From 2d to 3d geometry: discovering, conjecturing, proving

Daniela Ferrarello – Maria Flavia Mammana – Mario Pennisi
Dipartimento di Matematica e Informatica
Università di Catania, Italy
ferrarello@dmi.unict.it - fmammana@dmi.unict.it - pennisi@dmi.unict.it

Abstract
Geometry is offered a small time in high school teaching, and, always more often, 3d geometry is even not done at all. This is due to several factors, from drawing difficulties and visual difficulties to lack of time.
Here we present a classroom activity that moving from 2d geometry brings students into 3d geometry in a “natural way”. Students face with some surprising analogies between figures of the plane and figures of the space. And even more surprisingly, the analogy still holds in proving those properties. We will also present the results of the activity that has been experimented in different classes of some high schools in Sicily.

Introduction

We live in a three-dimensional world and we have difficulties in the study of three-dimensional geometry. The first approach to space geometry is already in childhood and without any difficulty: kindergarten teachers would speak of the ball, of the cube, of the cone to their children, associating the concepts with objects familiar to them- the ball or the cube children play with, or the ice cream cone. At this age children often call a circle a ball and a square a cube. There seems to be a "dominance" of space geometry on plane geometry. The learning process of the three-dimensional geometry then goes to elementary and middle school on the study of surfaces and volumes. So far, so good.
The main difficulties concern then the study of space geometry at high school, i.e. when proving activity really enters in students’ life. The three-dimensional Euclidean geometry is almost a taboo. This is because, even though we live in a three dimensional space, teaching / learning three dimensional geometry presents difficulties in graphic representation as well as in mental visualization: it is not easy to draw a three dimensional figure on a plane and it is not easy to imagine the mutual position of objects in space. In addition, although the students are familiar with the 3D world, they do not have as much familiarity with the proofs in 3D geometry. Space geometry is indeed part of school programs but, in the classroom activity, it is often relegated to the end of the year and therefore covered superficially at best, if not left out completely. Moreover, very simple problems on 3d geometry are assigned in Italian high school final exams, and quite often they are on volumes and surfaces. Teachers then prefer to spend time on something else other than 3d Euclidean geometry, forgetting how important it is for students’ education.
It is necessary to retrieve the interest of students and teachers to this topic, also taking into account the results of Italian students in national and international tests. The results highlight the weakness of our students on this topic, which is confirmed also by the difficulties that students have in the solution of 3d geometry problems at the math contest organized by our department (DMI), the Etniade (Aleo et al, 2011). The Etniade is a contest for students of the first two years of high school, taking place at the DMI for 23 years, with the participation of about 300 students of Eastern Sicily. Students who participate are the "best" of about 14,000 students and the problems that are hard to solve for them are the geometry ones!, In particular solid geometry.
Our proposal aims at retrieving students, and teachers, interest in solid geometry. The idea is inspired by an existing analogy in definitions and properties of figures of the plane, the quadrilaterals, and other of the space, the tetrahedra (Mammana, Micale & Pennisi, 2009).
In general, the activity starts by analyzing definitions and properties of suitable defined quadrilaterals and continues with the analogous definitions and properties of tetrahedra. The

“passage” from plane to space is done with the aid of a dynamic geometry software, Cabri Géomètre 3D.

