partial}\psi_1$ where $\psi_1 \in A^{p+1,q-1}(V)$. Then $\overline{\partial}(\partial\psi_1) = -\partial\overline{\partial}\psi_1 = -\partial^2\phi = 0$ so that $\overline{\partial}\psi_1$ is a ∂-closed form in $A^{p+2,q-1}(V)$. We claim that, in the diagram

$$\begin{tikzcd}
 & H^{q-2}(V,\Omega_c^{p+3}) \arrow[d, "\delta"] \\
H^q(V,\Omega^p) \arrow[ur, dashed, "\alpha_2"] \arrow[r, "\alpha_1"] \arrow[dr, "\beta_1"'] & H^{q-1}(V,\Omega_c^{p+2}) \arrow[d] \\
 & H^{q-1}(V,\Omega^{p+2}),
\end{tikzcd} \qquad (\mathrm{A}.8)_1$$

$\beta_1(\phi) = \partial\psi_1$. □

Proof. We give the argument for $q = 2$; this will illustrate how the general case works. Let then $\{U_\alpha\}$ be a suitable covering of V with nerve $\mathfrak{U}$, and denote by $C^q(\mathfrak{U}, S)(Z^q(\mathfrak{U}, S))$ the q-cochains (q-cocycles) for $\mathfrak{U}$ with coefficients in a sheaf S. Now $\phi \in Z^2(\mathfrak{U}, \Omega^p)$, and $\phi = \delta\xi_1$ for some $\xi_1 \in C^1(\mathfrak{U}, A^{p,0})$ ($A^{p,q}$ being the sheaf of $C^\infty(p,q)$ forms). Then $\overline{\partial}\xi_1 \in Z^1(\mathfrak{U}, A^{p,1})$ and $\overline{\partial}\xi_1 = \delta\xi_2$ for $\xi_2 \in C^0(\mathfrak{U}, A^{p,1})$. Now $\overline{\partial}\xi_2 \in Z^0(\mathfrak{U}, A^{p,2})$ and the global form $\xi \in A^{p,2}(V)$ defined by $\xi|U_\alpha = \overline{\partial}\xi_2$ is a Dolbeault representative in $H^2_{\overline{\partial}}(V, \Omega^p)$ of ϕ.

Clearly $\partial\xi \in A^{p+1,2}(V)$ is a Dolbeault representative of $\beta_0(\phi) \in H^2(V, \Omega^{p+1})$, and $\partial\xi = \overline{\partial}\psi_1$ for some $\psi_1 \in A^{p+1,1}(V)$. We want to find a Cêch cochain $\theta \in C^1(\mathfrak{U}, \Omega^{p+1})$ with $\delta\theta = \partial\phi$. To do this, we let $\zeta_2 = \partial\xi_2 + \psi_1 \in C^0(\mathfrak{U}, A^{p+1,1})$. Then $\overline{\partial}\zeta_2 = -\partial\xi + \overline{\partial}\psi = 0$ so that $\zeta_2 = \overline{\partial}\lambda_2$ for some $\lambda_2 \in C^0(\mathfrak{U}, A^{p+1,0})$. We let $\zeta_1 = \partial\xi_1 + \delta\lambda_2 \in$
186 $C^1(\mathfrak{U}, A^{p+1,0})$. Then $\delta\zeta_1 = \delta\partial\xi_1 = \partial\phi$, and $\overline{\partial}\zeta_1 = -\partial\overline{\partial}\xi_1 + \delta\overline{\partial}\lambda_2 = -\partial\overline{\partial}\xi_1 + \delta\zeta_2 = -\partial\delta\xi_2 + \partial\delta\xi_2 = 0$ so that $\theta = \zeta_1 \in C^1(\mathfrak{U}, \Omega^{p+1})$. In $(\mathrm{A}.8)_0$, $\alpha_1(\phi) \in H^1(V, \Omega_c^{p+2})$ is represented by $\partial\theta \in Z^1(\mathfrak{U}, \Omega_c^{p+1})$. Observe that $\delta\partial\theta = \delta\partial\zeta_1 = \delta(\partial^2\xi_1 + \delta\partial\lambda_2) = 0$.

We now want a Dolbeault representative for $\partial\theta \in Z^1(\mathfrak{U}, \Omega^{p+1})$. Since $\partial\theta = \delta\partial\lambda_2$ where $\partial\lambda_2 \in C^0(\mathfrak{U}, A^{p+2,0})$, such a representative is given by $\overline{\partial}\partial\lambda_2 \in Z^0(\mathfrak{U}, A^{p+2,1})$. But $\overline{\partial}\partial\lambda_2 = -\partial\overline{\partial}\lambda_2 = -\partial\zeta_2 =$

$-\partial(\partial\xi_2 + \psi_1) = -\partial\psi_1$; that is to say, $-\partial\psi_1$ is a Dolbeault representative of $\alpha_1(\phi) \in H^1(V, \Omega^{p+2})$, which was to be shown.

Now $\beta_1(\phi) = 0$ by the lemma on Kähler manifolds, and so $\partial\psi_1 = \bar{\partial}\psi_2$ where $\psi_2 \in A^{p+2,q-2}(V)$. Then $\bar{\partial}(\partial\psi_2) = -\partial\bar{\partial}\psi_2 = -\partial^2\psi_1 = 0$ so that $\partial\psi_2$ is a $\bar{\partial}$-closed form in $A^{p+3,q-2}(V)$. As before, we show that, in the diagram,

$$\begin{tikzcd} & H^{q-3}(V, \Omega_c^{p+4}) \arrow[d, "\delta"] \\ H^q(V, \Omega^p) \arrow[ur, "\alpha_3"] \arrow[r, "\alpha_2"] \arrow[dr, "\beta_2"'] & H^{q-2}(V, \Omega_c^{p+3}) \arrow[d] \\ & H^{q-2}(V, \Omega^{p+3}), \end{tikzcd} \tag{A.8$_2$}$$

$\beta_2(\phi) = \partial\psi_2$.

Inductively then we show that $\beta_k(\phi) = 0$ in $H^{q-k}(V, \Omega^{p+k+1})$ because $\beta_k(\phi) = \partial\psi_k$ for some $\psi_k \in A^{p+k,q-k}(V)$. At the last step, $\beta_q(\phi) \equiv 0$ because no holomorphic form on V can be ∂-exact. This completes the proof of (A.9), and hence of Theorem A.6. □

Examples. For $q = 0$, the sequence (A.7) becomes

$$0 \to H^0(V, \Omega_c^p) \to H^0(V, \Omega^p) \to 0, \tag{A.11}$$

which says that *every holomorphic p-form on V is closed* (theorem of Hodge).

For $p = 0$, (A.10) becomes:

$$0 \to H^{q-1}(V, \Omega_c^1) \to H^q(V, \mathbf{C}) \xrightarrow{\alpha} H^q(V, \mathcal{O}) \to 0, \tag{A1.12}$$

and α is just the projection onto $H_{\bar{\partial}}^{0,q}(V) \cong H^q(V, \mathcal{O})$ of a class $\phi \in$ **187**
$H^q(V, \mathbf{C})$. In particular, for $q = 1$, we have:

$$0 \to H^0(V, \Omega^1) \to H^1(V, \mathbf{C}) \to H^1(V, \mathcal{O}) \to 0. \tag{A.13}$$

As a final example, we let $H^1(V, \mathcal{O}^*)$ be the group of line bundles on V. Then we have a diagram

$$\begin{tikzcd}
0 \arrow[r] & H^0(V,\Omega^2) \arrow[r] & H^1(V,\Omega_c^1) \arrow[r] & H^1(V,\Omega^1) \arrow[r] & 0 \\
& & H^1(V,\mathcal{O}^*) \arrow[u, "d\log"] \arrow[ur, dashed, "c_1"'] & &
\end{tikzcd} \tag{A1.14}$$

(here c_1 is the usual *Chern class mapping*).

(b) What we want to show now is that there are natural injections

$$0 \to H^q(V, \Omega_c^p) \xrightarrow{\Delta} H^{p+q}(V, \mathbf{C}) \tag{A1.15}$$

such that

(i) the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & H^{p+q}(V,\mathbf{C}) \arrow[r, equal] & H^{p+q}(V,\mathbf{C}) \\
& \vdots \arrow[u, "\delta"] & \vdots \arrow[u, equal] \\
0 \arrow[r] & H^q(V,\Omega_c^p) \arrow[u, "\delta"] \arrow[r, "\Delta"] & H^{p+q}(V,\mathbf{C}) \arrow[u, equal] \\
0 \arrow[r] & H^{q-1}(V,\Omega_c^{p+1}) \arrow[u, "\delta"] \arrow[r, "\Delta"] & H^{p+q}(V,\mathbf{C}) \arrow[u, equal] \\
& \vdots \arrow[u, "\delta"] & \vdots \arrow[u, equal] \\
0 \arrow[r] & H^0(V,\Omega_c^{p+q}) \arrow[u, "\delta"] \arrow[r, "\Delta"] & H^{p+q}(V,\mathbf{C}); \arrow[u, equal] \\
& 0 \arrow[u] &
\end{tikzcd} \tag{A1.16}$$

188 (ii) the following diagram commutes:

$$\begin{tikzcd} H^q(V,\Omega_c^p) \arrow[r, "\Delta"] \arrow[d] & H^{p+q}(V,\mathbf{C}) \arrow[d] \\ H^q(V,\Omega^p) \arrow[r] & \mathbf{H}_{\bar\partial}^{p,q}(V), \end{tikzcd} \tag{A.17}$$

where $\mathbf{H}_{\bar\partial}^{p,q}(V)$ is the space of harmonic (p,q) forms;

(iii) In the filtration $\{F_m^{p+q}(V)\}$ of $H^{p+q}(V,\mathbf{C})$ arising from the spectral sequence of (A.1), $F_p^{p+q}(V)$ is the image of $H^q(V,\Omega_c^p)$; and is represented by a d-closed form $\phi \in \sum_{r\geqslant 0} A^{p+r,q-r}(V)$ defined modulo $d\psi$ where

$$\psi \in \sum_{r\geqslant 0} A^{p+r,q-r-1}(V) \quad \text{(c.f. (A.4))}. \tag{A.18}$$

Proof of (i). This is essentially a tautology; the vertical maps δ are injections by (A.7), and so the requirement of commutativity defines $\Delta : H^q(V,\Omega_c^p) \to H^{p+q}(V,\mathbf{C})$. For later use, it will be convenient to have a prescription for finding Δ, both in Cêch theory and using deRham, and so we now do this.

Let then $\{U_\alpha\}$ be a suitable covering of V with nerve $\mathfrak{U}$ and let $\phi \in H^q(V,\Omega_c^p)$. Then ϕ is defined by $\phi \in Z^q(\mathfrak{U},\Omega_c^p)$, and $\phi = d\psi_1$ for some $\psi_1 \in C^q(\mathfrak{U},\Omega^{p-1})$. Now $d\delta\psi_1 = \delta d\psi_1 = \delta\phi = 0$ so that $\phi_1 = \delta\psi_1 \in Z^{q+1}(\mathfrak{U},\Omega_c^{p-1})$. In fact, $\phi_1 = \delta(\phi)$ in (A1.16). Continuing, we get $\phi_2 \in Z^{q+2}(\mathfrak{U},\Omega_c^{p-2}),\ldots,$ on up to $\phi_p \in Z^{p+q}(\mathfrak{U},\mathbf{C})(\mathbf{C} = \Omega_c^0)$, where $\phi = \phi_0$, $\phi_k = \partial\psi_k$ with $\phi_{k-1} = d\psi_k(\psi_k \in C^{q+k-1}(\mathfrak{U},\Omega^{p-k}))$, and then $\Delta(\phi) = \phi_p$.

To find the deRham prescription for Δ, we let $A^{s,t}$ be the sheaf of C^∞ forms of type (s,t) on V and $B^{s,t} = \sum_{r\geqslant 0} A^{s+r,t-r}$. Also, $B_c^{s,t}$ will be the closed forms. Then $dB^{s,t} \subset B_c^{s,t+1}$, and we claim that we have exact sheaf sequences:

$$0 \to B_c^{s,t} \to B^{s,t} \xrightarrow{d} B_c^{s,t+1} \to 0. \tag{A1.19}$$

189 *Proof.* Let ϕ be a germ in $B_c^{s,t+1}$ and write $\phi = \sum_{r \geqslant 0} \phi_{s+r,t+1-r}$. Since $d\phi = 0$, $\overline{\partial}\phi_{s,t+1} = 0$ and so $\phi_{s,t+1} = \overline{\partial}\psi_{s,t}$. Then $\phi - d\psi_{s,t} \in B_c^{s+1,t}$, and continuing we find $\psi_{s,t}, \ldots, \psi_{s+t,0}$ with $\phi - d(\psi_{s,t} + \cdots + \psi_{s+t,0}) \in B_c^{s+t+1,0}$. But then $\phi - d(\psi_{s,t} + \cdots + \psi_{s+t,0})$ is a closed holomorphic $s+t+1$-form, and so $\phi - d(\psi_{s,t} + \cdots + \psi_{s+t,0}) = d\eta_{s+t,0}$; i.e. d is onto in (A1.19), which was to be shown.

The exact cohomology sequence of (A1.19) gives:

$$\left.\begin{array}{l} 0 \to H^r(V, B_c^{s,t+1}) \to H^{r+1}(V, B_c^{s,t}) \to 0 \quad (r \geqslant 1); \\ 0 \to H^0(V, B_c^{s,t+1})/dH^0(V, B^{s,t}) \to H^1(V, B_c^{s,t}) \to 0. \end{array}\right\} \tag{A.20}$$

Using these, we find the following diagram:

$$\begin{array}{ccc} H^q(V, \Omega_c^p) = H^q(V, B_c^{p,0}) & & \\ \| \wr & & \\ H^{q-1}(V, B_c^{p,1}) & & \\ \| \wr & & \\ \vdots & & \\ \| \wr & & \\ H^1(V, B_c^{p,q-1}) & \cong & H^0(V, B_c^{p,q})/dH^0(V, B^{p,q-1}); \end{array} \tag{A.21}$$

the composite in (A.21) gives

$$0 \to H^q(V, \Omega_c^p) \xrightarrow{\Delta} B_c^{p,q}(V)/dB^{p,q-1}(V) \to 0. \tag{A.22}$$

This Δ is just the deRham description of Δ in (A1.16), and by writing down (A.22) we have proved (iii) above. □

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STANDARD CONJECTURES ON ALGEBRAIC CYCLES

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1 Introduction. We state two conjectures on algebraic cycles, which arose from an attempt at understanding the conjectures of Weil on the ζ-functions of algebraic varieties. These are not really new, and they were worked out about three years ago independently by Bombieri and myself.

The first is an existence assertion for algebraic cycles (considerably weaker than the Tate conjectures), and is inspired by and formally analogous to Lefschetz's structure theorem on the cohomology of a smooth projective variety over the complex field.

The second is a statement of positivity, generalising Weil's well-known positivity theorem in the theory of abelian varieties. It is formally analogous to the famous Hodge inequalities, and is in fact a consequence of these in characteristic zero.

What remains to be proved of Weil's conjectures ? Before stating our conjectures, let us recall what remains to be proved in respect of the Weil conjectures, when approached through l-adic cohomology.

Let $X/\mathbf{F}_q$ be a smooth irreducible projective variety of dimension n over the finite field $\overline{\mathbf{F}}_q$ with q elements, and l a prime different from the characteristic. It has then been proved by M. Artin and myself that the Z-function of X can be expressed as

$$\begin{aligned} Z(t) &= \frac{L'(t)}{L(t)}, \\ L(t) &= \frac{L_0(t)L_2(t)\dots L_{2n}(t)}{L_1(t)L_3(t)\dots L_{2n-1}(t)}, \\ L_i(t) &= \frac{1}{P_i(t)}, \end{aligned}$$

194 where $P_i(t) = t^{\dim H^i(\overline{X})} Q_i(t^{-1})$, Q_i being the characteristic polynomial of the action of the Frobenius endomorphism of X on $H^i(\overline{X})$ (here H^i stands for the i^{th} l-adic cohomology group and $\overline{X}$ is deduced from X by base extension to the algebraic closure of $\mathbf{F}_q$). But it has not been proved so far that

(a) the $P_i(t)$ have integral coefficients, independent of $l(\neq \text{ char } \mathbf{F})$;

(b) the eigenvalues of the Frobenius endomorphism on $H^i(\overline{X})$, i.e., the reciprocals of the roots of $P_i(t)$, are of absolute value $q^{i/2}$.

Our first conjecture meets question (a). The first and second together would, by an idea essentially due to Serre [4], imply (b).

2 A weak form of conjecture 1.

From now on, we work with varieties over a ground field k which is algebraically closed and of arbitrary characteristic. Then (a) leads to the following question: If f is an endomorphism of a variety X/k and $l \neq \text{ char } k$, f induces

$$f^i : H^i(X) \to H^i(x),$$

and each of these f^i has a characteristic polynomial. *Are the coefficients of these polynomials rational integers, and are they independent of l ?* When X is smooth and proper of dimension n, the same question is meaningful when f is replaced by any cycle of dimension n in $X \times X$, considered as an algebraic correspondence.

In characteristic zero, one sees that this is so by using integral cohomology. If char $k > 0$, one feels certain that this is so, but this has not been proved so far.

Let us fix for simplicity an isomorphism

$$_{l^\infty}k^* \simeq \mathbf{Q}_l/\mathbf{Z}_l \qquad \text{(a heresy!).}$$

We then have a map

$$\text{cl} : \mathscr{Z}^i(X) \otimes_{\mathbf{Z}} \mathbf{Q} \to H_l^{2i}(X)$$

which associates to an algebraic cycle its cohomology class. We denote the image by $C_l^i(X)$, and refer to its elements as *algebraic cohomology classes.*

A known result, due to Dwork-Faton, shows that for the integrality question (not to speak of the independence of the characteristic polynomial of l), it suffices to prove that **195**

$$\operatorname{Tr} f_i^N \in \frac{1}{m}\mathbf{Z} \quad \text{for every} \quad N \geqslant 0,$$

where m is a fixed positive integer*. Now, the graph Γ_{f^N} in $X \times X$ of f^N defines a cohomology class on $X \times X$, and if the cohomology class Δ of the diagonal in $X \times X$ is written as

$$\Delta = \sum_0^n \pi_i$$

where π_i are the projections of Δ onto $H^i(X) \otimes H^{n-i}(X)$ for the canonical decomposition $H^n(X \times X) \simeq \sum_{i=0}^{n} H^i(X) \otimes H^{n-i}(X)$, a known calculation shows that

$$\operatorname{Tr}(f^N)_{H^i} = (-1)^i \operatorname{cl}(\Gamma_{f^N})\pi_i \in H^{4n}(X \times X) \approx \mathbf{Q}_l.$$

Assume that the π_i are algebraic. Then $\pi_i = \dfrac{1}{m} \operatorname{cl}(\Pi_i)$, where Π_i is an algebraic cycle, hence

$$\operatorname{Tr}(f^N)_{H^i} = (-1)^i(\Pi_i \cdot \Gamma_{f^N}) \in \frac{1}{m}\mathbf{Z}$$

and we are through.

Weak form of Conjecture 1. $(C(X))$: The elements π_i^l are algebraic, (and come from an element of $\mathscr{Z}^i(X) \otimes_{\mathbf{Z}} \mathbf{Q}$, which is independent of l).

N.B. 1. The statement in parenthesis is needed to establish the independence of P_i on l.

*This was pointed out to me by S. Kleimann.

2. If $C(X)$ and $C(Y)$ hold, $C(X \times Y)$ holds, and more generally, the Künneth components of any algebraic cohomology class on $X \times Y$ are algebraic.

3 The conjecture 1 (of Lefschetz type).

Let X be smooth and projective, and $\xi \in H^2(X)$ the class of a hyperplane section. Then we have a homomorphism

$$\cup\xi^{n-i} : H^i(X) \to H^{2n-i}(X) \quad (i \leqslant n). \tag{*}$$

196 It is expected (and has been established by Lefschetz [2], [5] over the complex field by transcendental methods) that this is an isomorphism for all characteristics. For $i = 2j$, we have the commutative square

$$\begin{tikzcd} H^{2j}(X) \arrow[r, "\xi^{n-2j}"] & H^{2n-2j}(X) \\ C^j(X) \arrow[u] \arrow[r] & C^{n-j}(X) \arrow[u] \end{tikzcd}$$

Our conjecture is then: $(A(X))$:

(a) (*) *is always an isomorphism (the mild form);*

(b) *if $i = 2j$, (*) induces an isomorphism (or equivalently, an epimorphism) $C^j(X) \to C^{n-j}(X)$.*

N.B. If $C^j(X)$ is assumed to be finite dimensional, (b) is equivalent to the assertion that $\dim C^{n-j}(X) \leqslant \dim C^j(X)$ (which in particular implies the equality of these dimensions in view of (a).

An equivalent formulation of the above conjecture (for all varieties X as above) is the following.

$(B(X))$: *The Λ-operation (c.f. [5]) of Hodge theory is algebraic.*

By this, we mean that there is an algebraic cohomology class λ in $H^*(X \times X)$ such that the map $\Lambda : H^*(X) \to H^*(X)$ is got by lifting a class from X to $X \times X$ by the first projection, cupping with λ and taking the image in $H^*(X)$ by the Gysin homomorphism associated to the second projection.

Note that $B(X) \Rightarrow A(X)$, since the algebraicity of Λ implies that of Λ^{n-i}, and Λ^{n-i} provides an inverse to $\cup\xi^{n-i} : H^i(X) \to H^{2n-i}(X)$. On the other hand, it is easy to show that $A(X \times X) \Rightarrow B(X)$ and this proves the equivalence of conjectures A and B.

The conjecture seems to be most amenable in the form B. Note that $B(X)$ is stable for products, hyperplane sections and specialisations. In particular, since it holds for projective space, it is also true for smooth **197**
varieties which are complete intersections in some projective space. (As a consequence, we deduce for such varieties the wished-for integrality theorem for the Z-function !). It is also verified for Grassmannians, and for abelian varieties (Liebermann [3]).

I have an idea of a possible approach to Conjecture B, which relies in turn on certain unsolved geometric questions, and which should be settled in any case.

Finally, we have the implication $B(X) \Rightarrow C(X)$ (first part), since the π_i can be expressed as polynomials with coefficients in $\mathbf{Q}$ of Λ and $L = \cup\xi$. To get the whole of $C(X)$, one should naturally assume further that there is an element of $\mathscr{Z}(X \times X) \otimes_{\mathbf{Z}} \mathbf{Q}$ which gives Λ for every l.

4 Conjecture 2 (of Hodge type). For any $i \leqslant n$, let $P^i(X)$ be the 'primitive part' of $H^i(X)$, that is, the kernel of $\cup\xi^{n-i+1} : H^i(X) \to H^{2n-i+2}(X)$, and put $C^j_{\mathrm{Pr}}(X) = P^{2j} \cap C^j(X)$. On $C^j_{\mathrm{Pr}}(X)$, we have a $\mathbf{Q}$-valued symmetric bilinear form given by

$$(x, y) \to (-1)^j K(x \cdot y \cdot \xi^{n-2j})$$

where K stands for the isomorphism $H^{2n}(X) \simeq \mathbf{Q}_l$. Our conjecture is then that

$(Hdg(X))$: *The above form is positive definite.*

One is easily reduced to the case when $\dim X = 2m$ is even, and $j = m$.

Remarks. (1) In characteristic zero, this follows readily from Hodge theory [5].

(2) $B(X)$ and $Hdg(X \times X)$ imply, by certain arguments of Weil and Serre, the following: if f is an endomorphism of X such that $f^*(\xi) = q \cdot \xi$ for some $q \in \mathbf{Q}$ (which is necessarily > 0), then the eigenvalues of $f_{H^i(X)}$ are algebraic integers of absolute value $q^{i/2}$. Thus, this implies all of Weil's conjectures.

(3) The conjecture $Hdg(X)$ together with $A(X)(a)$ (the Lefschetz conjecture in cohomology) implies that numerical equivalence of cy-
198 cles is the same as cohomological equivalence for any l-adic cohomology if and only if $A(X)$ holds.

Thus, we see that in characteristic 0, the conjecture $A(X)$ is equivalent to the well-known conjecture on the equality of cohomological equivalence and numerical equivalence.

(4) In view of (3), $B(X)$ and $Hdg(X)$ imply that numerical equivalence of cycles coincides with $\mathbf{Q}_l$-equivalence for any l. Further the natural map

$$Z^i(X) \underset{\mathbf{Z}}{\otimes} \mathbf{Q}_l \to H^i_l(X)$$

is a monomorphism, and in particular, we have

$$\dim_{\mathbf{Q}} C^i(X) \leqslant \dim_{\mathbf{Q}_l} H^i_l(X).$$

Note that for the deduction of this, we do not make use of the positivity of the form considered in $Hdg(X)$, but only the fact that it is non-degenerate.

Another consequence of $Hdg(X)$ and $B(X)$ is that the stronger version of $B(X)$, viz. that Λ comes from an algebraic cycle with rational coefficients *independent of* l, holds.

Conclusions. The proof of the two standard conjectures would yield results going considerably further than Weil's conjectures. They would form the basis of the so-called "theory of motives" which is a systematic theory of “arithmetic properties” of algebraic varieties, as embodied in their groups of classes of cycles for numerical equivalence. We have at present only a very small part of this theory in dimension one, as contained in the theory of abelian varieties.

Alongside the problem of resolution of singularities, the proof of the standard conjectures seems to me to be the most urgent task in algebraic geometry.

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FORMAL LINE BUNDLES ALONG EXCEPTIONAL LOCI

By Heisuke Hironaka

201 **Introduction.** If A is a noetherian ring with an ideal I, then we define the I-adic Henselization to be the limit of all those subrings of $\widehat{A}$ which are étale over A, where $\widehat{A}$ denotes the I-adic completion of A. This notion naturally globalizes itself. Namely, if X is a noetherian scheme with a closed subscheme Y, then the *Henselization of X along Y* is the local-ringed space $\widetilde{X}$ with a structural morphism $h : \widetilde{X} \to X$ such that $|\widetilde{X}| = |Y|$ and $\underline{O}_{\widetilde{X}}(U) =$ the $I_Y(U)$-adic Henselization of $\underline{O}_X(U)$ for every open affine subset U of $|Y|$, where $|\ |$ denotes the underlying topological space and I_Y the ideal sheaf of Y in $\underline{O}_X$. If $\widehat{X}$ is the completion (a formal scheme) of X along Y with the structural morphism $f : \widehat{X} \to X$, there exists a unique morphism $g : \widehat{X} \to \widetilde{X}$ such that $f = hg$. In this article, I present some general techniques for "equivalences of homomorphisms" with special short accounts in various special cases, and then briefly sketch a proof of the following algebraizability theorem : *Let k be a perfect field and $\pi : X \to X_0$ a proper morphism of algebraic schemes over k. Let $\widetilde{X}$ (resp. $\widehat{X}$) be the Henselization (resp. completion) of X along $\pi^{-1}(Y_0)$ with a closed subscheme Y_0 of X_0. If π induces an isomorphism $X - \pi^{-1}(Y_0) \xrightarrow{\sim} X_0 - Y_0$, then the natural morphism $g : \widetilde{X} \to \widetilde{X}$ induces an isomorphism $g_* : R^1 p(\underline{O}^*_{\widetilde{X}}) \xrightarrow{\sim} R^1 p(\underline{O}^*_{\widehat{X}})$, where p denotes the continuous map from $|\widetilde{X}| = |\widehat{X}| = |\pi^{-1}(Y_0)|$ to $|Y_0|$ induced by π.* In other words, if z is a closed point of Y_0, every line bundle on $\widehat{X}$ in a neighborhood of $\pi^{-1}(z)$ is derived from a line bundle on $\widetilde{X}$ in some neighborhood of $\pi^{-1}(z)$.

202 A *Henselian scheme* is, by definition, a local-ringed space S with a coherent sheaf of ideals J such that $(|S|, \underline{O}_S/J)$ is a noetherian scheme and S is locally everywhere isomorphic to a Henselization of a noetherian scheme. Such J (resp. the corresponding subscheme) is called a

defining ideal sheaf (resp. subscheme) of S. If S is a scheme (resp. Henselian scheme, resp. formal scheme) and I a coherent sheaf of ideals on S, then the *birational blowing-up* $\pi : T \to S$ of I is defined in the category of schemes (resp. Henselian schemes, resp. formal schemes, where morphisms are those of local-ringed spaces) is defined to be the one which has the universal mapping property : (i) $I\underline{O}_T$ is invertible as $\underline{O}_T$-module, and (ii) if $\pi' : T' \to S$ is any morphism with the property (i) and with a scheme (resp. Henselian scheme, resp. formal scheme) T', there exists a unique morphism $b : T' \to T$ with $\pi' = \pi b$. One can prove the existence in those categories. Now, let T be any noetherian scheme (resp. Henselian scheme, resp. formal scheme), and Y a noetherian scheme with a closed embedding : $Y \subset T$. Let $p : Y \to Y_0$ be any proper morphism of schemes. Then the *birational blowing-down along* p (in the respective category) means a "proper" morphism $\pi : T \to S$ (in the respective category) together with as embedding $Y_0 \subset S$ such that there exists a coherent ideal sheaf J on S which has the following properties : (1) $J \supset I^j$ for $j >> 0$, where I is the ideal sheaf of Y_0 in S, and (2) if $\alpha : T' \to T$ and $\beta : S' \to S$ are the birational blowing-up of the ideal sheaves $J\underline{O}_T$ and J, respectively, then the natural morphism $T' \to S'$ is an isomorphism. Now, given a noetherian scheme X and a closed subscheme Y of X, we let $\widetilde{X}$ (resp. $\widehat{X}$) denote the Henselization (resp. completion) of X along Y. Let $p : Y \to Y_0$ be a proper morphism of noetherian schemes. We then propose the following problem: *If there exists a birational blowing-down of* $\widehat{X}$ *along* p *in the category of formal schemes, does there follow the same of* $\widetilde{X}$ *along* p *in the category of Henselian schemes ?* For simplicity, let us consider the case in which X, Y, Y_0 are all algebraic schemes over a perfect field k. In this case, we can prove that, $\widehat{X} \to S$ being the formal birational blowing-
down, the Henselian blowing-down exists if and only if S is locally 203
everywhere algebraizable, i.e. isomorphic to completions of algebraic schemes over k. Clearly the problem is local in S. Suppose we have an algebraic scheme X_0 containing Y_0 in such a way that S is isomorphic to the completion of X_0 along Y_0. By a somewhat refined Chow's Lemma we may assume that there exists an ideal sheaf $\widehat{J}$ in S which contains $\widehat{I}^j$ for $j >> 0$, where $\widehat{I}$ = the ideal sheaf of Y_0 in S, and such

that $\widehat{X} \to S$ is the birational blowing-up of $\widehat{J}$. Clearly, $\widehat{J}$ is induced by an ideal sheaf J on X_0. Let $X' \to X_0$ be the birational blowing-up of J, and $\widetilde{X}$ (resp. $\widetilde{X}'$) the Henselization of X (resp. X') along Y (resp. the inverse image of Y_0). Then we can prove that $\widetilde{X}$ is isomorphic to $\widetilde{X}'$, using the above algebraizability theorem of line bundles and some techiniques of "equivalence of embeddings along exceptional subschemes.". All the details in these regards will be presented elsewhere. I like to note that recently M. Artin obtained an outstanding theorem in regard to "*étale approximations*", which produced a substantial progress in the above Blowing-down Problem as well as in many related problems.

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1 Derivatives of a map.

Let R be a commutative ring with unity, let A and B be two associative R-algebras with unity, and let E be an $(A, B)_R$-module, i.e. a left A-and right B-module in which the actions of A and B induce the same R-module structure. Then an endomorphism τ of E as an abelian group will be called a *$(A, B)_R$-module derivation* of E (into itself) if there exist ring derivations in the usual sense, say α (resp. β, resp. d) of A (resp. B, resp. R) into itself, such that

1.1.1 $\tau(aeb) = \alpha(a)eb + a\tau(e)b + ae\beta(b)$ for all $a \in A$, $e \in E$ and $b \in B$, and

1.1.2 $\alpha(ra) = d(r)a + r\alpha(a)$ and $\beta(rb) = d(r)b + r\beta(b)$ for all $r \in R$, $a \in A$ and $b \in B$.

204 **Remark 1.2.** If the actions of R, A, and B upon E are all faithful, then τ determines all the other α, β and d. In any case, under the conditions, we say that α, β and d are compatible with τ.

We are interested in applying the above definition to the following situation. Let L and L' be two R-modules, let $A = \mathrm{End}_R(L)$ and $B = \mathrm{Eng}_R(L')$, and let $E = \mathrm{Hom}_R(L', L)$. If D is an R-module of ring derivations of R into itself, then we obtain an R-module, denoted by

$\mathrm{Der}_D(L', L)$, which consists of all the $(A, B)_R$-module derivations of E which are compatible with the derivations in D. If L' and L are both finite free R-modules with fixed free bases, say $b' = (b'_1, \ldots, b'_r)$ and $b = (b_1, \ldots, b_s)$ respectively, then we can give explicit presentations to all the elements of $\mathrm{Der}_D(L', L)$. Namely, if $d \in D$ and $f = ((f_{ij})) \in E$, then we let $d_{b,b'}(f) = ((df_{ij})) \in E$, where the matrix presentation $((\quad))$ is given by means of the free bases b and b'. The following fact is then immediate from a well-known theorem about ring derivations of a full matrix algebra.

Theorem 1.3. *Let L, L', b, b', A, B and D be the same as above. Then every $(A, B)_R$-module derivation τ of $\mathrm{Hom}_R(L', L)$, compatible with $d \in D$, can be written as follows:*

$$\tau(e) = d_{b,b'}(e) + a_0 e - e b_0$$

for all $e \in \mathrm{Hom}_R(L', L)$, where $a_0 \in A$ and $b_0 \in B$.

As is easily seen, if we define $\alpha(a) = d_{b,b}(a) + a_0 a - a a_0$ for all $a \in A$ and $\beta(b) = d_{b',b'}(b) + b_0 b - b b_0$ for all $b \in B$, then α (resp. β) is ring derivation of A (resp. B) which is compatible with τ.

From now on, we assume that R is noetherian. Given a homomorphism $f : F' \to F$, of finite R-modules, we consider various *permissible squares* (p, α, β, f) *over* f, i.e. $p : L' \to L$, $\alpha : L' \to F'$ and $\beta : L \to F$ such that $\beta p = f \alpha$, that both α and β are surjective, that both L' and L are finite free R-modules and that $p(\mathrm{Ker}(\alpha)) = \mathrm{Ker}(\beta)$. We can prove

Theorem 1.4. *Let R, $f : F' \to F$, and D be the same as above. Let $h : F \to \overline{F}$ be a homomorphism of R-modules. Then there exists an R-submodule $B = B(f, h, D)$ of $\mathrm{Hom}_R(F', \overline{F})$ such that for every per-* 205
missible square (p, α, β, f) over f as above,

$$B = \alpha^{*-1}(h\beta)_*(\mathrm{Der}_D(L', L)p)$$

where $\alpha^* : \mathrm{Hom}_R(F', \overline{F}) \to \mathrm{Hom}_R(L', \overline{F})$ is induced by α, $(h\beta)_* : \mathrm{Hom}_R(L', L) \to \mathrm{Hom}_R(L', \overline{F})$ induced by $h\beta$, and $\mathrm{Der}_D(L', L)p$ is the R-submodule of $\mathrm{Hom}_R(L', L)$ of all the derivatives of p by the elements of $\mathrm{Der}_D(L', L)$.

Definition 1.5. If f, D and h are the same as in (1.4), then we define the obstruction module for (f,h,D) to be

$$T_R(f,h,D) = K(f,h)/B(f,h,D) \cap K(f,h)$$

where $K(f,h) = \mathrm{Ker}(\mathrm{Hom}_R(F',\overline{F}) \to \mathrm{Hom}_R(\mathrm{Ker}(f),\overline{F})$.

We shall be particularly interested in the two special cases, the one in which h is the natural homomorphism $F \to \mathrm{Coker}(f)$ and the other in which h is the identity automorphism of F. We write $T_R(f,D)$ for $T_R(f,h,D)$ in the former special case, and $T_R^*(f,D)$ in the latter. We also write $T_R(f)$ for $T_R(f,(0))$, and $T_R^*(f)$ for $T_R^*(f,(0))$.

Remark 1.5.1. It is easy to prove that if $hf = 0$, then $B(f,h,D)$ is contained in $K(f,h)$.

Remark 1.5.2. Assume that both F and F' are free. In virtue of (1.3), one can then find a canonical isomorphism

$$T_R(f) \xrightarrow{\sim} \mathrm{Ext}^1_R(E,E),$$

where $E = \mathrm{Coker}(f)$. Moreover, one can find a homomorphism of D into $\mathrm{Ext}^1_R(E,E)$ whose cokernel is $T_R(f,D)$. In particular, if $E = R/J$ with an ideal J, we have a canonical homomorphism $\beta : D \to \mathrm{Ext}^1_R(E,E)$ having the property. Namely, there is a canonical isomorphism $\mathrm{Hom}_R(J,E) \xrightarrow{\sim} \mathrm{Ext}^1_R(E,E)$, and an element of D induces an R-homomorphism from J to R/J. The canonical isomorphism $T_R(f,D) \xrightarrow{\sim} \mathrm{Coker}(\beta)$ is then induced by the obvious epimorphism

$$K(f,h) \to \mathrm{Ext}^1_R(E,E).$$

206 **Remark 1.5.3.** Let (p,α,β,f) be a permissible square over f as before. Then β induces an isomorphism from $\mathrm{Coker}(p)$ to $\mathrm{Coker}(f)$. If E denotes this cokernel, α induces a homomorphism from $\mathrm{Hom}_R(F',E)$ to $\mathrm{Hom}_R(L',E)$. We can prove that this homomorphism induces a monomorphism $m : T_R(f,D) \to T_R(p,D)$ and that m is an isomorphism if F and F' are projective. Let us say that two homomorphisms $f_i : F'_i \to F_i (i = 1,2)$ are *equivalent* to each other if they admit permissible

squares $(p, \alpha_i, \beta_i, f_i)$ with the same $p : L' \to L$. Let C be an equivalence class of such homomorphisms. Then $T_R(f, D)$ with $f \in C$ is independent of $f : F' \to F$, so long as F and F' are projective.

Remark 1.5.4. If x is a prime ideal in R (or any multiplicatively closed subset of R), then D generates an R_x-module of derivations of R_x into itself. Let D_x denote this module. Let $f_x : F'_x \to F_x$ and $h_x : F_x \to \overline{F}_x$ denote the localizations of f and h respectively. Then there exists a canonical isomorphism :

$$T_R(f, h, D)_x (= T_R(f, h, D) \otimes R_x) \xrightarrow{\sim} T_{R_x}(f_x, h_x, D_x).$$

Remark 1.5.5. Assume that both F and F' are free. Let $N(f) = \{\lambda \in \mathrm{Hom}_R(L', L) | \lambda(\mathrm{Ker}(f)) \subset \mathrm{Im}(f)\}$. Then the natural homomorphism $h : F \to E = \mathrm{Coker}(f)$ induces an epimorphism $N(f) \to K(f, h)$. This then induces an isomorphism $N(f)/B(f, \mathrm{id}, D) \to T_R(f, D)$. As $N(f) \supset K(f, \mathrm{id})$, we get a monomorphism $\omega : T^*_R(f, D) \to T_R(f, D)$ in general.

Remark 1.5.6. Let

$$F' \xrightarrow{f} F \xrightarrow{f_{r-1}} F_{r-2} \xrightarrow{f_{r-2}} \ldots \to F_0 \xrightarrow{f_0} G \to 0$$

be an exact sequence of R-modules, where r is an integer > 1 and all the F's (i.e., F', F, F_i, $0 \leqslant i \leqslant r-2$) are free. Take the case of $D = (0)$. Then we get a canonical isomorphism $T^*_R(f) \to \mathrm{Im}(\mathrm{Ext}^1_R(E, F) \to \mathrm{Ext}^1_R(E, E))$, with $E = \mathrm{Coker}(f)$, and the monomorphism ω of (1.5.5) in this case is nothing but inclusion into $\mathrm{Ext}^1_R(E, E)$ with respect to the isomorphism of (1.5.2). Moreover, we get a canonical isomorphism **207**
$T^*_R(f) \xrightarrow{\sim} \mathrm{Im}(\mathrm{Ext}^r_R(G, F) \to \mathrm{Ext}^r_R(G, E))$.

Remark 1.5.7. Let C be an equivalence class of homomorphisms in the sense of (1.5.3). Then, for any two f_1 and f_2 belonging to C, there exists a canonical isomorphism from $T^*_R(f_1, D)$ to $T^*_R(f_2, D)$, i.e. $T^*_R(f)$ with $f \in C$ is uniquely determined by C provided the f_i and f are homomorphisms of projective R-modules.

2 Two equivalence theorems of homomorphisms.

Let R be a Zariski ring with an ideal of definition H, and D an R-module of ring derivations of R. Let us assume :

2.1 There is given a group of ring automorphisms of R, denoted by (D), such that for every integer $j > 1$ and every $d \in H^j D$, there exists $\lambda \in (D)$ with $\lambda(r) \equiv r + d(r) \bmod H^{2j}$ for all $r \in R$.

When R is complete, $(\widehat{D})$ denotes the closure of (D) in $\mathrm{Aut}(R)$ with respect to the H-adic congruence topology.

If f and f' are two homomorphisms of R-modules from F' to F, then we ask whether there exists a (D)-*equivalence from* f *to* f', i.e. a triple (λ, α, β) with $\lambda \in (D)$ and λ-automorphisms α and β of F and F', respectively, such that $f'\alpha = \beta f$.

Let F be a finite R-module. We say that F is (D)-*rigid* (with respect to the H-adic topology in R) if the following condition is satisfied :

2.2 Let $b : L \to F$ be any epimorphism of R-modules with a finite free R-module L. Then one can find a pair of nonnegative integers (r_0, t_0) such that if α is a λ-automorphism of L, $\equiv \mathrm{id}_L \bmod H^j L$ with $j \geqslant t_0$ and with $\lambda \in (D)$, then there exists an R-automorphism α' of L, $\equiv \mathrm{id}_L \bmod H^{j-r_0} L$, such that $\alpha'\alpha$ induces a λ-automorphism of F, i.e. $\alpha'\alpha(\mathrm{Ker}(b)) = \mathrm{Ker}(b)$.

Note that the (D)-rigidity is trivial if (D) consists of only the identity. As to the other nontrivial cases, we have the following useful sufficient condition : If F is locally free on $\mathrm{Spec}(R) - \mathrm{Spec}(R/H)$, then it is (D)-rigid for any (D). In fact, we can prove

208 **Theorem 2.3.** *Let* $X = \mathrm{Spec}(R)$ *and* $Y = \mathrm{Spec}(R/H)$. *Let* L *be a finite* R*-module, locally free on* $X - Y$, *and* K *a submodule of* L *such that* L/K *is locally free on* $X - Y$. *Then there exists a pair of nonnegative integers* (t_0, r_0) *which has the following property. Let* K' *be any submodule of* L *such that* L/K' *is locally free on* $X - Y$ *and that* $\mathrm{rank}((L/K')_x) \geqslant \mathrm{rank}((L/K)_x)$ *for every* $x \in X - Y$. *If* $K' \equiv K \bmod H^j L$ *with* $j \geqslant t_0$, *then there exists an automorphism* σ *of the* R*-module* L *such that* $\sigma \equiv \mathrm{id}_L \bmod H^{j-r_0} L$ *and* $\sigma(K) = K'$.

We have two types of equivalence criteria, the one in terms of the obstruction module $T_R(f, D)$ and the other in terms of $T_R^*(f, D)$. Each of the two serves better than the other, depending upon the type of applications, as will be seen in the next section.

Equivalence Theorem I. *Let us assume that R is complete. Let $f : F' \to F$ be a homomorphism of finite R-modules, such that*

(i) *both F and F' are (D)-rigid, and*

(ii) $H^c T_R(f, D) = (0)$ *for all* $c >> 0$.

Then there exists a triple of nonnegative integers (s, t, r) which has the following property. Let us pick any integer $j \geqslant t$ and any homomorphism $f' : F' \to F$ such that

(a) $\mathrm{Ker}(f) \subset \mathrm{Ker}(f') + H^s F'$, *and*

(b) $f' \equiv f \bmod H^j F$.

Then there exists a $(\widehat{D})$-equivalence from f to f' which is congruent to the identity mod H^{j-r}.

The last congruence means, of course, that if (λ, α, β) is the (D)-equivalence then $\lambda \equiv \mathrm{id}_R \bmod H^{j-r}$, $\alpha \equiv \mathrm{id}_F \bmod H^{j-r}F$ and $\beta \equiv \mathrm{id}_{F'} \bmod H^{j-r}F'$.

Equivalence Theorem II. *Let us assume that R is complete. Let $f : F' \to F$ be a homomorphism of finite R-modules, such that*

(i) *f is injective,*

(ii) *both F and F' are (D)-rigid, and*

(iii) $H^c T_R^*(f, D) = (0)$ *for all* $c >> 0$.

Then there exists a pair of nonnegative integers (t, r) which has the **209**
following property. Let us pick any integer $j \geqslant t$ and any homomorphism $f' : F' \to F$ such that $f' \equiv f \bmod H^j F$. Then there exists a $(\widehat{D})$-equivalence from f to f' which is congruent to the identity mod H^{j-r}.

Remark 2.4.1. If (D) consists of only id_R, then the (D)-equivalence is nothing but a pair of R-automorphisms α and β with the commutativity. In this case, the equivalence theorems hold without the completeness of R.

Remark 2.4.2. Let us assume that both F and F' are locally free on $\mathrm{Spec}(R) - \mathrm{Spec}(R/H)$. Let us say that two homomorphisms of finite R-modules $f_i : F'_i \to F_i$, $i = 1, 2$, are f-equivalent to each other if there exist epimorphisms of finite free R-modules $e_i : L'_i \to L_i$, an isomorphism $b : F_1 \oplus L_1 \to F_2 \oplus L_2$ and an isomorphism $b' : F'_1 \oplus L'_1 \to F'_2 \oplus L'_2$ such that $(f_2 \oplus e_2)b' = b(f_1 \oplus e_1)$. Let C be the f-equivalence class of the given map f. Then (s, t, r) of E.Th.I (resp. (t, r) of E.Th.II) can be so chosen to have the property of the theorem not only for the given f but also for every $f_1 : F'_1 \to F_1$ belonging to C (and satisfying (i) of E.Th.II).

Remark 2.4.3. The equivalence theorems can be modified in a somewhat technical fashion so as to become more useful in a certain type of application. To be precise, let q be any nonzero element of R which is not a zero divisor of $\mathrm{Coker}(f)_x$ for any point x of $\mathrm{Spec}(R) - \mathrm{Spec}(R/H)$. Then, under the same assumptions of the respective E. Th.'s, we can choose (s, t, r) (resp. (t, r)) in such a way that : If f' satisfies the stronger congruence $f' \equiv f \bmod qH^jF$, instead of mod H^jF, then we can find a $(\widehat{D})$-equivalence from f' to f, $\equiv \mathrm{id} \bmod qH^{j-r}$. This modification of the E.Th.'s is used in establishing certain equivalence by a dimension-inductive method in terms of hyperplane sections.

3 Examples of applications.

Example I (Equivalence of Singularities). Let k be a noetherian ring
210 (for instance, a field). Let $R_0 = k[x] = k[x_1, \ldots, x_N]$, a polynomial ring of N variables over k. (In what follows, R_0 may be replaced by a convergent power series over an algebraically closed complete valued field.) Let R_1 be a ring of fractions of R_0 with respect to a multiplicatively closed subset of R_0, and H_1 a non-unit ideal in R_1. Let R be the H_1-adic completion of R_1, and $H = H_1R_1$. Let J be an ideal in R,

let $X = \operatorname{Spec}(R/J)$, let $Y = \operatorname{Spec}(R/J + H)$ and $\pi : X \to S$ be the projection map with $S = \operatorname{Spec}(k)$. We assume:

3.1 $X - Y$ is formally S-smooth, i.e. for every point x of $X - Y$, the $d \times d$-minors of the jacobian $\partial(f_1, \ldots, f_m)/\partial(x_1, \ldots, x_N)$ generate the unit ideal in the local ring $O_{X,x}$, where $J = (f_1, \ldots, f_m)R$ and d is the codimension of X in $\operatorname{Spec}(R)$ at x. If $X' = \operatorname{Spec}(R/J')$ with another ideal J' in R, then we ask if there exists a k-automorphism σ of R which induces an isomorphism from X to X'. For this purpose, we pick and fix an exact sequence

$$L_2 \xrightarrow{g} L_1 \xrightarrow{f} R \xrightarrow{h} R/J \to 0 \tag{3.2}$$

where L_i are finite free R-modules for $i = 1, 2$, $\operatorname{Im}(f) = J$ and h is the natural homomorphism. Let D be the R-module of derivations of the k-algebra R, which is generated by $\partial/\partial x_1, \ldots, \partial/\partial x_N$. We apply our equivalence theorem to this D and the map f. As was seen in (1.5.2), we have a canonical homomorphism $\beta : D \to \operatorname{Ext}^1_R(R/J, R/J)$ and an isomorphism $T_R(f, D) \xrightarrow{\sim} \operatorname{Coker}(\beta)$. Thus the obstruction module $T_R(f, D)$ is seen to be independent of the choice of (g, f) in (3.2). Moreover, as is easily seen, (3.1) is equivalent to saying that the localization of β, or

$$\beta_x : D_x \to \operatorname{Ext}^1_R(R/J, R/J)_x \quad (= \operatorname{Hom}_R(J/J^2, R/J)_x),$$

is surjective for every point x of $X - Y$. Hence it is also equivalent to

$$H^c T_R(f, D) = (0) \quad \text{for all} \quad c >> 0. \tag{3.3}$$

Therefore the following is a special case of E.Th. I.

Theorem 3.3. *Let the assumptions be the same as above. Then there exists a triple of nonnegative integers (s, t, r) which has the following property. Let j be any integer $\geqslant t$, and let $g' : L_2 \to L_1$ and* 211
$f' : L_1 \to R$ be any pair of homomorphisms such that (a) $f'g' = 0$, (b) $g' \equiv g \bmod H^s L_1$ *and* (c) $f' \equiv f \bmod H^j$. *Then there exists an automorphism of the k-algebra R which induces an isomorphism from X to $X' = \operatorname{Spec}(R/\operatorname{Im}(f'))$ and which is congruent to the identity* $\bmod H^{j-r}$.

Example II (Equivalence of Vector Bundles) Let R be any noetherian Zariski ring with an ideal of definition H. (For instance, $R = (R_0/J_0)(1+H_0)^{-1}$ with any pair of ideals J_0 and H_0 in the ring R_0). Let $X = \mathrm{Spec}(R)$ and $Y = \mathrm{Spec}(R/H)$. Let V be a vector bundle on $X - Y$, or a locally free sheaf on $X - Y$. Then there exists a finite R-module E which generates V on $X - Y$. Let us fix an exact sequence

$$L_2 \xrightarrow{g} L_1 \xrightarrow{f} L_0 \xrightarrow{h} E \to 0 \tag{3.4}$$

where the L_i are all free R-modules ($i = 0, 1, 2$). We apply our equivalence theorem to f with $D = (0)$. We have $T_R(f) = \mathrm{Ext}^1_R(E, E)$ by (1.5.1). Since E is locally free on $X - Y$, $\mathrm{Ext}^1_R(E, E)_x = (0)$ for all $x \in X - Y$. This implies

$$H^c T_R(f) = (0) \quad \text{for all} \quad c >> 0. \tag{3.5}$$

Thus we get the following special case of E.Th. I.

Theorem 3.6. *Let the assumptions be the same as above. Then there exists a triple of nonnegative integers (s, t, r) which has the following property. Let j be any integer $\geqslant t$, and let $g' : L_2 \to L_1$ and $f' : L_1 \to L_0$ be any pair of homomorphisms such that* (a) $f'g' = 0$, (b) $g' \equiv g \bmod H^s L_1$ *and* (c) $f' \equiv f \bmod H^j L_0$. *Then there exists an automorphism of the R-module L_0 which induces an isomorphism from V to V', with the locally free sheaf V' on $X - Y$ generated by* $\mathrm{Coker}(f')$, *and which is congruent to the identity* mod $H^{j-r} L_0$.

Remark 3.7. An important common feature of Theorems 3.3 and 3.6 is that, when the singularity or the vector bundle is represented by an R-valued point (f, g) in the affine algebraic scheme defined by the simultaneous quadratic equations $fg = 0$ (in terms of fixed free bases of
212 the L_i), all the approximate points (with respect to the H-adic topology of R) in the scheme represent the same singularity or the same vector bundle respectively.

Example III. Let R be a regular Zariski ring with an ideal of definition H. Let $\overline{R} = R/J$ with an ideal J, and $\overline{H} = H\overline{R}$. Let $\overline{E}$ be a finite $\overline{R}$-module. Let $Z = \mathrm{Spec}(R)$, $X = \mathrm{Spec}(\overline{R})$ and $Y = \mathrm{Spec}(\overline{R}/\overline{H})$. Let us assume :

3.8 X is locally a complete intersection of codimension e in Z at every point of $X - Y$, and

3.9 $\overline{E}$ is locally free on $X - Y$.

Let us take a resolution of $\overline{E}$ as an R-module by finite free R-modules: $\to L_p \xrightarrow{f_p} L_{p-1} \xrightarrow{f_{p-1}} \dots \xrightarrow{f_1} L_0 \xrightarrow{f_0} \overline{E} \to 0$. Then, by 1.5.6, $T_R^*(f_p)$ is isomorphic to the image of the natural homomorphism $\mathrm{Ext}_R^p(\overline{E}, L_{p-1}) \to \mathrm{Ext}_R^p(\overline{E}, E)$ with $E = \mathrm{Coker}(f_p)$. By (3.8), $\mathrm{Ext}_R^p(\overline{R}, R)_x = 0$ if $p \neq e$, and $= \overline{E}_x$ if $p = e$, for all points x of $Z - Y$. Thus, by (3.9), we get

3.10 For every positive $p \neq e$, $H^c T_R^*(f_p) = 0$ for all $c >> 0$.

Let $F' = \mathrm{Im}(f_p)$, $F = L_{p-1}$ and $f : F' \to F$ the inclusion. Then $T_R^*(f)$ is isomorphic to $T_R^*(f_p)$, and the following is a special case of E. Th. II.

Theorem 3.11. *Let the assumptions be the same as above, and let p be a positive integer $\neq e$. Then there exists a pair of nonnegative integers (t, r) such that if $f' : L_p \to L_{p-1}$ is any homomorphism with* $\mathrm{Ker}(f') \supset \mathrm{Ker}(f)$ *and with $f' \equiv f \bmod H^j L_{p-1}$ for an integer $j \geqslant t$, then there exists an equivalence from f' to f which is congruent to* $\mathrm{id} \bmod H^{j-r}$.

Example IV. Let us further specialize the situation of Example III and examine the case of $p = e$. Namely, we take (R, H) of Example I and assume (3.1) in addition to (3.8) and (3.9). Let D be the same as in Example I. We can then prove that $T_R^*(f_e, D)_x = 0$ for all $x \in Z - Y$, in the following two special cases.

Case (a) $e = 1$ and $\overline{E}$ has rank 1 on $X - Y$. (Or the case of a line bundle **213**
on a sliced hypersurface.)

Case (b) $e = 2$ and $L_0 = R$, so that $\overline{E} = R/J$. (Or the case of singularity of embedding codimension two.)

Again, as a corollary of E.Th.II, we obtain an equivalence theorem for f_e in these two special cases, in which (t, r) has the same property as that of (3.11) except that “equivalence” must be replaced by “$\mathrm{Aut}_k(R)$-equivalence”.

Remark 3.12. The equivalence theorems in Examples III and IV give us the following rather strong algebraizability theorem in the above special cases. Let the notation be the same as above and as in Example I. Assume the situation of either Case (a) or Case (b). Suppose $\overline{E}$ admits a free resolution of finite length. (This is always so, if R is local and regular.) Then, for every positive integer j, we can find a finite R_0-module $\overline{E}_0$ and an automorphism λ of R, $\equiv \mathrm{id}_R \bmod H^j$, such that $\overline{E}_0 \underset{R_0}{\otimes} R$ and $\overline{E} \underset{\lambda}{\otimes} R$ are isomorphic to each other as R-modules, where $\otimes_\lambda$ denotes the tensor product over R as R is viewed as R-algebra by λ. In fact, we can prove the algebraizability of the homomorphism f_p (or, $(\widehat{D})$-equivalence from f_p to a homomorphism obtained by the base extension $R_0 \to R$) by an obvious descending induction on p.

4 An algebraizability theorem of line bundles.

Let k be a perfect field, and R_0 a local ring of an algebraic scheme over k at a closed point. Let $X_0 = \mathrm{Spec}(R_0)$. Let Y_0 be a closed subscheme of X_0 defined by an ideal H_0 in R_0. Let R be the H_0-adic completion of R_0, and R' the H_0-adic Henselization of R_0, i.e. the limit of those sub-rings of R which are étale over R_0. Let $X = \mathrm{Spec}(R)$, $X' = \mathrm{Spec}(R')$, $Y = \mathrm{Spec}(R/H)$ with $H = H_0R$, and $Y' = \mathrm{Spec}(R'/H')$ with $H' = H_0R'$. We have natural morphisms $c : X \to X'$ and $c' : X' \to X_0$, which induce isomorphisms $Y \xrightarrow{\sim} Y'$ and $Y' \xrightarrow{\sim} Y_0$. Note that every subscheme D of X with $|D| \subset |Y|$ has an isomorphic image in X' and in X_0, where $|\quad|$
214 denotes the point-set. The following is the algebraizability theorem of line bundles along the exceptional locus of a birational morphism.

Theorem 4.1. *Let $\pi' : X_1' \to X'$ be a proper morphism which induces an isomorphism $X_1' \to \pi'^{-1}(Y') \xrightarrow{\sim} X' \to Y'$. Let $\widehat{X}_1$ be the completion (a formal scheme) of X_1' along $\pi'^{-1}(Y')$, and $h' : \widehat{X}_1 \to X_1'$ the natural morphism. Then h' induces an isomorphism*

$$h'^* : \mathrm{Pic}(X_1') \xrightarrow{\sim} \mathrm{Pic}(\widehat{X}_1).$$

If $\pi : X_1 \to X$ is the morphism obtained from π' by the base extension $c : X \to X'$, then we have a natural morphism $h : \widehat{X}_1 \to X_1$. By the GFGA theory of Grothendieck, this induces an isomorphism $h^* : \mathrm{Pic}(X_1) \xrightarrow{\sim} \mathrm{Pic}(\widehat{X}_1)$. Hence, (4.1) amounts to saying that the natural morphism $g : X_1 \to X_1'$ induces an isomorphism

$$g^* : \mathrm{Pic}(X_1') \xrightarrow{\sim} \mathrm{Pic}(X_1). \tag{4.1.1}$$

Let $U' = X' - Y'$ and $U = X - Y$. It is not hard to see that, for each individual (R_0, H_0), (4.1) for all π' as above is equivalent to the following

Theorem 4.2. *The morphism $c : X \to X'$ induces an isomorphism $\lambda : \mathrm{Pic}(U') \xrightarrow{\sim} \mathrm{Pic}(U)$.*

In fact, we can easily prove:

(i) If $\omega \in \mathrm{Pic}(U)$, then there exists a finite R-submodule E of $H^0(\omega)$ which generates the sheaf ω.

(ii) If (ω, E) is as above and if $\omega = \lambda(\omega')$ with $\omega' \in \mathrm{Pic}(U')$, then there exists a finite R'-submodule E' of $H^0(\omega')$ such that E is isomorphic to $E' \underset{R'}{\times} R$.

Now, to see the equivalence of (4.1) and (4.2), all we need is the following "Cramer's rule".

Remark 4.3. Quite generally, let E be a finite R-module which is locally free of rank r in $X - Y$, and $\underline{E}$ the coherent sheaf on X generated by E. 215
Assume that $\mathrm{Supp}(\underline{E})$ is equal to the closure of $X - Y$ in X. Let us pick an epimorphism $\alpha : L \to E$ with a free R-module of rank p. Let D be the subscheme of X defined by the annihilator in R of the cokernel of the natural homomorphism $(\wedge^{p-r}\mathrm{Ker}(\alpha)) \otimes (\wedge^r L) \to \wedge^p L$. Let $\pi : X_1 \to X$ be the birational blowing-up with center D, and let $\underline{E}_1$ be the image of the natural homomorphism $\pi^*(\underline{E}) \to i_* i^*(\pi^*(\underline{E}))$, where i is the inclusion $X_1 - \pi^{-1}(Y) \to X_1$. Then π induces an isomorphism $X_1 - \pi^{-1}(Y) \xrightarrow{\sim} X - Y$, $\pi^*(\underline{E}) \to \underline{E}_1$ is isomorphic in $X_1 - \pi^{-1}(Y)$, and $\underline{E}_1$ is locally free of rank r throughout X_1. Moreover, D has an

isomorphic image D' in X' and if $\pi' : X'_1 \to X'$ is the birational blowing-up with center D', then π' satisfies the assumptions in (4.1). Note that a birational blowing-up and a base extension commute if the latter is flat.

Let E be the same as in (4.3). Let E'_i, $i = 1, 2$, be finite R'-modules such that we have an isomorphism from $E'_i \underset{R'}{\otimes} R$ to E for each i. Then there exists an isomorphism from E'_1 to E'_2. In fact, since R is R'-flat, we have a natural isomorphism

$$\mathrm{Hom}_{R'}(E'_1, E'_2) \underset{R'}{\otimes} R \to \mathrm{Hom}_R(E, E)$$

with reference to the given isomorphisms. This means that id_E is arbitrarily approximated by the image of an element of $\mathrm{Hom}_{R'}(E'_1, E'_2)$. But, as is easily seen, any good approximation of id_E is an isomorphism itself. since R is faithfully R'-flat, this proves the existence of an isomorphism from E'_1 to E'_1. Let us remark that this proves the injectivity of λ of (4.2). We can also deduce from this, without much difficulty, that g^* of (4.1.1) (and hence h'^* of (4.1)) is also injective.

The essence of the theorems is the surjectivity of g^*, or the same of λ. As is seen in the arguments given above, this surjectivity is equivalent to the following

Theorem 4.4. *Let E be a finite R-module which is invertible on $X - Y$. Then there exists a finite R'-module E' such that $E' \underset{R'}{\otimes} R$ is isomorphic to E.*

216 We shall now indicate the key points in proving these theorems. As a whole, the proof is a combination of induction on $n = \dim R$ and the reduction to the case of (3.12)-(a).

Remark 4.5. Let the assumptions be the same as in (4.1) and in the immediately following paragraph. Let N' be any coherent ideal sheaf on X'_1 with $N'^2 = 0$. Let $\widehat{N} = N' \underline{O}_{\widehat{X}_1}$. Let X'_2 (resp. $\widehat{X}_2$) be the subscheme of X'_1 (resp. $\widehat{X}_1$) defined by N' (resp. $\widehat{N}$). Then we have the following natural commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & N' \arrow[r] \arrow[d] & \underline{O}^*_{X'_1} \arrow[r] \arrow[d] & \underline{O}^*_{X'_2} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \widehat{N} \arrow[r] & \underline{O}^*_{\widehat{X}_1} \arrow[r] & \underline{O}^*_{\widehat{X}_2} \arrow[r] & 0
\end{tikzcd}$$

which yields the following exact and commutative diagram.

$$\begin{tikzcd}
\arrow[r] & H^1(N') \arrow[r] \arrow[d] & \mathrm{Pic}(X'_1) \arrow[r] \arrow[d, "a_1"] & \mathrm{Pic}(X'_2) \arrow[r] \arrow[d, "a_2"] & H^2(N') \arrow[r] \arrow[d] & {} \\
\arrow[r] & H^1(\widehat{N}) \arrow[r] & \mathrm{Pic}(\widehat{X}_1) \arrow[r] & \mathrm{Pic}(\widehat{X}_2) \arrow[r] & H^2(\widehat{N}) \arrow[r] & {}
\end{tikzcd} \tag{4.5.1}$$

We have natural isomorphisms $H^i(N')\underset{R'}{\times}R \to H^i(\widehat{N})$ because R is R'-flat. Since $\mathrm{Supp}(H^i(N')) \subset |Y|$ for all $i > 0$, $H^i(N') \to H^i(N)$ are isomorphisms for $i = 1, 2$. Hence the surjectivity of a_2 implies the same of a_1.

Remark 4.6. By the standard amalgamation technique, one can easily reduce the proof of either one of the three theorems to the case of $\dim Y < n = \dim X$. Assuming this, let us try to prove (4.1) by induction on n. Let d be any element of H_0 such that $\dim R_0/dR_0 < n$. Let $\widehat{X}_1(j)$ (resp. $X'_1(j)$) be the subscheme of $\widehat{X}_1$ (resp. X'_1) defined by the ideal sheaf generated by d^{j+1}. Let $h'_j : \widehat{X}_1(j) \to X'_1(j)$ be the natural morphism. By induction assumption, we have isomorphisms $(h'_j)^* : \mathrm{Pic}(X'_1(j)) \xrightarrow{\sim} \mathrm{Pic}(\widehat{X}_1(j))$ for all j. In view of the cohomology sequences of (4.5.1) adapted to these cases, as the cohomology of coherent sheaves (those nilideal sheaves) is computable by any fixed open **217**
affine covering, we get canonical isomorphisms:

$$\varprojlim_j \mathrm{Pic}(\widehat{X}_1(j)) \to \mathrm{Pic}(\widehat{X}_1) \quad \text{and} \quad \varprojlim_j \mathrm{Pic}(X'_1(j)) \to \mathrm{Pic}(\widetilde{X}_1),$$

where $\widetilde{X}_1 = \varprojlim_j X'_1(j)$. Therefore, the natural morphism $\widehat{X}_1 \to \widetilde{X}_1$

induces an isomorphism $\mathrm{Pic}(\widetilde{X}_1) \to \mathrm{Pic}(\widehat{X}_1)$. In short, to prove (4.1) (or any of the other theorems), we may replace H_0 by any dR_0 as above.

Remark 4.7. To prove (4.4), we may assume that R_0 is reduced and $\dim R_0/H_0 < n = \dim R_0$. (See (4.5) and (4.2)). We can choose a system $z = (z_1, \dots, z_{n+1})$ with $z_i \in R_0$ and an element $d \in k[z]$ such that if S_0 is the local ring of $\mathrm{Spec}(k[z])$ which is dominated by R_0, then

(i) R_0 is a finite S_0-module,

(ii) $d \in H_0$ and $\dim R_0/dR_0 < n$,

(iii) $V_0 - W_0$ is of pure dimension and k-smooth, where $V_0 = \mathrm{Spec}(S_0)$ and $W_0 = \mathrm{Spec}(S_0/dS_0)$, and

(iv) the natural morphism $X_0 \to V_0$ induces an isomorphism $X_0 - \widetilde{Y}_0 \to V_0 - W_0$, where $\widetilde{Y}_0$ denotes the preimage of W_0 in X_0. Now, by (4.6), the proof of (4.4) is reduced to the case of $H_0 = dR_0$ and, in view of (4.2), to the case in which $X_0 = V_0$ and X_0 is the closure of $X_0 - Y_0$. In short, to prove (4.4), we may assume that R_0 is a local ring of a hypersurface in an affine space over k and that $X_0 - Y_0$ is k-smooth and dense in X_0.

