Simple Amazons endgames and their connection to Hamilton circuits in cubic subgrid graphs

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Abstract. Amazons is a young board game with simple rules and a high branching factor, which makes it a suitable test-bed for planning research. This paper considers the computational complexity of Amazons puzzles and restricted Amazons endgames. We first prove the NP-completeness of the Hamilton circuit problem for cubic subgraphs of the integer grid. This result is then used to show that solving Amazons puzzles is an NP-complete task and determining the winner of simple Amazons endgames is NP-equivalent.

Keywords: Amazons endgame, puzzle, NP-complete, planning

1 Introduction

The success of full–width search and total enumeration in certain combinatorial problems – such as Rubik's cube [8], Othello [1], checkers [11], and chess [3] – masks the lack of progress in the planning and reasoning departments. The consequences are apparent: in spite of vast hardware speed-ups, hardly any AI system can pass the Turing test except for very specialized tasks. In the domain of games the problem becomes evident if we increase the number of move choices from dozens to thousands. If a system uses sophisticated pruning techniques it may still find reasonable moves. However, we can easily turn up the heat by decreasing the impact of single moves (which increases the length of move sequences) or replacing slow turn based play by fast real-time action. At this point even the greatest systems using traditional approaches look pathetic compared to human abilities. Prominent examples are real-time war simulation games – such as Starcraft[1] – in which the computer AIs desperately try to coordinate combat units. Currently, their only way of winning against humans is by starting with a considerable material advantage or simply by cheating.

In order to push planning and reasoning research, we need to focus on tasks that require goal directed search in order to cope with vast state spaces. Moreover, the major goals should be simple enough to be in reach of current machine learning techniques. Finally, the tasks should be suited to human mental abilities because this is the current AI benchmark per se.

Amazons is a young board game that is beginning to attract researchers for these reasons. It is played on an $n \times n$ board (usually $n = 10$). Both players have four amazons. A move consists of picking an amazon to move like a chess queen

[1] Starcraft is a trademark of Blizzard Entertainment

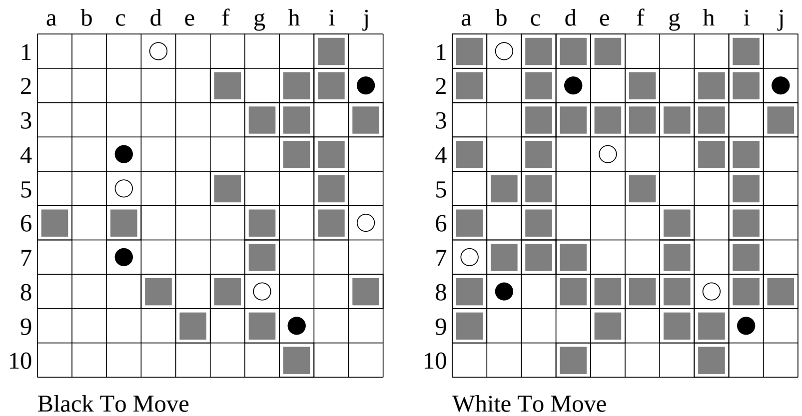


Fig. 1. Typical middle and endgame positions

and shooting an arrow in queen direction from the amazon's destination to an empty square. This square gets blocked for the remainder of the game and no amazon or arrow can pass it. Arrows are not allowed to pass amazons either and amazons can not be captured. Blocking squares is mandatory. The game proceeds in turns and the first player without any legal move loses. Figure 1 shows two typical Amazons positions. In the standard starting position all amazons are evenly distributed along the four edges. To make the game more interesting one can also place amazons randomly after blocking a small number of squares. Both players then estimate the final game outcome expressed as a move surplus of the player to move. The average estimate is used to assign colors and to determine the winner when the game is finished.

