The Three Body Problem, A Cambridge Mystery

Reviewed by Richard Montgomery

The Three Body Problem, A Cambridge Mystery
Catherine Shaw
Allison & Busby, March 2005
ISBN 0-749-08347-6
Paperback, US$9.95, 288 pages

We now have at least three books titled the *The Three Body Problem*: [Marchal], [Valtonen], and the book under review. In the last, three Cambridge mathematicians are murdered within a few weeks of each other. They were working on the three body problem and living in Cambridge, England, in the late 1880s. The heroine-detective is a school teacher with strong interests in mathematics.

I love Cambridge, England. I love a good murder mystery. I have spent a decade working on aspects of the three body problem. You might think that I would love this book.

But its style really put me off. Shaw wrote it as a series of letters from our heroine to her sister. The letters, particularly the first dozen or so, hit me as sickeningly sweet. I felt as if I were imprisoned in a Victorian room, overflowing with lacy curtains, pink floral wallpaper, fluffy pillows, and priceless porcelain dolls. I was tied to an ottoman, and my kidnapper was stuffing my mouth so full of crumpets and jam that I could not even scream.

But perhaps the style is accurate to the times. If you can get over the style (doubtless some will enjoy it) the book is historically accurate. The plot becomes entertaining. Cambridge, England, is sketched in glowing detail. The personalities and sociology of the mathematicians are well portrayed. The psychologies that drive some of our breed are set down with surprising accuracy. The egotism, thirst for fame, and arrogance found in our profession come across clearly, as do one or two of our admirable traits. You will see reflections of more than a few colleagues in the pages of this book.

The murders in the book were fueled by the King Oscar II Prize, a prestigious mathematics prize competition which had its beginnings in 1884. King Oscar II of Sweden and Norway put up the prize money. Mittag-Leffler (at the University of Stockholm) organized the competition. Poincaré won the competition in January of 1889, after the alleged murders in the book take place. The story of his win is even more remarkable than the fictional murders. Poincaré won the prize, everyone believing he had made a significant advance in resolving "the problem". His work was to be published as a memoir in the journal *Acta*, another King Oscar–Mittag-Leffler joint venture. In going over the memoir Phragmén found a small bothersome error. Poincaré had to look into the error. It grew, and the heart of his prize work unravelled. To get to the truth behind his error, Poincaré uncovered the first example of what is known today as "chaos" in nonlinear dynamics—specifically, of homoclinic tangles. But a limited number of copies of *Acta* with the mistaken manuscript were already in print and mailed out. There was even an attempted cover-up of the error. More than a year later the error finally stood corrected, and the correct manuscript was printed

Richard Montgomery is professor of mathematics at the University of California, Santa Cruz. His email address is rmont@math.ucsc.edu.

in *Acta*. Poincaré forked out his entire prize money and then some to pay for the double printing. What a great reminder to us all that real advances are often built on serious mistakes! (For details on this history see in particular the book [Barrow-Green]. Other sources are the website http://www-groups.dcs.st-and.ac.uk/history/Biographies/Poincare.html and the book [Diacu-Holmes].)

Shaw does a good job presenting the history and spirit of the three body problem during the time of the King Oscar II Prize. She does not present much mathematics, but what is there is right. She also goes into some detail regarding mathematics education, recounting a historical debate on the subject in which Cayley played a central role, a debate whose basic lines we hear repeated to this day.

I had a hard time getting past the first third of the book, but once I did the plot started to open up as the characters were fleshed out with all their obsessive flaws. I began to relax and enjoy the book. My wife, a much more experienced mystery reader than I, guessed the ending quickly. It involves a nice twist.

If you are still reading this review I will count you as a captive audience and go on for a bit about the mathematical three body problem. What exactly is "the problem"? Barrow-Green [Barrow-Green] states it succinctly: "Three particles move in space under their mutual gravitational attraction; given their initial conditions, determine their subsequent motion." But Poincaré's Oscar-winning work implies that this problem as stated is not solvable. The situation is similar to that of Galois's proof of the insolubility of the general quintic. Nonsolvability proofs usually do not end the story, but rather begin a much vaster story. The insolubility of the three body problem, through the mechanism of homoclinic tangles, and the techniques developed by Poincaré heralded the qualitative theory of dynamical systems, and with this theory the three body problem blossomed into a whole universe of problems.

I will sketch three open problems that fit under the umbrella of the three body problem. They concern the density, or ubiquity, of various types of solutions.