In this final situation, a proof of (4.4) can be derived from the algebraizability theorem for Case (a) of Ex. IV, or Remark 3.12. To see this, let T_0 be the local ring of the affine space of dimension $n+1$ which carries X_0, at the closed point of X_0. Let G_0 be the ideal in T_0 which corresponds to H_0 in R_0, T' the G_0-adic Henselization of T_0, and T the G_0-adic completion of T_0. Let $G = G_0 T$. Then the result of (3.12) (in
218 which R should be replaced by T) implies that, for every positive integer j, we can find an automorphism λ of T and a finite T_0-module E_0 such that $\lambda \equiv \mathrm{id}_T \bmod G^j$ and $E_0 \underset{T_0}{\otimes} T$ is isomorphic to $E \underset{\lambda}{\otimes} T$ as T-module, where E is view as T-module in an obvious way. Let $E'' = E_0 \underset{T_0}{\otimes} T'$. Let J' be the kernel of the epimorphism $T' \to R'$, and J'' the annihilator in T' of E''. Clearly $J'T$ is the annihilator of E. Thanks to the equivalence theorem (3.6), it is now sufficient to find an automorphism λ' of R', well

approximate to λ, such that $\lambda'(J'') = J'$. Namely, $E' = E'' \underset{\lambda'}{\otimes} T'$ then has the property of (4.4). The existence of λ' is easy enough to prove, because J' is generated by a single element whose gradient does not vanish at any point of $X' - Y'$. (This is essentially Hensel's lemma.)

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INVOLUTIONS AND SINGULARITIES

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Heinrich Behnke zum 70. *Geburtstag gewidmet.*

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1 Introduction. Let X be a compact oriented differentiable manifold without boundary of dimension $4k-1$ with $k \geqslant 1$. Let $T : X \to X$ be an orientation preserving fixed point free differentiable involution. In [7] an invariant $\alpha(X,T)$ was defined using a special case of the Atiyah-Bott-Singer fixed point theorem. If the disjoint union mX of m copies of X bounds a $4k$-dimensional compact oriented differentiable manifold N in such a way that T can be extended to an orientation preserving involution T_1 on N which may have fixed points, then

$$\alpha(X,T) = \frac{1}{m}(\tau(N,T_1) - \tau(\operatorname{Fix} T_1 \circ \operatorname{Fix} T_1)). \tag{1}$$

Here $\tau(N,T_1)$ is the signature of the quadratic form f_{T_1} defined over $H_{2k}(N,\mathbf{Q})$ by

$$f_{T_1}(x,y) = x \circ T_1 y$$

where "$\circ$" denotes the intersection number. $\tau(\operatorname{Fix} T_1 \circ \operatorname{Fix} T_1)$ is the signature of the "oriented self-intersection cobordism class" $\operatorname{Fix} T_1 \circ \operatorname{Fix} T_1$. According to Burdick [4] there exist N and T_1 with $m=2$.

In §2 we shall study a compact oriented manifold $\mathscr{D}$ whose boundary is $X - 2(X/T)$. This manifold $\mathscr{D}$ was first constructed by Dold [5]; we give a different description of it. Namely, $\mathscr{D}$ is a branched covering of degree 2 of $(X/T) \times I$, where I is the unit interval. The covering transformation is an orientation preserving involution T_1 of $\mathscr{D}$ which restricted to the boundary is T on X and the trivial involution on $2(X/T)$, and $\operatorname{Fix} T_1$ is the branching locus.

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We show that

$$\alpha(X,T) = \tau(\mathscr{D},T_1) = -\tau(\mathscr{D}),$$

where $\tau(\mathscr{D})$ is the signature of the $4k$-dimensional manifold $\mathscr{D}$. Thus **220**
$\alpha(X,T)$ is always an integer. The construction of $\mathscr{D}$ is closely related to Burdick's result on the oriented bordism group of B_{Z_2} and can in fact be used to prove it.

In [7] it was claimed that if X^{4k-1} is an integral homology sphere then $\tau(\mathscr{D}) = \pm\beta(X,T)$, where $\beta(X,T)$ is the Browder-Livesay invariant [3]. The proof was not carried through. It turns out that the definition of Browder-Livesay is also meaningful without assumptions on the homology of X. In §3 we shall prove

$$\beta(X,T) = -\tau(\mathscr{D}). \tag{3}$$

By (**??**), we obtain

$$\alpha(X,T) = \beta(X,T). \tag{4}$$

Looking at $\mathscr{D}$ as a branched covering of $(X/T) \times I$ has thus simplified considerably the proof of (4) envisaged in [7].

If $a = (a_0, a_1, \ldots, a_{2k}) \in \mathbf{Z}^{2k+1}$ with $a_j \geqslant 2$, then the affine algebraic variety

$$z_0^{a_0} + z_1^{a_1} + \cdots + z_{2k}^{a_{2k}} = 0 \tag{5}$$

has an isolated singularity at the origin whose "neighborhood boundary" is the Brieskorn manifold [1]

$$\Sigma_a^{4k-1} \subset C^{2k+1}$$

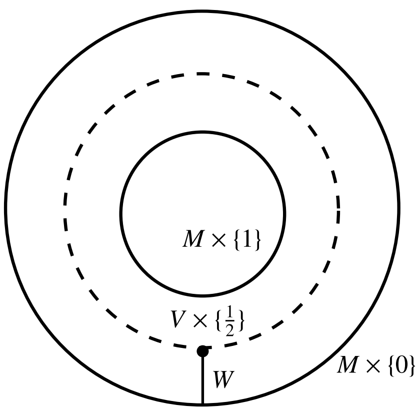
given by the equation (5) and

$$\sum_{i=0}^{2k} z_i \bar{z}_i = 1. \tag{6}$$

If all the a_j are odd, then $Tz = -z$ induces an orientation preserving fixed point free involution T_a on Σ_a. The calculation of $\alpha(\Sigma_a, T_a)$ is an open problem (compare [7]). This problem on isolated singularities is the justification for presenting our paper to a colloquium on algebraic geometry. In §4 we give the recipe for calculating $\alpha(\Sigma_a, T_a)$ for $k = 1$ in the case where the exponents a_0, a_1, a_2 are pairwise prime and odd.

221 **2 The Dold construction.** Let Y be a compact differentiable manifold without boundary and W a 1-codimensional compact submanifold with boundary ∂W. Then, as it is well known, one can construct a double covering of Y, branched at ∂W, by taking two copies of Y, "cutting" them along W and then identifying each boundary point of the cut in copy one with its opposite point in copy two. The same can be done if Y is a manifold with boundary and W intersects ∂Y transversally in a union of connected components of ∂W. The covering will then be branched at $\partial W - \partial W \cap \partial Y$.

We are interested in a very special case of this general situation. Let M be a compact differentiable manifold without boundary and V a closed submanifold without boundary of codimension 1 in M. Then we define $Y = M \times [0,1]$ and $W = V \times [0, \frac{1}{2}]$.



For the following we will need a detailed description of the double covering corresponding to $(M \times [0,1], V \times [0, \frac{1}{2}])$. The normal bundle of V in M defines a $\mathbf{Z}_2$-principal bundle $\widetilde{V}$ over V. If we "cut" M along V, we obtain a compact differentiable manifold C with boundary $\partial C = \widetilde{V}$. As a set, C is the disjoint union of $M - V$ and $\widetilde{V}$, and there is an obvious canonical way to introduce topology and differentiable structure in $(M - V) \cup \widetilde{V}$. Similarly, let C' be the disjoint union of $M \times [0,1] - V \times [0, \frac{1}{2})$ and $\widetilde{V} \times [0, \frac{1}{2})$, topologized in the canonical way. Then we consider

two copies C_1' and C_2' of C' and identify in their disjoint union each $x \in V \times \{\frac{1}{2}\} \subset C_1'$ with the corresponding point $x \in V \times \{\frac{1}{2}\} \subset C_2'$ and for $0 \leqslant t < \frac{1}{2}$ each point $v \in \widetilde{V} \times \{t\} \subset C_1'$ with the opposite point $-v \in \widetilde{V} \times \{t\} \subset C_2'$. Let $\mathscr{D}$ denote the resulting topological space **222**
and $\pi : \mathscr{D} \to M \times [0,1]$ the projection. Then C_1', C_2' and $V \times \{\frac{1}{2}\}$ are subspaces, and $\mathscr{D} - V \times \{\frac{1}{2}\}$ has a canonical structure as a differentiable manifold with boundary.

To introduce a differentiable structure on all of $\mathscr{D}$, we use a tubular neighbourhood of V in M. This may be given as a diffeomorphism

$$\kappa : \widetilde{V} \underset{Z_2}{\times} D^1 \to M$$

onto a closed neighbourhood of V in M, such that the restriction of κ to $\widetilde{V} \underset{Z_2}{\times} \{0\} = V$ is the inclusion $V \subset M$. Let Z_2 act on $D^2 \subset \mathbf{C}$ by complex conjugation. Then we get a tubular neighbourhood of $V \times \{\frac{1}{2}\}$ in $M \times [0,1]$

$$\begin{aligned} \lambda : \widetilde{V} \underset{Z_2}{\times} D^2 &\to M \times [0,1] \quad \text{by} \\ [v, x+iy] &\mapsto \left(\kappa(v,y), \frac{1}{2} + \frac{1}{4}x\right). \end{aligned} \tag{1}$$

Let the "projectin" $p : \widetilde{V} \underset{Z_2}{\times} D^2 \to \widetilde{V} \underset{Z_2}{\times} D^2$ be given on each fibre by $z \to z^2/|z|$. Then λp can be lifted to $\mathscr{D}$, which means that we can choose a map $\lambda_1 : \widetilde{V} \underset{Z_2}{\times} D^2 \to \mathscr{D}$ such that

$$\begin{tikzcd} \widetilde{V} \underset{Z_2}{\times} D^2 \arrow[r, "\lambda_1"] \arrow[d, "p"] & \mathscr{D} \arrow[d, "\pi"] \\ \widetilde{V} \underset{Z_2}{\times} D^2 \arrow[r, "\lambda"] & M \times [0,1] \end{tikzcd} \tag{2}$$

is commutative. Then there is exactly one differentiable structure on $\mathscr{D}$ for which λ_1 is a diffeomorphism onto a neighborhood of $V \times \{\frac{1}{2}\}$ in $\mathscr{D}$

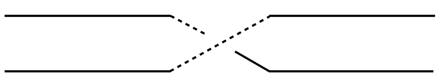
and which coincides on $\mathscr{D} - V \times \{\frac{1}{2}\}$ with the canonical structure. Up to diffeomorphism, of course, this structure does not depend on κ.

$\mathscr{D}$, then, is a double covering of $M \times [0, 1]$, branched at $V \times \{\frac{1}{2}\}$. The covering transformation on $\mathscr{D}$ shall be denoted by T_1. Note that on $\widetilde{V} \underset{Z_2}{\times} D^2$ (identified by λ_1 with a subset of $\mathscr{D}$) the transformation T_1 is given by $[v, z] \to [v, -z]$.

As a differentiable manifold, $\mathscr{D}$ is the same as the manifold constructed by Dold in his note [5].

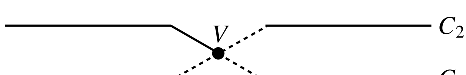
223 Now consider once more the differentiable manifold C with boundary $\partial C = \widetilde{V}$, which we obtained from M by cutting along V. Let $C_1 \cup C_2$ be the disjoint union of two copies of C. If we identify $x \in \widetilde{V}_1 \subset C_1$ with $-x \in \widetilde{V}_1 \subset C_1$ and $x \in \widetilde{V}_2$ with $-x \in \widetilde{V}_2$, we obtain from $C_1 \cup C_2$ the disjoint union of two copies of M :

If we identify $x \in \widetilde{V}_1 \subset C_1$ with $-x \in \widetilde{V}_2 \subset C_2$, we get a differentiable manifold which we denote by $\widetilde{M}$:



If we identify $x \in \widetilde{V}_1 \subset C_1$ with $x \in \widetilde{V}_2 \subset C_2$, then $C_1 \cup C_2$ becomes a closed manifold B (the usual "double" of C), and we use κ to introduce the differentiable structure on B :

If we, finally, identify for each $x \in \widetilde{V}$ all four points $x \in \widetilde{V}_1$, $-x \in \widetilde{V}_1$, $x \in \widetilde{V}_2$, $-x \in \widetilde{V}_2$ to one, then we obtain a topological space A :



(3)

Now obviously we have $2M = \pi^1(M \times \{1\})$, $A = \pi^{-1}(M \times \{\frac{1}{2}\})$ and $\widetilde{M} = \pi^{-1}(M \times \{0\})$, and by our choice of the differentiable structures of B and $\mathscr{D}$ (p in (2) is given by $z \to z^2/|z|$ instead of $z \to z^2$), the canonical map $B \to A$ defines an *immersion*

$$f : B \to \mathscr{D}.$$

It should be mentioned, perhaps, that for the same reason $\pi : \mathscr{D} \to M \times [0, 1]$ is not differentiable at $V \times \{\frac{1}{2}\}$.

Up to this point, we did not make any orientability assumptions. **224**
Considering now the "orientable case", we shall use the following convention : for orientable manifolds with boundary, we will always choose the orientations of the manifold and its boundary in such a way, that the orientation of the boundary, followed by the inwards pointing normal vector, gives the orientation of the manifold.

Now if X is any compact differentiable manifold without boundary and T a fixed point free involution on X with $X/T \cong M$, then (X, T) is equivariantly diffeomorphic to $(\widetilde{M}, T_1|\widetilde{M})$ for a suitably chosen $V \subset M$, and in fact our $(\widetilde{M}, T_1|\widetilde{M})$ plays the role of the (X, T) in §1. Therefore we will assume from now on, that $\widetilde{M}$ is *oriented* and $T_1|\widetilde{M}$ is *orientation preserving*. Let us also write T for $T_1|\widetilde{M}$.

Then the orientation of $\widetilde{M}$ defines an orientation of M and hence of C, and since $\widetilde{V} = \partial C$, an orientation of $\widetilde{V}$ is thus determined. Furthermore, the orientation of $\widetilde{M} \subset \partial\mathscr{D}$ induces an orientation of $\mathscr{D}$, relative to which

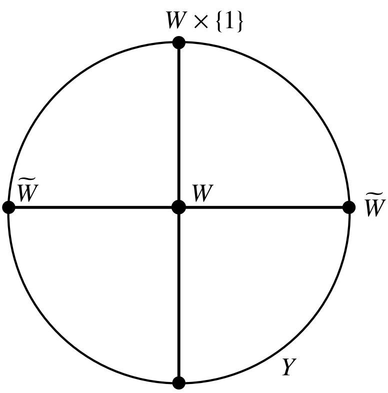
$$\partial\mathscr{D} = \widetilde{M} - 2M. \tag{4}$$

Clearly T_1 on $\mathscr{D}$ is orientation preserving, V may not be orientable, and $\widetilde{V} \to V$ is the orientation covering of V, because M is orientable.

The relation of the construction of $\mathscr{D}$ to the result of Burdick is the following. Let $\Omega_*(\mathbf{Z}_2)$ denote the bordism group of oriented manifolds with fixed point free orientation preserving involutions. Then we have homomorphisms

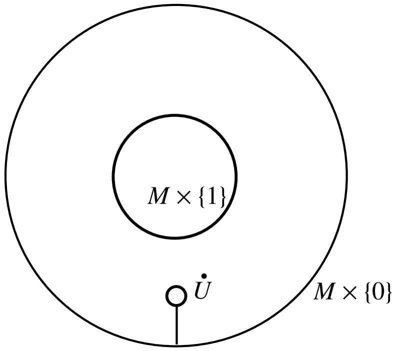
$$\Omega_n \oplus \mathfrak{N}_{n-1} \underset{j}{\overset{i}{\leftrightarrows}} \Omega_n(\mathbf{Z}_2)$$

as follows: if $[M] \in \Omega_n$ is represented by an oriented n-dimensional manifold M, the $i[M] \in \Omega_n(Z_2)$ is simply given by $2M$ with the trivial involution. Now let $[W]_2 \in \mathfrak{N}_{n-1}$ be represented by an $(n-1)$-dimensional manifold W, let $\widetilde{W} \to W$ denote the orientation covering, and let Y be the sphere bundle of the Whitney sum of the real line bundle over W associated to $\widetilde{W}$ and the trivial line bundle $W \times \mathbf{R}$:



225 The manifold Y is orientable, and we may orient Y at $\widetilde{W}$ by the canonical orientation of $\widetilde{W}$ followed by the normal vector pointing toward $W \times \{1\}$. Then we denote by $i(W)$ the oriented double covering of Y corresponding to $(Y, W \times \{1\})$, and we define $i[W] \in \Omega_n(\mathbf{Z}_2)$ to be represented by $i(W)$.

As already mentioned, any element of $\Omega_n(\mathbf{Z}_2)$ can be represented by the (unbranched) double covering $\widetilde{M}$ corresponding to some (M, V), and we define $j[\widetilde{W}, T] = [M] \oplus [V]_2$. Then i and j are well defined homomorphisms and clearly $j \circ i = \mathrm{Id}$, so i is injective. To show that i is also surjective, we have to construct for given (M, V) and $(n+1)$-dimensional oriented manifold $\mathscr{B}$ with boundary and with an orientation preserving fixed point free involution, such that equivariantly $\partial\mathscr{B} = \widetilde{M} - 2M - i(V)$. But such a manifold is given by $\mathscr{B} = \pi^{-1}(M \times [0,1] - U)$,



where U is the interior of the tubular neighborhood (1) of $V \times \{\frac{1}{2}\}$ in $M \times [0, 1$:

$$\begin{aligned}\partial\mathscr{B} &= \pi^{-1}(M \times \{0\}) \cup \pi^{-1}(M \times \{1\}) \cup \pi^{-1}(\dot{U}) \\ &= \widetilde{M} - 2M - i(V).\end{aligned}$$

Thus $i : \Omega_n \oplus \mathfrak{N}_{n-1} \to \Omega_n(\mathbf{Z}_2)$ is an isomorphism. Brudick uses in [4] **226**
essentially the same manifold $\mathscr{B}$ to prove the surjectivity of i.

We will now consider the invariant α and therefore assume that $\dim \widetilde{M} = 4k - 1$ with $k \geqslant 1$. First we remark, that for the trivial involution T on $2M$ the invariant α vanishes: since the nontrivial elements of Ω_{4k-1} are all of order two, there is an oriented X with $\partial X = 2M$. Let T' be the trivial involution on $2X$. Then $2\alpha(2M, T) = \tau(2X, T') = 0$, because it is the signature of a quadratic form which can be given by a matrix of the form

$$\begin{array}{|c|c|}\hline O & E \\ \hline E & O \\ \hline\end{array}$$

where E is a symmetric matrix. Hence it follows, that $\alpha(\partial\mathscr{D}, T_1|\partial\mathscr{D}) = \alpha(\widetilde{M}, T)$ and therefore by (1) of §1 we have

$$\alpha(\widetilde{M}, T) = \tau(\mathscr{D}, T_1) - \tau(\operatorname{Fix} T_1 \circ \operatorname{Fix} T_1). \tag{5}$$

Notice that here we apply the definition (1) of §1 of α in a case, where $\operatorname{Fix} T_1$ is not necessarily orientable, so that we have to use the Atiyah-Bott-Singer fixed point theorem also for the case of non-orientable fixed point sets.

Proposition. $\alpha(\widetilde{M}, T) = \tau(\mathscr{D}, T_1) = -\tau(\mathscr{D})$.

Proof. $\operatorname{Fix} T_1 \circ \operatorname{Fix} T_1 = 0 \in \Omega_*$, since the normal bundle of $\operatorname{Fix} T_1 = V \times \{\frac{1}{2}\}$ in $\mathscr{D}$ has a one-dimensional trivial subbundle. Therefore by (5), $\alpha(\widetilde{M}, T) = \tau(\mathscr{D}, T_1)$. To show that $\tau(\mathscr{D}, T_1) = -\tau(\mathscr{D})$, let again U denote our open tubular neighbourhood of $V \times \{\frac{1}{2}\}$ in $M \times [0,1]$, $\overline{U}$ its closure in $M \times [0,1]$ and correspondingly $U_1 = \pi^{-1}(U)$, $\overline{U}_1 = \pi^{-1}(\overline{U})$. Then $\tau(\overline{U}) = \tau(\overline{U}_1) = \tau(\overline{U}_1, T_1) = 0$, because $\overline{U}$ and $\overline{U}_1$ are disc bundles of vector bundles with a trivial summand and hence the zero section, which carries all the homology, can be deformed into a section
227 which is everywhere different from zero. Because of the additivity of the signature (compare (8) of [7]), we therefore have

$$\tau(\mathscr{D}, T_1) = \tau(\mathscr{D} - U_1, T_1).$$

But T_1 is fixed point free on $\mathscr{D} - U_1$, and hence we can apply formula (7) of [7], which is easy to prove and which relates the signature $\tau(M^{4k}, T)$ of a fixed point free involution with the signatures of M^{4k} and M^{4k}/T and we obtain

$$\begin{aligned} \tau(\mathscr{D}, T_1) = \tau(\mathscr{D} - U_1, T_1) &= 2\tau(M \times [0,1] - U) - \tau(\mathscr{D} - U_1) \\ &= 2\tau(M \times [0,1]) - \tau(\mathscr{D}). \end{aligned}$$

□

3 The Browder-Liversay invariant.

The involution on $\widetilde{V}$ which is given by $x \to -x$ shall also be denoted by T, because it is the restriction of T on $\widetilde{M}$ to $\widetilde{V}$, if we regard $\widetilde{V}$ via $\widetilde{V}_1 \subset C_1 \subset \widetilde{M}$ as a submanifold of $\widetilde{M}$. T is orientation reversing on $\widetilde{V}$, and since the intersection form $(x, y) \to x \circ y$ on $H_{2k-1}(\widetilde{V}, \mathbf{Q})$ is skew-symmetric, the quadratic form $(x, y) \to x \circ Ty$ is symmetric on $H_{2k-1}(\widetilde{V}, \mathbf{Q})$. Now we restrict this form to

$$L = \text{kernel of } H_{2k-1}(\widetilde{V}, \mathbf{Q}) \to H_{2k-1}(C, \mathbf{Q}), \tag{1}$$

where the homomorphism is induced by the inclusion $\widetilde{V} = \partial C \subset C$, and we denote by $\beta(\widetilde{M}, \widetilde{V}, T)$ the signature of this quadratic form on L. (If $\widetilde{M} = \Sigma^{4k-1}$ is a homotopy sphere, then $\beta(\widetilde{M}, \widetilde{V}, T)$ is by definition the *Browder-Livesay invariant* [3] $\sigma(\Sigma^{4k-1}, T)$ of the involution T on Σ^{4k-1}).

Theorem. $\alpha(\widetilde{M}, T) = \beta(\widetilde{M}, \widetilde{V}, T)$.

Thus $\beta(\widetilde{M}, T) = \beta(\widetilde{M}, \widetilde{V}, T)$ *is a well defined invariant of the oriented equivariant diffeomorphism class of* $(\widetilde{M}, T)$.

Proof of the Theorem. First notice, that the canonical deformation retraction of $M \times [0, 1]$ to $M \times \{\frac{1}{2}\}$ induces a deformation retraction of $\mathscr{D} = \pi^{-1}(M \times [0, 1])$ to $A = \pi^{-1}(M \times \{\frac{1}{2}\})$. To study $H_{2k}(A, Q)$, we consider the following part of a Mayer-Vietoris sequence for A (all homology with coefficients in Q): $H_{2k}(V) \oplus H_{2k}(C_1 \cup C_2) \xrightarrow{\phi} H_{2k}(A) \xrightarrow{\chi}$ 228

$$H_{2k-1}(\widetilde{V}_1 \cup \widetilde{V}_2) \xrightarrow{\psi} H_{2k-1}(V) \oplus H_{2k-1}(C_1 \cup C_2)$$

where $\widetilde{V}_1 \cup \widetilde{V}_2$ and $C_1 \cup C_2$ denote the disjoint unions, see figure (3) of §2.

In $H_{2k}(A)$ we have to consider the quadratic forms given by $(x, y) \to x \circ y$ and $(x, y) \to x \circ Ty$, where $\circ$ denotes the intersection number in $\mathscr{D}$. Now, the maps $V = V \times \{\frac{1}{2}\} \subset A$ and $C_1 \cup C_2 \to A$, which induce the homomorphism ϕ, are homotopic in $\mathscr{D}$ to maps into $\mathscr{D} - A$. Therefore if $x \in \operatorname{Im} \phi \subset H_{2k}(A)$ and y is any element of $H_{2k}(A)$, then $x \circ y = 0$. Thus if we denote

$$L' = H_{2k}(A)/\operatorname{Im} \phi, \tag{2}$$

then the quadratic forms $(x, y) \to x \circ y$ and $(x, y) \to x \circ Ty$ are well defined on L', and their signatures as forms on L' are $\tau(\mathscr{D})$ and $\tau(\mathscr{D}, T_1)$ respectively.

L' is isomorphic to the kernel of ψ, and hence we shall now take a closer look at $\ker\psi$. For this purpose we consider the Mayer-Vietoris sequences for $\widetilde{M}$ and B:

$$H_{2k}(\widetilde{M}) \xrightarrow{\tilde{\chi}} H_{2k-1}(\widetilde{V}_1 \cup \widetilde{V}_2) \xrightarrow{\tilde{\psi}} H_{2k-1}(\widetilde{V}) \oplus H_{2k-1}(C_1 \cup C_2)$$

$$H_{2k}(B) \xrightarrow{\chi^B} H_{2k-1}(\widetilde{V}_1 \cup \widetilde{V}_2) \xrightarrow{\psi^B} H_{2k-1}(\widetilde{V}) \oplus H_{2k-1}(C_1 \cup C_2).$$

In the sequence for $\widetilde{M}$, the homomorphism $H_{2k-1}(\widetilde{V}_1 \cup \widetilde{V}_2) \to H_{2k-1}(\widetilde{V})$ is induced by the identity $\widetilde{V}_1 \to \widetilde{V}$ on $\widetilde{V}_1$ and by the involution $T : \widetilde{V}_2 \to \widetilde{V}$ on $\widetilde{V}_2$, in the sequence for B however by the identity on both components. If we write $H_{2k-1}(\widetilde{V}_1 \cup \widetilde{V}_2)$ as $H_{2k-1}(\widetilde{V}) \oplus H_{2k-1}(\widetilde{V})$, the kernel of $H_{2k-1}(\widetilde{V}_1 \cup \widetilde{V}_2) \to H_{2k-1}(C_1 \cup C_2)$ is $L \oplus L$, and so we get:

$$\ker\tilde{\psi} = \{(a,b) \in L \oplus L | a + Tb = 0\},$$
$$\ker\psi^B = \{(a,b) \in L \oplus L | a + b = 0\}.$$

229 Let a be an element of $H_*(\widetilde{V}, \mathbf{Q})$. Then $a + Ta$ vanishes if and only if a is in the kernel of $H_*(\widetilde{V}, \mathbf{Q}) \to H_*(V, \mathbf{Q})$. Thus the kernel of ψ is

$$\ker\psi = \{(a,b) \in L \oplus L | a + Ta + b + Tb = 0\}.$$

$\ker\psi^B$ and $\ker\tilde{\psi}$ are subspaces of $\ker\psi$, and if we write (a,b) as

$$\left(\frac{a-b}{2}, \frac{b-a}{2}\right) + \left(\frac{a+b}{2}, \frac{a+b}{2}\right),$$

we see that in fact

$$\ker\psi = \ker\psi^B + \ker\tilde{\psi}. \tag{3}$$

By the isomorphism $L' \cong \ker\psi$, which is induced by χ, (3) becomes

$$L' = L^B + \widetilde{L},$$

where L^B denotes the subspace of L' corresponding to $\ker\psi^B$ under this isomorphism, and $\widetilde{L}$ corresponds to $\ker\tilde{\psi}$.

Let us first consider $\widetilde{L}$. Any element in $\widetilde{L}$ can be represented by an element $\widetilde{f}_*(x)$, where $x \in H_{2k}(\widetilde{M})$ and $\widetilde{f} : \widetilde{M} \to A$ is the canonical map:

$$\begin{array}{ccccc}
H_{2k}(\widetilde{M}) & \xrightarrow{\widetilde{\chi}} & H_{2k-1}(\widetilde{V}_1 \cup \widetilde{V}_2) & \xrightarrow{\widetilde{\psi}} & H_{2k-1}(\widetilde{V}) \oplus H_{2k-1}(C_1 \cup C_2) \\
\downarrow{\scriptstyle \widetilde{f}_*} & & \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow \\
H_{2k}(A) & \xrightarrow{\chi} & H_{2k-1}(\widetilde{V}_1 \cup \widetilde{V}_2) & \xrightarrow{\psi} & H_{2k-1}(V) \oplus H_{2k-1}(C_1 \cup C_2).
\end{array}$$

But $\widetilde{f}$ is homotopic in $\mathscr{D}$ to the inclusion $\widetilde{M} = \pi^{-1}(M \times \{0\}) \subset \mathscr{D}$, hence we have $L^B \circ L' = 0$. Therefore the quadratic forms on L' given by $(x, y) \to x \circ y$ and $(x, y) \to x \circ Ty$ can be restricted to L^B and their signatures will still be $\tau(\mathscr{D})$ and $\tau(\mathscr{D}, T_1)$.

Now, any element in $L^B \subset H_{2k}(A)/\mathrm{Im}\,\phi$ can be represented by an element $f_*^B(x)$, where $x \in H_{2k}(B)$ and $f^B : B \to A$ is the canonical map. Furthermore, χ induces an isomorphism between L^B and the "Browder-Livesay Module" L, because

$$L^B \underset{\cong}{\longrightarrow} \ker \psi^B = \{(a, -a) | a \in L\} \cong L.$$

Hence in view of the proposition in §2, out theorem would be proved if **230**
we can show that the following lemma is true.

Lemma. *Let x, $y \in H_{2k}(B)$ and $\overline{x} = f_*(x)$, $\overline{y} = f_*(y)$ the corresponding element under the homomorphism $f_* : H_{2k}(B) \to H_{2k}(\mathscr{D})$ induced by the canonical map $f : B \to A \subset \mathscr{D}$. By* (3)*, we have $\chi^B(x) = \chi(\overline{x}) = (a, -a)$ and $\chi^B(y) = \chi(\overline{y}) = (b, -b)$ for some a, $b \in L$. We claim:*

$$-\overline{x} \circ \overline{y} = a \circ Tb \tag{4}$$

Proof of the Lemma. First we note that we can make some simplifying assumptions on x and y. By a theorem of Thom ([9], p. 55), up to multiplication by an integer $\neq 0$, any integral homology class of a differentiable manifold can be realized by an oriented submanifold, and hence we may assume that x and y are given by oriented $2k$-dimensional submanifolds of B, which we will again denote by x and y. Of course x and y may be assumed to be transversal at $\widetilde{V} \subset B$. Then $\widetilde{V} \cap x$ and $\widetilde{V} \cap y$ are differentiable $(2k-1)$-dimensional orientable submanifolds of $\widetilde{V}$. We orient $\widetilde{V} \cap x$ (and similarly $\widetilde{V} \cap y$) as the boundary of the oriented manifold $C_1 \cap x$. Then $\widetilde{V} \cap x$ and $\widetilde{V} \cap y$ represent a and b in $H_{2k-1}(\widetilde{V})$,† and we shall now denote $\widetilde{V} \cap x$ by a and $\widetilde{V} \cap y$ by b.

Since in a neighborhood of $\widetilde{V}$, B is simply $\widetilde{V} \times \mathbf{R}$ and x is $a \times \mathbf{R}$, it is clear that any isotopy of a in $\widetilde{V}$ can be extended to an isotopy of x in B which is the identity outside a given neighborhood of $\widetilde{V}$ in B, such that x remains transversal to $\widetilde{V}$ during the isotopy. Therefore we may assume that the submanifold a of $\widetilde{V}$ is transversal to b and Tb.

There are all the preparations we have to make in B. Now let us immerse B into $\mathscr{D}$ and thus get immersions of x and y into $\mathscr{D}$ which will represent $\overline{x}$ and $\overline{y} \in H_{2k}(\mathscr{D})$. To obtain transversality of these im-
231 mersions however, we immerse x into $\mathscr{D}$ by the standard immersion $f : BtoA \subset \mathscr{D}$ and y by an immersion $f' : B \to \mathscr{D}$, which is different, but isotopic to f.

To define f', let $0 < \epsilon < \frac{1}{4}$ and choose a real-valued C^∞-function h on the interval $[0, 1]$ with $h(t) = t$ for $t < \frac{1}{2}\epsilon$, $h(t) = \epsilon$ for $t > \frac{1}{2}$ and $0 < h(t) \leqslant \epsilon$ for all other t. Using $\kappa : \widetilde{V} \underset{Z_2}{\times} D^1 \to M$, we get a function on $\kappa(\widetilde{V} \underset{Z_2}{\times} D^1) \subset M$ by $[v, t] \to h(|t|)$, which we now extend to a function $\overline{h}$ on M by defining $\overline{h}(p) = \epsilon$ for all $p \notin \kappa(\widetilde{V} \underset{Z_2}{\times} D^1)$. Then $M \to M \times [0, 1]$, given by $p \to (p, \frac{1}{2} + \overline{h}(p))$ is obviously covered by an immersion $f' : B \to \mathscr{D}$ which is isotopic to f.

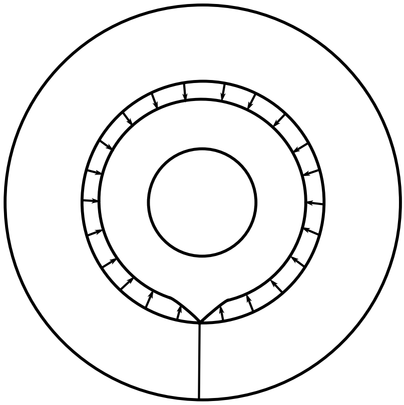
Then in fact the immersions $f : x \to \mathscr{D}$ and $f' : y \to \mathscr{D}$ are transversal to each other, and for $p \in x$, $q \in y$ we have

$$f(p) = f'(q) \Leftrightarrow p = q \in a \cap b \quad \text{or} \quad p = Tq \in a \cap Tb.$$

Looking now very carefully at all orientations involved, we obtain

$$-\overline{x} \circ \overline{y} = a \circ Tb + a \circ b. \tag{5}$$

†Or $-a$ and $-b$, but we may replace χ and χ^B in the Mayer-Vietoris sequences for A and B by $-\chi$ and $-\chi^B$, so let us assume that they represent a and b.



Recall that $\widetilde{V}$ is the boundary ∂C of the oriented manifold C and that a and b are in the kernel of $H_{2k-1}(\partial C) \to H_{2k-1}(C)$. Then the intersection *homology class* $s(a,b) \in H_0(\partial C)$ is in the kernel of $H_0(\partial C) \to H_0(C)$ (see Thom [8], Corollaire V.6, p. 173), and therefore the intersection *number* $a \circ b$ is zero, hence (5) becomes–$\overline{x} \circ \overline{y} = a \circ Tb$, and the lemma is proved.

4 Resolution of some singularities. For a tripel

$$a = (a_0, a_1, a_2)$$

of pairwise prime integers with $a_j \geqslant 2$ consider the variety $V_a \subset \mathbf{C}^3$ given by

$$z_0^{a_0} + z_1^{a_1} + z_2^{a_2} = 0. \tag{1}$$

The origin is the only singularity of V_a. We shall describe a resolution of this singularity. 232

Theorem. *There exist a complex surface (complex manifold of complex dimension* 2*) and a proper holomorphic map*

$$\phi : M_a \to V_a$$

such that the following is true:

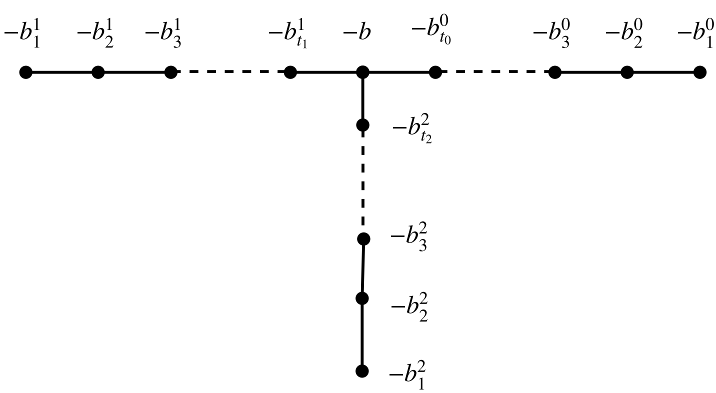
(i) $\phi : M_a - \phi^{-1}(0) \to V_a - \{0\}$ *is biholomorphic.*

(ii) $\phi^{-1}(0)$ *is a union of finitely many rational curves which are non-singularly imbedded in* M_a.

(iii) *The intersection of three of these curves is always empty. Two of these curves do not intersect or intersect transversally in exactly one point.*

(iv) *We introduce a finite graph* $\mathfrak{g}_a$ *in which the vertices correspond to the curves and in which two vertices are joined by an edge if and only if the corresponding curves intersect.* $\mathfrak{g}_a$ *is star-shaped with three rays.*

(v) *The graph* $\mathfrak{g}_a$ *will be weighted by attaching to each vertex the self-intersection number of the corresponding curve. This number is always negative. Thus* $\mathfrak{g}_a$ *looks as follows.*



233 (vi) $b = 1$ *or* $b = 2$; $b^j_i \geqslant 2$. *Let* q_0 *be determined by*

$$0 < q_0 < a_0 \quad \textit{and} \quad q_0 \equiv -a_1 a_2 \bmod a_0$$

and define q_1, q_2 *correspondingly. Let* q'_j *be given by*

$$0 < q'_j < a_j \quad \textit{and} \quad q_j q'_j \equiv 1 \bmod a_j.$$

Then the numbers b_i^j in the graph $\mathfrak{g}_a$ are obtained from the continued fractions

$$\frac{a_j}{q_j} = b_1^j - \cfrac{1}{b_2^j - \cfrac{1}{b_3^j - \cfrac{}{\ddots - \cfrac{1}{b_{t_j}^j}}}}.$$

(vii) *If the exponents a_0, a_1, a_2 are all odd, then*

$$b = 1 \iff q_0' + q_1' + q_2' \equiv 0 \bmod 2,$$
$$b = 2 \iff q_0' + q_1' + q_2' \equiv 1 \bmod 2.$$

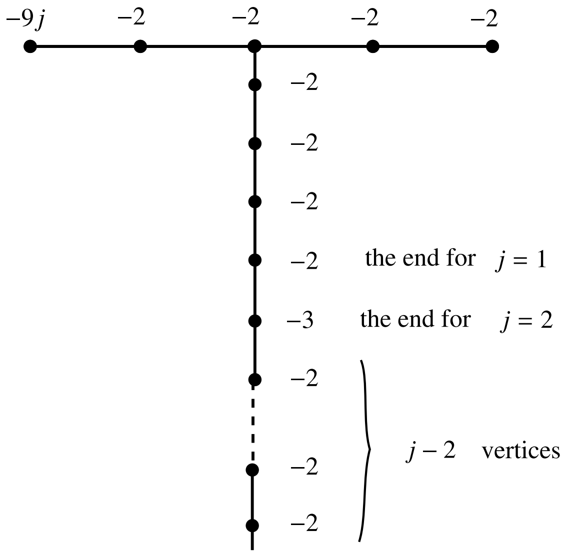
Before proving (i)-(vii) we study as an example

$$z_0^3 + z_1^{6j-1} + z_2^{18j-1} = 0. \tag{2}$$

We have

$$q_0 = q_0' = 2$$
$$q_1 = 4 \text{ for } j = 1 \text{ and } q_1 = 6j - 7 \text{ for } j \geqslant 2$$
$$q_1' = 5j - 1$$
$$q_2 = 2, q_2' = 9j.$$

By (vii) we get $b = 2$. The continued fractions for $\frac{3}{2}$, $\frac{5}{4}$ resp. $\frac{6j-1}{6j-7}$, $\frac{18j-1}{2}$ lead then to the graph



(11.1)

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Proof of the preceding theorem. We use the methods of [6]. The algebroid function

$$f = (-x_1^{a_1a_2} - x_2^{a_1a_2})^{1/a_0}$$

defines a branched covering $V_a^{(1)}$ of $\mathbf{C}^2$ (coordinates x_1, x_2 in $\mathbf{C}^2$). Blowing up the origin of $\mathbf{C}^2$ (compare [6], §1.3) gives a complex surface W with a non-singular rational curve $K \subset W$ of self-intersection number -1 and an algebroid function $\widetilde{f}$ on W branched along K and along a_1a_2 lines which intersect K in the a_1a_2 points of K satisfying

$$-x_1^{a_1a_2} - x_2^{a_1a_2} = 0 \tag{4}$$

where x_1, x_2 are now regarded as homogeneous coordinates of K. The algebroid function $\widetilde{f}$ defines a complex space $V_a^{(2)}$ lying branched over W with a_1a_2 singular points lying over the points of K defined by (4). In a neighborhood of such a point we have

$$\widetilde{f} = (\zeta_1\zeta_2^{a_1a_2})^{1/a_0} \tag{5}$$

where $\zeta_2 = 0$ is a suitable local equation for K and $\zeta_1 = 0$ for the line passing through the point and along which $V_a^{(2)}$ is branched over W. The 235
singularity of type (5) can be resolved according to [6], §3.4, where

$$\omega = (z_1 z_2^{n-q})^{1/n}, \quad (0 < q < n, (q, n) = 1) \tag{6}$$

was studied. In our case, we have

$$n = a_0 \quad \text{and} \quad q = q_0, \quad \text{see (vi) above,}$$

for all the $a_1 a_2$ singular points of $V_a^{(2)}$. The resolution gives a complex surface $V_a^{(3)}$ with the following property. The singularity of $V_a^{(1)}$ was blown up in a system of rational curves satisfying (iii) and represented by a star-shaped graph with $a_1 a_2$ rays of the same kind. The following diagram shows only one ray where the unweighted vertex represents the central curve $\widetilde{K}$ which under the natural projection $V_a^{(3)} \to W$ has K as bijective image

(7)

$V_a^{(1)}$ is of course just the affine variety

$$x_0^{a_0} + x_1^{a_1 a_2} + x_2^{a_1 a_2} = 0$$

which can be mapped onto V_a (see ((1)) by

$$(x_0, x_1, x_2) \to (z_0, z_1, z_2) = (x_0, x_1^{a_2}, x_2^{a_1}).$$

Denote by G the finite group of linear transformations

$$\begin{gathered} (x_1, x_2) \to (\epsilon_2 x_1, \epsilon_1 x_2) \quad \text{with} \quad \epsilon_2^{a_2} = \epsilon_1^{a_1} = 1. \\ V_a = V_a^{(1)}/G. \end{gathered} \tag{8}$$

Then the group G operates also on $V_a^{(3)}$. There are two fixed points, namely the points $0 = (0, 1)$ and $\infty = (1, 0)$ of $\widetilde{K} = K$ (with respect to the homogeneous coordinates x_1, x_2 on K). The $a_1 a_2$ points of $\widetilde{K}$

in which the curves with self-intersection number $-b^0_{t_0}$ of the a_1a_2 rays intersect $\widetilde{K}$ are an orbit under G. The a_1a_2 rays are all identified in $V_a^{(3)}/G$. Thus $V_a^{(3)}/G$ is a complex space with two singular points P_0, P_∞ corresponding to the fixed points. $V_a^{(3)}/G$ is thus obtained from V_a by blowing up the singular point in a system of t_0+1 rational curves showing the following intersection behaviour:

(9)

236 but where the vertex without weight represents a rational curve passing through the singular points P_0, P_∞.

We must find the representation of G in the tangent spaces of the fixed points $0=(0,1)$ and $\infty=(1,0)$. In the neighborhood of 0 we have local coordinates such that

$$y_1 = \frac{x_1}{x_2} \quad \text{and} \quad x_2 = y_2^{a_0}. \tag{10}$$

We consider G as the multiplicative group of all pairs (δ_2,δ_1) with $\delta_2^{a_2} = \delta_1^{a_1} = 1$ and put $\delta_1^{a_0} = \epsilon_1$ and $\delta_2 = \epsilon_2$ (see (8)). Then G operates in the neighborhood of the fixed point 0 as follows:

$$(y_1,y_2) \to (\delta_2\delta_1^{-a_0}y_1, \delta_1 y_2). \tag{11}$$

Thus P_0 is the quotient singularity with respect to the action (11). If we first take the quotient with respect to the subgroup G_0 of G given by $\delta_1 = 1$ we obtain a non-singular point which admits local coordinates (t_1,t_2) with

$$t_1 = y_1^{a_2} \quad \text{and} \quad t_2 = y_2. \tag{12}$$

Thus P_0 is the quotient singularity with respect to the action of G/G_0 which is the group of a_1-th roots of unity. By (11) and (12) for $\delta_1^{a_1} = 1$ the action is

$$(t_1,t_2) \to (\delta_1^{-a_0a_2}t_1, \delta_1 t_2) = (\delta_1^{q_1}t_1, \delta_1 t_2). \tag{13}$$

Looking at the invariants $\zeta_1 = t_1^{a_1}$, $\zeta_2 = t_2^{a_1}$ and $w = t_1 t_2^{a_1 - q_1}$ for which

$$w^{a_1} = \zeta_1 \zeta_2^{a_1 - q_1}$$

we see that P_0 is a singularity of type (6). We use [6], §3.4 (or [2], Satz 2.10) for P_0 and in the same way for P_∞ and have finished the proof except for the statements on b in (vi) and (vii). The surface M_a of the theorem is $V_a^{(3)}/G$ with P_0 and P_∞ resolved. The function f we started from gives rise to a holomorphic function on M_a. Using the formulas of [6], §3.4, we see that f has on the central curve of M_a the multiplicity $a_1 a_2$, and on the three curves intersecting the central curve the multiplicities

$$(a_1 a_2 q_0' + 1)/a_0, a_2 q_1', a_1 q_2'.$$

By [6], §1.4 (1), we obtain **237**

$$a_0 a_1 a_2 b = q_0' a_1 a_2 + q_1' a_0 a_2 + q_2' a_0 a_1 + 1.$$

Therefore

$$a_0 a_1 a_2 b < 3 a_0 a_1 a_2 \quad \text{and} \quad b = 1 \quad \text{or} \quad 2.$$

The congruence in (vii) also follows. This completes the proof.

Remark. Originally the theorem was proved by using the $\mathbf{C}^*$-action on the singularity (1) and deducing abstractly from this that the resolution must look as described. Brieskorn constructed the resolution explicitly by starting from $x_0^n + x_1^n + x_2^n (n = a_0 a_1 a_2)$ and then passing to a quotient. This is more symmetric. The method used in this paper has the advantage to give the theorem also for some other equations $z_0^{a_0} + h(z_1, z_2) = 0$ as was pointed out by Abhyankar in Bombay.

Now suppose moreover that the exponents a_0, a_1, a_2 are all odd. The explicit resolution shows that the involution $Tz = -z$ of $\mathbf{C}^3$ can be lifted to M_a. The lifted involution is also called T. It has no fixed points outside $\phi^{-1}(0)$. It carries all the rational curves of the graph $\mathfrak{g}_a$ over into themselves [7]. Thus T has the intersection points of two curves as fixed points. Let Fix T be the union of those curves which are pointwise fixed. Then Fix T is given by the following recipe.

Theorem. *For the involution T on $M_a(a_0, a_1, a_2$ odd) we have: The central curve belongs to* $\operatorname{Fix} T$. *If a curve is in* $\operatorname{Fix} T$, *then the curves intersecting it are not in* $\operatorname{Fix} T$. *If the curve C is not in* $\operatorname{Fix} T$ *and not an end curve of one of the three rays, then the following holds: If the self-intersection number $C \circ C$ is even, then the two curves intersecting C are both in* $\operatorname{Fix} T$ *or both not. If $C \circ C$ is odd, then one of the two curves intersecting C is in* $\operatorname{Fix} T$ *and one not. If C is an end curve of one of the three rays and if C is not in* $\operatorname{Fix} T$, *then $C \circ C$ is odd if and only if the curve intersecting C belongs to* $\operatorname{Fix} T$.

238 *Proof.* The involution can be followed through the whole resolution. It is the identity on the curve K. On the three singularities of type (6) the involution is given by $(z_1, z_2) \to (z_1, -z_2)$. Here z_1 and z_2 are not coordinate functions of C^3 as used in (1), but have the same meaning as in [6], §3.4. The theorem now follows from formula (8) in [6], §3.4. Compare also the lemma at the end of §6 of [7].

For a_0, a_1, a_2 pairwise prime and odd, we can now calculate the invariant α of the involution T_a on $\Sigma^3_{(a_0,a_1,a_2)}$ (see the Introduction). The quadratic form of the graph $\mathfrak{g}_a$ is negative-definite. Therefore ([7], §6)

$$\alpha(\Sigma^3_{(a_0,a_1,a_2)}, T_a) = -(t_0 + t_1 + t_2 + 1) - \operatorname{Fix} T \circ \operatorname{Fix} T. \tag{14}$$

Here $t_0 + t_1 + t_2 + 1$ is the number of vertices of $\mathfrak{g}_a$ whereas $\operatorname{Fix} T \circ \operatorname{Fix} T$ is of course the sum of the self-intersection numbers of the curves belonging to $\operatorname{Fix} T$. The calculation of α is a purely mechanical process by the two theorems of this §. The number α in (14) is always divisible by 8 (compare [7]) and for $(a_0, a_1, a_2) = (3, 6j - 1, 18j - 1)$ we get for α the value $8j$ (see (11.1)).

Observe that

$$\operatorname{Fix} T \circ C \equiv C \circ C \bmod 2 \tag{15}$$

for all curves in the graph $\mathfrak{g}_a$, a fact which is almost equivalent to our above recipe for $\operatorname{Fix} T$. The quadratic form of $\mathfrak{g}_a$ has determinant ± 1 because $\Sigma^3_{(a_0,a_1,a_2)}$ is for pairwise prime a_j an integral homology sphere ([1], [2], [7]). The divisibility of α by 8 is then a consequence of a well known theorem on quadratic forms.

The manifold Σ_a^{2n-1} (see the Introduction) is diffeomorphic to the manifold $\Sigma_a^{2n-1}(\epsilon)$ given by

$$z_0^{a_0} + \cdots + z_n^{a_n} = \epsilon \qquad (16)$$
$$\Sigma z_i \bar{z}_i = 1,$$

where ϵ is sufficiently small and not zero. $\Sigma_a^{2n-1}(\epsilon)$ bounds the manifold $N_a(\epsilon)$ given by

$$z_0^{a_0} + \cdots + z_n^{a_n} = \epsilon \qquad (17)$$
$$\Sigma z_i z_i \leqslant 1.$$

This fact apparently cannot be used to investigate the involution T_a in **239**
the case of odd exponents because then (17) is not invariant under T_a. If, however, the exponents are all even, then (17) is invariant under T_a and for $n = 2k$ the number $\alpha(\Sigma_a^{4k-1}, T_a)$ can be calculated using like Brieskorn [1] the results of Pham on $N_a(\epsilon)$. We get in this way □

Theorem. *Let* $a = (a_0, a_1, \ldots, a_{2k})$ *with* $a_i \equiv 0 \bmod 2$. *Then*

$$\alpha(\Sigma_a^{4k-1}, T_a) = \sum_j \epsilon(j)(-1)^{j_0+\cdots+j_{2k}}. \qquad (18)$$

The sum is over all $j = (j_0, j_1, \ldots, j_{2k}) \in Z^{2k+1}$ with $0 < j_r < a_r$ and $\epsilon(j)$ is 1, -1 or 0 depending upon whether the sum $\dfrac{j_0}{a_0} + \cdots + \dfrac{j_{2k}}{a_{2k}}$ lies strictly between 0 and 1mod 2, or strictly between 1 and 2mod 2, or is integral.

Remark. For simplicity the resolution was only constructed for the exponents a_0, a_1, a_2 being pairwise relatively prime. The resolution of the singularity

$$z_0^{a_0} + z_1^{a_1} + z_2^{a_2} = 0$$

can also be done in a similar way for arbitrary exponents and gives the following information.

Theorem. *If* $a_0 \equiv a_1 \equiv a_2 \bmod 2$ *and* d *is any integer* $\geqslant 1$, *then*

$$\alpha(\Sigma^3_{(da_0,da_1,da_2)}, T_{da}) = d\alpha(\Sigma^3_{(a_0,a_1,a_2)}, T_a) + d - 1.$$

For a_0, a_1, a_2 all odd and $d = 2$ we get

$$\alpha(\Sigma^3_{(a_0,a_1,a_2)}, T_a) = \frac{1}{2}(\alpha(\Sigma^3_{(2a_0,2a_1,2a_2)}, T_{2a}) - 1)$$

and therefore a method to calculate α also for odd exponents by formula (18).

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GEOMETRIC AND ANALYTIC METHODS IN THE THEORY OF THETA-FUNCTIONS*

By Jun-Ichi Igusa

241 It is well known that theta-functions appeared in the early nineteenth century in connection with elliptic functions. This theory, although it has apparent analytic features, is basically geometric. However, the Gauss proof of the transformation law for the Legendre modulus and Jacobi's application of theta-functions to number theory are not geometric. More precisely, if we start from the view-point that theta-functions give nice projective embeddings of a polarized abelian variety over **C**, we are in the geometric side of the theory. On the other hand, if we regard theta-functions as functions obtained by the summations of normal densities over a lattice in a vector space over **R**, we are in the analytic side of the theory. After this naive explanation of the title, we shall start discussing certain results in the theory of theta-functions.

1. Suppose that X is a vector space of dimension n over **C** and L a lattice in X. Then, a holomorphic function $x \to \theta(x)$ on X is called a *theta-function belonging to L* if it has the property that

$$\theta(x+\xi) = e(A_\xi(x) + b_\xi)\cdot\theta(x)$$

for every ξ in L with a **C**-linear form A_ξ on X and with a constant b_ξ both depending on ξ. The notation $e(b)$ stands for $\exp(2\pi i b)$. If the theta-function is not the constant zero, it determines a positive divisor D of the quotient $Q = X/L$, which is a complex torus. A fundamental existence theorem in the theory of theta-functions asserts that every positive divisor of Q can be obtained in this manner (cf. [23], [24]).

We observe that the function $(x,\eta) \to A_\eta(x)$ on $X \times L$ can be extended uniquely to an **R**-bilinear form on $X \times X$. Then, the bi-character

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f of $X \times X$ defined by

$$f(x,y) = e(A_y(x) - A_x(y))$$

takes the value 1 on $L \times L$ and also on the diagonal of $X \times X$. The divisor **242**
D is called non-degenerate if f is non-degenerate. In this case, f gives an autoduality of X, and the pair (Q, f) is called a polarized abelian variety over $\mathbf{C}$. Another fundamental theorem in the theory of theta-functions asserts that the vector space over $\mathbf{C}$ of theta-functions which have the same periodicity property as $\theta(x)^3$ gives rise to a projective embedding of this polarized abelian variety (c.f. [23], [24]). We shall consider the special case when f gives an autoduality of (X, L) in the sense that the annihilator L_* of L with respect to f coincides with L. The polarized abelian variety is then called *autodual*. In this case, we can choose a $\mathbf{Z}$-base $\xi_1, \ldots, \xi_{2n}$ of L so that we get

$$f\left(\sum_{i=1}^{2n} x_i\xi_i, \sum_{i=1}^{2n} y_i\xi_i\right) = e(x'\,{}^t y'' - x''\,{}^t y').$$

We are denoting by x' and x'' the line vectors determined by $x_1, \ldots, x_n$ and $x_{n+1}, \ldots, x_{2n}$; similarly for y' and y''. We observe that there exists a unique $\mathbf{C}$-base of X, and hence a unique $\mathbf{C}$-linear isomorphism $X \xrightarrow{\sim} \mathbf{C}^n$ mapping the colomn vector determined by $\xi_1, \ldots, \xi_{2n}$ to a $2n \times n$ matrix composed of an $n \times n$ matrix τ and 1_n. Then τ is necessarily a point of the Siegel upper-half plane $\mathfrak{S}_n$ and, if x is mapped to z, we have

$$\begin{aligned}\theta(x) =& e(\text{polynomial in } z \text{ of degree two})\\ &\cdot \sum_{p \in \mathbf{Z}^n} e\Big(\frac{1}{2}(p+m')\tau^t(p+m') + (p+m')^t(z+m'')\Big),\end{aligned}$$

in which m', m'' are elements of $\mathbf{R}^n$. We shall denote this series by $\theta_m(\tau, z)$ in which m is the line vector composed of m' and m''.

The above process of obtaining the *theta-function* $\theta_m(\tau, z)$ of *characteristic m* depends on the choice of the $\mathbf{Z}$-base $\xi_1, \ldots, \xi_{2n}$ of L. However, once it is chosen, the process is unique except for the fact that the

characteristic m is determined only up to an element of $\mathbf{Z}^{2n}$. Now, the passing to another $\mathbf{Z}$-base of L is given by a left multiplication of an element σ of $Sp_{2n}(\mathbf{Z})$ to the column vector determined by $\xi_1, \ldots, \xi_{2n}$. If
243 σ is composed of $n \times n$ submatrices $\alpha, \beta, \gamma, \delta$ the point τ^* of $\mathfrak{S}_n$ which corresponds to the new base is given by

$$\sigma \cdot \tau = (\alpha\tau + \beta)(\gamma\tau + \delta)^{-1}.$$

Furthermore, if $\theta_{m^*}(\tau^*, z^*)$ corresponds to the new base, the relation between $\theta_{m^*}(\tau^*, z^*)$ and $\theta_m(\tau, z)$ is known except for a certain eighth root of unity. This eighth root of unity has been calculated explicitly for some special σ, e.g. for those σ in which γ is non-degenerate. This is the classical *transformation low of theta-functions.* In particular, if we consider the theta-constants $\theta_m(\tau) = \theta_m(\tau, 0)$ for m in $\mathbf{Q}^{2n}$, the transformation law implies that any homogeneous polynomial of even degree, say $2k$, in the theta-constants defines a modular form of weight k belonging to some subgroup, say Γ, of $Sp_{2n}(\mathbf{Z})$ of finite index. We recall that a modular form of wight k belonging to Γ is a holomorphic function ψ on $\mathfrak{S}_n$ (plus a boundedness condition at infinity for $n = 1$) obeying the following transformation law:

$$\psi(\sigma \cdot \tau) = \det(\gamma\tau + \delta)^k \cdot \psi(\tau)$$

for every σ in Γ. The set of such modular forms forms a vector space $A(\Gamma)_k$ over $\mathbf{C}$, and the graded ring

$$A(\Gamma) = \bigoplus_{k \geqslant 0} A(\Gamma)_k$$

is called the *ring of modular forms belonging to* Γ.

Now, the above summarized theory does not give the precise nature of the subgroup Γ, nor does it provide information on the set of modular forms obtained from the theta-constants. We have given an almost satisfactory answer to these problems in [9], and it can be stated in the following way.

Let $\Gamma_n(l)$ denote the principal congruence group of level l and consider only those characteristics m satisfying $lm \equiv 0 \bmod 1$. Then, for any even level l, a monomial $\theta_{m_1} \dots \theta_{m_{2k}}$ defines a modular form belonging to $\Gamma_n(l)$ if and only if the quadratic form q on $\mathbf{R}^{2n}$ defined by

$$q(x) = (l/2)\left(\sum_{\alpha=1}^{2k}(x^t m_\alpha\right)^2 + (k/2)x'\,{}^t x'')$$

is $\mathbf{Z}$-valued on $\mathbf{Z}^{2n}$. Moreover, the integral closure within the field of 244
fractions of the ring generated over $\mathbf{C}$ by such monomials is precisely the ring $A(\Gamma_n(l))$.

The proof of the first part depends on the existence of an explicit transformation formula for $\theta_{m_1} \dots \theta_{m_{2k}}$ valid for every σ in $\Gamma_n(2)$. The proof of the second part depends on the theory of compactifications (c.f. [1], [4]) and on the Hilbert 'Nullstellensatz". We would like to call attention to the fact that this result connects the unknown ring $A(\Gamma_n(l))$ to an explicitly constructed ring of theta-constants. As an application, we have obtained the following theorem [12]:

There exists a ring homomorphism ρ from a subring of $A(\Gamma_n(1))$ to the ring of projective invariants of a binary form of degree $2n + 2$ such that ρ increases the weight by a $\frac{1}{2}n$ ratio. Moreover, an element of $A(\Gamma_n(1))$ is in the kernel of ρ if and only if it vanishes at every "hyperelliptic point" of $\mathfrak{S}_n$.

This subring contains all elements of even weights as well as all polynomials in the theta-constants whose characteristics m satisfy $2m \equiv 0 \bmod 1$ and which are contained in $A(\Gamma_n(1))$. It may be that $A(\Gamma_n(1))$ simply consists of such polynomials in the theta-constants. If we denote by $\mathscr{A}_n(l)$, for any even l, the ring generated over $\mathbf{C}$ by all monomials $\theta_{m_1} \dots \theta_{m_{2k}}$ satisfying $lm \equiv 0 \bmod 1$, this is certainly the case when $\mathscr{A}_n(2)$ is integrally closed. Now. Mumford informed us (in the fall of 1966) that $\mathscr{A}_1(4)$ is not integrally closed. Subsequently we verified that $\mathscr{A}_1(l)$ is integrally closed if and only if $l = 2$. On the other hand, we have shown in [8] that $\mathscr{A}_2(2)$ is integrally closed. Therefore, it is possible that $\mathscr{A}_n(2)$ is integrally closed for every n. We can propose the more general problem of whether we can explicitly give a set of generators for the integral closure of $\mathscr{A}_n(l)$. Concerning this problem, we would like to mention a result of Mumford to the effect that $\mathscr{A}_n(l)$ is integrally closed locally at every finite point. For this and for other important results, we refer to his paper [16]. Also, we would like to mention a relatively recent paper of Siegel that has appeared in the Göttingen Nachrichten [22, III].

245 For the application of the homomorphism ρ, it is useful to know that, if ψ is a cusp form, i.e. a modular form vanishing at infinity, at which ρ is defined, the image $\rho(\psi)$ is divisible by the discriminant of the binary form. Also, in the case when $n = 3$, the kernel of ρ is a principal ideal generated by a cusp form of weight 18. From this, we immediately get

$$\dim_{\mathbf{C}} A(\Gamma_n(1))_8 \leqslant 1$$

for $n = 1, 2, 3$. Actually we have an equality here because each coefficient ψ_k in

$$\Pi(t + (\theta_m)^8) = t^N + \psi_1 \cdot t^{N-1} + \cdots + \psi_N,$$

in which the product is taken over the $N = 2^{n-1}(2^n + 1)$ characteristics m satisfying $2m \equiv 0$, $2m'^t m'' \equiv 0 \bmod 1$, is an element of $A(\Gamma_n(1))_{4k}$ different from the constant zero for every n. Furthermore, by using the classical structure theorem for the ring of projective invariants of a binary sextic, we have reproduced our structure theorem of $A(\Gamma_2(1))$ in [12]. It seems possible to apply the same method to the case when $n = 3$ by using Shioda's result in [21] on the structure of the ring of projective invariants of a binary octavic.

We shall discuss a special but hopefully interesting application of what we have said so far to a conjecture made by Witt in [27]. He observed that the lattice in $\mathbf{R}^m$ generated by the root system D_m can be extended to a lattice on which $(\frac{1}{2})x^t x$ is $\mathbf{Z}$-valued if and only if m is a multiple of 8. In this case, there are two extensions, and they are conjugate by an isomorphism inducing an automorphism of D_m. The restriction of $(\frac{1}{2})x^t x$ to the extended lattice gives a positive, non-degenerate quadratic form (of discriminant 1) to which we can associate, for every given n, a theta-series called the class invariant. For $m = 8k$, this is an element of $A(\Gamma_n(1))_{4k}$ different from the constant zero. If we denote the first two elements by f_n, g_n, the conjecture is that $(f_3)^2 = g_3$. We note that $\dim_{\mathbf{C}} A(\Gamma_3(1))_8 \leqslant 1$ gives an affirmative answer to this conjecture. Also, M. Kneser has given another solution by a highly ingenious argument in [14].

Now, if we consider the difference $(f_4)^2 - g_4$, we get a cusp form of weight 8 for $n = 4$. According to the property of the homomorphism ρ, this cusp form vanishes at every hyperelliptic point. We may inquire whether it also vanishes at every "jacobian point". This question reminds us of an invariant discovered by Schottky which vanishes at every jacobian point (c.f. [19]). We can see easily that the Schottky invariant, denoted by J, is also a cusp form of weight 8 for $n = 4$. We may, therefore, inquire how the two cusp forms are related. The answer is given by the following identity: **246**

$$(f_4)^2 - g_4 = 2^{-2}3^2 5.7\text{-times } J.$$

Actually, this identity can be proved directly, and it provides a third solution for the Witt conjecture.

In this connection, we would like to mention that, as far as we can see it, Schottky did not prove the converse, i.e. the fact that the vanishing of J is "characteristic" for the jacobian point. In fact, he did not even attempt to prove it. However, it appears that this can be proved. A precise statement is that the divisor determined by J on the quotient $\Gamma_4(1)\backslash\mathfrak{S}_4$ is irreducible in the usual sense and it contains the set of jacobian points as a dense open subset. We shall publish the proofs for this and for the above mentioned identity in a separate paper.

2. We have considered the ring of modular forms so to speak algebraically. However, as we mentioned earlier, it depends on the possibility of compactifying the quotient $\Gamma\backslash\mathfrak{S}_n$ to a (normal) projective variety, say $\mathscr{S}(\Gamma)$. This is a natural approach to the investigation of modular forms. After all, to calculate the dimension of $A(\Gamma)_k$ is a problem of "Riemann-Roch type", and it was with this problem in mind that Satake first attempted to compactify the quotient $\Gamma_n(1)\backslash\mathfrak{S}_n$ in [18]. The theory of compactifications has been completed recently by Baily and Borel [2], and it may be just about time to examine the possibility of applying it to the determination of the dimension of $A(\Gamma)_k$.

247 Now, for such a purpose, it is desirable that $\mathscr{S}(\Gamma_n(l))$ becomes non-singular for a large l. However, the situation is exactly the opposite. In fact, every point of $\mathscr{S}(\Gamma_n(l)) - \Gamma_n(l)\backslash\mathfrak{S}_n$ is singular on $\mathscr{S}(\Gamma_n(l))$ except for a few cases and, for instance, the dimension of the Zariski tangent space tends to infinity with l (c.f. [10]). We may, therefore, inquire whether $\mathscr{S}(\Gamma_n(l))$ admits a "natural" desingularization. If the answer is affirmative, we may further hope to obtain the Riemann-Roch theorem for this non-singular model. It turns out that the problem is quite delicate and the whole situation seems to be still chaotic.

