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EXISTENCE OF PERIODIC WAVES FOR A PERTURBED QUINTIC BBM EQUATION

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Abstract. This paper dealt with the existence of periodic waves for a perturbed quintic BBM equation by using geometric singular perturbation theory. By analyzing the perturbations of the Hamiltonian vector field with a hyperelliptic Hamiltonian of degree six, we proved that periodic wave solutions persist for sufficiently small perturbation parameter. It is also proved that the wave speed $c_0(h)$ is decreasing on h by analyzing the ratio of Abelian integrals, where h is the energy level value. Moreover, the upper and lower bounds of the limit wave speed are given.

1. **Introduction.** Traveling waves in nonlinear wave equations can model many nonlinear complex phenomenon in physics, chemistry, biology, mechanics, optics, etc. There exist many shallow water wave models, such as the Korteweg-de Vries (KdV) [12], the Benjamin-Bona-Mahony [3], the Green-Naghdi [9] and more recently the Camassa-Holm [4] equations. All these model equations govern the asymptotic dynamics of wave profiles of long waves in shallow water. To solve the practical problems, many authors have tried to explain wave motions on a liquid layer over an inclined plane in fluid dynamics. Topper and Kawahara [18] proposed the following partial differential equation,

$$u_t + uu_x + \alpha u_{xx} + \beta u_{xxx} + \gamma u_{xxxx} = 0, \tag{1.1}$$

where the variable u means the hight of the wave at the point x and time t. The physical parameters α, β and γ are all positive. Here, the wave motion is assumed depending only on the gradient direction. If the inclined plane is infinitely long and the surface tension is relatively weak, the u_{xx} and u_{xxxx} terms are relatively small, then (1.1) can be considered as a 1-parameter equation by taking an transformed appropriate scale transformation of u, x and t. For example, (1.1) can be transformed to the following equation,

$$u_t + uu_x + u_{xxx} + \varepsilon(u_{xx} + u_{xxxx}) = 0, \tag{1.2}$$

In some physical circumstances, for example, correspond to the case ε is small. It is important in the sense of understanding the role of dispersion u_{xx} and dissipation u_{xxxx} in nonlinear wave systems. A year later, Ogawa [14] studied the existence

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of traveling waves of the perturbed KdV equation (1.2) and gave the relation between the amplitude and the wavelength. When the backward diffusion (u_{xx}) and dissipation (u_{xxxx}) terms are absent, the equation (1.2) is regarded as the KdV equation.

In general, perturbation theory in a dynamical system can be divided into three types: periodic or quasi-periodic forcing, singular perturbation and regular perturbation. If the perturbation term of a partial differential equation is quasi-periodic forcing, an infinite dimensional KAM theory is used to investigate dynamics of the system. This theory is an extension of the well-known classical KAM theory, which was established by Kolmogorov [11], Arnold [1] and Moser [13]. If the Kolmogorov non-degenerate condition is satisfied, the majority of cycle is persistent under perturbations.

When a perturbed system can be reduced to a singular perturbation system, the main problem is the existence of traveling wave solutions of the system. In order to deal with singular perturbations, a classic approach is to use the Fenichel's theory [8], which ensures the existence of an invariant manifold and then simplifies the problem to a regular perturbation system on the manifold. In these cases [7, 14, 15, 16], the perturbation always has only one or two terms with lower degrees on the invariant manifold.

As we all know, perturbations are usually not restricted on manifolds, therefore, there are very few problems that can be directly reduced to regularly perturbed systems. Furtheremore, there exist few mathematical tools available to study the dynamic behavior of perturbed systems, and the analysis and calculation based on this method can hardly be used to prove the existence of periodic waves yet.

Recently, Yan *et al.* [20] discussed a perturbed generalized KdV equation.

$$u_t + u^n u_x + u_{xxx} + \varepsilon(u_{xx} + u_{xxxx}) = 0. \tag{1.3}$$

When $\varepsilon = 0$ and $n = 2$, (1.3) becomes the generalized KdV equation. Using the geometric singular perturbation theory, they have proved that traveling wave solutions persist for sufficiently small perturbation parameter.

In 2005, Wazwaz [19] has studied some nonlinear dispersive generalized forms of the Benjamin-Bona-Mahony (BBM) equation.

$$(u^m)_t + (u^n)_x + (u^l)_{xxx} = 0. \tag{1.4}$$

The aim of [19] is to extend the work conducted by Rosenau [17] to make further progress in finding compactons of dispersive structures.

More recently, Chen *et al.*[5] investigated a perturbed generalized BBM equation for $m = 2$, $n = 3$ and $l = 1$, that is,

$$(u^2)_t + (u^3)_x + u_{xxx} + \varepsilon(u_{xx} + u_{xxxx}) = 0, \tag{1.5}$$

and established the existence of solitary waves and uniqueness of periodic waves. In this paper, the authors applied Picard-Fuchs equation to determine the periodic waves, and developed a good approach to prove that the dominating factor of the Abelian integral is monotonic. Using the same manner, Chen *et al.* [6] also proved that the perturbed defocusing mKdV equation,

$$u_t + (u^2)u_x + u_{xxx} + \varepsilon(u_{xx} + u_{xxxx}) = 0, \tag{1.6}$$

has a unique periodic wave. Both of the works [5] and [6] studied the perturbation problems restricted on the manifolds by using geometric singular perturbation theory. As we all know, a lot of work is done on the study of the generalized BBM equation. However, till now there are few works on the perturbed quintic BBM equation.

In this paper, we study the BBM equation (1.4) for $m = 3$, $n = 5$ and $l = 1$, that is, we consider a perturbed quintic BBM equation

$$(u^3)_t + (u^5)_x + u_{xxx} + \varepsilon(u_{xx} + u_{xxxx}) = 0, \tag{1.7}$$

where $\varepsilon > 0$ is a perturbation parameter. When $\varepsilon = 0$, Eq.(1.7) becomes the quintic BBM equation

$$(u^3)_t + (u^5)_x + u_{xxx} = 0. \tag{1.8}$$

By using the traveling coordinate $\xi = x - ct$, where c is speed, we insert the traveling wave $u = \varphi(\xi)$ into Eq.(1.7). It follows that

$$-3c\varphi^2(\xi)\varphi'(\xi) + 5\varphi^4(\xi)\varphi'(\xi) + \varphi'''(\xi) + \varepsilon[\varphi''(\xi) + \varphi''''(\xi)] = 0. \tag{1.9}$$

Integrating Eq.(1.9) and without loss of generality, we can put the integral constant to zero, then

$$-c\varphi^3(\xi) + \varphi^5(\xi) + \varphi''(\xi) + \varepsilon[\varphi'(\xi) + \varphi'''(\xi)] = 0. \tag{1.10}$$

After doing the suitable scale transformation $\xi = \tau/c$, $\varphi = \sqrt{c}z$ to (1.10), the final equation can be transformed to

$$-z^3(\tau) + z^5(\tau) + z''(\tau) + \varepsilon\left(\frac{1}{c}z'(\tau) + cz'''(\tau)\right) = 0. \tag{1.11}$$

If we have a solution $z(\tau)$ of (1.11) for $\varepsilon > 0$ and $c > 0$, then the corresponding $\varphi(\xi)$ is a traveling wave solution to Eq.(1.9), and, therefore, $u(x, t)$ is a traveling wave solution to the original equation (1.7).