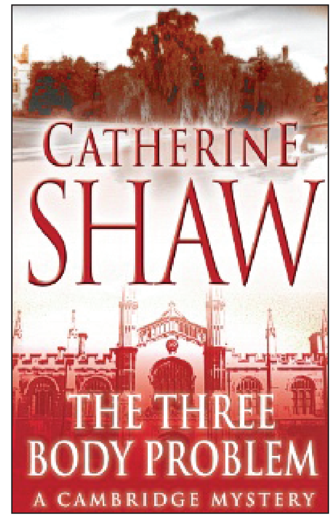
Probably most of us have heard of the Poincaré conjecture in topology. Poincaré has another, less well-known but more open, conjecture. Poincaré's "other conjecture" asserts that periodic orbits are topologically dense: within ϵ of any solution and for any bounded time interval, there is a periodic orbit that shadows the given solution to within ϵ over the given time interval. Poincaré's other conjecture and his faith in the importance of periodic orbits have stimulated an enormous amount of research, including the creation of Floer homology [Floer] to prove the Arnol′d conjecture concerning topological lower bounds on the number of periodic orbits and the various spectacular counterexamples of flows having no periodic orbits (e.g., [Kuperberg] and [Ginzburg]) .

As stated, this Poincaré conjecture is false for the three body problem, but for an easily avoided reason. Call a motion *bounded* if the distance between the three bodies remains bounded as a function of time, and *unbounded* otherwise. Periodic solutions are necessarily bounded. But there are open sets' worth of solutions, all of whose motions are unbounded. Indeed the three body problem has two invariants, or "constants of the motion": the energy and the (total) angular momentum. These are analytic functions on phase space that are constant on any solution curve. Every solution with positive or zero energy is unbounded. So in studying this Poincaré conjecture we should restrict ourselves to negative energy solutions. The standard way of properly restating Poincaré's "other conjecture" is that "the periodic solutions are dense within the bounded solutions."

The second open problem really should come *before* Poincaré's "other conjecture". Is it true that arbitrarily close to any bounded solution lies an unbounded solution? In other words, are the unbounded motions dense and consequently the bounded motions nowhere dense? If the answer were "yes" and if our universe consisted of only the Sun, the Earth, and the Moon moving in a bounded orbit according to Newton's laws, then by nudging the Moon with an arbitrarily small force, we could send one of the three bodies infinitely far from the other two. Michael Herman in his address at the 1998 International Congress of Mathematicians called this last question "the oldest problem in dynamical systems".

The third open problem is special to the planar three body problem. If collisions are excluded, then the configuration space for the problem is three copies of the plane minus collisions and has the homotopy type of a two-sphere minus three points times a circle. The circle is generated by rigidly rotating the triangle formed by the three bodies. If, following Poincaré as usual, we are interested in orbits that are not absolutely periodic but rather periodic modulo a rigid rotation, then we can drop the circle factor. We are now looking at free homotopy classes of curves on the two-sphere minus three points. Such classes are encoded by their eclipse (or syzygy) sequence. An eclipse is a collinear configuration of the three bodies. Noncollision eclipses come in three colors—1, 2, and 3, depending on which mass lies "in the middle" between the other two at eclipse. A solution then has an eclipse sequence, such as 1231212…, that lists the eclipses in their order of occurrence. Is every eclipse sequence realized by some solution? If not, which sequences are excluded? Does the set of realized eclipse sequences have positive density within the set of all possible eclipse sequences?