In order to explain some results in this direction, we recall that the boundary of a bounded symmetric domain has a stratification which is inherited by the standard compactification of its arithmetically defined quotient. We shall consider only such "absolute" stratification. The union of the first, the second,....strata is called the boundary of the compactification. Then we can state our results in the following way.

Suppose that D is isomorphic to a bounded symmetric domain, and convert the complexification of the connected component of $\mathrm{Aut}(D)$, up to an isogeny, into a linear algebraic group, say G, over $\mathbf{Q}$. Denote by Γ the principal congruence group of $G_{\mathbf{Z}}$ of level l. Then the monoidal transform, say $\mathscr{M}(\Gamma)$, of the standard compactification $\mathscr{S}(\Gamma)$ of $\Gamma\backslash D$ along its boundary, i.e. the blowing up of $\mathscr{S}(\Gamma)$ with respect to the sheaf of ideals defined by all cusp forms, is non-singular over the first strata for every l not smaller than a fixed integer. Moreover, the fiber of $\mathscr{M}(\Gamma) \to \mathscr{S}(\Gamma)$ at every point of the first strata is a polarized abelian variety. In the special case when $D = \mathfrak{S}_n$ and $G = Sp_{2n}(\mathbf{C})$, the monoidal transformation desingularizes up to the third strata with 3 as the fixed integer.

The proof of the first part is a refinement of the proof of a similar result in [10]. It was obtained (in the summer of 1966) with the help of A. Borel. The proof of the second part and the description of various fibers are in [11]. (The number 3 comes from the theorem on projective
248 embeddings of a polarized abelian variety and from the fact that $\Gamma_n(l)$ operates on $\mathfrak{S}_n$ without fixed points for $l \geqslant 3$). The basis of the proofs is the theory of Fourier-Jacobi series of Pyatetski-Shapiro [17]. In this connection, we would like to mention an imaginative paper by Gindikin

and Pyatetski-Shapiro [7]. However, their main result requires the existence of a non-singular blowing up of $\mathscr{S}(\Gamma)$ which, so to speak, coincides with the monoidal transformation along the first strata and which does not create a divisor over the higher strata. The existence of such blowing up is assured for $\Gamma = \Gamma_n(l)$ up to $n = 3$. In fact, the monoidal transformation has the required properties (c.f. [11]).

Because of the serious difficulty in constructing a natural desingularization, we may attempts to apply directly to $\mathscr{S}(\Gamma)$ the "Riemann-Roch theorem for normal varieties" proved first by Zariski. According to Eichler, his work on the "Riemann-Roch theorem" [6] contains additional material useful for this purpose. Eichler informed us (in the spring of 1967) that he can calculate, for instance, the dimension of $A(\Gamma_2(1))_k$ at least when k is a multiple of 6, and thus recover the structure theorem for $A(\Gamma_2(1))$.

3. We have so far discussed the geometric method in the theory of theta-functions. The basic features are that objects are "complex-analytic" if not algebraic. We shall now abandon this restriction and adopt a freer viewpoint. This has been provided by a recent work of A. Weil that has appeared in two papers ([25], [26]). We shall start by giving an outline of his first paper.

We take an arbitrary locally compact abelian group X and denote its dual by X^*. We shall denote by T the multiplicative group of complex numbers t satisfying $t\bar{t} = 1$ and by $(x, x^*) \to \langle x, x^* \rangle$ the bicharacter of $X \times X^*$ which puts X and X^* into duality. Then the regular representation of X and the Fourier transform of the regular representation of X^* satisfy the so-called Heisenberg commutation relation. Therefore the images of X and X^* by these representations generate a group $\mathbf{A}(X)$ of unitary operators with the group $\mathbf{T}$, of scalar multiplications by elements of T, as its center such that

$$\mathbf{A}(X)/\mathbf{T} \xrightarrow{\sim} X \times X^*,$$

the isomorphism being bicontinuous. Consider the normalizer $\mathbf{B}(X)$ **249**
of $\mathbf{A}(X)$ in $\mathrm{Aut}(L^2(X))$. Then the Mackey theorem [15] implies that $\mathbf{T}$ is the centralizer of $\mathbf{A}(X)$ in $\mathrm{Aut}(L^2(X))$ and that every bicontinuous automorphism of $\mathbf{A}(X)$ inducing the identity on $\mathbf{T}$ is the restriction to

$\mathbf{A}(X)$ of an inner automorphism of $\mathbf{B}(X)$. If $B(X)$ denotes the group of such automorphisms of $\mathbf{A}(X)$, we have

$$\mathbf{B}(X)/\mathbf{T} \xrightarrow{\sim} B(X),$$

the isomorphism being bicontinuous. On the other hand, if $\mathscr{S}(X)^*$ is the Schwarz-Bruhat space of X (c.f. [3]), Weil has shown that every $\mathbf{s}$ in $\mathbf{B}(X)$ gives a bicontinuous automorphism $\Phi \to \mathbf{s}\Phi$ of $\mathscr{S}(X)$. The proof is based on what might be called a five-step decomposition of $\Phi \to \mathbf{s}\Phi$, which comes from a work of Segal [20]. Now, if L is a closed subgroup of X, every Φ in $\mathscr{S}(X)$ gives rise to a function F^Φ on $\mathbf{B}(X)$ by the following integral

$$F^\Phi(\mathbf{s}) = \int_L (\mathbf{s}\Phi)(\xi)d\xi$$

taken with respect to the Haar measure $d\xi$ on L. Weil has show that F^Φ has a remarkable invariance property with respect to a certain subgroup of $\mathbf{B}(X)$ determined by L. Then he has specialized to the "arithmetic case" and proved the continuity of $(\mathbf{s}, \Phi) \to \mathbf{s}\Phi$ and $\mathbf{s} \to F^\Phi(\mathbf{s})$ restricting $\mathbf{s}$ to the metaplectic group, which is a fiber-product over $B(X)$ of $\mathbf{B}(X)$ and of an adelized algebraic group.

We shall now explain some supplements to the Weil theory and our generalization of theta-functions in [13]. For other group-theoretic treatment of theta-functions, we refer to Cartier [5].

If $\mathbf{s}$ is an element of $\mathbf{B}(X)$, it gives rise to a bicontinuous automorphism σ of $X \times X^*$ keeping

$$((x, x^*), (y, y^*)) \to \langle x, y^*\rangle\langle y, x^*\rangle^{-1}$$

invariant. The group $Sp(X)$ of such automorphisms of $X \times X^*$ is known as the symplectic group of X. The Weil theory implies that $\mathbf{s} \to \sigma$ gives rise to a continuous monomorphism

$$\mathbf{B}(X)/\mathbf{A}(X) \to Sp(X).$$

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*There should be no confusion between $\mathscr{S}(\Gamma)$ and $\mathscr{S}(X)$ because in the first case the group is non-commutative and in the second it is commutative.

We have shown that this monomorphism is surjective and bicontinuous. The topology of $Sp(X)$ is determined by the fact that the group of bicontinuous automorphisms of any locally compact group becomes a topological group by the (modified) compact open topology. We observe that neither $\mathbf{B}(X)$ nor $Sp(X)$ is locally bounded, in general. However, if G is a locally compact group and $G \to Sp(X)$, a continuous homomorphism, the fiber-product

$$\mathbf{B}(X)_G = \mathbf{B}(X) \underset{Sp(X)}{\times} G$$

is always locally compact. As for the continuity of $\mathbf{B}(X) \times \mathscr{S}(X) \to \mathscr{S}(X)$, it is false in general. However, if Σ is a locally compact subset of $\mathbf{B}(X)$, the induced mapping $\Sigma \times \mathscr{S}(X) \to \mathscr{S}(X)$ is continuous. In particular, the mapping $\mathbf{B}(X)_G \times \mathscr{S}(X) \to \mathscr{S}(X)$ defined by $((\mathbf{s}, g), \Phi) \to \mathbf{s}\Phi$ is continuous. Also, the function F^{Φ} is always continuous on $\mathbf{B}(X)$. In fact, it can be considered locally everywhere as a coimage of continuous functions on Lie groups.

The continuous function F^{Φ} on $\mathbf{B}(X)$ may be called an *automorphic function* because of its invariance property mentioned before. We may then define a theta-function as a special automorphic function. For this purpose, we observe that X can be decomposed into a product $X_0 \times \mathbf{R}^n$, in which X_0 is a closed subgroup of X with compact open subgroups. Although this decomposition is not intrinsic (except when X_0 is the union of totally disconnected compact open subgroups), the dimension n is unique and X_0 contains all compact subgroups of X. We consider a function $\Phi_0 \otimes \Phi_\infty$ on X defined by

$$\begin{aligned} \Phi_0 = \; & \text{the characteristic function of a compact open} \\ & \text{subgroup of } X_0 \\ \Phi_\infty(x_\infty) = \; & \exp(-\pi x_\infty{}^t x_\infty). \end{aligned}$$

Then finite linear combinations of elements of $\mathbf{A}(X_0)\Phi_0 \otimes \mathbf{B}(\mathbf{R}^n)\Phi_\infty$ form a dense subspace $\mathscr{G}(X)$ of $\mathscr{S}(X)$ which is $\mathbf{B}(X)$-invariant. Moreover $\mathscr{G}(X)$ is intrinsic. We take an element Φ of $\mathscr{G}(X)$ and call F^{Φ} a *theta-function* on $\mathbf{B}(X)$. Then, every automorphic function on $\mathbf{B}(X)$ can

be approximated uniformly on any compact subset of $\mathbf{B}(X)$ by a theta- 251
function. Also, we can restrict F^{Φ} to the fiber-product $\mathbf{B}(X)_G$. This procedure gives rise to various theta-functions.

On the other hand, if a tempered distribution I on X vanishes on $\mathscr{G}(X)$, it vanishes identically. Actually a smaller subspace than $\mathscr{G}(X)$ is dense in $\mathscr{S}(X)$. In fact, we can find a locally compact, solvable subgroup $\Sigma(X)$ of $\mathbf{B}(X)$ such that the vanishing of the continuous function $\mathbf{s} \to I(\mathbf{s}\Phi)$ on $\Sigma(X)$ is characteristic for the vanishing of I for $\Phi = \Phi_0 \otimes \Phi_\infty$. Consequently, we would have an identity $I_1 = I_2$ of tempered distributions I_1 and I_2 on X if they give rise to the same function on $\Sigma(X)$. It appears that these facts explain to some extent the rôles played by theta-functions in number theory. For instance, we can convince ourselves easily that the Siegel formula (for the orthogonal group) as formulated and proved by Weil [26] and the classical Siegel formula by Siegel [22] involving theta-series and Eisenstein series are equivalent. This does not mean that the Siegel formula for any given Φ in $\mathscr{S}(X)$ can be obtained linearly from the classical Siegel formula. The space $\mathscr{G}(X)$ is too small for this. We observe that, if we denote by $\mathscr{G}_k(X)$ the subspace of $\mathscr{S}(X)$ consisting of elements of $\mathscr{G}(X)$ multiplied by "polynomial functions" of degree k, it is also $\mathbf{B}(X)$-invariant. Such a space has appeared (in the arithmetic case) in the proof of the functional equation for the Hecke L-series. It seems that the meaning and the actual use of elements of $\mathscr{S}(X)$ not contained in the union of

$$\mathscr{G}(X) = \mathscr{G}_0(X) \subset \mathscr{G}_1(X) \subset \cdots$$

are things to be investigated in the future.

In rounding off our talk, we remark that, if we take a vector space over $\mathbf{R}$ as X and a lattice in X as L, the theta-function F^{Φ} becomes, up to an elementary factor, the theta-function that we have introduced in the beginning (with the understanding that the previous (X, L) is replaced by $(X \times X^*, L \times L_*)$). Moreover, the invariance property of F^{Φ} becomes the transformation law of theta-functions.

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ON SOME GROUPS RELATED TO CUBIC HYPERSURFACES

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255 **Introduction and motivation.** Let V be a nonsingular cubic curve in a projective plane over a field k. If the set $V(k)$ of its k-points is nonempty, one may endow V with a structure of an abelian variety over k, taking one of the points for identity. Then the group law on $V(k)$ may be easily described geometrically. The description is especially simple if identity is an inflexion point of V: then the sum of three points is identity if and only if they are collinear. It follows that the ternary relation $L(x, y, z)$ ("x, y, z are collinear") may be considered as the basic one for the theory of one-dimensional abelian varieties. This is the classical approach, which makes possible, for example, to give "elementary" proofs of Mordell-Weil theorem and Riemann conjecture for elliptic curves.

The ternary relation L is defined not only for cubic curves but, say, for all geometrically irreducible cubic hypersurfaces in a projective space. However, this relation was scarcely utilized for studying of arithmetic and geometric properties of these varieties. One of the main reasons of it was that one could not relate L to some standard algebraic structure: in fact, cubic hypersurfaces of dimension $\geqslant 2$ are definitely not group varieties.

In this talk we suggest and discuss two different ways to construct a group by means of the relation L.

The first way is to consider for any nonsingular point $x \in V(k)$ the birational automorphism $t_x : V \to V$, which in its existence domain is given by the relation $L(x, y, t_x(y))$. (In other words, t_x is "the reflection relatively to point x"). Then one may consider the group B, generated by automorphisms t_x for all $x \in V(k)$ (or for some subset of points). If $\dim V = 1$, this group B is easily seen to be isomorphic to the canonical
256 extension of $\mathbf{Z}_2$ by $V(k)$ with usual group law. In particular one can

reconstruct $V(k)$ from B. So one can hope that properties of the group B are of some interest in the general case.

The second way to construct a group is to imitate the one-dimensional definition, but applying it to some classes of points of V rather than the points themselves. Namely, consider a decomposition of $V(k)$ in disjoint subsets $V(k) = \cup C_i$, enjoying the following properties:

(a) if $L(x, y, z)$, where $x \in C_p$, $y \in C_q$, $z \in C_r$, then any of classes C_p, C_q, C_r is uniquely defined by the other two;

(b) on the set of classes $\{C_i\}$ there exists a structure of abelian group Γ of *period two* such that the relation "sum of three classes is equal to zero" coincides with the relation, induced by L on this set.

(All this simply means that one can add two classes, drawing a line through its representatives and then taking the class of the third point of intersection of this line with V. In fact, one must be a little bit more careful to avoid lines lying in V: c.f. the statement of Theorem 3 below.)

An example of such decomposition in case $\dim V = 1$ is given by the family of cosets $V(k)/2V(k)$. So the groups constructed by this procedure are similar to "weak Mordell-Weil groups". F. Châtelet has discovered nontrivial groups of this kind for certain singular cubic surfaces [1].

We shall describe now briefly our main results. The first section below contains complete definitions and statements, the second gives some ideas of proof.

The group B is studied in some detail for nonsingular cubic surfaces. In particular, we give a complete system of relations between maps t_x for x, not lying on the union of 27 lines of V. Besides, we prove, that for k-minimal surfaces such t_x together with the group of projective transformations of V generate the whole group of birational k-automorphisms of V (Theorems 1 and 2).

The main result on the group Γ of classes of points of $V(k)$ is proved for all dimensions and states the existence of the unique "finest" decomposition or "biggest" group Γ. In fact, we consider such decompositions **257** not only of the total set $V(k)$, but of any subset of it fulfilling some natural conditions. It happens then that, say, the identity class of any such decomposition may be decomposed again and so on. So one can construct for any V a sequence of 2-groups Γ_n which for $\dim V = 1$ is given by $\Gamma_n = 2^n V(k)/2^{n+1}V(k)$ (Theorem 3).

The last Theorem 4 shows a connection between groups B and Γ.

Main statements. Let V be a nonsingular cubic surface in $\mathbf{P}^3$, defined over a field k. All points we consider are geometric K-points for subfields K of an algebraic closure $\overline{k}$ of k.

A point $x \in V(\overline{k})$ is called *nonspecial*, if it does not belong to the union of 27 lines on $V \otimes \overline{k}$.

A pair of points $(x, y) \in V(\overline{k}) \times V(\overline{k})$ is called *nonspecial*, if points x, y are distinct and nonspecial and if the line in $\mathbf{P}^3$, containing x, y, is not tangent to V and is disjoint with any line of $V \otimes \overline{k}$.

For any point $x \in V(k)$ the birational k-automorphism $t_x : V \to V$ is defined (c.f. Introduction). If the point $t_x(y)$ is defined, we shall sometimes denote it $x \circ y$. If $x \circ y$ and $y \circ x$ are both defined, then $x \circ y = y \circ x$.

Theorem 1. *Let W be the group of projective automorphisms of $V \otimes \overline{k}$ over $\overline{k}$ and B the group of birational automorphisms generated by the reflections t_x relative to the nonspecial points $x \in V(\overline{k})$.*

Then the group, generated by W and B, is the semidirect product of these subgroups with normal subgroup B and the natural action

$$wt_xw^{-1} = t_{w(x)}, \quad w \in W, \quad x \in V(\overline{k}).$$

Moreover, the complete system of relations between the generators t_x of group B is generated by

$$t_x^2 = 1; \quad (t_xt_{x\circ y}t_y)^2 = 1 \tag{1}$$

for all nonspecial pairs (x, y) of points of $V(\overline{k})$.

258 In particular, it follows from relations (1), that if (x, y) is a nonspecial pair of points, defined and conjugate over a quadratic extension K/k, then the K-automorphism $t_xt_{x\circ y}t_y$ of $V \otimes K$ is obtained by

the ground field extension from some k-automorphism of V which we shall denote $s_{x,y}$.

Theorem 2. *Suppose moreover that k is perfect and the surface V is k-minimal (i.e. any birational k-morphism $V \to V'$ is an isomorphism). Then the group of birational k-automorphisms of V is generated by the subgroup of its projective automorphisms and by elements t_x, $s_{x,y}$ for all nonspecial points $x \in V(k)$ and for all nonspecial pairs of points (x, y), defined and conjugate over some quadratic extension of k.*

We note that in the paper [2] the following statement was proved: *two minimal cubic surfaces are birationally equivalent over k if and only if they are* (*projectively*) *isomorphic*. So Theorems 1 and 2 give us a fairly complete description of the category of such surfaces and birational applications. Note also an analogy with the category of one-dimensional abelian varieties. It suggests the desirability to investigate also rational applications of finite degree ("isogenies").

To compare the two-dimensional case with one-dimensional one note that if $\dim V = 1$, then

$$t_x^2 = 1, \quad (t_x t_y t_z)^2 = 1 \tag{2}$$

for *all triples* (x, y, z) of points of V. But even this is not a complete system of relations: there exist relations, depending on the structure of k and on the particular nature of some points. Thus in dimension 2 the properties of group B are less subtle.

The statement of Theorem 1 without essential changes should be valid for all dimensions $\geqslant 2$.

Now we shall state the main notions necessary to define the groups Γ.

Let V be a geometrically irreducible cubic hypersurface over a field k. Let $C \subset V(k)$ be a Zariski-dense set of points of V. We say that a subset $C' \subset C$ consists of *almost all* points of C, if C' contains the intersection of C with a Zariski-dense open subset of V.

The following definition describes a class of sets C, for which we **259**
can construct "group decompositions".

Definition. *$C \subset V(k)$ is called an admissible set, if the following two conditions are fulfilled.*

(a) *C is Zariski-dense and consists only of nonsingular points of V.*

(b) *Let $x \in C$. Then for almost all points $y \in C$ the point $t_x(y)$ is defined and belongs to C.*

The following sets, if they are dense, are admissible:

1. the set of all nonsingular points of V;
2. the set of hyperbolic points of a nonsingular cubic surface over $\mathbf{R}$, if it is a connected component of $V(\mathbf{R})$;
3. the minimal set, containing a given system of points and closed under the relation L.

Theorem 3. *Let $C \subset V(k)$ be an admissible set. Let $\Gamma(C)$ be the abelian group, generated by the family of symbols $C(x)$ for all $x \in C$, subject to following relations :*

$$2C(x) = 0, \quad C(x) + C(y) + C(z) = 0 \tag{3}$$

for all triples of different points x, y, $z \in C$, lying on a line not belonging to V. Then the map of sets

$$C \to \Gamma(C) : x \to C(x)$$

is surjective. Thus the group $\Gamma(C)$ is isomorphic to a group of classes of C under certain equivalence relation, with the composition law "drawing a line through two points".

Moreover, the union of classes, corresponding to any subgroup of $\Gamma(C)$, is an admissible set.

From this theorem it is clear that any different "group decomposition" of $\Gamma(C)$ (c.f. Introduction) can be obtained from the one constructed by collecting together cosets of some subgroup of $\Gamma(C)$.

As was stated, for $\dim V = 1$ and $C = V(k)$ we have $\Gamma(C) = V(k)/2V(k)$. A certain modification of this result remains valid in the dimension two.

260 **Theorem 4.** *Let C be an admissible subset of nonspecial points of a nonsingular cubic surface V. Let $B(C)$ be the group of birational automorphisms of V, generated by t_x, $x \in C$, and $B_0(C) \subset B(C)$ the normal subgroup generated by $s_{x,y}$ for all nonspecial pairs $(x, y) \in C \times C$. Then*

$$\Gamma(C) \simeq B(C)/B_0(C).$$

This is an easy consequence of the Theorem 1 and so probably generalizes to all dimensions $\geqslant 2$.

Our knowledge of groups $\Gamma(C)$ is very poor. Unlike groups B they depend on the constant field k in a subtle way. For example, if C consists of all nonsingular points of V in an algebraically closed field, the $\Gamma(C) = \{0\}$ (as is the one-dimensional case: the group of points of an abelian variety is divisible).

On the other hand, I can construct examples of nonsingular cubic surfaces V over number fields k such that the group $C(V(k))$ has arbitrarily many generators. I do not know whether it can be infinite. Perhaps for a reasonable class of varieties a kind of "weak Mordell-Weil theorem" is true.

We wish to formulate some more problems which arise naturally in connection with our construction. Beginning with some admissible set C_0, denote by $C_1 \subset C_0$ the identity class of $\Gamma(C_0)$. As C_1 is admisible, we can construct the identity class of $\Gamma(C_0)$, and so on. Let C_{i+1} be the identity class of $\Gamma(C_i)$; we get a sequence of sets of points $C_0 \supset C_1 \supset C_2 \supset \cdots$ and of groups $\Gamma(C_i)$.

Let k be a number field and $C_0 = V(k)$. Does the sequence $\{C_i\}$ stabilize? What is the intersection $\cap C_i$? (In the one-dimensional case it consists of points of odd order). Is it possible to "put together" all of groups $\Gamma(C_i)$ by constructing an universal group and say, a normal series of it, with factors $\Gamma(C_i)$?

Theorem 4 shows a certain approach to the last question. In fact, $\Gamma(C_i) = B(C_i)/B_0(C_i)$, where

$$B(C_0) \supset \cdots \supset B(C_i) \supset B_0(C_i) \supset B(C_{i+1}) \supset \cdots$$

But the "gaps" $B_0(C_i) \supset B(C_{i+1})$ are probably nontrivial.

261 One of the serious obstacles to studying $\Gamma(V(k))$ is the lack of a manageable criterion for a point to be in the identity class. More generally, one would like to have a substitute for the basic homomorphism

$$\delta : V(k)/2V(k) \to H^1(k,_2V(k)),$$

of one-dimensional case.

Ideas of proofs. Theorem 1 is proved by a refinement of methods, developed in the section 5 of [2], where a complete proof of Theorem 2 is given.

Let V be a (nonsingular) projective k-surface. Let $Z(V) = \varinjlim \mathrm{Pic}(V')$, where the system of groups Pic is indexed by the set of birational $\bar{k}$-morphisms $V' \to V \otimes \bar{k}$ with natural dominance relation. The group $Z(V)$ is endowed with the following structures all of which are induced from "finite levels".

1. $Z(V)$ is a G-module, where $G = \mathrm{Gal}(\bar{k}/k)$;
2. there is a G-invariant pairing "intersection index" $Z(V) \times Z(V) \to \mathbf{Z}$;
3. there is a G-invariant augmentation $Z(V) \to \mathbf{Z}$: "intersection index with the canonical class";
4. the cone of positive elements $Z^+(V)$;
5. the cone $Z^{++}(V)$, dual to $Z^+(V)$ relative to the pairing.

Now for any k-birational map $f : V' \to V$ an isomorphism $f^* : Z(V) \to Z(V')$ is defined, preserving all of the structures above, and $(fg)^* = f^*g^*$. In particular, the group of birational k-automorphisms of V is represented in the group $Z(V)$.

To use this representation effectively, one must look at the group $Z(V)$ differently.

Construct a prescheme $E(V) = (\cup V')/R$, the sum of all V', k-birationally mapped onto $V \otimes \bar{k}$, by the equivalence relation R, which **262**
patches together the biggest open subsets of V' and V'' isomorphic under the natural birational application $V' \longleftrightarrow V''$.

To understand better the architecture of $E(V)$, look at the simplest morphism $V' \to V$, contracting a line l onto a point x. Patching V' and V by identifying $V' \backslash l$ with $V' \backslash x$, we get a little bit of $E(V)$. If $k = \mathbf{C}$, l is the Riemann sphere; so the result of patching may be viewed as V complemented by a "bubble" blown up from the point $x \in V$. To get the whole $E(V)$ one must blow up bubbles from all points of V, then from all points of these bubbles, and so on. So I suggest to call this $E(V)$ "the bubble space" associated to V.

Now, this bubble space is connected with $Z(V)$ as follows. Each point of $Z(V)(\bar{k})$ defines its bubble to which in turn corresponds a certain element of $Z(V)$. So there is the natural homomorphism

$$\operatorname{Pic}(V \otimes \bar{k}) \oplus C^0(E(V)) \to Z(V)$$

where $C^0(V)$ is the free abelian group, generated by all $\bar{k}$-points of V.

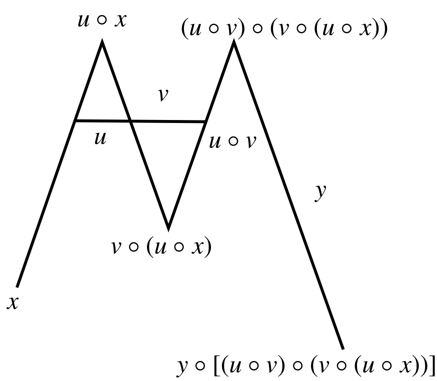
In fact it is an isomorphism. The structures on $Z(V)$, defined earlier, have a rather detailed description in this new setting. On the other hand this presentation of $Z(V)$ is well fit for calculations giving a nice "geometric" set of generators for $Z(V)$.

We omit these calculations which are rather long, especially those in the proof of Theorem 1. Note only some similarity with the description of all relations in Coxeter groups generated by reflections in an Euclidian space.

The proof of Theorem 3 is much more elementary. To prove the surjectivity of $C \to \Gamma(C)$ it is sufficient to find for any $x, y \in C$ a

third point $z \in C$ such that $C(x) + C(y) = C(z)$. We say that points x, $y \in C$ are *in general position*, if the line, containing them, is not tangent to V (and in particular does not belong to V) and the third intersection point of this line with V belongs to C. Then, by definition, $C(x) + C(y) = C(z)$. So one must consider the case, when x, y are not in
263 general position. One can find such points u, $v \in C$, that the following pairs of points are in general position:

$$(x, u); \quad (t_u(x), v); (t_v t_u(x), u \circ v); \quad (t_{u\circ v} t_v t_u(x), y).$$



So we are done:

$$C(x)+C(y) = C(x)+C(u)+C(v)+C(u\circ v)+C(y) = C(y\circ(t_{u\circ v} t_v t_u x)).$$

To find points (u, v) one may work in $V(k)$, because C is dense, and then one sees that such points exist on a plane section through (x, y).

The same argument shows the density and consequently the admissibility of a subgroup of $\Gamma(C)$.

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ON CANONICALLY POLARIZED VARIETIES

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Brief summary of notations and conventions. We shall follow basically notations and conventions of [21], [25], [32]. For basic results on specializations of cycles, we refer to [24]. When U is a complete variety, non-singular in codimension 1 and X a U-divisor, the module of functions g such that $\mathrm{div}(g) + X \succ 0$ will be denoted by $L(X)$. We shall denote by $l(X)$ the dimension of $L(X)$. The complete linear system determined by X will be denoted by $\Lambda(X)$. A finite set of functions (g_i) in $L(X)$ defines a rational map f of U into a projective space. f will be called a rational map of U defined by X. When (g_i) is a basis of $L(X)$, it will be called a *non-degenerate map*. X will be called *ample* if a non-degenerate f is a projective embedding. It will be called *non-degenerate* if a positive multiple of X is ample. (In the terminology of Grothendieck, these are called *very ample* and *ample*). Let W be the image of U by f and Γ the closure of the graph of f on $U \times W$. We shall denote by $\deg(f)$ the number $[\Gamma : W]$. For any U-cycle Y, we shall denote by $f(Y)$ the cycle $pr_2(\Gamma \cdot (Y \times W))$. We shall denote by $\mathfrak{K}(U)$ a *canonical divisor* of U. When X is a Cartier divisor on U, we shall denote by $\mathscr{L}(X)$ the invertible sheaf defined by X. When U is a subvariety of a projective space, C_U will denote a hyperplane section of U. When U is a polarized variety, a *basic polar divisor* will be denoted by X_U.

By an *algebraic family of positive cycles*, we shall understand the set of positive cycles in a projective space such that the set of Chow-points forms a *locally closed* subset of a projective space. By identifying these cycles with their Chow-points, some of the notations and results on points can be carried over to algebraic families and this will be done frequently.

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Finally, $\mathfrak{G}_l$, $\mathfrak{G}_a$, $\sim$ will denote respectively the group of divisors linearly equivalent to zero, the group of divisors algebraically equivalent to zero and the linear equivalence of divisors.

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Introduction. Let V^n be a polarized variety and X_V a basic polar divisor on V. Then the Euler-Poincaré characteristic $\chi(V, \mathscr{L}(mX_V))$ is a polynomial $P(m)$ in m. We have defined this to be the *Hilbert characteristic polynomial* of V (c.f. [16]). If d is the rank of V, i.e. $d = X_V^{(n)}$, any algebraic deformation of V of rank d has the same Hilbert characteristic polynomial $P(m)$ (c.f. [16]). As we pointed out in [16], if we can find a constant c, which depends only on $P(m)$, such that mX_V is ample for $m \geqslant c$, then the existence of a universal family of algebraic deformations of V of bounded ranks follows. The existence of such a constant is well-known in the case of curves and Abelian varieties. We solved this problem for $n = 2$ in [17] (compare [9], [10], [12]). But the complexity we encountered was of higher order of magnitude compared with the case of curves. The same seems to be the case for $n \geqslant 3$ when compared with the case $n = 2$. One of the main purposes of this paper is to solve the problem for $n = 3$ when V is "generic" in the sense that $\mathfrak{K}(V)$ is a non-degenerate polar divisor.

In general, let us consider the following problems.

(A_n) Find a constant c, which depends on the polynomial $P(x)$ only, such that $h^i(V, \mathscr{L}(mX_V)) = 0$ for $i > 0$ whenever $m \geqslant c$.

(A'_n) Find two constants c, c', which depend on $P(x)$ only, such that $h^i(V, \mathscr{L}(mX_V)) < c'$ for $i > 0$ whenever $m \geqslant c$.

(A''_n) Find two constants c, c', which depend on $P(x)$ only, such that $|l(mX_V) - P(m)| < c'$ whenever $m \geqslant c$.

(B_n) Find a constant c, which depends on $P(x)$ only, such that mX_V defines a birational map of V whenever $m \geqslant c$.

(C_n) Find a constant c, which depends on $P(x)$ only, such that mX_V is ample whenever $m \geqslant c$.

As we mentioned, what we are interested in is the solution of (C_n). But (A_n), (B_n) could be regarded as step-stones for this purpose. It is easy to see that the solution of (C_n) implies the solutions of (A_n) and (B_n). In
267 the case of characteristic zero and $\mathfrak{K}(V)$ a non-degenerate polar divisor, (A_n) can be solved easily (§1). Assuming that (A''_n) has a solution, we shall show that (B_n) has a solution for $n = 3$ when the characteristic is zero. Assuming that (A_n) and (B_n) have solutions and that $\mathfrak{K}(V)$ is a non-degenerate polar divisor, we shall show that (C_n) has a solution when the characteristic is zero. Hence (C_3) has a solution when the characteristic is zero and $\mathfrak{K}(V)$ is a non-degenerate polar divisor.

Chapter I. (A''_n) and (B_n)

1 Canonically polarized varieties. We shall first recall the definition of a polarized variety as modified in [16]. Let V^n be a complete non-singular variety and $\mathscr{M}$ a finite set of prime numbers consisting of the characteristic of the universal domain (or the characteristics of universal domains) and the prime divisors of the order of the torsion group of divisors of V. Let $\mathscr{X}$ be a set of V-divisors satisfying the following conditions: (a) $\mathscr{X}$ contains an ample divisor X; (b) a V-divisor Y is contained in $\mathscr{X}$ if and only if there is a pair (r, s) of integers, which are prime to members of $\mathscr{M}$, such that $rY \equiv sX \bmod \mathfrak{G}_a$. When there is a pair $(\mathscr{M}, \mathscr{X})$ satisfying the above conditions, $\mathscr{X}$ is called a *structure set* of polarization and $(V, \mathscr{X})$ a *polarized variety*. A divisor in $\mathscr{X}$ will be called a *polar divisor* of the polarized variety. There is a divisor X_V in $\mathscr{X}$ which has the following two properties: (a) a V-divisor Y is in $\mathscr{X}$ if and only if $Y \equiv rX_V \bmod \mathfrak{G}_)a$ where r is an integer which is prime to members of $\mathscr{M}$; (b) when Z is an ample polar divisor, there is a positive integer s such that $Z \equiv sX_V \bmod \mathfrak{G}_a$ (c.f. [16]). X_V is called a *basic polar divisor*. The self-intersection number of X_V is called the *rank* of the polarized variety. A polarized variety will be called a *canonically polarized variety* if $\mathfrak{K}(V)$ is a polar divisor.

Lemma 1. *Let V^n be a canonically polarized variety and $P(x) = \Sigma_{\gamma_{n-i}} x^i$ the Hilbert characteristic polynomial of V. Then $\mathfrak{K}(V) \equiv \rho X_V \mathrm{mod}\ \mathfrak{G}_a$ where ρ is a root of $P(x) - (-1)^n \gamma_n = 0$.*

Proof. It follows from Serre's duality theorem that

$$\chi(V, \mathscr{L}(mX_V)) = (-1)^n \chi(V, \mathscr{L}(\mathfrak{K}(V) - mX_V)) \quad \text{(c.f. [23]).}$$

We may replace $\mathfrak{K}(V)$ by ρX_V in this equality since χ is invariant under **268**
algebraic equivalence of divisors (c.f. [4], [16]). Then we get $P(m) = (-1)^n P(\rho - m)$. Setting $m = 0$, we get $(-1)^n P(\rho) = \gamma_n$. Our lemma is thereby proved. □

Proposition 1. *Let V be a canonically polarized variety in characteristic zero and $P(x)$ the Hilbert characteristic polynomial of V. Then there is a positive integer ρ_0, which depends on $P(x)$ only, such that $h^i(V, \mathscr{L}(Y)) = 0$ for $i > 0$ and $h^0(V, \mathscr{L}(Y)) = l(Y) > 0$ whenever $m \geqslant \rho_0$ and $Y \equiv mX_V \mathrm{mod}\ \mathfrak{G}_a$.*

Proof. Let γ_n be the constant term of $P(x)$ and s_0 the maximum of the roots of the equation $P(x) - (-1)^n \gamma_n = 0$. Let $\mathfrak{K}(V)$ be a canonical divisor of V and k an algebraically closed common field of rationality of V, X_V and $\mathfrak{K}(V)$. There is an irreducible algebraic family $\mathfrak{H}$ of positive divisors on V, defined over k, such that, for a fixed k-rational divisor C_0 in $\mathfrak{H}$, the classes of the $C - C_0$, $C \in \mathfrak{H}$, with respect to linear equivalence exhausts the points of the Picard variety of V (c.f. [15]). We shall show that $2s_0$ can serve as ρ_0.

Take t so that $t - (s_0 - \rho) = t' > 0$ and $t - 2(s_0 - \rho) = m > 0$ where $\mathfrak{K}(V) \equiv \rho X_V \mathrm{mod}\ \mathfrak{G}_a$. For any C' in $\mathfrak{H}$, $tX_V + C' - C_0 + \mathfrak{K}(V) \equiv (t' + s_0) X_V \mathrm{mod}\ \mathfrak{G}_a$ and $t' + s_0 > 0$. Hence $tX_V + C' - C_0 + \mathfrak{K}(V)$ is non-degenerate (c.f. [16], Th. 1) and the higher cohomology groups of the invertible sheaf $\mathscr{A}$ determined by $tX_V + C' - C_0 + 2\mathfrak{K}(V)$ vanish by Kodaira vanishing theorem (c.f. [11]). $tX_V + C' - C_0 + 2\mathfrak{K}(V) \equiv (m + 2s_0) X_V \mathrm{mod}\ \mathfrak{G}_a$ and $\chi(V, \mathscr{A}) = P(m + 2s_0)$ since χ is invariant by algebraic equivalence of divisors. Moreover, $P(m + 2s_0) > 0$ by our choice of m and s_0. It follows that $h^0(V, \mathscr{A} > 0$. Therefore, in order to complete our proof, it is enough to prove that a V-divisor Z such that $Z \equiv (m + 2s_0)X_V \bmod \mathfrak{G}_a$ has the property that $Z \sim tX_V + C' - C_0 + 2\mathfrak{K}(V)$ for some C' in $\mathfrak{H}$. Clearly, such Z is algebraically equivalent to $tX_V + 2\mathfrak{K}(V)$. Hence

$$Z - (tX_V + 2\mathfrak{K}(V)) \sim C' - C_0$$

for some C' in $\mathfrak{H}$. Our proposition is thereby proved. □

269 **2 Estimation of $l(C_U)$ on a projective surface.** Let V be a non-singular surface in a projective space and Γ a curve on V. Let $\mathfrak{R}$ be the intersection of local rings of Γ at the singular points of Γ. Using only those functions of Γ which are in $\mathfrak{R}$, we can define the concept of complete linear systems and associated sheaves, as on a non-singular curve. Let Γ' be a V-divisor such that $\Gamma' \sim \Gamma$, that Γ' and Γ intersect properly on V and that no singular point of Γ is a component of $\Gamma \cdot \Gamma'$. Similarly let $\mathfrak{K}(V)$ be such that $\mathfrak{K}(V)$ and Γ intersect properly on V and that no singular point of Γ is a component of $\Gamma \cdot \mathfrak{K}(V)$. Then $\Gamma \cdot (\Gamma' + \mathfrak{K}(V)) = \mathfrak{K}(\Gamma)$ is a canonical divisor of Γ, $p_a(\Gamma) = 1 + \frac{1}{2}\deg(\mathfrak{K}(\Gamma))$ and the generalized Riemann-Roch theorem states that $l(\mathfrak{m}) = \deg(\mathfrak{m}) - p_a(\Gamma) + 1 + l(\mathfrak{K}(\Gamma) = \mathfrak{m})$ for a Γ-divisor $\mathfrak{m}$ (c.f. [20], [22]).

If C is a complete non-singular curve, the theorem of Clifford states that $\deg(\mathfrak{m}) \geqslant 2l(\mathfrak{m}) - 2$ for a *special* C-divisor $\mathfrak{m}$. We shall first extend this to Γ.

Lemma 2. *Let V, Γ and $\mathfrak{K}(\Gamma)$ be as above and $\mathfrak{m}$ a special positive divisor on Γ (i.e. $l(\mathfrak{K}(\Gamma) - \mathfrak{m}) > 0$). Then* $\deg(\mathfrak{m}) \geqslant 2l(\mathfrak{m}) - 2$.

Proof. When $l(\mathfrak{m}) = 1$, our lemma is trivial since $\mathfrak{m}$ is positive. Therefore, we shall assume that $l(\mathfrak{m}) > 1$.

Let Γ^* be a normalization of Γ, α the birational morphism of Γ^* on Γ and T the graph of α. For any Γ-divisor $\mathfrak{a}$, we set $\mathfrak{a}^* = \alpha^{-1}(\mathfrak{a}) = pr_{\Gamma^*}((\mathfrak{a} \times \Gamma^*) \cdot T)$. When the f are elements of $L(\mathfrak{a})$, i.e. elements of $\mathfrak{R}$ such that $\operatorname{div}(f) + \mathfrak{a} > 0$, the $f \circ \alpha = f^*$ generate a module of functions on Γ^* which we shall denote by $\alpha^{-1}L(\mathfrak{a})$. The module $L(\mathfrak{K}(\Gamma) - \mathfrak{m})$ is not empty since $\mathfrak{m}$ is special. Hence it contains a function g in $\mathfrak{R}$. Let

$\operatorname{div}(g^*) = \mathfrak{m}^* + \mathfrak{n}^* - \mathfrak{K}(\Gamma)^*$. Let N be the submodule of functions f^* in $\alpha^{-1}L(\mathfrak{K}(\Gamma))$ defined by requiring f^* to pass through $\mathfrak{n}^*$, i.e. requiring f^* to satisfy: coefficient of x in $\operatorname{div}(f^*) \geqslant$ coefficient of x in $\mathfrak{n}^*$ for each component x of $\mathfrak{n}^*$. Let $\dim N = \dim L(\mathfrak{K}(\Gamma)) - t$. Then $\mathfrak{n}^*$ imposes t linearly independent conditions in $\alpha^{-1}L(\mathfrak{K}(\Gamma))$. When these t linear conditions are imposed on the vector subspace $\alpha^{-1}L(\mathfrak{K}(\Gamma) - \mathfrak{m})$, we get the vector space generated by g^* over the universal domain. It follows that $l(\mathfrak{K}(\Gamma) - \mathfrak{m}) - t \leqslant 1$, i.e. $t \geqslant l(\mathfrak{K}(\Gamma) - \mathfrak{m}) - 1$. Since $l(\mathfrak{K}(\Gamma)) = p_a(\Gamma)$, **270**
we then get $p_a(\Gamma) - \dim N \geqslant (\mathfrak{K}(\Gamma) - m) - 1$. $L(\mathfrak{m})$ has a basis (h_i) from $\mathfrak{R}$. The $h_i \cdot g$ are elements of $L(\mathfrak{K}(\Gamma))$ and $h_i^* \cdot g^* \in N$. Hence the multiplication by g^* defines an injection of $\alpha^{-1}L(\mathfrak{m})$ into N. It follows that $l(\mathfrak{m}) \leqslant \dim N$ and $p_a(\Gamma) - l(\mathfrak{m}) \geqslant l(\mathfrak{K}(\Gamma) - \mathfrak{m}) - 1$. By the generalized Riemann-Roch theorem, we have $l(\mathfrak{K}(\Gamma) - \mathfrak{m}) = l(\mathfrak{m}) - \deg(\mathfrak{m}) + p_a(\Gamma) - 1$. When this is substituted above, we get the required inequality. □

Proposition 2. *Let V be a non-singular surface in a projective space such that $p_g \geqslant 1$. Let C_0 be a curve on V such that the complete linear system $\Lambda(C_0)$ is without fixed point and that $C_0^{(2)} > 0$. Then* $\dim \Lambda(C_0) \leqslant \frac{1}{2}C_0^{(2)} + 1$.

Proof. If $\dim \Lambda(C_0) = 0$, there is nothing to prove. Therefore we shall assume that $\dim \Lambda(C_0) > 0$. Let $\mathfrak{R}$ be the intersection of local rings of C_0 at the singular points of C_0. There is a canonical divisor $\mathfrak{K}(V)$ of V whose support does not contain any singular point of C_0. Such $\mathfrak{K}(V)$ and C_0 intersect properly on V. There is a member C of $\Lambda(C_0)$ which does not contain any singular point of C_0. Such C and C_0 intersect properly on V. We have

$$p_a(C_0) = 1 + \frac{1}{2}\deg(C_0 \cdot (C + \mathfrak{K}(V))$$

and $C_0 \cdot (C + \mathfrak{K}(V)) = \mathfrak{k}$ is a canonical divisor of C_0. Let $\mathfrak{m} = C_0 \cdot C$. By the generalized Riemann-Roch theorem, we get $l(\mathfrak{m}) = \deg(\mathfrak{m}) - p_a(C_0) + 1 + l(\mathfrak{k} - \mathfrak{m})$ and $\mathfrak{k} - \mathfrak{m} = C_0 \cdot \mathfrak{K}(V)$. By our assumption there is a function f, other than 0, in the module $L(\mathfrak{K}(V))$. Since the support of $\mathfrak{K}(V)$ does not contain any singular point of C_0, f is regular at these points. Let f' be the function on C_0 induced by $f \cdot f'$ is then regular at every singular point of C_0 and is contained in $\mathfrak{R}$. Since $\operatorname{div}(f) + \mathfrak{K}(V) > 0$ it follows that $\operatorname{div}(f') + C_0.\ \mathfrak{K}(V) > 0$. Hence $f' \in L(C_0 \cdot \mathfrak{K}(V)) = L(\mathfrak{k} - \mathfrak{m})$, which proves that $l(\mathfrak{k} - \mathfrak{m}) > 0$ and that $\mathfrak{m}$ is a special C_0-divisor.

By Lemma 2 we have $\deg(\mathfrak{m}) = C_0^{(2)} \geqslant 2l(\mathfrak{m}) - 2$. Every function g in $L(C)$ induces a function g' on C_0 contained in $\mathfrak{R}$ since $C_0 \cdot C$ has no singular component on C_0. Hence $\operatorname{div}(g') + \mathfrak{m} > 0$ and it follows that
271 $g' \in L(\mathfrak{m})$. If $g' = 0$, $\operatorname{div}(g) = C_0 - C$. Consequently $l(\mathfrak{m}) \geqslant l(C) - 1$ and $C_0^{(2)} \geqslant 2l(C) - 4$. Our proposition follows at once from this.

We shall recall here the definition of the *effective geometric genus* of an algebraic variety W. Let W and W' be complete normal varieties and assume that there is a birational morphism of W' on W. Then $p_g(W) \geqslant p_g(W')$. Hence there is a complete normal variety W'', birationally equivalent to W, such that $p_q(W'')$ has the minimum value p_g among the birational class of W. This p_g is called the *effective geometric genus* of W. When W is non-singular, $p_g(W) = p_g$ (c.f. [13], [31]). □

Proposition 3. *Let U be an algebraic surface in a projective space and C_U a hyperslane section of U. Assume that the effective geometric genus p_g of U is at least* 1. *Let Λ be the linear system of hyperplane sections of U and denote by $C_U^{(2)}$ the degree of U. Then* $\dim \Lambda \leqslant \frac{1}{2} C_U^{(2)} + 1$.

Proof. Let V be a non-singular surface in a projective space and f a birational morphism of V on U (c.f. [34]). Let k be an algebraically closed common field of rationality of U, V and f. Let T be the graph of f on $V \times U$ and P the ambient projective space of U. We may assume that U is not contained in any hyperplane. Then T and $V \times H$ intersect properly on $V \times P$ for every hyperplane H. We set $f^{-1}(H) = \mathrm{pr}_V(T \cdot (V \times H))$. Let Λ^* be the set of $f^{-1}(H)$. It is a linear system on V. Since f is a morphism, it has no fixed point. Therefore it has no fixed component in particular. Let H be a generic hyperplane over k and $C_U = U \cdot H$. The $\mathrm{pr}_V(T \cdot (V \times H)) = \mathrm{pr}_V(T \cdot (V \times C_U))$ where the latter intersection-product is taken on $V \times U$ (c.f. [25], Chap. VIII). Setting

$f^{-1}(C_U) = \mathrm{pr}_V(T \cdot (V \times C_U))$, $f^{-1}(C_U)$ is a generic member of Λ^* over k. Since f is a birational transformation, every component of $f^{-1}(C_U)$ has to appear with coefficient 1. It follows that $f^{-1}(C_U)$ is an irreducible curve by the theorem of Bertini (c.f. [25], Chap. IX) and Λ^* has no fixed point. Moreover, $p_g(V) = p_g$ and $\dim \Lambda = \dim \Lambda^*$.

Let C_U and C'_U be two independent generic members of Λ over k. When Q is a component of $C_U \cap C'_U$, it is a generic point of U over k and is a proper component of multiplicity 1 on U. When that is so, we **272**
get $f^{-1}(C_U) \cdot f^{-1}(C'_U) = f^{-1}(C_U \cdot C'_U)$ (c.f. [25], Chap. VIII). Then $\deg(C_U \cdot C'_U) = \deg(f^{-1}(C_U) \cdot f^{-1}(C'_U))$ and our proposition follows from these and from Proposition 2. □

3 A discussion on fixed components. Let V^n be a complete variety, non-singular in codimension 1 and X a divisor on V. We denote by $\Lambda(X)_{\mathrm{red}}$ the reduced linear system determined uniquely by $\Lambda(X)$. Then $\Lambda(X) = \Lambda(X)_{\mathrm{red}} + F$ and F is called the *fixed part* of $\Lambda(X)$. A component of F is called *a fixed component* of $\Lambda(X)$.

Lemma 3. *Let V^n be a complete variety, non-singular in codimension* 1*, X a positive V-divisor and $F = \Sigma_1^l a_i F_i$ the fixed part of $\Lambda(X)$. Assume that $l((\alpha-1)X + F) > l((\alpha-1)X)$ for some positive integer $\alpha > 1$ and that $X \neq F$. Then we have the following results:* (a) *there is a positive divisor $F' = \Sigma_1^l a'_i F_i$ such that $F - F' \succ 0$, that $l((\alpha-1)X + F') = l((\alpha-1)X + F)$ and that $l((\alpha-1)X + F' - F_j) < l((\alpha-1)X + F)$ for all j with $F' - F_j \succ 0$;* (b) *let I be the set of indices i such that $a'_i \neq 0$. Then the F_i, $i \in I$, are not fixed components of $\Lambda((\alpha-1)X + F')$.*

Proof. (a) follows immediately from our assumption. Let k be an algebraically closed common field of rationality of V, X and for the components of F. Let L be a generic divisor of $\Lambda((\alpha-1)X + F')_{\mathrm{red}}$ over k. The fixed part of $\Lambda((\alpha-1)X + F')$ is obviously of the form $\Sigma_1^l b_s F_s$. Suppose that $b_i \neq 0$ for some $i \in I$. Then $L + \Sigma_1^l b_s F_s - F_i$ is positive and is a member of $\Lambda((\alpha-1)X + F' - F_i)$. Hence $l((\alpha-1)X + F') = l((\alpha-1)X + F' - F_i)$ which is contrary to our choice of F'. Our lemma is thereby proved. □

Lemma 4. *Using the same notations and assumptions of Lemma 3, let* $\beta = \Pi_I a'_i$. *Then there is the smallest positive integer* γ *satisfying* $\beta(a_i - a'_i) - \gamma a'_i \geqslant 0$ *for* $i \in I$ *and* $\beta(a_{i_0} - a'_{i_0}) - \gamma a'_{i_0} = 0$ *for some* $i_0 \in I$. *Moreover,* $\Lambda(\beta\alpha X + \gamma(\alpha - 1)X)$ *has the property that* F_{i_0} *is not a fixed component of it.*

Proof. Let Z be a generic divisor of $\Lambda(X)_{\mathrm{red}}$ over k. Then $\alpha X \sim (\alpha - 1)X + \Sigma_1^l a_i F_i + Z = ((\alpha - 1)X + \Sigma_I a'_i F_i) + \Sigma_I (a_i - a'_i)F_i + F'' + Z$
273 for some positive divisor F'' which does not contain the F_i, $i \in I$, as components. By the above lemma, $\Lambda((\alpha-1)X+\Sigma_I a'_i F_i)$ has the property that no F_i, $i \in I$, is a fixed component of it. Let m be a positive integer. Then $(\beta + m)((\alpha - 1)X + \Sigma_I a'_i F_i) + \beta Z + \Sigma_I(\beta a_i - \beta a'_i - m a'_i)F_i + \beta F'' \sim \beta\alpha X + m(\alpha - 1)X$. By what we have seen above. the F_i, $i \in I$, are not fixed components of the complete linear system determined by $(\beta + m)((\alpha - 1)X + \Sigma_I a'_i F_i) + \beta Z$. By the definition of β, the $\beta(a_i - a'_i)$ are divisible by the a'_i. Hence we can find the smallest positive integer γ as claimed in our lemma. Then there is an index $i_0 \in I$ such that F_{i_0} is not a component of $\Sigma_I(\beta a_i - \beta a'_i - \gamma a'_i)F_i$. From these our lemma follows at once. □

Proposition 4. *Let* V^n *be a complete non-singular variety,* X *a positive non-degenerate divisor on* V *and* F *the fixed part of* $\Lambda(X)$. *Assume that there is an integer* $\alpha > 1$ *such that* $l((\alpha - 1)X + F) > l((\alpha - 1)X)$ *and that* $X \neq F$. *Let* $d = X^{(n)}$ *and* $\mu(X, \alpha) = (d^d\alpha + d^{d+1}(\alpha - 1))!$. *Then there is a component* F_i *of* F *such that it is not a fixed component of the complete linear system determined by* $\mu(X, \alpha)X$.

Proof. We shall estimate β and γ of Lemma 4. Let Z be as in the proof of Lemma 4 and $I(\ ,\ )$ denote the intersection number. Since $X \sim Z + \Sigma_1^l a_i F_i$, $X^{(n)} = I(X^{(n-1)}, Z) + \Sigma_1^l a_i I(X^{(n-1)}, F_i) = d$. Hence $a_i > 0$ and $\Sigma_1^l a_i < d$. It follows that $\beta < \Pi_I a_i < d^d \cdot \gamma$ satisfies $\beta(a_i - a'_i) \geqslant \gamma a'_i$. Hence $\gamma \leqslant \beta(a_i - a'_i) < \beta a_i < d^{d+1}$. Our proposition now follows from these and from Lemma 4. □

4 Estimation of some intersection numbers and its application. Let W^n be a non-singular projective variety and T^{n-1} a subvariety of W. Let Y be a non-degenerate divisor on W and k a common field of rationality of W, T and Y. Let m be a positive integer such that $l(mY) > 1$ and $A_1, \dots, A_n n$ independent generic divisors of $\Lambda(mY)$ over k. Let the D_i be the proper components of $|A_1| \cap \dots \cap |A_s|$ on W and $a_i = \text{coef.}_{D_i}(A_1 \dots A_s)$. Let I be the set of indices i such that D_i contains a generic point of W over k. Then we define the symbol $I(A_1 \dots A_s/W, k)$ to denote $\Sigma_I a_i$. We shall denote by $(A_1 \dots A_s/W, k)$ the W-cycle $\Sigma_I a_i D_i$. Let the E_j be the proper components of $|A_1| \cap \dots \cap |A_s| \cap |T|$ on W and $b_j = \text{coef.}_{E_j}(A_1 \dots A_s \cdot T)$. Let J be the set of indices j such that E_j contains a generic point of T over k. Then we denote by $(A_1 \dots A_s \cdot T/T, k)$ and by $I(A_1 \dots A_s \cdot T/T, k)$ the W-cycle $\Sigma_J b_j E_j$ and the number $\Sigma_J b_j$ respectively. **274**

Lemma 5. *Let W^n be a non-singular projective variety and T^{n-1} a subvariety of W, both defined over a field k. Let Y be a non-degenerate W-divisor, rational over k, and m a positive integer such that $l(mY) > 1$. Let $A_1, \dots, A_s$, $s \leqslant n$, be s independent generic divisors of $\Lambda(mY)$ over k. Then we have the following inequalities:* (a) $I(A_1 \dots A_s/W, k) \leqslant m^s Y^{(n)}$; (b) $I(A_1 \dots A_s \cdot T/T, k) \leqslant m^s I(Y^{(n-1)}, T)$.

Proof. We shall prove only (b). (a) can be proved similarly. We set $\Sigma b_i E_i = A_1 * \dots * A_s * T$. If the b_i are zero for all i, our lemma is obviously true. Hence we shall assume that the b_i are positive.

Let r be a large positive integer such that $rm\,Y$ is ample and $C_1, \dots, C_s, C'_1, \dots, C'_{n-s-1} n-1$ independent generic divisors of $\Lambda(rm\,Y)$ over k. Since $(A_1 \dots A_s \cdot T/T, k) \prec A_1 * \dots * A_s * T$, it follows that $I(A_1 \dots A_s \cdot T/T, k) < \Sigma b_i$. Since Y is non-degenerate, $\Sigma b_i I(E_i, Y^{(n-s-1)}) \geqslant \Sigma b_i$. We have

$$(1/(rm)^{n-s-1}) \deg\{(A_1 * \dots * A_s * T) \cdot C'_1 \dots C'_{n-s-1}\} = \Sigma b_i I(E_i, Y^{(n-s-1)}).$$

The left hand side can be written as

$$(1/r^s (rm)^{n-s-1}) \deg\{(rA_1 * \dots * rA_s * T) C'_1 \dots C'_{n-s-1}\}.$$

The rA_i are members of $\Lambda(rm\ Y)$. Hence $((rA_i), T, (C'_j))$ is a specialization of $((C_i), T, (C'_j))$ over k. Let $\mathfrak{m} = C_1 \dots C_s \cdot T \cdot C'_1 \dots C'_{n-s-1}$ and $\mathfrak{m}'$ an arbitrary specialization of $\mathfrak{m}$ over k over the above specialization. Then, when Q is a component of $rA_1 * \dots * rA_s.\ T \cdot C'_1 \dots C'_{n-s-1}$ with the coefficient v, Q appears exactly v times in $\mathfrak{m}'$ by the compatibility of specializations with the operation of inter-section-product (c.f. [24]). It follows that $\deg(\mathfrak{m}) = (mr)^{n-1} I(Y^{(n-1)}, T) \geqslant \deg\{(rA_1 * \dots * rA_s * T)C'_1 \dots C'_{n-s-1}\}$. (b) follows easily from these. □

As an application of Lemma 5, we shall prove the following proposition which we shall need later.

275 **Proposition 5.** *Let W^n be a non-singular projective variety and Y a non-degenerate divisor on W. Let m be a positive integer such that $l(mY) > Y^{(n)} m^{n-1} + n - 1$. Let f be a non-degenerate rational map of W defined by mY. Then* $\deg(f) \neq 0$, *i.e. the image of W by f has dimension n.*

Proof. By our assumption, $l(mY) > Y^{(n)} m^s + s$ for $1 \leqslant s \leqslant n-1$. Let U^s be the image of W by f. Then $\Lambda(mY)_{\rm red}$ consists of $f^{-1}(H)$ where H denotes a hyperplane in the ambient space of U. Let k be a common field of rationality of W, Y and f and $A_1, \dots, A_s$ (resp. $B_1, \dots, B_s$) independent generic divisors of $\Lambda(mY)$ (resp. $\Lambda(mY)_{\rm red}$) over k. As is well known and easy to prove by means of the intersection theory. $\deg(U) \leqslant I(B_1 \dots B_s/W, k)$. Moreover, $I(B_1 \dots B_s/W, k) = I(A_1 \dots A_s/W, k)$ as can be seen easily. Then we get $\deg(U) \leqslant m^s Y^{(n)}$ by Lemma 5. Let Λ be the linear system of hyperplane sections of U. We have $\dim \Lambda \leqslant m^s Y^{(n)} + s - 1$ (c.f. [17]). On the other hand, $l(mY) = \dim \Lambda(mY) + 1 = \dim \Lambda(mY)_{\rm red} + 1 = \dim \Lambda + 1$. This contradicts our assumption if $s < n$. Our proposition is thereby proved. □

5 A solution of (B_3), (I).

First, we shall fix some notations which shall be used through the rest of this chapter. Let V^3 be a polarized variety of dimension 3 and $P(m) = \Sigma_0^3 \gamma_{3-i} m^i$ the Hilbert characteristic polynomial of V. Let $d = X_V^{(3)}$. As is well known, $\gamma_0 = d/3!$ As we pointed out in our introduction, *we shall solve* (B_3) *under the assumption that* (A_3'') *has a solution*. As we showed in Proposition 1, (A_3'') has a solution when V is canonically polarized and the characteristic is zero. Therefore, *we shall assume that there are two constants* c_0 *and* c, *which depend on the polynomial* $P(x)$ *only, such that* $|l(mX_V) - P(m)| < c$ *whenever* $m \geqslant c_0$. We shall denote by Σ the set of polarized varieties of dimension 3 such that $P(x)$ is their Hilbert characteristic polynomial. We shall use V to denote a "variable element" of Σ.

From our basic assumption, we can find a positive integer $e_0 \geqslant c_0$, which depends on $P(x)$ only, such that $l(mX_V) > dm^2 + 2$ for $m \geqslant e_0$. For such m *a non-degenerate rational map of* V *defined by* mX_V *maps* V **276**
generically onto a variety of dimension 3 by Proposition 5. Moreover, when that is so, $\Lambda(mX_V)_{\mathrm{red}}$ is irreducible, i.e. it contains an irreducible member (c.f. [25], Chap. IX).

The following lemma has been essentially proved in the course of the proof of Proposition 4.

Lemma 6. *Let* $r \geqslant e_0$ *and* $F' = \Sigma_1^{v} b_i F_i$ *the fixed part of* $\Lambda(rX_V)$. *Then* $v \leqslant r^3 d$, $\Sigma_1^{v} b_i < r^3 d$ *and* $b_i < r^3 d$.

When r is a positive integer, we shall set $P(rx) = P_r(x)$ to regard it as a polynomial in x. For two positive integers r and m, we set $\delta(r,m) = 3(\gamma_0 r^3)m^2 - (1/rd)m^2 + 2c + 2$; $\delta'(r,m) = 2(\gamma_0 r^3)m^2 + 2c + 2$.

Lemma 7. *When a positive integer* r *is given, it is possible to find a positive integer* α', *which depends on* $P(x)$ *and* r *only, such that* $P_r(m) - P_r(m-1) > \delta(r,m)$ *and* $> \delta'(r,m)$ *whenever* $m \geqslant \alpha'$.

Proof. $P_r(m) = \gamma_0 r^3 m^3 + \cdots$ and $P_r(m) - P_r(m-1) = 3(\gamma_0 r^3)m^2 + \cdots$ Therefore such a choice of α' is possible. □

Proposition 6. *There are positive integers* v, $\overline{v}$, *which depend on* $P(x)$ *only, and* e, α, *which depend on a member* V *of* Σ, *with the following properties :* (a) $e_0 \leqslant e \leqslant v$, $\alpha \leqslant \overline{v}$; (b) *when* $F = \Sigma_1^{l} a_i F_i$ *is the fixed part of* $\Lambda(e_0 X_V)$, *there is a positive integer* t *such that* $t \leqslant 1 < e_0^3 d$ *and that* $F_1, \ldots, F_t$ *are not fixed components of* $\Lambda(eX_V)$; (c) $P_e(m) - P_e(m-1) >$

$\delta(e,m)$ *and* $> \delta'(e,m)$ *for* $m \geqslant \alpha$*;* (d) *when* F^* *is the fixed part of* $\Lambda(eX_V)$*,* $l((\alpha-1)eX_V + F^*) = l((\alpha-1)eX_V)$*.*

Proof. We can choose α_0, depending on $P(x)$ only, such that $P_{e_0}(m) - P_{e_0}(m-1) > \delta(e_0, m)$ and $> \delta'(e_0.m)$ whenever $m \geqslant \alpha_0$ by Lemma 7. Assume that $l((\alpha_0 - 1)e_0X_V + F) > l((\alpha_0 - 1)e_0X_V)$. Then, after rearranging indices if necessary, we may assume that F_1 is not a fixed component of the complete linear system determined by $\mu(e_0X_V, \alpha_0)e_0X_V$ by Proposition 4, where

$$\mu(e_0X_V, \alpha_0) = (s^s\alpha_0 + s^{s+1}(\alpha_0 - 1)! \quad \text{and} \quad s = e_0^3 d.$$

Let $e_1 = e_0\mu(e_0X_V, \alpha_0)$ and F' the fixed part of $\Lambda(e_1X_V)$. By Lemma 7 we can find a positive integer α_1, depending on $P(x)$ only, so
277 that $P_{e_1}(m) - P_{e_1}(m-1) > \delta(e_1, m)$ and $> \delta'(e_1, m)$ whenever $m \geqslant \alpha_1$. Assume still that $l((\alpha_1 - 1)e_1X_V + F') > l((\alpha_1 - 1)e_1X_V)$. Then the complete linear system determined by $\mu(e_1X_V, \alpha_1)e_1X_V$ has not F_2 as a fixed component, after rearranging indices if necessary (c.f. Proposition 4).

Since $l < e_0^3 d$ by Lemma 6, this process has to terminate by at most $e_0^3 d - 1$ steps. Positive integers e_i, α_i we choose successively can be chosen so that they depend only on $P(x)$. To fix the idea, let us choose α_i as follows. Let r_i be the largest among the set of roots of the equations $P_{e_i}(x) - P_{e_i}(x-1) - \delta(e_i, x) = 0$, $P_{e_i}(x) - P_{e_i}(x-1) - \delta'(e_i, x) = 0$ and let $\alpha_i = |r_i| + 1$. Suppose that our process terminates when we reach to the pair (e_t, α_t). Then $t \leqslant l < e_0^3 d$. From the definition of the function μ (c.f. Proposition 4) and from our choice of the α_i, we can find easily upperbounds v, $\overline{v}$ for e_t, α_t which depend on $P(x)$ only and not on t. Moreover, when F'' is the fixed part of $\Lambda(e_tX_V)$, $F_1, \ldots, F_t$ are not components of F'' and $l((\alpha_t - 1)e_tX_V + F'') = l((\alpha_t - 1)e_tX_V)$ by our assumption made above. Our proposition follows from these at once. □

6 A solution of (B_3), (II). In this paragraph, we shall keep the notations of Proposition 6. Let k be an algebraically closed common field of rationality of V and X_V and T a generic divisor of $\Lambda(eX_V)_{\mathrm{red}}$

over k. $\Lambda(\alpha e X_V)$ is canonically associated to the module $L(\alpha e X_V)$. Let K be a field of rationality of T over k and L' the module of rational functions on T induced by $L(\alpha e X_V)$. Let g be a non-degenerate rational map of T defined by L', defined over K. Let U be the image of T by g. A non-degenerate rational map of V defined by meX_V maps V generically onto a variety of dimension 3 since $me \geqslant e \geqslant e_0$ by Proposition 5, whenever $m > 0$. Since T is a generic divisor of $\Lambda(eX_V)_{\mathrm{red}}$ over k, it follows that $\dim U = 2$. Let h be a non-degenerate rational map of T defined by the module of rational functions on T which is induced by $L(eX_V)$. Then the image W of T by h has dimension 2 also as we have seen above.

