The Ten Coolest Numbers

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This is an attempt to give a count-down of the top ten coolest numbers. Let's first concede that this is a highly subjective ordering — one person's $\sqrt{14.38}$ is another's $\frac{\pi^2}{6}$. The astute (or probably simply "awake") reader will notice, for example, a definite bias toward numbers interesting to a number theorist in the below list. (On the other hand, who better to gauge the coolness of numbers...) Let's begin by setting down some ground rules.

What's in the list?

What makes a number cool? I think a word that sums up the key characteristic of cool numbers is "canonicality." Numbers that appear in this list should be somehow fundamental to the nature of mathematics. They could represent a fundamental fact or theorem of mathematics, be the first instance of an amazing class of numbers, be omnipresent in modern mathematics, or simply have an eerily long list of interesting properties. Perhaps a more appropriate question to ask is the following:

What's *not* in the list?

There are some really awesome numbers that I didn't include in the list. I'll go through several examples to get a feel for what sorts of numbers don't fit the characteristics mentioned above. Shocking as it may seem, I first disqualify the constants appearing in Euler's formula $e^{i\pi} + 1 = 0$. This was a tough, and perhaps absurd, decision. Maybe these five (e, i, π, 1, and 0) belong at the top of the list, or perhaps they're just too fundamentally important to be considered exceptionally *cool.* Or perhaps it's just they're just so cliché'd that we'll get a significantly more interesting list by excluding them. Or maybe, just maybe, they're genuinely less cool than the numbers currently on the list.

Also disqualified are numbers whose primary significance is cultural, rather than mathematical: Despite being the answer to life, the universe, and everything, 42 is (comparatively) mathematically uninteresting. Similarly not included in the list were 867-5309, 666, 1337, Colbert numbers, and the first illegal prime number. Also disqualified were constants of nature like Newton's g and G, the fine structure constant, Avogadro's number, etc. Though these are undeniably numbers of great significance, their values are a) not precisely known, and b) frequently depend on a (somewhat) arbitrary choice of unit.

Finally, I disqualified numbers that were highly non-canonical in construction. For example, the prime constant and Champernowne's constant are both mathematically interesting, but only because they were, at least in an admittedly vague sense, constructed to be as such. Also along these lines are numbers like G63 and Skewe's constant, which while mathematically interesting because of roles they've played in proofs, are not inherently interesting in and of themselves.

That said, I felt free to ignore any of these disqualifications when I felt like it. I hope you enjoy the following list, and I welcome feedback.

Honorable Mentions

- 65,537 – This number is arguably the number with the most potential. It's currently the largest Fermat prime known. If it turns out to be *the* largest Fermat prime, it might earn itself a place on the list, by virtue of thus also being the largest prime value of n for which an n-gon is constructible using only a rule and compass.

- Conway's constant – The construction of the number can be found here. Though this number has some remarkable properties (not the least of which is being unexpectedly algebraic), it's completely non-canonical construction kept it from overtaking any of our list's current members.

- 1728 and 1729 – This pair just didn't have quite enough going for them to make it. 1728 is an important j-invariant of elliptic curves and a coefficient of the corresponding modular form, and is a perfect cube. 1729 happens to be the third Carmichael number, but the primary motivation for including 1729 is because of the mathematical folklore associated it to being the first *taxicab number*, making it more interesting (math-)historically than mathematically.

- 28 – Aside from being a perfect number, a fairly interesting fact in and of itself, the number 28 has some extra interesting "aliquot" properties that propel it beyond other perfect numbers. Specifically, the largest known collection of sociable numbers has cardinality 28, and though this might seem a silly feat in and of itself, the fact that sociable numbers and perfect numbers are so closely related may reveal something slightly more profound about 28 than it just being perfect.

