# Technische Universität München

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## The 'Princess and Monster' Game on an Interval


Bachelor's Thesis by Leonhard Geupel



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I hereby declare that I have written the Bachelor's Thesis on my own and have used no other than the stated sources and aids.

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# Abstract

This version of 'Princess and Monster' games at which the Monster seeks out for a Princess in a dark room, takes place on the interval $[-1, 1]$. It seems that this version had not been studied for a long time because it appears to be trivial. On the contrary it is difficult enough so that no one has been able to solve it so far. This work gives a short introduction into the world of 'Princess and Monster' games and presents the results of [1] by showing properties for optimal strategies of both players, as well as estimates on the value of the game. Finally it updates numerical findings of [2] which provides considerations on how to approximate a possible solution of the game.

# Zusammenfassung

Diese Version der 'Prinzessin und Monster' Spiele, bei denen das Monster nach einer Prinzessin in einem dunklen Raum sucht, findet auf dem Intervall $[-1, 1]$ statt. Es erweckt den Anschein, als wäre diese Version lange Zeit nicht studiert worden, weil sie trivial wirkt. Sie ist im Gegenteil schwierig genug, so dass bisher niemand im Stande war, sie zu lösen. Diese Arbeit gibt eine kurze Einführung in die Welt der 'Prinzessin und Monster' Spiele und stellt die Ergebnisse von [1] vor, in dem Eigenschaften von optimalen Strategien beider Spieler so wie Abschätzungen des Wertes des Spiels hergeleitet werden. Zuletzt werden numerische Erkenntnisse aus [2], das Überlegungen für die Approximation einer möglichen Lösung des Spiels bereitstellt, auf den neuesten Stand gebracht.

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# Contents

# 1 Introduction

## 1.1 Search games with mobile Hider on general spaces

A specific version of a search game is a search game whose so called Hider is mobile. In his book [3] Rufus Isaacs introduced such games under the name “Princess and Monster Games”. Here the Monster (Searcher) seeks out for the Princess (Hider) in a totally dark room, respectively any metric space $Q$ with metric $d_Q$ where nothing can be spotted apart from its boundary. This means that at no time the Monster can see the Princess and vice versa except for the time $T$ when they meet. Let $S = S(t)$ and $H = H(t)$ be continuous paths on $Q$ that describe the movement of both players at which the speed of the Monster is at most 1. Apart from that the Monster and the Princess can choose any starting point $S(0)$, respectively $H(0)$. Then the capture time, cost function or payoff $T$ for $S$ and $H$ is

$$T = C(S, H) = \min \{t : S(t) = H(t)\} \tag{1}$$

which has been shown to be reasonable even for not compact spaces in [6]. As the Monster wants to minimize $T$ and the Princess wants to maximize it, the game is called zero-sum. Therefore $T$ also “represent(s) the gain of the Hider” and “the loss (effort spent in searching) of the Searcher” [4].

To keep up the terminology of game theory, it is important to differentiate between available strategies of the Monster and the Princess. So the trajectories $S$ and $H$ are called pure strategies and their corresponding spaces are identified as follows: The pure strategy space $\mathcal{H}$ of the Hider consists of all continuous paths $H : [0,\infty] \mapsto Q$. Because of the limited speed of the Searcher, his pure strategy space $\mathcal{S}$ consists of all paths in $\mathcal{H}$ with Lipschitz constant 1, that is $\mathcal{S} = \{S : [0,\infty) \mapsto Q : d_Q(S(t), S(t')) \leq |\, t - t'|, \forall\, t, t' \geq 0\}$. Due to the fact that noone knows the chosen strategy of the opponent or just to assure oneself against the worst case, it is reasonable to choose the value of a strategy as the worst possible payoff for that strategy, that is $V(S) = \sup\limits_{H \in \mathcal{H}} C(S, H)$ $\forall\, S \in \mathcal{S}$ and $V(H) = \inf\limits_{S \in \mathcal{S}} C(S, H)\ \forall\, H \in \mathcal{H}$. The pure value is the value of the game achieved only via pure strategies and is $V = \inf\limits_{S \in S} V(S) = \sup\limits_{H \in \mathcal{H}} V(H)$. Sometimes there is only a value that is abitrary close to the pure value. But if the pure value exists, then there is for any $\varepsilon > 0$ a such called $\varepsilon$-optimal pure strategy whose value is worse only about $\varepsilon$. Or more exactly, if $S_\varepsilon$ is an $\varepsilon$-optimal pure searcher strategy and $H_\varepsilon$ an $\varepsilon$-optimal pure hider strategy, then $V(S_\varepsilon) < (1 + \varepsilon)V$ and $V(H_\varepsilon) > (1 - \varepsilon)V$ holds. Thus $V$ is the best guaranteed payoff for both players when they use both one fixed trajectory. But in general it doesn’t suffice to use only pure strategies, because mostly there is no pure strategy that dominates all other pure strategies. So the Monster and the Princess have to make probabilistic choices among pure strategies which are called mixed strategies. Mathematically spoken, a mixed strategy for a player with pure strategy space $\mathcal{X}$ is the probability distribution over $\mathcal{X}$. Of course, any pure strategy can be regarded as mixed strategy which concentrates only on that specific pure strategy. But as those mixed strategies are usually highly randomized, this is one reason why search games aren't easy to solve. "Our feeling is that just how the players use chance to pick their paths is secondary; probably any haphazard meandering will do about as well as any other" [3]. Focusing therefore only on mixed strategies from now on, the capture time can't be a fixed cost anymore, but an expected cost. Let $s$ be either a mass function or a density function of a mixed searcher strategy and $h$ be either a mass function or a density function of a mixed hider strategy depending on $\mathcal{S}$ and $\mathcal{H}$ in each case being finite or not. For short, we call $s$ and $h$ mixed strategies, too. Let moreover $x(S,H) = s(S) \cdot h(H)\ \forall S \in \mathcal{S},\ \forall H \in \mathcal{H}$. Then the expected capture time $T^*$ reads depending on $\mathcal{S} \times \mathcal{H}$ being finite or not as:

$$T^* = c(s,h) = \sum_{(S,H) \in \mathcal{S} \times \mathcal{H}} x(S,H) \cdot C(S,H) \tag{2}$$

or

$$T^* = c(s,h) = \int_{\mathcal{S} \times \mathcal{H}} x(S,H) \cdot C(S,H) d(S,H) \tag{3}$$