We shall establish some inequalities.

$$\dim T_{r_T}\Lambda(\alpha e X_V) > \frac{1}{2}e^3\alpha^2 d - (1/ed)\alpha^2 + 1. \tag{6.1}$$

In fact, $\dim T r_T \Lambda(\alpha e X_V) = l(\alpha e X_V) - l(\alpha e X_V - T) - 1 = l(\alpha e X_V) -$ **278**
$l((\alpha-1)eX_V + F^*) - 1 = l(\alpha e X_V) - l((\alpha-1)eX_V) - 1$ by our choice of α and e (c.f. Proposition 6). By Proposition 6, (c) and by the equality $\gamma_0 = d/6$, $l(\alpha e X_V) - l((\alpha-1)eX_V) > P_e(\alpha) - P_e(\alpha-1) - 2c > \frac{1}{2}e^3\alpha^2 d - (1/ed)\alpha^2 + 2$. Our inequality is thereby proved.

The following formulas are well known and easy to prove by means of the intersection theory (c.f. [25], Chap. VII).

$$\begin{aligned} \deg(g)\cdot\deg(U) &= I(A_1\cdot A_2\cdot T/T, K),\\ \deg(h)\cdot\deg(W) &= I(B_1\cdot B_2\cdot T/T, K), \end{aligned} \tag{6.2}$$

where A_1, A_2 (resp. B_1, B_2) are independent generic divisors of $\Lambda(\alpha e X_V)$ (resp. $\Lambda(eX_V)$) over K.

Next we shall find an upperbound for $\deg(U)$.

$$\begin{aligned} &\deg(U) \leqslant (1/\deg(g))(e^3\alpha^2 d - e^2\alpha^2) \quad \text{if} \quad F^* \neq 0; \qquad (6.3)\\ &\deg(U) \leqslant (1/\deg(g))e^3\alpha^2 d \quad \text{if} \quad F^* = 0. \end{aligned}$$

In fact, $A_i \sim \alpha e X_V$ and X_V is non-degenerate. Applying Lemma 5 to our case, we get $I(A_1\cdot A_2\cdot T/T, K) \leqslant e^2\alpha^2 I(X_V^{(2)}, T)$ and $T + F^* \sim eX_V$.

We have $I(X_V^{(2)}, T) = I(X_V^{(2)}, eX_V) - I(X_V^{(2)}, F^*) = ed - I(X_V^{(2)}, F^*)$. (6.3) follows from these.

Let Λ be the linear system of hyperplane sections of U. From (6.3) and from Theorem 3 in [17], we immediately get

$$\dim \Lambda \leqslant (1/\deg(g))(e^3\alpha^2 d - e^2\alpha^2) + 1 \quad \text{if} \quad F^* \neq 0;$$
$$\dim \Lambda \leqslant (1/\deg(g))e^3\alpha^2 d + 1 \quad \text{if} \quad F^* = 0. \tag{6.4}$$

In order to estimate an upper bound for $\deg(g)$, it is enough to do so for $\deg(h)$. By doing so, we shall get

$$\deg(g) \leqslant e^3 d. \tag{6.5}$$

From (6.2) we get $\deg(h) \leqslant I(B_1 \cdot B_2 \cdot T/T, K)$. By Lemma 5, the latter is bounded by $e^2 I(X_V^{(2)}, T)$. This, in turn, is bounded by $e^2 I(X_V^{(2)}, T + F^*) = e^3 d$. This proves our inequality since $\deg(g) \leqslant \deg(h)$.

Combining (6.4) and (6.5), we get

$$\dim \Lambda \leqslant (1/\deg(g))e^3\alpha^2 d - (1/ed)\alpha^2 + 1 \quad \text{if} \quad F^* \neq 0. \tag{6.6}$$

279 As is well known, $\dim Tr_T \Lambda(\alpha e X) = \dim \Lambda$. Consider now the case $F^* \neq 0$ first. From (6.1) and (6.6), we get $(1/\deg(g))e^3\alpha^2 d - (1/ed)\alpha^2 + 1 > \frac{1}{2}e^3\alpha^2 d - (1/ed)\alpha^2 + 1$. Hence $\deg(g) < 2$ and conseqently $\deg(g) = 1$ and g is birational. From the definition of g, it follows that a non-degenerate rational map f of V defined by $\alpha e X_V$ induces on T a birational map. When that is so, f is a birational map since T is a generic divisor of the linear system $\Lambda(eX_V)_{\text{red}}$ of positive dimension over k, a proof of which will be left as an exercise to the reader.

Next, consider the case when $F^* = 0$. By Proposition 6, we have $P_e(\alpha) - P_e(\alpha - 1) > \delta'(e, \alpha)$. Then we get $\dim Tr_T \Lambda(\alpha e X_V) > (1/3)e^3\alpha^2 d + 1$, a proof of which is quite similar to that of (6.1). Combining this with (6.4), we get $(1/\deg(g))e^3\alpha^2 d > (1/3)e^3\alpha^2 d$. Hence $\deg(g) < 3$. This proves that $\deg(f) \leqslant 2$, a proof of which will be left to the reader again.

By Proposition 6, e is bounded by v and α is bounded by $\overline{v}$. Let $\rho_1 = (v\,\overline{v})!$ We shall summarize the results of this paragraph as follows.

Proposition 7. *There is a constant ρ_1, which depends on $P(x)$ only, such that the following properties hold:* (a) *if $\Lambda(eX_V)$ has a fixed component, $\rho_1 X_V$ defines a birational transformation of V;* (b) *if $\Lambda(eX_V)$ has no fixed component, a non-degenerate rational map of V defined by $\rho_1 X_V$ is of degree at most 2;* (c) *ρ_1 is divisible by e and α.*

7 A solution of (B_3), (III).

Proposition 7 solves our problem in the case when $\Lambda(eX_V)$ has a fixed component. In order to treat the other case, we shall review the concept of minimum sums of linear systems and fix some notations.

In general, let W be a variety and $\mathscr{M}$, $\mathscr{N}$ two modules of rational functions of finite dimensions on W. We shall denote by $\Lambda(\mathscr{M})$, $\Lambda(\mathscr{N})$ the reduced linear systems on W defined by these two modules. Let $\mathscr{R}$ be the module generated by $f \cdot g$ with $f \in \mathscr{M}$ and $g \in \mathscr{N}$. Then the reduced linear system $\Lambda(\mathscr{R})$ is known as the *minimum sum* of $\Lambda(\mathscr{M})$ and $\Lambda(\mathscr{N})$. We shall denote this minimum sum by $\Lambda(\mathscr{M}) \oplus \Lambda(\mathscr{N})$. When Λ is a reduced linear system on W and Λ' the minimum sum of r **280**
linear systems equal to Λ, we shall write $\oplus^r \Lambda$ for Λ'.

Let W' be another variety and β a rational map of W' into W. Let Γ be the closure of the graph of β on $W' \times W$ and Λ a linear system of divisors on W. Let Λ' be the set of W'-divisors $\beta^{-1}(Z) = \mathrm{pr}_{W'}(\Gamma \cdot (W' \times Z))$ with $Z \in \Lambda$. Then Λ' is a linear system of W'-divisors and this will be denoted by $\beta^{-1}(\Lambda)$.

Lemma 7. *Let f be a non-degenerate rational map of V defined by $\rho_1 X_V$ and W the image of V by f. When N is the dimension of the ambient space of W, the following inequalities hold:*

$$P(\rho_1) - c - 1 \leqslant N \leqslant P(\rho_1) + c - 1; \ \ \deg(W) \leqslant \rho_1^3 d.$$

Proof. $N = l(\rho_1 X_V) - 1$. Hence the first inequality follows from (A_3''). Let k be a common field of rationality of V, X_V and f and Z_1, Z_2, Z_3 independent generic divisors of $\Lambda(\rho_1 X_V)$ over k. Then $\deg(W) \leqslant I(Z_1 \cdot Z_2 \cdot Z_3/V, k) \leqslant \rho_1^3 d$ by Lemma 5. Our lemma is thereby proved. □

Corollary. *Let k_0 be the algebraic closure of the prime field. There is a finite union of irreducible algebraic families of irreducible varieties* *in projective spaces, all defined over k_0, such that, when $V \in \Sigma$ and when f, W are as in our lemma, W is a member of at least one of the irreducible families.*

Proof. This follows at once from our lemma and from the main theorem on Chow-forms (c.f. [3]). □

In the following lemma, *we shall assume that global resolution, dominance and birational resolution in the sense of Abhyankar hold for algebraic varieties of dimension n* (c.f. [35]). These hold for characteristic zero (c.f. [5]) and for algebraic varieties of dimension 3 when the characteristic is different from 2, 3 and 5 (c.f. [35]).

Lemma 8. *Let $\mathfrak{F}$ be an irreducible algebraic family of irreducible varieties in a projective space and k an algebraically closed field over which $\mathfrak{F}$ is defined. Then $\mathfrak{F}$ can be written as a finite union $\bigcup_j \mathfrak{F}_j$ of irreducible*
281 *algebraic families, all defined over k, with the following properties:* (a) *$\mathfrak{F}_i \cap \mathfrak{F}_j = \varnothing$ whenever $i \neq j$;* (b) *for each j, there is an irreducible algebraic family $\mathfrak{H}_j$ of non-singular varieties in a projective space, defined over k;* (c) *when W_i is a generic member of $\mathfrak{F}_i$ over k, there is a generic member W_i^* of $\mathfrak{H}_i$ over k and a birational morphism ϕ_i of W_i^* on W_i such that W_i^*, ϕ_i are defined over an algebraic extension of the smallest field of definition of W_i over k;* (d) *when W_i' is a member of $\mathfrak{F}_i$, Γ_i the graph of ϕ_i and when $(\Gamma_i', W_i^{*'})$ is an arbitrary specialization of (Γ_i, W_i^*) over k over the specialization $W_i \to W_i'$ ref. k, $W_i^{*'}$ is a member of $\mathfrak{H}_i$ and Γ_i' is the graph of a birational morphism ϕ_i' of $W_i^{*'}$ on W_i;* (e) *when C_i' is a generic hyperplane section of W_i' over a filed of definition of W_i', ${\phi'}_i^{-1}(C_i')$ is non-singular.*

Proof. Let W be a generic member of $\mathfrak{F}$ over k and K the smallest field of definition of W over k. There is a non-singular variety W^* in a projective space and a birational morphism ϕ of W^* on W, both defined over $\overline{K}$. For the sake of simplicity, replace W^* by the graph of ϕ. Then ϕ is simply the projection map. Therefore, we can identify the graph of ϕ with W^*. A multiple projective space can be identified with a non-singular subvariety of a projective space by the standard process. Chow-points

of positive cycles in a multiple projective space can then be defined by means of the above process. Let C be a hyperplane section of W, rational over K. W^* can be chosen in such a way that $\phi^{-1}(C) = U$ is non-singular. Let w, w^*, u be respectively the Chow-points of W, W^*, U and $\overline{T}$, $\overline{T}^*$ the locus of w, (w^*, u) over k. An open subset T of $\overline{T}$ over k is the Chow-variety of $\mathfrak{F}$. The set T^* of points on $\overline{T}^*$ corresponding to pairs of non-singular varieties is k-open on $\overline{T}^*$ as can be verified without much difficulty. Let Z be the locus of (w, w^*, u) over k on $T \times T^*$. Let T_1 be the set of points of T over which Z is complete (i.e. proper). Then T_1 is non-empty and k-open (c.f. [25], Chap. VII, Cor., Prop. 12). The set-theoretic projection of the restriction Z_1 of Z on $T_1 \times T^*$ on T_1 contains a non-empty k-open set F_1. Let H_1 be the locus of w^* over k. Then the families $\mathfrak{F}_1$, $\mathfrak{H}_1$ defined by F_1, H_1 satisfy (b), (c), (d) and (e) of our lemma which is not difficult to verify. $T_1 - F_1$ can be written as a finite union of locally closed irreducible subvarieties of T, defined **282**
over k, such that no two distinct components have a point in common. Then we repeat the above for each irreducible component to obtain the lemma. □

Corollary. *There are two finite sets of irreducible algebraic families* $\{\mathfrak{F}_i\}$, $\{\mathfrak{H}_i\}$ *with the following properties :* (a) *when* $V \in \Sigma$ *and* f *a non-degenerate rational map of* V *defined by* $\rho_1 X_V$, *there is an index* i *such that the image* W *of* V *by* f *is a member of* $\mathfrak{F}_i$; (b) *every member of the* $\mathfrak{H}_i$ *is a non-singular subvariety of a projective space;* (c) *the* $\mathfrak{F}_i$ *and the* $\mathfrak{H}_i$ *satisfy* (c), (d) *and* (e) *of our lemma.*

Proof. This follows from the Corollary of Lemma 7, Lemma 8 and from the basic assumptions on resolutions of singularities. □

Lemma 9. *Let the characteristic be zero,* $V \in \Sigma$ *and* f *a non-degenerate rational map of* V *defined by* $\rho_1 X_V$. *Let* W *be the image of* V *by* f. *Then there is a non-singular projective variety* W^* *and a birational morphism* ϕ *of* W^* *on* W *with the following properties:* (a) *when* k *is a common field of rationality of* V, X_V, f *and* ϕ *and when* C_W *is a generic hyperplane section of* W *over* k, $\phi^{-1}(C_W)$ *is non-singular;* (b) *when* C_W

and C'_W are independent generic over k, $\phi^{-1}(C_W) \cdot \phi^{-1}(C'_W)$ is non-singular; (c) *there is a positive integer π, which depends only on $P(x)$, such that $|p_a(\phi^{-1}(C_W))| < \pi$.*

Proof. When a non-singular subvariety of a projective space is specialized to another such variety over a discrete valuation ring, the virtual arithmetic genus is not changed (c.f. [2], [4]). Take the $\mathfrak{F}_i$, $\mathfrak{H}_i$ as in the Corollary of Lemma 9 and take W^* from a suitable $\mathfrak{H}_i$. Then there is a birational morphism ϕ of W^* on W, satisfying (a). (c) follows from (a) when we take the above remark into account. (a) and (b) follow easily also from the theorem of Bertini on variable singularities since the characteristic is zero. □

Lemma 10. *There is a constant ρ_2, which depends on $P(x)$ only, such that $m\rho_1 X_V$ has the following properties for $m \geqslant \rho_2$, provided that it does not define a birational map and the characteristic is zero:* (a) *when f' is a non-degenerate rational map of V defined by $m\rho_1 X_V$, k' a field of*
283 *rationality of V and X_V and T a generic divisor of $\oplus^m \Lambda(\rho_1 X_V)$ over k', T is irreducible and the effective geometric genus of the proper transform of T by f' is at least 2;* (b) $\deg(f') = 2$ *and f' induces on T a rational map of degree* 2*.*

Proof. Let f, W, W^*, ϕ, k be as in Lemma 10. Let C_W be generic over k and $U = \phi^{-1}(C_W)$. Let U' be a generic specialization of U over k, other than U. By the modular property of p_a[†], we get $p_a(mU) = mp_a(U) + \Sigma_1^{m-1} p_a(sU' \cdot U)$. Applying the modular property again to $p_a(sU' \cdot U)$ on U, which is non-singular by Lemma 10 we get the following equality:

$$sp_a(U' \cdot U) + \frac{1}{2}s(s-1)(U' \cdot U)^{(2)} - s - 1 = p_a(sU' \cdot U).$$

From the definition of U, it is clear that $U^{(3)} > 0$ on W^*. Hence $(U' \cdot U)^{(2)} = U^{(3)} > 0$. Moreover $|p_a(U)| < \pi$, by Lemma 10. Using these

[†]This can be proved exactly in the same way as Lemma 5 of [31] because of our Lemma 10.

and $\Sigma_1^{m-1} s(s-1) = (m-1)m(2m-1)/6$, we get

$$p_a(mU) > -m(\pi+1) - m(m-1) + (m-1)m(2m-1)/12 + 1.$$

We can find a positive integer ρ_2, which depends on $P(x)$ only, such that the right hand side of the above inequality is at least 2 whenever $m \geqslant \rho_2$. When that is so, any member A of $\Lambda(mU)$ satisfies $p_a(A) > 1$ since the virtual arithmetic genus of divisors is invariant with respect to linear equivalence.

Let Λ be the linear system of hyperplane sections of W. Clearly $\phi^{-1}(\oplus^m \Lambda) = \phi^m \phi^{-1}(\Lambda)$ and the latter contains a non-singular member A by the theorem of Bertini on variable singularities. Let p_g, p_a, q denote respectively the effective geometric genus, effective arithmetic genus and the irregularity of A. Then $q = p_g - p_a$ and $p_a(A) = p_a$. Since $q \geqslant 0$, it follows that $p_g > 1$ whenever $m \geqslant \rho_2$. When C_m is a generic divisor of $\oplus^m \Lambda$ over k, we can take for A the variety $\phi^{-1}(C_m)$. Therefore, the effective geometric genus of C_m is at least 2 when $m \geqslant \rho_2$.

Assume that f' is not birational for some $m \geqslant \rho_2$ and rational over k. Then $\Lambda(m\rho_1 X_V)$ has no fixed component and $\deg(f') = 2$ by Proposition 7. By the same proposition, the same is true for $\Lambda(\rho_1 X_V)$ and f. **284**
Let W' be the image of V by f'. Since $\deg(f) = \deg(f')$, there is a birational transformation h between W' and W such that $f = h \circ f'$ holds generically. Then $f^{-1}(C_m) = T$ is irreducible (c.f. [25], Chap. IX) and is a generic divisor of $\oplus^m f^{-1}(\Lambda) = \oplus^m \Lambda(\rho_1 X_V)$ over k. Let L be the proper transform of T by f'. C_m and L are birationally corresponding subvarieties of W and W' by h^{-1} and, when that is so, the effective geometric genus of L is at least 2. Our lemma follows easily from this.

Let now f' denote a non-degenerate rational map of V defined by $\rho_2\rho_1 X_V$, and *assume that f' is not birational*. By Proposition 7, deg $(f') = 2$ and $\Lambda(\rho_2\rho_1 X_V)$ has no fixed component. By Lemma 11, the complete linear system contains a linear pencil whose generic divisor T has the property that its proper transform D by f' has the effective geometric genus which is at least 2.

Let f be a non-degenerate rational map of V into a projective space defined by $m\rho_2\rho_1 X_V$ and *assume that f has still the property that* deg

$(f) = 2$. Let E be the proper transform of T by f and g the rational map induced on T by f. Then D and E are clearly birationally equivalent and the effective geometric genus of E is at least 2. As in (6.1), $\dim Tr_T \Lambda(m\rho_2\rho_1 X_V) = l(mT) - l(mT - T) - 1 > P_{\rho_1\rho_2}(m) - P_{\rho_1\rho_2}(m - 1) - 2c - 1$. The leading coefficient of the right hand side of the above inequality is given by $\frac{1}{2}(\rho_1\rho_2)^3 d$. □

Let K be the smallest field of rationality of T over k and Z_1, Z_2 two independent generic divisors of $\Lambda(mT)$ over K. Then exactly as in (6.2), we get $\deg(g)\deg(E) = I(Z_1 \cdot Z_2 \cdot T/T, K)$. By Lemma 5, the latter is bounded by $(\rho_1\rho_2)^3 m^2 d$. By Lemma 11 and by our assumption, $\deg(g) = 2$. Hence $\deg(E) \leqslant \frac{1}{2}(\rho_1\rho_2)^3 m^2 d$. Let Λ be the linear system of hyperplane sections of E. By Proposition 3, $\dim \Lambda \leqslant \frac{1}{4}(\rho_1\rho_2)^3 m^2 d + 1$. Since g is defined by $Tr_T\Lambda(mT)$, it follows that $\dim Tr_T\Lambda(mT) = \dim \Lambda$. Therefore,

$$P_{\rho_1\rho_2}(m) - P_{\rho_1\rho_2}(m-1) - 2c - 1 < \frac{1}{4}(\rho_1\rho_2)^3 m^2 d + 1.$$

Since the leading coefficient of the left hand side is $\frac{1}{2}(\rho_1\rho_2)^3 d$, we can find a constant ρ_3, which depends on $P(x)$ only, such that the above inequality does not hold for $m \geqslant \rho_3$. For such m, g and hence f has
285 to be birational. Setting $\frac{1}{2}\rho_4 = \rho_1\rho_2\rho_3$ and combining the above result with that of Proposition 7, we get

Theorem 1. *Let the characteristic be zero, V^3 a polarized variety, $P(m) = \chi(V, \mathscr{L}(mX_V))$ and assume that (A_3'') is true. Then there is a constant ρ_4, which depends on $P(x)$ only, such that mX_V defines a birational transformation of V when $m \geqslant \rho_4$.*

Corollary. *Let the characteristic be zero and V^3 be canonically polarized. Then (B_3) is true.*

Chapter II. The Problem (C_n).

In this chapter, we shall solve (C_n) for canonically polarized varieties V^n under the following assumptions: *(A_n) and (B_n) are true; theorems on dominance and birational resolution in the sense of Abhyankar hold for dimension n.* As we remarked already, this is the case when the characteristic is zero (c.f. [5]) or when $n = 1, 2, 3$ if the characteristics 2, 3 and 5 are excluded for $n = 3$ (c.f. [35]).

8 Preliminary lemmas.

Lemma 11. *Let U and U' be two non-singular subvarieties of projective spaces and g a birational transformation between U and U'. Then we have the following results:* (a) $g(\mathfrak{K}(U)) + E' \sim \mathfrak{K}(U')$ *where E' is a positive U'-divisor whose components are exceptional divisors for g^{-1};* (b) $l(m\mathfrak{K}(U)) = l(m\mathfrak{K}(U'))$ *for all positive integers m;* (c) $\Lambda(m\mathfrak{K}(U')) = \Lambda(g(m\mathfrak{K}(U))) + mE'$ *for all positive integers m.*‡

Proof. These results are well known for characteristic zero. (b) and (c) are easy consequences of (a). (a) can be proved as in [33] using fundamental results on monoidal transformations (c.f. [29], [33]) and the theorem of dominance. □

Lemma 12. *Let U be a non-singular subvariety of a projective space such that $C_U \sim m\mathfrak{K}(U)$ for some positive integer m. Let U' be a non-singular subvariety of a projective space, birationally equivalent to U.* **286**
Then $m\mathfrak{R}(U')$ defines a non-degenerate birational map h' of U', mapping U' generically onto a non-singular subvariety U^* of a projective space such that $C_{U^*} \sim m\mathfrak{K}(U^*)$. Moreover, U and U^* are isomorphic.

Proof. Let g be a birational transformation between U and U'. Then $g(C_U) \sim g(m\mathfrak{K}(U))$ and $\Lambda(g(\mathfrak{K}(U))) + mE' = \Lambda(m\mathfrak{K}(U'))$ where E' is a positive U'-divisor whose components are exceptional divisors for g^{-1} by Lemma 12. Assume first that the set of hyperplane sections of U

‡A subvariety of codimension 1 of U' is called exceptional for g^{-1} if the proper transform of it by g^{-1} is a subvariety of U of codimension at least 2.

forms a complete linear system. Then $g(C_U)$ is irreducible for general C_U (c.f. [25], Chap. IX). Hence mE' is the fixed part of $\Lambda(m\mathfrak{K}(U'))$. Since $l(m\mathfrak{K}(U)) = l(m\mathfrak{K}(U'))$ by Lemma 12, it follows that all members of $\Lambda(g(m\mathfrak{K}(U)))$ are of the form $g(C_U)$. This proves that g^{-1} is a non-degenerate rational map defined by $m\mathfrak{K}(U')$. If our assumption does not hold for U, apply a non-degenerate map of U, defined by C_U, to U. This amp is obviously an isomorphism and the image of U by this clearly satisfies our assumption. □

Lemma 13. *Let U^n (resp. U'^n) be a complete non-singular variety, $\mathfrak{K}(U)$ (resp. $\mathfrak{K}(U')$) a canonical divisor of U (resp. U') and $\mathfrak{O}$ a discrete valuation ring with the quotient field k_0 and the residue field k_0'. Assume that U, $\mathfrak{K}(U)$ are rational over k_0 and that $(U', \mathfrak{K}(U'))$ is a specialization of $(U, \mathfrak{K}(U))$ over $\mathfrak{O}$. Assume further that the following conditions are satisfied:* (i) *there is a positive integer m_0 such that $l(m_0\mathfrak{K}(U)) = l(m_0\mathfrak{K}(U'))$;* (ii) *a non-degenerate rational map h (resp. h') defined by $m_0\mathfrak{K}(U)$ (resp. $m_0\mathfrak{K}(U')$) is birational;* (iii) *$h'(U') = W'$ is non-singular and $C_{W'} \sim m_0\mathfrak{K}(W')$. Then the following two statements are equivalent:* (a) *There is a birational map g of U, between U and a non-singular subvariety W^* of a projective space such that $C_{W^*} \sim t\mathfrak{K}(W^*)$ for some positive integer t and $\mathfrak{K}(W')^{(n)} = \mathfrak{K}(W^*)^{(n)}$;* (b) $\deg(h(U)) = \deg(h'(U'))$. *Moreover, when* (a) *or* (b) *is satisfied, $h(U) = W$ is non-singular, $C_W \sim m_0\mathfrak{K}(W)$ and $\mathfrak{K}(W)^{(n)} = \mathfrak{K}(W')^{(n)}$.*

Proof. First assume (a). h is uniquely determined by $m_0\mathfrak{K}(U)$ up to a
287 projective transformation. Therefore we get $\deg(h(U)) \geqslant \deg(h'(U'))$ by Proposition 2.1 of the Appendix since specializations are compatible with the operation of algebraic projection (c.f. ([24]). Let the Z_i be n independent generic divisors of $\Lambda(m_0\mathfrak{K}(U))$ over k_0. Then $\deg(h(U)) = I(Z_1 \dots Z_n/U, k_0)$ since h is birational. Let $d_0 = \mathfrak{K}(W')^{(n)} = \mathfrak{K}(W^*)^{(n)}$. Then $\deg(h'(U')) = {m_0}^n d_0$ by (iii). Hence $\deg(h(U)) \geqslant {m_0}^n d_0$. By Lemma 11, $\Lambda(g(m\mathfrak{K}(U))) + mE^* = \Lambda(m\mathfrak{K}(W^*))$ for all positive m where E^* is as described in the lemma. Let L be a common field of rationality of W^* and g over k_0 and the Y_i (resp. Y_i^*) n independent generic divisors of $\Lambda(m\mathfrak{K}(U))$ (resp. $\Lambda(m\mathfrak{K}(W^*))$) over L. Then we

have $I(Y_1 \dots Y_n/U, L) = I(Y_1^* \dots Y_n^*/W^*, L)$. By Lemma 5,

$$I(Y_1^* \dots Y_n^*/W^*, L) \leqslant m^n d_0.$$

Setting $m = m_0$, we therefore get $I(Y_1 \dots Y_n/U, L) \leqslant {m_0}^n d_0$. The left hand side of this is obviously $I(Z_1 \dots Z_n/U, k_0)$. Combining the two inequalities we obtained, we get $\deg(h(U)) = {m_0}^n d_0 = \deg(h'(U'))$. Hence (a) implies (b).

Now we assume (b). Let $W = h(U)$, $C = C_W$, $C' = C_{W'}$. By Proposition 2.1 of the Appendix and by the compatibility of specializations with the operation of algebraic projection, we get $(U, \mathfrak{K}(U), W) \to (U', \mathfrak{K}(U'), W')$ ref. $\mathfrak{O}$. Since W' is non-singular, W is non-singular too. Since h is defined by $m_0\mathfrak{K}(U)$, there is a positive U-divisor F such that $h^{-1}(C) + F \sim m_0\mathfrak{K}(U)$. Hence there is a positive divisor T with $h(m_0\mathfrak{K}(U)) \sim C + T$. There is a positive divisor E such that $C+T+E \sim m_0\mathfrak{K}(W)$ by Lemma 11. Let $C''+T'+E'$ be a specialization of $C + T + E$ over $\mathfrak{O}$ over the specialization under consideration. Since linear equivalence is preserved by specializations (c.f. [24]), $C' \sim C''$ and $C' + T' + E' \sim m_0\mathfrak{K}(W')$ (c.f. Lemma 1.1 of the Appendix; U, U' are clearly non-ruled since $l(m\mathfrak{K}(U))$, $l(m\mathfrak{K}(U'))$ are positive for large m). Since $m_0\mathfrak{K}(W') \sim C'$ by (iii), it follows that T', E' are positive and $T' + E' \sim 0$. This proves that $T = E = 0$ and $m_0\mathfrak{K}(W) \sim C$. Our lemma is thereby proved. □

9 A proof of (C_n). In order to solve (C_n), we shall fix some notation. We shall denote by Σ the *set of canonically polarized varieties with the fixed Hilbert characteristic polynomial* $P(x)$ and by V^n a "variable element" of Σ. As we have shown in Lemma 1, there is a root ρ of the equation $P(x) - (-1)^n\gamma_n = 0$ such that $\mathfrak{K}(V) \equiv \rho X_V \bmod G_a$. 288
Then Σ can be expressed as a union of subspaces Σ_ρ corresponding to ρ. In order to solve (C_n), we may restrict our attention to Σ_ρ. From our basic assumptions stated at the beginning of this chapter, there is a constant ρ_5, which depends on $P(x)$ only, having the following properties: (a) higher cohomology groups of $\mathscr{L}(Y)$ vanish and $l(Y) > 0$ whenever $Y \equiv mX_V \bmod \mathfrak{G}_a$ and $m \geqslant \rho_5$; (b) such Y defines a birational transformation of V. When $V \in \Sigma_\rho$ and when X_V is replaced by $\mathfrak{K}(V)$, (a) and

(b) still hold since $\mathfrak{K}(V) \equiv \rho X_V \text{mod } G_a$ and ρ is a positive integer by the definition of a basic polar divisor. From now on, we shall restrict ourselves to the study of Σ_ρ and V shall denote a "variable element" of this set. We set $d_0 = \mathfrak{K}(V)^{(n)}$ and $\rho_6 = \rho \cdot \rho_5$.

Let f be a non-degenerate birational map of V defined by $\rho_5\mathfrak{K}(V)$. f maps V into the projective space of dimension $P(\rho_6) - 1$ and the degree of the image is bounded by ${\rho_6}^n X_V^{(n)} = {\rho_5}^n d_0$, which follows easily from Lemma 5. When we do this for each member of Σ_ρ, we see that each such image is contained in a finite union $\mathfrak{F}$ of irreducible algebraic families of irreducible varieties by the main theorem on Chow-forms (c.f. [3]). Let $\mathfrak{A}'$ be the set of images of members of Σ_ρ in $\mathfrak{F}$ thus obtained. Applying Lemma 8 to $\mathfrak{F}$, we get immediately the following results.

Lemma 14. *There are finite unions $\bigcup_i \mathfrak{F}_i$ and $\bigcup_i \mathfrak{H}_i$ of irreducible algebraic families of irreducible varieties in projective spaces with the following properties:* (a) *each $\mathfrak{F}_i$ contains some members of $\mathfrak{A}'$ and $\mathfrak{A}'$ is contained in $\bigcup_i \mathfrak{F}_i$;* (b) *$\bigcup_i \mathfrak{H}_i$ consists of non-singular varieties;* (c) *when W is a member of $\mathfrak{F}_i$, there is a member U of $\mathfrak{H}_i$ and a birational morphism of U on W.*

Let $\mathfrak{H}$ be a finite union of irreducible algebraic families of positive cycles in projective spaces and $\mathfrak{u}$ a set of members of $\mathfrak{H}$. We shall say that $\mathfrak{H}$ is $\mathfrak{u}$-*admissible* if each component family of $\mathfrak{H}$ contains some members of $\mathfrak{u}$. We shall denote by $\mathfrak{A}$ the set of members U of $\bigcup_i \mathfrak{H}_i$ such
289 that there is a birational morphism of U on a member of $\mathfrak{A}'$. Then $\bigcup_i \mathfrak{H}_i$ is $\mathfrak{A}$-admissible by Lemma 15. Elements U of $\mathfrak{A}$ satisfy the following three conditions (c.f. Lemma 13):

(I) $m\mathfrak{K}(U)$ defines a birational transformation f_m of U for large m;

(II) $f_m(U)$ is non-singular for large m;

(III) $l((\rho_5 + m)\mathfrak{K}(U)) = P_\rho(\rho_5 + m)$ for $m > 0$.

We shall find a subset of $\bigcup_i \mathfrak{H}_i$, containing $\mathfrak{A}$, satisfying the above three conditions, which can be expressed as a finite union of irreducible algebraic families of non-singular varieites. From this we shall recover *members of Σ_ρ in some definite projective space, up to isomorphisms, in such a way that hyperplane sections are some fixed multiple of canonical divisors.* This is the main idea of the rest of this paragraph.

Lemma 15. *There is a finite union $\bigcup_i \mathfrak{I}_j$ of irreducible algebraic families contained in $\bigcup_i \mathfrak{H}_i$ having the following properties:* (a) *$\bigcup_j \mathfrak{I}_j$ is $\mathfrak{A}$-admissible;* (b) *every member of $\bigcup_j \mathfrak{I}_j$ satisfies the condition* (III).

Proof. Since $\bigcup_i \mathfrak{H}_i$ is $\mathfrak{A}$-admissible, there is a member V of Σ_ρ and a member U of $\mathfrak{H}_i$ such that V and U are birationally equivalent. Then $l((\rho_5 + m)\mathfrak{K}(V)) = l((\rho_5 + m)\mathfrak{K}(U))$ for all positive integers m by Lemma 12. Setting $m_0 = \rho_5$, $m_i = \rho_5 + i$, $q_{m_i} = P_\rho(\rho_5 + i)$ and then $q_{m_i} = P_\rho(\rho_5 + i) + 1$ in the Corollary to Proposition 1.1 in the Appendix, we get our lemma easily. □

Lemma 16. *There is a finite union $\bigcup_i \mathfrak{I}_i$ of irreducible algebraic families contained in $\bigcup_i \mathfrak{I}_j$ having the following properties:* (a) *$\bigcup_i \mathfrak{I}_i$ is $\mathfrak{A}$-admissible;* (b) *every member of $\bigcup_i \mathfrak{I}_i$ satisfies the condition* (I).

Proof. Let k be a common field of definition of the component families $\mathfrak{I}_i$. Let $U_0 \in \mathfrak{I}_j \cap \mathfrak{A}$ and U a generic member of $\mathfrak{I}_j$ over k. $m\mathfrak{K}(U_0)$ defines a birational map for large m by Lemma 12. Therefore $m\mathfrak{K}(U)$ defines a rational map f of U such that f does not decrease the dimension by Lemma 1.1 and the Corollary to Proposition 2.1 of the Appendix. Then we see that $m\mathfrak{K}(U)$ defines a birational map for large m which is an easy consequence of the technique of normalization in a finite algebraic extension of the function field (c.f. [25], Appendix I).

Fix a positive integer m_0 such that $m_0\mathfrak{K}(U)$, $m_0\mathfrak{K}(U_0)$ both define bi- **290**
rational transformations. Then $m_0\mathfrak{K}(U)$ is, in particular, linearly equivalent to a positive U-divisor Y. Consider an algebraic family with divisors over $\overline{k}$ such that (U, Y) is a generic element of it over $\overline{k}$ and apply Proposition 2.2 and its Corollary 3 of the Appendix to it (c.f. also Lemma 1.1 of the Appendix). Then we see that there is an irreducible algebraic family $\mathfrak{I}'_j$ such that the Chow-variety of $\mathfrak{I}'_j$ is k-open on that of $\mathfrak{I}_j$ and that $m_0\mathfrak{K}(U')$ defines a birational map of U' whenever U' is in $\mathfrak{I}'_j \cdot \mathfrak{I}_j - \mathfrak{I}'_j$ is

a finite union of irreducible algebraic families. Apply the above procedure to all those components of $\mathfrak{I}_j - \mathfrak{I}'_j$ which contain some members of $\mathfrak{A}$. This process cannot continue indefinitely. Doing the same for each $\mathfrak{I}_j$, we get easily our lemma. □

Lemma 17. *Let $\mathfrak{K}$ be a finite union of irreducible algebraic families in projective spaces and $\mathfrak{B}$ a set of members of $\mathfrak{K}$. Assume that the following conditions are satisfied :* (i) *$\mathfrak{K}$ consists of non-singular varieties;* (ii) *$\mathfrak{K}$ is $\mathfrak{B}$-admissible;* (iii) *$\mathfrak{B}$ is a subset of $\mathfrak{A}$;* (iv) *every member of $\mathfrak{K}$ satisfies the conditions* (I) *and* (III). *Then there is a finite union $\mathfrak{K}^*$ of irreducible algebraic families, contained in $\mathfrak{K}$, having the following properties:* (a) *$\mathfrak{K}^*$ is $\mathfrak{B}$-admissible;* (b) *for each component $\mathfrak{K}^*_i$ cf $\mathfrak{K}^*$, there is a $U_i \in \mathfrak{K}^*_i \cap \mathfrak{B}$ and a positive integer $m_i \geqslant \rho_5$ such that $m_i\mathfrak{K}(U_i)$ defines a non-degenerate birational map h_i of U_i such that $h_i(U_i) = W_i$ is non-singular and that $C_{W_i} \sim m_i\mathfrak{K}(W_i)$;* (c) *when U is a generic member of $\mathfrak{K}^*_i$ over a common field k of definition of the $\mathfrak{K}^*_i$, $m_i\mathfrak{K}(U)$ defines a non-degenerate birational map h of U such that* $\deg(W_i) = \deg(W)$, *where $W = h(U)$;* (d) *for each member U' of $\mathfrak{K}^*_i$, $m_i\mathfrak{K}(U')$ defines a birational map.*

Proof. We proceed by induction on the dimension of $\mathfrak{K}$. When the dimension of $\mathfrak{K}$ is zero, our lemma is trivial. Therefore we assume that our lemma is true for dimension up to $s-1$ and set $\dim \mathfrak{K} = s$. In order to prove our lemma, it is clearly enough to do so when $\mathfrak{K}$ is an irreducible algebraic family.

Let Y be a positive divisor on U such that $Y \sim \rho_5\mathfrak{K}(U)$, where U denotes a generic member of $\mathfrak{K}$ over k. Then we consider an algebraic
291 family with divisors defined over $\bar{k}$ such that (U, Y) is a generic member of it over $\bar{k}$ and apply Corollary 2 to Proposition 2.2 in the Appendix to our situation. By doing so, we can find a positive integer m_0 such that $m\mathfrak{K}(U')$ defines a birational map for every member U' of $\mathfrak{K}$ whenever $m \geqslant m_0$.

Let U_1 be a member of $\mathfrak{B}$. Since $\mathfrak{B}$ is contained in $\mathfrak{A}$, there is a positive integer $m_1 \geqslant \rho_5, m_0$ with the following properties: $m_1\mathfrak{K}(U_1)$ defines a non-degenerate birational map h_1 of U_1 and $W_1 = h_1(U_1)$ is non-singular; $C_{W_1} \sim m_1\mathfrak{K}(W_1)$ (c.f. Lemma 13). Let h be a non-degenerate birational map defined by $m_1\mathfrak{K}(U)$. Since h is determined uniquely by $m_1\mathfrak{K}(U)$ up to a projective transformation, we see that $\deg(h(U)) \geqslant \deg(h_1(U_1)) = \deg(W_1)$, $W_1 = h_1(U_1)$, by applying Proposition 2.1 of the Appendix and using the compatibility of specializations with the operations of algebraic projection.

Let $\mathfrak{K}'$ be the set of members U' of $\mathfrak{K}$ with the following properties: when h' is a non-degenerate birational map of U' defined by $m_1\mathfrak{K}(U')$, then $\deg(h'(U')) \leqslant \deg(W_1)$. *We claim that $\mathfrak{K}' \supset \mathfrak{B}$ and is a finite union of irreducible algebraic families.* We consider the same algebraic family with divisors as above, which is defined over $\bar{k}$, having (U, Y) as a generic element over $\bar{k}$. In applying Corollary 1 to Proposition 2.2 in the Appendix to our situation, we let $\deg(h_1(U_1)) = s_0$. In view of Lemma 1.1 of the Appendix, it is then easy to see that the set of Chow-points of members of $\mathfrak{K}'$ is a closed subset of that of $\mathfrak{K}$ over k. Let U'' be a member of $\mathfrak{B}$. Then it is contained in $\mathfrak{A}$ and there is a member V'' of Σ_ρ such that there is a birational map f'' of V'', mapping V'' generically onto U''. Let k' be a common field of rationality of U'', V'' and f over k. By Lemma 11, $\Lambda(f''(m_1\mathfrak{K}(V''))) + m_1E = \Lambda(m_1\mathfrak{K}(U''))$ where E is a positive U''-divisor whose components are exceptional divisors for f''^{-1}. Let h'' be a non-degenerate rational map of U'' defined by $m_1\mathfrak{K}(U'')$ and the Z_i (resp. Z'_i) independent generic divisors of $\Lambda(m_1\mathfrak{K}(V''))$ (resp. $\Lambda(m_1\mathfrak{K}(U''))$) over k'. Then the above relation between two complete linear systems show that $I(Z_1 \dots Z_n/V'', k') = I(Z'_1 \dots Z'_n/U'', k')$. Moreover, h'' is birational by our choice of m_1, $\deg(h''(U'')) \leqslant I(Z'_1 \dots Z'_n/U'', k')$ and **292**
$I(Z_1 \dots Z_n/V'', k') \leqslant {m_1}^n d_0$ by Lemma 5. It follows that $\deg(h''(U'')) \leqslant {m_1}^n d_0$. On the other hand, U_1 is the underlying variety of some member of Σ_ρ by Lemma 13. Consequently $\deg(W_1) = m_1^n d_0$ by the same lemma. This proves that U'' is contained in $\mathfrak{K}'$. Our contention is thereby proved. □

Let $\mathfrak{K}''$ be the union of those components of $\mathfrak{K}'$ which contain U_1. Denote by U'' now a generic member of a component of $\mathfrak{K}''$ over $\bar{k}$. As before, from Proposition 2.1 and from the compatibility of specializations with the operation of algebraic projection, we see that $\deg(h''(U''))$

$\geqslant \deg(h_1(U_1))$ and consequently $\deg(h''(U'')) = \deg(h_1(U_1))$. Moreover, $\mathfrak{K}''$ is $\mathfrak{K}'' \cap \mathfrak{B}$-admissible.

We have $\mathfrak{B} = (\mathfrak{K}'' \cap \mathfrak{B}) + ((\mathfrak{K}' - \mathfrak{K}'') \cap \mathfrak{B})$. $\mathfrak{K}' - \mathfrak{K}''$ is a finite union of irreducible algebraic families. Call $\mathfrak{K}_1$ the union of those components of $\mathfrak{K}' - \mathfrak{K}''$ which contain some members of $\mathfrak{B}$ and set $\mathfrak{B}_1 = \mathfrak{B} \cap \mathfrak{K}_1$. We have $\dim \mathfrak{K} > \dim \mathfrak{K}_1$. By our induction assumption, (a), (b), (c), (d) are satisfied by $\mathfrak{K}_1$, $\mathfrak{B}_1$. We have shown that $\mathfrak{K}''$, $\mathfrak{K}'' \cap \mathfrak{B}$ satisfy these too by Lemma 14. Thus our lemma is proved.

From our lemma, Lemma 14 and from Lemma 1.1 of the Appendix, we get

Corollary 1. *U and h in* (c) *of our lemma further satisfy the following properties : $W = h(U)$ is non-singular, $C_W \sim m_i\mathfrak{K}(W)$ and $\mathfrak{K}(W)^{(n)} = \mathfrak{K}(W_i)^{(n)} = d_0$.*

Corollary 2. *In our Lemma 17 and Corollary 1 above, m_i may be replaced by a positive integer which is a multiple of m_i.*

Proof. This is an easy consequence of Lemma 11. □

Theorem 2. *Let V^n be a canonically polarized variety and $P(x)$ its Hilbert characteristic polynomial. Assume that (A_n) and (B_n) have solutions for V and that theorems on dominance and birational resolution in the sense of Abhyankar hold for dimension n. Then there is a constant ρ_7 which depends on $P(x)$ only such that $\rho_7 X_V$ defines a non-degenerate projective embedding of V.*

293 *Proof.* As we pointed out at the beginning of this paragraph, it is enough to prove this for $V \in \Sigma_\rho$. By Lemmas 15, 15, 16, the finite union $\mathfrak{J}$ of irreducible families constructed in Lemma 16, together with $\mathfrak{A}$, satisfies the requirements of Lemma 17. Therefore, there is a finite union $\mathfrak{M}$ of irreducible algebraic families satisfying the conclusions of Lemma 17. For the sake of simplicity, we shall say that a *non-singular projective variety D has a property* (*) *with respect to t'* if $t'\mathfrak{K}(D)$ defines a non-degenerate birational map h of D such that $A = h(D)$ is non-singular, is the underlying variety of a member of Σ and that $C_A \sim t'\mathfrak{K}(A)$.

Let the $\mathfrak{M}_i$ be the components of $\mathfrak{M}$ and the U_i, m_i as in Lemma 17, (b) ($\mathfrak{K}_i^*$ in the Lemma is our $\mathfrak{M}_i$). By Corollary 2 of Lemma 17, m_i may be replaced by $t = \Pi_i m_i$ or by any positive multiple of t. Let k be an algebraically closed common field of rationality of the $\mathfrak{M}_i$ and U a generic member of $\mathfrak{M}_i$ over k. U has the property (*) with respect to t by Lemma 17 and its corollaries. Let $U' \in \mathfrak{M}_i$ and h' a non-degenerate birational map defined by $t\mathfrak{K}(U')$. Assume that h' has the properties that $\deg(h'(U')) = \deg(h(U))$ and that $h'(U')$ is non-singular. h is uniquely determined by $t\mathfrak{K}(U)$ up to a projective transformation. Therefore, we may assume without loss of generality transformation. Therefore, we may assume without loss of generality that $W' = h'(U')$ is a specialization of $W = h(U)$ over k by Lemma 1.1 and Proposition 2.1 of the Appendix, since specializations are compatible with the operation of algebraic projection. Since self-intersection numbers and linear equivalence are preserved by specializations, it follows that W' has also the property (*) with respect to t (c.f. Lemma 1.1 of the Appendix and [2]).

Let Y be a member of $\Lambda(t\mathfrak{K}(U))$ and consider an algebraic family with divisors, defined over k, with a generic element (U, Y) over k. We apply Corollary 3 to Proposition 2.2 in the Appendix to this. By doing so, we can find an irreducible algebraic family $\mathfrak{M}_i'$ of non-singular varieties, having the following properties: (a) the Chow variety of $\mathfrak{M}_i'$ is k-open on that of $\mathfrak{M}_i$; (b) when $U' \in \mathfrak{M}_i'$, $t\mathfrak{K}(U')$ defines a non-degenerate birational map of U' such that $h'(U')$ is non-singular; (c) $\deg(h'(U')) = \deg(h(U))$. Let $\mathfrak{M}' = \bigcup_i \mathfrak{M}_i'$ and $\mathfrak{A}^* = \mathfrak{A} \cap \mathfrak{M}'$. As we **294**
have shown above $U' \in \mathfrak{A}^*$ has the property (*) with respect to t.

Let $\mathfrak{B} = (\mathfrak{M} - \mathfrak{M}') \cap \mathfrak{A} \cdot \mathfrak{M} - \mathfrak{M}'$ is a finite union of irreducible algebraic families. When we remove from it those components which do not contain members of $\mathfrak{B}$, we get a finite union $\mathfrak{N}$ of irreducible algebraic families, which is contained in $\mathfrak{M}$, $\mathfrak{B}$-admissible and satisfies $\dim \mathfrak{M} > \dim \mathfrak{N}$. When we apply our process to $\mathfrak{N}$ and $\mathfrak{B}$ and continue it, applying Lemma 17 and its corollaries, it has to terminate by a finite number of steps. Consequently, we can find a positive integer t' such that a member of $\mathfrak{B}$ has the property (*) with respect to t'. When we set $\rho_7 = t \cdot t' \cdot \rho$, this constant satisfies the requirements of our theorem. □

Corollary. *Let the characteristic be zero, V^3 a canonically polarized variety and $P(x)$ the Hilbert characteristic polynomial of V. Then (C_3) is true for V^3 and $P(x)$.*

Proof. This follows easily from our theorem, Theorem 1 and from Proposition 1. □

Appendix

1 **Lemma 1.1.** *Let U^n and U'^n be non-singular and non-ruled subvarieties of projective spaces such that U' is a specialization of U over a discrete valuation ring $\mathfrak{O}$. Let $\mathfrak{K}(U)$ be a canonical divisor of U and (U', Y) a specialization of $(U, \mathfrak{K}(U))$ over $\mathfrak{O}$. Then Y is a canonical divisor of U'.*

Proof. When $n = 1$, the complete linear system of canonical divisors on U (resp. U') is characterized by the fact that it is a complete linear system of positive divisor of degree $2g - 2$ and dimension at least $g - 1$. Hence our lemma is easily seen to be true in this case.

Assume that our lemma is true for dimensions up to $n - 1$. Let k (resp. k') be a common field of rationality of U and $\mathfrak{K}(U)$ (resp. U' and
295 Y) and C, C^* (resp. C', C'^*) independent generic hypersurface sections of U (resp. U') over k (resp. k'). Then (U', Y, C', C'^*) is a specialization of $(U, \mathfrak{K}(U), C, C^*)$ over $\mathfrak{O}$. $C \cdot (C^* + \mathfrak{K}(U))$ is a canonical divisor of C (c.f. [31]) and this has the unique specialization $C' \cdot (C'^* + Y)$ over the above specialization with reference to $\mathfrak{O}$, since specializations and intersection-product are compatible operations. It follows that $C' \cdot (C'^* + Y)$ is a canonical divisor of C'. When $\mathfrak{K}(U')$ is a canonical divisor of U', rational over k', $C' \cdot (C'^* + Y) \sim C' \cdot (C'^* + \mathfrak{K}(U'))$. C' is a generic hypersurface section of U' over a filed of rationality of C'^* over k'. Moreover, when the degree of the hypersurface is at least two, a generic linear pencil contained in the linear system of hypersurface

sections consists of irreducible divisors (c.f. [19]). It follows that $Y \sim \mathfrak{K}(U')$ by an equivalence criterion of Weil (c.f. [27], Th. 2).[§]

We shall consider a family of varieties with divisors on them. Let $\mathfrak{A}$ be an irreducible algebraic family of subvarieties in a projective space, A the Chow-variety of it and a a generic point of A over a field of definition k of A, corresponding to a variety $U(a)$. Let $X(a)$ be a divisor on $U(a)$ and $X(a) = X(a)^{+} - X(a)^{-}$ the reduced expression for $X(a)$ where $X(a)^{+}$, $X(a)^{-}$ are both positive divisors. Let u (resp. v) be the Chow-point of $X(a)^{+}$ (resp. $X(a)^{-}$) and $z = (u, v)$. Let A' be the locus of (a, z) over $\bar{k}$. When (a', z') is a point of A', a' defines a cycle $U(a')$ in the projective space uniquely such that the support of Y is contained in the support of $U(a')$. When every member of $\mathfrak{A}$ is irreducible. A' defines an irreducible family whose member is a variety with a chain of codimension 1 on it. We shall call this an *irreducible family of varieties with chains of codimension* 1. When k' is a field of definition for A', we shall call k' a field of definition or rationality of the family. □

Proposition 1.1. *Let $\mathfrak{A}'$ be an irreducible algebraic family of non-singular varieties $U(a')$ with divisors $X(a')$ and $\{q_{m_i}\}$ an increasing sequence of positive integers starting with $q_{m_0} > 1$. Assume that there* 296
is a member $(U(a_0), X(a_0))$ such that $l(m_i X(a_0)) \geqslant q_{m_i}$ for all i. Let $(U(a), X(a))$ be a generic member of $\mathfrak{A}'$ over a common field k of rationality of $\mathfrak{A}'$, a_0 and assume that $l(m_i X(a)) \geqslant q_{m_i}$ for $0 \leqslant i \leqslant s-1$ but $l(m_s X(a)) < q_{m_s}$. Then, there is a finite union $\mathfrak{E}$ of irreducible families, defined over $\bar{k}$ and contained in $\mathfrak{A}'$, such that a member $(U(a'), X(a'))$ of $\mathfrak{A}'$ is in $\mathfrak{E}$ if and only if $l(m_i X(a')) \geqslant q_{m_i}$ for $0 \leqslant i \leqslant s$.

We shall prove this by a series of lemmas.

Let $\dim U(a_0) = n$ and $H_1, \ldots, H_{n-1}$ independent generic hypersurfaces of degree t over k. Let $H^{(1)} = H_1 \ldots H_{n-1}$ and $H^{(1)}, \ldots, H^{(r)}$ r independent generic specializations of $H^{(1)}$ over k. For each point a' of

[§] In [27], Th. 2, it is claimed that $Y - \mathfrak{K}(U' \sim \Sigma m_i T_i$ where the T_i are some subvarieties of U'. But these T_i are components of reducible members of such a pencil contained in the linear system of hyperplane secitons. We can eliminate them using linear systems of hypersurface sections.

A', we set $U(a') \cdot H^{(i)} = C(a')_i$ whenever the intersection is proper. We take r and t sufficiently large.

Lemma 1.2. *Let B be the set of points a' on A' satisfying the following conditions:* (i) *$C(a')_i$ is defined for all i;* (ii) *the $C(a')_i$ are non-singular for all i;* (iii) *$X(a')$ and $C(a')_i$ intersect properly on $U(a')$ for all i. Let k' be an algebraically closed common field of rationality for the $H^{(i)}$ over k. Then B is a k'-open subset of A'.*

Proof. These are well-known and easy exercises. Therefore, we shall omit a proof. □

We shall show that the set of points a' on B such that $l(m_s X(a')) \geqslant q_{m_s}$ forms a k'-closed subset of B. We can cover A' by open sets B by changing the $H^{(i)}$. Therefore our problem is reduced to the similar problem on the family defined by B. In order to solve our problem on this family we may replace B by a variety with a proper and surjective morphism on it. Therefore, we may assume without loss of generality that the $C(a')_i$ carry rational points over $k'(a')$.

For each a' in B, let $J(a')_i$ be the Jacobian variety of $C(a')_i$ and $\Gamma(a')_i$ the graph of the canonical map $\phi(a')_i$ of $C(a')_i$ into $J(a')_i$. We assume that these are constructed by the method of Chow so that these are compatible with specializations (c.f. [1], [8]). In order to simplify the notations, we simply denote by $\Im(Y \cdot C(a')_i)$ the *Abelian sum* of
297 $Y \cdot C(a')_i$ on $J(a')_i$, whenever Y is a $U(a')$-divisor such that $Y \cdot C(a')_i$ is defined. It should be pointed out here that the $J(a')_i$ and the $\Gamma(a')_i$ are rational over $k'(a')$.

Let P be the ambient projective space of the $U(a)$ and F^* the closed subset of a projective space, consisting of Chow-points of positive cycles in P which have the same dimension and degree as members of $\Lambda(m_s X(a_0))$. There is a closed subset T^* of $B \times F^*$ such that a point (a, y) of $B \times F^*$ is in T^* if and only if $U(a)$ carries the cycle $Y(y)$ defined by y (c.f. [3]). Let F be the geometric projection of T^* on F^* and $T = B \times F \cap T^*$. T is a k'-closed subset of $B \times F$. Let a be a generic point of B over k'. Since the $J(a')_i$ are defined over $k'(a')$ for $a' \in B$, there is a subvariety Z of $B \times \Pi_i P_i$, where the P_i are ambient spaces for the $J(a)_i$, such that $Z(a) = \Pi_i J(a)_i$ (c.f. [25], Chap. VIII).

Lemma 1.3. *Let U and U' be non-singular subvarieties of a projective space such that U' is a specialization of U over a field k. Let X (resp. X') be a divisor on U (resp. U') such that (U', X') is a specialization of (U, X) over k. Let u' be a given point of U'. Then there are divisors D, E (resp. D', E') on U (resp. U') with the following properties:* (a) *$X \sim D - E$ on U and $X' \sim D' - E'$ on U';* (b) *the supports of D', E' do not contain u';* (c) *(U', X', D', E') is a specialization of (U, X, D, E) over k.*

Proof. Let C (resp. C') be a hypersurface section of U (resp. U') by a hypersurface of degree m. Then, as is well known, $X+C$ (resp. $X'+C'$) is ample on U (resp. U') and $l(X+C) = l(X'+C')$, $l(C) = l(C')$ when m is sufficiently large (c.f. [25], Chap. IX, [31], [21], [4]). Denote by $G(*)$ the support of the Chow-variety of the complete linear system determined by $*$. Since linear equivalence is preserved by specializations (c.f. [24]), it follows that $(U', X', G(X'+C'), G(C'))$ is a specialization of $(U, X, G(X+C), G(C))$ over k. When a point x' in $G(X'+C')$ and a point y' in $G(C)$ are given, there is a point x in $G(X+C)$ and a point y in $G(C)$ such that $(x, y) \to (x', y')$ ref. k over the above specialization. We can choose x', y' so that the corresponding divisors D', E' do not pass through u'. Since $X' \sim X' + C' - C'$ and $X \sim X + C - C$, our lemma follows at once from the above observations. □

Corollary. *Let T_α be a component of T and (a, y) a generic point of T_α over k'. There is a rational map f_α of T_α into Z such that $f_\alpha(a, y) = (a, \ldots, \mathscr{S}(Y(y) \cdot C(a)_i), \ldots)$.*[¶] *Moreover, f_α satisfies the following conditions:* (a) *when $(a', y') \in T_\alpha$, $f_\alpha(a, y)$ has a unique specialization (a', Q') over k' over $(a, y) \to (a', y')$ ref. k';* (b) *when Y' is a $U(a')$-divisor such that $Y' \sim Y(y')$ and that Y' and the $C(a')_i$ intersect properly on $U(a')$, $Q' = (a', \ldots, \mathscr{S}(Y', C(a')_i), \ldots)$;* (c) *the locus L of $(a, \ldots, \mathscr{S}(m_s X(a) \cdot C(a)_i), \ldots)$ over k' is a subvariety of Z and con-* 298

[¶] $Y(y)$ denotes the divisor defined by y.

tains $(a', \ldots, \mathscr{S}(m_s X(a') \cdot C(a)_i), \ldots)$ *whenever* $a' \in B$ *and the latter is a unique specialization of the former over* k', *over* $a \to a'$ ref. k'.

Proof. This follows easily from Lemma 1.3, from the compatibility of specializations with the Chow-construction of Jacobian varieties, the operation of intersection-product (c.f. [24]), the Abelian sums and from the invariance of linear equivalence by specializations. □

Let W^*_α be the closure of the graph of f_α on $B \times F \times Z$, W^* the union of the W^*_α and $W = W^* \cap B \times F \times L$. W is a k'-closed subset of $B \times F \times L$.

Lemma 1.4. *The set* E' *of points* $a' \in B$ *such that* $W \cap a' \times F \times L$ *has component of dimension at least* $q_{m_s} - 1$ *forms a* k'*-closed proper subset of* B.

Proof. When a is a generic point of B over k', the projection of the intersection on F is the support of the Chow-variety of $\Lambda(m_s X(a))$ (c.f. [14], [26]). Then our lemma follows at once from [27], Lemma 7, applied to $B \times F \times \overline{L}$ where $\overline{L}$ denotes the closure of L in its ambient space. □

Lemma 1.5. *Let* $a' \in B$ *such that* $l(m_s X(a')) < q_{m_s}$. *Then* $a' \notin E'$.

Proof. Assume the contrary. Then the intersection $W \cap a' \times F \times L$ contains a component $a' \times D$ of dimension at least $q_{m_s} - 1$ by Cor., Lemma 1.3. Let k'' be an algebraically closed field, containing k', over which D is defined and Q' a generic point of D over k''. It is of the form (y', e') where $e' = (a', \ldots, \mathscr{S}(m_s X(a') \cdot C(a')_i), \ldots)$ (c.f. Cor.,
299 Lemma 1.3). Since r is sufficiently large, there is an index i such that $H^{(i)}$ is generic over $k(a', y')$ (c.f. [26], Lemma 9). Then Cor., Lemma 1.3 implies that $\mathscr{S}(m_s X(a') \cdot C(a')_i) = \mathscr{S}(Y(y') \cdot C(a')_i)$. Since t is sufficiently large and since $H^{(i)}$ is generic over $k(a', y')$, it follows that $m_s X(a') \sim Y(y')$ by an equivalence criterion of Weil (c.f. [27], Th. 2; see also the footnote for Lemma 1.1). Hence $l(m_s X(a')) \geqslant q_{m_s}$ and this contradicts our assumption. □

Lemma 1.6. *Let $a' \in B$ such that $l(m_s X(a') \in) \geqslant q_{m_s}$. Then $a' \in E'$.*

Proof. Let D' be the Chow-variety of the complete linear system $\Lambda(sX(a'))$. Then $\dim D' \geqslant q_{m_s} - 1 > 0$. Let k'' be an algebraically closed field, containing k', over which D' is rational. Let y' be a generic point of D' over k''. Then (a', y') is contained in some component T_α of T. Let (a, y) be a generic point of T_α over k''. Let

$$e' = (a', \ldots, \mathscr{S}(m_s X(a') \cdot C(a')_i), \ldots)$$

and $e = (a, \ldots, \mathscr{S}(Y(y) \cdot C(a)_i), \ldots)$. Then $(a, y, e) \to (a', y', e')$ ref. k'' by Cor., Lemma 1.3. Hence (a', y', e') is a point of W. It follows that $W \cap a' \times F \times L$ contains $a' \times D' \times e'$ and a' is contained in E'. Our lemma is thereby proved. □

As we have pointed out, Lemmas 1.4, 1.5, 1.6 prove our proposition.

Corollary to Proposition 1.1. *Let the notations and assumptions be as in our proposition. There is a finite union $\mathfrak{E}$ of irreducible families, defined over $\bar{k}$ and contained in $\mathfrak{A}'$, such that a member $(U(a'), X(a'))$ of $\mathfrak{A}'$ is in $\mathfrak{E}$ if and only if $l(mX(a')) \geqslant q_m$ for all m.*

2

Proposition 2.1. *Let V^n (resp. V'^n) be a complete abstract variety, non-singular in codimension 1, and X (resp. X') a divisor on V (resp. V'). Let k be a common field of rationality of V and X, $\mathfrak{O}$ a discrete valuation ring of k and assume that (V', X') is a specialization of (V, X) over $\mathfrak{O}$ and that $l(X) = l(X')$. Let Γ' be the closure of the graph of a non-degenerate rational map of V' defined by X'. Then, there is a field K over k, a discrete valuation ring $\mathfrak{O}'$ of K dominating $\mathfrak{O}$ and the closure Γ of the graph of a non-degenerate rational map of V defined by X such* 300
that $(V', X', \Gamma' + Z')$ is a specialization of (V, X, Γ) over $\mathfrak{O}'$, where Z' is such that $pr_{V'} Z' = 0$.

Proof. Let k' be the residue field of $\mathfrak{O}$. Since X' is rational over k', $\Lambda(X')$ is defined over k'. Let $g'_1 = 1, g'_2, \ldots, g'_{N+1}$ be functions on V' which define Γ'. From Lemmas 4 and 5, [16], we can see easily that there is a filed K over k, a discrete valuation ring $\mathfrak{O}'$ of K which dominates $\mathfrak{O}$ and a basis (g_i) of $L(X)$ over K such that $(V', X', (g'_i))$ is a specialization of $(V, X, (g_i))$ over $\mathfrak{O}'$. Let Γ be the closure of the graph of a non-degenerate rational map of V defined by X, determined in terms of (g_i). Let T be a specialization of Γ over $\mathfrak{O}'$. It is clear and easy to see that Γ' is contained in the support of T. Therefore, Γ' is a component of T. When that is so, our proposition follows from the compatibility of specializations with the operation of algebraic projection (c.f. [24]).

In the discussions which follow, we shall need the following definition. Let U and W be two abstract varieties, f a rational map of U into W and U' a subvariety of U along which f is defined. Let f' be the restriction of f on U' and W' the geometric image of U' by f'. We shall denote by $f[U']$ the variety W' if $\dim U' = \dim W'$ and 0 otherwise. □

Corollary. *Notations and assumptions being the same as in our propositions, let f (resp. f') be a non-degenerate rational map of V (resp. V') defined by X (resp. X'). When $f'[V'] \neq 0$, then $f[V] \neq 0$.*

Proof. Let k' be the residue field of $\mathfrak{O}$ and Q' a generic point of V' over k'. Then there are n independent generic divisors X'_i of $\Lambda(X')$ over k' such that Q' is a proper point of intersection of $\bigcap_i X'_i$. Let the X_i be n independent generic divisors of $\Lambda(X)$ over k and $\mathfrak{O}^*$ a discrete valuation ring, dominating $\mathfrak{O}$, such that $(V', X', (X'_i))$ is a specialization of $(V, X, (X_i))$ over $\mathfrak{O}^*$ (c.f. [16]). By the compatibility of specializations with the operation of intersection-product (c.f. [24], in particular, Th. 11, Th. 17), there is a point Q in V such that it is a proper component of $\bigcap_i X_i$ and that Q' is a specialization of Q over the above specialization with reference to $\mathfrak{O}^*$. This proves our corollary. □

301 We shall consider again, as in §1, an algebraic family (irreducible) $\mathfrak{A}'$ with divisors in a projective space. We shall assume that every member $(U(a), X(a))$ satisfies the conditions that $U(a)$ *is non-singular in codimension* 1 *and that* $X(a)$ *is a positive divisor on* $U(a)$. Therefore, a is a pair of the Chow-point of $U(a)$ and that of $X(a)$. Let k be an algebraically closed field of rationality of $\mathfrak{A}'$ and A' the locus of a over k, where a corresponds to a generic member of $\mathfrak{A}'$ over k.

Proposition 2.2. *$\mathfrak{A}'$ and A' being as above, assume that the following conditions are satisfied:* (i) *when $a' \in A'$, then $l(X(a')) = l(X(a))$ where a is a generic point of A' over k;* (ii) *when f is a non-degenerate rational map of $U(a)$ defined by $X(a)$, then $f[U(a)] \neq 0$. Then the set E of points a' in A' such that a non-degenerate rational map f' of $U(a')$ defined by $X(a')$ has the property* $\deg(f'[U(a')]) < s = \deg(f[U(a)])$ *is a k-closed subset of A'.*

Proof. Since $X(a)$ is rational over $k(a)$, there is a non-degenerate rational map f of $U(a)$, defined by $X(a)$, which is defined over $k(a)$. Let Γ be the closure of the graph of f and t the Chow-point of Γ. We shall denote Γ by $\Gamma(t)$. Let w be the Chow-point of $f[U(a)]$. We shall denote $f[U(a)]$ also by $W(w)$. Let T (resp. W) be the locus of t (resp. w) over k and D the locus of (a, t, w) over k. D is then a subvariety of $A' \times T \times W$.

Let W_0 be the set of points w' such that the corresponding $W(w')$ with the Chow-point w' is irreducible and not contained in any hyper-plane. Let T_0 be the set of points t' such that the corresponding $\Gamma(t')$ with the Chow-point t' is irreducible and $D_0 = D \cap A' \times T_0 \times W_0$. As is well known, W_0 is k-open on W and T_0 is k-open on T. Hence D_0 is a closed subvariety of $A' \times T_0 \times W_0$, defined over k. The set-theoretic projection of D_0 on A' contains a k-open subset of A' (c.f. [28]). Let D' be the largest k-open subset of A' contained in this projection.