When $\varepsilon = 0$, Eq. (1.11) becomes an unperturbed system

$$-z^3(\tau) + z^5(\tau) + z''(\tau) = 0, \tag{1.12}$$

whose solutions are traveling wave solutions to the quintic BBM equation. Eq.(1.12) has an equivalent form

$$\begin{cases} \frac{\mathrm{d}z}{\mathrm{d}\tau} = y, \\ \frac{\mathrm{d}y}{\mathrm{d}\tau} = z^3 - z^5. \end{cases} \tag{1.13}$$

It is easy to see that system (1.13) has three equilibrium points $O(0, 0)$, $P_1(1, 0)$ and $P_2(-1, 0)$. The origin $O(0, 0)$ is a nilpotent saddle of third order, $P_1(1, 0)$ and $P_2(-1, 0)$ are two centers. Thus system (1.13) belongs to the so called "a double cuspidal loop" [2] case. The phase portrait of system (1.13) is given as Figure 1.

The system (1.13) is a Hamiltonian system with Hamiltonian of degree six

$$H(z, y) = -\frac{1}{2}y^2 - \frac{1}{6}z^6 + \frac{1}{4}z^4,$$

satisfying $H(\pm 1, 0) = 1/12$, and $H(0, 0) = 0$. Now, we consider a level curve of the form $H = h$. If h satisfies $0 < h < 1/12$, then it corresponds to two periodic orbits Γ_h of (1.13). If $h = 0$, it includes two homoclinic orbits. If $h < 0$, then it represents a big periodic orbit surrounding the three singular points . Therefore we can parameterize these traveling wave solutions to (1.12) by h. Using this parametrization, we can describe the existence result of periodic wave solutions of (1.7) as the following theorem.

Theorem 1.1. *For the perturbed quintic BBM equation(1.7), the following results holds.*

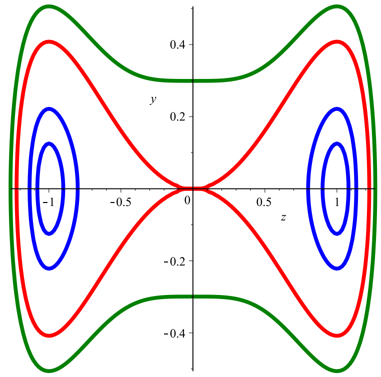

FIGURE 1. The phase portrait of system (1.13).

(1) For any sufficiently small $\varepsilon > 0$, there exists $\varepsilon^* > 0$, $\varepsilon \in [0, \varepsilon^*]$ and $h \in [\delta, 1/12]$, Eq.(1.7) has two periodic wave solutions

$$u_{\pm} = \pm cz(\varepsilon, h, c, \tau),$$

where $c = c(\varepsilon, h)$, and $z(\varepsilon, h, c, \tau)$ is a solution of Eq.(1.11), and δ is a small positive number.

(2) $c = c(\varepsilon, h)$ is a smooth function of ε and h, and converges to $c_0(h)$ as $\varepsilon \to 0$, where $c_0(h)$ is a smooth decreasing function for $h \in (0, 1/12]$, and,

$$\frac{\sqrt{2}}{2} \leq c_0(h) < \frac{8}{9}, \quad \lim_{h\to 0} c_0(h) = \frac{8}{9}, \quad \lim_{h\to \frac{1}{12}} c_0(h) = \frac{\sqrt{2}}{2}.$$

(3) When $\varepsilon \to 0$, $z(\varepsilon, h, c, \tau)$ converges to $z(h, c_0(h), \tau)$ uniformly in τ, where $z(h, c_0(h), \tau)$ is a solution of system (1.13) on the level curve $H = h$.

The goal of this paper is to study the existence of periodic waves for Eq.(1.7). To obtain our main result, it is necessary to compute the Abelian integral (3.2)(see Section 3) with two ratios. The chief difficulty lies in proving the monotonicity of the ratio of Abelian integral $Z(k)$. We will finish the task in Section 3.

The remaining part is organized as follows. Section 2 is devoted to investigate the existence of periodic wave solutions of Eq.(1.7) by using the geometric singular perturbation theory. In Section 3, the Abelian integral theory is used to analysis the limit speed and the monotonicity of the wave speed $c_0(k)$, and the main results are proved.

2. **Perturbation analysis.** In this section, our purpose is to find the periodic solutions to Eq.(1.11) by using the geometric singular perturbation theory and regular perturbation analysis for a Hamiltonian system.

Firstly, we introduce the singular perturbation theorem on invariant manifolds which comes from Theorem 9.1 in Fenichel [8]. For convenience, we use a version of this theorem due to Theorem 1-2 in Jones [10].

Consider the system

$$\begin{cases} \dot{x} = f(x, y, \varepsilon), \\ \dot{y} = \varepsilon g(x, y, \varepsilon), \end{cases} \tag{2.1}$$

where the $\prime$ means differentiation with respect to t. $x \in R^n$, $y \in R^l$ and ε is a real parameter. We shall compile three hypotheses about the system (2.1).

(H1) The functions f and g are both assumed to be C^∞ on a set $U \times I$, where $U \subset R^N$ is open, with $N = n + l$, and I is an open interval, containing 0.

We are given an l-dimensional manifold, possibly with boundary, M_0 which is contained in the set $\{f(x, y, 0) = 0\}$. The manifold M_0 is said to be normally hyperbolic if the linearization of (2.1) at each point in M_0 has exactly $dim(M_0)$ eigenvalues on the imaginary axis $\Re(\lambda) = 0$. The assumption (H2) is given as follows.