Anyway, the values of the mixed strategies and the value of the game can be easily transferred from the values declared in conjunction with pure strategies. $v(s) = \sup_h c(s,h)$ and $v(h) = \inf_s c(s,h)$ denote the value of a mixed searcher strategy $s$, respectively of a mixed hider strategy $h$. Hence $v(s) \geq c(s,h) \geq v(h)$ for any $s$ and for any $h$. If equality holds, the corresponding mixed strategies $s$ and $h$ are called optimal strategies while the coinciding values of both strategies form the value of the game $v$.

$$v = \inf_s v(s) = \sup_h v(h) \tag{4}$$

In this case, there also exist for any $\varepsilon$ so called $\varepsilon$-optimal strategies $s_\varepsilon$ for the Searcher and $h_\varepsilon$ for the Hider, which satisfy

$$v(s_\varepsilon) \leq (1+\varepsilon)v \tag{5}$$

and

$$v(h_\varepsilon) \geq (1-\varepsilon)v \tag{6}$$

After clarifying the basic terms of search games with mobile Hider on general, metric spaces, this work introduces its mainpart, the Monster and the Princess on an Interval, whose present results are carried out after [1] and [2] in the following.

## 1.2 The game on an interval

According to [1] the minimax theorem of Alpern and Gal [6] can be used to prove the existence of $v$ for any $Q$. Moreover it shows that there is also an optimal mixed searcher strategy as well as an $\varepsilon$-optimal mixed hider strategy. Nevertheless these Princess and Monster Games are so difficult that none but one have been solved meaning that the value of these game is still unknown, let alone the strategies of both players. Of the easiest spaces $Q$, speaking of the circle and the interval, only one has been solved yet: the circle. The other one appeared to be trivial and had not even been studied for a long time. It seemed that on the interval $I = [-1, 1]$ the Monster simply starts with probability $\frac{1}{2}$ at one of both ends ($-1$ or $1$) and straightly grazes the whole interval for the Princess with maximal speed 1 until it reaches the other end. Against this, the best what the Princess could do is an $\varepsilon$-optimal strategy: She waits e.g. at 0 until time $1-\varepsilon$, then goes equiprobably to either end. As this strategy works for any $\varepsilon > 0$ which can be arbitrary close to 0, it is ignored when calculating payoffs now and in the rest of this work when using $\varepsilon$-optimal strategies. From there the two possible capture times are 1 and 2. This gives the following expected capture time $T^* = \frac{1}{2}(\frac{1}{2}(1+2)) + \frac{1}{2}(\frac{1}{2}(2+1)) = \frac{3}{2}$. As said, optimal mixed strategies are rather highly randomized. The Monster can maybe expect a such intelligent move of the Princess and expand his strategy by adding a strategy that counters her loitering in the middle of the interval from time to time. Later it will be shown that for this reason the value of the game is by all means not $\frac{3}{2}$, but smaller and therefore not trivial. Transforming a handful of pure strategies into a matrix game and solving it as linear program after [7] will also give a lower bound for $v$. By including a bit more complex strategies with continuous initial distribution for both players, the sharper estimate $1 < \frac{15}{11} < v < \frac{13}{9} < \frac{3}{2}$ achieved in [1], will precisely be recomputed. At last this work offers a numerical approach like in [2] that calculates an approximated value of a restricted version of the game via strategies used to get the above-mentioned estimate. The so found value of this special game is $\approx 1.3727$, but together with the associated optimal strategies of both players, the results of this work differ from those in [2]. However, it is at first necessary to talk about the properties of the optimal, respectively $\varepsilon$-optimal strategies of the Monster and the Princess to restrict the strategies to be regarded. Here [1] serves again as standard. Then it is possible to construct and to comprehend strategies that help to contribute to the solution of the game.

# 2 Properties of the game

In this chapter it will be shown that every optimal searcher strategy as well as every ($\varepsilon$-)optimal hider strategy must have got particular properties on the one hand. But on the other hand there also exists of eventually more possible ($\varepsilon$-)optimal strategies always at least one strategy with other certain characteristics. By trying to construct optimal strategies by oneself – and that is the only thing these properties are important for the rest of this work –, it suffices to consider last named strategies for which all of the following rules apply: First of all, it is clear without proof that searcher paths have to cover the entire interval or hiding at some end for example gives a payoff that is infinite. Furthermore it is obvious that the Monster and the Princess equiprobably use pure strategies and their symmetric counterparts such that the mixed strategies of both players are invariant under the reflection $\phi(x) = -x$. Such mixed strategies are called symmetric. A pure strategy that is symmetric to $S \in \mathcal{S}$, respectively $H \in \mathcal{H}$, is denoted as $-S$, respectively $-H$. Then it will be proofed that a Searcher reaching an end of the interval should go directly to the other end. In contrast, a Hider that reaches an end, should stay there until the end of the game. Besides, the Hider should never move faster than the maximal velocity of the Searcher which is 1. Without losing the $\varepsilon$-optimality of a mixed hider strategy, it is possible to consider that it only consists of a finite number of pure strategies. Then it will be shown that it is sufficient for the Searcher to also use only a finite number of pure strategies which can exactly be determined when the hider strategy is known once. This fact will later be very important for finding lower bounds on the value of the game. It will also justify that the Searcher has to utilize his maximal speed 1 the whole time. At last there exists an $\varepsilon$-optimal hider mixed strategy in which the pure strategies do not interesect. A consequence of this will be that a Hider will only loiter around either in the positive part of the interval $[0, 1]$ or in its negative part $[-1, 0]$. But at first here are a few definitions that make it easier to handle the proofs of above-named properties mathematically.