Let $a' \in D'$. There is a point $(a', t', w') \in D_0$. By our choice of W_0, T_0 and D_0, $\Gamma(t')$ is irreducible, $W(w')$ is irreducible, $\deg(W(w')) = \deg(W(w)) = s$ and $W(w')$ is not contained in any hyper-plane. More- **302**
over, $(U(a'), X(a'), \Gamma(t'), W(w'))$ is a specialization of $(U(a), X(a), \Gamma(t), W(w))$ over k. Since linear equivalence is preserved by specializations and since specializations are compatible with the operations of intersection-product and algebraic projection (c.f. [24]), it follows that $\Gamma(t)'$ is the closure of the graph of a non-degenerate map of $U(a')$ determined by $X(a')$ and $\mathrm{pr}_2\,\Gamma(t') = mW(w')$ if $pr_2\Gamma(t) = mW(w)$. Thus a point of E cannot be contained in D'.

$A' - D'$ is a k-closed subset of A'. Let A'' be a component of it and a' a generic point of A'' over k. Let f' be a non-degenerate rational map of $U(a')$ defined by $X(a')$ and assume that $f'[U(a')] = 0$. If A'' has another point a'', let f'' be a similar map of $U(a'')$ defined by $X(a'')$. We consider a curve C on A'' which contains a' and a''. The existence of such a curve is well known and easy to prove by using the theorem of Bertini. Normalizing C and localizing it at a point corresponding to a'', we apply the result of Proposition 2.1. Then we see that $f''[U(a'')] = 0$ since specializations are compatible with the operation of algebraic projection. Assume this time that $f'[U(a')] \neq 0$. Consider a curve C on A' which contains a and a' and proceed as above. Then we see that $\deg(f[U(a)]) = s \geqslant \deg(f'[U(a')])$. When a'', f'' are as above, we see also that $\deg(f'[U(a')]) \geqslant \deg(f''[U(a'')])$ by the same technique. Therefore, choosing only those A'' such that $\deg(f'[U(a')]) = s$ and repeating the above process, we get our proposition easily. □

Corollary 1. *Let $s_0 \leqslant s$ be a non-negative integer. Then the set E_{s_0} of points a' of A' such that a non-degenerate rational map f' defined by $X(a')$ has the property* $\deg(f'[U(a')]) \leqslant s_0$ *is a k-closed subset of A'.*

Proof. This follows easily from our proposition. □

Corollary 2. *With the same notations and assumptions of our proposition, assume further that f is a birational map. Then there is a k-open subset A'_0 of A' such that the following conditions are satisfied by points a' of A'_0: When f' is a non-degenerate map of $U(a')$ defined by $X(a')$, f' is a birational map and* $\deg(f(U(a))) = \deg(f'(U(a')))$.

Proof. Using the same notations of the proof of our proposition, let $a' \in$
303 D'. Then a point $(a', t', w') \in D_0$ was such that $\Gamma(t')$ is irreducible, $W(w')$ is irreducible, $\deg(W(w)) = \deg(W(w')) = s$ and $W(w')$ is not contained in any hyperplane. Then $\Gamma(t')$ is the closure of the graph of a birational map defined by $X(a')$. Therefore, it is easy to see that D' satisfies our requirement as A'_0. □

Corollary 3. *With the same notations and assumptions of our proposition, assume further that f is a birational map and that $f(U(a))$ is non-singular. Then there is a k-open subset A'_0 of A' such that the following* *conditions are satisfied by points a' of A'_0: When f' is a non-degenerate rational map of $U(a')$ defined by $X(a')$, f' is a birational map, $f'(U(a'))$ is non-singular and that* $\deg(f(U(a))) = \deg(f'(U(a')))$.

Proof. In the proofs of our proposition and corollary, above take W_0 to be the set of points w' such that $W(w')$ is irreducible, non-singular and not contained in any hyperplane. W_0 is also a k-open subset of W. The rest of our proof will then be exactly the same as that of the above corollary. □

Corollary 4. *With the same notations and assumptions of our proposition, assume that f is not birational. Then there is a k-open subset A''_0 of points a' of A' with the following property: When f' is a non-degenerate rational map defined by $X(a')$, f' is not birational and* $\deg(f[U(a)]) = \deg(f'[U(a')])$.

Proof. The proof of Corollary 2 above goes through almost word for word when we make the following change: (i) "birational" should be changed to "not birational". It should be noted that $\mathrm{pr}_2\,\Gamma(t) = mW(w)$, $\mathrm{pr}_2\,\Gamma(t') = mW(w')$ and $m > 1$ in the proof of our proposition since f in our case is not birational. □

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BI-EXTENSIONS OF FORMAL GROUPS

By David Mumford

In the Colloquium itself, I announced that all abelian varieties can 307
be lifted to characteristic zero. The proof of this, as sketched there, is roughly as follows.

(i) It suffices to prove that every char p abelian variety is a specialization of a char p abelian variety with multiplicative formal group (an "ordinary" abelian variety), since Serre (unpublished) has shown that these admit liftings.

(ii) A preliminary reduction of the problem was made to abelian varieties X such that the invariant

$$\alpha(X) = \dim_k \operatorname{Hom}(\alpha_p, X)$$

is 1.

(iii) A method was found to construct deformations of a polarized abelian variety from deformations of its polarized Dieudonné module.

(iv) Finally, some simple deformations of polarized Dieudonné modules were constructed to establish the result.

However, it seems premature to give this proof here, since the basic method used in (iii) promises to give much fuller information on the local structure of the formal moduli space of a polarized abelian variety, and this would make my *ad hoc* method obsolete. I want instead to give some basic information on the main new technical tool which is used in (iii).

1 Cartier's result. In the note [1], Cartier has announced a module-theoretic classification of formal groups over arbitrary ground-rings R. We require only the special case where $p = 0$ in R, which is foreshadowed in Dieudonn'es original paper [2], before the category men got a hold of it, modifying the technique until the restriction "R = perfect field" came to seem essential.

308 **Definition.** *Let R be a ring of characteristic p. Let $W(R)$ be the ring of Witt vectors over R, and let*

$$\begin{aligned}(a_0, a_1, a_2, \ldots)^\sigma &= (a_0^p, a_1^p, a_2^p, \ldots),\\ (a_0, a_1, a_2, \ldots)^t &= (0, a_0, a_1, \ldots).\end{aligned}$$

Then A_R will denote the ring

$$W(R)[[V]][F]$$

modulo the relations:

(a) $FV = p$,

(b) $VaF = a^t$,

(c) $Fa = a^\sigma F$,

(d) $aV = Va^\sigma$,

for all $a \in W(R)$.

Theorem (Dieudonné-Cartier). *There is a covariant equivalence of categories between*

(A) *the category of commutative formal groups Φ over R, and*

(B) *the category of left A_R-modules M such that*

(a) $\bigcap_i V^i M = (0)$,

(b) $Vm = 0 \Rightarrow m = 0$, *all $m \in M$,*

(c) *M/VM is a free R-module of finite rank.*

The correspondence between these 2 categories can be set up as follows. Recall first that a formal group Φ/R (by which we mean a set of n power series $\phi_i(x_1, \dots, x_n; y_1, \dots, y_n)$, $1 \leqslant i \leqslant n$, satisfying the usual identities, c.f. Lazard [3]) defines a covariant functor F_Φ from R-algebbras S to groups : i.e. $\forall\, S/R$,

$$F_\Phi(S) = \{(a_1, \dots, a_n) | a_i \in S, a_i \text{ nilpotent}\}$$

where

$$\begin{aligned}&(a_1, \dots, a_n) \cdot (b_1, \dots, b_n)\\ &= (\phi_1(a_1, \dots, a_n; b_1, \dots, b_n), \dots, \phi_n(a_1, \dots, a_n; b_1, \dots, b_n)).\end{aligned}$$

N. B. In what follows, we will often call the functor F_Φ instead of the power series Φ the formal group, for simplicity.

Let $\widehat{W}$ be the functor 309

$$\begin{cases} \widehat{\mathbf{W}} = \{(a_0, a_1, \dots) | a_i \in S, a_i \text{ nilpotent, almost all } a_i = 0\}, \\ \text{gp law } = \text{Witt vector addition.} \end{cases}$$

Then we attach to the commutative formal group Φ the set

$$M = \mathrm{Hom}_{\text{gp. functors}/R}(\widehat{W}, F_\Phi),$$

and since $A_R \cong \mathrm{Hom}(\widehat{\mathbf{W}}, \widehat{\mathbf{W}})^0$, we can endow M with the structure of left A_R-module. Conversely, to go in the other direction, first note that any A_R-module M as in the theorem can be resolved:

$$0 \to A_R^n \xrightarrow{\beta} A_R^n \xrightarrow{\alpha} M \to 0. \tag{*}$$

In fact, choose $m_1, \dots, m_n \in M$ whose images mod VM are a basis of M/VM as R-module. Define

$$\alpha(P_1, \dots, P_n) = \sum_{i=1}^n P_i m_i.$$

It is easy to check that Fm_i can be expanded in the form $\sum_{j=1}^{n} Q_{ij}(V)m_j$, Q_{ij} a power series in V with coefficients in $W(R)$. Define

$$\beta(P_1,\ldots,P_n) = \left(\sum_{i=1}^{n} P_i \cdot Q_{i1} - \delta_{i1}F,\ldots,\sum_{i=1}^{n} P_i \cdot Q_{in} - \delta_{in}F\right).$$

It is not hard to check that (*) is exact. Then β defines a monomorphism of group functors $\beta^* : (\widehat{W})^n \to (\widehat{W})^n$, and let F be the quotient functor $(\widehat{W})^n/\beta^*(\widehat{W})^n$. Then F is isomorphic to F_Φ for one and-up to canonical isomorphism-only one formal group Φ.

Moreover, we get a resolution of the functor F_Φ:

$$0 \to (\widehat{W})^n \xrightarrow{\beta^*} (\widehat{W})^n \to F_\Phi \to 0.$$

When R is a perfect field, the above correspondence can be extended to an analogous correspondence between p-divisible groups over R and
310 $W(R)[F,V]$-modules of suitable type (c.f. [4], [5]). However, it does not seem likely at present that such an extension exists for non-perfect R's. This is a key point.

2 Bi-extensions of abelian groups.

Let A, B, C be 3 abelian groups. A bi-extension of $B \times C$ by A will denote a set G on which A acts freely, together with a map

$$G \xrightarrow{\pi} B \times C$$

making $B \times C$ into the quotient G/A, together with 2 laws of composition:

$$+_1 : G \underset{B}{\times} G \to G \quad ; \quad +_2 : G \underset{C}{\times} G \to G$$

where, by definition ($\mathrm{def}\,\|$),

$G \underset{B}{\times} G = \{(g_1,g_2) | \pi(g_1), \pi(g_2)$ have same B-component$\}$

$G \underset{C}{\times} G = \{(g_1,g_2) | \pi(g_1), \pi(g_2)$ have some C-component$\}$

These are subject to the requirement:

(i) for all $b \in B$, $G'_b = \pi^{-1}(b \times C)$ is an abelian group under $+_1$, π is a surjective homomorphism of G'_b onto C, and via the action of A on G'_b, A is isomorphic to the kernel of π;

(ii) for all $c \in C$, $G^2_c = \pi^{-1}(B \times c)$ is an abelian group under $+_2$, π is a surjective homomorphism of G^2_c onto B, and via the action of A on G^2_c, A is isomorphic to the kernel of π;

(iii) given $x, y, u, v \in G$ such that

$$\begin{aligned}\pi(x) &= (b_1, c_1)\\ \pi(y) &= (b_1, c_2)\\ \pi(u) &= (b_2, c_1)\\ \pi(v) &= (b_2, c_2),\end{aligned}$$

then

$$(x +_1 y) +_2 (u +_1 v) = (x +_2 u) +_1 (y +_2 v).$$

This may seem like rather a mess, but please consider the moti- **311**
vating example: let X be an abelian variety over an algebraically closed field k, let $\widehat{X}$ be its dual, and let P be the universal, or Poincaré, line bundle on over $X \times \widehat{X}$. Then P_k, the underlying set of closed points of P, is a bi-extension of $X_k \times \widehat{X}_k$ by k^*!

Notice that if G is a bi-extension of $B \times C$ by A, then $\pi^{-1}(B \times 0)$ splits canonically into $A \times B$, and $\pi^{-1}(0 \times C)$ splits canonically into $A \times C$. In fact, we can lift B to $\pi^{-1}(B \times 0)$ by mapping $b \in B$ to the element of G which is the identity in $\pi^{-1}(b \times C)$; and we can lift C to $\pi^{-1}(0 \times C)$ by mapping $c \in C$ to the element of G which is the identity in $\pi^{-1}(B \times c)$.

Bi-extensions can be conveniently described by co-cycles: choose a (set-theoretic) section

$$\begin{tikzcd} G \arrow[r, "\pi"'] & B \times C \arrow[l, bend right, "s"'] \end{tikzcd}$$

Via s and the action of A on G, we construct an isomorphism

$$G \cong A \times B \times C$$

such that the action of A on G corresponds to the action of A on $A \times B \times C$ which is just addition of A-components, leaving the B-and C-components fixed. Then $+_1$ and $+_2$ go over into laws of composition on $A \times B \times C$ given by:

$$\begin{aligned}(a, b, c) +_1 (a', b, c') &= (a + a' + \phi(b; c, c'), b, c + c')\\ (a, b, c) +_2 (a', b', c) &= (a + a' + \psi(b, b'; c), b + b', c).\end{aligned}$$

For $+_1$, $+_2$ to be abelian group laws, we need:

(a) $$\phi(b; c + c', c'') + \phi(b; c, c') = \phi(b; c, c' + c'') + \phi(b; c', c'')$$
$$\phi(b; c, c') = \phi(b; c', c);$$

(b) $$\psi(b + b', b''; c) + \psi(b, b'; c) = \psi(b, b' + b''; c) + \psi(b', b''; c)$$
$$\psi(b, b'; c) = \psi(b', b; c).$$

The final restriction comes out as:

312 (c) $$\phi(b + b'; c, c') - \phi(b; c, c') - \phi(b'; c, c')$$
$$= \psi(b, b'; c + c') - \psi(b, b'; c) - \psi(b, b'; c').$$

What are the co-boundaries? If you alter s by adding to it a map $\rho : B \times C \to A$, then you check that the new ϕ', ψ' are related to the old ones by

$$\begin{aligned}\phi'(b; c, c') - \phi(b; c, c') &= \rho(b, c + c') - \rho(b, c) - \rho(b, c')\\ \psi'(b, b'; c) - \psi(b, b'; c) &= \rho(b + b', c) - \rho(b, c) - \rho(b, c).\end{aligned}$$

Using this explicit expression by co-cycles and co-boundaries, it is clear that the set of all bi-extensions of $B \times C$ by A forms itself an abelian group, which we will denote

$$\text{Bi-ext}\ (B \times C, A).$$

It is also clear, either from the definition or via co-cycles, that Bi-ext is a covariant functor in A, and a contravariant functor in B and C.

3 Bi-extensions of group-functors.

Definition. *If F, G, H are 3 covariant functors from the category of R-algebras to the category of abelian groups, a bi-extension of $G \times H$ by F is a fourth functor K such that for every R-algebra S, $K(S)$ is a bi-extension of $G(S) \times H(S)$ by $F(S)$ and for every R-homomorphism $S_1 \to S_2$, the map $K(S_1) \to K(S_2)$ is a homomorphism of bi-extensions (in the obvious sense). In particular, if F, G, H are formal groups, this gives us a bi-extension of formal groups.*

If F, G, H are formal groups, it is easy again to compute the bi-extensions K by power series co-cycles. In fact, one merely has to check that:

(i) there is a functorial section

$$\begin{tikzcd} K \arrow[r, "\pi"'] & G \times H \arrow[l, bend right, "s"'] \end{tikzcd}$$

(this follows using the "smoothness" of the functor F, i.e. $F(S) \to$ **313**
$F(S/I)$ is surjective if I is a nilpotent ideal);

(ii) any morphism of functors from one product of formal groups to another such product is given explicitly by a set of power series over R in the appropriate variables.

In fact, we will be exclusively interested in the case where $F = \widehat{\mathbf{G}}_m$ is the formal multiplicative group; that is

$$\widehat{\mathbf{G}}_m(S) = \left\{ \begin{array}{l} \text{Units in } S \text{ of form } 1 + x,\ x \text{ nilpotent,} \\ \text{composed via multiplication.} \end{array} \right\}$$

Then if G and H are formal groups in variables $x_1, \ldots, x_n$ and $y_1, \ldots, y_m$, a bi-extension of $G \times H$ by $\widehat{\mathbf{G}}_m$ is given by 2 power series

$$\sigma(x_1, \ldots, x_n; y_1, \ldots, y_m, y'_1, \ldots, y'_m), \tau(x_1, \ldots, x_n, x'_1, \ldots, x'_n; y_1, \ldots, y_m)$$

with constant terms 1 such that - abbreviating n-tuples and m-tuples:

$$\begin{aligned}
&\sigma(x;\Phi(y,y'),y'')\cdot\sigma(x;y,y') = \sigma(x;y,\Phi(y',y''))\cdot\sigma(x;y',y'')\\
&\sigma(x;y,y') = \sigma(x;y',y)\\
&\tau(\Psi(x,x'),x'',y)\cdot\tau(x,x';y) = \tau(x,\Psi(x',x'');y)\cdot\tau(x',x'';y)\\
&\tau(x,x';y) = \tau(x',x;y)\\
&\sigma(\Psi(x,x');y,y')\cdot\sigma(x;y,y')^{-1}\cdot\sigma(x';y,y')^{-1} = \tau(x,x';\Phi(y,y'))\cdot\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\tau(x,x';y)^{-1}\cdot\tau(x,x';y')^{-1},
\end{aligned}$$

if Φ, Ψ are the group laws of G and H respectively.

We want one slightly non-trivial fact about general bi-extensions. This result gives essentially the method for computing Bi-ext's via resolutions.

Proposition 1. *Let E, G, G' be abelian group functors as above. Suppose*

$$\begin{gathered}
0 \to F_1 \to F_0 \to G \to 0\\
0 \to F_1' \to F_0' \to G' \to 0
\end{gathered}$$

are 2 exact sequences of such functors. Then

$$\begin{aligned}
&\mathrm{Ker}\{\text{Bi-ext}\,(G\times G',E) \to \text{Bi-ext}\,(F_0\times F_0',E)\}\\
&\cong \frac{\{(f,g)\,|\,f: F_0\times F_1' \to E \text{ and } g: F_1\times F_0' \to E \textit{ bi-homomorphisms } \operatorname{res} f = \operatorname{res} g \text{ on } F_1\times F_1'\}}{\{(f,g)\,|\,\exists h: F_0\times F_0' \to E \textit{ bi-homomorphism, } f \text{ and } g \text{ restrictions of } h\}}
\end{aligned}$$

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The proof goes along these lines: let H be a bi-extension of $G\times G'$ by E. If it lies in the above kernel, then the induced bi-extension of $F_0\times F_0'$ is trivial:

$$H \underset{(G\times G')}{\times} (F_0\times F_0') \cong E\times F_0\times F_0'.$$

Consider the equivalence relation on the functor $E\times F_0\times F_0'$ induced by the mapping of it onto H. It comes out that there are maps $f: F_0\times F_1' \to E$, $g: F_1\times F_0' \to E$ such that this equivalence relation is generated by

$$(a, b, c) \sim (a + f(b, \overline{c}), b, c + \overline{c}), a \in E(S), b \in F_0(S)$$
$$c \in F_0'(S), \overline{c} \in F_1'(S). \quad (15.1)$$

and

$$(a, b, c) \sim (a + g(\overline{b}, c), b + \overline{b}, c), a \in E(S), b \in F_0(S)$$
$$b \in F_1(S), c \in F_0'(S). \quad (15.2)$$

Moreover, f and g have to be bi-homomorphisms with res $f =$ res g on $F_1 \times F_1'$. Conversely, given such g and g, define the functor H to be the quotient of $E \times F_0 \times F_0'$ by the above equivalence relation. H turns out to be a bi-extension. Finally, the triviality of H can be seen to be equivalent to f and g being the restrictions of a bi-homomorphism $h : F_0 \times F_0' \to E$.

4 Bi-extensions of $\widehat{W}$.

Proposition 2. Bi $-$ ext$(\widehat{\mathbf{W}} \times \widehat{\mathbf{W}}, \widehat{\mathbf{G}}_m) = (0)$.

Proof. Consider functors F from (R-algebras) to (abelian groups) which are isomorphic as set functors to D^I, where

$$D^I(S) = \{(a_i) | a_i \in S, \text{ all } i \in I, a_i \text{ nilpotent, almost all } a_i = 0\}$$

and where I is an indexing set which is either finite or countably infinite. Note that all our functors are of this type. Then I claim that for all R of char p, all such F, there is a canonical retraction p_F:

$$\begin{tikzcd}
\underset{\text{set-functors}}{\text{Hom}}(\widehat{\mathbf{W}}, F) \arrow[r, bend left, "p_F"] & \underset{\text{gp-functors}}{\text{Hom}}(\widehat{\mathbf{W}}, F) \arrow[l, "\text{inclusion}"]
\end{tikzcd}$$

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which is functorial both with respect to (15.1) any homomorphism $F \to G$, and (15.2) base changes $R_1 \to R_2$.

The construction of p_F is based on Theorem 1 of Cartier's note [1]. Let $\widehat{W}^*$ be the full Witt group functor (i.e. based on all positive integers, rather than powers of p), and let $i : D \to \widehat{\mathbf{W}}^*$ be the canonical inclusion used in [1]. Then Theorem 1 asserts that for all formal groups F, every morphism $\phi : D \to F$ extends uniquely to a homomorphism $u : \widehat{\mathbf{W}}^* \to F$.

$$\begin{tikzcd} D \arrow[r, "\phi"] \arrow[d, hook, "i"'] & F \\ \widehat{\mathbf{W}}^* \arrow[ur, dashed, "u"'] & \end{tikzcd}$$

Cartier informs me that this theorem extends to all F's of our type. On the other hand, $\widehat{\mathbf{W}}$, over a ring of char p, is a direct summand of $\widehat{\mathbf{W}}^*$:

$$\begin{tikzcd} \widehat{\mathbf{W}}^* \arrow[r, shift right, "\pi"'] & \widehat{\mathbf{W}}. \arrow[l, shift right, "j"'] \end{tikzcd}$$

Construct p_F as follows: given $f : \widehat{\mathbf{W}} \to F$, let $\phi =$ res to D of $f \circ \pi$; let $u =$ extension of ϕ to a homomorphism u; let $p_F(f) = u \circ j$.

Now let F be a bi-extension of $\widehat{\mathbf{W}} \times \widehat{\mathbf{W}}$ by $\widehat{\mathbf{G}}_m$. For every R-algebra S and every $a \in \widehat{\mathbf{W}}(S)$, let F'_a (resp. F''_a) denote the fibre functor of F over $\{a\} \times \widehat{\mathbf{W}}$ (resp. $\widehat{\mathbf{W}} \times \{a\}$) (i.e. $F_a(T) = \{b \in F(T) | 1^{\text{st}}$ (resp. 2^{nd}) component of $\pi(b)$ is induced by a via $S \to T\}$). Then F'_a and F''_a are
316 group functors of the good type extending $\widehat{\mathbf{W}}$ by $\widehat{\mathbf{G}}_m$ over ground ring S. Now since $\widehat{\mathbf{G}}_m$ is smooth, one can choose a section s to π:

$$\begin{tikzcd} F \arrow[r, "\pi"'] & (\widehat{\mathbf{W}} \times \widehat{\mathbf{W}}) \arrow[l, bend right, "s"'] \end{tikzcd}$$

s restricts to morphisms $s_a : \hat{W}/S \to F'_a$, for all $a \in \widehat{\mathbf{W}}(S)$. Take $p_{F'_a}(s_a)$. As a varies, these fit together into a new section $p'(s)$ to π. But $p'(s)$ is now a homomorphism with respect to addition into the 2^{nd} variable, i.e.

$$p'(s)(u, v) +_1 p'(s)(u, v') = p'(s)(u, v + v'). \qquad (*)'$$

Now switch the 2 factors : $p'(s)$ restricts to morphism $p'(s)_a : \widehat{\mathbf{W}}/S \rightarrow F''_a$, for all $a \in \widehat{\mathbf{W}}(S)$. Take $p_{F''_a}(p'(s)_a)$. As a varies, these fit together into a new section $p''(p'(s))$ to π.

Then this satisfies :

$$p''(p'(s))(u,v) +_2 p''(p'(s))(u',v) = p''(p'(s))(u+u',v). \qquad (*)''$$

But now, using the functoriality of p, and the property of bi-extensions linking $+_1$ and $+_2$, it falls out that $p''(p'(s))$ still has property $(*)'$ enjoyed by $p'(s)$! So $p''(p'(s))$ preserves both group laws and splits the extension F. $\square$

Definition. *$\overline{A}_R$ will denote the ring $W(R)[[F,V]]$ modulo the relations*

(a) $FV = p$

(b) $VaF = a'$

(c) $Fa = a^\sigma F$

(d) $aV = Va^\sigma$, *all* $a \in W(R)$.

Every element in this ring can be expanded uniquely in the form:

$$P = a_0 + \sum_{i=1}^{\infty} V^i a_i + \sum_{i=1}^{\infty} a_{-i} F^i.$$

For every such P, let

$$P^* = a_0 + \sum_{i=1}^{\infty} a_i F^i + \sum_{i=1}^{\infty} V^i a_{-i}.$$

Then $*$ is an anti-automorphism of $\overline{A}_k$ of order 2. We shall consider $\overline{A}_R$ **317**
as an $A_R \times A_R$-module via

$$(P,Q)\cdot x = P\cdot x\cdot Q^*. \qquad (*)$$

Proposition 3. Bi-hom$_R(\widehat{\mathbf{W}} \times \widehat{\mathbf{W}}, \widehat{\mathbf{G}}_m) \cong \overline{A}_R$.
Moreover, since $A_R = \mathrm{Hom}_R(\widehat{\mathbf{W}}, \widehat{\mathbf{W}})^0$, the left-hand side is an $A_R \times A_R$-module; under the above isomorphism, this structure corresponds to the $A_R \times A_R$-module structure on $\overline{A}_R$ defined by $(*)$.

Proof. Cartier [1] has shown that for all R, the Artin-Hasse exponential defines isomorphisms

$$\mathrm{Hom}_R(\widehat{\mathbf{W}}, \widehat{\mathbf{G}}_m) \cong \mathbf{W}(R)$$

where $\mathbf{W}$ is the full Witt functor

$$\begin{cases} \mathbf{W}(R) = \{(a_0, a_1, \ldots) | a_i \in R\} \\ \text{group law = addition of Witt vectors.} \end{cases}$$

Therefore,

$$\text{Bi-Hom}_R(\widehat{\mathbf{W}} \times \widehat{\mathbf{W}}, \widehat{G}_m) \cong \mathrm{Hom}_R(\widehat{\mathbf{W}}, \mathbf{W}).$$

Define a homomorphism

$$\begin{array}{ll} & \overline{A}_R \xrightarrow{\phi} \mathrm{Hom}_R(\widehat{\mathbf{W}}, \mathbf{W}) \\ \text{by} & P \to \text{the map } [b \mapsto P(b)]. \end{array}$$

Here $P(b)$ means that V and F operate on Witt vectors in the usual way: note that the doubly infinite series P operators on b since b has only a finite number of components and all are nilpotent, whereas $P(b)$ is allowed to have all components non-zero.

Let

$$\widehat{\mathbf{W}}_n(R) = \{(a_0, a_1, \ldots) | a_i^{p^n} = 0, \text{ all } i; \text{ almost all } a_i = 0\}.$$

Notice that

$$\mathrm{Hom}_R(\widehat{\mathbf{W}}, \mathbf{W}) \cong \varprojlim_n \mathrm{Hom}_R(\widehat{\mathbf{W}}_n, \mathbf{W}),$$

318 and that ϕ factors through maps

$$\overline{A}_R / \overline{A}_R \cdot F^n \xrightarrow{\phi_n} \mathrm{Hom}_R(\widehat{\mathbf{W}}_n, \mathbf{W}).$$

It suffices to show that ϕ_n is an isomorphism for all n. But for $n = 1$, $\overline{A}_R / \overline{A}_R \cdot F \cong R[[V]]$, while

$$\mathrm{Hom}_R(\widehat{\mathbf{W}}_1, \mathbf{W}) \cong \mathrm{Hom}_{p\text{-Lie algebras}}(\mathrm{Lie}(\widehat{\mathbf{W}}, \mathrm{Lie}(\mathbf{W})).$$

Also $\mathrm{Lie}(\widehat{\mathbf{W}})$ is the free R-module on generators $\widehat{e}_0$, $\widehat{e}_1$, $\widehat{e}_2, \ldots$ with $\widehat{e}_i^{(p)} = \widehat{e}_{i+1}$; and $\mathrm{Lie}(\mathbf{W})$ is the R-module of all expressions $\sum_{i=0}^{\infty} a_i e_i$, $a_i \in R$, with same p^{th} power map. Moreover $\sum_{i=0}^{\infty} V^i a_i \in R[[V]]$ goes via ϕ_1 to the lie algebra map taking $\widehat{e}_0$ to $\sum_{i=0}^{\infty} a_i e_i$. Thus ϕ_1 is an isomorphism. Now use induction on n, and the exact sequences

$$0 \to \widehat{\mathbf{W}}_{n-1} \to \widehat{\mathbf{W}}_n \xrightarrow{F^{n-1}} \widehat{\mathbf{W}}_1 \to 0.$$

This leads to the diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_R(\widehat{\mathbf{W}}_1, \mathbf{W}) \arrow[r, "\circ F^{n-1}"] & \mathrm{Hom}_R(\widehat{\mathbf{W}}_n, \mathbf{W}) \arrow[r] & \mathrm{Hom}_R(\widehat{\mathbf{W}}_{n-1}, \mathbf{W}) & \\
0 \arrow[r] & \overline{A}_R/\overline{A}_R \cdot F \arrow[r, "\times F^{n-1}"] \arrow[u, "\phi_1"'] & \overline{A}_R/\overline{A}_R \cdot F^n \arrow[r] \arrow[u, "\phi_n"'] & \overline{A}_R/\overline{A}_R \cdot F^{n-1} \arrow[r] \arrow[u, "\phi_{n-1}"'] & 0.
\end{tikzcd}$$

The bottom line is easily seen to the exact, so if ϕ_1 and ϕ_{n-1} are isomorphisms, the diagram implies that ϕ_n is an epimorphism. □

Corollary. *Let F_1 and F_2 be group functors isomorphic to $(\widehat{\mathbf{W}})^{n_i}$ for some n_1, n_2. Let $M_i = \mathrm{Hom}_R(\widehat{\mathbf{W}}, F_i)$ be the corresponding finitely generated, free A_R-module. Then there is a $1-1$ correspondence between bi-homomorphisms*

$$B : F_1 \times F_2 \to \widehat{\mathbf{G}}_m$$

and maps

$$\beta : M_1 \times M_2 \to \overline{A}_R,$$

bi-linear in the following sense: **319**

$$\beta(Pm, Qn) = P \cdot (m, n) \cdot Q^*$$

(all $m \in M_1$, $n \in M_2$, P, $Q \in A_R$).

5 Applications. Putting Propositions 1, 2 and 3 together, we conclude the followng

Corollary. *(a) Let Φ, Ψ be formal groups over R.*

(b) Let M, N be the corresponding Dieudonné modules.

(c) Let

$$0 \to F_1 \to F_0 \to M \to 0$$
$$0 \to G_1 \to G_0 \to N \to 0$$

be resolutions of M and N by finitely generated, free A_R-modules. Then the group $\mathrm{Bi}-\mathrm{ext}_R(\Phi \times \Psi, \widehat{\mathbf{G}}_m)$ *of bi-extensions of formal groups can be computed as the set of pairs of bi-linear maps:*

$$\beta : F_0 \times G_1 \to \overline{A}_R,$$
$$\gamma : F_1 \times G_0 \to \overline{A}_R,$$

such that $\beta = \gamma$ on $F_1 \times G_1$, taken modulo restrictions of bi-linear maps $\alpha : F_0 \times G_0 \to \overline{A}_R$.

In another direction, bi-extensions can be linked to p-divisible groups, as defined by Take [6].

Proposition 4. *Let F and F' be formal groups over a* char p *ring R. Assume that the subgroups G_n(resp. G'_n) = $\mathrm{Ker}(p^n$ in $F(resp\ F'))$ form p-divisible groups over R(i.e. F and F' are "equi-dimensional", or of "finite height"). Then there is a $1-1$ correspondence between* (1) *bi-extensions of $F \times F'$ by $\widehat{G}_m$ and* (2) *sets of bi-homomorphisms β_n : $G_n \times G'_n \to \mu_{p^n}$, such that for all $x \in G_{n+1}(S)$, $y \in G'_{n+1}(S)$,*

$$\beta_n(px, py) = \beta_{n+1}(x, y)^p.$$

320 *Proof.* We will use descent theory and existence of quotients by finite, flat equivalence relations: c.f. Raynaud's article in the same volume as Tate's talk [6]. Starting with the β_n's, let L_n be the quotient functor in the flat topology of $\widehat{\mathbf{G}}_m \times G_n \times G'_{2n}$ by the equivalence relation:

$$(\lambda, x, y) \sim (\lambda \cdot \beta_n(x, b), x, y + b)$$

where $\lambda \in \widehat{\mathbf{G}}_m(S)$, $x \in G_n(S)$, $y \in G'_{2n}(S)$, $b \in G'_n(S)$. Then L_n is a bi-extension of $G_n \times G'_n$ by $\widehat{G}_m$. Moreover, L_n is a subfunctor of L_{n+1}, so if we let L be the direct limit of the functor L_n, then L is a bi-extension of $F \times F'$ by $\widehat{\mathbf{G}}_m$.

Conversely, if we start with L, let L_n be the restriction of L over $G_n \times G'_n$. In the diagram

$$\begin{tikzcd} & L_n \arrow[d, "\pi"] \\ G_n \times G'_{2n} \arrow[ur, dashed, "\phi"] \arrow[r, "1\times p^r"'] & G_n \times G'_n, \end{tikzcd}$$

I want to define a canonical map ϕ which is a homomorphism in both variables, i.e. which splits the induced bi-extension over $G_n \times G'_{2n}$. Suppose $x \in G_n(S), y \in G'_n(S)$ for some R-algebra S. Choose $z_1 \in L(S)$ such that $\pi(z_1) = (x, y)$. If we add z_1 to itself p^n times in the 1st variable, we obtain a point:

$$[p^n]_{+_1}(z_1) = z_2$$
$$\pi(z_2) = (0, y).$$

But $\pi^{-1}((0 \times F')$ is canonically isomorphic to $\widehat{\mathbf{G}}_m \times (0) \times F'$, so $z_2 = (\lambda, 0, y)$, some $\lambda \in \widehat{\mathbf{G}}_m(S)$. Now choose a finite flat S-algebra S' such that $\lambda = \mu^{p_n}$ for some $\mu \in \widehat{\mathbf{G}}_m(S')$. Letting z_1 also denote the element of $L(S')$ induced by z_1, define $z'_1 = \mu^{-1} \cdot z_1$. This is a new point of L over (x, y), which now satisfies $[p^n]_{+_1}(z'_1) = (1, 0, y)$. Now add z'_1 to itself p^n times in the 2nd variable. This gives a point

$$[p^n]_{+_2}(z'_1) = z'_3 \in L^*_n(S'),$$
$$\pi(z'_3) = (x, p^n y).$$

Clearly, z'_3 is independent of the choice of μ, so by descent theory, z'_3 **321**
must be induced by a unique element $z_3 \in L_n(S)$. Define $\phi(x, y) = z_3$. It is easy to check that ϕ is a homomorphism in both variables.

We can use ϕ to set up a fibre product diagram:

$$\begin{tikzcd} \widehat{\mathbf{G}}_m \times G_n \times G'_{2n} \arrow[r, "\alpha"] \arrow[d, "\pi"] & L_n \arrow[d, "\pi"] \\ G_n \times G'_{2n} \arrow[r, "(1\times p^n)"'] & G_n \times G'_n \end{tikzcd}$$

where α is a homomorphism of bi-extensions. Since p^n is faithfully flat, so is α, and L_n is therefore the quotient of $\widehat{\mathbf{G}}_m \times G_n \times G'_{2n}$ by a suitable flat equivalence relation. For every $x \in G_n(S)$, $y \in G'_{2n}(S)$, $b \in G'_n(S)$ and $\lambda \in \widehat{\mathbf{G}}_m(S)$, there is a unique element $\beta_n(x, y, b, \lambda) \in \widehat{\mathbf{G}}_m(S)$ such that

$$\alpha((\lambda, x, y)) = \alpha((\lambda \cdot \beta_n(x, y, b, \lambda), x, y + b)$$

and this function β_n describes the equivalence relation. Using the fact that α is a homomorphism of bi-extensions, we deduce

(1) that β_n does not depend on λ,

(2) $\beta_n(x, y, b) \cdot \beta_n(x, y + b, b') = \beta_n(x, y, b + b')$ (via associativity of equivalence relation),

(3) $\beta_n(x, y, b) \cdot \beta_n(x', y, b) = \beta_n(x + x', y, b)$ (α preserves $+_1$),

(4) $\beta_n(x, y, b) \cdot \beta_n(x, y', b') = \beta_n(x, y + y', b + b')$ (α preserves $+_2$).

By (4) and (2) with $b = y' = 0$,

$$\beta_n(x, y, 0) \cdot \beta_n(x, 0, b') = \beta_n(x, y, b') = \beta_n(x, y, 0) \cdot \beta_n(x, y, b'),$$

hence β_n is independent of y too. Then (3) and (4) show that β_n is a
322 bi-homomorphism, so L_n is constructed from a β_n as required. We leave it to the reader to check that if we start from a set of β_n's, and construct a bi-extension L, then the above procedure leads you back to these same β_n's. □

I think that with these results, bi-extensions can be applied to the problem of determining the local structure of the moduli space of polarized abelian varieties.

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SOME QUESTIONS ON RATIONAL ACTIONS OF GROUPS

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323 The contents are divided into two parts. In Part I, we discuss the rings of invariants of a finite group in a noetherian ring. In Part II, we raise some questions on rational actions of groups, mostly connected algebraic groups. By a ring, we mean a commutative ring with identity. By a subring, we mean a subring having common identity.

Part I.

0 We discuss here the following question.

Question 0.0. Let R be a noetherian ring and let G be a finite group acting on R. Let A be the ring of invariants in R. Is A noetherian ?

Unfortunately, the answer is not affirmative in general as will be shown later by counter-examples. Since the examples which we have non-normal, we raise a question.

Question 0.1. Assume, in Question 0.0, R is a direct sum of normal rings. Is then A noetherian ?

We shall begin with some simple cases. We maintain the meanings of R, G, A of Question 0.0.

Proposition 0.2. *If the order g of G is not divisible by the characteristic of any residue class field of R, in other words, if g is a unit in R, then A is noetherian.*

Proof. If $h_1, \ldots, h_s \in A$ and if $f \in (\Sigma h_i R) \cap A$, then $f = \Sigma h_i r_i (r_i \in R)$. Then $gf = \sum_{\sigma \in G} \sigma f = \sum_i \sum_{\sigma \in G} h_i(\sigma r_i)$, and $f = \sum_i h_i (g^{-1} \sum_{\sigma \in G} \sigma r_i) \in \Sigma h_i A$. Thus we have $(\Sigma h_i R) \cap A = \Sigma h_i A$. From this the assertion follows easily. □

Proposition 0.3. *If R is a Dedekind domain, then A is also a Dedekind domain, and R is a finite A-module.* **324**

The proof is obvious in view of the following well known lemma (see for instance [L]).

Lemma 0.4. *Let A' be a normal ring and let k' be an integral extension of A' in an algebraic extension L of the field of quotients K of A'. Assume that $a \in R'$ generates L over K. Let $f(x)$ be the irreducible monic polynomial for a over A'. Then letting d be one of* (i) *discriminant of $f(x)$ and* (ii) *$df(a)/dx$, we have $dR' \subseteq A'[a]$.*

Another easy case is:

Remark 0.5. If R is a ring of quotients of a finitely generated ring R_0 over a subring F of A and if F is pseudo-geometric, then A is a ring of quotients of a finitely generated ring A_0 over F, hence A is noetherian.

As our example below (see the proof of Proposition 0.11) shows, Question 0.0 is not affirmative even if we assume that R is a pseudo-geometric local integral domain of Krull dimension 1, whose derived normal ring is a valuation ring: this fact shows that:

Remark 0.6. Assume that a subring S of R is G-stable and that B is the ring of G-invariants in S. Even if R is a discrete valuation ring of the field of quotients of S and is a finite S-module, A may not be a finite B-module.

On the other hand, one can show:

Remark 0.7. If, for a subring S of a noetherian ring R, R is a finite S-module, then S is noetherian. (Proof of this remark will be published somewhere else.)

Therefore the writer believes it is an important question to ask for reasonable sufficient conditions for R to be a finite A-module.

Now we are going to construct counter-examples to the question.

Proposition 0.8. *Let F be a field of characteristic $p \neq 0$ and let x_1, $x_2, \ldots$ infinitely many indeterminates. Consider the derivation $D = \sum_{i=1} x_i^{s_i} \dfrac{\partial}{\partial x_i}$ of the field $K = F(x_1, \ldots, x_n, \ldots)$, such that* (i) *each s_i is* **325**

a non-negative integer $\equiv 0$, 1 *modulo p, either* 0 *or greater than* $p-1$ *and* (ii) *infinite number of* s_i *are* $\equiv 1$ *modulo p. Let C be the field of constants with respect to D. Then* $[K : C] = \infty$.

Proof. For simplicity, we assume that $s_i = p+1$ for $i = r, r+1, \ldots$. We show that $x_r, x_{r+1}, \ldots$ are linearly independent over C. For, if $\sum_{i \geqslant r} x_i c_i = 0 (c_i \in C)$, then by the operation of D, we have $\Sigma x_i^{s_i} c_i = 0$ which can be written $\Sigma x_i (x_i^p c_i) = 0$. Since $x_i^p c_i \in C$, we have got another linear relation, and we get a contradiction. □

Proposition 0.9. *Let K be a field of characteristic* $p \neq 0$. *Let y be an element defined by* $y^2 = 0$. *Consider the ring* $R = K[y] = K + yK$. *Let D be a derivation of K. Then the map* $\sigma : f + yg \to f + yg + y\,Df$ *gives an automorphism of R and* $\sigma^p = 1$.

Proof is easy and we omit it.

Now we have

Proposition 0.10. *In the question stated at the beginning, even if R is an artinian ring, A can be non-noetherian.*

Proof. Let K, C and D be as in Proposition 0.8 and then let y, σ be as in Proposition 0.9. For $G = \{1, \sigma, \ldots, \sigma^{p-1}\}$, $A = \{f + yg | Df = 0\} = C + yK$. Since $[K : C] = \infty$, A is not noetherian. □

Proposition 0.11. *In the question, even if R is assumed to be a pseudo-geometric local integral domain of Krull dimension* 1, *A can be non-noetherian.*

Proof. Let F be a field of characteristic $p \neq 0$ and let y, z_1, $z_2, \ldots$ be infinitely many indeterminates. Set $K^* = F(z_1, z_2, \ldots)$ and $V = K^*[y]_{(y)}$. Then V is a discrete valuation ring and has an automorphism σ such that $\sigma z_1 = z_1 + y$ and σ fixes every element of $F[z_2, z_3, \ldots, y] \cdot \sigma^p = 1$. We set

$$x_1 = z_1^{2p}, x_i = z_i + y^p z_i^{p+1} z_i^p (i \geqslant 2), w_1 = y^{2p}, w_i = y^{2p} z_i (i \geqslant 2).$$

326 Then $\sigma z_i = z_i + w_i$ and $\sigma w_i = w_i$. Thus $G = \{1, \sigma, \ldots, \sigma^{p-1}\}$ acts on the ring $R' = F[x_1, x_2, \ldots, w_1, w_2, \ldots]$. Set $R = R'_{yV\cap R'}$. Then G acts on R. The ring of invariants A is of the form $A'_{yV\cap A'}$ with $A' = A \cap R'$. We observe elements of A'. It is of the form $f(x) + \Sigma w_i t_i(x) +$ (terms of higher degree in w). Invariance implies that $\Sigma w_i \dfrac{\partial f(x)}{\partial x_i} \equiv 0 (\text{mod } y^{2p+1}V)$. This implies that, denoting by D the derivation $\dfrac{\partial}{\partial x_1} + \sum_{i\geqslant 2} x_i^{p+1} \dfrac{\partial}{\partial x_i}$ of $K = F(x_1, x_2, \ldots)$, $Df(x) = 0$. Therefore $A'/yV \cap A'$ is contained in the field of constants with respect to this D. Thus, Proposition 0.8 shows that $[R/yV \cap R : A/yV \cap A] = \infty$ and that the sequence of ideals $(y^{2p+1}V \cap A) + \sum_{i=2}^{2} w_i A (n = 2, 3, \ldots)$ gives an infinite ascending chain of ideals. Thus A is not noetherian. That R is a pseudogeometric local integral domain of Krull dimension 1 follows from the fact that $R \supset F(x_1, x_2, \ldots)[y^{2p}]$. □

Remark 0.12. The examples above can be modified to be examples in case of unequal characteristics. In the first example, R is such that (i) characteristic is p^2, (ii) $R/PR = K$. In the latter example, let y be $p^{1/2p}$.

At the end of this Part I, we raise the following question in view of our construction of these counter-examples.

Question 0.13. Let R be a noetherian ring and let S be a subring such that R is integral over S. Assume that for every prime ideal P of E, the ring R/P is an almost finite integral extension of $S/(P \cap S)$ and that there is a non-zero-divisor $d \in S$ such that $dR \subseteq S$. Is S noetherian ?

We note that the following fact can be proved easily.

Remark 0.14. Question 0.13 is affirmative if R is either an artinian ring or an integral domain of Krull dimension 1, even if we do not assume the existence of d. Without assuming the existence of d, one can have a counter-example in case R is a normal local domain of Krull dimension 2. (In [L], there is an example of a local domain, say B, of Krull 327 dimension 2 such that there is a non-noetherian ring S between B and its derived normal ring R. These S and R give a counter-example.)

Part II.

1 Let G be a group acting on a function field K over an algebraically closed ground field k.* We say that the action is rational if there is a pair of an algebraic group G^* and a model V of K, both defined over k, such that (i) G is a subgroup of G^* and (ii) the action of G is induced by a rational action of G^* on V. Thus we are practically thinking of rational actions of algebraic groups.

At first, we discuss the choice of V. Namely, we fix a group G, which may be assumed to be algebraic, and a function field K over an algebraically closed field k such that G is acting rationally on K. Then there may be many models V of K which satisfy the requirement in the above definition.

Proposition 1.1. *When a V satisfies the requirement, then so does the derived normal model of V.*

The proof is easy.

Proposition 1.2. *If a quasi-affine variety V satisfies the requirement, then there is an affine model V' of K which satisfies the requirement.*

Proof. Let R be the ring of elements of K which are everywhere regular on V. Then the rationality of the action of G on V implies that $\sum_{\sigma \in G} (\sigma f)k$ is a finite k-module for every $f \in R([F])$. Let $f_1, \ldots, f_n$ be elements of R such that $K = k(f_1, \ldots, f_n)$ and let $g_1, \ldots, g_s$ be a linearly independent base of $\sum_{i=1}^{n} \sum_{\sigma \in G} (\sigma f_i)k$. Then the affine model V' defined by $k[g_1, \ldots, g_s]$ is the desired variety. □

*Though many of our discussions can be adapted to the case where k is a ground ring, we assume that k is an algebraically closed field for the sake of simplicity.

Proposition 1.3. *If an affine variety V satisfies the requirement, then there is a projective model V' of K which satisfies the requirement.* 328

Proof. As is seen by the proof above, we may assume that the affine ring R of V is such that $R = k[g_1, \ldots, g_s]$, where $\sum_{i=1}^{s} g_i k$ is a representation module of G. Then the projective variety V' with generic point $(1, g_1, \ldots, g_s)$ is the desired variety. □

Remark 1.4. In the case above, the action of G is practically that of a linear group.

It was proved by Kambayashi ([K]) that

Proposition 1.5. *If G is a linear group and V is a complete variety, then there is a projective model V' of K which satisfies the requirement (and such that every element of G defines a linear transformation on V').*

These results suggest to us the following question.

Question 1.6. Does the rationality of the action of G imply the existence of a projective model of K which satisfies the requirement? How good can the singularity of such a model be?

In connection with this question, we raise

Question 1.7. Let G be a connected linear group acting rationally on a normal abstract variety V. Let L be a linear system on V. Does it follow that there is a linear system L^* on V which contains all $\sigma L (\sigma \in G)$?

If this question has an affirmative answer, then at least for linear groups, Question 1.6 has an affirmative answer. Note that Question 1.7 is affirmative if V is complete ([K]).

On the other hand, if there is a quasi-affine variety V which satisfies the requirement, then for every model V' of K satisfying the requirement, it is true that the orbit of a generic point of V' is quasi-affine. Thus, even if G is a connected linear algebraic group, if, for instance, the isotropy group (=stabilizer) of a generic point contains a Borel subgroup of G, then there cannot be any quasi-affine model of K satisfying the requirement (unless the action of G is trivial). Therefore we raise 329

Question 1.8. Assume that the orbit of a generic point of a V is quasi-affine. Does this imply that there is an affine model of K which satisfies the requirement ?

2 We observe the subgroup generated by two algebraic groups acting on a function field. More precisely, let G and H be subgroups of the automorphism group $\mathrm{Aut}_k K$ of the function field K over the group field k. We shall show by an example that

Proposition 2.1. *Even if G and H are linear algebraic groups, which are isomorphic to the additive group G_a of k and acting rationally on K, the subgroup $G \vee H$ generated by G and H (in $\mathrm{Aut}_k K$) may not have any rational action on K.*

Example. Let a and b be non-zero elements of k and let K_0, x, y be such that $ax^2 + by^2 = 1$, $K_0 = k(x, y)$ and trans. $\deg_k K_0 = 1$. We assume here that k is not of characteristic 2. Let (z, w) be a copy of (x, y) over k and let $K = k(x, y, z, w) =$ (quotient field of $k(x, y) \underset{k}{\otimes} k(z, w)$). Set $t = (y - w)/(x - z)$. Then $K = k(x, y, t) = k(z, w, t)$. We note the relation :

$$\begin{pmatrix} x \\ y \end{pmatrix} = F_t \begin{pmatrix} z \\ w \end{pmatrix}$$

with

$$F_t = \frac{1}{bt^2 + a} \begin{pmatrix} bt^2 - a & -2bt \\ -2at & a - bt^2 \end{pmatrix}.$$

$\mathrm{Aut}_k K$ contains the following subgroups G and H :

$$G = \{\sigma_c \in \mathrm{Aut}_{k(x,y)} K | c \in k, \sigma_c t = t + c\} \cong G_a,$$

$$H = \{\tau_c \in \mathrm{Aut}_{k(z,w)} K | c \in K, \tau_c t = t + c\} \cong G_a.$$

$\mathrm{Aut}_k K$ has an element ρ such that $\rho^2 = 1$, $\rho x = z$, $y = w$. Then $H = \rho^{-1} G_\rho$. G acts rationally on the affine model of K defined by $k[x, y, t]$ and H acts rationally on the affine model of K defined by $k[z, w, t]$. Thus G and H act rationally on K. For $c_i \in k$, we observe the element
330 ment $\tau_{c_1}\rho\tau_{c_2}\rho \ldots \rho\tau_c\rho_s$; let us denote this element by $[c_1, \ldots, c_s]$. Then

$[c]\binom{z}{w} = \tau_c\binom{x}{y} = F_{t+c}\binom{z}{w}$ Note that if $\alpha\binom{z}{w} = F_t^*\binom{z}{w}$ for $\alpha \in \operatorname{Aut} K$ and with $F_t^* \in GL(2, k(t))$, then

$$([c]\alpha)\begin{pmatrix} z \\ w \end{pmatrix} = F_{t+c}^* F_t \begin{pmatrix} z \\ w \end{pmatrix}$$

Thus, to each $[c_1, \ldots, c_s]$ there corresponds a matrix in $GL(2, k(t))$. In view of this correspondence, one can see easily that the dimension of the algebraic thick set $(\rho G)^n = \rho G \rho G \ldots \rho G$ tends to infinity with n.

Remark 2.2. Similar example is given so that G and H are isomorphic to the multiplicative group of k, by changing $\sigma_c t = t + c$ and $\tau_c t = t + c$ to $\sigma_c t = ct$ and $\tau_c t = ct$ respectively.

Now we raise

Question 2.3. Give good conditions for connected algebraic subgroups G and H of $\operatorname{Aut}_k K$ so that $G \vee H$ is algebraic.

3

Let G be an algebraic group acting on a variety V. Then there may be fixed points of G on V. In particular, if G is linear and if V is complete, then there is at least one fixed point ([B]). The following fact was noticed by Dr. John Forgarty.

Proposition 3.1. *If G is a connected unipotent linear group and if V is a projective variety, then the set F of fixed points on V is connected. More generally, if W is a connected closed set in a projective variety and if G is a connected unipotent linear group which acts rationally on W, then the set F of fixed points of G on W is connected.*

Proof. We shall prove the last statement by induction on $\dim G$. Then we may assume that $\dim G = 1$, i.e. G is isomorphic to the additive group of k. Thus, in the following until we finish the proof of the proposition, we assume that G is the additive group of k and that varieties and curves are projective ones. □

Lemma 3.2. *Under the circumstances, let C be an irreducible curve on which G acts rationally. If there are two fixed (mutually different) points on C, then every point of C is a fixed point.*

331 *Proof.* G can be imbedded in a projective line L biregularly. Then L consists of G and a point. If $P \in C$ is not fixed, then $C - GP$ is a point, which is not the case. □

Corollary 3.3. *Under the circumstances, let C be a connected reducible curve on which G acts rationally. Let C' be an irreducible component of C. If either there are two points (mutually different) on C' which are on some other components of C or there is a fixed point P on C' which is not on any other irreducible component of C, then every point of C' is fixed.*

The proof is easy because (i) since G is connected, every component of C is G-stable and therefore (ii) every point which is common to some mutually different irreducible components of C is a fixed point.

Corollary 3.4. *Under the circumstances, let C be a connected curve on which G acts rationally. Then the set F_0 of fixed points on C is connected.*

Proof. If C is irreducible, then either F_0 consists of a point or $F_0 = C$, and the assertion holds good in this case. We assume now that C is reducible. If $P \in C$ is not fixed, then let C' be the irreducible component of C which carries P. C' carries only one fixed point, say Q. Then every component of C, which has a common point with C', goes through Q. Therefore $C - GP$ is a connected curve, whose set of fixed points is F_0. Thus we finish the proof by induction on the number of irreducible components of C.

Now we go back to the proof of Proposition 3.1. Let $W_i (i = 1, \dots, n)$ be the irreducible components of W. Since G is connected solvable, $W_i \cap W_j$ contains a fixed point P_{ij}, unless $W_i \cap W_j$ is empty. If one knows that every $F \cap W_i$ is connected, then the existence of P_{ij} shows the connectedness of F. Thus we may assume that W is irreducible. Let P^* be generic point of W and let P be a point of F. If P^* is fixed, then every point of W is fixed, and our assertion is obvious in this case. Therefore we assume that P^* is not a fixed point. Let $\overline{C}$ be the closure of GP^*. Then $\overline{C} - GP^*$ consists of a point, say Q^*. Consider a specializa-
332 tion of $(\overline{C}, Q^*)$ with reference to $P^* \to P$: let $(\overline{C}, Q^*, P^*) \to (C, Q, P)$

be such a specialization. The locus D of Q^* (i.e. the subvariety of W having Q^* as its generic point) consists only of fixed points. Q lies on $D \cap C$. By the connectedness theorem, C is connected, whence $F \cap C$ is connected by Corollary 3.4. Thus F contains a connected subset containing P and Q^*. Since this is true for every $P \in F$, we complete the proof. □

On the other hand, it is obvious that

Proposition 3.5. *If G is a connected linear algebraic group whose radical is unipotent, acting on a projective space rationally as a group of linear transformations, then the set of fixed points forms a linear subvariety.*

Now our question is

Question 3.6. Find a good theorem including Proposition 3.1 and 3.5.

In connection with this question, we give an example.

Example 3.7. There is a pair of a semi-simple linear algebraic group G and a connected closed set V in a projective space $\mathbf{P}$ such that (i) G acts rationally on $\mathbf{P}$ as a group of linear transformations, (ii) $GV = V$, i.e. V is G-stable and (iii) the set F of fixed points on V is not connected.

The construction of the example. Let n be an arbitrary natural number and let $G = GL(n+1, k)$. Each $\sigma \in G$ defines a linear transformation on the projective space $\mathbf{P}$ of dimension $n+2$ defined by the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sigma \end{bmatrix}.$$

A point $(a_0, \ldots, a_{n+2})$ is a fixed point if and only if $a_2 = \ldots = a_{n+2} = 0$. Let V be the algebraic set defined by $X_0 X_1 = 0$. V is a connected and $GV = V$. But V has only two fixed points $(1, 0, \ldots, 0)$ and $(0, 1, 0, \ldots, 0)$.

4 We assume here that G is a connected linear group acting rationally on a projective variety V. Let P^* be a generic point of V and let D^* be **333**
the closure of GP^*. Then we can think of the Chow point Q^* of D^*. We raise a question.

Question 4.1. Is the function field K of V purely transcendental over $k(Q^*)$? In other words, is D^* rational (in the strong sense over $k(Q^*)$?

Since G is linear, it is obvious that K is uni-rational over $k(Q^*)$.

Proposition 4.2. *If G is the additive group of k, then the answer is affirmative.*

Proof. The assertion is obvious if P^* is a fixed point. In the other case, D^* has a unique fixed point, which must be rational over $k(Q^*)$. Therefore D^* must be rational over $k(Q^*)$. □

5 In this last section, we add some questions related to the Mumford Conjecture. As was proved by Dr. Seshadri, the Mumford Conjecture on the rational representation of linear algebraic groups is true for $SL(2,k)$. Let us call a linear algebraic group "semi-reductive" if the statement of the Mumford Conjecture holds good for the group. Then the following three propositions are well known.

Proposition 5.1. *If a linear algebraic group G is semi-reductive, then* (i) *so is every normal subgroup of G and every homomorphic image of G (by a rational homomorphism) and* (ii) *the radical of G is a torus group. Conversely, when N is a normal subgroup of a linear algebraic group G, if both N and G/N are semi-reductive, then G is semi-reductive.*

Proposition 5.2. *Finite groups and torus groups are semi-reductive.*

Proposition 5.3. *If k is of characteristic zero, then a linear algebraic group G is semi-reductive if and only if its radical is a torus group.*

The Mumford Conjecture itself is a hard question. The writer feels that if the following two questions have affirmative answers, then it may help our observation on the conjecture.

Question 5.4. Let G be a connected, semi-simple semi-reductive linear algebraic group. Then its connected algebraic subgroup H is semi-reductive if the following conditions are satisfied : **334**

(i) H is semi-simple and (ii) G/H is affine.

Question 5.5. Let G be a connected semi-simple algebraic linear group such that every proper closed normal subgroup is finite. Then there is a pair of a natural number n and a linear algebraic group G^* such that (i) G and G^* have finite normal subgroups N and N^* such that $G/N = G^*/N^*$ and (ii) G^* is a subgroup of $GL(n,k)$ and (iii) $GL(n,k)/G^*$ is affine.

Note that (1) if the Mumford Conjecture has an affirmative answer, then these two questions have affirmative answers and (2) if these questions have affirmative answers, then we have only to prove the Mumford Conjecture for $SL(n,k)$ for each natural number n.

Added in Proof: Question 0.1 has been answered negatively by K. R. Nagarajan, Groups acting on noetherian rings, *Nieuw Archief voor Wiskunde* (3) XIV (1968), 25-29. (Though his proof contains an error, the example is good.)

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VECTOR BUNDLES ON CURVES

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335 **1 Introduction.** We shall review 'in this paper some aspects of the theory of vector bundles on algebraic curves with particular reference to the explicit determination of the moduli varieties of vector bundles of rank 2 on a curve of genus 2 (see [3]). Later we prove, using these results, the non-existence of (algebraic) Poincaré families parametrised by non-empty Zariski open subsets of the moduli space of vector bundles of rank 2 and degree 0 on a curve of genus 2 [Theorem, §3]. This result is of interest in view of the following facts :

(i) there do exist such families when the rank and degree are coprime;

(ii) in general (i.e. even if the degree and rank are not coprime) every stable point has a neighbourhood in the usual topology parametrising a holomorphic Poincaré family of vector bundles;

(iii) there exists always a Poincaré family of *projective* bundles parametrised by the open set of stable bundles.

The essential point in the proof of the non-existence of Poincaré families is to show that a certain projective bundle, which arises geometrically in the theory of quadratic complexes, does not come from a vector bundle. The reduction to the geometric problem is found in §7. The geometric problem, which is independent of the theory of vector bundles, is explained in §5 and the solution is found in §8.

The idea of reducing this question to the geometric problem arose in our discussions with Professor D. Mumford, to whom our warmest thanks are due.

*Presented by M. S. Narasimhan.

2 The moduli variety $U(n, d)$. Let X be a compact Riemann surface or equivalently a complete non-singular irreducible algebraic curve defined over $\mathbf{C}$. We shall assume that the genus g of X is $\geqslant 2$. 336
If $W(\neq 0)$ is a vector bundle (algebraic) on X we define $\mu(W)$ to be the rational number degree $W/\operatorname{rank} W$. A vector bundle W will be called *stable* (resp. *semi-stable*) if for every proper sub-bundle V of W we have $\mu(V) < \mu(W)$ (resp. $\mu(V) \leqslant \mu(W)$). D. Mumford proved that the isomorphism classes of stable bundles of rank n and degree d on X form a non-singular quasi-projective algebraic variety (of dimension $n^2(g-1)+1$).

A characterisation of stable bundles in terms of irreducible unitary representations of certain discrete groups was given by M. S. Narasimhan and C. S. Seshadri [4]. This result implies that the space of stable bundles of rank n and degree d is compact if $(n, d) = 1$ and that a vector bundle of degree 0 is stable if and only if it arises from an irreducible unitary representation of the fundamental group of X. Moreover two such stable bundles are isomorphic if and only if the corresponding unitary representations are equivalent. These results suggest a natural compactification of the space of stable bundles, namely the space of bundles given by all unitary representations (not necessarily irreducible) of a given type.

C. S. Seshari in [7] proved that this natural compactification is a projective variety. More precisely, Seshadri proved the following. Let W be a semi-stable vector bundle on X. Then W has a strictly decreasing filtration

$$W = W_0 \supset W_1 \supset \cdots \supset W_n = (0)$$

such that, for $1 \leqslant i \leqslant n$, W_i/W_{i-1} is a stable vector bundle with $\mu(W_{i-1}/W_i) = \mu(W)$. Moreover the bundle $\operatorname{Gr} W = \bigoplus_{i=1}^{n} W_{i-1}/W_i$ is determined by W upto isomorphism. We say that two semi-stable bundles W_1 and W_2 are *S-equivalent* if $\operatorname{Gr} W_1 \approx \operatorname{Gr} W_2$. Obviously two stable bundles are S-equivalent if and only if they are isomorphic. It is proved in [7] that there is a unique structure of a normal projective variety $U(n, d)$ on the set of S-equivalence classes of semi-stable vector

bundles of rank n and degree d on X such that the following property holds: if $\{W_t\}_{t\in T}$ is an algebraic (resp. holomorphic) family of semi-
337 stable vector bundles of rank n and degree d parametrised by an algebraic (resp. a complex) space T, then the mapping $T \to U(n,d)$ sending t to the S-equivalence class of W_t is a morphism.

Regarding the singularities of the varieties $U(n,d)$ we have the following result [3].

Theorem 2.1. *The set of non-singular points of $U(n,d)$ is precisely the set of stable points in $U(n,d)$ except when $g = 2$, $n = 2$ and d even.*

It is easy to see that the above characterisation breaks down in the exceptional case. It will follow from the results quoted in §4, that when $g = 2$, d even, the variety $U(2,d)$ is actually non-singular.

Now let L be a line bundle of degree d. Let $U_L(n,d)$ be the subspace of $U(n,d)$ corresponding to vector bundles with the determinantal bundle isomorphic to L. It is easy to see [4, §3] that all stable vector bundles V in $U_L(n,d)$ can be obtained as extensions

$$0 \to E \to V \to (\det E)^{-1} \otimes L \to 0,$$

where E is a suitably chosen vector bundle, depending only on $U_L(n,d)$. Let U be the Zariski open subset of $H^1(X, \mathrm{Hom}(L,E) \otimes \det E)$ corresponding to stable bundles. Then the natural morphism $U \to U_L(n,d)$ given by the universal property has as image the set of stable points of $U_L(n,d)$. This shows that the varieties $U_L(n,d)$ are unirational.

By a refinement of the above, it has been shown that the variety $U_L(n,d)$ is even rational if $d \equiv \pm 1 (\mathrm{mod}\, n)$. The rationality of these varieties in general is not known.

3 Poincaré families.

The next problem in the theory of vector bundles is the construction of universal (Poincaré) families of bundles on X parametrised by $U(n,d)$. The existence of such a universal bundle is well-known in the case $n = 1$.

Definition. *Let Ω be a non-empty Zariski open subset of $U(n,d)$ or $U_L(n,d)$. A Poincaré family of vector bundles on X parametrised by Ω*

338 *is an algebraic vector bundle P on $\Omega \times X$ such that for any $\omega \in \Omega$ the bundle on X obtained by restricting P to $\omega \times X$ is in the S-equivalence class ω. The bundle P will be called a Poincaré bundle.*

The following theorem has been proved independently by D. Mumford, S. Ramanan and C. S. Seshadri.

Theorem. *If n and d are coprime, there is a Poincaré bundle on $U(n,d) \times X$.*

However we prove, in contrast, the

Main Theorem. *Let X be a compact Riemann surface of genus 2. Then there exists no algebraic Poincaré family parametrised by any non-empty Zariski open subset of $U(2,0)$.*

The theorem will be proved in §8. In the next sections we recall some results on vector bundles on a curve of genus 2 which will be used in the proof.

4 Vector bundles of rank 2 and degree 0 on a curve of genus 2.

Theorem 4.1. *Let X be of genus 2 and S be the space of S-equivalence classes of semi-stable bundles of rank 2 with trivial determinant on X. Let J^1 be the variety of equivalence classes of line bundles of degree 1 on X and Θ the divisor on J^1 defined by the natural imbedding of X in J^1. Then S is canonically isomorphic to the projective space $\mathbf{P}$ of positive divisors on J^1 linearly equivalent to 2Θ.*

For the proof see [3], §6.

Remarks. (i) The space S is identified with the set of isomorphism classes of bundles of rank 2 and trivial determinant which are either stable or are of the form $j \oplus j^{-1}$, where j is a line bundle of degree 0. The space of non-stable bundles in S, which is isomorphic to the quotient of the Jacobian J of X by the canonical involution of J, gets imbedded in $\mathbf{P}$ as a Kummer surface.

