SHORTER NOTICES.

Leçons sur le Calcul des Variations. Par J. HADAMARD. Recueillies par M. FRÉCHET. Tome Premier. Paris, Hermann, 1910. 8vo. viii+520 pp. 18 francs.

A NEW aspect is given to the calculus of variations as it is presented by Hadamard in his emphasis upon the functional character of the whole problem. While this aspect of the subject has been mentioned by several writers before Hadamard's treatise appeared, it is an innovation in a formal treatment of the entire subject. The ideas of the functional calculus do not appear on the surface of things in a manner offensive to the traditions, but they are subconsciously present throughout the book, as is explained by Hadamard himself in the preface; and they crop out at the surface occasionally.

Briefly stated, any line integral between fixed end points, for example, is a function of the path of integration, in the sense that when that path is given, the integral is determined. From this standpoint, a problem of the calculus of variations is a problem in finding the maximum or minimum values of a functional expression, the independent variable being a *curve*. Of course, we wish also to determine the value of the independent variable—that is the curve—for which the extreme value occurs.

Since the book is now well known to all who are interested in the calculus of variations, this review must be a retrospect and an appreciation rather than an analysis. Much as the reviewer regrets the obstacles which have prevented an earlier review, this changed viewpoint is not without advantages.

Hadamard at once commanded respectful attention everywhere; an immediate review would not have increased nor diminished the desire for instant examination of the work. The reception the book has received throughout the world fully justifies this sweeping statement. Thus Carathéodory in the *Bulletin des Sciences Mathématiques* (volume 34, pages 124–142) calls it an epoch-making work (un livre qui marquera une date dans l'histoire du sujet). I will content myself with noting briefly the plan of organization and what appear to be the guiding motives.

The book proper is preceded by an introduction on extrema of functions and on the properties of differential equations, and is followed by an appendix on implicit functions. These treatments are auxiliary and consider only questions necessary for the developments of the book itself.

The body of the work is divided into three main divisions, called books. In the first book, the task attempted is only the setting of the problem. This is done, to be sure, in a fashion not at all naive. In fact, the difficulties peculiar to the subject, such as the familiar distinction between strong and weak extrema, the generalized notion of neighborhood, and so on, are fundamentally interesting from the standpoint of the functional calculus; every point of this character is discussed in full, and in a most interesting manner.

In the second book, the conditions of the first order—those associated commonly with the first variation—are discussed, together with generalizations of the problem. Here again, questions of peculiar fundamental difficulty are always discussed carefully as a consideration of the first moment. Thus the disagreement between the problem stated in homogeneous form and that stated in non-homogeneous form, which is not always clearly recognized, is made vividly clear. The preliminary Euler condition, the first variation, the fundamental lemma, conjugate points, transversals, and so on, are discussed, usually without great variation from methods now grown classical.

Especial emphasis is laid on practical applications, too often much neglected. Attention is called particularly to No. 140, page 151, and to No. 314, page 381, in which Hadamard sets forth the claim that the methods of mechanics and those of the treatment of geodetic lines by Darboux not only constitute a special case but practically forecast in a trustworthy manner the entire theory. Not only these broad claims but also profuse individual problems characterize the entire book as one of more than usual direct practical bearing.

In the later chapters of this second book are considered many generalizations of the first problem: the case of several variables, the case on a surface, the case of variable end-points, the isoperimetric problem, and a detailed account of Mayer's problem.

This last—the problem of Mayer—is a second instance of a treatment of functional expressions, the first instance being the ordinary problem of the calculus of variations. It is used as a point of departure for a brief but illuminating chapter on the functional calculus, which is the real motive of the entire book, as explained above. The ideas of Volterra are first outlined, including the notion of a derivative of a functional expression. Then the concept of a linear functional expression is introduced, and some of the results of Hadamard, Fréchet and others are expounded. An interesting discussion of this part is given by Carathéodory in his review mentioned above.

The third book consists of a discussion of the final conditions of all types, for the general case of a "free" extremum. This discussion is more closely traditional, perhaps, though it is by no means slavish in following the methods of previous writers in detail. The second variation, the conditions of Jacobi, the fundamental Weierstrass theory, the methods of Hilbert and Kneser, are discussed fully; and such other problems as the case of discontinuous solutions (Carathéodory) and the existence of an absolute extremum (Osgood, Hilbert) are given in satisfactory completeness.

In all, the treatment is certainly well planned and well balanced. Due emphasis is given to generalizations, but the simplest forms of the problem predominate, and the treatment is therefore not inordinately complicated. The work of many authors is presented in a thoroughly digested form, and in a manner which is at once comprehensive and comprehensible; the student is given a well-rounded view of the entire subject. The special interests of the author are limited to their proper proportions as compared with the work of others.

Hadamard's treatise has already affected the development of the calculus of variations and that of the functional calculus; its influence on future developments should be profound.

E. R. HEDRICK.

Les Principes de l'Analyse mathématique. Par PIERRE BOUTROUX. Exposé historique et critique. Tome premier. Paris, Hermann, 1914. 8vo. xi + 547 pp.

THIS work is designed for those who desire a comprehensive view of mathematics, for the purpose of becoming acquainted with its intrinsic significance, and its historical evolution. The object of the book is to exhibit the facts of mathematics,