**Definition 1.** Let $\mathcal{X}$ be a closed pure strategy space consisting of continuous pure strategies of a player. Such a player with pure strategy $X \in \mathcal{X}$ **runs** or is called **running** at time $t$ if he or she moves with velocity 1 at time $t$. If it is clear at which time interval the player runs, the player just runs or is just called running.

*Remark* 2. As there exists a pure strategy space consisting of smooth (continuously differentiable) paths and that is dense in $\mathcal{X}$, it is legitimate to allow for smooth paths only without changing the value of the game. This implies $|X'(t)| = 1$ for a player with strategy $X$ that is running at time $t$.

**Definition 3.** A pure strategy $S \in \mathcal{S}$ is called **end-reflecting** if the player using this strategy turns after reaching an end and runs to the other end. This implies $\forall\, t_0 \in [0,\infty)$ that $|S(t_0)| = 1 \Rightarrow |S(t) - S(t_0)| = t - t_0\ \forall\, t$ with $t_0 \leq t \leq t_0 + 2$.

**Definition 4.** A pure strategy $H \in \mathcal{H}$ is called **end-absorbing** if the player using this strategy remains on the spot after reaching an end. This implies $\forall\, t_0 \in [0,\infty)$ that $|H(t_0)| = 1 \Rightarrow H(t) = H(t_0)\ \forall\, t \geq t_0$.

Of the following lemmas the first three affect the nature of the paths picked by the players, the rest concern mixed strategies as a whole.

**Proposition 5.** *Every pure searcher strategy $S \in \mathcal{S}$ is dominated by one which is end-reflecting.*

*Proof.* It is perspicuous that if the Monster has reached the end and has not found the Princess yet, he will find the Princess independent of any strategy she uses, as quickly as possible by running straight to the other end. To give a proper proof, suppose $S$ is not end-reflecting and let it reach $+1$ at first time $t_0$. Define $S^*$ as $S$ up to time $t_0$, but then let it be end-reflecting, thus $S^* = 1-(t-t_0)\ \forall t$ with $t_0 \leq t \leq t_0+2$. Let $H \in \mathcal{H}$ be arbitrary. If $C(S,H) \leq t_0$, then $C(S^*,H) = C(S,H)$. If for some $t_1 > t_0$ $C(S,H) = t_1$ holds, then the definition of $T$ immediately gives $S(t_1) - H(t_1) = 0$, and since the player with strategy $S^*$ is running, it yields $S(t) \geq S^*(t)\ \forall t \geq t_0$. Therefore we have $S^*(t_1) - H(t_1) \leq 0$. On the other hand by considering $H(t) \leq 1\ \forall\, t \in [0,\infty)$, we obtain $S^*(t_0) - H(t_0) = 1 - H(t_0) \geq 0$. As $S^* - H$ being a continuous function, the Intermediate Value Theorem applies such that the Monster using $S^*$ finds the Princess between $t_0$ and $t_1$. Thus in all cases $S^*$ satisfies $C(S^*,H) \leq C(S,H)$ and so $S^*$ dominates $S$. ☐

**Proposition 6.** *Every pure hider strategy $H \in \mathcal{H}$ is dominated by one which is end-absorbing.*

*Proof.* Again our intuition tells us that this is true because once the Princess has reached an end, it is better to stay there or she will only meet the Monster halfway. A proper proof of this fact is achieved like the proof above: Let $H$ be not end-absorbing and let it reach $+1$ at first time $t_0$. Define $H^*$ agreeing with $H$ up to time $t_0$, but as end-absorbing strategy and the Hider using it stays at $+1$. Consider any $S \in S$. Similarly to above we may also consider $C(S,H^*) = t_1 > t_0$ due to the fact that if $C(S,H) \leq t_0$, then $C(S,H^*) = C(S,H)$. All in all this yields $S(t_1) = 1 \geq H(t_1)$ and $S(t_0) < 1 = H(t_0)$ for a continuous $S$. By the Intermediate Value Theorem it follows that the Searcher finds the Hider between $t_0$ and $t_1$ and therefore $C(S,H^*) \geq C(S,H)$ holds. As $S$ was arbitrary, $H^*$ dominates $H$. ☐

**Proposition 7.** *Every smooth pure hider strategy $H$ in $\mathcal{H}$ is dominated by one which is in $\mathcal{S} = \{S : [0, \infty) \mapsto I = [-1, 1] : |S(t) - S(t')| \leq |\, t - t'|, \forall\, t, t' \geq 0\}$. That means that no Hider should move with greater velocity than* 1.