(ii) This theorem shows in particular that S is non-singular. It follows easily from this that $U(2,0)$ is non-singular if $g = 2$. In fact, $U(2,0)$ is isomorphic to the variety of positive divisors algebraically equivalent to 2Θ, which is a projective bundle over the Jacobian.

339 (iii) This theorem suggests a close connection between $U(2,0)$ and the variety of positive divisors on the Jacobian algebraically equivalent to 2Θ, when g is arbitrary. This relationship has been studied when $g = 3$ and will be published elsewhere.

5 Quadratic complexes and related projective bundles.

Before stating the next theorem it is convenient to recall certain notations connected with a quadratic complex of lines in a three dimensional projective space. For more details see [3].

Let R be a four dimensional vector space over $\mathbf{C}$. Then the Grassmannian of lines G in the projective space $P(R)$ is naturally embedded as a quadric in $P(\overset{2}{\wedge} R)$. Consider the tautological exact sequence

$$0 \to L^{-1} \to R \to F \to 0$$

of vector bundles on $P(R)$ where L is the hyperplane bundle on $P(R)$. This leads to an exact sequence

$$0 \to F \otimes L^{-1} \to \overset{2}{\wedge} R \to \overset{2}{\wedge} F \to 0.$$

This induces an injection $P(F \otimes L^{-1}) \to P(\overset{2}{\wedge} R) \times P(R)$; the image is contained in $G \times P(R)$ and is the incidence correspondence between lines and points in $P(R)$. Consider the diagram

$$\begin{tikzcd} & P(F \otimes L^{-1}) \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ G & & P(R) \end{tikzcd}$$

The map p_1 is a fibration with projective lines as fibres, associated to the universal vector bundle on G. For $\omega \in P(R)$, $p_2^{-1}(x)$ is mapped isomorphically by p_1 onto a plane contained in G. A quadratic complex of lines is simply an element of $PH^0(G, H^2)$, where H is the restriction to G of the hyperplane line bundle on $P(\overset{2}{\wedge} R)$.

A *generic* quadratic complex in $P(R)$ is a subvariety Q of G defined by equations of the form **340**

$$\begin{cases} \sum_{i=1}^{6} x_i^2 = 0, \\ \sum_{i=1}^{6} \lambda_i x_i^2 = 0, \ \lambda_i \text{ distinct}, \ \lambda_i \in \mathbf{C}, \end{cases}$$

with respect to a suitable coordinate system in $P(\overset{2}{\wedge} R)$, where $\sum_{i=1}^{6} x_i^2 = 0$ defines the Grassmannian. Let $Y = p_1^{-1}(Q)$. We then have a diagram

$$\begin{tikzcd} & Y \arrow[dl, "q_1"'] \arrow[dr, "q_2"] & \\ Q & & P(R) \end{tikzcd}$$

where q_1 and q_2 are surjective. For $\omega \in P(R)$, $q_2^{-1}(\omega$ is imbedded in the plane $p_2^{-1}(\omega)$ as a conic. A point $\omega \in P(R)$ where $q_2^{-1}(\omega)$ is a singular conic (i.e. a pair of lines) is called a singular point of the quadratic complex Q. The locus $\mathscr{K}$ of singular points in $P(R)$ is a quartic surface with 16 nodes viz. a Kummer surface. Thus if Ω is a Zariski open subset of $P(R) - \mathscr{K}$, the restriction of q_2 to $q_2^{-1}(\Omega)$ is a projective bundle over Ω. The geometric problem referred to in the introduction is whether this projective bundle is associated to an algebraic vector bundle. We shall show in §8 that this is not the case. In view of the results of §7, this will prove the main theorem.

6 Vector bundles of rank 2 and degree 1 on a curve of genus 2. It has been shown by P. E. Newstead [6] that the space of stable bundles of rank 2 with determinant isomorphic to a fixed line bundle of degree -1 on a curve of genus 2, is isomorphic to the intersection of two quadrics in a 5-dimensional projective space. The following theorem, which is proved in [3], is a canonical version of this result which
341 brings out at the same time the relationship between vector bundles (of rank 2) of degree 0 and -1. This relationship is of importance in the proof of non-existence of Poincaré families.

Theorem 6.1. (i) *Let X be of genus 2 and x a non-Weierstrass point of X (i.e. a point not fixed by the canonical rational involution on X). Let $S_{1,x}$ denote the variety of isomorphism classes of stable bundles of rank 2 and determinant isomorphic to L_x^{-1}, where L_x is the line bundle determined by x. Let $\mathbf{P}$ be the projective space defined in Theorem 4.1 and G the Grassmannian of lines in $\mathbf{P}$. Then $S_{1,x}$ is canonically isomorphic to the intersection Q of G and another quadric in the ambient 5-dimensional projective space.*

(ii) *The quadratic complex Q is generic and the singular locus of Q is the Kummer surface $\mathscr{K}$ in $\mathbf{P}$ corresponding to non-stable bundles in S.*

(iii) *With the identifications of S with $\mathbf{P}$ and $S_{1,x}$ with Q, the projective bundle on $S-\mathscr{K}$ defined by the quadratic complex Q (see §5) is just the subvariety of $S-\mathscr{K}\times S_{1,x}$ consisting of pairs (w,v) with $H^0(X,\mathrm{Hom}(V,W))\neq 0$, where V (resp. W) is in the class v (resp. w).*

(i) and (ii) have been explicitly proved in [3], Theorem 4, §9. It has been proved there that if $v\in S_{1,x}$ and Λ_v the line in $\mathbf{P}$ defined by v, then a point $w\in\mathbf{P}$ belongs to Λ_v if and only if $H^0(X,\mathrm{Hom}(V,W))\neq 0$ where V(resp. W) is a bundle in the class v(resp. w), (see §9 of [3]). (iii) is only a restatement of the above.

Remark. One can show that the space of lines on the intersection Q of the two quadrics is isomorphic to the Jacobian of X[3, 6]. This result is to be compared with the following theorem of D. Mumford and P. E. Newstead [2]. Let X be of genus $g \geqslant 2$, and $U'(2,1)$ be the subspace of $U(2,1)$ consisting of bundles with a fixed determinant. Then the intermediary Jacobian of $U'(2,1)$, corresponding to the third cohomology group of $U'(2,1)$, is isomorphic to the Jacobian of X. The Betti numbers of $U'(2,1)$ are determined in [5].

7 Reduction of the Main Theorem to a geometric problem.

Lemma 7.1. *Let W be a stable bundle of rank 2 and trivial determinant. Let $x \in X$. Let $\mathbf{O}_x = \mathbf{O}_X/\mathfrak{m}_x$ be the structure sheaf of x.*

(i) *If V is a stable bundle of rank 2 and determinant L_x^{-1} and $f : V \to$* **342**
W a non-zero homomorphism, then we have an exact sequence

$$0 \to \mathbf{V} \xrightarrow{\mathbf{f}} \mathbf{W} \to \mathbf{O}_x \to 0.$$

Moreover $\dim H^0(X, \mathrm{Hom}(V, W)) \leqslant 1$.

(ii) *If $\mathbf{W} \to \mathbf{O}_x$ is a non-zero homomorphism, then the kernel is a locally free sheaf of rank 2, whose associated vector bundle is a stable bundle with determinant L_x^{-1}.*

Proof. (i) It is clear that f must be of maximal rank; for, otherwise the line sub-bundle of W generated by the image of f would have degree $\geqslant 0$, since V is stable. Now the induced map $\overset{2}{\wedge} f : \overset{2}{\wedge} V \to \overset{2}{\wedge} W$ is non-zero and hence can vanish only at x (with multiplicity 1). Hence f is of maximal rank at all points except x and f is of rank 1 at x. This proves the first part of (i). Now suppose f and g are two linearly independent homomorphisms from V to W; choose $y \in X$, $y \neq x$, and let f_y, g_y be the homomorphisms $V_y \to W_y$ induced by f and g on the fibres of V and W at y. Then there exist λ, $\mu \in \mathbf{C}$, $(\lambda, \mu) \neq (0,0)$ such that $\lambda f_y + \mu g_y$ is not an isomorphism. Then $\lambda f + \mu g$ would be a non-zero homomorphism $V \to W$ which is not of maximal rank at y. This is impossible by earlier remarks.

(ii) Let V be the vector bundle determined by the kernel. It is clear that $\det V = L_x^{-1}$. To show that V is stable we have only to show that V contains no line subbundle of degree $\geqslant 0$. If L were a line subbundle of V of degree $\geqslant 0$, there would be a non-zero homomorphism $L \to W$, which is impossible since W is stable of degree 0.

Let $p : P \to \Omega \times X$ be a Poincaré bundle on $\Omega \times X$, where Ω is an open subset of S (see Theorem 4.1) consisting of stable points. Let $x \in X$ and let $\mathbf{O}_x = \mathbf{O}_X/\mathfrak{m}_x$ be the structure sheaf of the point x. Then the sheaf $\mathscr{H}\mathrm{om}(\mathbf{P}, p^*{}_X\mathbf{O}_x)$ on $\Omega \times X$ is p_Ω flat. Moreover, for each $\omega \in \Omega$

$$\begin{aligned}
&\dim H^0(\omega \times X, \mathscr{H}\mathrm{om}(\mathbf{P}, p^*{}_X\mathbf{O}_x)|_{\omega\times X})\\
&= \dim H^0(\omega \times X, \mathscr{H}\mathrm{om}(\mathbf{P}|_{\omega\times X}, \mathbf{O}_x))\\
&= \dim P^*_{(\omega,x)}\\
&= 2.
\end{aligned}$$

343 Hence by [1], the direct image $(p_\Omega)_* \operatorname{Hom}(\mathbf{P}, p^*\mathbf{O}_x)$ is a locally free sheaf on Ω and consequently defines a vector bundle E on Ω. $\square$

Proposition 7.1. *There is a morphism*

$$P(E) \to \Omega \times S_{1,x}$$

such that the diagram

$$\begin{tikzcd}
P(E) \arrow[rr] \arrow[dr] & & \Omega \times S_{1,x} \arrow[dl] \\
& \Omega &
\end{tikzcd}$$

is commutative. Moreover this morphism is an isomorphism onto the subvariety of pairs (W, V) such that $H^0(X, \operatorname{Hom}(V, W)) \neq 0$, $V \in S_{1,x}$, $W \in \Omega$.

Proof. Consider on $\Omega \times X$ the sheaf $\mathscr{G} = \mathscr{H}om(\mathbf{P}, p_X^*\mathbf{O}_x)$. Then we have clearly the canonical isomorphisms

$$p_*(\mathbf{T} \otimes p^*\mathscr{G}) \approx p_*(\mathbf{T}) \otimes \mathscr{G} \approx p_\Omega^*(\mathbf{E})^* \times \mathscr{G},$$

where $p : P(E) \times X \to \Omega \times X$ is the natural projection and T is the tautological hyperplane bundle in $P(E) \times X$. Moreover, the direct image of $p_\Omega^*(\mathbf{E}^*) \times \mathscr{G}$ on Ω is isomorphic to $\mathbf{E}^* \times p_{\Omega_*}(\mathscr{G}) \approx \mathbf{E}^* \otimes \mathbf{E}$. Hence $H^0(P(E) \times X, (\mathbf{T} \otimes p^*\mathscr{G})) \approx H^0(\Omega, E^* \otimes \mathbf{E})$. Hence the canonical element of $H^0(\Omega, \mathbf{E}^* \otimes \mathbf{E})$ (viz. the identity endomorphism of E) gives rise to an element of $H^0(P(E) \times X, \mathbf{T} \otimes p^*\mathscr{G})$. In other words, we have a canonical homomorphism $p^*\mathbf{P} \to p_X^*(\mathbf{O}_x) \otimes \mathbf{T}$ of sheaves on $P(E) \times X$. Consider the commutative diagram

$$\begin{tikzcd} P(E) \times X \arrow[r] \arrow[d, "p"] & P(E) \arrow[d, "p"] \\ \Omega \times X \arrow[r] & \Omega \end{tikzcd}$$

The direct image of $\mathbf{T} \otimes p^*(\mathscr{G})$ on $P(E)$ is simply $\mathbf{T} \otimes p^*(\mathbf{E})$, where T **344**
also denotes the tautological bundle on $P(E)$, and the canonical element in $H^0(P(E) \times X, \mathbf{T} \otimes p^*(\mathscr{G}))$ defined above is given by the tautological element of $H^0(P(E), \mathbf{T} \otimes p^*(\mathbf{E}))$. From this we see that for $f \in P(E)$, the restriction of the homomorphism $p^*(\mathbf{P}) \to p_X^*(\mathbf{O}_x) \otimes T$ to $f \times X$ can be described as follows. The restriction of $p^*(P)$ to $f \times X$ is the restriction of P to $p(f) \times X$ and hence is a stable vector bundle W with trivial determinant. Moreover f gives rise to a 1-dimensional subspace of $H^0(X, \mathscr{H}om(\mathbf{W}, \mathbf{O}_x))$. Any non-zero element in this 1-dimensional space gives rise to a surjective homomorphism of $p^*\mathbf{P}|f \times X = \mathbf{W}$ into $p_X^*\mathbf{O}_x \times \mathbf{T}|_{f \times X} \approx \mathbf{O}_x$. This homomorphism (upto a non-zero scalar) is the restriction of the canonical element. In particular it follows that the canonical homomorphism $p^*(\mathbf{P}) \to p_X^*(\mathbf{O}_x) \otimes \mathbf{T}$ is surjective. Moreover since $p_X^*(\mathbf{O}_x) \otimes \mathbf{T}$ has a locally free resolution of length 1 we see that the kernel of the homomorphism $p^*(\mathbf{P}) \to p^*(\mathbf{O}_x) \otimes \mathbf{T}$ is locally free. Let F be the vector bundle on $P(E) \times X$ associated to the kernel. $\square$

Lemma 7.2. *The restriction of the vector bundle F to $f \times X$, $f \in P(E)$ is a stable vector bundle of rank 2 and determinant L_x^{-1}.*

In view of our earlier identification the lemma follows from Lemma 7.1.

We now complete the proof of the proposition. By Lemma 7.2 and the universal property of $S_{1,x}$ we have a morphism $q : P(E) \to S_{1,x}$. Then the morphism $(p, q) : P(E) \to \Omega \times S_{1,x}$ satisfies the conditions of the proposition, in view of Lemma 7.1. The morphism is an isomorphism onto the subvariety described in Proposition 7.1, as this subvariety is non-singular by Theorem 6.1.

From Proposition 7.1 and Theorem 6.1 we have immediately the

Corollary. *If there is a Poincaré family on an open subset Ω of the set of stable points in S, then the projective bundle on Ω defined by the quadratic complex $Q = S_{1,x}$ is associated to a vector bundle.*

8 Proof of the Main Theorem. Solution of the geometric problem.

It is easy to see that if there is a Poincaré family
345 parametrised by a Zariski open subset of $U(2, 0)$, there would exist a Poincaré family parametrised by a Zariski open subset of the space of stable points in S. In view of the Corollary of Proposition 7.1, the main theorem in §3 follows from

Proposition 8.1. *With the notation of §5, let Ω be a Zariski open subset of $P(R) - \mathscr{K}$. Let $q_2 : q_2^{-1}(\Omega) \to \Omega$ be the projective bundle defined in §5. Then there is no algebraic vector bundle on Ω to which this projective bundle is associated.*

Proof. If there is such a vector bundle there would exist a Zariski open set Ω' of Ω and a section σ over Ω' of the projective bundle $q_2^{-1}(\Omega) \to \Omega$. Let D be the Zariski closure of $\sigma(\Omega')$ in Y. Then D is a divisor of Y and, since Y is non-singular, D defines a line bundle L_D on Y. The restriction of the first Chern class of L_D to a fibre $q_2^{-1}(\omega)$, $\omega \in \Omega'$, is the fundamental class of the fibre. On the other hand, we shall show that every element of $H^2(Y, \mathbf{Z})$ restricts to an even multiple of the fundamental class of $q_2^{-1}(\omega)$ in $H^2(q_2^{-1}(\omega), \mathbf{Z})$; this contradiction would prove the proposition. We have the commutative diagram

$$\begin{array}{ccc}
H^2(P(F \otimes L^{-1}), \mathbf{Z}) & \longrightarrow & H^2(p_2^{-1}(\omega), \mathbf{Z}) \approx H^2(\mathbf{P}^2, \mathbf{Z}) \\
\downarrow & & \downarrow \\
H^2(Y, \mathbf{Z}) & \longrightarrow & H^2(q_2^{-1}(\omega), \mathbf{Z}),
\end{array}$$

with the notation of §5. We first note that the canonical mapping $H^2(G, \mathbf{Z}) \to H^2(Q, \mathbf{Z})$ is an isomorphism, by Lefschetz's theorem on hypersurface sections. Moreover since $p_1 : P(F \otimes L^{-1}) \to G$ (resp. $q : Y \to Q$) is the projective bundle associated to a vector bundle, $H^2(P(F \otimes L^{-1})\mathbf{Z})$ (resp. $H^2(Y, \mathbf{Z})$) is generated by the first Chern class of the tautological line bundle of the fibration $P(F \otimes L^{-1}) \to G$ (resp. $Y \to Q$) and by $p_2^*(H^2(G, \mathbf{Z}))$ (resp. $q_2^* H^2(Q, \mathbf{Z})$). Since this tautological line bundle on $P(F \otimes L^{-1})$ restricts to the tautological line bundle of the fibration $Y \to Q$ and $H^2(G, \mathbf{Z}) \to H^2(Q, \mathbf{Z})$ is an isomorphism, it follows that $H^2(P(F \times L^{-1}), \mathbf{Z}) \to H^2(Y, \mathbf{Z})$ is surjective. Now from the commutativity of the diagram we see that image **346**
$H^2(Y, \mathbf{Z}) \to H^2(q_2^{-1}(\omega), \mathbf{Z})$ is contained in the image

$$H^2(p_2^{-1}(\omega), \mathbf{Z}) \to H^2(q_2^{-1}(\omega), \mathbf{Z}).$$

But $q_2^{-1}(\omega)$ is imbedded in the plane $p_2^{-1}(\omega)$ as a conic and hence the image $H^2(Y, \mathbf{Z}) \to H^2(q_2^{-1}(\omega), \mathbf{Z})$ consists of even multiples of the fundamental class of $q_2^{-1}(\omega)$. □

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MUMFORD'S CONJECTURE FOR $GL(2)$ AND APPLICATIONS

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In [12], it was shown that on a smooth projective curve X of genus $\geqslant 2$ over $\mathbf{C}$, there is a natural structure of a normal projective variety on the isomorphic classes of unitary vector bundles of a fixed rank. This can also be given a purely algebraic formulation, namely that on the classes of semi-stable vector bundles of a fixed rank and degree zero, under a certain equivalence relation, there is a natural structure of a normal projective variety when X is defined over $\mathbf{C}$. In fact this was used in [12]. It is then natural to ask whether this algebraic result holds good in arbitrary characteristic. The main obstacles to extending the proof of [12] to arbitrary characteristic are as follows: **347**

(1) to carry over the results of Mumford (obtained in characteristic 0) on quotient spaces of the N-fold product of Grassmannians for the canonical diagonal action of the full linear group (c.f. §4, Chap. 4, [5]), to arbitrary characteristic, and

(2) to find a substitute for unitary representations which have been used in [12], mainly to show that the varieties in question are complete.

It is not hard to see how to set about (2). One has to show that a certain morphism is proper (see §3, Lemma 2). This is not difficult but requires some careful analysis and it is an improvement upon some of the arguments in [12]. The difficulty (1) appears to be more basic. If Mumford's conjecture generalizing complete reducibility to reductive groups in arbitrary characteristic (cf. §1, Def. 3) is solved for all special linear groups, (1) would follow. In this we have partial success, namely we solve Mumford's conjecture for $GL(2)$, which allows us to solve (1) for the case of a product of Grassmannians of two planes. Consequently the results of [12] carry over to the case of vector bundles of *rank* 2 in arbitrary characteristic.

348 The proof of Mumford's conjecture for $GL(2)$ is rather elementary and we give it in §1. As for applications to vector bundles, only the solution of (2) above is given in detail (§3, Lemma 2, (3)). The other points are only sketched and proofs for most of these can be found in [5] or [12].

The algebraic schemes that we consider are supposed to be defined over an algebraically closed field **K** and of finite type over **K**. The points of an algebraic scheme are the geometric points in **K** and the algebraic groups considered are *reduced* algebraic group schemes. By a rational representation of an algebraic group G in a finite dimensional vector space. V, we mean a homomorphism $\rho : G \to \operatorname{Aut} V$ of algebraic groups.

1 Mumford's conjecture for $GL(2)$.

Definition 1. *An algebraic group G is said to be reductive if it is affine and* $\operatorname{rad} G$ *(radical of G) is a torus, i.e. a product of multiplication groups.*

Definition 2. *An algebraic group G is said to be linearly reductive if it is affine and every rational representation of G in a finite dimensional vector space is completely reducible.*

It is a classical result of H. Weyl that if the characteristic of the base field is zero, every reductive group is linearly reductive. A torus group is easily seen to be linearly reductive in arbitrary characteristic. If the characteristic p of the base field is *not zero*, there are not many more linearly reductive groups other than the torus groups; in fact, there is the following result due to Nagata: an algebraic group G is linearly reductive if and only if the connected component G^0 of G through identity is a torus and the order of the finite group G/G^0 is prime to p (c.f. [6]).

It is proved easily that an affine algebraic group G is linearly reductive if and only if any one of the two following properties holds :

(1) for every rational representation of G in a finite dimensional vec-

tor space V and a one dimensional G-invariant linear subspace V_0 of V, there exists a G-invariant linear subspace V_1 of V such that 349
$V = V_0 \oplus V_1$;

(2) for every rational representation of G in a finite dimensional vector space V and a G-invariant point $v \in V$, $v \neq 0$, there exists a G-invariant linear form f on V such that $f(v) \neq 0$.

Definition 3. *An algebraic group G is said to be geometrically reductive if it is affine and for every rational representation of G in a finite dimensional vector space V and a G-invariant point $v \in V$, $v \neq 0$, there exists a G-invariant polynomial f on V such that $f(v) = 1$ and $f(0) = 0$ or, equivalently, there is a G-invariant homogeneous form f on V such that $f(v) = 1$.*

Let G be a geometrically reductive algebraic group acting on an affine algebraic scheme X (we can even take X to be an arbitrary affine scheme over the base field $\mathbf{K}$, i.e. not necessarily of finite type over $\mathbf{K}$) and X_1, X_2 two G-invariant closed subsets of X such that $X_1 \cap X_2$ is empty. Then there exists a G-invariant $f \in A(X = \operatorname{Spec} A)$ such that $f(X_1) = 0$ and $f(X_2) = 1$. This is proved easily as follows : there exists an element $g \in A$ (not necessarily G-invariant) such that $g(X_1) = 0$ and $g(X_2) = 1$. Now the translates of g by elements of G span a *finite*-dimensional G-invariant linear subspace W of A. For every $h \in W$, $h(X_1) = 0$ and $h(X_2)$ is a constant. We have a canonical rational representation of G on W and therefore also on the dual W^* of W. The canonical inclusion $W \subset A$ defines a G-morphism $\phi : X \to W^*$ of X into the affine scheme W^* (to be strict the affine scheme whose set of geometric points is W^*) and we have $\phi(X_1) = 0$ and $\phi(X_2) = w$, $w \neq 0$. Now by the geometric reductivity of G, there exists h in the coordinate ring of W^* such that $h(0) = 0$ and $h(w) = 1$. Now if f is the image of h in A by the canonical homomorphism of the coordinate ring of W^* in A, then f has the required properties.

The following statements are proved easily.

(1) G is geometrically reductive if and only if for every rational representation of G in a finite-dimensional vector space V and a *semi-*

invariant point $v \in V$, $v \neq 0$ (i.e. the one-dimensional linear sub-
350 space of V spanned by v is G-invariant), there is a semi-invariant homogenous form f on V such that $f(v) = 1$.

(2) G is geometrically reductive if and only if for every rational representation of G in a finite-dimensional vector space V, a G-invariant linear subspace V_0 of V of codimension one and X_0 an element of V such that X_0 and V_0 span V and X_0 is G-invariant modulo V_0, there exists a G-invariant $F \in S_m(V)$ (m^{th} symmetric power) for some $m \geqslant 1$, such that F is monic in X_0 when F is written with respect to a basis $X_0, X_1, \ldots, X_n \in V$, $X_i \in V_0$, $i \geqslant 1$.

(3) Let N be a normal algebraic subgroup of an affine algebraic group G such that N and G/N are geometrically reductive. Then G is geometrically reductive. In particular, a finite product of geometrically reductive groups is geometrically reductive.

(4) Let G be a reductive group. Then G is geometrically reductive if and only if $G/\operatorname{rad} G$ is so.

(5) A linearly reductive group is geometrically reductive. A finite group is geometrically reductive.

The conjecture of Mumford states that a reductive group is geometrically reductive (c.f. Preface, [3]). On the other hand it can be shown that a geometrically reductive group is necessarily reductive (c.f. [8]).

Theorem 1. *The full linear group $GL(2)$ of 2×2 matrices is geometrically reductive.*

Proof. Let G be an affine algebraic group and ρ, ρ' rational representations of G in finite-dimensional vector spaces W, W' respectively. Let $\phi : W \to W'$ be a homomorphism of G-modules and w, w' semi-invariant points of W, W' respectively such that $w' = \phi(w)$, $w' \neq 0$. Now if there is a semi-invariant polynomial f on W' such that $f(w') = 1$ and $f(0) = 0$, then there is a semi-invariant polynomial g on W such that $g(w) = 1$ and $g(0) = 0$; in fact we can take g to be the image of

f under the canonical homomorphism induced by ϕ of the coordinate ring of W' into that of W. Using this simple remark, the proof of the 351
geometric reductivity of $GL(2)$ can be divided into the following steps.

(1) It is a well-known fact (c.f. §1, exposé 4, Prop. 4, [2]) that if G is an affine algebraic group and ρ a rational representation of G in a finite dimensional vector space W, then the G-module W can be imbedded as a submodule of A^n (n-fold direct sum of A), where A is a submodule of the coordinate ring of G, considered as a G-module for the regular representation (we should fix the right or the left regular representation). Thus to prove geometric reductivity of G, we have only to consider submodules A of the coordinate ring of G such that there exists a semi-invariant $a \in A$, $a \neq 0$.

(2) Let $G = GL(n)$, R the coordinate ring of G and (X_{ij}), $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n$, the canonical coordinate functions on G. The linear space generated by X_{ij} is a G-module and we can identify it with the G-module $V^n = V \oplus \cdots \oplus V$ (n times), where V is an n-dimensional vector space and G is represented as $\operatorname{Aut} V$. Let ξ be the function $\det |X_{ij}|$ and L the 1-dimensional G-submodule of R spanned by ξ. Now if W is a finite-dimensional linear subspace of R, there exists an integer $m \geqslant 1$ such that for any $g \in W$, $g\xi^m$ is a polynomial in (X_{ij}). A polynomial in (X_{ij}) can be uniquely expressed as a sum of multihomogenous forms in the sets of variables

$$Y_1 = (X_{11}, X_{21}, \ldots, X_{n1}),\ Y_2 = (X_{12}, X_{22}, \ldots, X_{n2}), \ldots$$
$$Y_n = (X_{1n}, X_{2n}, \ldots, X_{nn})(Y_i - i^{\text{th}} \text{ column of } (X_{ij})).$$

The space of multihomogenous forms in (X_{ij}) of degree m_i in Y_i can be identified with the G-module $W(m_1, \ldots, m_n)$, where

$$W(m_1, \ldots, m_n) = \bigoplus_{i=1}^{n} S^{m_i}(V)(S^{m_i}(V) - m_i^{\text{th}} \text{ symmetric power of } V).$$

Thus if W is a finite dimensional G-invariant linear subspace of R, $W \otimes L^{(m)}$ can be embedded as a G-submodule of a finite direct

sum of G-modules of the type $W(m_1, \dots, m_n)$, where $L^{(m)}$ denotes the 1-dimensional G-module $L \otimes \cdots \otimes L$ (m times). Thus to prove the geometric reductivity of $GL(n)$, it suffices to consider the G-
352 modules of the form $W(m_1, \dots, m_n)$ such that there is a non-zero semi-invariant element in it.

Now it is easy to see that $W(m_1, \dots, m_n)$ has a non-zero semi-invariant element v if and only if $m_1 = m_2 = \dots = m_n = m$ and then that v is in the 1-dimensional linear subspace spanned by ξ^m ($\xi = \det|X_{ij}|$). This is an immediate consequence of the following remarks:

(i) every 1-dimensional G-module (given by a rational representation) is isomorphic to $L^{(n)}$ for some $n \in \mathbf{Z}$ and

(ii) the only G-invariant elements of R are the scalars.

Thus to prove the geometric reductivity of $GL(n)$, we have only to consider the G-modules $W(m)$,

$$W(m) = W(m, \dots, m) = \otimes S^m(V) (n\text{-fold tensor product of } S^m(V))$$

with the semi-invariant element being ξ^m, $\xi = \det|X_{ij}|$.

(iii) Let $G = GL(2)$. Let $J : W(m) \to S^{2m}(V)$ be the canonical homomorphism, where for an element f in $W(m)$ being considered as a multi-homogeneous polynomial of degree m in $Y_1 = (X_{11}, X_{21})$, $Y_2 = (X_{12}, X_{22})$, $j(f)$ is the homogeneous polynomial of degree $2m$ in two variables obtained by setting $Y_1 = Y_2$. Now j is a G-homomorphism. Let $\theta_{m-1} : W(m-1) \to W(m)$ be the homomorphism defined by $\theta_{m-1}(f) = f\xi$, $f \in W(m-1)$. Now θ_{m-1} is a homomorphism of the underlying $SL(2)$ modules and it "differs" from a $GL(2)$ homomorphism only upto a character of $GL(2)$. Consider the following sequence

$$0 \to W(m-1) \xrightarrow{\theta_{m-1}} W(m) \xrightarrow{j} S^{2m}(V) \to 0. \qquad (*)$$

We claim that this sequence is exact. It is clear that θ_{m-1} is injective. Further the kernel of j consists precisely of those polynomials f in (X_{ij}) which belong to $W(m)$ and such that f vanishes when we set (X_{ij}) to be a singular matrix. Therefore $f = g\xi$, which means that $\ker j = \theta_{m-1}W(m-1)$. Now $\dim W(m) = (m+1)^2$, $\dim W(m-1) = m^2$ and $\dim S^{2m}(V) = (2m+1)$, so that $\dim W(m) = \dim W(m-1) + \dim S^{2m}(V)$. From this one concludes that (*) is exact.

We shall now show that the exact sequence (*) has a "quasi- 353
splitting", i.e. there is a closed G-invariant subvariety of $W(m)$ such that the canonical morphism of this subvariety into $S^{2m}(V)$ is surjective and *quasi-finite* i.e. every fibre under this morphism consists only of a finite number of points. Let D_m be the subset of $W(m)$ consisting of decomposable tensors, i.e. $D_m = \{f | f = g \otimes h, g, h \in S^m(V)\}$. Then D_m is obviously a G-invariant subset of $W(m)$. We have a canonical morphism

$$\Psi : S^m(V) \times S^m(V) \to S^m(V) \otimes S^m(V) = W(m)$$

and $D_m = \Psi(S^m(V) \times S^m(V))$. From the fact that Ψ is bilinear, we see that D_m is the cone over the image of Ψ', where Ψ' is the canonical morphism

$$\Psi' : \mathbf{P}(S^m(V)) \times \mathbf{P}(S^m(V)) \to \mathbf{P}(W(m))$$

induced by Ψ, $\mathbf{P}$ indicating the associated projective spaces. It follows now that D_m is a *closed* G-invariant subvariety of $W(m)$. The morphism $j_1 : D_m \to S^{2m}(V)$ induced by j is surjective, because every homogeneous form in two variables over an algebraically closed field can be written as a product of linear forms, in particular as a product of two homogeneous forms of degree m. We see also easily that $j_1 : D_m \to S^{2m}(V)$ is quasi-finite (it can also be shown without much difficulty that j_1 is proper so that j_1 is indeed a *finite* morphism but we do not make use of it in the sequel). An element $f \otimes g \in D_m$ becomes zero when we set $Y_1 = Y_2$ if and only if f and g are zero, i.e. we have $D_m \cap \theta_{m-1}(W(m-1)) = (0)$.

(iv) Let $G = GL(2)$. We shall now show by induction on m, that there exists a closed G-invariant subvariety H_m of $W(m)$ passing through 0 and not through ξ^m. This will imply that $GL(2)$ is geometrically reductive.

For $m = 0$, the assertion is trivial. Let H_{m-1} be a homogeneous G-invariant hypersurface of $W(m-1)$ not passing through ξ_{m-1}. Let H_m be the join of $\theta_{m-1}(H_{m-1})$ and D_m, i.e.

$$H_m = \{\lambda + \mu | \lambda \in \theta_{m-1}(H_{m-1}), \mu \in D_m\}.$$

354 We shall now show that H_m is a homogeneous G-invariant hypersurface of $W(m)$ not passing through ξ^m. It is immediate that ξ^m is not in H_m for if $\xi^m = \lambda + \mu$, λ in $\theta_{m-1}(H_{m-1})$, $\mu \in D_m$, then by setting $Y_1 = Y_2$ since ξ^m and λ become zero, we conclude that μ becomes zero. As remarked before, this implies that μ itself is zero so that $\xi^m \in \theta_{m-1}(H_{m-1})$. It would then follow that $\xi^{m-1} \in H_{m-1}$, which leads to a contradiction so that we conclude that ξ^m is not in H_m. The subset H_m is G-invariant and also invariant under homothecy. Thus to complete the proof of our assertion it suffices to show that H_m is closed and of codimension one in $W(m)$. This is an immediate consequence of the following lemma, since H_m is the join of the two homogeneous subvarieties $\theta_{m-1}(W(m-1))$ and D_m whose common intersection is (0). $\square$

Lemma 1. *Let Q_1, Q_2 be closed subvarieties of a projective space* $\mathbf{P}$ *such that $Q_1 \cap Q_2$ is empty. Then the join Q of Q_1 and Q_2 is a closed subvariety of* $\mathbf{P}$ *and* $\dim Q = \dim Q_1 + \dim Q_2 + 1$.

Proof of Lemma. Let Δ be the diagonal in $\mathbf{P} \times \mathbf{P}$ and $R = (\mathbf{P} \times \mathbf{P} - \Delta)$. If $r = (p_1 p_2)$, $p_i \in \mathbf{P}$, let $L(r)$ be the line in $\mathbf{P}$ joining p_1 and p_2. Then the mapping $r \to L(r)$ defines a correspondence between R and $\mathbf{P}$, and it is seen easily that this is defined by a *closed* subvariety of $R \times \mathbf{P}$. Since $Q_1 \cap Q_2$ is empty, we have $Q_1 \times Q_2 \subset R$. Let $\Gamma_1 = \mathrm{pr}_1^{-1}(Q_1 \times Q_2)$, pr_1 being the canonical projection of $R \times \mathbf{P}$ onto the first factor. Now the join $Q = \mathrm{pr}_2(\Gamma_1)$, pr_2 being the projection of $R \times \mathbf{P}$ onto the second factor. Since $Q_1 \times Q_2$ is complete, it follows that Q is a closed subvariety of $\mathbf{P}$.

We see that $\dim Q_1 + \dim Q_2 \leqslant \dim Q \leqslant \dim Q_1 + \dim Q_2 + 1$. Therefore to show that $\dim Q = Q \dim Q_1 + \dim Q_2 + 1$, it suffices to

show that $\dim Q \geqslant \dim Q_1 + \dim Q_2 + 1$. Since $Q_1 \cap Q_2$ is empty, we cannot have $\dim \mathbf{P} = \dim Q_1 + \dim Q_2$. If $\dim \mathbf{P} = \dim Q_1 + \dim Q_2 + 1$, we see that the lemma is true in this case. Suppose then that $\dim \mathbf{P} > \dim Q_1 + \dim Q_2 + 1$. Then there is a point $p \in \mathbf{P}$ which is not in the join Q of Q_1 and Q_2. Let us now project Q_1 Q_2 and Q from p in a hyperplane H not passing through p. Let Q'_1, Q'_2 and Q' be the images of Q_1, Q_2 and Q respectively in H. Then Q'_1, Q'_2 and Q' are isomorphic to Q_1, Q_2 and **355**
Q respectively. Further $Q'_1 \cap Q'_2$ is empty and Q' is the join of Q'_1 and Q'_2 in H. This process reduces the dimension of the ambient projective space by one. By a repetition of this procedure, we are finally reduced to the case $\dim \mathbf{P} = \dim Q_1 + \dim Q_2 + 1$, in which case the lemma is true as remarked before. This completes the proof of the lemma and consequently the proof of theorem is now complete.

Corollary. *A finite product of algebraic groups of the type $GL(2)$, $SL(2)$ or torus group is geometrically reductive.*

Remarks. (1) That H is a closed G-invariant subset of codimension one in $W(m)$ (in the above proof) can also be done by showing that the morphism $\phi : W(m-1) \times D_m \to W(m)$, defined by $\phi(w, d) = w + d$ is a surjective finite morphism. The proof that $\dim Q = \dim Q_1 + \dim Q_2 + 1$ in the above lemma, is due to C. P. Ramanujam.

(2) In characteristic 2, the geometric reductivity of $GL(2)$ was proved by Oda (c.f. [9]).

(3) The above proof gives also an analogue of geometric reductivity of $GL(2)$ over $\mathbf{Z}$ and consequently for more general ground rings as well.

(4) M. S. Raghunathan has pointed out another proof of the existence of a hypersurface in $W(m)(G = GL(2))$ with the required properties. We have an isomorphism of the $GL(2)$-modules V and V^* (V^* dual of V). If $m = p^\alpha - 1$, α a positive integer, p being the characteristic of the ground field, he points out that

$S^m(V) \approx S^m(V^*) \approx (S^m(V))^*$ as $GL(2)$-modules. Therefore in this case

$$W(m) \approx \mathrm{Hom}(S^m(V), S^m(V))$$

and then the hypersurface defined by endomorphisms with zero determinant will have the required properties. Since we can find arbitrarily large integers of the form $p^\alpha - 1$, the existence of the required hypersurface for arbitrary m follows easily.

2 Quotient spaces.

Definition 4. Let X be an algebraic scheme on which an affine algebraic
356 group G operates. A morphism $\phi : X \to Y$ of algebraic schemes is said to be a good quotient (of X modulo G) if it has the following properties :

(1) ϕ is a surjective affine morphism and is G-invariant; (2) $\phi_*(\mathbf{O}_X)^G = \mathbf{O}_Y$ and (3) if X_1, X_2 are closed G-invariant subsets of X such that $X_1 \cap X_2$ is empty, then $\phi(X_1)$, $\phi(X_2)$ are closed and $\phi(X_1) \cap \phi(X_2)$ is empty. We say that ϕ is a good affine quotient if ϕ is a good quotient and Y is affine.

The first two conditions are equivalent to the following : ϕ is surjective and for every affine open subset U of Y, $\phi^{-1}(U)$ is affine and G-invariant and the coordinate ring of U can be identified with the G-invariant subring of $\phi^{-1}(U)$. We see then that if ϕ is a good affine quotient, X is also affine. The following properties of good quotients are proved quite easily.

(1) The property of being a good quotient is *local* with respect to the base scheme, i.e. ϕ is a good quotient if and only if there is an open covering $\{U_i\}$ of Y such that every $V_i = \phi^{-1}(U_i)$ is G-invariant and the induced morphism $\phi_i : V_i \to U_i$ is a good quotient.

(2) A good quotient is also a *categorical quotient*, i.e. if $\psi : X \to Z$ is a G-invariant morphism, there is a unique morphism $\nu : Y \to Z$ such that $\nu \circ \phi = \psi$.

(3) *Transitivity properties.* Let X be an a one algebraic scheme on which an affine algebraic group G operates. Let N be a normal closed subgroup of G and H the affine algebraic group G/N. Suppose that $\phi_1 : X \to Y$ is a good quotient (resp. good affine quotient) of X modulo N. Then we have the following.

(a) The action of G goes down into an action of H on Y.

(b) If $\phi_2 : Y \to Z$ is a good quotient (resp. good affine quotient) of Y modulo H, then $\phi_2 \circ \phi_1 : X \to Z$ is a good quotient (resp. good affine quotient) of X modulo G.

$$\begin{tikzcd} X \arrow[rr, "\text{good}"] \arrow[dr, dashed] & & Y \arrow[dl, "\text{good}"] \\ & Z & \end{tikzcd}$$

(c) If $\phi : X \to Z$ is a good quotient (resp. good affine quotient) of X modulo G, there is a canonical morphism $\phi_2 : Y \to Z$ such that $\phi = \phi_2 \circ \phi_1$ and ϕ_2 is a good quotient (resp. good affine quotient of Y modulo H. **357**

$$\begin{tikzcd} X \arrow[rr, "\text{good}"] \arrow[dr, "\text{good}"'] & & Y \arrow[dl] \\ & Z & \end{tikzcd}$$

(4) If $\phi : X \to Y$ is a good quotient (modulo G), Z a *normal* algebraic variety on which G operates and $j : Z \to X$ a proper, injective G-morphism, then Z has a good quotient modulo G; in fact it can be identified with the normalisation of the reduced subvariety $(\phi \circ j)(Z)$ in a suitable finite extension.

The basic existence theorem on good quotients is the following.

Theorem 2. *Let $X = \operatorname{Spec} A$ be an affine algebraic scheme on which a geometrically reductive algebraic group G operates. Let $Y = \operatorname{Spec} A^G$ (A^G invariant subring of A) and $\phi : X \to Y$ the canonical morphism induced by $A^G \subset A$. Then ϕ is a good affine quotient.*

For the proof of this theorem, the only non-trivial point is to check that Y is an *algebraic* scheme, i.e. A^G is an algebra of finite type over $\mathbf{K}$ and this is assured by a theorem of Nagata, namely that if A is a $\mathbf{K}$-algebra of finite type on which a geometrically reductive group G operates (rationally), then A^G is also a $\mathbf{K}$-algebra of finite type (c.f. Main theorem, [6]). The other properties for ϕ to be a good quotient, are verified quite easily.

Definition 5. *Let X be a closed subscheme of the projective space $\mathbf{P}_n$ of dimension n. An action of an affine algebraic group G on X is said to be linear if it comes from a rational representation of G in the affine scheme $\mathbf{A}_{n+1}$ of dimension $(n+1)$.*

The above definition means that we have an action of G on $\mathbf{A}_{n+1} = \operatorname{Spec} \mathbf{K}[X_1, \dots, X_{n+1}]$ given by a rational representation of G on $\mathbf{A}_{n+1}$ and that if $\mathfrak{a}$ is the graded ideal of $\mathbf{K}[X_1, \dots, X_{n+1}]$ defining X, then $\mathfrak{a}$ is
358 left invariant by G. We have $X = \operatorname{Proj} R$, $R = \mathbf{K}[X_1, \dots, X_{n+1}]/\mathfrak{a}$. We denote by $\widehat{X}$ the cone over X ($\widehat{X} = \operatorname{Spec} R$) and by (0) the vertex of the cone $\widehat{X}$. The action of G lifts to an action on $\widehat{X}$ and this action and the canonical action of $\mathbf{G}_m$ on $\widehat{X}$ (homothecy) commute. We observe that the canonical morphism $p : \widehat{X} - (0) \to X$ is a principal fibre space with structure group $\mathbf{G}_m$ and that p is a good quotient (modulo $\mathbf{G}_m$).

Definition 5. *Let X be a closed subscheme of $\mathbf{P}_n$ and let there be given a linear action of an affine algebraic group G on X. A point $x \in X$ is said to be semi-stable if for some $\widehat{x} \in \widehat{X} - (0)$ over x, the closure (in $\widehat{X}$) of the G-orbit through $\widehat{x}$ does not pass through (0). A point $x \in X$ is said to be stable (to be more precise, properly stable) if for some $\widehat{x} \in \widehat{X} - (0)$ over x, the orbit morphism $\psi_{\widehat{x}} : G \to \widehat{X}$ defined by $g \to \widehat{x} \circ g$ is proper. We denote by X^{ss} (resp. X^s) the set of semi-stable (resp. stable) points of X.*

We have now the following

Theorem 3. *Let X be a closed subscheme of $\mathbf{P}_n$ defined by a graded ideal $\mathfrak{a}$ of $\mathbf{K}[X_1, \ldots, X_{n+1}]$ so that $X = \operatorname{Proj} R$, $R = \mathbf{K}[X_1, \ldots, X_{n+1}]/\mathfrak{a}$. Let there be given a linear action of an affine algebraic group G on X, $Y = \operatorname{Proj} R^G$ and $\phi : X \to Y$ the canonical rational morphism defined by the inclusion $R^G \subset R$. Suppose that G is geometrically reductive or, more generally, that the cone $\widehat{X}$ over X has a good affine quotient modulo G (c.f. Theorem 1). Then we have the following:*

(1) *$x \in X^{ss}$ if and only if there is a homogeneous G-invariant element $f \in R_+$(R_+ being the subring of R generated by homogeneous elements of degree $\geqslant 1$) such that $f(x) \neq 0$ (in particular, X^{ss} is open in X and ϕ is defined at $x \in X^{ss}$).*

(2) *$\phi : X^{ss} \to Y$ is a good quotient and Y is a projective algebraic scheme.*

(3) *X^s is a ϕ-saturated open subset, i.e. there exists an open subset Y^s of Y such that $X^s = \phi^{-1}(Y^s)$ and $\phi : X^s \to Y^s$ is a geometric quotient, i.e. distinct orbits of X^s go into distinct points of Y^s.*

This theorem is proved quite easily. **359**

Let $H_{p,r}(E)$ denote the Grassmannian of r-dimensional quotient linear spaces of a p-dimensional vector space E. We have a canonical immersion of $H_{p,r}(E)$ in the projective space associated to $\overset{p-r}{\wedge} E$ and if $X = H^N_{p,r}(E)$ denotes the N-fold product of $H_{p,r}(E)$, we have a canonical projective immersion of X, namely the Serge imbedding associated to the canonical projective imbedding of $H_{p,r}(E)$. There is a natural action of $GL(E) = \operatorname{Aut} E$ on $H_{p,r}(E)$ and this induces a natural action (the diagonal action) of $GL(E)$ on $H^N_{p,r}(E)$. The restriction of this action to the subgroup $G = SL(E)$ is a linear action with respect to the canonical projective imbedding of X. We denote by X^{ss} (resp. X^s) the set of semi-stable (resp. stable) points of X with respect to the canonical projective imbedding of X. Let R be the projective coordinate ring of X, $Y = \operatorname{Proj} R^G$, $\widehat{X}$ the cone over X and $\phi : X \to Y$ the canonical rational morphism as in Theorem 3 above. Then the result to be applied for the classification of vector bundles on an algebraic curve is as follows.

Theorem 4. *Let $X = H^N_{p,r}(E)$ with $1 \leqslant r \leqslant 2$. Then for the canonical action of $G = SL(E)$ on the cone $\widehat{X}$ over X, $\widehat{X}$ has a good affine quotient modulo G so that by Theorem 3, the rational morphism $\phi : X \to Y$ has the properties* (1), (2) *and* (3) *of Theorem 3; in particular, $\phi : X^{ss} \to Y$ is a good quotient and Y is a projective algebraic scheme.*

Further for $x \in X$, $x = \{E_i\}_{1 \leqslant i \leqslant N}$, E_i a quotient linear space of dimension r of E, $x \in X^{ss}$ (resp. X^s) if and only if for every linear subspace (resp. proper linear subspace) F of E, if F_i denotes the canonical image of F in E_i, we have

$$\frac{\frac{1}{N}\sum_{i=1}^{N} \dim F_i}{r} \geqslant \frac{\dim F}{p} (resp. >)$$

Indication of proof. Let W be the space of $(p \times r)$ matrices. Then we have canonical commuting operations of $GL(p)$ and $GL(r)$ on W. Let W^N be the N-fold product of W and σ_1 the canonical diagonal action
360 of $GL(p)$ on W^N. Let σ be the induced action of $SL(p)$ on W^N. We have a natural action τ_1 of $GL(r)^N$ on W^N. Let H be the subgroup of $GL(r)^N$ defined by elements $(g_1, \ldots, g_N)$ such that $\prod_{i=1}^{N} \det g_i = 1$ and τ the restriction of the action τ_1 to H. We note that $H/(SL(r))^N$ is a torus group. Therefore H is geometrically reductive since $1 \leqslant r \leqslant 2$. Then in view of Theorem 3 and the *transitivity* properties of good quotients, for proving the first part of the theorem, it suffices to show that a good quotient of W^N exists, respectively for the actions of $SL(p)$ and H, and that the good quotient of W^N modulo H can be identified with the cone $\widehat{X}$ over X.

$$\begin{tikzcd}
W^N \arrow[r, "H"', "\text{good}"] \arrow[d, "\text{good}"', "SL(p)"] \arrow[dr, "\text{good}"] & \widehat{X} \arrow[d, "SL(p)"] \\
(*) \arrow[r, "H", "\text{good}"'] & \widehat{Y}
\end{tikzcd}$$

These last two statements follow easily from the facts that for arbitrary r (i.e. without assuming $1 \leqslant r \leqslant 2$), a good quotient of W modulo the canonical action of $SL(r)$ exists and that it can be identified with the cone over $H_{p,r}(E)$. These facts can be checked explicitly, using a result of Igusa that $H_{p,r}(E)$ is projectively normal (c.f. [4]).

The proof of the last part of the theorem is the same as in §4, Chap. 4, [5] and we remark that for this it is not necessary to suppose that $1 \leqslant r \leqslant 2$. It should be noted that our definition of stable and semi-stable points differs, *a priori*, from that of [4], when the group is not geometrically reductive and that the computations of §4, [4] hold in arbitrary characteristic for *reductive* groups provided we take the definition of stable and semi-stable points in our sense.

3 Vector bundles over a smooth projective curve.

Let X be a smooth projective curve over $\mathbf{K}$. Let us suppose that the *genus g of X is* $\geqslant 2$. By a vector bundle V over X, we mean an algebraic vector bundle; we denote by $d(V)$ the degree of V and by $r(V)$ the rank **361**
of V. We fix a very ample line bundle L on X, let $l = d(L)$. If V is a vector bundle (resp. coherent sheaf) on X, we denote by $V(m)$, the vector bundle (resp. coherent sheaf) $V \otimes L^m$, where L^m denotes the m-fold tensor product of L. If F is a coherent sheaf on X, the *Hilbert polynomial $P = P(F, m)$* of F is a polynomial in m with rational coefficients, defined by

$P(m) = P(F, m) = \chi(F(m)) = \dim H^0(F(m)) - \dim H^1(F(m))$. If F is the coherent sheaf associated to a vector bundle V, we have

$$P(m) = d(V(m)) - r(V)(g-1) = d(V) + r(V)(ml - g + 1).$$

We recall that a vector bundle V on X is said to be *semi-stable* (resp. *stable*) if for every sub-bundle W of V (resp. proper sub-bundle W of V), we have

$$r(V)d(W) \leqslant r(W)d(V) (\text{resp. } r(V)d(W) < r(W)d(V)).$$

Let α be a positive rational number and $\mathbf{S}(\alpha)$ the category of semi-stable vector bundles V on X such that $d(V) = \alpha r(V)$. Then $\mathbf{S}(\alpha)$ is an abelian category and the Jordan-Hölder theorem holds in this category (c.f. Prop. 3.1, [12] and Prop. 1, [10]). For $V \in \mathbf{S}(\alpha)$, we denote by gr V the associated graded object; now gr V is a direct sum of stable bundles W such that $d(W) = \alpha r(W)$ (we note that gr V is not a well-determined object of $\mathbf{S}(\alpha)$, it is determined only upto isomorphism). Let $\mathbf{S}(\alpha, r)$ be the sub-category of $\mathbf{S}(\alpha)$ consisting of $V \in \mathbf{S}(\alpha)$ such that $r(V) = r$. It can be proved that $\mathbf{S}(\alpha, r)$ is *bounded*, i.e. there is an algebraic family of vector bundles on X such that every $V \in \mathbf{S}(\alpha, r)$ is found in this family (upto isomorphism). We can then find an integer m such that $H^0(V(m))$ generates $V(m)$ and $H^1(V(m)) = 0$ for all $V \in \mathbf{S}(\alpha, r)$. *We fix such an integer m in the sequel.* Let E be the trivial vector bundle on X of rank

$$p = r(\alpha + lm - g + 1).$$

If $V \in \mathbf{S}(\alpha, r)$, then $\dim H^0(V(m)) = p$, $V(m)$ is a quotient bundle of E and the Hilbert polynomial P of $W = V(m)$ is given by $P(n) = r(\alpha + lm + ln - g + 1)$, $P(0) = p$. The Hilbert polynomial is the same for
362 all $V(m)$, $V \in \mathbf{S}(\alpha, r)$. Let $Q(E/P) = \mathrm{Quot}(E/P)$ be the Grothendieck scheme of all $\beta : E \to F$, where F is a coherent sheaf on X; β makes F a quotient of E and the Hilbert polynomial of F is the above P; then $Q(E/P)$ is a projective algebraic scheme (c.f. Theorem 3.2, [3]). If $q \in Q(E/P)$, we denote by F_q the coherent sheaf which is a quotient of E, represented by q. Let R be the subset of $Q(E/P)$ determined by points $q \in Q(E/P)$ such that (i) F_q is locally free and (ii) the canonical mapping $\beta_q : E \to H^0(F_q)$ is surjective. If follows easily that for $q \in R$, β_q is indeed an isomorphism and that $H^1(F_q) = 0$. It can be shown that R is an *open, smooth and irreducible* subscheme of $Q(E/P)$ of dimension $(p^2 - 1) + (r^2(g-1) + 1)$ invariant under the canonical operation of Aut E on $Q(E/P)$ and that for $q_1, q_2 \in R$, F_{q_1} is isomorphic to F_{q_2} if and only if q_1, q_2 lie in the same orbit under $GL(E) = \mathrm{Aut}\, E$ (c.f. §6, [12] and §5 a, [10]). for $q \in R$, F_q is locally free and is therefore the sheaf of germs of a vector bundle; let R^{ss} (resp. R^s) denote the subset of R consisting of q such that (the bundle associated to) F_q is semi-stable (resp. stable). Let ıı be an ordered set of N distinct points $P_1, \ldots, P_N$ on X. Let $\tau_i : R \to H_{p,r}(E)$ be the morphism into the Grassmannian of r-dimensional quotient linear spaces of E (considered canonically as a

vector space of dimension p), which assigns to $q \in R$, the fibre at P_i of the vector bundle associated to F_q, considered canonically as a quotient linear space of E. Let

$$\tau : R \to H^N_{p,r}(E)$$

be the $GL(E)$-morphism defined by $\tau = \{\tau_i\}_{1 \leqslant i \leqslant N}$. Then we have the following basic

Lemma 2. *Given the category $\mathbf{S}(\alpha, r)$, we can then find an integer m and an ordered set $\mathfrak{n}$ of N points on X as above such that the morphism $\tau : R \to H^N_{p,r}(E) = Z$ has the following properties:*

(1) *τ is injective;*

(2) *$\tau(R^{ss}) \subset Z^{ss}$ and for $q \in R^{ss}$, $\tau(q)$ is stable if and only if F_q is a stable vector bundle;*

(3) *the induced morphism $\tau : R^{ss} \to Z^{ss}$ is proper.*

Remark. It can indeed be shown that $\tau : R^{ss} \to Z^{ss}$ is a closed immer- **363**
sion for a suitable choice of m and $\mathfrak{n}$.

Excepting (3), the other assertions have been proved before (§7, [12]). We shall now give a proof of (3).

Let R_1 be the subset of $Q(E/P)$ consisting of points $q \in Q(E/P)$ such that the corresponding coherent sheaf F_q is locally free. Then $R \subset R_1$ and R_1 is an open subscheme of $Q(E/P)$ invariant under $GL(E)$ (c.f. Prop. 6.1, [12]). Let $\mathfrak{n}$ be an ordered set of N points $P_1, \ldots, P_N$ on the curve X. Let $\tau_i : R_1 \to H_{p,r}(E)$ be the morphism (extending the above τ_i) into the Grassmannian of r-dimensional quotient linear spaces of E which assigns to $q \in R_1$, the fibre of the vector bundle associated to F_q at the point P_i, considered canonically as a quotient linear space of E. Let $\tau : R_1 \to H^N_{p,r}(E)$ be the morphism defined by $\tau = \{\tau_i\}_{1 \leqslant i \leqslant N}$.

We shall now extend the morphism $\tau : R_1 \to H^N_{p,r}(E)$ to a *multi-valued* (set) mapping of $Q(E/P)$ into $H^N_{p,r}(E)$ and we shall denote this extension by $\Phi = \{\Phi_i\}_{1 \leqslant i \leqslant N}$. Suppose now that for $q \in Q(E/P)$, F_q is not locally free. Then we have $F_q = V_q(m) \oplus T_q$, where T_q is a torsion sheaf and V_q is locally free (this decomposition holds because

X is a non-singular curve). Suppose that $P_i \notin$ Support of T_q. We then define $\Phi_i(q) \in H_{p,r}(E)$ as the fibre of the bundle $V_q(m)$ at P_i considered canonically as a quotient linear space of dimension r of E. Suppose that $P_i \in$ Support of T_q; we then define $\Phi_i(q)$ to be any point of $H_{p,r}(E)$. We thus obtain a multivalued (set) mapping $\Phi_i : Q(E/P) \to H_{p,r}(E)$ and we define $\Phi = \{\Phi_i\}_{1 \leqslant i \leqslant N}$. We claim now that Φ_i is a morphism in a neighbourhood of $q \in Q(E/P)$ if and only if $P_i \notin$ Support of T_q. For this it suffices to show that given $q_0 \in Q(E/P)$ such that $P_i \notin$ Support of T_{q_0}, there is a neighbourhood U of q_0 such that $P_i \notin$ Support of T_q for any q in U. We observe that F_{q_0} is locally free in a neighbourhood of P_i and therefore if F is the coherent sheaf on $X \times Q(E/P)$, which is a quotient of E and defines the family $\{F_q\}$, it follows by Lemma 6.1, [12], that F_q is locally free in a neighbourhood of $(P_i \times q_0) \in X \times Q(E/P)$. From this the existence of a neighbourhood U as required above follows
364 easily and our claim is proved. It is now immediate that the graph of Φ_i in $Q(E/P) \times H_{p,r}(E)$ is closed and that it contains the closure of the graph of $\tau_i : R_1 \to H_{p,r}(E)$ in $Q(E/P) \times H_{p,r}(E)$. From this it follows easily that the graph of Φ in $Q(E/P) \times H^N_{p,r}(E)$ contains the closure of the graph of $\tau : R_1 \to H^N_{p,r}(E)$ in $Q(E/P) \times H_{p,r}(E)$. Then we claim that

Claim (A). m and N can be so chosen that for $q \in Q(E/P)$, $\Phi(q)$ is semi-stable (resp. stable) if and only if $q \in R^{ss}$ (resp. R^s).

Let us first show how (A) implies (3) of Lemma 2. Let us denote by the same letter Φ, the graph of the multivalued set mapping $\Phi : Q(E/P) \to H^N_{p,r}(E)$. Let Γ be the graph of the morphism $\tau : R^{ss} \to H_{p,r}(E)^{ss}$ and Ψ the closure of Γ in $Q(E/P) \times H^N_{p,r}(E)$. Now (A) implies that $\Phi \supset \Psi$ and that

$$\Phi \cap (Q(E/P) \times H^N_{p,r}(E)^{ss}) = \Gamma.$$

This implies that

$$\Psi \cap (Q(E/P) \times H^N_{p,r}(E)^{ss}) = \Gamma$$

since $\Phi \supset \Psi \supset \Gamma$. Since Ψ is closed, this means that Γ, which by definition is closed in $R^{ss} \times H^N_{p,r}(E)^{ss}$, is also closed in $Q(E/P) \times H^N_{p,r}(E)^{ss}$.

Since $Q(E/P)$ is projective, in particular complete, the canonical projection of $Q(E/P) \times H^N_{p,r}(E)^{ss}$ onto $H^N_{p,r}(E)^{ss}$ is proper and this implies that

$$\tau : R^{ss} \to H^N_{p,r}(E)^{ss}$$

is proper.

We shall now prove ((A)). In view of (2) of Lemma 2 which has been proved in §7, [12], it suffices to prove the following:

Claim (B). m and N can be so chosen that for $q \in Q(E/P)$, $q \notin R^{ss}$, $\Phi(q)$ is not a semi-stable point of $H^N_{p,r}(E)$.

Let $\mathbf{F}(r)$ denote the category of all indecomposable vector bundles V on X such that $r(V) \leqslant r$ and $d(V) \geqslant -\operatorname{gr}(V)$. From the fact that the family of all indecomposable vector bundles on X of a given rank and degree is bounded (c.f. p. 426, Th. 3, [1]), it is deduced easily that there is an integer m_0 such that for $m \geqslant m_0$, $\forall V \in \mathbf{F}(r)$, $H^1(V(m)) = 0$ and **365**
$H^0(V(m))$ generates $V(m)$ (i.e. the canonical mapping of $H^0(V(m))$ onto the fibre of $V(m)$ at every point of X is surjective). In the following we fix a positive integer m such that $m \geqslant m_0$.

If $q \in Q(E/P)$, we have $F_q = \mathbf{V}_q(m) \oplus T_q$, where T_q is a torsion sheaf and $\mathbf{V}_q$ is the coherent sheaf associated to a vector bundle V_q. We denote by p_1 the natural projection of $H^0(F_q)$ onto $H^0(V_q(m))$. For proving ((B)), we require the following :

Claim (C). If $q \notin R^{ss}$, there is a proper linear subspace K of E (i.e. $K \neq E$, $K \neq (0)$) and a sub-bundle $W_q(m)$ of $V_q(m)$ ($W_q(m)$ could reduce to 0) such that

(i) $(p_1 \circ \beta_q)(K) \subset H^0(W_q(m))$ and $(p_1 \circ \beta_q)(K)$ generates $W_q(m)$ generically (i.e. there is at least one point P of X such that $(p_1 \circ \beta_q)(K)$ generates the fibre of $W_q(m)$ at P; we recall that β_q is the canonical mapping $E \to H^0(F_q)$) and,

(ii) $$\frac{r(W_q(m))}{\dim K} - \frac{r}{p} < 0.$$

We shall now prove ((C)) and this proof is divided into two cases.

Case (i) $q \notin R$. Suppose that $\mathrm{Ker}(p_1 \circ \beta_q) \neq 0$. Then we set $K = \mathrm{Ker}(p_1 \circ \beta_q)$ and $W_q(m) = (0)$. Then K generates $W_q(m)$ and the inequality (b) is obviously satisfied.

Suppose then that $\mathrm{Ker}(p_1 \circ \beta_q) = 0$, i.e. $p_1 \circ \beta_q : E \to H^0(Vq(m))$ is injective. Suppose further that for every indecomposable component $V_i(m)$ of $V_q(m)$, we have

$$d(V_i) \geqslant -\mathrm{gr}(V_i).$$

Then by our choice of m, we have $H^1(V_q(m)) = 0$ and $H^0(V_q(m))$ generates $V_q(m)$. For the torsion sheaf T_q, we have $T_q(n) = T_q$ for all n and $H^1(T_q) = 0$. It follows then that $H^1(F_q(n)) = 0$ for every $n \geqslant 0$. Then we have $P(n) = H^0(F_q(n))$ for every $n \geqslant 0$; in particular $p = \dim H^0(F_q)$. But since $p_1 \circ \beta_q : E \to H^0(V_q(m))$ is injective and $p = \dim E$, we conclude that $H^0(T_q) = 0$. Since T_q is a torsion sheaf,
366 this implies that $T_q = (0)$, i.e. that F_q is locally free. Further it follows that $\beta_q : E \to H^0(F_q)$ is an isomorphism so that $q \in R$, which is a contradiction.

We can therefore suppose that there is at least one indecomposable component $V_i(m)$ of $V_q(m)$ such that

$$d(V_i) < -gr(V_i).$$

Let $V_q(m) = W_1(m) \oplus W_2(m)$ such that for every indecomposable component $U(m)$ of $W_1(m)$, we have $d(U) \geqslant -gr(U)$ and for every indecomposable component $S(m)$ of $W_2(m)$, we have $d(S) < -gr(S)$. We note that since F_q is a quotient of E and $F_q = \mathbf{V}_q(m) \oplus T_q$, $V_q(m)$ is generated by its global sections; consequently $W_1(m)$ and $W_2(m)$ are also generated respectively by their global sections. If G is a vector bundle on X generated generically by $H^0(G)$, it can be shown easily (c.f. Lemma 7.2, [12]) that

$$\dim H^0(G) \leqslant d(G) + r(G)$$

and by applying this it follows easily that

$$\dim H^0(W_2(m)) < r(W_2)(lm - g + 1).$$

We see then that there is a linear subspace K of $E(\approx H^0(E))$ such that $(p_1 \circ \beta_q)(K) \subset H^0(W_1(m))$ and

$$\dim K > p - r(W_2)(lm - g + 1) = r(W_1)(lm - g + 1) = \frac{r(W_1)}{r} p.$$

This shows that

$$\frac{r}{p} > \frac{r(W_1(m))}{\dim K}.$$

Let $W_q(m)$ be the sub-bundle of $W_1(m)$ generated generically by K (through $p_1 \circ \beta_q$). Then we have

$$\frac{r(W_q(m))}{\dim K} < \frac{r(W_1(m))}{\dim K}.$$

Therefore, we have

$$\frac{r(W_q(m))}{\dim K} - \frac{r}{p} < 0.$$

This proves ((C)) in Case (i).

Case (ii) $q \in R$, $q \notin R^{ss}$. We have $F_q = \mathbf{V}_q(m)$, V_q being not semi- 367
stable. We see easily that there exists a *stable* sub-bundle $W_q(m)$ of $V_q(m)$ such that

$$\frac{d(W_q)}{r(W_q)} > \frac{d(V_q)}{r(V_q)} = \alpha > 0.$$

The bundle W_q is indecomposable and therefore $W_q \in \mathbf{F}(r)$. Therefore by our choice of m, $H^1(W_q(m)) = 0$ and $H^0(W_q(m))$ generates $W_q(m)$. We have also $H^1(V_q(m)) = 0$ and $\beta_q : E \to H^0(V_q(m))$ is an isomorphism. We set $K = H^0(W_q(m))$. Then by applying the Riemann-Roch theorem, we get

$$\frac{\dim K}{r(W_q)} = \frac{d(W_q(m))}{r(V_q)} - g + 1 = \frac{d(W_q)}{r(W_q)} + ml - g + 1,$$

$$\frac{p}{r} = \frac{d(V_q(m))}{r(V_q)} - (g - 1) = \frac{d(V_q)}{r(V_q)} + ml - g + 1.$$

Since $\dfrac{d(W_q)}{r(W_q)} > \dfrac{d(V_q)}{r(V_q)}$, we get $\dfrac{r(W_q)}{\dim K} - \dfrac{r}{p} < 0$. This completes the proof of ((C)).