- 4 – The problem with 4 is the difficulty in distinguishing between cool properties of 2 and cool properties of 4. It is unclear, for example, to which of them we attribute the trivial but not uninteresting relations

  $$4 = 2^2 = 2 \cdot 2 = 2 + 2 = 2 \uparrow 2 = 2 \uparrow\uparrow 2 = 2 \uparrow\uparrow\uparrow 2 = \cdots,$$

  the last few entries using Knuth's up-arrow notation. More significantly, 4 is of obvious prominence in the "4-squares theorem" and "4-color theorem." These are both remarkable results, but of debatable canonicality (see Waring numbers and chromatic numbers respectively for natural generalizations). These facts along would probably not merit inclusion even in the honorable mentions section, but 4 does have at least one particularly poignant claim to fame: It is the unique n such that $\mathbb{R}^n$ admits more than one differential structure, and indeed admits uncountably many so. That $\mathbb{R}^4$ (and 4-dimensional geometry in general) seems to persistently crop up as a pathology in differential geometry is certainly cause for intrigue.

- Chaitin's Constant $\Omega \approx$??? - The question marks themselves form part of the reason this constant could be included, Ω being an example of a number which is definable but not computable. Chaitin's constant can loosely be described as the probability that a Turing machine will halt on a randomly-provided string. There is no doubt that such a constant would represent something fundamental, but there are some unfortunate ambiguities in the definition, largely stemming from the ambiguity in ordering/encoding the set of all Turing machines. Alternate encodings define different constants, and it's difficult to say that any particular encoding is more canonical than any other.

#10) The Golden Ratio, ϕ

This was a tough one. Yes, it's cool that it satisfies the property that its reciprocal is one less than it, but this merely reflects that it's a root of the wholly generic polynomial $x^2 - x - 1 = 0$. Yes, it's cool that it may have an aesthetic quality revered by the Greeks, but this is void from consideration for being non-mathematical (and quite possibly bogus). Only slightly less canonical is that it gives the limiting ratio of subsequent Fibonacci numbers. Redeeming it, however, is its appearance in describing *all* "Fibonacci-like" sequences, and its being the solution to two sort mildly canonical operations:

$$\phi = \sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots}}} \qquad \text{and} \qquad \phi = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\ddots}}}}.$$

The latter of these is particularly interesting since the approximability of an irrational number by rationals is closely tied to the largeness of the coefficients in its continued fractions, earning the golden ratio the superlative of being the *most* irrational (in the sense of being least approximable) real number.

#9) 691

The prime number 691 made it on here for a couple of reasons: First, it's prime, but more importantly, it's the first example of an *irregular* prime, a class of primes of immense importance in algebraic number theory. (A word of caution: it's not the *smallest* irregular prime, but it's the one that corresponds to the earliest Bernoulli number, B_{12}, so 691 is only "first" in that sense). It also shows up in the coefficients of every non-constant Fourier coefficient in the q-expansion of the Eisenstein modular form $E_{12}(z)$, a fact closely related to Ramanujan's congruence relations (modulo 691) for the arithmetic function $\sigma_{11}(n) := \sum_{d|n} d^{11}$. Further testimony to its arithmetic significance is its seemingly magical appearance in the algebraic K-theory of the integers: Soulé has discovered an element of order 691 in the K-group $K_{22}(\mathbb{Z})$, a group whose torsion is otherwise very mysterious.

#8) 78,557

The number 78,557 is here to represent an amazing class of numbers called *Sierpinski* numbers, defined to be numbers k such that $2^n k + 1$ is composite for *every* $n \geq 1$. That such numbers exist is flabbergasting...we know from Dirichlet's theorem that primes occur infinitely often in non-trivial arithmetic sequences. Though the sequence formed by $78557 \cdot 2^n + 1$ isn't arithmetic, it certainly doesn't behave multiplicatively either, and there's no apparent reason why there shouldn't be a large (or infinite) number of primes in *every* such sequence. This notwithstanding, Sierpinski's composite number theorem proves there are in fact *infinitely* many odd such numbers k. As a small disclaimer, though it's proven that 78,557 is indeed a Sierpinski number, it is not quite yet known that it is the smallest. There are exactly 6 numbers smaller than 78,557 not yet known to be non-Sierpinski (for the curious, they are 10223, 21181, 22699, 24737, 55459 and 67607).