(H2) The set M_0 is a compact manifold, possibly with boundary, and is normally hyperbolic relative to the system (2.1) $|_{\varepsilon=0}$.

In order to significantly simplify the notation, we shall restrict attention to the case that M_0 is given as the graph of a function of x in terms of y. That is we assume there is a function $h^0(y)$, defined for $y \in K$, with K being a compact domain in R^l, and so that $M_0 = \{(x, y) : x = h^0(y)\}$. Thus, consider $x = h^0(y)$ wherein $y \in K$ and make the following assumption.

(H3) The set M_0 is given as the graph of the C^∞ function $h^0(y)$ for any $y \in K$. The set K is a compact, simply connected domain whose boundary is an $(l-1)$ -dimensional C^∞ submanifold.

The following two lemmas are crucial for our analysis.

Lemma 2.1. [10] *Under the hypotheses (H1)-(H2), if $\varepsilon > 0$, but sufficiently small, there exists a manifold M_ε that lies within $O(\varepsilon)$ of M_0 and is diffeomorphic to M_0. Moreover it is locally invariant under the flow of (2.1), and C^r, including in ε, for any $0 < r < +\infty$.*

Lemma 2.2. [10] *Under the hypotheses (H1)-(H3), if $\varepsilon > 0$ is sufficiently small, there is a function $x = h^\varepsilon(y)$, defined on K, so that the graph*

$$M_\varepsilon = \{(x, y) \mid x = h^\varepsilon(y)\},$$

is locally invariant under (2.1). Moreover h^ε is C^r, for any $0 < r < +\infty$, jointly in y and ε.

Now we recall system (1.11), which is equivalent to

$$\begin{cases} \frac{\mathrm{d}z}{\mathrm{d}\tau} = y, \\ \frac{\mathrm{d}y}{\mathrm{d}\tau} = w, \\ \varepsilon c \frac{\mathrm{d}w}{\mathrm{d}\tau} = z^3 - z^5 - w - \frac{\varepsilon}{c} y. \end{cases} \tag{2.2}$$

When $\varepsilon > 0$, (2.2) defines a system whose solutions evolve in the three dimensional (z, y, w) phase space. Note that in this phase space, system (2.2) admits the following three singular points $(0, 0, 0)$, $(1, 0, 0)$ and $(-1, 0, 0)$.

By making time scale transformation $\sigma = \tau/\varepsilon$, (2.2) becomes

$$\begin{cases} \frac{\mathrm{d}z}{\mathrm{d}\sigma} = \varepsilon y, \\ \frac{\mathrm{d}y}{\mathrm{d}\sigma} = \varepsilon w, \\ c \frac{\mathrm{d}w}{\mathrm{d}\sigma} = z^3 - z^5 - w - \frac{\varepsilon}{c} y. \end{cases} \tag{2.3}$$

When $\varepsilon > 0$, (2.2) and (2.3) are equivalent. Generally, (2.2) is called the slow system, and (2.3) is called the fast system.

Consider the slow system (2.2) with $\varepsilon = 0$. The critical manifold M_0 is given by the set

$$M_0 = \{(z, y, w) \in \mathbb{R}^3 \mid w = z^3 - z^5, (z, y) \in K\},$$

where $K = \{(y,z) : 0 < \delta \leq -1/2y^2 - 1/6z^6 + 1/4z^4 \leq 1/12\}$, δ is a small positive number. Obviously, the set K is a compact, simply connected domain. The assumption (H3) holds.

For M_0 to be normally hyperbolic, we must show that the linearization of the fast system, restricted to M_0, has exactly $dim M_0$ eigenvalues on the imaginary axis, with the remainder of the system hyperbolic. Note that the linearization of (2.3) restricted to M_0 is given by the following matrix

$$C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{c}(3z^2 - 5z^4) & 0 & -\frac{1}{c} \end{bmatrix}.$$

It can be easily seen that the matrix C has three eigenvalues $\lambda_1 = 0, \lambda_2 = 0, \lambda_3 = -1/c$, with λ_1 and λ_2 being on the imaginary axis , thus we can conclude that M_0 is normally hyperbolic. Obviously, the assumption (H2) holds. Consequently, it follows from Lemma 2.2 that, for $\varepsilon > 0$ sufficiently small, there exists a two-dimensional submanifold M_ε of $\mathbb{R}^3$ within the Hausdorff distance ε of M_0 and which is invariant under the flow of system (2.2).

Let

$$M_\varepsilon = \{(z,y,w) \in R^3 : w = z^3 - z^5 + \zeta(z,y,\varepsilon), (z,y) \in K\}, \tag{2.4}$$

where $\zeta(z,y,\varepsilon)$ depends smoothly on z, y, ε, and satisfies $\zeta(z,y,0) = 0$. We can expand $\zeta(z,y,\varepsilon)$ in ε as follows

$$\zeta(z,y,\varepsilon) = \varepsilon\zeta_1(z,y) + O(\varepsilon^2). \tag{2.5}$$

Substituting (2.4) and (2.5) into the third equality of slow system (2.2), we can get

$$\varepsilon c(3z^2 - 5z^4)y + O(\varepsilon^2) = -\varepsilon\zeta_1(z,y) - \frac{\varepsilon}{c}y + O(\varepsilon^2). \tag{2.6}$$

Comparing the coefficients of ε, we can get

$$\zeta_1(z,y) = c\left[(5z^4 - 3z^2)y - \frac{1}{c^2}y\right].$$

Therefore, the dynamics on the slow manifold M_ε for system (2.2) is determined by

$$\begin{cases} \frac{\mathrm{d}z}{\mathrm{d}\tau} = y, \\ \frac{\mathrm{d}y}{\mathrm{d}\tau} = z^3 - z^5 + \varepsilon c\left[(5z^4 - 3z^2)y - \frac{1}{c^2}y\right] + O(\varepsilon^2). \end{cases} \tag{2.7}$$

which is a regular perturbation problem. By Lemma 7.3 in [21] $O(0,0)$ is still a saddle point in system (2.7).