Amazons strategy is based on mobility and territory. Its high branching factor (more than 2000 opening moves on a 10x10 board!) limits the scope of full-width search and known forward pruning techniques considerably. State space sizes of PSPACE-complete puzzles like Sokoban [2] or RushHour [4] are also huge. The difference, however, is that the length of Amazons games is limited by the board size whereas move sequences in the mentioned puzzles can have exponential length. This makes solving hard instances of those puzzles less attractive for human players. Compared with the Asian board game Go, Amazons shares the property that in endgames the position gets split into separate subgames. This allows combinatorial game theory to step in and provide means of finding optimal moves faster than traditional approaches. On the other hand, the notorious problem of evaluating Go positions statically [9] does not seem to have an Amazons counterpart. As shown in past computer Amazons tournaments and in computer games against advanced human players, evaluations based on square-access-distance lead to reasonable (but still far from perfect) play.

In this paper we consider the computational complexity of solving *simple* Amazons puzzles and endgames. In these games amazons of equal color are located in their own, entirely sealed off territories. Thus, both opponents are separated and the winner is determined by the total number of moves each player can make in her own territories and whose turn it is. Because this scenario often occurs in actual games it would be helpful to incorporate automatic endgame scorers into Amazons game servers (e.g. the Generic Game Server (GGS) at `telnet://ftp.nj.nec.com:5000`), which quickly shortcut boring straight-forward move sequences. It turns out, however, that determining the winner even of simple Amazons endgames in general is NP-equivalent. This means that most likely there is no fast algorithm for solving the general problem, and we have to rely on clever heuristics to find (approximate) solutions to small problem instances in limited time.

In what follows, we first show that the Hamilton circuit problem and related problems are NP-complete for cubic subgraphs of the integer grid. We then use these results to prove that deciding whether an amazon can make a certain number of moves in a given board region is an NP-complete task, too. Finally, we conclude that simple Amazons endgames are NP-equivalent and motivate future Amazons research.

2 Hamilton circuits in cubic subgrid graphs

Definition 1. *Let G^∞ be the infinite graph consisting of all points of the plane with integer coordinates and edges connecting points with Euclidean distance one. Finite subgraphs of G^∞ are called* <u>*subgrid graphs*</u>*. Subgrid graphs with nodes of degree at most three are called* <u>*cubic subgrid graphs*</u>*.* <u>*Grid graphs*</u> *are finite node induced subgraphs of G^∞.*

In [6] it is shown that the Hamilton circuit problem for grid graphs is NP-complete. Nodes in grid graphs can have degree four, which makes this result impractical for proving the hardness of Amazons problems. This is because there is no easy way of modeling 4-way intersections that can be traversed only once – as we will see later. However, the proof ideas in [6] can be refined such that the reduction leads to cubic subgrid graphs which can be modeled by Amazons positions without much difficulty.

Theorem 1. *The set HC3G of all cubic subgrid graphs with a Hamilton circuit is NP-complete.*

Proof. Guessing a potential Hamilton circuit in a given cubic subgrid graph and verifying it in polynomial time shows that HC3G belongs to NP. In what follows we show that the set HCB3P of bipartite cubic planar graphs with a Hamilton circuit can be polynomial time reduced to HC3G. This concludes the proof because HCB3P is known to be NP-complete [10, 6].

Mapping a given bipartite cubic planar graph G into a cubic subgrid graph G_3 while preserving the Hamilton circuit property is a three step process illustrated in Figure 2: M_1 transforms G into a cubic orthogonal drawing. This task can be accomplished in linear time and space as shown in [7]. Cubic orthogonal drawings are not necessarily cubic subgrid graphs because node connections

G (bipartite cubic planar graph)

$\underset{M_1}{\longmapsto}$ G_1 (cubic orthogonal drawing + intermediate nodes)

$\underset{M_2}{\longmapsto}$ G_2 (parity preserving cubic subgrid graph)

$\underset{M_3}{\longmapsto}$ G_3 (Hamilton circuit preserving cubic subgrid graph)

