Ferrari's Solution of a Quartic Equation

1 Introduction

Example 1. Factorise $x^4 + 9$.

Solution. We use the fact that $M^2 - N^2 = (M+N)(M-N)$.
First note that $(x^2+3)^2 = x^4 + 6x^2 + 9$, and so

$$\begin{aligned} (x^2+3)^2 - 6x^2 &= x^4 + 9 \\ (x^2+3)^2 - (\sqrt{6}x)^2 &= x^4 + 9 \\ ((x^2+3) + \sqrt{6}x)((x^2+3) - \sqrt{6}x) &= x^4 + 9 \\ (x^2 + \sqrt{6}x + 3)(x^2 - \sqrt{6}x + 3) &= x^4 + 9. \end{aligned}$$

The quadratics can't be factorised any further, and so the full factorisation is

$$x^4 + 9 = (x^2 + \sqrt{6}x + 3)(x^2 - \sqrt{6}x + 3).$$

2 Factorising a quartic

Now consider the general quartic $x^4 + 2ax^3 + bx^2 + 2cx + d$. The key to factorising this quartic is to find numbers A, B, and C such that

$$x^4 + 2ax^3 + bx^2 + 2cx + d \equiv (x^2 + ax + A)^2 - (Bx + C)^2$$

and then to use the fact that $M^2 - N^2 = (M+N)(M-N)$.

Example 2. Solve the quartic equation $x^4 + 2x^3 - x^2 - 2x - 3 = 0$ for x.

Solution. Note that, for our quartic, we have $a = 1$, and so we must find numbers A, B, and C such that

$$x^4 + 2x^3 - x^2 - 2x - 3 \equiv (x^2 + x + A)^2 - (Bx + C)^2.$$

Expanding the brackets on the right hand side gives

$$\begin{aligned} (x^2+x+A)^2 &= (x^2+x+A)(x^2+x+A) \\ &= x^4 + x^3 + Ax^2 + x^3 + x^2 + Ax + Ax^2 + Ax + A^2 \\ &= x^4 + 2x^3 + (2A+1)x^2 + 2Ax + A^2 \end{aligned}$$

and $(Bx+C)^2 = B^2x^2 + 2BCx + C^2$. Now

$$\begin{aligned} (x^2+x+A)^2 - (Bx+C)^2 &= x^4 + 2x^3 + (2A+1)x^2 + 2Ax + A^2 - (B^2x^2 + 2BCx + C^2) \\ &= x^4 + 2x^3 + (2A + 1 - B^2)x^2 + 2(A - BC)x + A^2 - C^2 \end{aligned}$$

Therefore, we must find numbers A, B, and C such that

$$x^4 + 2x^3 - x^2 - 2x - 3 \equiv x^4 + 2x^3 + (2A + 1 - B^2)x^2 + 2(A - BC)x + A^2 - C^2.$$

By comparing coefficients, we see that

$$\begin{cases} -1 = 2A + 1 - B^2 & \Leftrightarrow & B^2 = 2A + 2 & (1) \\ -1 = A - BC & \Leftrightarrow & BC = A + 1 & (2) \\ -3 = A^2 - C^2 & \Leftrightarrow & C^2 = A^2 + 3 & (3) \end{cases}$$

Now (1) $\times$ (3) gives $B^2C^2 = (2A + 2)(A^2 + 3) = 2(A + 1)(A^2 + 3)$ and $(2)^2$ gives $(BC)^2 = (A + 1)^2$. We can now eliminate $(BC)^2$ to obtain

$$\begin{aligned} (A + 1)^2 &= 2(A + 1)(A^2 + 3) \\ 2(A + 1)(A^2 + 3) - (A + 1)^2 &= 0 \\ (A + 1)(2(A^2 + 3) - (A + 1)) &= 0 \\ (A + 1)(2A^2 + 6 - A - 1) &= 0 \\ (A + 1)(2A^2 - A + 5) &= 0. \end{aligned}$$