Now we can easily check if a periodic orbit persists or not as follows. By symmetry, we only need to check the orbit with $z > 0$. First of all, we should pay attention to the dynamics of the unperturbed system (1.13), which can be understood by the level curve of H. Fix an initial data $(\alpha, 0)$ with $0 < \alpha < 1$. Now let $(z(\tau), y(\tau))$ be the solution of (2.7) with $(z,y)(0) = (\alpha, 0)$. Then there exist $\tau_1 > 0$ and $\tau_2 < 0$, so that

$$y(\tau) > 0 \quad for \quad 0 < \tau < \tau_1, \quad y(\tau_1) = 0,$$

and

$$y(\tau) < 0 \quad for \quad \tau_2 < \tau < 0, \quad y(\tau_2) = 0.$$

Let us define a function Φ as follows

$$\Phi(\alpha, c, \varepsilon) = \int_{\tau_2}^{\tau_1} \dot{H}(z,y)\mathrm{d}\tau.$$

Here,

$$\dot{H}(z,y) = -\varepsilon c\left[(5z^4 - 3z^2)y^2 - \frac{1}{c^2}y^2\right] + O(\varepsilon^2).$$

$\Phi(\alpha, c, \varepsilon)$ denotes difference of the level between the two points on the z-axis;

$$\Phi(\alpha, c, \varepsilon) = H(z(\tau_1), y(\tau_1)) - H(z(\tau_2), y(\tau_2)).$$

Hence, $\Phi(\alpha, c, \varepsilon) = 0$ if and only if $z(\tau)$ is a periodic solution of (2.7), that is, our goal is to solve $\Phi = 0$. Since $\Phi(\alpha, c, 0) = 0$, we have

$$\Phi(\alpha, c, \varepsilon) = \varepsilon\tilde{\Phi}(\alpha, c, \varepsilon).$$

When $\varepsilon \to 0$, $\tilde{\Phi}(\alpha, c, \varepsilon)$ has a limit

$$\tilde{\Phi}_0(\alpha, c) = \lim_{\varepsilon\to 0}\tilde{\Phi}(\alpha, c, \varepsilon) = c\int_{\tau_2}^{\tau_1}\left[(5z_0^4 - 3z_0^2)y_0^2 - \frac{1}{c^2}y_0^2\right]\mathrm{d}\tau.$$

Here, (z_0, y_0) is a solution of (1.13) and this integral is performed on a level curve $H = H(\alpha, 0) \in (0, 1/12)$. Since,

$$\int_{\tau_2}^{\tau_1} z_0^2 {z_0'}^2\mathrm{d}\tau = -2\int_{\tau_2}^{\tau_1} z_0^2 {z_0'}^2\mathrm{d}\tau - \int_{\tau_2}^{\tau_1} z_0^3 z_0''\mathrm{d}\tau.$$

i.e.

$$\int_{\tau_2}^{\tau_1} z_0^2 {z_0'}^2\mathrm{d}\tau = -\frac{1}{3}\int_{\tau_2}^{\tau_1} z_0^3 z_0''\mathrm{d}\tau.$$

Similarly,

$$\int_{\tau_2}^{\tau_1} z_0^4 {z_0'}^2\mathrm{d}\tau = -\frac{1}{5}\int_{\tau_2}^{\tau_1} z_0^5 z_0''\mathrm{d}\tau.$$

Then we have

$$\begin{aligned}\int_{\tau_2}^{\tau_1}[(5z_0^4 - 3z_0^2)y_0^2 - \frac{1}{c^2}y_0^2]\mathrm{d}\tau &= -\int_{\tau_2}^{\tau_1} z_0^5 z_0''\mathrm{d}\tau + \int_{\tau_2}^{\tau_1} z_0^3 z_0''\mathrm{d}\tau - \int_{\tau_2}^{\tau_1}\frac{1}{c^2}{z_0'}^2\mathrm{d}\tau.\\ &= \int_{\tau_2}^{\tau_1} {z_0''}^2\mathrm{d}\tau - \int_{\tau_2}^{\tau_1}\frac{1}{c^2}{z_0'}^2\mathrm{d}\tau.\end{aligned}$$

Therefore,

$$\tilde{\Phi}_0(\alpha, c) = c\left(\int_{\tau_2}^{\tau_1} {z_0''}^2\mathrm{d}\tau - \int_{\tau_2}^{\tau_1}\frac{1}{c^2}{z_0'}^2\mathrm{d}\tau\right) = \frac{1}{c}\left(c^2\int_{\tau_2}^{\tau_1} {z_0''}^2\mathrm{d}\tau - \int_{\tau_2}^{\tau_1} {z_0'}^2\mathrm{d}\tau\right).$$

And we have to determine the limit speed c_0 by

$$c_0^2\int_{\tau_2}^{\tau_1} {z_0''}^2\mathrm{d}\tau - \int_{\tau_2}^{\tau_1} {z_0'}^2\mathrm{d}\tau = 0. \tag{2.8}$$

3. **Analysis by the Abelian integral theory.** In this section, we focus on calculating the limit speed c_0 with h and prove our main theorem. Here, we suppose that $z(\tau)$ is a solution of (1.12).

Firstly, let Q and R be

$$Q = \frac{1}{2}\int_{\tau_2}^{\tau_1} {z''}^2\mathrm{d}\tau, \qquad R = \frac{1}{2}\int_{\tau_2}^{\tau_1} {z'}^2\mathrm{d}\tau,$$

Eq.(2.8) becomes $c_0^2 Q - R = 0$. In what follows, we will give specific expressions for Q and R. And when $0 \le k = 2h < 1/6$,

$$-\frac{1}{3}z^6 + \frac{1}{2}z^4 = k. \tag{3.1}$$

Let $\alpha(k)$ and $\beta(k)$ be the two non-negative real roots of (3.1), where $0 \le \alpha(k) < \beta(k)$. Here, as mentioned above, the orbit $(z(\tau), y(\tau))$ is on the level curve $H = h = k/2$, where $y = \mathrm{d}z/\mathrm{d}\tau$, therefore, we have

$$Q = \int_{\alpha}^{\beta} \frac{(-z^5+z^3)^2}{E(z)}\mathrm{d}z, \quad R = \int_{\alpha}^{\beta} E(z)\mathrm{d}z,$$

by system (1.13). Here, $E(z) = \sqrt{\frac{1}{2}z^4 - \frac{1}{3}z^6 - k}$.