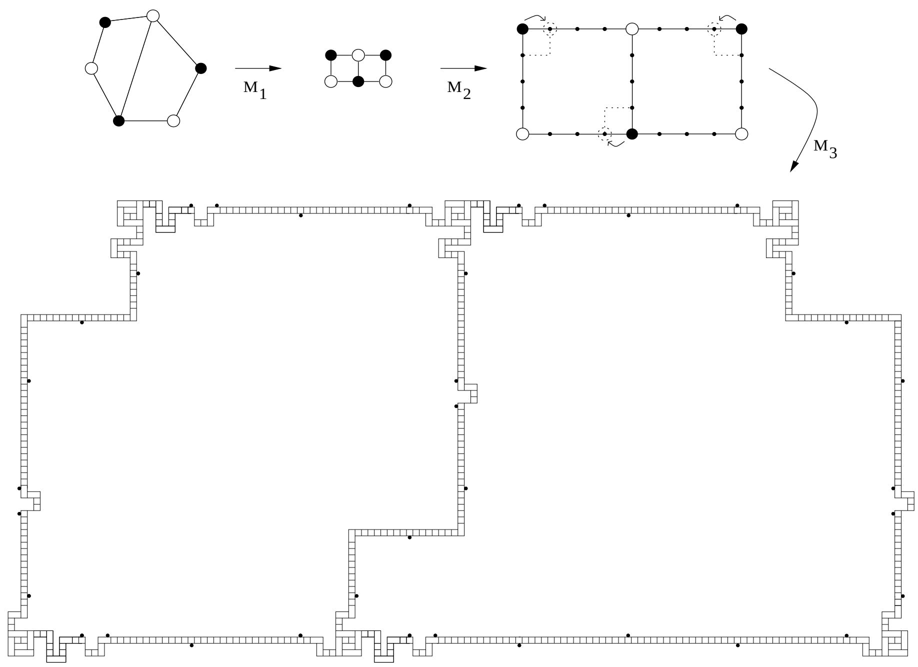


Fig. 2. Transformation example

may be longer than the unit grid length. Adding the missing intermediate nodes solves this problem. In general, however, the resulting graph does not preserve the Hamilton circuit property. To save this property, a second mapping, M_2, scales up the augmented orthogonal drawing by a factor of four first. Then – if necessary – it moves images of G nodes one grid position to the left or right and reconnects the edges to adjust the parity $(x(v)+y(v) \bmod 2)$ with respect to the original node partition (G is bipartite). Thus, in G_2 the images of the original nodes are connected by simple paths of odd length. This is necessary for applying the last transformation, M_3, which replaces all nodes of G_2 by (adjusted) copies of the 17x17 cluster and strips shown in Figure 3. Original nodes of degree two are replaced by clusters from which one tentacle has been removed (w.l.o.g. there are no nodes of degree one). Each component has some outgoing edges marked with black dots. When connecting components the respective markers have to match. The odd distance of original node images in G_2 ensures a unique matching.

Finally, one reflector gadget (shown in Figure 3) is placed in each component connecting strip. The resulting graph G_3 is a cubic subgrid graph because all nodes in the clusters and strips have degree at most three and connecting the components does not increase degrees. Since the entire graph transformation obviously can be computed in polynomial time, the proof rests on showing

$$G \text{ has a Hamilton circuit } \iff G_3 \text{ has a Hamilton circuit.}$$

"$\Rightarrow$": Starting with a Hamilton circuit p in G we construct a Hamilton circuit in G_3 by traversing strips and clusters as follows: beginning at corners of the cluster cores (Figure 4), strips and tentacles corresponding to edges in p are traversed by