#7) $\frac{\pi^2}{6}$

Perhaps the first striking thing about this number is that it is the sum of the reciprocals of the positive integer squares:

$$1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} + \cdots = \frac{\pi^2}{6}.$$

Though the choice of 2 here for the exponent is somewhat non-canonical (i.e. we've just noted that $\zeta(2) = \frac{\pi^2}{6}$, where ζ stands for the Riemann zeta function), and that this is largely interesting for math-historical reasons (it was the first sum of this type that Euler computed), we can at least include it here to represent the amazing array of numbers of the form $\zeta(n)$ for n a positive integer at least 2. This class of numbers incorporates two amazing and seemingly disparate collections, depending on whether n is even (in which case $\zeta(n)$ is known to be an explicit rational multiple of π^n) or odd (in which case extremely little is known, even for $\zeta(3)$). Finally, there's something slightly canonical about the fact that its reciprocal, $\frac{6}{\pi^2}$, gives the "probability" (in a suitably-defined sense) that two randomly chosen positive integers are relatively prime.

#6) Feigenbaum's constant, $\delta \approx 4.669201...$

This one's a little technical, but there's nothing fancy going on. Consider an iterative procedure where you begin with some real value of x, say between 0 and 1, and you plug it into a logicstic equation $f(x) = 2x(1-x)$. Then you take the result of that calculation, and plug *that* back into $f(x)$, to obtain $f(f(x))$. Now repeat, computing $f(f(f(x)))$, and $f(f(f(f(x))))$, etc. Go ahead, pick an x and do it. It may be pleasing, if not mind-boggling, that your sequence of outputs steadily approached $\frac{1}{2}$, and perhaps only a mild shock (it is not a hard exercise) that this happens whichever $x \in (0,1)$ you had picked to start with. Now we change the game a little – by replacing the value "2" in the definition of $f(x)$ and replace it with a parameter ρ which we will begin to modify. If we increase ρ to anything less than 3, we see roughly the same phenomena – all values tend, under this iteration, to a common value. When ρ increases past 3, however, so we consider iterating the function $f(x) = 3.2x(1-x)$, we see something strange and new appear: We find $f(0.5130) = .7995$ and $f(.7995) = .5130$, and that plugging an arbitrary starting value of x eventually leads the sequence of outputs to bouncing between these two values. Increase ρ by another to .449, we find that all of a sudden orbits can now oscillate between *four* distinct values instead of just 2. When we increase ρ by another .095, we begin to see orbits of 8 values, instead of 4. And this continues, soon hereafter seeing orbits of size 16, 32, etc.

But these critical values of ρ seem pretty random – can we predict when we expect to see the number of orbits double? Remarkably, yes, at least in the long run – the ratio of increases in ρ needed to double the number of orbits (e.g., $\frac{.449}{.095} = 4.726$) approaches a limit, dubbed Feigenbaum's constant $\delta \approx 4.669....$ That changes in orbit behavior are forseeable is a remarkable fact, and is a crucial step towards being able to predict the onset of impredictability in dynamical systems. You mild shock from above should be upgraded to moderate shock at this point.

But wait, there's more! The real selling point of this number is that this phenomena *has almost nothing to do with our starting function* $f(x)$, other than it being quadratic in nature (formally, having a single quadratic maximum). That all such dynamical systems bifurcate towards chaos at exactly the same is astounding, making moderate shock rather insufficient. Finally, we mention that though the quadratic assumption on $f(x)$ seem rather strong, there are different Feigenbaum-type constants for cubics, quadrics, etc. (all equally remarkable), and so Feigenbaum's δ constant above is the first in this beautiful class of chaotic numbers.