*Proof.* We construct a pure strategy $H^*$ for a Princess who follows a Princess using $H$, but only moves with a velocity which is at most 1. The idea behind this is that on the one hand a Princess using $H^*$ can't be caught by a Monster from behind and on the other hand she will not crash into the Monster before the Princess using $H$. So it is sufficient to proof the last named fact which goes by a handful of following, valid assumptions: Let $t_0$ be the first time that the Princess with $H$ is moving faster than speed 1. As $H$ is smooth and not in $\mathcal{S}$, the definition $t_0 = \inf\, \{t \in [0, \infty) : |H'(t)| > 1\} < \infty$ as well as its finiteness is justified. Define $H^*(t) = H(t)\ \forall\, t \leq t_0$ and $\forall\, t$ with $\tau \geq t > t_0$ the Princess using $H^*$ continues to move with speed 1 in the same direction as the Princess using $H$ until they meet again at time $\tau$ because of the boundedness of the interval. Suppose that hiders with strategy $H$ and $H^*$ are in different locations when the Hider with strategy $H$ is found at time $C(S, H)$ by the Searcher using any strategy $S \in \mathcal{S}$ or $C(S, H^*) = C(S, H)$ holds. Furthermore suppose that the capture also occurs before time $\tau$ or else let $t_1 = \inf\, \{t \in [\tau, \infty) : |H'(t)| > 1\}$ and repeat the construction inductively. This yields $t_0 < C(S, H) < \tau$. As there always exist symmetric pure strategies in every $\varepsilon$-optimal mixed hider strategy, let $H$ be the one of both symmetric variants that moves right at time $t_0$ such that $H^*(t) < H(t)\ \forall\, t \in (t_0, \tau)$ holds. Since $C(S, H) \in (t_0, \tau)$ we also have $H^*(C(S, H)) < H(C(S, H)) = S(C(S, H))$. Moreover the Searcher using $S$ moves with bounded speed and therefore it yields $H^*(t) < S(t)\ \forall\, t \in (t_0, C(S, H))$ resulting into $C(S, H^*) > C(S, H)$ when putting all cases together. ☐

The following lemmas now concern mixed strategies, probability measures on the Borel $\sigma$-algebra of $\mathcal{S}$, respectively any subset that is dense in $\mathcal{S}$.

**Proposition 8.** *For any $\varepsilon > 0$ there exists an $\varepsilon$-optimal hider mixed strategy that consists of a finite number of pure strategies.*

*Proof.* Discretize the interval into a large, but finite number of grid points and consider polygonal paths on this grid that approximate the pure strategies of the Princess. Then the set of all those polygonal paths – let's call it $\mathcal{P}$ – is also finite and as the grid points of the discretization can be arbitrary close to each other, $\mathcal{P}$ is dense in the pure strategy space $\mathcal{S}$ of the Princess. Let $S, H \in \mathcal{S}$ be arbitrary and $P \in \mathcal{P}$ denotes the approximation of $H$. While a Princess using $P$ is limited to a grid, it yields $C(S, P) \leq C(S, H)$ for a unlimited Monster using $S$. Since $C(S, H)$ is lower semi-continuous as a function on $S$ and $H$, it follows $C(S, P) \uparrow C(S, H)$ for an increasing number of grid points. Thus, replacing $\mathcal{H}$ by $\mathcal{P}$ in the definition of the value of the game (2), respectively (3), shows that the $\varepsilon$-optimality is preserved. ☐

**Proposition 9.** *Suppose the Hider restricts his mixed strategy on a finite set of pure strategies. Then the optimal response by the Searcher is to concentrate on only a finite set of pure strategies as well. In particular, he should run to one possible Hider after another until all possible Hiders using different pure strategies have been caught. This means for such a pure searcher strategy $S$ and for pure hider strategies $H_j$ with numbering $j \in 1, \dots, J$ such that the capture times are $C(S, H_j) = t_j$ with $0 = t_1 \leq t_2 \leq \dots \leq t_J$:*

$$S'(t) = sign\,(H_{j+1}(t_j) - S(t_j))\;\forall\, t \in (t_j, t_{j+1}), j \in 1, \dots, J-1 \tag{7}$$

*Proof.* Let $S^*$ be any pure searcher strategy which is an optimal response to the hider mixed strategy consisting of $H_j, j \in 1, \dots, J$ so that $C(S^*, H_j) = t_j^*$ nondecreasing in $j$. Moreover suppose it fails (7) for some $j$. Let $k$ be the smallest such $j$ for it. Again as there exist symmetric pure strategies, let $S^*$ be the trajectory with $S^*(t_k^*) < H_{k+1}(t_k^*)$ without loss of generality. Define also a new pure searcher strategy $S$ with $C(S, H_j) = t_j\;\forall\, j \in 1, \dots, J$. The Searcher using it first follows the Searcher with $S^*$ until time $t_k^*$ when he continues to use (7) by running right. Therefore he will be the first one meeting $H_j$ and since $H_j \in \mathcal{S}$, he can follow up the Hider using $H_j$ until this Hider is found by the Searcher with $S^*$ at time $t_{k+1}^*$. After that the Searcher with strategy $S$ can agree with $S^*$ once more until the next time when $S^*$ fails (7) again. This yields the function

$$S(t) = \begin{cases} S^*(t), & for\ t \leq t_k^* \\ S^*(t_k) + (t - t_k), & for\ t_k^* \leq t \leq t_{k+1} \\ H_{j+1}(t), & for\ t_{k+1} \leq t \leq t_{k+1}^* \\ S^*(t), & for\ t_{k+1}^* \leq t \end{cases}$$

Thus it is clear that $t_j = t_j^*\;\forall\, j \in [J] \setminus \{k+1\}$ and $t_{k+1} < t_{k+1}^*$ holds. Hence $S^*$ is not an optimal response to the mixed hider strategy. □

**Corollary 10.** *If a Searcher is using an optimal strategy, then he runs all the time.*

**Definition 11.** A pair of pure hider strategies $H_1, H_2$ is called **non-crossing** if $H_1(t) \leq H_2(t)\;\forall\, t \geq 0$ holds. If the inequality holds strictly, $H_1$ and $H_2$ are called **non-intersecting**.