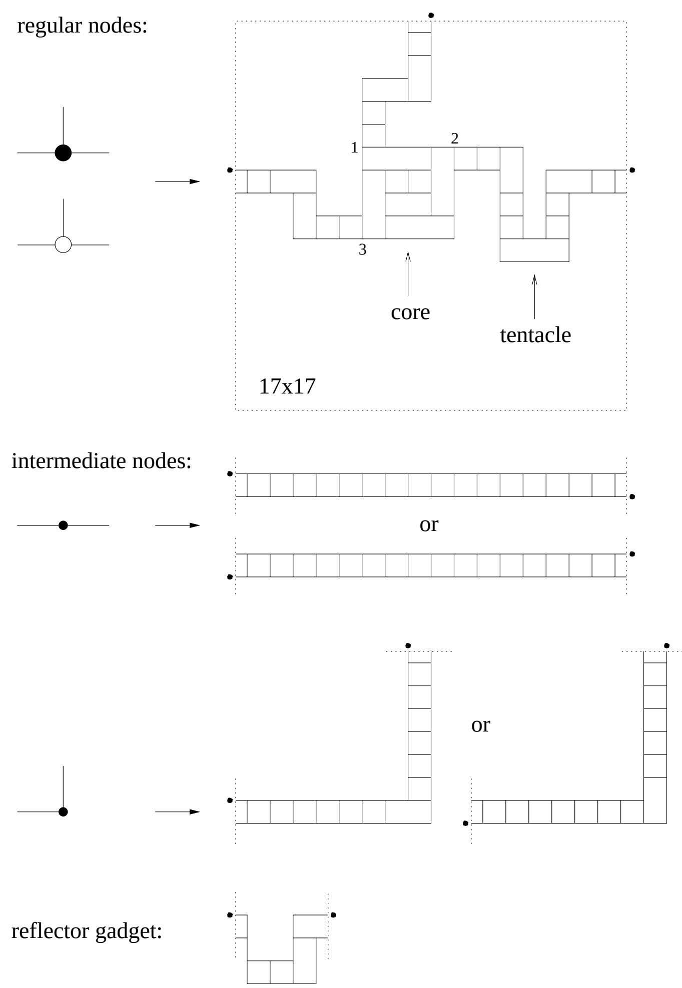
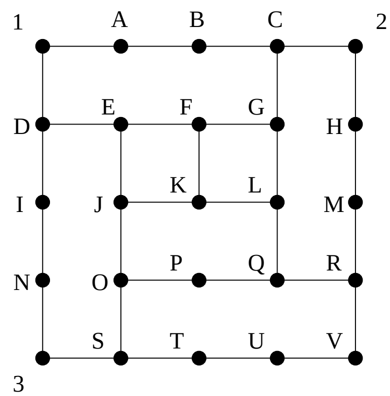


Fig. 3. Transformation M_3



Hamilton paths:

1→2: 1-A-B-C-G-L-Q-P-O-J-K-F-E-D-I-N-3-S-T-U-V-R-M-H-2
1→3: 1-A-B-C-2-H-M-R-V-U-T-S-O-P-Q-L-G-F-K-J-E-D-I-N-3
2→3: 2-H-M-R-V-U-T-S-O-P-Q-L-K-J-E-F-G-C-B-A-1-D-I-N-3

Fig. 4. The cluster core and its Hamilton paths

battlements paths (Figure 5). The remaining ones are covered by parallel paths (N.B.: the component markers indicate edges visited by battlements paths. Their positions determine the edges along the strips and tentacles that can be omitted to ensure that degrees do not exceed three). It remains to connect the nodes in the cluster cores. The core has been designed in such a way that a Hamilton path exists between each pair of three corners (and all nodes again have degree at most three). Thus, the two battlements paths ending in corners of each core can be connected by Hamilton paths. If parallel paths originate from the third corner of some cores, the corresponding corner edges have to be removed from the inter-core Hamilton paths. The result is a Hamilton circuit in G_3.