Theoretical framework

The classroom activity we are going to present has been developed within a double laboratory course given at the University of Catania, in the Mathematics and Computer Science Department, in the Piano Lauree Scientifiche project. This project aims to improve the relationship between students and basic scientific subjects: chemistry, physics, mathematics and material sciences. The choice we made in the last years was to work with the teachers in preparing all the material. In fact, we decided not to have a “traditional course” for in service teachers, but rather we moved in the direction of Communities of Practice (CoP). For the teachers' true professional growth, "training about innovation", which acquaints them with new practices to apply, is not enough; "training for innovation" is needed, which places the teachers in conditions to improve their own instructional/educational methodology. Growth of professional knowledge and competence is a process tied not only to the capacity to recognize and resolve problems in the professional context in which they arise but especially the capacity to reflect on one's own actions and on the possibility to do so within a group of individuals who practice the same activity and share the same operative path, indeed a community of practice. Creating environments in which members of the group share experience and knowledge developed in their own professional practice and discuss with others the difficulties encountered and the solutions adopted to overcome them, means that the teacher can be an aware experimenter of novelty, establishing himself as a researcher. It is possible in this way to truly share not only the product but also the process that leads to a result with the active participation and criticism of each (Mammana et al, 2009).

The activity with the teachers is perfectly contextualized in the theoretical framework of the TPACK, Technological Pedagogical Content Knowledge, that proposes the interplay of the three different types of knowledge: Content (math content, in this case), Pedagogy and Technology. The goal is not to add technology to traditional approach, but to understand how the knowledge of technology can be used to get into the subjects and develop them, and to understand how it can support and improve learning.

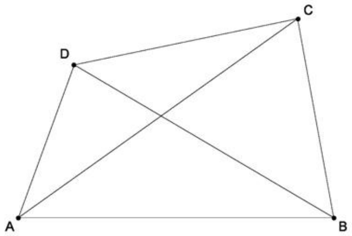
Moreover, we decided to have the classroom activity with the students structured as Mathematics Laboratories in the sense of Curricula UMI (MPI 2003). In this kind of activity students are given a task, not too hard neither too difficult, and, guided by the teachers, they are called to work by themselves for solving the task. The idea is “not to give the whole secret once in a time, but to make the students guess it. Make them discover by themselves the most possible.” (Polya, 1962).

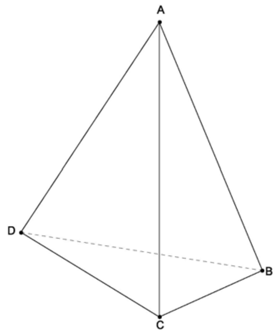
In general, the activity with teachers and the lab with students aim to a didactics thought as the intersection between the teaching action of the teacher and the the learner, in line with the concept of enactivism (Rossi, 2011).

The topic

Sometimes you can discover that apparently different things are more similar than you think, just changing your perspective. This happens with a quadrilateral and a tetrahedron: students can judge them as belonging to different fields (2d and 3d geometry), but surprisingly they can discover so many analogies between the two geometrical objects.

The crucial point is to see a tetrahedron coming from a quadrilateral, by “extracting” a vertex from the plane into the space. Of course we need to uniform some definitions in order to have a perfect matching: first, while in a conventional quadrilateral we have 4 sides and two diagonals, in a tetrahedron we have 6 edges. So we need to define a “non conventional” quadrilateral with 6 edges; this is possible because 6 are the possible couples of 4 vertices, i.e. we consider as edges of a quadrilateral not only conventional sides, but also diagonals. This is sufficient to construct an expressive analogy between 4 vertices objects in two dimensions and 4 vertices objects in three dimensions.





In the following table we summarize the analogies we handled in our activity.