**Proposition 12.** *Let a mixed hider strategy $h$ consist of two equal used pure hider strategies $H_1$ and $H_2$. Define new pure hider strategies $H_1 \wedge H_2 = \min\limits_{t \geq 0}\,\{H_1(t), H_2(t)\}$ and $H_1 \vee H_2 = \max\limits_{t \geq 0}\,\{H_1(t), H_2(t)\}$. Then $h$ is dominated by a strategy that mixes $H_1 \wedge H_2$ and $H_1 \vee H_2$ with equal probability. Consequently, any finite mixed hider strategy may be assumed to consist of non-crossing pure strategies.*

*Proof.* According to its definition, $H_1 \wedge H_2$ and $H_1 \vee H_2$ form a set that coincides with $\{H_1(t), H_2(t)\}$ for all $t$. Let a Searcher using a pure strategy $S$ catch the first of the two Hiders with the original strategies $H_1$ and $H_2$ – let it be $H_1$ without loss of generality – at time $t_1$. Therefore he also catches the first of the two hiders using $H_1 \wedge H_2$ and $H_1 \vee H_2$ at time $t_1$. As regarding symmetric strategies, let the Searcher with $S$ approach from the left side and thus, catch $H_1 \wedge H_2$ at time $t_1$. Consequently we obtain $S(t_1) = H_1 \wedge H_2(t) = H_1(t_1) \leq H_2(t_1) = H_1 \vee H_2(t_1)$. Together with $H_2(t) \leq H_1 \vee H_2(t)\ \forall t \geq 0$, it yields $C(S, H_2) \leq C(S, H_1 \vee H_2)$ and so $H_1 \vee H_2$ dominates $H_2$ while $H_1 \wedge H_2$ and $H_1$ are at least equally good. Since $H_1 \wedge H_2$ and $H_1 \vee H_2$ are obviously non-crossing, we can therefore construct a finite mixed hider strategy out of several non-crossing pure strategies. □

*Remark* 13. As a finite collection of non-crossing paths can be approximated arbitrary closely by a collection of non-intersecting paths, any finite mixed hider strategy may be assumed to consist of non-intersecting pure strategies.

To recapitulate, in the following part of this work only mixed strategies have to be considered that are finite, symmetric and additionally non-intersecting, if it is about mixed hider strategies. Before proceeding with showing the non-triviality of the game in the next section, there is just only one more short corollary left that instantaneously follows for mixed hider strategies being non-intersecting and symmetric at the same time.

**Proposition 14.** *Any pure hider strategy $H$ in a non-intersecting, symmetric mixed hider strategy is contained in either one half of the interval which means $H(t) \in [-1, 0]$ or $H(t) \in [0, 1]$ for all $t \geq 0$.*

*Proof.* As $H$ is used by the Princess, so is its symmetric part $-H$. $H(t) = 0$ for any $t$ would imply an intersection of $H$ and $-H$, unless $H$ is immobile and stays at 0 for the rest of the game. But when $H(t) \neq 0\ \forall t \geq 0$, neither $H$ nor $-H$ can traverse the middle of the interval. In both cases, the claim holds. □

# 3 Analytic study of the game on the interval

## 3.1 The non-triviality of the game

When looking at the 'Princess and Monster' game on an interval, there are several strategies for both players that appear to be obviously optimal. If they were, the game would be considered to be trivial as one may when hearing the first time of this problem. One component of such 'obviously optimal' strategies has already been investigated in this work, that is the so called sweeper strategy, let it denote with $A$.

In this pure strategy the Monster, respectively 'the sweeper', starts at 1 and runs to $-1$, such that $A(t) = 1 - t \;\forall\, t \in [0,2]$ or symmetrically $-A(t) = -1 + t \;\forall\, t \in [0,2]$ yields. Apparently it also possesses the property gained in Proposition 5. An $\varepsilon$-optimal response of the Princess is to loiter in the middle of the interval until time $1-\varepsilon$ and then go to $+1$, respectively $-1$ and is denoted by $F$, respectively $-F$. Describing both strategies as trajectories that satisfy the relevant properties corresponding to Proposition 6, 7 and 14, this gives $F(t) = \begin{cases} \max\limits_{t\geq 0}\{0, -1+\varepsilon+t\}, & for\; 0 \leq t \leq 2-\varepsilon \\ 1, & for\; t > 2-\varepsilon \end{cases}$, respectively $-F(t) = \begin{cases} \min\limits_{t\geq 0}\{0, +1-\varepsilon-t\}, & for\; 0 \leq t \leq 2-\varepsilon \\ -1, & for\; t > 2-\varepsilon \end{cases}$. Mixing $F$ and $-F$ with equal probability then yields as shown the expected payoff $\frac{3}{2}$. The reason why this response is optimal and why therefore we can put $v \leq \frac{3}{2}$, is that a Monster mixing $A$ and $-A$ will equiprobably find any Princess either in the first half of the interval or her symmetric counterpart in the second half such that in the worst case one capture time is 1 and the other one is 2. Another strategy seeming to be part of an optimal strategy and also fulfilling all corresponding properties of pure hider strategies, is the strategy $E$ whose user hides at $+1$ for the rest of the game. So for all $t \geq 0$ we obtain $E(t) = 1$, respectively $-E(t) = -1$. The strategy which mixes $E$ and $-E$ with equal probability, is optimally countered according to Proposition 9 by the mixed strategy consisting of the equiprobable usage of the sweeper strategies $A$ and $-A$. As every pure strategy is used with equal probability here, it suffices to concentrate on one of two symmetric pure strategies for one player when computing the expected payoff $T^*$. This means $T^* = \frac{1}{2}C(S,E) + \frac{1}{2}C(-S,E) = \frac{1}{2}C(S,E) + \frac{1}{2}C(S,-E) = \frac{1}{2}(0+2) = 1$. Altogether we obtain $1 \leq v \leq \frac{3}{2}$. To show the non-triviality of the game, it is already enough to proof $1 < v < \frac{3}{2}$. Therefore this work starts with rather easy strategies for both players to get the estimate $1 < \frac{97}{95} < v < \frac{47}{32} < \frac{3}{2}$. Then some strategies exerting themselves for a sharper estimate with continuous initial distributions, are presented.