"⇐": Given a Hamilton circuit in G_3, we claim that tentacles and strips covered by parallel paths can be removed while maintaining a Hamilton circuit in the remainder of the graph. Once all these strips and tentacles have been removed from G_3, Hamilton paths in cores and battlements paths remain. These form a Hamilton circuit which corresponds to a Hamilton circuit in G because each cluster is connected to two neighboring clusters and clusters are the images of the original nodes of G. Figure 6 illustrates the parallel path scenario. To maintain a

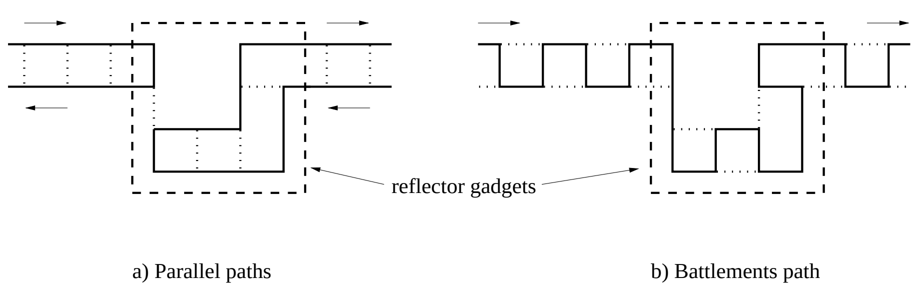


Fig. 5. The two ways of covering strips

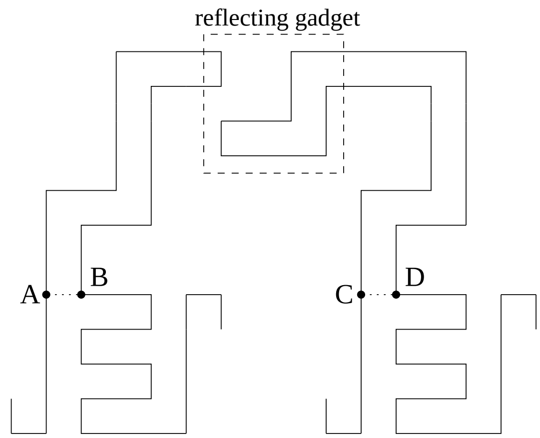


Fig. 6. Parallel path scenario

Hamilton circuit the parallel paths are replaced by the edges (A,B) and (C,D). At this point it is important to note that reflecting gadgets are necessary to prevent parallel paths (A..D) (B..C) which would invalidate this part of the proof.

□

Definition 2. *A collision path in a graph is an edge disjoint path* $v_0 e_1 v_1 e_2 ... e_l v_l$ *with at most one node repetition (i.e.* $\exists\ i, j$ *with* $v_i = v_j$ *and* $i \neq j$*) which ends right after the repetition, if there is one.*

Collision path examples are shown in Figure 7a).

Corollary 1. *The sets of all cubic subgrid graphs G with the following properties are NP-complete:*

a) G has a Hamilton path with specified endpoints
b) G has a Hamilton path
c) G has a collision path of length $|V_G| - 1$ *with specified starting point*
d) G has a collision path of length $|V_G| - 1$