TABLE OF THE ANALOGIES

QUADRILATERALS	TETRAHEDRA
Q is a ***convex quadrilateral*** with ***vertices*** *A, B, C, D*. The points A, B, C, D are such that any three of them are non-collinear. The vertices detect six segments AB, BC, CD, DA, AC, BD, that are called ***edges***. The edges of ***Q*** are the four sides and the two diagonals. Two ***edges*** are said to be ***opposite*** if they do not have common vertices. They are opposite edges: AB and CD, BC and DA, AC and BD, that is, either two opposite sides or the two diagonals. We call ***faces*** of ***Q*** the triangles determined by three vertices of ***Q***. There are four faces: ABC, BCD, CDA, DAB. A ***vertex*** and a ***face*** are said to be ***opposite*** if the vertex does not belong to the face. For each vertex there is one and only one opposite face.	***T*** is a ***tetrahedron*** with ***vertices*** *A, B, C, D*. The points A, B, C, D are non coplanar. The vertices detect six segments AB, BC, CD, DA, AC, BD, that are called ***edges***. Two ***edges*** are said to be ***opposite*** if they do not have common vertices. They are opposite edges: AB and CD, BC and DA, AC and BD. We call ***faces*** of ***T*** the triangles determined by three vertices of ***T***. There are four faces: ABC, BCD, CDA, DAB. A ***vertex*** and a ***face*** are said to be ***opposite*** if the vertex does not belong to the face. For each vertex there is one and only one opposite face.
The segment joining the midpoints of two opposite edges of ***Q*** is called ***bimedian*** of ***Q***. ***Q*** has three bimedians, two relative to a pair of opposite sides and one relative to the diagonals. ***Theorem 1. The three bimedians of a quadrilateral all pass through one point.*** The point G common to the three bimedians of ***Q*** is called the *centroid* of ***Q***. ***Theorem 2. The centroid bisects each bimedian.***	The segment joining the midpoints of two opposite edges of ***T*** is called ***bimedian*** of ***T***. ***T*** has three bimedians. ***Theorem 1. The three bimedians of a tetrahedron all pass through one point.*** The point G common to the three bimedians of ***T*** is called the *centroid* of ***T***. ***Theorem 2. The centroid bisects each bimedian.***

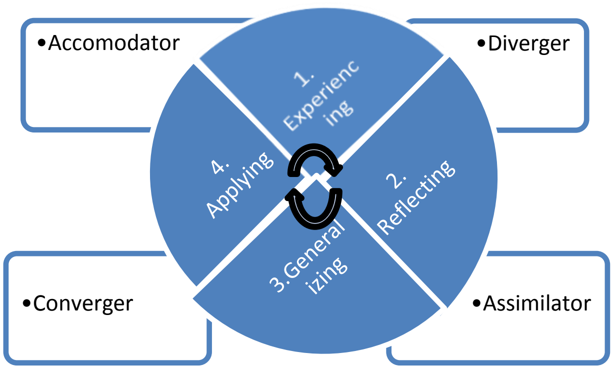
The segment joining a vertex of ***Q*** with the centroid of the opposite face is called ***median*** of ***Q***. ***Q*** has four medians. ***Theorem 3. The four medians of a quadrilateral meet in its centroid.*** ***Theorem 4. The centroid of a quadrilateral divides each median in the ratio 1:3, the longer segment being on the side of the vertex of Q.***	The segment joining a vertex of ***T*** with the centroid of the opposite face is called ***median*** of ***T***. ***T*** has four medians. ***Theorem 3. The four medians of a tetrahedron meet in its centroid (Commandino's Theorem).*** ***Theorem 4. The centroid of a tetrahedron divides each median in the ratio 1:3, the longer segment being on the side of the vertex of T.***

Proofs in space of the theorem above are very similar to proofs in plane. Other analogies and proofs of the properties can be found in (Mammana, et al 2009).

The activity

The format we used to carry out this activity is the double lab described in (Ferrarello, Mammana and Pennisi, 2013): the first laboratory involves teachers and its aim is to build teaching materials starting from contents given by tutors (the authors of the paper) and a prototype of worksheet to fill with the content. This phase is very important because teachers keep in touch, collaborate, have time and space to get advises from each other and work together, as their students do in the second lab.

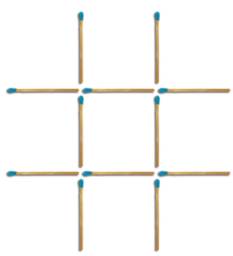
The second lab involves students and the modality is not different from the previous one, in fact students learn in a collaborative environment, they not only learn the contents, but also learn how to discover math properties by themselves; in fact worksheets we are going to introduce in the next, are not exhaustive, but have to be filled, giving to every single student time and space to observe, to think, to guessing properties, to generalize, to conjecture and finally to prove, what is denied to students who passively receive a frontal lectures.



