Week 1 - Superfluidity

I. REVIEW OF BEC

Thinking again about a a ton of bosons in a box, the state equation concerning number was:

$$\frac{N}{V} = \frac{1}{V}\sum_{\vec{k}} \frac{1}{e^{\epsilon_k \beta}/\zeta - 1} \tag{1}$$

with

$$\zeta = e^{\mu\beta}. \tag{2}$$

The $k = 0$ term in the sum is the only one with potential to diverge as the temperature goes to zero. All other terms have a finite energy, and therefore the denominator can never vanish. While keeping the $k = 0$ term in tact, the continuum approximation for the rest:

$$\frac{N}{V} = \frac{1}{V}\frac{\zeta}{1-\zeta} + \int\limits_{\vec{k}\neq 0} \frac{k^2 dk}{2\pi^2}\frac{1}{e^{\epsilon_k\beta}/\zeta - 1} \tag{3}$$

With the help of our previous analysis we obtain:

$$\rightarrow \frac{1}{V}\frac{\zeta}{1-\zeta} + \frac{1}{\lambda_T^3} g_{3/2}(\zeta) \tag{4}$$

(Concentrating on 3-d). Assuming now that

$$V \rightarrow \infty \tag{5}$$

when the density reaches the maximum value from below:

$$n_c = \frac{1}{\lambda_T^3} g_{3/2}(1) = \left(\frac{2\pi m}{h^2}\right)^{3/2} T^{3/2} \overbrace{g_{3/2}(1)}^{2.612} \tag{6}$$

ζ reaches one, and the first term is undetermined, the second term saturates at n_c. Say that the indeterminate term is n_0, and we see that it is determined by the remainder of bosons, not hosted by the nonzero energy states:

$$n_0 = n - n_c = n - \frac{2.612}{h^3}(2\pi m)^{3/2} T^{3/2} \tag{7}$$

This is Bose-Einstein condensation, and the bosons in the lowest state - are called the condensate. From the erspective of changing temperature, we said that at:

$$T = T_{BEC} = \frac{h^2}{2\pi m}\left(\frac{n}{2.612}\right)^{2/3} \tag{8}$$

The occupation of the $k = 0$ state, n_0, starts increasing:

$$n_0 = \left(\frac{2\pi m}{h^2}\right)^{3/2} 2.612\left(T_{BEC}^{3/2} - T^{3/2}\right). \tag{9}$$

II. BEC OF HELIUM

Long after the liquification of Helium, it was discovered in 1938 by Kapitsa that Helium undergoes a phase transition into a different liquid phase - He II. The phase transition was best observed by measuring the heat capacity of the liquid.

This phase transition was accompanied by a divergence of the heat capacity, and took place at 2.172K. The new phase exhibited a range of very curious behavior -

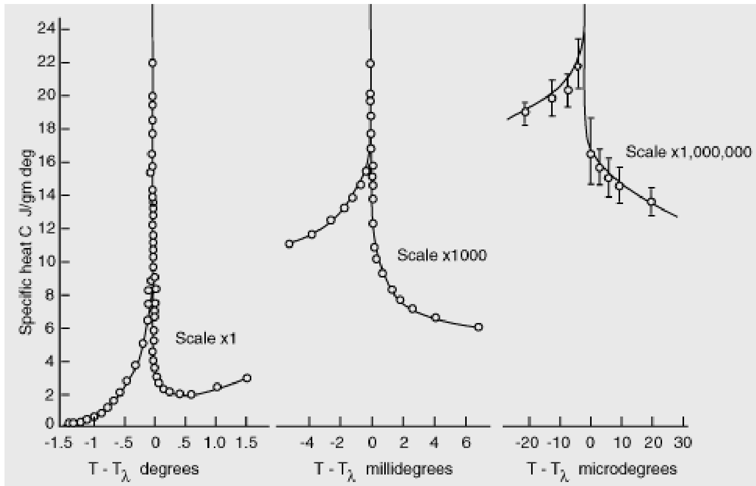

FIG. 1: Lambda point of He. The heat capacity diverges at $T_\lambda = 2.172K$. The plot shows the divergence in several resolution levels.

- It would leap out of a vessel where it is contained.
- Put a tube in the liquid; if you shine light at the liquid, helium would start sprouting out of the tube creating a fountain.