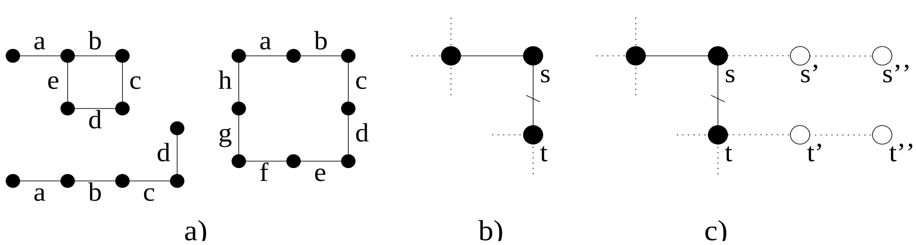


Fig. 7. Collision paths and graph adjustments

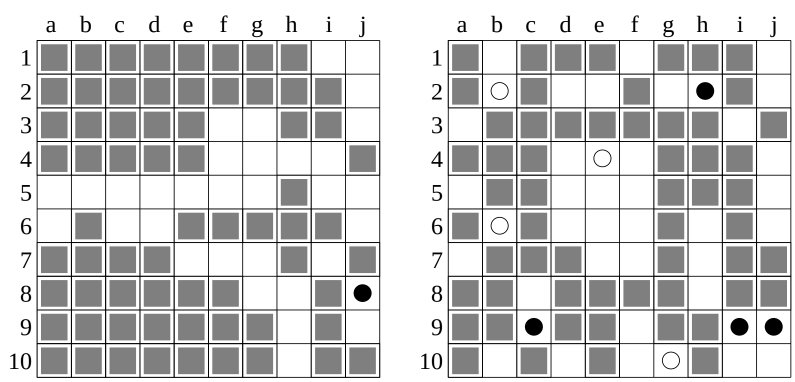
Proof. In all cases the NP membership is obvious. NP-hardness is shown by reducing HC3G.

Cases a) & b): Let G be a finite connected cubic subgrid graph without nodes of degree one. Then G has a "corner" node s of degree two (Figure 7b), i.e. s has no upper neighbor and there are no nodes in G to the right of s and its lower neighbor t. Such a node s can be found by first maximizing x coordinates of nodes in G and then maximizing the y coordinates on the resulting vertical line. Thus, G has a Hamilton circuit if and only if $G-(s,t)$ has a Hamilton path with endpoints s and t. Moreover, if two nodes s' and t' are added and connected to s resp. t (Figure 7c), it follows that G has a Hamilton circuit if and only if there is a Hamilton path in $G-(s,t)+(s,s')+(t,t')$.

Cases c) & d): In these cases we extend G by four nodes s', s'', t', and t'' (Figure 7c) to form a new graph G'. There are no paths with a collision in G' of length $|V_{G'}|$ because collisions can only occur in the G part. However, if there is a collision in this part, the path ends there and its length is less than $|V_{G'}|$. Thus, the only collision paths of length $|V_{G'}|-1$ are Hamilton paths from s'' to t''. This shows that G has a Hamilton circuit if and only if G' has a collision path of length $|V_{G'}|-1$ (d). Since s'' is start or endpoint of all such paths, c) follows as well. □

3 Simple Amazons endgames

Definition 3. *A set of (vertically, horizontally, or diagonally) connected empty squares that is entirely surrounded by blocked squares or board edges together with amazons of one color placed inside the region is called an Amazons puzzle. An Amazons puzzle solution is a move sequence of maximum length.*



Is there a move sequence that fills the entire region? (yes)

Black To Move — Who wins? (Black)

Fig. 8. An Amazons puzzle and a simple Amazons endgame

Definition 4. *Simple Amazons endgames are sequences of puzzles for amazons of both colors. Black is to move first. Black wins the simple Amazons endgame if the total solution length of Black's puzzles is greater than White's. Otherwise, Black loses (Figure 8 illustrates both definitions).*

Theorem 2. *The set* $AP := \{(p, b) \mid$ *Amazons puzzle* p *has solution length at least* $b\}$ *is NP-complete.*

Proof. We note that for a given position and solution length a move sequence can be guessed and verified in polynomial time. Hence, AP is an element of NP. We show AP's NP-hardness by mapping cubic subgrid graphs G into pairs (p, b) such that:

G has a collision path of length $n-1$ ($n = |V_G|$) starting in a specified node s if and only if the amazon can make at least b moves in position p. (*)