#5) The Monster, $|M| =$ 808017424794512875886459904961710757005754368000000000

The above integer is the size of *the monster group* M, the largest of the sporadic groups. This gives it a relatively high degree of canonicality. It's unclear (at least to me) why there should be *any* sporadic groups, or why, given that they exist, there should only be finitely many. Since there *is*, however, there must be something fairly special about the largest possible one.

Also contributing to this number's rank on this list is the remarkable properties of the monster group itself, which has been realized (or rather, was constructed as) a group of rotations in 196,883-dimensional space, representing in some sense a *limit* to the amount of symmetry such a space can possess.

#4) The Euler-Mascheroni Constant, $\gamma \approx 0.577215\ldots$

One of the most amazing facts from elementary calculus is that the harmonic series diverges, but that if you put an exponent on the denominators even just a *hair* above 1, the result is a convergent sequence. A refined statement says that the partial sums of the harmonic series grow like $\ln(n)$, and a further refinement says that the error of this approximation approaches our constant:

$$\lim_{n\to\infty} 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln(n) = \gamma.$$

This seems to represent something fundamental about the harmonic series, and thus of integers themselves. Finally, perhaps due to importance inherited from the crucially important harmonic series, the Euler-Mascheroni constant appears magically in wondrous formulas spread all throughout modern analysis. For some idea of γ's ability to pop up in unforeseen places, see the MathWorld entry on the Euler-Mascheroni constant.

#3) Khinchin's constant, $K \approx 2.685252\ldots$

For a real number x, we define a "geometric mean function" $f(x)$ by

$$f(x) = \lim_{n\to\infty} (a_1 \cdots a_n)^{1/n},$$

where the a_i are the terms of the simple continued fraction expansion of x. By nothing short of a miracle of mathematics, this function of x is almost everywhere (i.e. everywhere except for a set of measure 0) *independent of* x!!! In other words, except for a "small" number of exceptions, this function $f(x)$ always outputs the same value, dubbed Khinchin's constant and denoted by K. It's hard to impress upon a casual reader just how astounding this is, but consider the following: *Any* infinite collection of non-negative integers $a_0, a_1, \ldots$ forms a continued fraction, and indeed each continued fraction gives an infinite collection of that form. That the partial geometric means of these sequences is *almost everywhere constant* tells us a great deal about the distribution of sequences showing up as continued fraction sequences, in turn revealing something very fundamental about the structure of real numbers.

#2) The Oddest Prime: 2

This number caused quite a bit of controversy in discussions leading up to the construction of this list. The question here is canonicality. The first argument of "It's the only even prime" is merely a re-wording of "It's the only prime divisible by 2," which could uniquely characterizes *any* prime (e.g. 5 is the only prime divisible by 5, etc.). Of debatable canonicality is the immensely prevalent notion of "working in binary." To a computer scientist, this may seem extremely canonical, but to a mathematician, it may simply be an (not quite) arbitrary choice of a finite field over which to work. There may even be some merit to a more philosophical argument based on the (somewhat inane, but also somehow deep) argument that it is the smallest integer bigger than 1, and thus represents plurality, dichotomy, choice, etc. Leaving these aside, 2 does have some genuinely interesting mathematical features. For instance:

- The (somewhat canonical) field of real numbers $\mathbb{R}$ has index 2 in its algebraic closure $\mathbb{C}$. This gives that the Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ is finite or order 2 – particularly amazing since this is *the only possible order of a finite non-trivial absolute Galois group* (by the Artin-Schreier theorem).
- The factor 2π (or more frequently, its inverse $\frac{1}{2\pi}$) is prevalent enough in complex analysis, plane geometry, Fourier analysis, and even quantum mechanics (considering the simplicity of formulas using the reduced Planck's constant $\hbar := h/2\pi$) that I've heard people lament that π should have been defined to be twice its current value.
- By Fermat's Last Theorem, it's the *only* prime number p for which $x^p + y^p = z^p$ has any rational solutions. While this particular Diophantine equation might not be particularly canonical, the extreme significance of the mathematics behind its proof merited its inclusion in this sub-list.
- Fields of characteristic 2 have the property that all of their elements are their own negatives, a fact which is simultaneously useful (frequently simplifying calculations) and annoying (frequently messing up the cleanest statements of a particular theorem). This is part of a general meta-mathematical observation that the case $p = 2$ very frequently must be dealt with separately than all other primes. It is not uncommon to see papers reproving a result for $p = 2$ that was previously known for all other p.
- It's the size of the group of units $\{\pm 1\}$ in the integers $\mathbb{Z}$, and the group of roots of unity in $\mathbb{Q}$, meaning that (among many other things) the Kummer extensions of $\mathbb{Q}$ are exactly the quadratic extensions.