In fact, both of these effects are due to *zero viscosity* of the fluid.
The density of fluid helium at atmospheric pressure is roughly:

$$n = 2.2 \cdot 10^{28} \frac{1}{m^3} \tag{10}$$

When we plug this to the formula for T_{BEC} we obtain:

$$T_{BEC} = \frac{h^2}{2\pi m} \left(\frac{n}{2.612} \right)^{2/3} = 3.16K \tag{11}$$

with $m = 4 \cdot 1.6 \cdot 10^{-27} kg$.
The superfluid transition of Helium is not so far away.

III. SUPERFLUIDITY AND THE ROTON MINIMUM

what did we forget in the BEC treatment? The answer is - *interactions*. We are dealing with a really dense gas, and interactions must come into play.
The main place where interactions show up is the excitation spectrum. Without interactions, the excitations out of the superfluid are just the one-particle momentum states:

$$\vec{k} \tag{12}$$

with:

$$\epsilon_{\vec{k}} = \frac{\hbar^2 k^2}{2m}. \tag{13}$$

But when there are interactions, this can not be so - when particles move relative to eachother, they pull with them a whole cloud, in a motion very much like that of sound. We can no longer really describe the excitations of the fluid in terms of sinlge particle motion. Instead, we need to describe the motion in terms of **collective modes**. The low lying energy excitations can still be described by $\vec{k}$ and $\epsilon_{\vec{k}}$, but now the spectrum is linear rather than quadratic at low energy. It will also have a speed of sound - $c = 239m/s$.
Like phonons in a solid, sounds waves of the liquid will also have a rather strange dispersion, that will not be forever linear. For the interacting liquid, there is a very special length scale. In fact, the only available length scale is

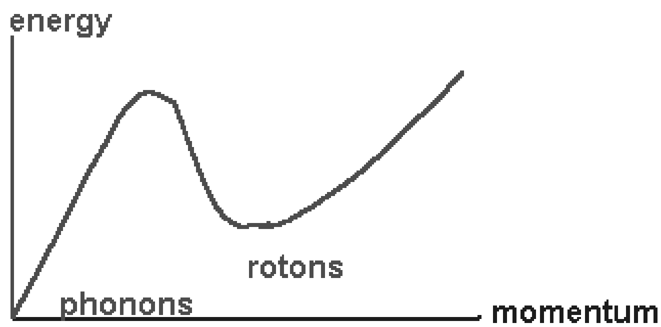

FIG. 2: SF helium excitation spectrum

the interatomic distance. Since the atoms of the fluid are strongly interacting, a competing state to the liquid state is always the solid state. A perturbation in the density, or a density wave with the wave length of the interatomic distance probes this tendency of the liquid to form a solid, and is therefore favored energetically. This is expressed as a distinct minimum in the dispersion of the sound wave, called the roton minimum.

This minimum indeed occurs at wavelengths

$$\lambda_0 \approx 3.5\text{Å} \tag{14}$$

which is roughly the density, with a gap of

$$\Delta = 8.65K. \tag{15}$$

The roton minimum was hypothesized by Landau, and independently, by our very own Richard Feynman.

Note that if you could tune the interactions in situ, upon making them more strong, you would eventually expect solidification. But before, you'll see a reduction in the energy of the roton minumum. When the roton minimum hits zero, the liquid turns solid.

IV. LANDAU CRITERION FOR SUPERFLUIDITY

So far it is not clear why a moving superfluid doesn't dissipate its kinetic energy. The spectrum of excitations is not gapped (which would be a sufficient condition for superflow), even though the number of low lying excitations is decreased relative to a non-interacting BEC. Let us now derive the Landau criterion for a superflow, and his expression for a critical velocity.

Consider a condensate sitting in a pipe. Now boost it in the +x direction with velocity v. Imperfections in the surface of the pipe are supposed to slow down the liquid by back-scattering particles particles.

Landau: In order to have viscosity, a backscattering should *reduce* the energy in the moving condensate.

In the liquid's rest frame, backscattering induces an excitation with momentum $-k\hat{x}$, and energy $\epsilon_{\vec{k}}$. In this frame the liquid is at rest and therefore has zero energy.