For convenience, we represent Q and R by the following integrals:

$$J_n(k) = \int_{\alpha}^{\beta} z^n E(z)\mathrm{d}z, \qquad n = 0, 1, 2, \cdots.$$

Then it meets

$$\int_{\alpha}^{\beta} \frac{z^n}{E(z)}\mathrm{d}z = -2J_n'(k).$$

Therefore, Q and R are represent as follows:

$$R = J_0(k), \quad Q = \int_{\alpha}^{\beta} \frac{(-z^5+z^3)^2}{E(z)}\mathrm{d}z = -2J_6'(k) + 4J_8'(k) - 2J_{10}'(k).$$

By Green formula, we have $R(k) = J_0(k) = \int_{\alpha}^{\beta} E(z)\mathrm{d}z = \int_{\alpha}^{\beta} y\mathrm{d}z = \iint_{int\Gamma_h} \mathrm{d}z\mathrm{d}y > 0$. By the first equality of system (2.2), $T = 2\int_0^{\frac{T}{2}} \mathrm{d}\tau = 2\int_{\alpha}^{\beta} 1/y\mathrm{d}z = -4J_0'(k) > 0$, then $J_0'(k) < 0$. Let $Z(k) = Q(k)/R(k)$, since the function $Z(k)$ about variable k is continuous , then we have the following proposition.

Proposition 3.1. *For $0 < k < \frac{1}{6}$, $Z'(k) > 0$. Moreover,*

$$\frac{81}{64} < Z(k) < 2, \quad \lim_{k\to 0} Z(k) = \frac{81}{64}, \qquad \textit{and} \qquad \lim_{k\to\frac{1}{6}} Z(k) = 2.$$

To prove this proposition, we need the following lemmas. Firstly, let us study the basic properties of J_0, J_2 and J_4 by the following lemmas.

Lemma 3.1. *Let $B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx, p > 0, q > 0$ be the Beta function. Then we have*

$$J_0(0) = \frac{3\sqrt{3}}{8}B\left(\frac{3}{2},\frac{3}{2}\right), \quad J_2(0) = \frac{9\sqrt{3}}{16}B\left(\frac{3}{2},\frac{5}{2}\right), \quad J_4(0) = \frac{27\sqrt{3}}{32}B\left(\frac{3}{2},\frac{7}{2}\right).$$

Moreover,

$$\frac{J_2(0)}{J_0(0)} = \frac{3}{4}, \qquad \frac{J_4(0)}{J_0(0)} = \frac{45}{64}.$$

Proof. Since $\alpha(0) = 0$, $\beta(0) = \frac{\sqrt{6}}{2}$, we can get

$$J_0(0) = \int_0^{\frac{\sqrt{6}}{2}} z^2\sqrt{\frac{1}{2} - \frac{1}{3}z^2}\mathrm{d}z = \frac{\sqrt{2}}{2}\int_0^{\frac{\sqrt{6}}{2}} z^2\sqrt{1 - \frac{2}{3}z^2}\mathrm{d}z.$$

Let $1 - \frac{2}{3}z^2 = t$, then $z^2 = \frac{3}{2}(1-t)$, $\mathrm{d}z = -\frac{\sqrt{6}}{4}(1-t)^{-\frac{1}{2}}\mathrm{d}t$, we can get

$$J_0(0) = \frac{\sqrt{2}}{2}\int_0^{\frac{\sqrt{6}}{2}} z^2\sqrt{1 - \frac{2}{3}z^2}\mathrm{d}z = \frac{3\sqrt{3}}{8}\int_0^1 (1-t)^{\frac{1}{2}}t^{\frac{1}{2}}\mathrm{d}t = \frac{3\sqrt{3}}{8}B\left(\frac{3}{2},\frac{3}{2}\right).$$

Similarly,

$$J_2(0) = \frac{9\sqrt{3}}{16}B\left(\frac{3}{2},\frac{5}{2}\right), \quad J_4(0) = \frac{27\sqrt{3}}{32}B\left(\frac{3}{2},\frac{7}{2}\right).$$

Since

$$B(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)},\qquad \Gamma(s+1)=s\Gamma(s).$$

Here $\Gamma(s)=\int_0^1 x^{s-1}e^{-x}\mathrm{d}x+\int_1^{+\infty}x^{s-1}e^{-x}\mathrm{d}x,\quad s>0$ is the Gamma function, we have

$$\frac{J_2(0)}{J_0(0)}=\frac{\frac{9\sqrt{3}}{16}B(\frac{3}{2},\frac{5}{2})}{\frac{3\sqrt{3}}{8}B(\frac{3}{2},\frac{3}{2})}=\frac{3}{2}\times\frac{\Gamma(\frac{3}{2})\Gamma(\frac{5}{2})}{\Gamma(4)}\times\frac{\Gamma(3)}{\Gamma(\frac{3}{2})\Gamma(\frac{3}{2})}=\frac{3}{2}\times\frac{\frac{3}{2}\times\Gamma(\frac{3}{2})\Gamma(3)}{3\times\Gamma(\frac{3}{2})\Gamma(3)}=\frac{3}{4}.$$

Similarly,

$$\frac{J_4(0)}{J_0(0)}=\frac{45}{64}.$$

□

Lemma 3.2. $\lim\limits_{k\to1/6}\frac{J_2(k)}{J_0(k)}=\lim\limits_{k\to1/6}\frac{J_4(k)}{J_0(k)}=1.$

Proof. By mean value theorem for integrals, we have

$$\lim_{k\to1/6}\frac{J_2(k)}{J_0(k)}=\lim_{z\to1}z^2=1.$$

Similarly,

$$\lim_{k\to1/6}\frac{J_4(k)}{J_0(k)}=\lim_{z\to1}z^4=1.$$

□

Lemma 3.3.

$$\begin{pmatrix}J_0\\J_2\\J_4\end{pmatrix}=\begin{pmatrix}3/2k & 0 & -1/4\\ 1/8k & k & -3/16\\ 15/128k & 3/16k & 3/4k-45/256\end{pmatrix}\begin{pmatrix}J_0'\\J_2'\\J_4'\end{pmatrix}.$$

Proof. Since $E^2=1/2z^4-1/3z^6-k$, we can get $E\mathrm{d}E/\mathrm{d}z=z^3-z^5$. J_0 can be calculated as follows.

$$\begin{aligned}J_0=\int_\alpha^\beta E\mathrm{d}z&=\int_\alpha^\beta E^2\frac{\mathrm{d}z}{E}\\&=\int_\alpha^\beta\left(\frac{1}{2}z^4-\frac{1}{3}z^6-k\right)\frac{\mathrm{d}z}{E}\\&=\int_\alpha^\beta\left[\frac{1}{2}z^4-\frac{1}{3}z\left(z^3-E\frac{\mathrm{d}E}{\mathrm{d}z}\right)-k\right]\frac{\mathrm{d}z}{E}\\&=\int_\alpha^\beta\left(\frac{1}{6}z^4-k\right)\frac{\mathrm{d}z}{E}+\frac{1}{3}\int_\alpha^\beta z\mathrm{d}E\\&=-\frac{1}{3}J_4'-\frac{1}{3}J_0+2kJ_0',\end{aligned}$$

that is,

$$J_0=\frac{3}{2}kJ_0'-\frac{1}{4}J_4'.$$

On the other hand, J_2, J_4 is calculated by the same method as J_0.