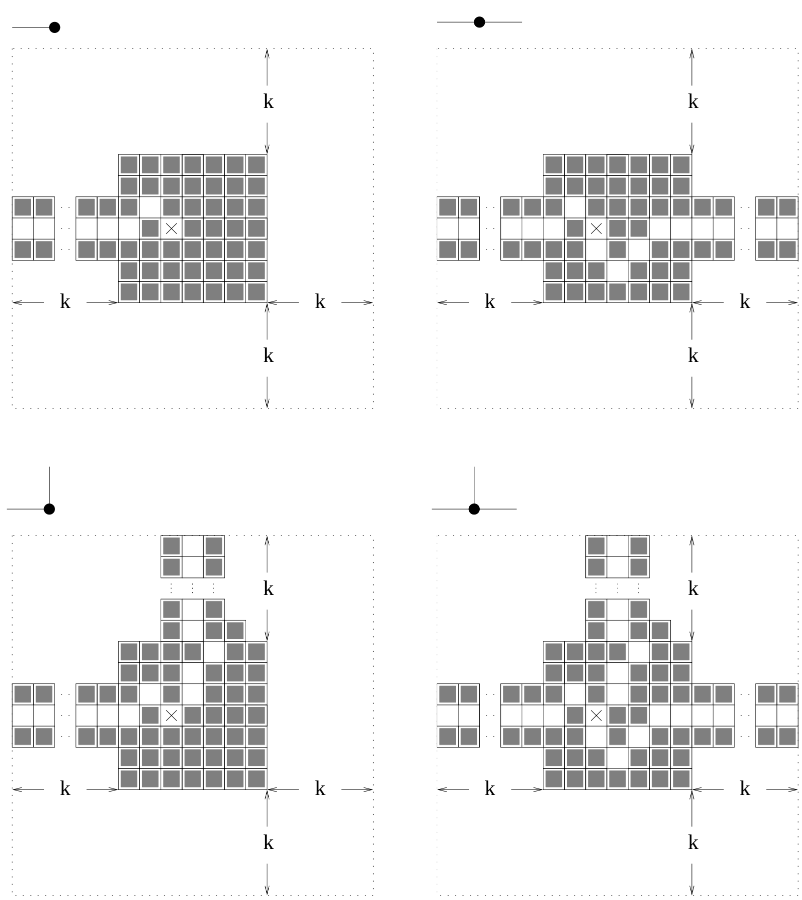


Fig. 9. Mapping parts of a cubic subgrid graph into Amazons board regions

Nodes and their connections are mapped into board regions according to Figure 9. The amazon is placed on the marked center square in the image of the starting node. Figure 10 illustrates the transformation that defines p. The regions have been designed such that the amazon on her way from region to region must visit and block the marked central squares, and corridors can only be traversed once. Thus, the sequence of visited regions corresponds to a collision path in G starting with s.

Let m be the maximum number of moves the amazon can make in position p and l the maximum length of collision paths in G. We pick corridor length $12n$ (i.e. $k = 6n$) and claim

$$l \geq n - 1 \iff m \geq 12(n^2 - n). \tag{1}$$

The theorem follows by setting the move threshold b to $12(n^2 - n)$ in (*).

To prove (1) we consider upper and lower bounds on the number of moves in a maximum move sequence corresponding to a collision path of length l. Clearly, $m \geq l(2k + 1)$ holds because the amazon can traverse at least the corridors square by square. On the other hand, $m \leq l(C + 2k + 1) + C$, where C is the maximum number of empty squares in the 7x7 region centers. Inserting $C = 11$ and $k = 6n$ leads to $m \geq l(12n + 1)$ and $m \leq l(12n + 12) + 11$. Therefore, we can conclude

$$\begin{aligned} l \geq n - 1 \;&\Rightarrow\; m \geq (n-1)(12n+1) &&= 12n^2 - 11n - 1 \\ l \leq n - 2 \;&\Rightarrow\; m \leq (n-2)(12n+12) + 11 &&= 12n^2 - 12n - 13, \end{aligned}$$