Now, there is the Galilean transformation. Let's go aside for a second and consider a particle moving with velocity $-u$ relative to the fluid, this particle representes the excitation. Its energy in the fluid's frame of reference is

$$\epsilon_u = \frac{1}{2}mu^2 \tag{16}$$

But now, since this $-u$ is relative to a frame that moves withvelocity v to the right, the energy of the body is then actually:

$$\frac{1}{2}m(v-u)^2 = \frac{1}{2}mv^2 + \epsilon_u - vp_u \tag{17}$$

where

$$p_u = mu \tag{18}$$

Coming back to the superfluid, the backscattered excitation has energy $\epsilon_{\vec{k}}$ in the rest fram of the liquid. In the moving frame it has:

$$\epsilon' = \frac{1}{2}mv^2 + \epsilon_{\vec{k}} - \hbar kv \tag{19}$$

but from this expression we should really eliminate the term $\frac{1}{2}mv^2$ since it is already included in the total kinetic energy of the liquid. Finally, the change of energy in the rest frame of the pipe is:

$$\Delta E = \epsilon_{\vec{k}} - \hbar kv < 0 \tag{20}$$

And this has to be smaller than zero if indeed the superfluid passed some of its energy to the pipe, or to heat. This we write as:

$$v > v_c = \frac{\epsilon_{\vec{k}}}{\hbar k} \tag{21}$$

This is the critical velocity of the superfluid. If the fluid moves at speeds lower than this, no backscattering occurs - and the fluid continues to flow forever, essentially.

What is the critical velocity for helium? most likely it is determined by the roton minimum. For the roton parameters this is

$$v_c = 60m/s \tag{22}$$

Looks good. Except that experiments showed a range of critical velocities which is much much smaller - on the order of cm/s. The Landau criterion looks very nice - but it is still missing something..

V. QUANTUM MECHANICS OF BOSE CONDENSATES

BEC is a strictly quantum phenomena. A crucial factor is that a macroscopic number of particles choose to crowd the $k = 0$ state.

How do we write a many-body wavefunction for the condensate?

$$\psi\left(r_1, r_2, r_3, \ldots, r_N\right) \tag{23}$$

The starting point must be the one-particle zero-energy wave function:

$$\psi(r) = 1 \tag{24}$$

The many body wave function is presumably just a product of $\psi(r)$:

$$\psi\left(r_1, r_2, r_3, \ldots, r_N\right) = \prod_{i=1}^{N} \psi\left(r_i\right) = 1. \tag{25}$$

This is a totally boring wavefunction. To spice it up a bit, recall that wave functions are defined upto a constant of modulu 1. We could have also written:

$$\psi(r) = e^{-i\phi} \tag{26}$$

The many-body wave function of a condensate is then:

$$\psi\left(r_1, r_2, r_3, \ldots, r_N\right) = \prod_{i=1}^{N} \psi\left(r_i\right) = e^{-iN\phi} \tag{27}$$

Better. If we allow the phase ϕ to be determined, we can get a lot of information. In fact, let us promote ϕ to a level of an operator - just like $\hat{N}$ is.

If we want to find the number of atoms in the condensate we write:

$$\hat{N}\left|\Psi\right\rangle = -\frac{1}{i}\frac{\partial}{\partial\hat{\phi}}\left|\Psi\right\rangle = N\left|\Psi\right\rangle \tag{28}$$

Thus we can identify the number operator with a derivative with respect to ϕ:

$$\hat{N} = -\frac{1}{i}\frac{\partial}{\partial\hat{\phi}} \tag{29}$$

which can also be written in the familiar form using a commutator:

$$\left[\hat{N},\, \hat{\phi}\right] = i. \tag{30}$$

The wave function we wrote for the condensate was:

$$|\Psi\rangle \to e^{-iNphi} \tag{31}$$

a number. If we now turn ϕ into an operator, then this formula should be considered as a wave function for ϕ, and then, if we can write a commutator between $\hat{N}$ and $\hat{\phi}$, then we should also say that there is an uncertainty relation:

$$\Delta\phi \cdot \Delta N \geq \frac{1}{2}. \tag{32}$$

We were discussing a system with a fixed number of particles, N, and therefore the wave function is delocalized in terms of ϕ: $|\psi(\phi)|^2 = |e^{-iN\phi}|^2 = 1$. When we factor in interactions to the treatment of the BEC, we see that in fact - BEC also implies that the wave function becomes more close to an Eigen-function of ϕ, and not of number.

This follows from the fact that in a condensate, the number of particles at zero energy should heavily fluctuate - it is indetermined by Stat. Mech., and we got it to be determined by a knowledge of the density. But this should only be considered an average value of the density. What would a ϕ eigenfunction would look like when decomposed in the number-basis?

$$|\phi = \phi_0\rangle = \sum_N e^{-iN(\hat{\phi}-\phi_0)} \tag{33}$$

this sum would give a delta-function (roughly), and we see that the number is completely undetermined. We can make it better by constructing a wave-packet:

$$|\phi = \phi_0\rangle = \sum_N e^{-iN(\hat{\phi}-\phi_0)-(N-N_0)^2/2\sigma} \tag{34}$$

but this is exactly an expression for the uncertainty principle - phase-number. This formalism was constructed by P.W. Anderson.