The whole activity is divided in five phases, according to the theory of David Kolb (Kolb & Fry, 1975):

1. Experiencing - *Concrete experience*
2. Reflecting - *Reflective observation*
3. Generalizing - *Abstract conceptualization*
4. Applying - *Active experimentation*
5. Re-experiencing

In such a way all the learning styles adopted by Kolb are kept: the converger and diverger thought types, as well as the assimilator and accommodator cognitive processes.

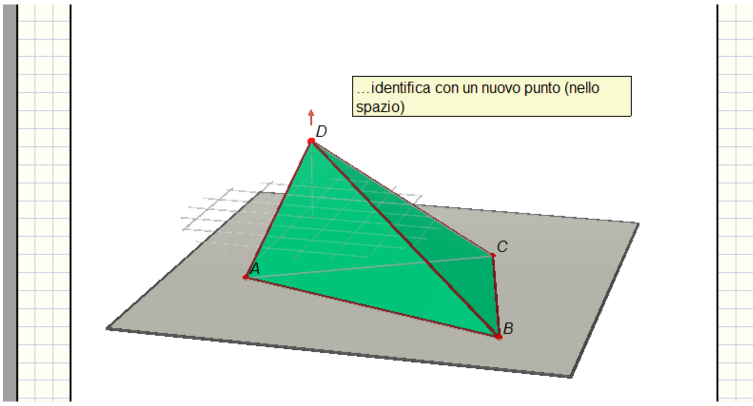


In phase 1, Experiencing, we give preliminary problems to solve by changing position to some matches, like the following: "Can you obtain 3 congruent squares by moving only four matches?"

And the final problem was the famous problem given by Einstein "Can you construct four equilateral triangles with six matches?", as a provocation, that is solvable in the space, by a tetrahedron.

Phase 2, Reflecting, and 3, Generalizing, are realized by worksheets (those built up by teachers in the first lab), that guide the students to pose problems, by stimulating observation of some objects to be manipulated in the DGS (phase 2) and to generalize, by conjecturing properties and proving them (phase 3).

Phase 4, Applying, is achieved by exercises on quadrilaterals and tetrahedra, whose solutions request the theorems the students proved in the previous phase. These problems, some proposed by the tutors and some given by teachers, often require a certain practice with the software (especially Cabri 3d).
In the final phase, Re-experiencing, we asked students to solve the problem presented at the beginning of the activity (the Einstein problem solved by a tetrahedron) and invite students to resort their creative ability to produce something to present to all the students in the final meeting.
The core of the activity, at least for the time spent, focus on phases 2 and 3, i.e. on worksheets. Worksheet prototype is described in (Ferrarello & Mammana, 2012): briefly, there is a two columns structure; in the first column an action is declared and in the second one it is is explicitly made. Some word is written in bold italic font, this means that word is exactly the name of the correspondent command in the DGS; in such a way students know which command they should use, without wasting time in looking for the right one.
Each worksheet has dots to be filled by students, but not all the worksheets have the same space to be filled. In fact once a property is get in the plane, we leave students more free to guess and above all to prove the analogous in the space with less details, hoping they are able to handle space geometry by using analogy.
The whole activity is based on the use of a DGS on the one hand and on the "analogy" between figures on the other hand. These two "ingredients" will guide the students during the activity, discovering, conjecturing and proving.
We divided worksheets in Q-sheets and T-sheets, meaning, respectively, worksheets on quadrilaterals in plane and on tetrahedra in space. At the end of each space worksheet there is a "Send to your notebook" action, asking students to write the analogies they have found in a "table of the analogies", similar to the one we gave in the previous section. The notebook should be papery or electronic (for instance a .doc file).
Worksheet 1Q and 1T aim to give basic definitions, given in the previous paragraph. These worksheets are pretty simple, there is nothing to prove, no property is to be found, but they are so important because they are responsible to set up the analogy in students minds. Here students see something as something else (Ferrara & Mammana, in press), and by seeing a tetrahedron like a sort of "evolution" of a quadrilateral they start to think they could traduce properties in the plane to a higher level.
1Q-sheet guides students to revisit quadrilaterals with the new definitions of edges, faces, opposite elements. It could be done also with a simple picture in the paper, but the sheet asks to do it with the DGS Cabri II plus. The reason of this choice is that the twin worksheet 1T, working with Cabri 3d, starts from a quadrilateral drawn in the plane. In such a way students handle with the same modality the two objects quadrilaterals met in 1Q and 1T. Then, in 1T, after constructing a "quadrilateral in the space" i.e. a quadrilateral staying in the base plane, there is the crucial action: ***Redefinition***. The worksheet asks to use the Cabri command Redefinition to drag a vertex of the quadrilateral into the space turning the quadrilateral into a tetrahedron.