### 3.1.1 A simple searcher strategy

To build a better searcher strategy such that $T^* < \frac{3}{2}$, let's have a look at $A$ and $-A$ versus $F$ and $-F$ once more. As revealed in the introduction of this work, the Monster can improve his strategy by adding new pure strategies. Now that we have the helpful proposition 9, it is known what at least a pair of missing strategies should fulfill and that is to start at 0. Let therefore be $B$ and $-B$ be such a pair that punishs loitering hiders in the middle of the interval. More specifically, a Monster using $B$ starts at 0, runs to the left until it meets the sweeper using $-A$ at time $\frac{1}{2}$ and then joins him, but

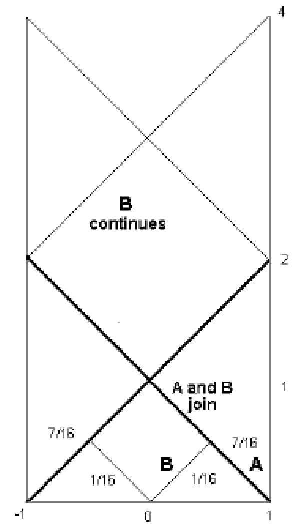


Figure 1: The searcher strategy for $V < \frac{47}{32}$ in a space-time diagram (adopted from [1])

after reaching 1 the Monster has to run back to $-1$ by Proposition 5. The symmetric strategy $-B$ does the same, but there the Monster runs first right and then joins the sweeper with $A$. The mixed strategy of the Monster now goes in the following way: It uses strategies $\pm A$ each with probability $\frac{7}{16}$ and $\pm B$ each with probability $\frac{1}{16}$. Those strategies are drawn in a space-time diagram $[-1, 1] \times [0, \infty)$, Fig. 1 (see also [1]). Search paths are depicted here as lines of slope $\pm 1$ with $A$ and $-A$ being the thicker lines. This new mixed searcher strategy brings a slight improvement as shown below.

**Lemma 15.** *If the Monster uses the mixed strategy described above, then the expected payoff $\frac{47}{32}$ is guaranteed against any strategy of the Princess. Therefore $v \le \frac{47}{32} = 1.46875$ holds.*

*Proof.* Let $H$ be any pure hider strategy and let $P(t)$ be the probability that the hider has been caught since time $t$ by any Monster using one of its pure searcher strategies. Then we consider two cases: (i) $|H(\frac{1}{2})| \le \frac{1}{2}$, and (ii) $|H(\frac{1}{2})| > \frac{1}{2}$.

(i) In this case we may once again assume because of symmetry that $0 \le H(\frac{1}{2}) \le \frac{1}{2}$. So therefore the Princess using $H$ has been caught by the Monster with $-B$ at the latest at time $\frac{1}{2}$ such that $P(\frac{1}{2}) = \frac{1}{16}$. Then the Princess can maximally avoid the rest of the possible opponents until time 1. She chooses to meet the slight less likely appearing Monster using $A$ in the middle first, then $\varepsilon$-optimally escapes Monsters with $-A$ and $-B$ by running to 1 until time 2. Thus, it yields $P(1) = \frac{8}{16}$, $P(2) = 1$ and $T^* \le \frac{1}{16} \cdot \frac{1}{2} + \frac{7}{16} \cdot 1 + \frac{8}{16} \cdot 2 = \frac{47}{32}$.

(ii) The same trick as in (i) let us assume $H(\frac{1}{2}) > \frac{1}{2}$ without loss of generality. Hence the Princess using $H$ has met the Monster with $A$ until time $\frac{1}{2}$ while $\varepsilon$-optimally escaping the Monster with $-B$ such that $P(\frac{1}{2}) = \frac{7}{16}$ yields. Then she can maximally avoid Monsters with $B$ and $-A$ that joined themselves until time 2 and the Monster with $-B$ until time 4 by hiding at the end 1. Consequently, we have $P(2) = \frac{15}{16}$, $P(4) = 1$ and therefore $T^* = \frac{7}{16} \cdot \frac{1}{2} + \frac{8}{16} \cdot 2 + \frac{1}{16} \cdot 4 = \frac{47}{32}$. □

We proceed with a lower bound for $v$. In exchange it is preliminarily important to know about the relationship of matrix games and search games and how to solve such matrix games.