$$
\begin{aligned}
J_2 = \int_\alpha^\beta Ez^2 \mathrm{d}z = & \int_\alpha^\beta z^2 E^2 \frac{\mathrm{d}z}{E} \\
= & \int_\alpha^\beta z^2 \left(\frac{1}{2}z^4 - \frac{1}{3}z^6 - k\right) \frac{\mathrm{d}z}{E} \\
= & \int_\alpha^\beta \left(z^3 - E\frac{\mathrm{d}E}{\mathrm{d}z}\right)\left(-\frac{1}{3}z^3 + \frac{1}{2}z\right)\frac{\mathrm{d}z}{E} + 2kJ_2' \\
= & \int_\alpha^\beta \left(\frac{1}{2}z^4 - \frac{1}{3}z^6 - k\right) \frac{\mathrm{d}z}{E} + 2kJ_2' - 2kJ_0' \\
& + \frac{1}{3}\int_\alpha^\beta z^3 \mathrm{d}E - \frac{1}{2}\int_\alpha^\beta z \mathrm{d}E \\
= & J_0 + 2kJ_2' - 2kJ_0' + \frac{1}{2}J_0 - J_2,
\end{aligned}
$$

that is,

$$J_2 = \frac{1}{8}kJ_0' + kJ_2' - \frac{3}{16}J_4'.$$

Similarly, a direct computation shows that

$$
\begin{aligned}
J_4 = \int_\alpha^\beta Ez^4 \mathrm{d}z = & \int_\alpha^\beta z^4 E^2 \frac{\mathrm{d}z}{E} \\
= & \int_\alpha^\beta z^4 \left(\frac{1}{2}z^4 - \frac{1}{3}z^6 - k\right) \frac{\mathrm{d}z}{E} \\
= & \int_\alpha^\beta \left(z^3 - E\frac{\mathrm{d}E}{z}\right)\left(-\frac{1}{3}z^5 + \frac{1}{2}z^3\right)\frac{\mathrm{d}z}{E} + 2kJ_4' \\
= & \int_\alpha^\beta \left(\frac{1}{2}z^6 - \frac{1}{3}z^8 - z^2 k\right) \frac{\mathrm{d}z}{E} + 2kJ_4' - 2kJ_2' \\
& + \frac{1}{3}\int_\alpha^\beta z^5 \mathrm{d}E - \frac{1}{2}\int_\alpha^\beta z^3 \mathrm{d}E \\
= & J_2 + 2kJ_4' - 2kJ_2' + \frac{3}{2}J_2 - \frac{5}{3}J_4,
\end{aligned}
$$

that is,

$$J_4 = \frac{15}{128}kJ_0' + \frac{3}{16}kJ_2' + \left(\frac{3k}{4} - \frac{45}{256}\right)J_4'.$$

This proves the lemma. ∎

By Lemma 3.3, we can get the following lemma.

Lemma 3.4. *J_0, J_2 and J_4 satisfy the Picard-Fuchs equation*

$$
\begin{pmatrix} J_0' \\ J_2' \\ J_4' \end{pmatrix} = \frac{1}{\Delta}\begin{pmatrix} 4k - \frac{3}{4} & -\frac{1}{4} & \frac{4}{3} \\ -\frac{1}{2}k & 6k - \frac{5}{4} & \frac{4}{3} \\ -\frac{1}{2}k & -\frac{3}{2}k & 8k \end{pmatrix}\begin{pmatrix} J_0 \\ J_2 \\ J_4 \end{pmatrix},
$$

where $\Delta = k(6k-1)$.

Lemma 3.5. *$J_n, n = 6, 8, 10$ can be represented only by J_0, J_2 and J_4 as follows.*

$$J_6 = \frac{21}{20}J_4 - \frac{3}{10}kJ_0,$$
$$J_8 = \frac{189}{160}J_4 - \frac{3k}{4}J_2 - \frac{27}{80}kJ_0,$$
$$J_{10} = \left(\frac{6237}{4480} - \frac{15}{14}k\right)J_4 - \frac{99}{112}kJ_2 - \frac{891}{2240}kJ_0.$$

Proof. A direct computation shows that

$$\begin{aligned} J_6 = \int_\alpha^\beta Ez^6\mathrm{d}z = \int_\alpha^\beta z\left(z^3 - E\frac{\mathrm{d}E}{\mathrm{d}z}\right)E\mathrm{d}z \\ &= J_4 - \int_\alpha^\beta z\left(\frac{1}{2}z^4 - \frac{1}{3}z^6 - k\right)\mathrm{d}E \\ &= J_4 - \frac{7}{3}J_6 + \frac{5}{2}J_4 - kJ_0 \\ &= -\frac{7}{3}J_6 + \frac{7}{2}J_4 - kJ_0, \end{aligned}$$

which implies the first equality of this lemma. By the same manner, the last two equalities can be obtained. □

To analyze Q and R, we can represent them by J_0, J_2 and J_4. Applying Lemma 3.4 and Lemma 3.5, we get $R = J_0$, $R' = J_0'$, and

$$\begin{aligned} Q &= -2J_{10}' + 4J_8' - 2J_6' \\ &= -2\left(\frac{6237}{4480}J_4' - \frac{15}{14}kJ_4' - \frac{99}{112}kJ_2' - \frac{891}{2240}kJ_0' - \frac{15}{14}J_4 - \frac{99}{112}J_2 - \frac{891}{2240}J_0\right) \\ &\quad + 4\left(\frac{189}{160}J_4' - \frac{3k}{4}J_2' - \frac{27}{80}kJ_0' - \frac{3}{4}J_2 - \frac{27}{80}J_0\right) - 2\left(\frac{21}{20}J_4' - \frac{3}{10}kJ_0' - \frac{3}{10}J_0\right) \\ &= \left(-\frac{357}{2240} + \frac{15}{7}k\right)J_4' - \frac{69}{56}kJ_2' + \frac{51}{1120}kJ_0' + \frac{15}{7}J_4 - \frac{69}{56}J_2 + \frac{51}{1120}J_0 \\ &= \left(-\frac{357}{2240} + \frac{15}{7}k\right)\left(-\frac{1}{2\Delta}kJ_0 - \frac{3}{2\Delta}kJ_2 + \frac{8k}{\Delta}J_4\right) + \frac{15}{7}J_4 - \frac{69}{56}J_2 + \frac{51}{1120}J_0 \\ &\quad - \frac{69}{56}k\left(-\frac{k}{2\Delta}J_0 + \frac{24k-5}{4\Delta}J_2 + \frac{4}{3\Delta}J_4\right) + \frac{51}{1120}k\left(\frac{16k-3}{4\Delta}J_0 - \frac{1}{4\Delta}J_2 + \frac{4}{3\Delta}J_4\right) \\ &= 2\left(\frac{10}{7} + \frac{15}{14}\right)J_4 - \left(\frac{99}{56} + \frac{69}{56}\right)J_2 \\ &= 5J_4 - 3J_2. \end{aligned}$$