Returning to the second and "oldest problem", why can't we answer it numerically? Couldn't we take high-precision numerical code, run it for a long time, and simply "see" stability or instablity? Indeed, based on such numerical experiments, most practicing astrophysicists weigh in on the side of "nowhere dense", i.e., of eventual escape. Statistically, it appears that most orbits are unbounded; see, for example, Chapter 7 of [Valtonen]. But such numerical experiments will never lead to proofs or even to very convincing arguments concerning the oldest problem. The difficulty of turning such experiments into proofs lies in the KAM (Kolmogorov-Arnol′d-Moser) theorem and the nature of Arnol′d diffusion. In celestial mechanics, and for Hamiltonian systems generally, there are almost never solutions that are stable in the standard sense found in a first course on dynamical systems or ordinary differential equations. Instead, we must live with the much weaker "KAM stability". The KAM theorem asserts that if a periodic orbit is linearly stable (the linearized flow about the orbit is stable) and if an additional "twist" condition (on higher derivatives of the flow along the orbit) is satisfied, then that solution is surrounded by a family of invariant tori. These tori are the famed KAM tori. The flow on each torus is quasi-periodic, and each has half the dimension of the underlying phase space. But the tori do not fill up phase space near the orbit. A neighborhood of the orbit is homeomorphic to $D^n \times T^n$, where D^n is the n-disc and T^n the n-dimensional torus. The integer $2n$ is the dimension of phase space, and n is called the "number of degrees of freedom". But the KAM tori make up a subset of the form $C \times T^n \subset D^n \times T^n$ in this neighborhood, where $C \subset D^n$ is a Cantor set which is nowhere dense but has positive measure. If $n = 2$, then the KAM tori force stability, for the KAM tori have dimension 2 and lie within the three-manifold of constant energy. There these tori surround the original orbit and so topologically block nearby solutions from escaping. But as soon as $n > 2$ no such topological blocking occurs. Solutions may "leak around" the KAM tori. This leaking is Arnol′d diffusion. If the answer to the oldest problem is "yes, the bounded orbits are nowhere dense," then Arnol′d diffusion must be in play. Why can't we see the Arnol′d diffusion numerically? First, the density of the KAM tori approaches 1 as the original orbit is approached. Secondly, "Nekhoroshev estimates" guarantee that the "time of escape" for any Arnol′d diffusing orbit goes like $p(\epsilon) = exp(-1/\epsilon)$, where ϵ is the distance of its initial condition from the original orbit. This $p(\epsilon)$ is a flat function: its Taylor expansion is identically zero. In practical terms, it signals that escape is extremely slow: it may take longer than the history of the universe for our escaping Moon to move one lunar radius away from the Earth.



I am guessing I was offered the privilege of reviewing this book because of the footnote you will find on its last page. That footnote directs you to a javascript movie made by University of California, Santa Cruz, computer science professor Charlie McDowell and includes a reference to my own work with Alain Chenciner on the three body problem. This work was the rediscovery and rigorous existence proof of an orbit now known as "the eight". This orbit was first found numerically by Cris Moore [Moore] in a beautiful and refreshingly short paper. Alain Chenciner and I [Chenciner] rediscovered Moore's eight and detailed its symmetry and variational properties. Carles Simó showed (numerically) that the eight is "KAM stable". (I am writing this review while at Simó's 60th birthday conference. Happy Birthday, Carles!)

As a graduate student I was sure I would never work on the three body problem. So many famous dead mathematicians had worked on it. How could I compete with all these old greats? And at least as important, how could I work in an area where most of the real experts were long gone? Wouldn't it be like working in a morgue? But I have found an active research community in mathematical celestial mechanics and a horde of hard problems that are very much alive. This murder mystery brought back some of my old misgivings about the three body problem and about the competitive culture within mathematics generally. Once I got over the sticky sweetness of the style, it gave me a nice puzzle and a good sense of the times, of the prize, and of Cambridge.

This book might make a nice summer read if you want a break from research or reviewing, or a good birthday or Christmas present for a colleague or friend with an interest in math.

References

[Barrow-Green] June Barrow-Green, *Poincaré and the Three Body Problem*, in History of Mathematics, vol. 11, Amer. Math. Soc., Providence, RI, and London Mathematical Society, London, 1997.

[Chenciner] Alain Chenciner and Richard Montgomery, A remarkable periodic solution of the three-body problem in the case of equal masses, *Ann. of Math.* **152**, no. 3 (2000), 881–901.

[Diacu-Holmes] Florin Diacu and Philip Holmes, *Celestial Encounters. The Origins of Chaos and Stability*, Princeton University Press, Princeton, NJ, 1996.

[Floer] Andreas Floer, Morse theory for fixed points of symplectic diffeomorphisms, *Bull. Amer. Math. Soc.* **16**, no. 2 (1987), 279–28.

[Ginzburg] Viktor Ginzburg and Başak Z. Gürel, A C^2-smooth counterexample to the Hamiltonian Seifert conjecture in R^4, *Ann. of Math.* **158**, no. 3 (2003), 953–76.

[Kuperberg] Krystyna Kuperberg, Counterexamples to the Seifert conjecture, *Proceedings of the International Congress of Mathematicians*, Vol. II (Berlin, 1998).

[Marchal] Christian Marchal, *The three-body problem*, Studies in Astronautics, vol. 4, Elsevier Science Publishers, B.V., Amsterdam, 1990.

[Moore] Cristopher Moore, Braids in classical gravity, *Phys. Rev. Lett.*, **70**, no. 24 (1993), 3675–79.

[Valtonen] Mauri Valtonen and Hannu Karttunen, *The Three-Body Problem*, Cambridge University Press, 2006.