### 3.1.2 Matrix games

In general a matrix game is a game in which two rivals play against each other by picking either a column or a row of a $m \times n$ matrix $G = (g_{i,j})_{i=1,\dots,m,j=1,\dots,n}$. The first player secretly chooses a column $j \in [n]$ whereas the second player secretly chooses a row $i \in [m]$. Both players get the payoff $g_{i,j}$ at which the aim of player one is to maximize his expected payoff and the aim of player two is to minimize it for any length of time, respectively any number of repetition of the game. This is achieved for both players by using mixed strategies. In particular, if you exchange player one with the Princess and player two with the Monster, you get a restricted version of the Monster and the Princess game in which exactly $m$ pure strategies are available for the Monster and exactly $n$ pure strategies are available for the Princess. This means row $i$ stands for the $i$-th pure strategy $H_j$ of the Princess, column $j$ stands for the $j$-th pure strategy $S_j$ of the Monster and $g_{i,j}$ is $g_{i,j} = C(S_j, H_j)$. The tricky thing is that a matrix game only works with a finite number of pure strategies, but after Proposition 8 and 9, it suffices to concentrate on finite mixed strategies for the Princess as well for the Monster. So theoretically if we had a finite, $\varepsilon$-optimal mixed strategy for the Princess, we could construct an optimal response of the Monster with the help of Proposition 9 and therefore the solution of the matrix game that inherits all those pure strategies, would give the value of the game. At least we can construct for any hider strategy an optimal response of the Monster and a solution of this matrix game would then give the value of the corresponding hider strategy which is by definition a lower bound for the value of the game. But the question that remains is, how to solve such a matrix game? Now having the problem in form of a matrix $G$, the answer is to embed $G$ into a linear problem (LP for short) which can be solved by several algorithms like the Simplex algorithm. The Princess for example wants a mixed strategy $h$ in form of a vector $p = (h(H_1), \dots, h(H_1))^T$ with a guaranteed and maximized payoff $z$. As $h$ resembles a mass function, it has to satisfy $h(H_j) \geq 0 \; \forall \, j \in [n]$ and $\sum_{j=1}^{n} h(H_j) = 1$. Because $z$ is guaranteed, the strategy $h$ should ensure that its payoff is at least as good as $z$ for all strategies of the Monster. This yields $\sum_{i=1}^{n} x_i \cdot g_{ij} \geq z \; \forall \, j \in [n]$. Let $1_n = (1, \dots, 1)^T \in R^n$

and $0_n = (0,\ldots,0)^T \in R^n$. Maximizing $z$ while transforming these conditions into several matrix multiplications, gives the following LP:

$$
\begin{aligned}
z^* := \max_{(p^T,\, z)^T \in \mathbb{R}^{n+1}} & \; z \\
\text{such that} \quad -Gp + z \cdot 1_m & \le 0_m \\
1_n^T p & = 1 \\
p & \ge 0_n
\end{aligned}
\iff
\begin{aligned}
\max_{(p^T,\, z)^T \in \mathbb{R}^{n+1}} & \begin{pmatrix} 0_n \\ 1 \end{pmatrix}^T \begin{pmatrix} p \\ z \end{pmatrix} \\
\text{such that} \quad \begin{pmatrix} -G & 1_m \end{pmatrix} \begin{pmatrix} p \\ z \end{pmatrix} & \le 0_m \\
\begin{pmatrix} 1_n^T & 0 \end{pmatrix} \begin{pmatrix} p \\ z \end{pmatrix} & = 1 \\
p & \ge 0_n
\end{aligned}
\tag{8}
$$

Consider now the dual problem with $q \in \mathbb{R}^m$, $w_1 \in \mathbb{R}$, $w_2 \in \mathbb{R}$, $s \in \mathbb{R}^n$ and $I_n$ being the $n \times n$ unit matrix:

$$
\begin{aligned}
\min_{(q^T,\, s^T,\, w_1,\, w_2)^T \in \mathbb{R}^{m+n+2}} & \begin{pmatrix} 0_m \\ 0_n \\ 1 \\ -1 \end{pmatrix}^T \begin{pmatrix} q \\ s \\ w_1 \\ w_2 \end{pmatrix} \\
\text{such that} \quad \begin{pmatrix} -G & 1_m \\ -I_n & 0_n \\ 1_n^T & 0 \\ -1_n^T & 0 \end{pmatrix}^T \begin{pmatrix} q \\ s \\ w_1 \\ w_2 \end{pmatrix} & = \begin{pmatrix} 0_n \\ 1 \end{pmatrix} \qquad \iff \\
q & \ge 0_m \\
s & \ge 0_n \\
w_1, w_2 & \ge 0
\end{aligned}
$$

$$
\begin{aligned}
\min_{(q^T,\, s^T,\, w_1,\, w_2)^T \in \mathbb{R}^{m+n+2}} & \; w_1 - w_2 \\
\text{s. t.} \quad -G^T q - s + w_1 \cdot 1_n - w_2 \cdot 1_n & = 0_n \\
1_m^T q & = 1 \\
q & \ge 0_m \\
s & \ge 0_n \\
w_1, w_2 & \ge 0
\end{aligned}
\;\underset{s \ge 0}{\overset{w = w_1 - w_2}{\iff}}\;
\begin{aligned}
\min_{(q^T,\, w)^T \in \mathbb{R}^{m+1}} & \; w =: w^* \\
\text{s.t.} \quad -G^T q + w \cdot 1_n & \ge 0_n \\
1_m^T q & = 1 \\
q & \ge 0_m
\end{aligned}
\tag{9}
$$

As understanding $q$ as a vector containing a mixed strategy of the Monster and $w$ as a payoff that should be an upper bound for every possible payoff the Monster can achieve and that should be minimal, the dual problem (9) exactly provides what the Monster wants to do. Now able to solve the value of the matrix game $z^* = w^*$and its corresponding optimal mixed strategies, the next section shows how to compute a lower bound for $v$ like described above.