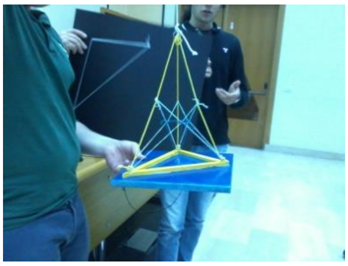
Worksheets of the group 2 and 3 are divided in two parts In the first part definitions are given, then proper Cabri actions are asked to construct the objects whose definitions were given, finally

properties are conjectured by proper explorations. In the second part properties are written and students are asked to prove them.
We choose to divide these worksheets in two parts because in such a way students can't read the statement of the theorem until they completed the conjecture part. So they are free in the activity of conjecturing, the only suggestions are given by observation and exploration of the objects dragged into the DGS sheet.
We decide to give complete statement of the theorems (in the second part) in such a way students have a well written property and can write up correctly every statement in their notebook.
As mentioned, in the second parts students are guided to prove theorems, but there is a difference between proofs in 2 d and in 3d: proofs about quadrilaterals are more guided, while proofs about tetrahedra are almost free, because we want to see if students are able to work in 3 dimensions by using what learnt in 2 dimensions, by using analogy. 2Q-sheet and 2T-sheet give the definitions of bimedians and guide students in the discovering of theorems 1 and 2, i.e. the concurrency of bimedians in a point G (the centroid), which bisects every bimedian. Finally students are guided to prove (as discussed above).
3Q-sheet and 3T-sheet give the definitions of medians and guide students to discover theorems 3 and 4, i.e. the concurrency of medians in a point G (the centroid), which divides each median in the ratio 1:3, and then they are asked to prove them.

Results and products

The teachers expressed their considerations on the classroom activity by completing a "Logbook", where they were described modalities and the organization of the work, the students behaviour and the results obtained both from the motivation point of view (attitude, interest, commitment) that the cognitive point of view (an increase in the level of learning). In addition, a final questionnaire allowed them to express their final evaluation of the activity. Here is a summary emerged from an examination of "logbooks" and questionnaires (in quotes some comments from the teachers).
All teachers recognized that "this methodology based on the analogy between the figures of the plane and figures of the space foster the study of three dimensional geometry because the proofs in space are easier if you make similar proofs in plan first" and also allows "to do not deal with them separately ".
Regarding the use of Cabri teachers have spoken unanimously, with considerations such as: "The use of Cabri in the activity was indispensable" and "definitely [...] favours the process of teaching-learning. ". "The problem remains, however, that both software are not free." Some teacher also noted that "often the students were much more interested in the software itself, its potentials, than the subject matter."
With regard to the methodology used in the classroom, teachers have expressed that "The methodology used –the mathematics laboratory- with worksheets to be completed with the use of Cabri in the two versions, has led the students, even the less able ones, to face geometry topics otherwise difficult to understand. Additionally, in dealing with particular difficulties, the collaboration and the comparison with students certainly favoured the construction of knowledge in a different way "; while "the worksheets used led 'by hand' the students to face challenging topics. Moreover, "this new methodology has made them (students) more free, with respect to the usual classroom activity, therefore they learned by playing."
With regard to the involvement of students emerged, except in one case, that " The students have responded well to the proposed activity. They participated with interest and felt involved in the teaching proposal. In fact, though the activity was not during school hours student's attendance has remained good "and this is true also for " the case of those students less inclined to discipline.". In addition, "The discovery and study of software has greatly stimulated them to go beyond what was required and in some cases to conjecture new properties."
Finally, we report some considerations on the validity of the proposed activities. "The activity made easier student's approach to 3d-geometry. In particular, they understood the difference between