A. Josephson relations

Time to ask about the behavior of $\hat{\phi}$. The first question is how does it evolve in time.

Time dependence of $\hat{\phi}$

It is a phase associated with the zero energy state of a single particle, but is also an operator that relates to the wave function of the condensate. If indeed an operator, it may have a nontrivial time evolution. By Heisenberg equation of motion:

$$\frac{d\hat{\phi}}{dt} = -\frac{i}{\hbar}\left[\hat{\mathcal{H}},\, \hat{\phi}\right] \tag{35}$$

But we can also write:

$$\hat{\phi} = \frac{1}{i}\frac{\partial}{\partial N} \tag{36}$$

then we get:

$$\to \frac{1}{\hbar}\left(\frac{\partial\hat{\mathcal{H}}}{\partial N}\right) = \frac{1}{\hbar}qV \tag{37}$$

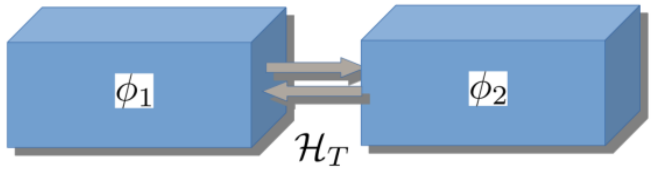

FIG. 3: Two condensates connected through a tunneling contact. This setup allows the probing of the dynamics of the superfluid phases in the two condensates.

where qV is the total potential energy (written to immitate an electric potential) that affects each particle in the bose gas. In a non-interacting gas we would have:

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}_0 + \hat{N} qV \tag{38}$$

where qV is just external fields.

So ϕ functions just like the phase of an ordinary time-dependent wave-function. And we obtained the **Josephson relation**:

$$\dot{\hat{\phi}} = \frac{qV}{\hbar}. \tag{39}$$

The right-hand-side also defines the Josephson frequency. It allows us to find out the charge of the particles condensing. This is the ultimate way in which it was confirmed that Cooper pairs with charge 2e were responsible for superconductivity.

Space dependence of $\hat{\phi}$.

The other possible depenedence is space dependence. What happens if, say, we have *two* condensates, with differnt superfluid-phases:

$$\hat{\phi}\,|1\rangle = \phi_1 \quad \hat{\phi}\,|2\rangle = \phi_2 \tag{40}$$

Let's put them in contact. This means that we are allowing the exchange of particles between the two condensates. The hamiltonian than becomes:

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}_1 + \hat{\mathcal{H}}_2 + \hat{\mathcal{H}}_T \tag{41}$$

where $\hat{\mathcal{H}}_T$ is the *tunneling Hamiltonian.* This part of the hamiltonian, need's to deliver particles from condensate 1 to 2 and vice versa. How do we do that? What quantum operator increases the number of particles in condensate 1 by 1?

$$e^{-i\hat{\phi}_1} \tag{42}$$

How do we know? First, because this way multiplying $e^{-i\hat{\phi}}$ by $|N\rangle = e^{-iN\hat{\phi}}$ gives $|N\rangle = e^{-i(N+1)\hat{\phi}}$ Another way of looking at it is: The differential representation of $\hat{\phi}$:

$$e^{-i\hat{\phi}_1} = e^{-1\cdot\frac{\partial}{\partial N}} \tag{43}$$

This is just the translation operator of $\hat{N}$, just like $e^{ia\hat{p}}$ is the translation operator for a location x. Now, the wave function in the $\hat{N}$ basis for a number state N_0 is:

$$\psi(\hat{N}) = \delta_{\hat{N},N_0} \tag{44}$$

and:

$$e^{-1\cdot\frac{\partial}{\partial N}}\psi(\hat{N}) = \delta_{\hat{N-1},N_0} = \delta_{\hat{N},N_0+1}. \tag{45}$$

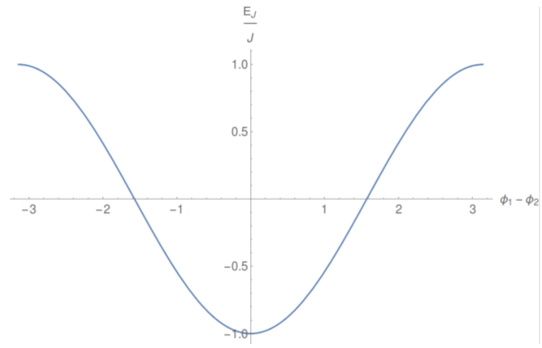

FIG. 4: The Josephson energy of the tunneling element between the two condensates. Note that the minimum energy is for the two phases being the same.