We shall now show that ((C)) implies ((B)). Let $q \in Q(E/P)$, $q \notin R^{ss}$. If L is a subspace of E, we denote by L_i the canonical image of L in the quotient linear space of E represented by $\Phi_i(q)$. Let

$$\rho(L) = \frac{\frac{1}{N}\sum\limits_{i=1}^{N} \dim L_i}{\dim L} - \frac{r}{p}.$$

Then ((B)) would follow if we show that there is a proper subspace K of E such that $\rho(K) < 0$ (see the last assertion of Th. 4). Take a proper linear subspace K of E as provided by ((C)) above. Then we have

$$\rho(K) = \frac{r(W_q(m))}{\dim K} - \frac{r}{p} < 0.$$

We have

$$|\mu(K) - \rho(K)| \leqslant \sum_{i=1}^{N} |r(W_q) - \dim K_i| \tag{a}$$

368 since $\dim K \geqslant 1$. Now to estimate the right side, we should consider those i for which $r(W_q) - \dim K_i$ could be different from zero. This could occur for i such that $P_i \in$ Support of T_q or $P_i \notin$ Support of T_q. Suppose that $P_i \in$ Support of T_q and $r(W_q) - \dim K_i \neq 0$. Then K does *not* generate the fibre of $W_q(m)$ at P_i. The number of distinct points of X where K does not generate the fibre of $W_q(m)$ is at most $d(W_q(m))$ (c.f. Lemma 7.1, [12]). From these facts, we deduce that

$$|\mu(K) - \rho(K)| \leqslant \frac{r(d(W_q(m))) + \text{Card}(\text{Support of } T_q)}{N}. \tag{b}$$

Now for $F_q = \mathbf{V}_q(m) \oplus T_q$, by applying the Riemann-Roch theorem, we get for sufficiently large n that

$$\dim H^0(F_q(n)) = \lambda + nrl + \dim H^0(T_q) - r(g-1)$$

where $d(V_q(m)) = \lambda > 0$ (λ is positive because $V_q(m)$ is generated by its global sections). On the other hand, for sufficiently large n,

$$\dim H^0(F_q(n)) = P(n) = r(\alpha + lm + ln - g + 1).$$

Therefore we obtain that

$$\dim H^0(T_q) + \lambda = r(ml + \alpha). \tag{c}$$

Since Card (Support of T_q) $\leqslant \dim H^0(T_q)$ and $\lambda \leqslant 0$, we get that

$$\text{Card}(\text{Support of } T_q) \leqslant r(ml + \alpha). \tag{d}$$

We note that the family of vector bundles $\{V_q(m)\}$, $q \in Q(E/P)$ is a bounded family. This could be seen as follows. The degree of every indecomposable component of $V_q(m)$ is positive, in particular, bounded below, because $V_q(m)$ is generated by global sections. On the other hand from ((C)) above we see that

$$\lambda + d(V_q(m)) \leqslant r(ml + \alpha),$$

i.e. the degree of $V_q(m)$ is bounded above. From these facts, it follows that the degrees of every indecomposable component of $V_q(m)$ are both bounded below and above. This implies that $\{V_q(m)\}$ is a bounded family (c.f. p. 426. Th. 3, [1]). Now $W_q(m)$ is generated generically by K (through $p_1 \circ \beta_q$) and therefore by its global sections as well. As we just saw for the case of $V_q(m)$, it follows that the degrees of all the inde- **369**
composable components of $W_q(m)$ are bounded below. Then since the family $\{W_q(m)\}$, $q \in Q(E/P)$ is a family of sub-bundles of the bounded family $\{V_q(m)\}$, $q \in Q(E/P)$, it can be proved without much difficulty that the degrees of the indecomposable components of $W_q(m)$ are also bounded above (c.f. Prop. 11.1, [11]). It follows then, as we just saw for the case of $V_q(m)$, that $\{W_q(m)\}$, $q \in Q(E/P)$, is a bounded family. In particular, there is an absolute positive constant θ such that

$$d(W_q(m)) \leqslant \theta.$$

Looking at the inequalities (b), (c), and (d), we get that

$$|\mu(K) - \rho(K)| \leqslant r(\theta + r(ml + \alpha)).$$

Suppose that $N \geqslant 2p^2 r(\theta + r(ml + \alpha))$. Then we have

$$|\mu(K) - \rho(K)| \leqslant \frac{1}{2p^2}.$$

On the other hand, since $\dim K \leqslant p$ and $\mu(K) < 0$, we have

$$-\mu(K) = |\mu(K)| = \left| \frac{r}{p} - \frac{r(W_q)}{\dim K} \right| \geqslant \frac{1}{p^2}.$$

We have

$$-\rho(K) = -\mu(K) - (\rho(K) - \mu(K)).$$

Therefore we get

$$-\rho(K) \geqslant -\mu(K) - |\mu(K) - \rho(K)|$$

which gives

$$-\rho(K) \geqslant \frac{1}{p^2} - \frac{1}{2p^2} = \frac{1}{2p^2} > 0.$$

Thus we have proved that if $q \notin R^{ss}$ and $N \geqslant 2p^2(\theta + r(ml + \alpha))$, then there exists a proper linear subspace K of E such that $\rho(K) < 0$. This completes the proof of ((B)) and thus (3) of Lemma 2 is proved.

Let us now take in the above lemma $r = 2$, i.e. we consider semi-stable vector bundles V of rank 2 such that $\alpha = d(V)/r(V)$. Then Z^{ss} has a good quotient (modulo $SL(E)$) and the quotient is in fact a projective
370 variety (c.f. §2, Theorem 4). Since R^{ss} is a smooth variety, in particular normal, then by the properties of good quotients, it follows that R^{ss} has a good quotient $\phi : R^{ss} \to T$ modulo $GL(E)$ (equivalently $SL(E)$ or $PGL(E)$) such that T is projective. It is checked easily that R^s is *non-empty* and that the closures of the $GL(E)$-orbits through $q_1, q_2 \in R^{ss}$ intersect if and only if $\operatorname{gr} F_{q_1} = \operatorname{gr} F_{q_2}$. It follows then that T can be identified naturally with the classes of vector bundles in $\mathbf{S}(\alpha, 2)$ under

the equivalence relation $V_1, V_2 \in \mathbf{S}(\alpha, 2)$, $V_1 \sim V_2$ if and only if $\mathrm{gr}\, V_1 = \mathrm{gr}\, V_2$ and that $\dim T = (4g - 3)$. It can also be seen easily that T has a weak universal mapping property (*coarse moduli scheme* in the sense of Def. 5.6, Chap. 5, [4]). Thus we get the following

Theorem 5. *Let U_α be the equivalence classes of semi-stable X of rank 2 and degree 2α under the equivalence relation $V_1 \sim V_2$ if and only if $\mathrm{gr}\, V_1 = \mathrm{gr}\, V_2 (\alpha = 0$ or $\frac{1}{2})$. Then there exists a structure of a normal projective variety on U_α, uniquely determined by the following properties:*

(1) *given an algebraic family of vector bundles $\{V_t\}$, $t \in T$, of rank 2 and degree 2α on X, parametrized by an algebraic scheme T, the canonical mapping $T \to U_\alpha$, defined by $t \to \mathrm{gr}\, V_t$ is a morphism;*

(2) *given another structure U' on U_α having the property (1), the canonical mapping $U_\alpha \to U'$ is a morphism.*

Remark. It can be shown that U is *smooth* when $\alpha = \frac{1}{2}$.

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THE UNIPOTENT VARIETY OF A SEMISIMPLE GROUP

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IF G is a connected semisimple linear algebraic group over a field 373
of characteristic 0, one easily sees that the variety V of unipotent elements of V is isomorphic to the variety $\mathfrak{V}$ of nilpotent elements of its Lie algebra $\mathfrak{g}$; moreover one can choose the isomorphism so as to be compatible with the canonical actions of G on V and $\mathfrak{V}$. In this note the analogous situation in characteristic $p > 0$ will be discussed. We have to restrict p to be "good" for G (see 0.3). This is not so surprising, since in "bad" characteristics there are anomalies in the behaviour of unipotent elements.

Due to technical difficulties, we cannot prove the *isomorphism* of V and $\mathfrak{V}$ for $p > 0$, but only a slightly weaker result (Theorem 3.1). This is, however, sufficient for the applications which are discussed in §4.

0 Notations and recollections.

0.1 k denotes a field, $\bar{k}$ an algebraic closure of k and k_s a separable closure. p is the characteristic of k.

An algebraic variety V defined over k (or a k-variety) is a scheme which is of finite type and absolutely reduced over k. $V(k)$ denotes its set of k-rational points. We may and shall identify V with $V(\bar{k})$, or $V(k_s)$.

An algebraic group H defined over k (or a k-group) will mean here a *linear* algebraic group, i.e. an affine group scheme, of finite type and smooth over k. H^0 denotes the identity component of H.

The Lie algebra of an algebraic group H will be denoted by the corresponding German letter $\mathfrak{h}$. H acts on $\mathfrak{h}$ via the adjoint representation Ad. If $x \in H$, $Z(x)$ denotes the centralizer of x in H, $\mathfrak{z}(x) = \{X \in \mathfrak{h} | \mathbf{Ad}(x)X = X\}$ the centralizer of x in $\mathfrak{h}$. If $X \in \mathfrak{h}$, $Z(X) = \{x \in H | \mathbf{Ad}(x)X = X\}$ is the centralizer of X in H, $\mathfrak{z}(X)$ its centralizer in $\mathfrak{h}$ 374 and $N(X) = \{x \in H | \mathbf{Ad}(x)X \in \bar{k}^* X\}$ its normaliser in H.

0.2 G always denotes a connected semisimple linear algebraic group, defined over a field k.

Let T be a maximal torus of G, B a Borel subgroup containing T and U the unipotent part of B. Denote by Σ the set of roots of G with respect to T. B determines an order on Σ, let Σ^+ be the set of positive roots. We denote by $r_1, \dots, r_l$ the corresponding simple roots. For $r \in \Sigma$ there is an isomorphism x_r of the additive group $\mathbf{G}_a$ onto a closed subgroup G_r of G, such that

$$t.x_r(\xi).t^{-1} = x_r(t^r \xi) \quad (\xi \in \bar{k}),$$

where t^r denotes the value of the character r of T in $t \in T$. U is generated by the G_r with $r > 0$. G_r and G_{-r} generate a subgroup P_r which is connected semisimple of type $\mathbf{A}_1$. $X_r \in \mathfrak{g}$ will denote a nonzero tangent vector to G_r. We will say that G is simple if Σ is irreducible.

0.3 Let G be simple. Then there is a unique highest root r in Σ, for the given order. Express r as an integral linear combination of the r_i. The characteristic p of k is called *bad* for G, if p is a prime number dividing one of the coefficients in this expression. Otherwise p is called *good* for G. If p is good and if moreover p does not divide the order of the centre of the simply connected covering of G, then p is called *very good* for G. $p = 0$ is always very good. For the simple types, the bad $p > 0$ are:

$\mathbf{A}_l$: none; $\mathbf{B}_l$, $\mathbf{C}_l$, $\mathbf{D}_l$: $p = 2$; $\mathbf{E}_6$, $\mathbf{E}_7$, $\mathbf{F}_4$, $\mathbf{G}_2$: $p = 2, 3$; $\mathbf{E}_8$: $p = 2, 3, 5$. A good p is very good, unless G is of type $\mathbf{A}_l$ and p divides $l+1$. If G is arbitrary, p is defined to be good or very good for G if it is so far all simple normal subgroups of G.

0.4 $x \in G$ is called regular, if $\dim Z(x)$ equals rank G. We shall have to make extensive use of the properties of regular unipotent elements of G, which are established in [14], [15].

1 The unipotent variety of G. Let $V(G)$ (or V, if no confusion can arise) denote the set of unipotent elements of G. Then $V(G)$

is closed in G. We call $V(G)$ the *unipotent variety* of G. The following 375
result justifies this name.

Proposition 1.1. *$V(G)$ is an irreducible closed subvariety of G of dimension* $\dim G - \operatorname{rank} G$. *$G$ acts on $V(G)$ by inner automorphisms. $V(G)$ and the action of G on it are defined over k.*

Except for the last statement, this contained in ([14], 4.4, p. 131). We sketch the proof since we have to refer to it later on.

Suppose first that G is quasi-split over k. Let B be a Borel subgroup of G, which is defined over k. Then U is also defined over k. Consider the subset W of $G/B \times G$, consisting of the (gB, x) such that $g^{-1}xg \in U$. This is a closed subvariety of $G/B \times G$ ([5], exp. 6, p. 12, 1.13). It follows from loc. cit. that W is irreducible and defined over k. V is the projection of W onto the second factor of $G/B \times G$, let $\pi : W \to V$ be the corresponding morphism. That V has the asserted dimension is proved in ([14], loc. cit). G/B being a projective k-variety, π is proper, defined over k. V is closed in G and defined over k.

Let G act on $G/B \times G$ by $(h, (gB, x)) \to (hgB, hxh^{-1})$. This action is defined over k, W is stable and π is a G-equivariant k-morphism $W \mapsto V$, G acting on V as in the statement of the proposition.

If k is arbitrary, G splits over k_s (see [2], 8.3 for example). It follows that V is defined over k_s. Since V is clearly stable under $\Gamma = \mathrm{Gal}(k_s/k)$, it is defined over k. W is also defined over k, moreover if $s \in \Gamma$ there exists $g_s \in G(k_s)$ such that

$$^{s}B = g_s B g_s^{-1}, \; ^{s}U = g_s U g_s^{-1}.$$

Define a new action $(s, w) \mapsto {}_s w$ of Γ on W by ${}_s(gB, x) = ({}^s g \cdot g_s B, {}^s x)$. W is stable under this action of Γ, hence this defines a structure of k-variety on W, such that the projection $\pi : W \to V$ is a G-equivariant k-morphism.

If G and G' are two semisimple k-groups, and f a k-homomorphism, there exists an induced k-morphism $V(f) : V(G) \to V(G')$.

Proposition 1.2.

(i) *If f is a separable central isogeny, then $V(f)$ is an isomorphism, compatible with the actions of G and G',* 376 (ii) *$V(G \times G')$ is isomorphic to $V(G) \times V(G')$ as a k-variety, the isomorphism being compatible with the actions of G, G', $G \times G'$.*

The proof of Proposition 1.2 is easy.

The following results are essentially contained in [15].

Proposition 1.3.

(i) *$V(G)$ is nonsingular in codimension* 1*;*

(ii) *$V(G)$ is normal if G is simply connected, or if p does not divide the order of the centre of the simple connected covering of G.*

We may assume k to be algebraically closed. In V we have the open subvariety O of the regular elements. O is an orbit of G and all its elements are simple points of V (see [15], 1.2, 1.5 for these statements). (i) then follows from the fact, proved loc. cit. (6.11 e), that the irreducible components of $V - O$ have codimension $\geqslant 2$.

If G is simply connected, then by ([15], 6.1, 8.1) V is a complete intersection in G. The first statement in (ii) then follows from known normality criteria (e.g. [7], iv, 5.8.6, p. 108) and the second one is a consequence of 1.2 (i).

We now prove some properties of the proper k-morphism $\pi : W \to V$, introduced in the proof of 1.1.

Proposition 1.4. *If G is adjoint, then π is birational.*

Since a regular unipotent element is contained in exactly one Borel subgroup, π is bijective on $\pi^{-1}(O)$ (O denoting as before the variety of regular elements). 1.4 will then follow, if we show that π is separable (see e.g. [7], III, 4.3.7, p. 133). We may assume k to be algebraically closed. Let ϕ be the morphism $G \times U \to G$ such that $\phi(g, u) = gug^{-1}$. From the definition of π it follows that there exists a morphism $\psi : G \times U \to W$ such that $\phi = \pi \circ \psi$. To prove the separability of π, it suffices to prove that in some point $a \in G \times U$, the tangent map $(d\phi)_a : T(G \times U)_a \to T(G)_{\phi(a)}$ has image of dimension $\dim V$.

Let u be a regular unipotent element in U. We will take $a = (e, u)$. Identify the tangent space $T(G \times U)_a$ with $\mathfrak{g} \oplus \mathfrak{u}$ (via a right translation

with (e, u)), identify $T(G)_{\phi(a)}$ with $\mathfrak{g}$. Then $(d\phi)_a$ becomes the homomorphism $\alpha : \mathfrak{g} \oplus \mathfrak{u} \to \mathfrak{g}$, which sends (X, U) into $(1 - \mathbf{Ad}(u)^{-1})X + U$. **377**
It follows that $\alpha(\mathfrak{g} \oplus \mathfrak{u})$ contains $(\mathbf{Ad}(u) - 1)\mathfrak{g}$, which, by ([15], 4.3, p. 58) has dimension equal to $\dim V$. (Since G is adjoint, the $\mathfrak{z}$ of loc. cit. is now the zero element). This implies the result.

Remark. By dimensions one finds also that $\mathfrak{u} \subset (\mathbf{Ad}(u) - 1)\mathfrak{g}$.

Proposition 1.5. *The fibres of π are connected.*

We may assume k to be algebraically closed. In view of the definition of W this can be stated in another way, namely that, B denoting some Borel subgroup of G, the fixed point set in G/B of any unipotent element $g \in G$ is connected. Identifying G/B with the variety of Borel subgroups of G, 1.5 can also be interpreted as follows: the closed subvariety of G/B consisting of the Borel subgroups containing g, is connected. 1.5 follows, if G is simply connected, from 1.3 (ii) and 1.4 by Zariski's connectedness theorem (see [7], 4.3.7, p. 133). The general case then follows at once, since central isogenies do not affect the statement.

Another proof of 1.5 was given by J. Tits. Since his method of proof will be useful in §2, we will reproduce his proof here. We interpret G/B as the variety of Borel subgroups of G. Let $g \in B$ be a unipotent element of G, let B' be another Borel subgroup containing g. By Bruhat's lemma, $B \cap B'$ contains a maximal torus T. Let N be its normalizer and $\mathscr{W} = N/T$ be the Weyl group. For $w \in \mathscr{W}$, denote by n_w a representative in N. There exists then $w \in \mathscr{W}$ such that $B' = n_w B n_w^{-1}$. B determines an order on the root system Σ, let $w_1, \ldots, w_l$ ($l = \operatorname{rank} G$) be the reflections in $\mathscr{W}$ defined by the corresponding simple roots. Since the w_i generate $\mathscr{W}$, we can write w as a product $w = w_{i_1} \ldots w_{i_t}$. We take t as small as possible. Put $v_0 = 1$, $v_h = w_{i_1} \ldots w_{i_h} (1 \leqslant h \leqslant t)$, $B_h = n_{v_h} B n_{v_h}^{-1}$, so that $B_0 = B$, $B_t = B'$. Let Σ_h denote the set of $r \in \Sigma$ such that $r > 0$, $v_h r < 0$. It is known that the minimality of t implies that $\Sigma_h \subset \Sigma_{h+1}$ (this follows e.g. from [5], p. 14-06, lemma). The intersection $B_0 \cap B_h$ is generated by T and the subgroups G_r (see 0.2) with $r \notin \Sigma_h$ ([5], exp. 13, No. 2). Hence $B_h \supset B_0 \cap B_t = B \cap B'$, in particular g belongs to all B_h. Let $X \subset G/B$

378 be the variety of Borel subgroups containing g. It suffices to show that we can connect B_h and B_{h+1} inside X by a projective line.

Put $u = v_h w_{h+1} v_h^{-1}$, then $B_{h+1} = n_u B_h n_u^{-1}$, moreover u is a reflection in a simple root for the order on Σ determined by B_h. Changing the notation, we are reduced to proving that B and B' are connected inside X by a projective line, if w is a reflection in a *simple* root $r > 0$. Then let P_r be the subgroup of G generated by G_r and G_{-r}. P_r is of type $\mathbf{A}_1$ and we may take $n_w \in P_r$. One easily checks that $hBh^{-1} \supset B \cap B'$ for all $h \in P_r$. $P_r \cap B$ is a Borel subgroup of P_r. Let $\psi : P_r/P_r \cap B \to G/B$ be the canonical immersion. $L = P_r/P_r \cap B$ is a projective line and $\psi(L)$ contains both B and B'. This establishes our assertion.

2 The nilpotent variety of G.

We discuss now the Lie algebra analogues of the results of §1. We recall that an element $X \in \mathfrak{g}$ is called *nilpotent* if it is tangent to a unipotent subgroup of G. Equivalently, X is nilpotent if it is represented by a nilpotent matrix in any matrix realization of G (see [1], §1, pp. 26-27). Let $\mathfrak{V}(G)$ (or $\mathfrak{V}$) denote the set of nilpotent elements in $\mathfrak{g}$, $\mathfrak{g}$ being endowed with the obvious structure of affine space over k. Then V is a closed subset of $\mathfrak{g}$.

Proposition 2.1. $\mathfrak{V}(G)$ *is an irreducible closed subvariety of* $\mathfrak{g}$ *of dimension* $\dim G - \operatorname{rank} G$. *$G$ acts on* $\mathfrak{V}(G)$ *via the adjoint representation of G.* $\mathfrak{V}(G)$ *and the action of G on it are defined over k.*

The proof is similar to that of 1.1. First let G be quasi-split over k. We use the notations of the proof of 1.1. Instead of W, we consider now the closed subvariety $\mathfrak{W}$ of $G/B \times \mathfrak{g}$, consisting of the (gB, X) such that $\mathbf{Ad}(g)^{-1}X \in \mathfrak{u}$. G acts on $\mathfrak{W}$ by

$$(h, (gB, X)) \mapsto (hgB, \mathbf{Ad}(h)X).$$

The projection of $G/B \times \mathfrak{g}$ onto its second factor induces a G-equivariant proper morphism $\tau : \mathfrak{W} \to \mathfrak{V}$. The argument parallels now that of the proof of 1.1 (see [2], §2, where a similar situation is discussed).

379 If $f : G \to G'$ is a k-homomorphism of semisimple k-groups, there exists an induced k-morphism $\mathfrak{V}(f) : \mathfrak{V}(G) \to \mathfrak{V}(G')$ (by [2], 3.1).

Proposition 2.2.

(i) *If f is a separable isogeny, then $\mathfrak{B}(f)$ is an isomorphism, compatible with the actions of G and G';*

(ii) *$\mathfrak{B}(G \times G')$ is isomorphic to $\mathfrak{B}(G) \times \mathfrak{B}(G')$ as a k-variety, the isomorphism being compatible with the actions of the G, G', $G \times G'$.*

The proof is left to the reader.

1.3 can only be partially extended to $\mathfrak{B}$.

Proposition 2.3. *Let p be good for G. Then $\mathfrak{B}(G)$ is nonsingular in codimension* 1.

This will not be needed, so we only indicate briefly how this can be proved. If p is good, there exist in $\mathfrak{g}$ regular nilpotent elements (by [14], 5.9 b, p. 138, this is also a consequence of [10], 5.3, p.8). The orbit $\mathfrak{O}$ in $\mathfrak{B}$ of such an element is open and consists of nonsingular points. One then uses the method of [15] to prove that all irreducible components of $\mathfrak{B} - \mathfrak{O}$ have codimensions at least 2.

It is likely that $\mathfrak{B}(G)$ is normal if G is simply connected and p is good. However we are not able to prove this. A proof along the lines of that of 1.3 would require the analogue of 1.3 (ii). In characteristic 0 this is a result of Kostant ([9]). For a proof of the corresponding fact in positive characteristics it seems that one needs detailed information about the ring of G-invariant polynomial functions on $\mathfrak{g}$.

If the normality of $\mathfrak{B}$ were known. Theorem 3.1 could be ameliorated and its proof could be simplified.

Proposition 2.4. *Suppose that p is very good for G. Then $\tau : \mathfrak{W} \to \mathfrak{B}$ is birational.*

The proof is similar to that of 1.4. Instead of results on regular unipotent elements, one now uses those on regular nilpotent elements of $\mathfrak{g}$, which are discussed in ([14], 4, p. 138).

Proposition 2.5. *The fibres of* $\tau : \mathfrak{W} \to \mathfrak{B}$ *are connected.*

380 A proof based on Zariski's connectedness theorem, as in 1.5, cannot be given here since we cannot use normality of $\mathfrak{B}$. But Tits' proof works in the case of $\mathfrak{B}$ and carries over with some obvious modifications.

3 Relation between V and $\mathfrak{B}$.

In this number we shall prove the following theorem.

Theorem 3.1. *Suppose that* G *is simply connected and that* p *is good for* G. *Then there exists a* G*-equivariant* k*-morphism* $f : V \to \mathfrak{B}$, *which induces a homeomorphism* $V(\bar{k}) \to \mathfrak{B}(\bar{k})$.

The normality of $\mathfrak{B}$ would imply that f is an isomorphism. However 3.1 is already sufficient for the applications we want to make. In characteristic 0, one easily gives a proof of 3.1, using the logarithm in some matrix realization of G. For the proof we need a number of auxiliary results. The first three give some rationality results on regular unipotents and nilpotents.

Proposition 3.2. *Suppose that* G *is adjoint and that* p *is very good for* G. *Let* X *be a regular nilpotent element of* $\mathfrak{g}(k)$. *Then its centralizer* $Z(X)$ *is connected, defined over* k *and is a* k*-split unipotent group.*

Recall that a connected unipotent k-group is called k-split, if there exists a composition series of connected k-groups, such that the successive quotients are k-isomorphic to $\mathbf{G}_a$ (see [11], p. 97). That $Z(X)$ is connected unipotent is proved in ([14], 5.9b, p. 138). Len $N(X)$ be the normalizer of X in G (see 0.1). Under our assumptions, $N(X)$ is also defined over k ([10], 6.7, p. 11). Moreover, $N(X)$ is connected. In fact, if S is a maximal torus of its identity component $N(X)^0$, then for any $g \in N(X)$, there exists $s \in S$ such that $\mathbf{Ad}(g)X = \mathbf{Ad}(s)X$, whence $N(X) \subset S \cdot Z(X) \subset N(X)^0$, since $Z(X)$ is connected.

Now $N(X)$ contains a maximal torus S which is defined over k (by a theorem of Rosenlicht-Grothendieck, see [1]). Define a character a of S by $\mathbf{Ad}(s)X = s^a X$. Then a is clearly defined over k; moreover since $Z(X)$ is unipotent and since $N(X)/Z(X)$ has dimension 1, we have
381 that $\dim S = 1$. It follows that S is k-split. S acts on $Z(X)$ by inner

automorphisms. We claim that S acts without fixed points. To prove this, we may assume k algebraically closed, moreover it suffices to prove that S acts without fixed points on the Lie algebra $\mathfrak{z}$ of $Z(X)$ ([3], 10.1, p. 127).

Since all regular nilpotents are conjugate ([14], 5.9c, p. 130), it suffices to prove the assertion for a particular X. We may take, with the notations of 0.2, $X = \sum_{i=1}^{l} X_{r_i}$ (loc. cit. p. 138). Then one may take for S the subtorus of T, which is the identity component of the intersection of the kernels of all $r_i - r_j$. This S acts without fixed points on $\mathfrak{u}$, hence also on $\mathfrak{z}$, since $\mathfrak{z} \subset \mathfrak{u}$ ([14], 5.3, p. 138). By the conjugacy of maximal tori, the assertion now follows for an arbitrary maximal torus of $N(X)$.

S acting without fixed points on $Z(X)$, it follows that $Z(X)$ is k-split ([2], 9.12). This concludes the proof of 3.2.

The following result generalizes ([13], 4.14, p. 135).

Corollary 3.3. *Under the assumptions of 3.2, let Y be another regular unipotent in $\mathfrak{g}(k)$. Then there exists $g \in G(k)$ such that $Y = \mathbf{Ad}(g)X$.*

Let $P = \{g \in G | \mathbf{Ad}(g)X = Y\}$. P is defined over k. This is proved in the same way as the fact that $Z(X)$ is defined over k ([10], 6.7, p. 11, see [2], 6.13 for a similar situation). P is a principal homogeneous space of the k-split unipotent group $Z(X)$, hence P has a k-rational point g ([13], III-8, Prop. 6), which has the required property.

Proposition 3.4. *Suppose that G is quasi-split over k. Then*

(i) *$G(k)$ contains a regular unipotent element;*

(ii) *if p is good, $\mathfrak{g}(k)$ contains a regular nilpotent element.*

Replacing G by its simply connected covering (which is defined over k) we may assume G to be simply connected. Then we also may assume G to be simple over k and even absolutely simple ([18], 3.1.2, p. 6).

(i) First assume G is not of type $\mathbf{A}_l$ (l even). Let B be a Borel subgroup of G which is defined over k. With the notations of 0.2, we take $x = \prod_{i=1}^{l} x_{r_i}(\xi_i)$, where the order of the product and the **382**

$\xi_i \in k_s^*$ are chosen such that $x \in G(k)$. This is possible, see ([15], proof of 9.4, p. 72 for a similar situation). If G is of type $\mathbf{A}_l$ (l even) a slightly different argument is needed, similar to the one of (loc. cit. 9.11, p. 74). One could also prove (i) in that case by an explicit check in case G is a special unitary group.

The proof of (ii) is similar (but simpler). Take as regular nilpotent $X = \sum_{i=1}^{l} \xi_i X_{r_i}$, with suitable $\xi_i \in k_s^*$. X is regular by ([14], p. 138).

In the next result we shall be dealing with the unipotent part U of a Borel subgroup of a k-split G, and with its Lie algebra. Notations being as in 0.2, we have the following formula,

$$x_r(\xi)x_s(\eta)x_r(\xi)^{-1}x_s(\eta)^{-1} = \prod_{i,j>0} x_{ir+js}(C_{ijrs}\xi^i\eta^j)(\xi,\eta \in k), \quad (19.1)$$

where r, $s \in \Sigma$, $r + s \neq 0$. The product is taken over the integral linear combinations of r, s which are in Σ, and the C_{ijrs} are integers. We presuppose a labelling of the roots in taking the product, the labelling being such that the roots with lower height come first. The height of a positive root $r = \sum_{i=1}^{l} n_i(r)r_i$ is defined as $h(r) = \sum_{i=1} n_i(r)$. Now (19.1) shows that there exists a groupscheme U_0 over $\mathbf{Z}$, such that $U = U_0 \underset{\mathbf{Z}}{\times} k$. The same is true for B, so that $B = B_0 \underset{\mathbf{Z}}{\times} k$. U_0 is isomorphic, as a scheme, to an affine space over $\mathbf{Z}$, B_0 is isomorphic as a scheme to $U_0 \times \mathbf{G}_m^1$. B_0 acts on U_0.

For simplicity, we shall identify $U_0(B_0)$ here with the sets $U_0(K)$ ($B_0(K)$) of points with values in some algebraically closed filed $K \supset \mathbf{Z}$, likewise for $\mathbf{G}_a$.

Let s be the product of the bad primes for G, let $R = \mathbf{Z}_s$ be the ring of fractions $n/s^k (n \in \mathbf{Z})$. Put $U_1 = U_0 \underset{\mathbf{Z}}{\times} R$, $B_1 = U_0 \underset{\mathbf{Z}}{\times} R$. The homomorphism $x_r : G_a \to U$ comes from a homomorphism of group
383 schemes over $\mathbf{Z} : \mathbf{G}_a \to U_0$, which leads to a homomorphism over $R : \mathbf{G}_a \to U_l$. The latter one will also be denoted by x_r, the image of x_r is also denoted by G_r. Let $\mathfrak{u}_1$ be the Lie algebra of U_1. It is a

free R-module, having a basis consisting of elements tangent to the G_r. We denote these basis elements by X_r, as in 0.2. We endow $\mathfrak{u}_1$ with its canonical structure of affine space over R.

After these preparations, we can state the next result.

Proposition 3.5. *There exists a B_1-equivariant isomorphism of R-schemes $\phi : U_1 \to \mathfrak{u}_1$.*

This is proved by exploiting the argument of ([14], pp. 133-134) used to determine the centralizer of regular unipotent and nilpotent elements in good characteristics.

Define $v \in U_1(R)$ by $v = \prod_{i=1}^{l} x_{r_i}(1)$. Then the argument of loc. cit. extends to the present case and shows that the centralizer Z of v in U_l is a closed sub-groupscheme of U_1, isomorphic as a scheme to 1-dimensional affine space over R. Moreover, since $Z \underset{R}{\times} K$ is commutative ([8], 5.8, p. 1003) it follows that Z is commutative.

We claim that there is a homomorphism of R-groupschemes $\psi : \mathbf{G}_a \to Z$, such that $\psi(\xi) = \prod_r x_r(F_r(\xi))(\xi \in l)$, where F_r is a polynomial in $R[T]$ such that $F_{r_i} = T(1 \leqslant i \leqslant l)$. This can be proved by the method of ([14], pp. 133-134), defining F_r by induction on the height of r. It follows, that the Lie algebra $\mathfrak{z}$ of Z contains an element of the form $X = \sum_{r>0} \xi_r X_r$, with $\xi_r \in R$ and $\xi_{r_i} = 1(1 \leqslant i \leqslant l)$. Since X is in the Lie algebra of the commutative group scheme Z, we have $\mathbf{Ad}(Z)X = X$.

But the same which has been said above about the centralizer of v applies to the centralizer Z_1 of X: this is also a closed sub-group scheme of U_1, isomorphic to l-dimensional affine space over R (since the argument of [14] applies also to nilpotent elements like X).

Z_1 and Z have the same Krull dimension $l + 1$. But since Z_1 is a closed subscheme of Z_1, we must have $Z = Z_1$. Let F be the closed subscheme of U_1 consisting of the $\prod_{r>0} x_r(\xi_r)$ such that $\xi_{r_i} = 1$ $(1 \leqslant i \leqslant$
$l)$. Using again the method of ([14], p. 133) one defines a morphism **384**
$\chi : F \to U_1$, such that $\chi(x)v\chi(x)^{-1} = x(x \in F)$.

Let O be the open subscheme of U_1 consisting of the $\prod_{r>0} x_r(\xi_r)$ such that $\xi_{r_i} \neq 0 (1 \leqslant i \leqslant l)$. χ is easily seen to extend to a morphism $\chi : O \to B_1$ such that $\chi(x)v\chi(x)^{-1} = x (x \in O)$. It follows that O is the orbit of v under B_1. Define a morphism $\phi : O \to \mathfrak{u}_1$ by

$$\phi(bvb^{-1}) = \mathbf{Ad}(b)X \quad (b \in B_1). \tag{19.2}$$

From the preceding remarks it follows that ϕ is well-defined, moreover $\phi(\prod_{r>0} x_r(\xi_r))$ is a polynomial function in $\xi_r, \xi_{r_i}^{-1} (1 \leqslant i \leqslant l)$.

We want to show that ϕ extends to a morphism $\phi : U_1 \to \mathfrak{u}_1$, satisfying (19.2). Now there is a $B_1 \underset{R}{\times} K$-equivariant K-morphism ϕ_1 : $U_1 \underset{R}{\times} K \to u_1 \underset{R}{\times} K$ given by the logarithm is a suitable matrix realization of the algebraic group $U_1 \underset{R}{\times} K$. $\phi \underset{R}{\times}$ id extends to a $B_1 \times K$-equivariant K-morphism of an open set of $U_1 \times K$ which contains v into $\mathfrak{u}_1 \times K$.

We have that $\phi_1(v)$ and $\phi \times \mathrm{id}(v)$ are conjugate in $U_1(K)$ (by [14], 5.3, 5.9 c pp. 137-138). But since ϕ_1 is completely determined by $\phi_1(v)$, we have that $\phi \times \mathrm{id} = \mathbf{Ad}(b) \circ \phi_1$, for suitable $b \in B_1(k)$. It follows that $\phi \times$ id can be extended to all of $U_1 \times K$. Hence ϕ can be extended to a morphism $U_1 \to \mathfrak{u}_1$, as desired.

So we have a B_1-equivariant R-morphism $\phi : U_1 \to \mathfrak{u}_1$, with $\phi(v) = X$. Reversing the roles of U_1 and $\mathfrak{u}_1$, one gets in the same manner a B_1-equivariant R-morphism $\phi' : \mathfrak{u}_1 \to U_1$ such that $\phi'(X) = v$. But then ϕ and ϕ' are inverses, so that is an isomorphism. This concludes the proof of 3.5.

Remark. The analogue of 3.5, with R replaced by $\mathbf{Z}$ and U_1 by U_0, $\mathfrak{u}_1$ by $\mathfrak{u}_0$, is false. In fact, this would imply that, over any field, the centralizer of a regular unipotent element would be connected. This is not true in bad characteristics ([14], 4.12, p. 134).

We can now prove 3.1. First let G be split over k. We use the notations of 0.2. From 3.5 we get a B-equivariant k-isomorphism $\lambda : U \to \mathfrak{u}$. Let $W, \pi; \mathfrak{W}, \tau$ be as in §§1 and 2. Then

$$\theta : (gB, x) \to (gB, \mathbf{Ad}(g)\lambda(g^{-1}xg))$$

385 defines a G-equivariant k-isomorphism of W onto $\mathfrak{W}$. Denote by $\mathcal{O}_V$ the sheaf of local rings on V. Since π and τ are proper, we can apply Grothendieck's connectedness theorem ([7], III, 4.3.1, p. 130). Using 1.3(ii) and 1.4 we find that the direct image $\pi_*(\mathcal{O}_W) = \mathcal{O}_V$. Let $\mathfrak{W} \xrightarrow{\tau_1} \mathfrak{V}' \xrightarrow{\tau_2} \mathfrak{V}$ be the Stein factorization of τ (loc.cit. p.131). Then $(\tau_1)_*(\mathcal{O}_{\mathfrak{W}}) = \mathcal{O}_{\mathfrak{V}'}$ and τ'_2 is finite. The definition of $\mathfrak{V}'$ ([7], III, p. 131) shows that G acts on it, and that τ_1, τ_2 are G-equivariant. V and $\mathfrak{V}$ are affine varieties (1.1 and 2.1). Also, since $\mathfrak{V}'$ is finite over $\mathfrak{V}$, $\mathfrak{V}'$ is affine ([7], III, 4.4.2, p. 136). It follows then from the definition of direct image that the ring of sections $\Gamma(W, \mathcal{O}_W)$ is isomorphic to $\Gamma(V, \mathcal{O}_V)$, likewise that $\Gamma(\mathfrak{W}, \mathcal{O}_{\mathfrak{W}})$ is isomorphic to $\Gamma(\mathfrak{V}, \mathcal{O}_{\mathfrak{V}'})$, these isomorphisms being compatible with the canonical actions of G. This is obvious in the first case, and in the second case it follows again from the definition of $\mathfrak{V}'$. But V and $\mathfrak{V}'$ being affine. $\Gamma(V, \mathcal{O}_V)$ and $\Gamma(\mathfrak{V}', \mathcal{O}_{\mathfrak{V}'})$ determine V and $\mathfrak{V}'$ completely. Also, W and $\mathfrak{W}$ are isomorphic via θ. Putting this together, we get a G-equivariant k-isomorphism $\mu : V \to \mathfrak{V}'$.

By ([7], III, 4.3.3, p.131), for any $x \in \mathfrak{V}$, the number of connected components of $\tau^{-1}(x)$ equals the number of points of $(\tau_2)^{-1}(y)$. By 2.5 this implies that τ_2 is bijective on $\mathfrak{V}'(\bar{k})$. Then $f = \tau_2 \circ \mu$ satisfies the requirements of 3.1.

Notice that f is not unique, but is completely determined by $f(v)$, where v is a given regular unipotent element. We now turn to the case that G is arbitrary, not necessarily split over k. G being simply connected, we may as well suppose that G is absolutely simple ([10], 3.1.2, p. 46). We first dispose of the case that G is of type $\mathbf{A}$. Then G is a k-form of $\mathbf{SL}_n$. Now there is, in the case of the split group of type $\mathbf{SL}_n$, a very simple argument to prove 3.1. Identifying in that case G and $\mathfrak{g}$ with subsets of a matrix algebra, V becomes the set of unipotent matrices, V that of nilpotent matrices and we can take $f(v) = v - 1$.

If we have another k-form G of $\mathbf{SL}_n$, then it is obtained from $\mathbf{SL}_n$ by **386**
a twist using a cohomology class in $H^1(k, \mathrm{Aut}\,\mathbf{SL}_n)$. The corresponding form $\mathfrak{g}$ is obtained from $\mathfrak{sl}_n$ by the same twist. The above f then clearly induces an isomorphism $V \to \mathfrak{V}$ having the required properties.

We may now assume G to be absolutely simple, but not of type $\mathbf{A}$. Then if p is good for G, it is also very good. Suppose that G is quasi-split over k. Let $g \in G(k)$ be regular unipotent, let $X \in \mathfrak{g}(k)$ be regular nilpotent (they exist by 3.4). Since G splits over k_s, we have, by the first part of the proof, a G-equivariant k_s-isomorphism $f : V \to \mathfrak{B}$. By 3.3 there exists $h \in G(k_s)$ such that $\mathbf{Ad}(h)f'(v) = X$. But then $f = f' \circ \mathbf{Ad}(h)$ is G-equivariant, defined over k_s and satisfies ${}^s f = f$ for all $s \in \mathrm{Gal}(k_s/k)$. Hence f is defined over k. Finally, an arbitrary k-form G is obtained by an inner twist from a quasi-split k-form G (this is implicit, for example, in [18], 3). Let f_1 have the required properties for G_1. One easily checks then that f_1 determines an f having the properties of 3.1. This concludes the proof of 3.1.

Corollary 3.6. *Suppose that G is adjoint and that p is very good for G. Let x be a regular unipotent element of $G(k)$. Then its centralizer $Z(x)$ is connected, defined over k and is a k-split unipotent group.*

Let f be as in 3.1. Then $X = f(x)$ is a regular nilpotent element in $\mathfrak{g}$. We have $Z(x) = Z(X)$. The assertion then follows from 3.2.

Corollary 3.7. *Under the assumptions of 3.6, let y be another regular unipotent element in $G(k)$. Then there exists $g \in G(k)$ such that $y = gxg^{-1}$.*

The proof is similar to that of 3.3.

Remark 3.8. The condition in 3.6 and 3.7 that p be a very good prime cannot be relaxed. As an example, consider the case where $G = \mathbf{PSL}_2$ and where k is a non-perfect field of characteristic 2. The ring of regular functions of $\mathbf{SL}_2$ being identified to $A = k[X, Y, Z, U]/(XU - YZ - 1)$, that of G is isomorphic to the subring of A generated by the products of
387 an even number of variables. Hence one can identify $\mathbf{PSL}_2(k)$ with the subgroup of $\mathbf{SL}_2(\bar{k})$, consisting of the matrices a such that $\rho a \in GL_2(k)$ for some $\rho \in \bar{k}$ with $\rho^2 \in k^*$.

In our situation, let $\rho \in \bar{k}$, $\rho^2 \in k^*$. Then $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & \rho \\ \rho^{-1} & 0 \end{pmatrix}$ are both regular unipotents in $\mathbf{PSL}_2(k)$, but it is easily checked that they are not conjugate by an element of $\mathbf{PSL}_2(k)$. On the other hand, if k is perfect, 3.6 and 3.7 are already true if p is good. But if p is bad, both 3.6 and 3.7 are false (see [14], 4.14, p. 135 for the first statement and

4.12, p. 134, 4. 15c, p. 136 for the others).

Corollary 3.9. *With the notations of 3.1, we have $f(e) = O$.*

For e is the only unipotent element of G in the centre of G and O is the only nilpotent element of $\mathfrak{g}$ invariant under $\mathbf{Ad}(G)$ (the last assertion follows for example, by using the fact that any nilpotent element is contained in the Lie algebra of a Borel subgroup).

4 Applications.

First we give some applications of 3.1 to rationality problems.

Proposition 4.1. *Let k be a finite field with q elements. Suppose that G is simply connected and that p is good for G. Then the number of nilpotent elements in $\mathfrak{g}(k)$ is $q^{\dim G - \operatorname{rank} G}$.*

Steinberg has proved that the number of unipotent elements in $G(k)$ is $q^{\dim G - \operatorname{rank} G}$ ([16], 15.3, p. 98). The assertion then follows from 3.1.

Proposition 4.2. *Suppose that p is very good for G. Then the following conditions are equivalent:*

(i) *G is anisotropic;*

(ii) *$G(k)$ does not contain unipotent elements $\neq e$.*

We recall that G is called anisotropic, if G does not contain a non-trivial k-subtorus S, which is k-split, i.e. k-isomorphic to a product of multiplicative groups.

If $G(k)$ contains a unipotent $\neq e$, then 3.1 and 3.8 imply that $\mathfrak{g}(k)$ contains a nilpotent $\neq O$. One then argues as in ([10], 6.8, p. 11) to show that G contains a k-split sub-torus S. Hence (i) $\Rightarrow$ (ii). Conversely, if **388**
G contains such a subtorus S, then G has a proper parabolic k-subgroup ([3], 4.17, p. 92). Its unipotent radical R is a k-split unipotent group ([3], 3.18, p. 82) and it follows that $R(k) \neq \{e\}$, so that G has a rational unipotent $\neq e$.

For perfect k and good p, 4.2 was proved in ([10], 6.3, p. 10). More general results were announced in [17].

Proposition 4.3. *Suppose that p is good for G. For any $g \in G$,* $\dim G - \dim Z(g)$ *is even.*

This was conjectured for arbitrary p in ([15], 3.10, p. 56) and is known to be true in characteristic 0 ([9], Prop. 15, p. 364).

We may assume that k is algebraically closed and G simply connected. Let $g = g_s g_u$ be the decomposition of G into its semisimple and unipotent parts. Since $Z(g_s)$ is reductive and of the same rank on G, it follows readily that $\dim G - \dim Z(g_s)$ is even. Because $Z(g)$ is the centralizer of g_u in $Z(g_s)$, it follows that it suffices to consider the case that g is unipotent. Then 3.1 implies, that 4.2 is equivalent to the assertion that $\dim G - \dim Z(X)$ is even for any nilpotent $X \in \mathfrak{g}$. But $\dim Z(X)$ equals the dimension of the Lie algebra centralizer $\mathfrak{z}(X)$ of X ([10], 6.6, p. 11). So we have to prove that $\dim \mathfrak{g} - \dim \mathfrak{z}(X)$ is even if X is nilpotent in g. This we do by an adaptation of the method used in characteristic 0 ([9], loc. cit.), even for arbitrary X. We use the following lemma.

Lemma 4.4.

(i) *Suppose G is simple, not of type $\mathbf{A}_n$. If p is good for G, there exists a nondegenerate, symmetric bilinear form F on $\mathfrak{g}$, which is invariant under* $\mathbf{Ad}(G)$.

(ii) *There exists a nondegenerate symmetric bilinear form F on $\mathfrak{gl}_l(k)$, which is invariant under* $\mathbf{Ad}GL_l(k)$.

Proof of 4.4 (i) If G is of type $\mathbf{B}_l$, $\mathbf{C}_l$, $\mathbf{D}_l$, then $p \neq 2$ and we can represent G as a group of orthogonal or symplectic matrices in a vector space A, and $\mathfrak{g}$ by a Lie algebra of skewsymmetric linear transformations with respect to the corresponding symmetric or skewsymmetric bilinear form on A. $F(X, Y) = \mathrm{Tr}(XY)$ satisfies then our conditions.

389 If G is of type $\mathbf{E}_6$, $\mathbf{E}_7$, $\mathbf{E}_8$, $\mathbf{F}_4$, $\mathbf{G}_2$, the Killing form on $\mathfrak{g}$ is nondegenerate if p is good ([12], p. 551) and can be taken as our F.

(ii) $F(X, Y) = \mathrm{Tr}(XY)$ satisfies our conditions.

To finish the proof of 4.3, we can assume G to be simple. Let $X \in \mathfrak{g}$. First if G is not of type $\mathbf{A}_l$, we let F be as in 4.4 (i). consider the skewsymmetric bilinear form F_1 on $\mathfrak{g}$ defined by $F_1(Y,Z) = F([XY],Z)$. Then $\mathfrak{z}(X) = \{Y \in \mathfrak{g} | F_1(Y,Z) = 0 \text{ for all } Y \in \mathfrak{g}\}$. Since the rank of F_1 is even, $\dim \mathfrak{g} - \dim \mathfrak{z}(X)$ is even, which is what we wanted to prove. If G is of type $\mathbf{A}_l$, we apply the same argument, however not for $\mathfrak{sl}_{l+1}$ but for $\mathfrak{gl}_{l+1}$, using 4.3 (ii).

Proposition 4.5. *Suppose that G is adjoint and that p is good for G. Let g be a unipotent element of G. Then g lies in the identity component $Z(g)^0$ of its centralizer $Z(g)$.*

Let $f : V \to \mathfrak{B}$ be as in 3.1. Put $X = f(g)$, let $A = f^{-1}(\bar{k}X)$. Since f is a homeomorphism, this is a closed connected subset of V, containing (by 3.8) e and g. Moreover, since $Z(X) = Z(g)$, we find from the G-equivariance of f that $A \subset Z(g)$. It follows that $g \in Z(g)^0$.

Remark. In bad characteristics the assertion of 4.5 is not true ([14], 4.12, p. 134).

The number of unipotent conjugacy classes in G (resp. of nilpotent conjugacy classes in $\mathfrak{g}$) has been proved to be finite in good characteristics by Richardson ([10], 5.2, 5.3, p. 8). By 3.1, these two numbers are equal. In characteristic 0, there is a bijection of the set of unipotent conjugacy classes in G onto the set of conjugacy classes (under inner automorphisms) of 3-dimensional simple subgroups of G (see e.g. [8], 3.7, p. 988). Representatitives for the classes of such subgroups are known (see [6]). In characteristic $p > 0$ it is not advisable to work with 3-dimensional subgroups, one has to deal then directly with the unipotent elements. We will discuss this in another paper. Here we only want to point out one consequence of 3.1. We define a unipotent element $g \in G$ to be *semi-regular* if its centralizer $Z(g)$ is the product of the center of G with a unipotent group. Regular unipotent elements are semi-regular (as **390**
follows from [15], 3.1, 3.2, 3.3, pp. 54-55). The converse, however, fails already in characteristic 0. In that case, it has been proved by Dynkin ([6], 9.2, p. 169 and 9.3, p. 170) that for G simple semi-regular implies regular if and only if G is of type $\mathbf{A}_l$, $\mathbf{B}_l$, $\mathbf{C}_l$, $\mathbf{F}_4$, $\mathbf{G}_2$. The result we want to prove is the following one, which extends ([14], 4.11, p. 134).

Proposition 4.6. *Suppose that G is adjoint and that p is good for G. Let g be a semi-regular unipotent element of G. Then the centralizer $Z(g)$ is connected.*

By 3.1, $Z(g) \subset V$ is homeomorphic to the variety A of fixed points of $\mathbf{Ad}(g)$ in $\mathfrak{B}$. We claim that A is the set of fixed points of $\mathbf{Ad}(g)$ in the whole of $\mathfrak{g}$. For let $X \in \mathfrak{g}$, $\mathbf{Ad}(g)X = X$. Let $X = X_s + X_n$ be the Jordan decomposition of X([1], 1.3, p. 27), then $\mathbf{Ad}(g)X_s = X_s$. But this means that X_s in the Lie algebra of $Z(g)$ ([10], 6.6, p. 11). $Z(g)^0$ being unipotent, it follows that $X_s = 0$. This establishes our claim. It follows that A is a linear subspace of $\mathfrak{g}$, so that it is connected, hence so is $Z(g)$.

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BASE CHANGE FOR TWISTED INVERSE IMAGE OF COHERENT SHEAVES

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1 Existence Theorem. Let X/k be a complete smooth algebraic variety of dimension n over a field k and ω_X be the sheaf of differentials of degree n on X. There exists a canonical morphism:

$$\int\limits_{X/k} : H^n(X, \omega_X) \to k,$$

such that for any quasi-coherent sheaf on X, the induced map:

$$\mathrm{Ext}^{n-p}(X; F, \omega_X) \to H^p(X, F)^*$$

(*means dual over k) is an isomorphism for all p.

Let $X \to Y$ be an immersion of schemes which is regular, i.e. defined locally by a regular sequence of n parameters. Let $\mathbf{I}$ be the sheaf of ideals on Y defining X and $N_{X/Y} = (\overset{n}{\wedge}\mathbf{I}/\mathbf{I}^2)^{-1}$ the inverse of the highest exterior power of the cotangent sheaf. For any quasi-coherent sheaf F on X and any quasi-coherent sheaf G on Y, there exist canonical isomorphisms:

$$\mathrm{Ext}^{p-n}(X; F \underset{O_Y}{\otimes} N_{X/Y}) \simeq \mathrm{Ext}^p(Y; F, G),$$

for all p.

These two results are special cases of Grothendieck duality theory developed by Hartshorne in [1].

We use freely the notation of [1] and unless otherwise stated, the terminology of [1]. The general duality theorem can be summarized as follows.

Theorem 1 (Existence Theorem). *Let $f : X \to Y$ be a proper morphism of noetherian schemes of finite Krull dimension. There exists an exact functor*

$$f^! : D^+_{qc}(Y) \to D^+_{qc}(X)$$

and a morphism of functors

$$\int_f : Rf_* f^! \to \mathrm{id}$$

(denoted by Tr_f *in [1]) such that for any* $F \in D_{qc}(X)$ *and any* $G \in$ **394**
$D^+_{qc}(Y)$ *the morphism induced by* $\int$ *:*

$$\mathrm{Ext}^p(X; F, f'(G) \to \mathrm{Ext}^p(Y : Rf_*F, G)$$

are isomorphisms for all p.

It should be noted that the pair consisting of the functor $f^!$ and the morphism of functors $\int_f$ is unique up to unique isomorphism. An immediate consequence of the existence theorem is that if $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$ are proper morphisms of noetherian schemes of finite Krull dimension, then there exists a canonical isomorphism $f^! g^! \cong (gf)^!$. The functor $f^!$ is called the *twisted inverse image* functor.

Theorem 1 is proved in a slightly weaker form under somewhat more restricted hypotheses in ([1], chap. VII). Of course, Hartshorne gives in [1] an explicit description of the functor $f^!$. However, starting from this explicit description, the proof of Theorem 1 is rather long and leads to many verifications of compatibility.

In [1] Appendix, P. Deligne has pointed out that the existence theorem can be proved simply and directly. Using only the existence theorem, he has also proved Corollary 1 below. We would like to show that the results of [1], except the theory of dualizing and residual complexes, are easy consequences of the existence theorem.

2 Base change Theorem.

Theorem 2. *Let*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

be a cartesian square of noetherian schemes of finite Krull dimension where f is proper and g flat. Then the canonical morphism

$$g' * f^! \to f'^! g^* \tag{20.1}$$

395 *is an isomorphism. In particular, the functor $f^!$ is local on Y.*

Let us first indicate some corollaries.

Corollary 1. *The functor $f^!$ is local on X in the following sense. Let*

$$\begin{tikzcd} & U \arrow[dl, hook', "i_1"'] \arrow[dr, hook, "i_2"] & \\ X_1 \arrow[dr, "f_1"'] & & X_2 \arrow[dl, "f_2"] \\ & Y & \end{tikzcd}$$

be a commutative diagram, where Y is noetherian of finite Krull dimension, f_1 and f_2 are proper and i_1 and i_2 are open immersions. For any G in $D^+_{qc}(Y)$, there exists a canonical isomorphism

$$f_1^! G/U \simeq f_2^! G/U.$$

Proof. Using a closure of U in the fiber product $X_1 \underset{Y}{\times} X_2$ and the isomorphisms of composition of twisted inverse images, we are reduced to study the case $X_1 = Y$, $f_1 = \mathrm{id}_Y$. Let us consider the fiber product

$$\begin{tikzcd} U \arrow[d, hook, "i_1"'] & X_2 \underset{Y}{\times} U \arrow[l, "P_2"'] \arrow[d, hook, "P_1"] & U \arrow[l, "s"'] \\ Y & X_2 \arrow[l, "f_2"] & \end{tikzcd}$$

and the section $s : U \to X_2 \underset{Y}{\times} Y$ of the second projection p_2 defined by the open immersion $i_2 : U \hookrightarrow X_2$. Since s is an open and closed immersion, one has a canonical isomorphism $s^! \simeq s^*$. Applying the base change theorem, we obtain a canonical isomorphism

$$p_2^! i_1^* G \simeq p_1^* f_2^! G,$$

and applying the functor $s^! \simeq s^*$ to both sides we obtain an isomorphism

$$s^! p_2^! i_1^* G \simeq s^* p_1^* f_2^! G.$$

However $s^! p_2^!$ is isomorphic to the identity and therefore we have an isomorphism 396

$$G/U \simeq f_2^! G/U. \qquad \square$$

Lemma 1. *If $G \in D^+_{qc}(Y)$ has coherent cohomology, then $f^! G$ also has coherent cohomology. The functor $f^!$ carries direct sums into direct sums. Let $U \subset X \xrightarrow{f} Y$ be an open set in X on which f is of finite flat dimension (finite tor-dimension in the terminology of [1].) Then if G is a bounded complex on Y the complex $f^! G/U$ is also bounded (actually the functor $G \to f^! G/U$ is "way out" in the terminology of [1]).*

Proof. Since the statements are local on X and on Y, and since they are "stable under composition" we are reduced at once to the following two cases : case (1) f is a closed immersion and case (2) Y is affine, $X = P_1(Y)$ and f is the canonical morphism. In those two cases the verification is easy. $\square$

Corollary 2. *If $f : X \to Y$ is of finite flat dimension, then there exists for any $G \in D^b_{qc}(Y)$ a canonical isomorphism*

$$f^! G \xleftarrow{\sim} f^!(O_Y) \underset{O_X}{\overset{\mathbf{L}}{\bigotimes}} \mathbf{L} f^* G.^\dagger \tag{2.1}$$

If further $f^!(O_Y)$ is of finite flat amplitude, this isomorphism holds for all $G \in D^+_{qc}(Y)$.

Proof. The morphism (2.1) is defined by the universal property of $f^!$ (Theorem 1) and the projection formula (in a form slightly stronger than II. 5.6 in [1]). To prove that is an isomorphism, the lemma on way out functors (I.7 in [1]) is used. □

Corollary 3. *Let $R^{\bullet}_{Y}$ be a dualizing complex on Y (V. 2 in [1]). Then $f^! R^{\bullet}_{Y} = R^{\bullet}_{X}$ is a dualizing complex on X. Denote by D_Y, D_X the corresponding dualizing functors in $D_c(Y)$ and $D_c(X)$ respectively. For any $G \in D^-_c(Y)$ there exists a canonical isomorphism:*

$$f^! D_Y G \simeq D_X \mathbf{L} f^* G. \tag{3.1}$$

397 In particular, for any $G \in D^+_c(Y)$, there exists a canonical isomrophism:

$$f^! G \simeq D_X \mathbf{L} f^* D_Y G. \tag{3.2}$$

Proof. The first statement is local on Y and on X and is "stable under composition". Therefore we are reduced to proving it in the two cases noted in the proof of Lemma 1. In those cases the verification is easy (for the case of a closed immersion use V.2.4. in [1]; a similar proof can be given in the case of the canonical morphism $P_1(Y) \to Y$). Now the isomorphism (3.1) is a formal consequence of Theorem 1 and the projection formula. The isomorphism (3.2) is deduced from (3.1) via the defining property of a dualizing complex. □

So far we have used Theorem 2 only in the case when g is an open immersion. We will use Theorem 2 in the case of a smooth morphism g for the proof of Theorem 3 below.

Proposition 1. *Let $f : X \to Y$ be a regular immersion, defined locally by an O_Y-sequence $t_1, \ldots, t_n$. The Koszul complex built on $t_1, \ldots, t_n$*

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† $\overset{\mathbf{L}}{\otimes}$ denotes the total tensor product, i.e. the tensor product in the derived category.

defines locally an isomorphism

$$f^!(O_Y) \simeq N_{X/Y}[-n],$$

where $N_{X/Y}$ is the inverse of the highest exterior power of the cotangent sheaf. This isomorphism does not depend on the choice of the parameters $t_1, \ldots, t_n$ and thus defines a canonical global isomorphism.

Proof. See [1], III. 7.2. □

Theorem 3. *Let $f : X \to Y$ be a proper morphism of noetherian schemes of finite Krull dimension and $U \subset X$ an open subscheme of X smooth over Y of relative dimension n. Then, there exists a canonical isomorphism*

$$f^!(O_Y)/U \simeq \omega_{U/Y}[n],^{\dagger}$$

where $\omega_{U/Y}$ is the sheaf of relative differentials of degree n on U.

Proof. Consider the diagram

$$\begin{tikzcd} U \arrow[r, "\Delta"] & U \underset{Y}{\times} X \arrow[r, "p_2"] \arrow[d, "p_1"'] & X \arrow[d, "f"] \\ & U \arrow[r, "f/U"] & Y \end{tikzcd}$$

where p_1 and p_2 are the projections and Δ the diagonal. Using Theorem 2, we have a canonical isomorphism $\mathbf{L}p_2^* f^! O_Y \simeq p_1^! O_U$ and applying the functor $\Delta^!$ we get an isomorphism $\Delta^! \mathbf{L}p_2^* f^! O_Y \simeq \Delta^! p_1^! O_U$. But $p_1 \Delta$ is the identity morphism, hence we get $\Delta^! \mathbf{L}p_2^* f^! O_Y \simeq O_U$. Using Corollary 2 we obtain $\Delta^!(O_{U \underset{Y}{\times} X}) \overset{\mathbf{L}}{\underset{O_U}{\otimes}} \mathbf{L}\Delta^* \mathbf{L}p_2^* f^! O_Y \simeq O_U$. The morphism 398

†If $A = (A^i, d_A^i)$ is a complex and n an integer, $A[n]$ denotes the complex $A[n]^i = A^{n+i}, d^i_{A[n]} = (-1)^n d_A^{i+n}$.

$p_2\Delta : U \to X$ is the canonical injection. Therefore $\mathbf{L}\Delta^*\mathbf{L}p_2^* f^! O_Y \simeq f^! O_Y/U$. Now, using Proposition 1 we obtain an isomorphism :

$$N_{U/U\underset{Y}{\times}X}[-n] \overset{\mathbf{L}}{\underset{O_U}{\bigotimes}} f^! O_Y/U \simeq O_U$$

However $N_{U/U\underset{Y}{\times}X}[-n]$ is an invertible sheaf whose inverse is $\omega_{U/Y}$. Therefore we get an isomorphism

$$f^! O_Y/U \simeq \omega_{U/Y}[n].$$

Let $f : X \to Y$ be a proper and smooth morphism of noetherian schemes with $\dim(X/Y) = n$. We have an isomorphism $f^! O_Y \simeq \omega_{X/Y}[n]$. Hence the morphism $\int_f : Rf_* f^! O_Y \to O_Y$ which defines the duality in Theorem 1 induces and is uniquely determined by a morphism denoted once again by

$$\int_f : R^n f_* \omega_{X/Y} \to O_Y.$$

It remains to describe this latter morphism. Using the base change theorem (see Remark (1) at the end) one sees at once that it is enough to describe it when Y is the spectrum of a noetherian complete local ring A. Let $Z \xrightarrow{i} X$ be a closed subscheme of X, finite over Y and defined locally by an O_X-sequence. Let $g : Z \to Y$ be the composed morphism fi. The canonical isomorphism of composition of twisted inverse images
399 $i^!(\omega_{X/Y}[n]) = \mathrm{Ext}^n_{O_X}(O_Z, \omega_{X/Y}) \simeq g^! O_Y$ composed with the integral $\int_g : \Gamma(Z, g^! O_Y) \to A$ determines a morphism called the *residue map* :

$$\mathrm{Res}_Z : \mathrm{Ext}^n(X; O_Z, \omega_{X/Y}) \to A.$$

Furthermore the canonical morphism

$$\mathrm{Ext}^n(X; O_Z, \omega_{X/Y}) \xrightarrow{m_Z} H^n(X, \omega_{X/Y})$$

is embedded into a commutative diagram:

$$\begin{tikzcd}
\mathrm{Ext}^n(X; O_Z, \omega_{X/Y}) \arrow[d, "m_Z"'] \arrow[dr, "\mathrm{Res}_Z"] & \\
H^n(X, \omega_{X/Y}) \arrow[r, "\int_f"] & A
\end{tikzcd}$$

□

Proposition 2. *For any closed subscheme Z of X finite and étale over $Y = \mathrm{spec}(A)$, which intersects non-trivially all the connected components of X, the morphism m_Z is surjective.*

Proof. Decomposing X into its connected components, we may assume that X is connected. Using the Stein factorisation of f and the fact that Y is the spectrum of a complete local ring, we see that the closed fiber is also connected. By Nakayama's lemma and the base change property of the n-th direct image, we are reduced to proving the corresponding statement when $Y = \mathrm{spec}(k)$ where k is a field. By the duality theorem for f, the map m_Z can be interpreted as the canonical map

$$\Gamma(Z, O_Z)^* \to \Gamma(X, O_X)^*$$

which is the dual of the restriction map

$$\rho : \Gamma(X, O_X) \to \Gamma(Z, O_Z).$$

(Here, * means the dual over k). However $\Gamma(X, O_X)$ is a field (actually a separable finite extension of k), and therefore ρ is injective.

Since there is always a subscheme Z of X fulfilling the hypotheses of Proposition 2, this proposition says in other words that any integral can be computed by residues.

The residue map $\mathrm{Res}_Z : \mathrm{Ext}^n(X; O_Z, \omega_{X/Y}) \to A$ however, is com- **400**
pletely described by the *residue symbol* ([1], III. 9.). Choosing $t_1, \ldots, t_n$ an O_X-sequence of parameters which generate the ideal of Z locally around a closed point $z_0 \in Z$ and ω a differential form of degree n on

X defined in a neighbourhood U of z_0, the Koszul complex built over $t_1, \ldots, t_n$ defines an element

$$\begin{bmatrix} \omega \\ t_1, \ldots, t_n \end{bmatrix} \in \mathrm{Ext}^n(X; O_Z, \omega_{X/Y}),$$

and every element of $\mathrm{Ext}^n(X; O_Z, \omega_{X/Y})$ can be obtained as a sum of such elements. Applying the residue map we get an element

$$\mathrm{Res}_Z \begin{bmatrix} \omega \\ t_1, \ldots, t_n \end{bmatrix} \in A$$

which we denote simply by $\mathrm{Res}_{z_0} \begin{bmatrix} \omega \\ t_1, \ldots, t_n \end{bmatrix}$. With the aid of Theorems 1 and 2, it can be shown that this residue symbol has the following properties.

(R0) The residue symbol is A-linear in ω.

(R1) If $s_i = \Sigma c_{ij} t_j$ then $\mathrm{Res}_{z_0} \begin{bmatrix} \omega \\ t_1, \ldots, t_n \end{bmatrix} = \mathrm{Res}_{z_0} \begin{bmatrix} \det(c_{ij})\omega \\ s_1, \ldots, s_n \end{bmatrix}$.

(R2) The formation of the residue symbol commutes with any base change.

(R3) If the morphism $g : Z \to Y$ is an isomorphism at z_0 then

$$\begin{aligned} \mathrm{Res}_{z_0} \begin{bmatrix} dt_1 \wedge \ldots \wedge dt_n \\ t_1^{k_1}, \ldots, t_n^{k_n} \end{bmatrix} &= 1 \ \text{ if } \ k_1 = \ldots = k_n = 1, \\ &= 0 \ \text{ otherwise.} \end{aligned}$$

(R4) If $\omega \in \Gamma(U, \Sigma t_i \omega_{X/Y})$, then $\mathrm{Res}_{z_0} \begin{bmatrix} \omega \\ t_1, \ldots, t_n \end{bmatrix} = 0$.