conjecture and proof" and then "The activity made the study of the geometry of space more interesting because of a better visualization in space and because of a more active approach of the student."
From the cognitive point of view this activity has increased the ability to "frame different situations in the same logical scheme", even if some teacher proposes that "it should be more investigated whether the proposed topics were sufficiently captured."
The teachers appreciated the fact that they have been directly involved in the design and implementation of the worksheets. It was particularly appreciated the collaboration with tutors, considered "interesting, productive and profitable." In particular, one teacher pointed out that "producing the material to be use in class has been helpful, because often you underestimate if a problem is well-posed or not." The collaboration between teachers has been positive, giving rise of a "fully collaborative relationship."
At the end of the activity we organized a final meeting, where students presented their own product inspired by the topics: above all they did 3d real objects representing tetrahedra and the discovered properties by means of straws, rubber bands, paper and also laser. They were very happy to produce something thought and realized by themselves.


References

[1] Aleo, M.A., Fasciano M.C., Ferrarello D., Inturri, A., Mammana,M.F., Micale, B., Pappalardo, V., Pennisi, M. (2011). 20 anni di Etniade. Casa editrice La Tecnica della Scuola, Catania-Italy.

[2] Chiappini, G. (2007). *Il laboratorio didattico di matematica: riferimenti teorici per la costruzione*, Innovazione educativa, Inserto allegato al numero 8, Ottobre 2007.

[3] Ferrara, F., Mammana, M.F. (2014). Seeing in space is difficult: an approach to 3d geometry through a dge. Research Report in the *Proceedings of the 38th Conference of the International Group for the Psychology of Mathematics Education.* Vancouver, Canada.

[4] Ferrarello, D., Mammana, M.F. (2012). Training teachers to teach with technologies. In Proceedings of La didactique des mathématiques: approches et enjeux. Parigi, 31 May -2 June 2012.

[5] Ferrarello, D., Mammana, M.F., Pennisi, M (2013).Teaching by doing. In proceedings of CIEAEM 64, Torino, July 2013.

[6] Kolb. D. A. and Fry, R. (1975). Toward an applied theory of experiential learning. in C. Cooper (ed.). *Theories of Group Process,* London: John Wiley.

[7] Margarone, D., Mammana, M.F., Micale, B., Pennisi, M., Pluchino, S. (2009). Dai quadrilateri ai tetraedri: alla scoperta di sorprendenti analogie. Casa editrice La Tecnica della Scuola, Catania – Italy.

[8] Mammana, M.F. Micale, B., Pennisi, M. (2009). Quadrilaterals and tetrahedra. iJMEST. Vol.40, N.6 (12), pp. 817-828(12).

[9] MPI (2003). *Matematica 2003. La matematica per il cittadino*. Matteoni stampatore, Lucca

[10] Polya, G. (1962). Mathematical discovery: on understanding, learning and teaching problem solving. John Wiley & Sons.

[11] Rossi, P.G. (2011). Didattica Enattiva. Complessità, teorie dell'azione, professionalità docente. Franco Angeli, Milano.