Confusing, but correct...

This allows us to write $\hat{\mathcal{H}}_T$:

$$\hat{\mathcal{H}}_T = -\frac{J}{2}\left(e^{i\hat{\phi}_1 - i\hat{\phi}_2} + e^{-i\hat{\phi}_1 - i\hat{\phi}_2}\right) = -J\cos(\hat{\phi}_1 - \hat{\phi}_2) \tag{46}$$

When the codensates in 1 and 2 are well formed, and the phases ϕ_1 and ϕ_2, are well defined, the tunneling term brings about an amazing consequence - the Josephosn current. The change in the number of particles in condensates 1 and 2 are:

$$\left\langle \frac{dN_1}{dt} \right\rangle = -\left\langle \frac{dN_2}{dt} \right\rangle = -\frac{i}{\hbar}\left\langle \left[\hat{\mathcal{H}},\, N_1\right] \right\rangle = -\hbar \left\langle \left[\frac{\partial}{\partial \hat{\phi}_1}, \hat{\mathcal{H}}_T\right] \right\rangle \tag{47}$$

but this is just:

$$\left\langle \frac{dN_2}{dt} \right\rangle = I_{12} = \left\langle \frac{\partial \hat{\mathcal{H}}_T}{\partial \hat{\phi}_1} \right\rangle = \frac{J}{\hbar}\sin(\hat{\phi}_1 - \hat{\phi}_2) \tag{48}$$

Mind you - this predicts a current which can flow as long as there are particles in the two condensates, and without any potential differences between them! This is the Josphson supercurrent - and it also occurs in superconductors. In fact - that is where it was discovered. Once again, a superconductor is considered as a condensate of Cooper-pairs - two electrons that get together to form a boson.

B. Superfluid velocity and quantization of angular momentum

Since the phase angle ϕ represents a uniform phase for the particles in the condensate we could say that if it has a finite gradient, the gradient should be:

$$\hbar \nabla \phi = p = mv \tag{49}$$

But this phase describes all the particles in the superfluid, and therefore, this velocity is the velocity of the superfluid:

$$v_s = \frac{\hbar}{m}\nabla\phi. \tag{50}$$

This result seems rather innocent when put in the context of single body quantum mechanics. But it has quite a few rather surprising concequences not there for jusf one particles. One is the quantization of angular momentum.

If we take a single particle, and put it in a ring, its wave function will have to be periodic along the ring. This implies:

$$|\psi\rangle = e^{i\phi} = e^{in2\pi x/L} \tag{51}$$

where $L = 2\pi R$. The phase of the particle has to come an integer number of full circles around the ring. Therefore the velocity of the particle is:

$$v = \frac{\hbar}{m}\frac{n}{R} \tag{52}$$

Which implies:

$$mvR = n\hbar \tag{53}$$

This is the quantization of angular momentum for a quantum particle. Only in the case of the superfluid *all* particles participate in this motion. Therefore when a superfluid is placed in a ring, it can only move with angular momenta:

$$L = N\hbar \cdot n \tag{54}$$

don't be confused - the little n is the number of windings that the phase does around the ring, and the big N is the number of particles in the condensate. This could be a **huge** angular momentum.

What about the current? The density of the particles is $\rho = N/2\pi R$, The current is then:

$$I = \rho v_s = N\frac{1}{2\pi R}\frac{\hbar}{m}n\frac{1}{R} = N\frac{n\hbar}{2\pi m R^2} \tag{55}$$

But now, suppose we have all these paticles, moving with all this angular momentum. How can we stop them? In order for them to stop something very drastic needs to happen - *all* the particles in the condensate need to change their angular momentum from $n\hbar$ to zero. This is Nearly impossible - if we think about it as a tunneling process, this process will have an amplitude that is exponentially suppressed with the number of particles, since it involves a tunneling amplitude for each one. This very special state is a macroscopic quantum state, and therefore very protected from quantum fluctuations.

If the ring is not think, however, this process could happen using vortices.