We have a cubic equation for A. We only need to find one solution, so choose $A = -1$. Now (1) gives $B^2 = 0 \Rightarrow B = 0$ and (3) gives $C^2 = (-1)^2 + 3 = 4 \Rightarrow C = \pm 2$. Since we only choose one solution, we choose $C = 2$.
Now

$$\begin{aligned} x^4 + 2x^3 - x^2 - 2x - 3 &\equiv (x^2 + x + A)^2 - (Bx + C)^2 \\ &= (x^2 + x - 1)^2 - (0x + 2)^2 \\ &= (x^2 + x - 1 + 2)(x^2 + x - 1 - 2) \\ &= (x^2 + x + 1)(x^2 + x - 3) \end{aligned}$$

Hence

$$\begin{aligned} x^4 + 2x^3 - x^2 - 2x - 3 &= 0 \\ \Leftrightarrow (x^2 + x + 1)(x^2 + x - 3) &= 0 \\ \Leftrightarrow (x^2 + x + 1) = 0 \quad &\text{or} \quad (x^2 + x - 3) = 0 \\ \Leftrightarrow (x^2 + x - 3) &= 0 \\ \Leftrightarrow x &= \frac{-1 \pm \sqrt{1 - 4(1)(-3)}}{2} = \frac{-1 \pm \sqrt{13}}{2} \end{aligned}$$

3 The general case

Given a general quartic $x^4 + 2ax^3 + bx^2 + 2cx + d$, we need to find numbers A, B, and C such that

$$x^4 + 2ax^3 + bx^2 + 2cx + d \equiv (x^2 + ax + A)^2 - (Bx + C)^2.$$

When we expand the brackets on the right-hand side, we get

$$x^4 + 2ax^3 + bx^2 + 2cx + d = x^4 + 2ax^3 + (2A + a^2 - B^2)x^2 + 2(A - BC)x + A^2 - C^2.$$

By comparing coefficients, we see that

$$\left\{ \begin{array}{lllllll} b & = & 2A + a^2 - B^2 & \Leftrightarrow & B^2 & = & 2A + a^2 - b \quad (1) \\ c & = & A - BC & \Leftrightarrow & BC & = & A - c \quad (2) \\ d & = & A^2 - C^2 & \Leftrightarrow & C^2 & = & A^2 - d \quad (3) \end{array} \right.$$

Now (1) $\times$ (3) gives $B^2C^2 = (2A + a^2 - b)(A^2 - d)$ and $(2)^2$ gives $(BC)^2 = (A - c)^2$. We can now eliminate $(BC)^2$ to obtain

$$(A - c)^2 = (2A + a^2 - b)(A^2 - d)$$

We have a cubic equation for A, which we can solve by using the cubic formula. Since a cubic equation has three solutions, we can normally choose A to be one of three values: if possible choose A so that B^2 and C^2 are positive.
Once we have found A, B, and C, then, by using the fact that

$$(x^2 + ax + A)^2 - (Bx + C)^2 = [x^2 + ax + A + (Bx + C)][x^2 + ax + A - (Bx + C)]$$

we can factorise the quartic into a product of two quadratics, and hence we can fully factorise the quartic by factorising the two quadratics.

4 The quintic and above

A quintic is a polynomial of degree 5. An obvious question to ask is if there is a formula for solving the general quintic equation $ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0$.
Consider the formula for solving a quadratic equation:

$$ax^2 + bx + c = 0 \Leftrightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Notice that the formula is built up from the coefficients a, b and c. The same is true for cubic and quartics: we build the solution starting from the coefficients, and then using the operations of addition, subtraction, multiplication, division, and n^{th} roots. This process of finding the solution is called a *solution by radicals.*
In 1824, a mathematician called Abel proved that the general quintic equation is not solvable by using radicals, and the same is true for even higher degree polynomial equations.
An example is $x^5 - 6x + 3 = 0$: this equation has five solutions, but we can't build the solutions starting from the numbers 1, -6 and 3.
Hence

$$\boxed{\text{there is no “formula” to solve the general quintic } ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0}$$