It is not difficult to show that there exists only one residue symbol which possesses the properties (R0) to (R4) [2]. □

3 Proof of Theorem 2. We keep the notations of Theorem 2. First we have to make explicit the canonical morphism

$$g'^* f^! \to f'^! g^*.$$

There are apparently two ways to define such a morphism.

First Definition. The functor $Rg_* : D^+_{qc}(Y') \to D^+_{qc}(Y)$ is right adjoint **401**
to the functor g^*. We have therefore adjunction morphisms denoted by Φ_g : id $\to Rg_* g^*$, $\Psi_g : g^* Rg_* \to$ id. Since g is flat, we have a base change isomorphism for the total direct image $\sigma : g^* Rf_* \to Rf'_* g'^*$. Taking the right adjoint of both sides we get an isomorphism $\tau : Rg'_* f'^! \xrightarrow{\sim} f^! Rg_*$. We can define a canonical morphism by composing the following morphisms :

$$g'^* f^! \xrightarrow{g'^* f^! \circ \Phi_g} g'^* f^! Rg_* g^* \xrightarrow[\sim]{g'^* \circ \tau \circ g^*} g'^* Rg'_* f'^! g^* \xrightarrow{\Psi_{g'} \circ f'^! g^*} f'^! g^*.$$

Second Definition. By Theorem 1, the functor $f^!$ is right adjoint to the functor Rf_*. Therefore we have adjunction morphisms denoted by cotr_f : id $\to f^! Rf_*$ and $\int_f : Rf_* f^! \to$ id. We can define a canonical morphism by composing the following morphisms :

$$g'^* f^! \xrightarrow{\mathrm{cotr}_{f'} \circ g'^* f^!} f'^! Rf'_* g'^* f^! \xrightarrow[\sim]{f'^! \circ \sigma \circ f^!} f'^! g^* Rf_* f^! \xrightarrow{f'^! g^* \circ \int_f} f'^! g^*.$$

Fortunately those definitions yield the same morphism denoted by

$$c_g : g'^* f^! \to f'^! g^*$$

as a result from a general lemma on adjoint functors. The morphisms c_g verify the usual cocycle property with respect to the composition of base change. It follows at once that we have only to prove that c_g is an isomorphism in the two following cases : case (1) the morphism g is an open immersion and case (2) the morphism g is affine.

Case 1. The Morphism g is an open immersion.

We need the following lemma.

Lemma 2. *Let X be a noetherian scheme, $i : U \to X$ an open immersion, m a sheaf of ideals on X such that the support of $O_X/m = X - U$. Then for any $G \in D^+_{qc}(X)$ and for any integer p, the canonical morphisms*

$$\varinjlim_n \operatorname{Ext}^p(X; m^n, G) \to H^p(U, G),$$

$$\varinjlim_n \mathscr{E}\mathrm{xt}^p(m^n, G) \to R^p i_* i^* G,$$

402 *are isomorphisms.*

Furthermore if F is a complex of sheaves which is bounded above and which has coherent cohomology then the canonical morhisms

$$\varinjlim_n \operatorname{Ext}^p(X; F \overset{\mathbf{L}}{\bigotimes} m^n, G) \to \operatorname{Ext}^p(U; F/U, G/U)$$

are isomorphisms.

Proof. This is the “derived version” of [1] app. Prop. 4. □

To prove Theorem 2 in that case, we use the first definition of the canonical morphism $c_g : g'^* f^! \to f'^! g^*$. Since two fo the three morphisms defining c_g are isomorphisms, it is enough to prove that the morphism

$$g'^* f^! \xrightarrow{g'^* f^! \circ \Phi_g} g'^* f^! R g_* g^*,$$

is an isomorphism, i.e. to prove that for any $G \in D^+_{qc}(Y)$ and any open subset V in X' the morphism

$$f^! \circ \Phi_g : f^! G \to f^! R g_* g^* G$$

induces isomorphisms :

$$w : H^p(V, f' G) \xrightarrow{\sim} H^p(V, f^! R g_* g^* G).$$

Let m be a sheaf of ideals on Y such that $\operatorname{supp}(O_Y/m) = Y - Y'$ and I a sheaf of ideals on X such that $\operatorname{supp}(O_X/I) = X - V$. The canonical morphism $G \to Rg_*g^*G$ factors through

$$G \to R\mathscr{H}\mathrm{om}(m^n, G) \to Rg_*g^*G.^\dagger$$

We have therefore a commutative diagram

$$\begin{tikzcd}
\varinjlim\limits_r \operatorname{Ext}^p(X; I^r, f^!G) \arrow[r, "u"] \arrow[dd] & \varinjlim\limits_r \varinjlim\limits_n \operatorname{Ext}^p(X; I^r, f^!R\mathscr{H}\mathrm{om}(m^n, G)) \arrow[d, "v"] \\
 & \varinjlim\limits_r \operatorname{Ext}^p(X; I^r, f^!Rg_*g^*G) \arrow[d] \\
H^p(V, f^!G) \arrow[r, "W"] & H^p(V, f^!Rg_*g^*G)
\end{tikzcd}$$

Since the verticle maps are isomorphisms by Lemma 2, it is enough to **403**
show that u and v are isomorphisms. For r fixed however, the map

$$v_{n,r} : \varinjlim_n \operatorname{Ext}^p(X; I^r, f^!R\mathscr{H}\mathrm{om}(m^n, G))\, \operatorname{Ext}^p(X, I^r, f^!Rg_*g^*G)$$

is isomorphic, by Theorem 1, to the map

$$\varinjlim_n \operatorname{Ext}^p(Y; Rf_*I^r, R\mathscr{H}\mathrm{om}(m^n, G)) \to \operatorname{Ext}^p(Y; Rf_*I^r, Rg_*g^*G),$$

which is in turn isomorphic to the map

$$\varinjlim_n \operatorname{Ext}^p(Y; Rf_*I^r \overset{L}{\bigotimes} m^n, G) \to \operatorname{Ext}^p(Y; Rf_*I^r/Y', G/Y').$$

Since f is proper, the complex Rf_*I^r has *coherent cohomology*. Therefore by Lemma 2, this latter map is bijective. Hence v is an isomorphism.

$^\dagger R\mathscr{H}\mathrm{om}$ is the total derived functor of the functor $\mathscr{H}\mathrm{om}$: the sheaf of homomorphisms.

For r and n fixed, the map

$$\mathrm{Ext}^p(X; I^r, f^!G) \to \mathrm{Ext}^p(X; I^r, f^! R\mathscr{H}om(m^n, G))$$

is isomorphic, by Theorem 1, to the map

$$\mathrm{Ext}^p(Y; Rf_* I^r, G) \to \mathrm{Ext}^p(Y; Rf_* I^r, R\mathscr{H}om(m^n, G)),$$

which is in turn isomorphic to the map

$$\mathrm{Ext}^p(Y; Rf_* I^r, G) \to \mathrm{Ext}^p(Y; Rf_* I^r \overset{\mathbf{L}}{\bigotimes} m^n, G).$$

The projection formula, yields an isomorphism

$$Rf_* I^r \overset{\mathbf{L}}{\bigotimes} m^n \to Rf_*(I^r \overset{\mathbf{L}}{\bigotimes} \mathbf{L}f^* m^n).$$

404 Therefore, once again applying Theorem 1, the map $u_{n,r}$ is isomorphic to the map

$$\mathrm{Ext}^p(X; I^r, f^!G) \to \mathrm{Ext}^p(X; I^r \overset{\mathbf{L}}{\bigotimes} \mathbf{L}f^* m^n, f^!G),$$

induced by the canonical morphism $\mathbf{L}f^* m^n \to O_X$. Going up to the limit on r, we obtain by Lemma 2 the map

$$H^p(V, f^!G) \to \mathrm{Ext}^p(V, \mathbf{L}f^* m^n/V, f^!V).$$

Since V is contained in X', the complex $\mathbf{L}f^* m^n/V$ is canonically isomorphic to O_X/V and therefore this latter map is bijective. Hence u is an isomorphism. This concludes the proof of Theorem 2 in Case 1.

Case 2. The morphism g is affine.

First we need two propositions.

Proposition 3 (Local form of Theorem 1). *Let $X \xrightarrow{f} Y$ be a proper morphism where Y is noetherian of finite Krull dimension. For any $F \in D^-_{qc}(X)$ and any $G \in D^+_{qc}(Y)$ the composed morphism*

$$Rf_* R\mathscr{H}om(F, f^!G) \xrightarrow{[Rf_*]} R\mathscr{H}om(Rf_* F, Rf_* f^! G) \xrightarrow{\text{“}\int_f''\text{”}} R\mathscr{H}om(Rf_* F, G)$$

is an isomorphism. (The morphism $[Rf_*]$ is obtained by sheafifying the functorial morphism)

$$R\mathrm{Hom}(F, f^! G) \to R\mathrm{Hom}(Rf_* F, Rf_* f^! G).$$

Proof. This is a formal consequence of Theorem 1 and the projection formula. □

Proposition 4. *Let $g : Y' \to Y$ be a flat morphism where Y is a noetherian, $M \in D_c^-(Y)$ a complex bounded above with coherent cohomology, $N \in D^+(Y)$ a complex of sheaves which is bounded below. The canonical morphism*

$$g^* R\mathscr{H}\mathrm{om}(M, N) \to R\mathscr{H}\mathrm{om}(g^* M, g^* N)$$

is an isomorphism.

Proof. See [1], II. 5.8. □

We now prove Theorem 2 in the second case. We proceed in three steps.

Step A. *Let F be a sheaf on X (not necessarily quasi-coherent). For* **405**
any $G \in D_{qc}^+(Y)$ denote by

$$l_g(F, G) : Rf'_* R\mathscr{H}\mathrm{om}(g'^* F, g'^* f^! G) \to Rf'_* R\mathscr{H}\mathrm{om}(g'^* F, f'^! g^* G)$$

the morphism in $D(X')$ induced by c_g. Then $l_g(F, G)$ is an isomorphism whenever F is coherent.

Proof. We have a commutative diagram

$$\begin{tikzcd}
Rf_* R\mathscr{H}\mathrm{om}(g'^{*F}, g'^* f^! G) \arrow[r, "l_g(F,G)"] \arrow[d, "{[Rf'_*]}"] & Rf_* R\mathscr{H}\mathrm{om}(g'^* F, f'^! g^* G) \arrow[d, "{[Rf'_*]}"] \\
R\mathscr{H}\mathrm{om}(Rf'_* g'^{*F}, Rf'_* g'^* f^! G) \arrow[r, "Rf'_* \circ c_g"] \arrow[dr, dashed, "\mu"'] & R\mathscr{H}\mathrm{om}(Rf_* g^* F, Rf_* f^! g^* G) \arrow[d, "``\textstyle\int_{f'}\text{''}"] \\
 & R\mathscr{H}\mathrm{om}(Rf'_* g'^* F, g^* G)
\end{tikzcd}$$

where μ is the only morphism which makes the diagram commutative. By Proposition 404, we know that the composed vertical morphism on the right is an isomorphism. Hence it is enough to prove that the composed morphism

$$\begin{array}{ccc} Rf'_* \mathscr{H}\mathrm{om}(g'^{*F}, g'^* f^! G) & \xrightarrow{[Rf'_*]} & R\mathscr{H}\mathrm{om}(Rf'_* g'^* F, Rf'_* g^* f^! G) \\ & & \downarrow \mu \\ & & R\mathscr{H}\mathrm{om}(Rf'_* g'^* F, g^* G) \end{array} \tag{*}$$

is an isomorphism.

Using the second definition of c_g, it is easily checked that the morphism μ is induced by the composed morphism

$$Rf'_* g'^* f^! G \xrightarrow{\sigma \circ f^!} g^* Rf_* f^! G \xrightarrow{g^* \circ \int_f} g^* G.$$

406 Using this description of the morphism μ, the Proposition 4 for g and g' and the base change isomorphism $\sigma : g^* Rf_* \to Rf'_* g'^*$, it can be seen that the morphism (*) above is isomorphic to the morphism

$$\begin{aligned} g^* Rf_* R\mathscr{H}\mathrm{om}(F, f^! G) &\xrightarrow{g^* \circ [Rf_*]} g^* R\mathscr{H}\mathrm{om}(Rf_* F, Rf_* f^! G) \\ &\xrightarrow{g^* \circ \int_f} g^* R\mathscr{H}\mathrm{om}(Rf_* F, G), \end{aligned}$$

which is, by Proposition 404, an isomorphism. □

Step B. *For any open set V in X, the morphism*

$$\begin{aligned} l_g(O_V, G) : Rf'_* R\mathscr{H}\mathrm{om}(g'^* O_V, g^* f^! G) \\ \to Rf'_* R\mathscr{H}\mathrm{om}(g'^* O_V, f^! g^* G)^\dagger \end{aligned}$$

is an isomorphism.

† The sheaf O_V is the characteristic sheaf of the open set V: the sheaf equal to O_X on V extended by zero outside V.

Proof. Let m be a sheaf of ideals on X such that $\operatorname{supp}(O_X/m) = X - V$. We are going to approximate the sheaf O_V by the pro-object m^r, $r \in \mathbf{N}$. Since g' is flat, $g'^* m$ is a sheaf of ideals on X such that $\operatorname{supp}(O_{X'}/g'^* m) = X' - g'^{-1}(V)$. By Lemma 2 we know that for any $q \in \mathbf{Z}$, the canonical morphisms

$$\varinjlim_r \operatorname{Ext}^q(g'^* m^r, g'^* f^! G) \to \operatorname{Ext}^q(g'^* O_V, g'^* f^! G)$$

$$\varinjlim_r \operatorname{Ext}^q(g'^* m^r, f'^! g^* G) \to \operatorname{Ext}^q(g'^* O_V, f'^! g^* G)$$

are isomorphisms. Furthermore the space X' is noetherian; thus the functor Rf_* commutes with directed limits. Hence the canonical morphisms

$$\varinjlim_r R^p f'_* \operatorname{Ext}^q(g'^* m^r, f'^! g^* G) \to R^p f'_* \operatorname{Ext}^q(g'^* O_V, f^! g^* G)$$

$$\varinjlim_r R^p f'_* \operatorname{Ext}^q(g'^* m^r, g'^* f^! G) \to R^p f'_* \operatorname{Ext}^q(g'^* O_V, g'^* f^! G)$$

are isomorphisms. Therefore the hypercohomology spectral sequences show that for any $n \in \mathbf{Z}$, the canonical morphisms

$$\varinjlim_r \mathscr{H}^n Rf'_* R\mathscr{H}om(g'^* m^n, f^! g^* G) \to \mathscr{H}^n Rf'_* R\mathscr{H}om(g'^* O_V, f'^! g^* G)$$

$$\varinjlim_r \mathscr{H}^n Rf'_* R\mathscr{H}om(g'^* m^n, g^* f^! G) \to \mathscr{H}^n Rf^!_* R\mathscr{H}om(g'^* O_V, g'^* f^! G)$$

are isomorphisms. Since for any r the morphism $l_g(m^r, G)$ is an isomor- **407**
phism (Step A), the morphism $l_g(O_V, G)$ is also an isomorphism. □

Step C. Since g is an affine morphism, the scheme X' can be covered by affine open subspaces which are inverse images by g' of affine open subspaces of X. Therefore, to show that c_g is an isomorphism, it is enough to show that for any affine open set V in X and any $n \in \mathbf{Z}$, the maps induced by c_g :

$$H^n(g'^{-1}(V), g'^* f^! G) \to H^n(g'^{-1}(V), f'^! g^* G) \qquad (**)$$

are isomorphisms. Denote by $i : g^{-1}(O_V) \to X'$ the open immersion. For any complex of sheaves M on X' (bounded below) we have canonical isomorphism

$$R\mathscr{H}\mathrm{om}(g'^*(O_V), M) \simeq Ri_*i^*M.$$

Applying Rf'_* we get an isomorphism

$$Rf'_*R\mathscr{H}\mathrm{om}(g'^*(O_V), M) \xrightarrow{\sim} Rf'_*Ri_*i^*M,$$

and applying the functor $R\Gamma(Y, \quad)$, the derived functor of the functor global section on Y, we get an isomorphism

$$R\Gamma(Y', Rf_*R\mathscr{H}\mathrm{om}(g'^*(O_V), M)) \xrightarrow{\sim} R\Gamma(Y', Rf_*Ri_*i^*M).$$

The composition of direct image functors yields an isomorphism

$$R\Gamma(Y', Rf_*Ri_*i^*M) \xrightarrow{\sim} R\Gamma(g'^{-1}(V), M);$$

thus we have an isomorphism

$$R\Gamma(Y', Rf_*R\mathscr{H}\mathrm{om}(g'^*(O_V), M)) \xrightarrow{\sim} R\Gamma(g'^{-1}(V), M).$$

Therefore applying $R\Gamma(Y', \quad)$ to both sides of $l_g(O_V, G)$, we obtain an isomorphism induced by c_g :

$$R\Gamma(g^{-1}(V), g'^*f^!G) \xrightarrow{\sim} R\Gamma(g^{-1}(V), f'^!g^*G),$$

and taking the n-th cohomology of both complexes we obtain the morphisms (**) which are hence isomorphisms. This concludes the proof of Theorem 2.

408 **Remarks.** (1) For the sake of simplicity we have not stated Theorem 2 completely. It should be completed by a description of the behaviour of the integration map under base change.

(2) One can prove the base change theorem (Theorem 2) when g is a morphism of finite flat amplitude under an hypothesis of cohomological transversality, namely: for any couple of points $y' \in Y$ and $x \in X$ such that $g(y') = f(x)$ and for any $n > 0$ $\mathrm{Tor}_n^{f(x)}(O_{y'}, O_x) = 0$.

(3) In the context of Étale cohomology, one can prove a base change theorem for the twisted inverse image by the same method when g is a smooth morphism, the main point being to have a proposition analogous to Proposition 4 which in the case of g smooth is a consequence of the base change theorem under smooth morphisms for direct images.

References

[1] R. Hartshorne : *Residues and Duality :* Lecture Notes, Springer Verlag.

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ZETA-FUNCTIONS AND MELLIN TRANSFORMS

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409 Classically, the concept of Mellin transform serves to relate Dirichlet series with automorphic functions. Recent developments indicate that this seemingly special device lends itself to broad generalizations, which promise to be of great importance for number-theory and group-theory. My purpose in this lecture is to discuss a typical example, arising from a specific number-theoretical problem.

By an A-field, I understand either an algebraic number-field or a function-field of dimension 1 over a finite filed of constants. Such fields, also sometimes called "global fields", are those for which one can build up a classfield theory and the theory of L-functions; these topics are treated in my book *Basic Number Theory* ([3]; henceforth quoted as BNT), and the notations in that book will be used freely here. In particular, if k is an A-field, its adele ring and its idele group will be denoted by k_A and by $k_A^\times$, respectively; I shall write $|z|$, instead of $|z|_A$, for the module of an idele z.

Write $\mathfrak{M}$ for the free group generated by the finite places of k; this will be written multiplicatively; it may be identified in an obvious manner with the group $I(k)$ of the fractional ideals of k, if k is an algebraic number-field, and with the group $D(k)$ of the divisors of k, if k is a function-field (except that $D(k)$ is written additively). For each finite place v of k, write $\mathfrak{p}_v$ for the corresponding generator of $\mathfrak{M}$; then we define a morphism μ of $k_A^\times$ onto $\mathfrak{M}$ by assigning, to each idele $z = (z_v)$, the element $\mu(z) = \Pi \mathfrak{p}_v^{n(v)}$ of $\mathfrak{M}$, where $n(v) = \mathrm{ord}(z_v)$ and the product is taken over all the finite places of k. If $\mathfrak{m} = \mu(z)$, we write $|\mathfrak{m}| = \Pi |z_v|_v$, the product being taken over the same places; thus we have $|\mathfrak{m}| = \mathfrak{N}(\mathfrak{m})^{-1}$ if k is an algebraic number-field, $\mathfrak{N}$ denoting the norm of an ideal in the usual sense, and $|\mathfrak{m}| = q^{-\deg(\mathfrak{m})}$ if k is a function-field, q being the number of elements of the field of constants of k (i.e., of the largest finite field in k). We say that $\mathfrak{m} = \Pi \mathfrak{p}_v^{n(v)}$ is integral if

$n(v) \geqslant 0$ for all v, and we write $\mathfrak{M}_+$ for the set (or semigroup) of all **410**
such elements of $\mathfrak{M}$; clearly $|\mathfrak{m}| \leqslant 1$ if $\mathfrak{m}$ is in $\mathfrak{M}_+$, and $|\mathfrak{m}| < 1$ if at the same time $\mathfrak{m} \neq 1$.

By a *Dirichlet series belonging to k*, we will understand any *formal* series L, with complex-valued coefficients, of the form

$$L(s) = \Sigma c(\mathfrak{m})|\mathfrak{m}|^s \tag{1}$$

where the sum is taken over all integral elements $\mathfrak{m}$ of $\mathfrak{M}$, i.e. over all $\mathfrak{m} \in \mathfrak{M}_+$. Such series make up a ring (addition and multiplication being defined formally in the obvious manner); the invertible ones, in that ring, are those for which the constant term $c(1)$ is not 0. Set-theoretically, one may identify this ring with the set of all mappings $\mathfrak{m} \to c(\mathfrak{m})$ of $\mathfrak{M}_+$ into the field $\mathbf{C}$ of complex numbers; it will always be understood that such a mapping, when it arises in connexion with a Dirichlet series, is extended to $\mathfrak{m}$ by putting $c(\mathfrak{m}) = 0$ whenever $\mathfrak{m}$ is not integral. The series (1) is absolutely convergent in some half-plane $\mathrm{Re}(s) > \sigma$ if and only if there is $\alpha \in \mathbf{R}$ such that $c(\mathfrak{m}) = O(|\mathfrak{m}|^{-\alpha})$; then it determines a holomorphic function in that half-plane; this will be so for all the Dirichlet series to be considered here. However, the knowledge of that function does *not* determine the coefficients $c(\mathfrak{m})$ uniquely, except when $k = \mathbf{Q}$, so that it does not determine the Dirichlet series (1) in the sense in which we use the word here. A case of particular importance is that in which the function given by (1) in its half-plane of absolute convergence can be continued analytically, as a holomorphic or as a meromorphic function, throughout the whole s-plane; then the latter function is also usually denoted by $L(s)$.

Let v be a finite place of k, and let $\mathfrak{p}_v$ be as above. We will say that the series L given by (1) is *eulerian* at v if it can be written in the form

$$(1 + c_1|\mathfrak{p}_v|^s + \cdots + c_m|\mathfrak{p}_v|^{ms})^{-1} \cdot \Sigma c(\mathfrak{m})|\mathfrak{m}|^s,$$

where the sum in the last factor is taken over all the elements $\mathfrak{m}$ of $\mathfrak{M}_+$ which are disjoint from $\mathfrak{p}_v$ (i.e. which belong to the subgroup of $\mathfrak{M}$ generated by the generators of $\mathfrak{M}$ other than $\mathfrak{p}_v$). The first factor in the same product is then called the *eulerian factor of L at v*. The above **411** condition can also be expressed by saying that there is a polynomial $P(T) = 1 + c_1 T + \cdots + c_m T^m$ such that, if we expand $P(T)^{-1}$ in a power-series $\sum_0^\infty c'_i T^i$, we have, whenever $\mathfrak{m}$ is in $\mathfrak{M}_+$ and disjoint from $\mathfrak{p}_v$, $c(\mathfrak{m}\mathfrak{p}_v^i) = c(\mathfrak{m})c'_i$ for all $i \geqslant 0$. We say that L is *eulerian* if it is so at all finite places of k.

Let ω be any character or "quasicharacter" of the ideal group $k_A^\times$, trivial on $k^\times$. It is well known that one can associate with it can eulerian Dirichlet series

$$L(s, \omega) = \sum \omega(\mathfrak{m})|\mathfrak{m}|^s = \prod_v (1 - \omega(\mathfrak{p}_v)|\mathfrak{p}_v|^s)^{-1}, \tag{2}$$

known as the L-series attached to the "Grössencharakter" defined by ω; its functional equation, which is due to Hecke, is as follows. For each infinite place w of k, write the quasicharacter ω_w induced by ω on $k_w^\times$ in the form $x \to x^{-A}|x|^s w$, with $A = 0$ or 1, if $k_w = \mathbf{R}$, and $z \to z^{-A}\bar{z}^{-B}(z\bar{z})^s w$, with inf $(A, B) = 0$, if $k_w = \mathbf{C}$. Write $G_1(s) = \pi^{-s/2}\Gamma(s/2)$, $G_2(s) = (2\pi)^{1-s}\Gamma(s)$, $G_w = G_1$ or G_2 according as w is real or imaginary, and put

$$\Lambda(s, \omega) = L(s, \omega) \prod_w G_w(s + s_w),$$

where the product is taken over the infinite places of k. Define the constant $\kappa = \kappa(\omega)$ and the idele b as in Proposition 14, Chapter VII-7, of BNT (page 132); we recall that, if d is a "differental idele" (cf. BNT, page 113) attached to the "basic character" of k_A used in the construction of $\kappa(\omega)$, and if $f(\omega) = (f_v)$ is an idele such that f_v is 1 at all infinite places and all places where ω is unramified, and otherwise has an order equal to that of the conductor of ω, then we can take $b = f(\omega)d$. That being so, the functional equation is

$$\Lambda(s, \omega) = \kappa(\omega) \cdot \omega(f(\omega)d)|f(\omega)d|^{s-1/2}\Lambda(1 - s, \omega^{-1}). \tag{3}$$

Let now L be again the Dirichlet series defined by (1); with it, we associate the family of Dirichlet series L_ω given by

$$L_\omega(s) = \sum c(\mathfrak{m})\omega(\mathfrak{m})|\mathfrak{m}|^s \tag{4}$$

for all choices of the quasicharacter ω of $k_A^\times/k^\times$, $\omega(\mathfrak{m})$ being as in (2). Some recent work of mine (c.f. [2]) and some related unpublished work **412**
by Langlands and by Jacquet† has shown that the knowledge of the functional equation, not only for L, but also at the same time for "sufficiently many" of the series L_ω provides us with valuable information about L and its possible relationship to automorphic functions of certain types. In particular, this is so whenever L is the zeta-function of an elliptic curve E over k, provided E is such that the functional equations for the series L_ω can effectively be computed. Unfortunately there are not as many such curves as one could wish; as "experimental material", I have been able to use only the following: (a) in characteristic 0, all the curves E with complex multiplication; their zeta-functions have been obtained by Deuring; (b) also in characteristic 0, some curves, uniformized by suitable types of automorphic functions, which can be treated by the methods of Eichler and Shimura; a typical example is the curve belonging to the congruence subgroup $\Gamma_0(11)$ of the modular group, whose equation, due to Fricke‡, is $Y^2 = 1 - 20X + 56X^2 - 44X^3$ (Tate has observed that it is isogenous to the curve $Y^2 - Y = X^3 - X^2$); (c) in any characteristic $p \geqslant 3$, any curve E of the form $wY^2 = X^3 + aX^2 + bX + c$ where $Y^2 = X^3 + aX^2 + bX + c$ is the equation of an elliptic curve E_0 over the field of constants k_0 of k, and w is in $k^\times$ and not in $(k^\times)^2 k_0^\times$. All these examples exhibit some common features, which can hardly fail to be significant and will now be described.

For the definition of the zeta-function $L(s)$ of the elliptic curve E over k, the reader is referred to [2]; there it is given only for $k = \mathbf{Q}$, but in such terms that its extension to the general case is immediate and requires no comment. It is eulerian. Also the conductor of E is

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†That work is still in progress. No attempt will be made here to describe its scope, but the reader should know that I have freely drawn upon it; my indebtedness will soon, I hope, be made apparent by their publication. In particular, my definition of the Mellin transform when k is not totally real is based on Langlands' more general "local functional equation" for $GL(2, \mathbf{C})$, even though it is also implicit in some earlier work of Maass (c.f. [1], pages 79-80).

‡C.f. F. Klein und R. Fricke, *Theorie der elliptischen Modulfunktionen,* Bd. II, Leipzig 1892, page 436.

413 to be defined as explained in [2]; it is an integral element $\mathfrak{a}$ of $\mathfrak{M}$; we will write $a = (a_v)$ for an idele such that $\mathfrak{a} = \mu(a)$ and that $a_v = 1$ whenever v is not one of the finite places occurring in $\mathfrak{a}$. For the examples quoted above, the zeta-functions are as follows: (a) if E has complex multiplication, and k' is the field generated over k by any one of the complex multiplcations of E, $L(s)$ is an L-series over k', with a "Grössencharakter", if $k' \neq k$, and the product of two such series if $k' = k$; (b) for Fricke's curve belonging to $\Gamma_0(11)$, Eichler has shown that the zeta-function is the Mellin transform of the cusp-form belonging to that same group; the curve's conductor is 11; (c) in the last example, let χ be the character belonging to the quadratic extension $k(w^{1/2})$ of k, and let q^α, q^β be the roots of the zeta-function of the curve E_0 over k_0; then the zeta-function of E is $L(s-\alpha,\chi)L(s-\beta,\chi)$.

In all these examples, one finds that the functional equation for L_ω has a simple form whenever the conductor $\mathfrak{f} = \mu(f(\omega))$ of ω is disjoint from the conductor $\mathfrak{a}$ of the given curve E, and that it is then as follows. For each infinite place ω of k, define s_w, A, B by means of ω, as explained above in describing the functional equation (3) for $L(s,w)$. Put $\mathfrak{G}_w(s) = G_2(s+s_w-A)$ if w is real; put $\mathfrak{G}_w(s) = G_2(s+s_w)^2$ if w is imaginary and $A = B = 0$, and $\mathfrak{G}_w(s) = G_2(s+s_w)\cdot G_2(s+s_w-1)$ if w is imaginary and $A+B > 0$. Put $\Lambda_\omega(s) = L_\omega(s)\cdot \Pi\mathfrak{G}_w(s)$, the product being taken over all the infinite places of k. Call R the number of such places where $A = 0$ (if the place is real) or $A = B = 0$ (if it is imaginary). Then :

$$\Lambda_\omega(s) = \pm(-1)^R\kappa(\omega)^2\omega(af(\omega)^2d^2)|af(\omega)^2d^2|^{s-1}\Lambda_{\omega^{-1}}(2-s), \quad (5)$$

where the sign $\pm$ is independent of ω, and notations are as in (3).

For $k = \mathbf{Q}$, it has been shown in [2] that L must then be the Mellin transform of a modular form belonging to the congruence subgroup $\Gamma_0(a)$ of the modular group. Our purpose is now to indicate that similar results hold true in general.

Once for all, we choose a "basic" character ψ of k_A, trivial on k and not on k_A, and a "differential idele" $d = (d_v)$ attached to ψ; we may choose ψ so that $d_w = 1$ for every infinite place w of k (c.f. BNT, 414 Chapter VIII-4, Proposition 12, p. 156; this determines ψ uniquely if k is of characteristic 0); we will assume that it has been so chosen.

We write G for $GL(2)$, so that G_k is $GL(2,k)$; as usual, we write then G_v, G_A for $GL(2,k_v)$, $GL(2,k_A)$. We identify the center of G with the "multiplicative group" $GL(1)$, hence the centers of G_k, G_v, G_A with $k^\times$, $k_v^\times$, $k_A^\times$, respectively, by the isomorphism $z \to z \cdot 1_2$. *All functions to be considered on any one of the groups G_v, G_A will be understood to be constant on cosets modulo the center*, so that they are actually functions on the corresponding projective groups. It is nevertheless preferable to operate in $GL(2)$, since our results can easily be extended to functions with the property $f(gz) = f(g)\omega(z)$, where ω is a given character of the center, and these useful generalizations can best be expressed in terms of $GL(2)$. By an *automorphic function*, we will always understand a continuous function on G_A, left-invariant under G_k (and, as stated above, invariant under the center $k_A^\times$ of G_A), with values of $\mathbf{C}$ or in a vector-space of finite dimension over $\mathbf{C}$; this general concept will be further restricted as the need may arise.

For a matrix of the form $\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}$, we write (x,y); we write B for the group of such matrices (and B_k, B_v, B_A for the corresponding subgroups of G_k, G_v, G_A). The group $B \cdot G_m$, with $G_m = GL(1)$, consists of the matrices $\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}$, and $G/(B \cdot G_m)$ may be identified in an obvious manner with the projective line D. In particular, $G_A/(B_A \cdot k_A^\times)$ is the "adelized projective line" D_A; it is compact, and its "rational points" (i.e. the "rational projective line" D_k) are everywhere dense in it. It amounts to the same to say that $G_k \cdot B_A \cdot k_A^\times$ is dense in G_A. Consequently, an automorphic function on G_A is uniquely determined by its values on B_A. Let Φ be such a function; call F the function induced on B_A by Φ; F is left-invariant under B_k, and in particular under $(1,\eta)$ for each $\eta \in k$, so that, for each $x \in k_A^\times$, the function $y \to F(x,y)$ on k_A can be expanded in Fourier series on k_A/k. Using the basic character ψ, and making use of the fact that F is also left-invariant under $(\xi,0)$ for all $\xi \in k^\times$, one 415 finds at once that this Fourier series may be written as

$$F(x,y) = f_0(x) + \sum_{\xi \in k^\times} f_1(\xi x)\psi(\xi y), \tag{6}$$

where f_0, f_1 are the functions on $k_A^\times$ respectively given by

$$f_0(x) = \int_{k_A/k} F(x,y)dy, \ f_1(x) = \int_{k_A/k} F(x,y)\psi(-y)dy.$$

We have $f_0(\xi x) = f_0(x)$ for all $\xi \in k^\times$; we will say that Φ is *B-cuspidal* if $f_0 = 0$.

Conversely, when such a Fourier series is given, the function F defined by it on B_A is left-invariant under B_k and may therefore be extended to a function of $G_k \cdot B_A \cdot k_A^\times$, left-invariant under G_k (and, as is always assumed, invariant under $k_A^\times$), and the question arises whether this can be extended by continuity to G_A. In order to give a partial answer to this question, we must first narrow down the kind of automorphic function which we wish to consider.

We first choose an element $\mathfrak{a}$ of $\mathfrak{M}_+$, which will play the role of a conductor, and, as before, an idele $a = (a_v)$ such that $\mathfrak{a} = \mu(a)$ and that $a_v = 1$ whenever v does not occur in $\mathfrak{a}$. Also, write $\mathbf{d} = (\mathbf{d}_v)$ for the element $(d, 0)$ of B_A, d being the differental idele chosen above. At each finite place v of k, the group $GL(2, r_v) = M_2(r_v)^\times$ is a maximal compact subgroup of G_v, consisting of the matrices $\begin{pmatrix} x & y \\ z & t \end{pmatrix}$, with coefficients x, y, z, t in the maximal compact subring r_v of k_v, such that $|xt - yz|_v = 1$ (i.e. that $xt - yz$ is in $r_v^\times$); then $\mathbf{d}_v^{-1} \cdot M_2(r_v)^\times \cdot \mathbf{d}_v$ is also such a subgroup of G_v, consisting of the matrices $\begin{pmatrix} x & d_v^{-1} y \\ d_v z & t \end{pmatrix}$, where x, y, z, t are as before. We will write $\mathfrak{K}_v$ for the subgroup of the latter group, consisting of the matrices of that form with $z \in a_v r_v$; this is a compact open subgroup of G_v, equal to $M_2(r_v)^\times$ at all the finite places which do not occur in $\mu(ad)$. On the other hand, we take for $\mathfrak{K}_w$ the orthogonal group $O(2)$ in 2 variables if w is real, and the unitary group $U(2)$ if

w is imaginary. Then the product $\mathfrak{K} = \Pi\mathfrak{K}_v$, taken over all the places
416 of k, defines a compact subgroup $\mathfrak{K}$ of G_A; it is open in G_A if k is of characteristic $p > 1$, but not otherwise. We have $G_v = B_v \cdot k_v^\times \cdot \mathfrak{K}_v$ for all places v of k, except those occurring in $\mathfrak{a}$; consequently, $B_A \cdot k_A^\times \cdot \mathfrak{K}$ is open in G_A.

We also introduce an element $\mathbf{a} = (\mathbf{a}_v)$ of G_A, which we define by putting $\mathbf{a}_v = \mathbf{d}_v^{-1} \cdot \begin{pmatrix} 0 & -1 \\ a_v & 0 \end{pmatrix} \cdot \mathbf{d}_v$ for v finite, and $\mathbf{a}_w = 1_2$ for w infinite. Clearly $\mathfrak{K}\mathbf{a} = \mathbf{a}\mathfrak{K}$.

The automorphic functions Φ which we wish to consider are to be right-invariant under $\mathfrak{K}_v$ for every finite place v of k; thus, if k is of characteristic $p > 1$, they are right-invariant under $\mathfrak{K}$, hence locally constant. Clearly, if Φ has that property, the same is true of the function Φ' given by $\Phi'(g) = \Phi(g\mathbf{a})$. If k is of characteristic $p > 1$, we take our functions Φ to be complex-valued. If k is a number-field, our purposes require that they take their values in suitable vector-spaces, that they transform according to given representations of the groups $\mathfrak{K}_w$ at the infinite places w of k, and that, at those places, they satisfy additional conditions to be described now.

It is well-known that, if $k_w = \mathbf{R}$ (resp. $\mathbf{C}$), the "Riemannian symmetric space" $G_w/k_w^\times\mathfrak{K}_w$ may be identified with the hyperbolic space of dimension 2 (resp. 3), i.e. with the Poincaré half-plane (resp. half-space). This can be done as follows. Let B_w^+ be the subgroup of B_w consisting of the matrices $b = (p, y)$ with $p \in \mathbf{R}_+^\times$ (i.e. $p \in \mathbf{R}$, $p > 0$) and $y \in k_w$. Every element g of G_w can be writeen as $g = bz\mathfrak{k}$ with $b \in B_w^+$, $z \in k_w^\times$, $\mathfrak{k} \in \mathfrak{K}_w$; here $b = (p, y)$, and $z\mathfrak{k}$, are uniquely determined by g. We identify $G_w/k_w^\times\mathfrak{K}_w$ with the Poincaré half-plane (resp. half-space) $H_w = \mathbf{R}_+^\times \times k_w$ by taking, as the canonical mapping of G_w onto $G_w/k_w^\times\mathfrak{K}_w$, the mapping ϕ_w of G_w onto H_w given by $\phi_w(g) = (p, y)$ for $g = bz\mathfrak{k}$, $b = (p, y)$ as above. The invariant Riemannian metric in H_w is the one given by $ds^2 = p^{-2}(dp^2 + dyd\bar{y})$. On H_w, consider the differential forms which are left-invariant under B_w^+; a basis for these consists of the forms $\alpha_1 = p^{-1}(dp + idy)$, $\alpha_2 = p^{-1}(dp - idy)$ if $k_w = \mathbf{R}$, and of $\alpha_1 = p^{-1}dy$, $\alpha_2 = p^{-1}dp$, $\alpha_3 = p^{-1}d\bar{y}$ if $k_w = \mathbf{C}$. Writing E'_w for the vector-space $M_{2,1}(\mathbf{C})$ resp. $M_{3,1}(\mathbf{C})$ of column-vectors (with 2 resp. 3

rows) over $\mathbf{C}$, we will denote by α_w the vector-valued differential form
417 on H_w, with values in E'_w, whose components are α_1, α_2 resp. α_1, α_2, α_3. One can then describe the action of $k_w^\times \mathfrak{K}_w$ on these forms by writing

$$\alpha_w(\phi_w(z\mathfrak{k}b)) = \mathfrak{M}_w(\mathfrak{k}\alpha_w(b),$$

where $\mathfrak{M}_w$ is a representation of $\mathfrak{K}_w$ in the space E'_w; for $k_w = \mathbf{R}$, for instance, this is given by

$$\mathfrak{M}_w\left(\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\right) = \begin{pmatrix} e^{-2i\theta} & 0 \\ 0 & e^{2i\theta} \end{pmatrix}, \ \mathfrak{M}_w\left(\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

A basis for the left-invariant differential forms on G_w which are 0 on $k_w^\times \mathfrak{K}_w$ is then given by the components of the vector-valued form

$$\beta_w(g) = \mathfrak{M}_w(\mathfrak{k})^{-1}\alpha_w(\phi_w(g))$$

where $\mathfrak{k}$ is given, as above, by $g = bz\mathfrak{k}$.

Now write E_w for the space of row-vectors $M_{1,2}(\mathbf{C})$ resp. $M_{1,3}(\mathbf{C})$; we regard this as the dual space to E'_w, the bilinear form $e \cdot e'$ being defined by matrix multiplication for $e \in E_w$, $e' \in E'_w$. Then, if h is an E_w-valued function on G_w, $h \cdot \beta_w$ is a complex-valued differential form on G_w; it is the inverse image under ϕ_w of a differential form on H_w if and only if $h(gz\mathfrak{k}) = h(g)\mathfrak{M}_w(\mathfrak{k})$ for all $g \in G_w$, $z \in k_w^\times$, $\mathfrak{k} \in \mathfrak{K}_w$; when that is so, h is uniquely determined by its restriction $(p, y) \to h(p, y)$ to B_w^+. We will say that h, or its restriction to B_w^+, is *harmonic* if $h \cdot \beta_w$ is the inverse image under ϕ_w of a harmonic differential form on the Riemannian space H_w, or, what amounts to the same, if h has the property just stated and if $h(p, y) \cdot \alpha_w$ is harmonic on H_w. For $k_w = \mathbf{R}$, this is so if and only if the two components of the vector-valued function $p^{-1}h(p, y)$ on H_w are respectively holomorphic for the complex structures defined on H_w by the complex coordinates $p \pm iy$. We will say that h is *regularly harmonic* if it is harmonic and if $h(p, y) = O(p^N)$ for some N when $p \to +\infty$, uniformly in y on every compact subset of k_w. If h is harmonic, so is $g \to h(g_0 g)$ for every $g_0 \in G_w$, since the Riemannian structure of H_w is invariant under G_w and since the form β_w is left-invariant on G_w. If h is regularly harmonic, so is $g \to h(b_0 g)$ for every $b_0 \in B_w^+$. It is easily seen that there is, up to a constant factor, only one regularly harmonic function $\mathbf{h}_w$ such that

$$\mathbf{h}_w((1,y)g) = \psi_w(y)\mathbf{h}_w(g)$$

for all $y \in k_w$, $g \in G_w$; this given by **418**

$$\mathbf{h}_w(p,y) = \psi_w(y)\mathbf{h}_w(p),$$
$$\mathbf{h}_w(p) = p \cdot (e^{-2\pi p}, 0) \text{ if } k_w = \mathbf{R},$$
$$\mathbf{h}_w(p) = p^2 \cdot (K_1(4\pi p), -2iK_0(4\pi p), K_1(4\pi p)) \text{ if } k_w = \mathbf{C},$$

where K_0, K_1 are the classical Hankel functions*. For any $x \in k_w^\times$, we write $\mathbf{h}_w(x)$ instead of $\mathbf{h}_w((x,0))$.

It is essential to note that $\mathbf{h}_w$ satisfies a "local functional equation", which, following Langlands, we can formulate as follows. Let ω be a quasicharacter of $k_w^\times$; as before, we write it in the form $x \to x^{-A}|x|^s$ with $A = 0$ or 1, if $k_w = \mathbf{R}$, and $z \to z^{-A}\bar{z}^{-B}(z\bar{z})^s$ with $\inf(A,B) = 0$, if $k_w = \mathbf{C}$. For $k_w = \mathbf{R}$, put $\mathscr{G}_w(\omega) = G_2(1+s-A)$; for $k_w = \mathbf{C}$, put $\mathscr{G}_w(\omega) = G_2(s+1)^2$ if $A = B = 0$, and $\mathscr{G}_w(\omega) = G_2(s)G_2(s+1)$ otherwise. Write j for the matrix $j = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, and put, for $g \in G_w$:

$$I_\omega(g,\omega) = \int_{k_w^\times} \mathbf{h}_w((z,0)g)\omega(z)d_z^\times, \tag{7}$$

where $d^\times z$ is a Haar measure on $k_w^\times$; this is convergent for $\mathrm{Re}(s)$ large. Then the functional equation is

$$\mathscr{G}_w(\omega)^{-1}I_w(g,\omega) = (-1)^\rho \mathscr{G}_w(\omega^{-1})^{-1}I_w(j^{-1}g,\omega^{-1}) \tag{8}$$

with $\rho = 1$ if $k_w = \mathbf{R}$, or if $k_w = \mathbf{C}$ and $A = B = 0$, and $\rho = A + B$ if $k_w = \mathbf{C}$ and $A + B > 0$. By (8), we mean that both sides, for given A, B, g, can be continued analytically, as holomorphic functions of s, over the whole s-plane, and are then equal. This can of course be verified by a

*Cf. G. N. Watson, *A treatise on the theory of Bessel function,* 2nd. ed., Cambridge 1952, page 78.

straightforward calculation for $k_w = \mathbf{R}$. A similar calculation for $k_w = \mathbf{C}$ might not be quite so easy. Both cases, however, are included in more general results of Langlands; moreover, a simple proof for (8) itself in the case $k_w = \mathbf{C}$, communicated to me by Jacquet, is now available. It
419 will be noticed that the gamma factors in (8) are essentially the same as those occurring in (5).

Now we write k_∞, $k_\infty^\times$, G_∞, $\mathfrak{K}_\infty$, H_∞, etc., for the products Πk_w, $\Pi k_w^\times$, ΠG_w, $\Pi \mathfrak{K}_w$, ΠH_w, etc., taken over the infinite places of k. We write E_∞, E'_∞ for the tensor-products $\otimes E_w$, $\otimes E'_w$, taken over the same places; these may be regarded as dual to each other. Then $\beta_\infty = \otimes \beta_w$ is a left-invariant differential form on G_∞ with values in E'_∞; its degree is equal to the number r of infinite places of k; if h is any function on G_∞ with values in E_∞, $h \cdot \beta_\infty$ is then a complex-values differential form of degree r on G_∞. We will say that h is *harmonic* if $h \cdot \beta_\infty$ is the inverse image of a harmonic differential form on H_∞; writing $p = (p_w)$, $y = (y_w)$ for elements of $(\mathbf{R}_+^\times)^r$ and k_∞, so that (p, y) is an element of H_∞, we will say that the harmonic function h is *regularly harmonic* if there is N such that $h(p, y) = O(p_w^N)$ for each w when $p_w \to +\infty$, uniformly over compact sets with respect to all variables except p_w. Up to a constant factor, the only regularly harmonic function $\mathbf{h}_\infty$ such that

$$\mathbf{h}_\infty((1, y)g) = \psi_\infty(y)\mathbf{h}_\infty(g)$$

for all $y \in k_\infty$, $g \in G_\infty$ is given by $\mathbf{h}_\infty(g) = \otimes \mathbf{h}_w(g_w)$ for $g = (g_w)$.

We will say that a continuous function Φ on G_A, with values in E_∞, is a harmonic automorphic function with the conductor $\mathfrak{a}$, or, more briefly, that it is $(h, \mathfrak{a})$-*automorphic* if it is left-invariant under G_k, invariant under $k_A^\times$, right-invariant under $\mathfrak{K}_v$ for every finite place v of k, and if, for every $g_0 \in G_A$, the function on G_∞ given for $g \in G_\infty$ by $g \to \Phi(g_0 g)$ is harmonic; if k is not of characteristic 0, the latter condition is empty, and we take $E_\infty = \mathbf{C}$. The function Φ' given by $\Phi'(g) = \Phi(g\mathbf{a})$ is then also $(h, \mathfrak{g})$-automorphic. For such a function Φ, we shall now consider more closely the Fourier series defined by (6). As Φ is harmonic on G_∞ and right-invariant under $\mathfrak{K}_v$ for every finite v, the same is true of the functions

$$\Phi_0(g) = \int\limits_{k_A/k} \Phi((1,y)g)dy, \Phi_1(g) = \int\limits_{k_A/k} \Phi((1,y)g)\psi(-y)dy$$

whose restrictions to B_A are $F_0(x,y) = f_0(x)$, $F_1(x,y) = f_1(x)\psi(y)$, where f_0, f_1 are as in (6). In particular, for every finite v, F_1 is right-invariant under the group $B_v \cap \mathfrak{K}_v$, hence under all matrices $(u,0)$ with **420**
$u \in r_v^\times$, and all matrices $(1, d_v^{-1}z)$ with $z \in r_v$; in view of the definition of the idele $d = (d_v)$, the latter fact means that $f_1(x) = 0$ unless $x_v \in r_v$ for all finite v, i.e. unless the element $\mathfrak{m} = \mu(x)$ of $\mathfrak{M}$ is integral; the former fact means that $f_1(x)$ depends only upon $\mathfrak{m}$ and upon the components x_w of x at the infinite places w of k. Putting $x_\infty = (x_w)$, we can therefore write $f_1(x) = f_1(\mathfrak{m}, x_\infty)$, and this is 0 unless $\mathfrak{m}$ is in $\mathfrak{M}_+$. For similar reasons, we can write $f_0(x) = f_0(\mathfrak{m}, x_\infty)$.

If k is of characteristic $p > 1$, this can be written $f_1(x) = f_1(\mathfrak{m})$, $f_0(x) = f_0(\mathfrak{m})$. As $f_1(\mathfrak{m})$ is 0 unless $\mathfrak{m}$ is in $\mathfrak{M}_+$, only finitely many terms of the Fourier series (6) can be $\neq 0$ for each (x,y); they are all 0, except possibly $f_0(x)$, if $|x| > 1$, since this implies $|\xi x| > 1$ for all $\xi \in k^\times$. On the other hand, if k is of characteristic 0, the convergence of the Fourier series follows from the fact that Φ, being harmonic, must be analytic in $g_\infty = (g_w)$.

Now we add three more conditions for Φ:

(I) Φ should be B-cuspidal, i.e. f_0 should be 0.

(II) If k is of characteristic 0, Φ should be regularly harmonic on G_∞, when the coordinates g_v at the finite places are kept constant. Then the same is true of Φ_1; in view of what we have found above, this implies that $f_1(\mathfrak{m}, x_\infty)$ is a constant scalar multiple of $\mathbf{h}_\infty(x_\infty)$ for every $\mathfrak{m}$, so that we can write

$$f_1(\mathfrak{m}, x_\infty) = c(\mathfrak{m})\mathbf{h}_\infty(x_\infty),$$

where $c(\mathfrak{m})$ is a complex-valued function on $\mathfrak{M}$, equal to 0 outside $\mathfrak{M}_+$. In the case of characteristic $p > 1$, we write $c(\mathfrak{m}) = f_1(\mathfrak{m})$.

(III) We assume that $c(\mathfrak{m}) = O(|\mathfrak{m}|^{-\alpha})$ for some α; (I) and (II) being assumed, this implies that $F(x, y) = O(|x|^{-1-\alpha})$ for $|x| \leqslant 1$, uniformly in y. Conversely, if $F(x, y) = O(|x|^{-\beta})$ for $|x| \leqslant 1$, uniformly in y, for some β, we have $c(\mathfrak{m}) = O(|\mathfrak{m}|^{-\beta})$.

Clearly (III) amounts to saying that the Dirichlet series (1) with the coefficients $c(\mathfrak{m})$ is absolutely convergent in some half-plane. This may be regarded as the Mellin transform of Φ. It is more appropriate for our purposes, however, to use that name for the series

$$Z(\omega) = \sum c(\mathfrak{m})\omega(\mathfrak{m}), \tag{9}$$

421 where ω is a quasicharacter of $k_A^\times/k^\times$, and $\omega(\mathfrak{m})$ is as in (2). For $s \in \mathbf{C}$, we write ω_s for the quasicharacter $\omega_s(z) = |z|^s$, and, for every quasicharacter ω, we define $\sigma = \sigma(\omega)$ by $|\omega(z)| = |z|^\sigma$, i.e. $|\omega| = \omega_\sigma$ (where $|\quad|$ in the left-hand side is the ordinary absolute value $|t| = (t\bar{t})^{\frac{1}{2}}$ for $t \in \mathbf{C}$). Then our condition (III) implies that (9) is absolutely convergent for $\sigma(\omega) > 1 + \alpha$. If we replace ω by $\omega \cdot \omega_s$ in (9), (9) becomes the same as the series (4); in other words, the knowledge of the function Z given by (9) on the set of all the quasi-characters of $k_A^\times/k^\times$ is equivalent to that of all the functions given by (4). As before, we define $Z(\omega)$ by analytic continuation in the s-plane, whenever possible, when it is not absolutely convergent.

Conversely, let the coefficients $c(\mathfrak{m})$ be given for $\mathfrak{m} \in \mathfrak{M}_+$; assume (III), and put $c(\mathfrak{m}) = 0$ outside $\mathfrak{M}_+$. Let Z be defined by (9); at the same time, define f_1 on $k_A^\times$ by putting $f_1(x) = c(\mathfrak{m})$ with $\mathfrak{m} = \mu(x)$ if k is of characteristic $p > 1$, and $f_1(x) = c(\mathfrak{m})\mathbf{h}_\infty(x_\infty)$ otherwise, with $x_\infty = (x_w)$; put $f_0(x) = 0$, and define $F(x, y)$ by the Fourier series (6), whose convergence follows at once from (III) and the definition of $\mathbf{h}_\infty$ if k is of characteristic 0, and is obvious otherwise. As we have said, the question arises now whether F can be extended to a continuous function Φ on G_A, left-invariant under G_k (and invariant under $k_A^\times$); if so, we may then ask whether this is an $(h, \mathfrak{a})$-automorphic function, which clearly must then satisfy (I) and (III) and is easily shown to satisfy (II). In that case we say that Φ and the series Z given by (9) are the *Mellin transforms* of each other.

We are now able to state our main results.

Theorem 1. *Let Φ be an $(h, \mathfrak{a})$-automorphic function on G_A; let Φ' be the $(h, \mathfrak{a})$-automorphic function given by $\Phi'(g) = \Phi(g\mathbf{a})$. Assume that Φ and Φ' satisfy* (I), (II), (III). *Call Z the series* (9) *derived from Φ as explained above, and Z' the series similarly derived from Φ'. Then, for all the quasicharacters ω whose conductor is disjoint from $\mathfrak{a}$, we have*

$$Z(\omega)\Pi\mathcal{G}_w(\omega) = (-1)^{r-R}\kappa(\omega)^2\omega(af(\omega)^2d^2)Z'(\omega^{-1})\Pi\mathcal{G}_w(\omega^{-1}). \quad (10)$$

Moreover, if Z is eulerian at any finite place v of k, not occurring in $\mathfrak{a}$, Z' is also eulerian there, and they have the same eulerian factor at v, 422
which is of the form

$$(1 - c|\mathfrak{p}_v|^s + |\mathfrak{p}_v|^{1+2s})^{-1} \quad (11)$$

with $c = c(\mathfrak{p}_v)$.

In (10), the two products are taken over the infinite places w of k, $\mathfrak{G}_w$ being as in (8); r is the number of such places; $\kappa(\omega)$ and $f(\omega)$ are as in (3) and (5), and R as in (5). Moreover, by (10), we mean that, if $\omega \cdot \omega_s$ is substituted for ω, both sides can be continued analytically as holomorphic functions of s in the whole s-plane, bounded in every strip $\sigma \leqslant \mathrm{Re}(s) \leqslant \sigma'$, and that they are equal; (10) and similar formulas should also be understood in that same sense in what follows.

It is worth noting* that, for Z to be eulerian at v in Theorem 1, it is necessary and sufficient that Φ should be an eigenfunction of the "Hecke operator" T_v which maps every function Φ on G_A onto the function $T_v\Phi$ given by

$$(T_v\Phi)(g) = \int_{\mathfrak{K}_v} \Phi(g\mathfrak{k} \cdot (\pi, 0))d\mathfrak{k},$$

where $d\mathfrak{k}$ is a Haar measure in $\mathfrak{K}_v$, and π is a prime element of k_v. More precisely, take $d\mathfrak{k}$ so that the measure of $\mathfrak{K}_v$ is 1; then, if $T_v\Phi = \lambda\Phi$, one finds, by taking $g = (x, y)$ in the above formula and expressing $\Phi(x, y)$ by (6), that Z has the eulerian factor (11) at v, with $c = (1 + |\mathfrak{p}_v|)\lambda$. We

*I owe this observation to Jacquet.

also note that here T_v generates the Hecke algebra for G_v, so that Φ is then an eigenfunction for all the operators in that algebra.

Theorem 2. *Let a series Z be given by* (9)*, and let Z' be a similar series; assume that both satisfy* (III)*. Let $\mathfrak{s}$ be a finite set of finite places of k, containing all the places which occur in $\mathfrak{a}$. Assume that Z, Z' are eulerian at every finite place of k outside $\mathfrak{s}$, with the same eulerian factor of the form* (11)*; also, assume* (10)*, in the sense explained above, for all the quasicharacters ω whose conductor is disjoint from $\mathfrak{s}$. Then*
423 *there is an $(h, \mathfrak{a})$-automorphic function Φ on G_A, satisfying* (I), (II), (III)*, such that Z and Z' are the Mellin transforms of Φ, and of the function $\Phi'(g) = \Phi(g\mathbf{a})$, respectively.*

There is no doubt that the assumptions in Theorem 2 are much more stringent than they need be. For $k = \mathbf{Q}$, it has been found in [2] that the eulerian property is not required at all; in the general case, it might perhaps be enough to postulate it at some suitable finite set of places. For $k = \mathbf{Q}$, the functional equation has to be assumed only for a rather restricted set of characters (those mentioned in [2], Satz 2), or even for a finite set of characters, depending upon $\mathfrak{a}$, when $\mathfrak{a}$ is given (since Hecke's group $\Gamma_0(A)$ is finitely generated). It seems quite possible that some such results may be true in general. One will also observe that, for $k = \mathbf{Q}$, Theorems 1 and 2 correspond merely to the case $\epsilon = 1$ of the results obtained in [2]; there is no difficulty in extending them so as to cover the case where ϵ is arbitrary; then, if they apply to two series Z, Z', and to the conductor $\mathfrak{a}$, they also apply to any pair Z_1, Z'_1 given by $Z_1(\omega) = Z(\chi\omega)$, $Z'_1(\omega) = Z'(\chi^{-1}\omega)$, where χ is any quasicharacter whose conductor $\mathfrak{f}(\chi)$ is disjoint from $\mathfrak{a}$; the conductor for the latter pair is $\mathfrak{a}_1 = \mathfrak{a}\mathfrak{f}(\chi)^2$. Leaving those topics aside, we shall not sketch briefly the proof the Theorems 1 and 2.

Consider first the question raised by Theorem 2. Starting from the series Z, we construct a function F on B_A by means of (6) as explained above; we construct F' similarly, starting from Z'. For these to be the restrictions to B_A of two $(h, \mathfrak{a})$-automorphic functions Φ, Φ' related to each other by $\Phi'(g) = \Phi(g\mathbf{a})$, it is obviously necessary that one should have $F(b) = F'(b')\mathfrak{M}_\infty(\mathfrak{k}_\infty)$, with $\mathfrak{M}_\infty = \otimes\mathfrak{M}_w$, whenever $b = jb'\mathfrak{k}z\mathbf{a}$

with $j = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \in G_k$, $\mathfrak{k} = (\mathfrak{k}_v) \in \mathfrak{K}$, $z \in k_A^\times$. By using the fact that G_k is the union of $B_k \cdot k^\times$ and of $B_k j B_k \cdot k^\times$, one shows that this condition is sufficient. Clearly, it is not affected if one restricts b, b' to a subset $\mathfrak{B}$ of B_A containing a full set of representatives of the right cosets in B_A modulo $B_A \cap \mathfrak{K}$. For $\mathfrak{B}$, we choose the set consisting of the elements (xfd, xe) with $x \in k_A^\times$, $f = (f_v) \in k_A^\times$, $e = (e_v) \in k_A$, with f and e restricted as follows. For each infinite place w, we take $f_w > 0$ and $f_w^2 + e_w \overline{e}_w = 1$. For each finite place v, we take f_v, e_v in r_v, with $f_v \neq 0$ **424**
and $\sup(|f_v|_v, |e_v|_v) = 1$. Then we call $\mathfrak{f} = \mu(f)$ the conductor of the element $b = (xfd, xe)$ of $\mathfrak{B}$. Take two such elements $b = (xfd, xe)$, $b' = (x'f'd, x'e')$, such that $b = jb'\mathfrak{k}z\mathbf{a}$ with $\mathfrak{k} \in \mathfrak{K}$, $z \in k_A^\times$; it is easily seen that they must have the same conductor $\mathfrak{f}$, and that this is disjoint from $\mathfrak{a}$; moreover, when x, f, e are given, one may choose x', f', e', $\mathfrak{k}$, z so that $f' = f$, that e', $\mathfrak{k}$, z are uniquely determined in terms of f and e, and that $x' = ax^{-1}$. Therefore the condition to be fulfilled can be written as

$$F(xfd, xe) = F'(ax^{-1}fd, ax^{-1}e')\mathfrak{M}_\infty(\mathfrak{k}_\infty), \tag{12}$$

with e' uniquely determined in terms of f, e, and $\mathfrak{k}_\infty$ in terms of f_∞, e_∞. Actually, one finds that it is enough, for Φ and Φ' to exist as required, that this should be so when $\mathfrak{f}$ is disjoint, not merely from $\mathfrak{a}$, but from any fixed set $\mathfrak{s}$ of places, containing $\mathfrak{a}$, provided it is finite, or at least provided its complement is "not too small" in a suitable sense. We must now seek to express (12) in terms of the original series Z, Z'.

In order to do this, we multiply (12) with an arbitrary quasi-character ω, and write *formally* the integrals of both sides over $k_A^\times/k^\times$. This, taken literally, is meaningless, since it leads to divergent integrals; leaving this aspect aside for the moment, we note first that, if we replace F in the left-hand side by the Fourier series which defines it, that side may be formally rewritten as

$$\sum c(\mathfrak{m}) \int \mathbf{h}_\infty(x_\infty)\psi(xef^{-1}d^{-1})\omega(xf^{-1}d^{-1})d^\times x, \tag{13}$$

where $d^\times x$ is the Haar measure in $k_A^\times$, and the integral, in the term corresponding to $\mathfrak{m}$, is taken over the subset of $k_A^\times$ determined by $\mu(x) = \mathfrak{m}$;

this is a coset of the kernel of μ, i.e. of the open subgroup $k_\infty^\times \times \Pi r_v^\times$ of $k_A^\times$. These integrals are easily calculated (by means of Proposition 14, Chapter VII-7, of BNT, page 132) in terms of the product $J = \Pi I_w((1, e_w), \omega_w)$, where the I_w are as defined in (7); they converge for $\sigma(\omega)$ large enough. One sees at once that they are 0 for all $\mathfrak{m}$ unless the conductor $\mathfrak{f}(\omega)$ of ω divides $\mathfrak{f} = \mu(f)$. If $\mathfrak{f}(\omega) = \mathfrak{f}$, one finds that (13) is no other than $J \cdot Z(\omega)$, up to a simple scalar factor. A similar formal
425 calculation for the right-hand side of (12) transforms it into the product of a scalar factor, of an integral similar to J, and of $Z'(\omega^{-1})$; comparing both sides and taking (8) into account, one gets the functional equation (10), for which we will now write $E(\omega)$. If we do not assume $\mathfrak{f}(\omega) = \mathfrak{f}$, but merely $\mathfrak{f} = \mathfrak{f}(\omega)\mathfrak{f}_1$ with $\mathfrak{f}_1 \in \mathfrak{M}_+$, the same procedure leads to a similar equation $E_1(\omega)$ connecting two Dirichlet series $Z_1(\omega)$, $Z_1'(\omega)$ whose coefficients depend only upon $\mathfrak{f}_1$ and the coefficients of Z and of Z', respectively.

If k is of characteristic $p > 1$, there is no difficulty in replacing the above formal argument by a correct proof. The same can be achieved for characteristic 0 by a straightforward application of Hecke's lemma (c.f. [2], page 149). The conclusion in both cases is that the validity of the equations $E_1(\omega)$ for all divisors $\mathfrak{f}_1$ of $\mathfrak{f}$ and all quasicharacters ω with the conductor $\mathfrak{f}\mathfrak{f}_1^{-1}$ is necessary and sufficient for (12) to hold for all e and all x, when $\mathfrak{f}$ is given. This proves Theorem 1 except for the last part, which one obtains easily by comparing the equations $E(\omega)$ and $E_1(\omega)$ for $\mathfrak{f}_1 = \mathfrak{p}_v$ when the eulerian property is postulated for Z at v. On the other hand, we see now, in view of what was said above, that, when Z and Z' are given, Φ and Φ' exist as required provided the functional equations $E(\omega)$, $E_1(\omega)$ are satisfied whenever $\mathfrak{f}(\omega)$ and $\mathfrak{f}_1$ are both disjoint from the given set $\mathfrak{s}$. If one assumes that Z, Z' are eulerian at each one of the places occurring in $\mathfrak{f}_1$, with an eulerian factor of the form (11), one finds that $Z_1(\omega)$, $Z_1'(\omega)$ differ from $Z(\omega)$, $Z'(\omega)$ only by an "elementary" factor and that $E_1(\omega)$ is a consequence of $E(\omega)$. This proves Theorem 2.

Examples for Dirichlet series satisfying the conditions in Theorem 2 are given, as we have seen, by the zeta-functions of elliptic curves (taking $Z(\omega) = L_\omega(1)$, $Z'(\omega) = \pm L_\omega(1)$, where $L(s)$ is the zeta-function)

in the cases (a), (b), (c) where these can be effectively computed; other similar examples, not arising from elliptic curves, can easily be constructed, as Hecke L-functions over quadratic extensions of k, or products of two such functions over k. Jacquet has pointed out that, when Z is a product of two Hecke L-functions, the automorphic function Φ is an Eisenstein series; this is the case in example (c), and in (a) when k **426**
contains the complex multiplications of the curve E; it cannot happen (according to [2], Satz 2) when $k = \mathbf{Q}$.

If k is a number-field with r infinite places, and if the zeta-function of an elliptic curve E over k satisfies the assumptions in Theorem 2, that theorem associates with it the differential form $\Phi \cdot \beta_\infty$ of degree r; since it is locally constant with respect to the coordinates at the finite places, it may be regarded as a harmonic differential form of degree r on the union of a certain finite number of copies (depending on the class-number of k) of the Riemannian symmetric space H_∞ belonging to G_∞. For $k = \mathbf{Q}$, some examples suggest that the periods of that form may be no other than those of the differential form of the first kind belonging to E. In the general case, one can at least hope to discover a relation between the periods of $\Phi \cdot \beta_\infty$ and those of the differential form of the first kind on E and on its conjugates over $\mathbf{Q}$. When k is of characteristic $p > 1$, however, Φ is a scalar complex-valued function on the discrete space $G_k \backslash G_A / \mathfrak{R} k_A^\times$, and it seems hard even to imagine a connexion between this and the curve E, closer than the one given by the definitoin of Φ in terms of Z.

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