Next, we investigate the property of

$$Z = \frac{Q}{R} = 5\frac{J_4}{J_0} - 3\frac{J_2}{J_0}. \tag{3.2}$$

Let

$$\tilde{x} = \frac{J_2}{J_0},\ x^* = \frac{J_2'}{J_0'};\quad \tilde{y} = \frac{J_4}{J_0},\ y^* = \frac{J_4'}{J_0'}.$$

Then we have the following Lemma.

Lemma 3.6. *For $0 < k_i < 1/6$, $i = 1, 2$, if $\tilde{x}'(k_1) = 0$ and $\tilde{y}'(k_2) = 0$, then $3/4 < \tilde{x}(k_1) < 1$, $45/64 < \tilde{y}(k_2) < 1$.*

Proof. By Lemma 3.3, we get

$$\frac{3}{4}J_0 - J_2 = kJ_0' - kJ_2',$$

i.e.

$$\frac{3}{4} - \frac{J_2}{J_0} = k\frac{J_0'}{J_0}\left(1 - \frac{J_2'}{J_0'}\right).$$

If $\tilde{x}'(k_1) = 0$, then $\tilde{x} = x^* = J_2'/J_0'$ and

$$\frac{3}{4} - \tilde{x}(k_1) = k\frac{J_0'(k_1)}{J_0(k_1)}(1 - \tilde{x}(k_1)).$$

Since $J_0'(k_1)/J_0(k_1) < 0$, $0 < k < 1/6$, then

$$\left(\frac{3}{4} - \tilde{x}(k_1)\right)(1 - \tilde{x}(k_1)) < 0,$$

i.e.

$$\frac{3}{4} < \tilde{x}(k_1) < 1.$$

In the same manner, we can get $45/64 < \tilde{y}(k_2) < 1$. □

By Lemma 3.1, Lemma 3.2 and Lemma 3.6, we can easily get the following lemma.

Lemma 3.7. *For* $0 \le k \le 1/6$ *, we have*

$$\frac{3}{4} \le \tilde{x}(k) \le 1, \quad \frac{45}{64} \le \tilde{y}(k) \le 1.$$

Lemma 3.8. *For* $0 < k < 1/6$ *, we have* $Z'(k) > 0$.

Proof. By equality (3.2), we have

$$\begin{aligned} Z'(k) &= \left[5\frac{J_4(k)}{J_0(k)} - 3\frac{J_2(k)}{J_0(k)}\right]' \\ &= \frac{1}{{J_0}^2(k)}\left[5(J_0(k){J_4}'(k) - J_4(k){J_0}'(k)) + 3(J_2(k){J_0}'(k) - J_0(k){J_2}'(k))\right] \\ &= \frac{1}{\Delta}\Bigg[-k - \frac{3}{4}(\frac{J_2(k)}{J_0(k)})^2 - \frac{20}{3}(\frac{J_4(k)}{J_0(k)})^2 + \frac{21}{4}\frac{J_2(k)}{J_0(k)}\cdot\frac{J_4(k)}{J_0(k)} \\ &\quad + (20k - \frac{1}{4})\frac{J_4(k)}{J_0(k)} + (\frac{45k}{2} - 6)\frac{J_2(k)}{J_0(k)}\Bigg] \\ &= \frac{1}{\Delta}\left[-k - \frac{3}{4}\tilde{x}^2 - \frac{20}{3}\tilde{y}^2 + \frac{21}{4}\tilde{x}\tilde{y} + (\frac{45k}{2} - 6)\tilde{x} + (20k - \frac{1}{4})\tilde{y}\right] \\ &= \frac{1}{\Delta}\cdot F(\tilde{x}, \tilde{y}, k). \end{aligned}$$

where $F(\tilde{x}, \tilde{y}, k) = -k - 3/4\tilde{x}^2 - 20/3\tilde{y}^2 + 21/4\tilde{x}\tilde{y} + (45k/2 - 6)\tilde{x} + (20k - 1/4)\tilde{y}$.

In what follows we are going to determine the sign of $F(\tilde{x}, \tilde{y}, k)$. Firstly, we study the maximum or minimum values of the continuous function $F(\tilde{x}, \tilde{y}, k)$. Since $F(\tilde{x}, \tilde{y}, k)$ is differentiable, $F(\tilde{x}, \tilde{y}, k)$ has a maximum point or minimum point at either the point $M(\tilde{x}_0, \tilde{y}_0, k_0)$ for which satisfies the $\partial F/\partial\tilde{x} = \partial F/\partial\tilde{y} = \partial F/\partial k = 0$ or the points on the boundary. Since

$$\begin{cases} \frac{\partial F}{\partial \tilde{x}} = -\frac{3}{2}\tilde{x} + \frac{21}{4}\tilde{y} + \frac{45}{2}k - 6, \\ \frac{\partial F}{\partial \tilde{y}} = \frac{21}{4}\tilde{x} - \frac{40}{3}\tilde{y} + 20k - \frac{1}{4}, \\ \frac{\partial F}{\partial k} = -1 + \frac{45}{2}\tilde{x} + 20\tilde{y}. \end{cases}$$

By direct computation,

$$\begin{cases} \tilde{x}_0 = -\frac{251}{1610}, \\ \tilde{y}_0 = \frac{2903}{12880}, \\ k_0 = \frac{78703}{386400}. \end{cases}$$

By Lemma 3.7, it is easy to find that $M(\tilde{x}_0, \tilde{y}_0, k_0)$ is not in the interior of three-dimensional cuboid domains

$$ABCDEFGH : \{\frac{3}{4} \le \tilde{x} \le 1, \quad \frac{45}{64} \le \tilde{y} \le 1, \quad 0 \le k \le \frac{1}{6}\}.$$