- If nothing else, it is certainly the first prime, and should at least be included for being the first representative of such an amazing class of numbers.

Finally, it is the only number on this list which occupies its own ranking.

#1) 163

Well, we've come down to it, this author's (perhaps not-so-) humble opinion of the coolest number in existence. Though a seemingly unlikely candidate, I hope to argue that 163 satisfies so many eerily related properties as to earn this title. I'll begin with something that most number theorists already know about this number – it is the largest value of d such that the number field $\mathbb{Q}(\sqrt{-d})$ has class number 1, meaning that its ring of integers is a unique factorization domain. The issue of factorization in quadratic fields, and of number fields in general, is (or, at least, has historically been) one of the principal driving forces of algebraic number theory, and to be able to pinpoint the end of perfect factorization in the quadratic imaginary case like this seems at least arguably fundamental. But even if you don't care about factorization in number fields, the above fact has some amazing repercussions to more basic number theory. The two following facts in particular jump out:

- $e^{\pi\sqrt{163}}$ is within 10^{-13} of an integer.
- The polynomial $f(x) = x^2 - x + 41$, which has discriminant -163, has the property that for integers $0 \leq x \leq 40$, $f(x)$ is prime.

Both of these are tied intimately (the former using deep properties of the j-function, the latter using relatively simple arguments concerning the splitting of primes in number fields) to the above quadratic imaginary number field having class number 1. Further, since $\mathbb{Q}(\sqrt{-163})$ is the *last* such field, the two listed properties are in some sense the best possible. Along a similar vein, $p = 163$ is the largest prime such that there exists an elliptic curve E over $\mathbb{Q}$ with an isogeny of degree p, which in turn makes $N = 163$ the last N such that the modular curve $Y_0(N)$ has $Y_0(N)(\mathbb{Q}) \neq \emptyset$.

Most striking to me, however, is the amazing frequency with which 163 shows up in a wide variety of class number problems. In addition to being the *last* value of d such that $\mathbb{Q}(\sqrt{-d})$ has class number 1 (the Heegner-Stark theorem, tremendously significant in it own right), it is the *first* value of p such that $\mathbb{Q}(\zeta_p + \zeta_p^{-1})$ (the maximal real subfield of the p-th cyclotomic field) has class number *greater* than 1. That 163 appears as the last instance of a quadratic field having unique factorization, and the first instance of a real cyclotomic field *not* having unique factorization, seems too remarkable to be coincidental. This is (maybe) further substantiated by a couple of other factoids:

- Hasse asked for an example of a prime and an extension such that the prime splits completely into divisors which *do not* lie in a cyclic subgroup of the class group. The first such example is any prime less than 163 which splits completely in the cubic field generated by the polynomial $x^3 = 11x^2 + 14x + 1$. This field has discriminant 163^2. (See Shanks' *The Simplest Cubic Fields*).
- The maximal conductor of an imaginary abelian number field of class number 1 corresponds to the field $\mathbb{Q}(\sqrt{-67}, \sqrt{-163})$, which has conductor $10921 = 67 \cdot 163$.

It is unclear the extent to which these additional arithmetical properties reflect deeper properties of the j-function or other modular forms, and remains a wide open field of study.

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