from which (1) follows. □

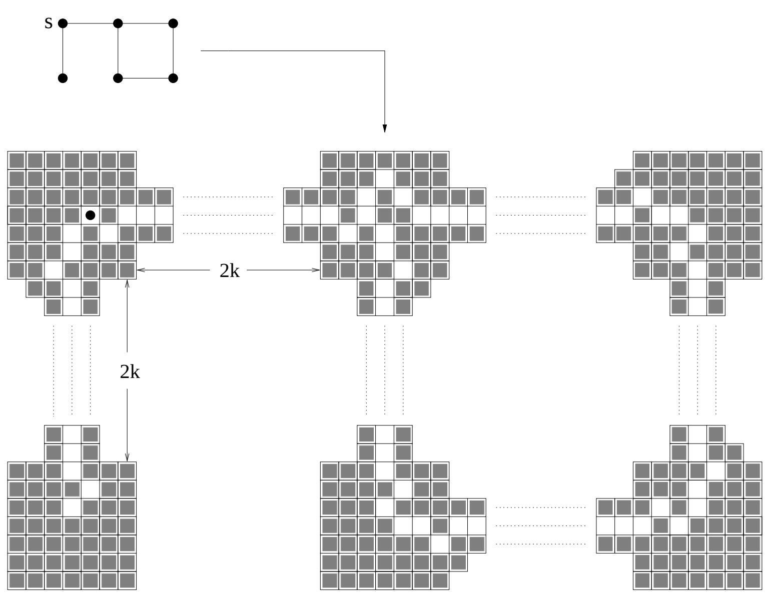
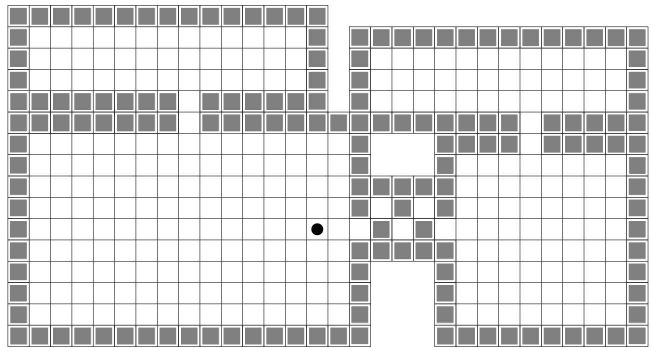


Fig. 10. Mapping example



Is there a move sequence that fills the entire region? (yes)

Fig. 11. Easy for humans – hard for short-sighted programs

Corollary 2. *The set SAE* $:= \{p \mid$ *Black wins simple Amazons endgame* $p\}$ *is NP-equivalent. Therefore, determining the winner of simple Amazons endgames in polynomial time is possible if and only if* $P = NP$.

Proof. (Following the terminology established in [5], a set S is called NP-equivalent if there are two NP-complete sets A and B with $A \propto_T S$ and $S \propto_T B$, where $\propto_T$ denotes oracle Turing reducibility in polynomial time) We show AP $\propto_T$ SAE $\propto_T$ AP.

AP $\propto_T$ SAE: from a given Amazons puzzle p and move limit b we construct a simple Amazons endgame by adding a strip of b empty squares that is surrounded by blocked squares. We place a white amazon on this strip and use black amazons in the puzzle region. It is easy to see that p has solution length $\geq b$, if and only if Black wins the endgame.

SAE $\propto_T$ AP: the solution length of each puzzle component of a given endgame can be found in polynomial time by a binary search that is guided by constant time queries of the AP oracle. The winner can then be determined by comparing both players' total solution length. □

4 Outlook

We have shown the hardness of Amazons puzzles by reducing an NP-complete graph problem. Since most computer scientists accept P $\neq$ NP as a working hypothesis, this result can be regarded as a "poor man's lower bound," meaning that (most likely) there is no polynomial time algorithm that can solve arbitrary Amazons puzzles. This limitation, however, applies only to the general problem. In particular, there exist types of large puzzles that humans find easy to solve, whereas current programs – being notoriously short-sighted – do not (Figure 11). This example clearly demonstrates the necessity of planning and reasoning in AI systems. Amazons, with its simple rule set and large branching factor, is therefore an ideal test-bed for future research on single agent and adversarial planning.

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