Therefore, the extreme value of $F(\tilde{x}, \tilde{y}, k)$ must be on the boundary. For convenience, the six plane of the cuboid can be expressed as the following forms

$$ABCD := \{k = 0, \frac{3}{4} \le \tilde{x} \le 1, \quad \frac{45}{64} \le \tilde{y} \le 1\}; \quad EFGH := \{k = \frac{1}{6}, \frac{3}{4} \le \tilde{x} \le 1, \quad \frac{45}{64} \le \tilde{y} \le 1\};$$

$$ADHE := \{\tilde{x} = \frac{3}{4}, 0 \le k \le \frac{1}{6}, \quad \frac{45}{64} \le \tilde{y} \le 1\}; \quad BCGH := \{\tilde{x} = 1, \quad 0 \le k \le \frac{1}{6}, \quad \frac{45}{64} \le \tilde{y} \le 1\};$$

$$ABFE := \{\tilde{y} = \frac{45}{64}, 0 \le k \le \frac{1}{6}, \quad \frac{3}{4} \le \tilde{x} \le 1\}; \quad DCGH := \{\tilde{y} = 1, \quad 0 \le k \le \frac{1}{6}, \quad \frac{3}{4} \le \tilde{x} \le 1\}.$$

On one hand, we consider the rectangular plane $ABCD$, then we have

$$F(\tilde{x}, \tilde{y}, 0) = -\frac{20}{3}\tilde{y}^2 + \left(\frac{21}{4}\tilde{x} - \frac{1}{4}\right)\tilde{y} - \frac{3}{4}\tilde{x}^2 - 6\tilde{x}.$$

For $3/4 \le \tilde{x} \le 1$, the discriminan of the above equation about variable $\tilde{y}$ is $(21/4\tilde{x} - 1/4)^2 - 20\tilde{x}^2 - 160\tilde{x}$, which sign is negative, then for any $\tilde{y}$, we have $F(\tilde{x}, \tilde{y}, 0) < 0$. By the same argument, we know that $F(\tilde{x}, \tilde{y}, 1/6) < 0$ is also true at the rectangular plane $EFGH$.

On the other hand, we consider the rectangular plane $ADHE$, then we have

$$\begin{aligned} F\left(\frac{3}{4}, \tilde{y}, k\right) &= \frac{127}{8}k - \frac{20}{3}\tilde{y}^2 - \frac{315}{64} + 20\tilde{y}k + \frac{59}{16}\tilde{y} \\ &= \left(\frac{127}{8} + 20\tilde{y}\right)k - \frac{20}{3}\tilde{y}^2 - \frac{315}{64} + \frac{59}{16}\tilde{y}. \end{aligned}$$

Since $127/8 + 20\tilde{y} > 0$, then on $0 < k < 1/6$, the function $F(k)$ about variable k is monotonically increasing. Therefore, we only need to prove $F(3/4, \tilde{y}, 1/6) < 0$. Direct calculation shows that

$$F\left(\frac{3}{4}, \tilde{y}, \frac{1}{6}\right) = -\frac{20}{3}\tilde{y}^2 + \frac{337}{48}\tilde{y} - \frac{437}{192}.$$

Since $-20/3 < 0$ and the discriminan of the equation is negative, then for any $\tilde{y}$, we have $F(3/4, \tilde{y}, 1/6) < 0$. Therefore, at the rectangular plane $ADHE$, we have $F(3/4, \tilde{y}, k) < 0$. By the same arguments, on any $0 < k < 1/6$, $F(\tilde{x}, \tilde{y}, k) < 0$ is also satisfied at rectangular plane of $BCGH, ABFE$, and $DCGH$.

In summary, on the boundary of the three-dimensional rectangular $ABCDEFGH$, we have $F(\tilde{x}, \tilde{y}, k) < 0$. Since $\Delta < 0$, then

$$Z'(k) = \frac{1}{\Delta} \cdot F(\tilde{x}, \tilde{y}, k) > 0.$$

That is, the monotonicity of $Z(k)$ is increasing monotonically on any $0 < k < 1/6$. This completes the proof of Lemma 3.8. □

Proof of Proposition 3.1. By Lemma 3.1 and Lemma 3.2, we have

$$\lim_{k\to 0} Z(k) = \lim_{k\to 0}\left(5\frac{J_4(k)}{J_0(k)} - 3\frac{J_2(k)}{J_0(k)}\right) = \frac{81}{64},$$

$$\lim_{k\to \frac{1}{6}} Z(k) = \lim_{k\to \frac{1}{6}}\left(5\frac{J_4(k)}{J_0(k)} - 3\frac{J_2(k)}{J_0(k)}\right) = 2.$$

Since $Z(k)$ is increasing by Lemma 3.8, then we can obtain $81/64 < Z(k) < 2$. This completes the proof of Proposition 3.1. □

By Eq.(2.8) in section 2 and Proposition 3.1, we have $c_0'(k) < 0$. Since $k \in [2\delta, 1/6] \subset (0, 1/6]$, we obtain $c_0(\delta) \to \frac{8}{9}$ as $\delta \to 0$. Therefore we have the following lemma.

Lemma 3.9. *For $0 < k \le 1/6$, (z_0, c_0) satisfies the limit speed condition (2.8). Moreover, $c_0'(k) < 0$, and*

$$\frac{\sqrt{2}}{2} \le c_0(k) < \frac{8}{9}, \quad \lim_{k\to 0} c_0(k) = \frac{8}{9}, \quad \lim_{k\to \frac{1}{6}} c_0(k) = \frac{\sqrt{2}}{2}.$$

Proof of Theorem 1.1. In Lemma 3.9, we have proved the Theorem 1.1(2). Since $\frac{\partial\tilde{\Phi}}{\partial c}(\alpha(k), c_0, 0) = \int z_0''^2 \mathrm{d}\tau + \int 1/c^2 z_0'^2 \mathrm{d}\tau > 0$, we can solve the equation $\tilde{\Phi} = 0$ by the implicit function theorem for $k \in [2\delta, 1/6]$. That is, there exists a unique smooth function $c_k(\varepsilon) = c(\varepsilon, k)$ and $\varepsilon \in (0, \varepsilon^*)$ so that

$$\tilde{\Phi}(\alpha(k), c(\varepsilon, k), \varepsilon) = 0 \quad for \quad 2\delta \le k \le \frac{1}{6}, \quad 0 < \varepsilon < \varepsilon^*.$$

where $k = 2h$, therefore we get the Theorem 1.1(1) and (3). □

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