### 3.1.3 A simple hider strategy

Let the strategies $A$, $B$, $E$ and $F$ be as before. Once more the already existing pure strategies will be enriched, this time starting with the strategies of the Princess. At this she may for example consider the $\varepsilon$-optimal strategies that let her start at $\pm\frac{1}{2}$ which lies in between the starting points of $\pm A$ and $\pm B$ and then remain there until $\frac{1}{2}-\varepsilon$. After ensuring a payoff of at least $\frac{1}{2}$, she can still delay the capture by either a Monster using $\pm A$ or a Monster using $\pm B$ by either running to the middle and back until the end or by running directly to the nearest end. Denote those strategies $X$ and $Y$ in same order. Together with their symmetric counterparts they are drawn in thick lines in Fig. 2 on the left side (see also [1]). Note that $X$ and $Y$ do not cross the centre like all paths of $\varepsilon$-optimal mixed hider strategies that fulfill the property of Proposition 14. If the Princess now mixes $\{\pm E, \pm F, \pm, X, \pm Y\}$, the Monster has to start in 0, $\pm\frac{1}{2}$ or $\pm 1$ and then run between all possible hider paths according to Proposition 9. When starting at an end, there is only one reasonable option for the Monster and that is to use the sweeper strategy $\pm A$. If it starts in 0, then it uses either $\pm B$ or to catch a nearer Princess using either $\pm E$ or $\pm Y$, it can in contrast run to an end and then to the other one. Denote this strategy with $M$. The last two possible pairs of strategies of the Monster are obtained when it starts in $\pm\frac{1}{2}$. Before turning in opposite direction, it runs to the nearest end first, 'strategy $\pm C$' or to the remote end, 'strategy $\pm D$'. Strictly speaking, you have to consider two more pairs of strategies starting in $\pm\frac{1}{2}$. One that brings the Monster to the centre and back and one that brings the Monster to $\pm\frac{1}{4}$ to additionally catch a Hider with $\pm X$ before turning. By computing the capture times, it is clear that both strategies are dominated by $\pm D$, though. Otherwise, note that there aren't more pure searcher strategies. That is because the Searcher mostly catches multiple possible Hiders, above all by running to an end. These strategies of the Monster can also be found in Fig. 2, this time on the right hand side.

**Lemma 16.** *If the Princess and the Monster mix all their above named strategies, then the solution of the corresponding matrix game yields the lower bound $v \geq \frac{97}{75} = 1.2933$.*

*Proof.* A game matrix $G$ equipped with the expected capture time of $\{S_i, -S_i\}$ versus $\{H_j, -H_j\}$ for the $i$-th pure searcher strategy $\pm S_i$ and the $j$-th pure hider strategy $\pm H_j$ for all rows $i$ and all columns $j$, is in this case with ignoring $\varepsilon$:

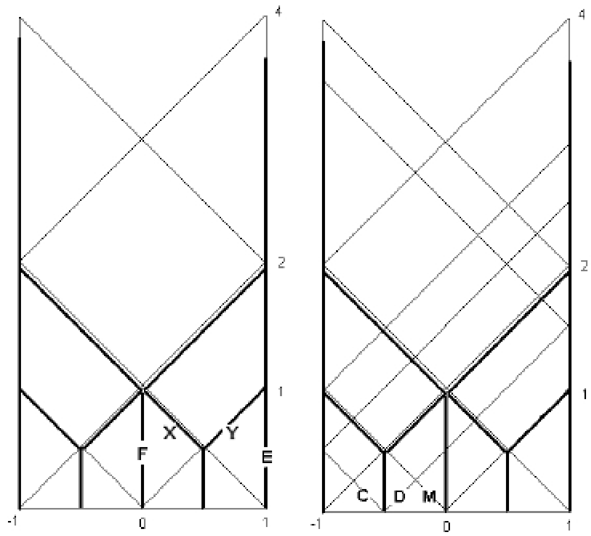


Figure 2: Left: $\{\pm E, \pm F, \pm, X, \pm Y\}$ is depicted thick against $\{\pm A, \pm B\}$; Right: $C$, $D$ and $M$ are added due to lucidity without their symmetric parts. (adopted from [1])

$$G = \begin{array}{cc} & \begin{matrix} \pm E & \pm F & \pm X & \pm Y \end{matrix} \\ \begin{matrix} \pm A \\ \pm B \\ \pm C \\ \pm D \\ \pm M \end{matrix} & \begin{pmatrix} \frac{1}{2}(0+2) & \frac{1}{2}(1+2) & \frac{1}{2}(1+2) & \frac{1}{2}(\frac{1}{2}+2) \\ \frac{1}{2}(2+4) & \frac{1}{2}(0+0) & \frac{1}{2}(\frac{1}{2}+2) & \frac{1}{2}(2+4) \\ \frac{1}{2}(\frac{1}{2}+\frac{5}{2}) & \frac{1}{2}(\frac{5}{4}+\frac{5}{2}) & \frac{1}{2}(0+\frac{5}{2}) & \frac{1}{2}(0+\frac{5}{2}) \\ \frac{1}{2}(\frac{3}{2}+\frac{7}{2}) & \frac{1}{2}(\frac{1}{2}+\frac{1}{2}) & \frac{1}{2}(0+\frac{3}{4}) & \frac{1}{2}(0+\frac{3}{2}) \\ \frac{1}{2}(1+3) & \frac{1}{2}(0+0) & \frac{1}{2}(\frac{1}{2}+3) & \frac{1}{2}(1+3) \end{pmatrix} \end{array} = \begin{array}{cc} & \begin{matrix} \pm E & \pm F & \pm X & \pm Y \end{matrix} \\ \begin{matrix} \pm A \\ \pm B \\ \pm C \\ \pm D \\ \pm M \end{matrix} & \begin{pmatrix} 1 & \frac{3}{2} & \frac{3}{2} & \frac{5}{4} \\ 3 & 0 & \frac{5}{4} & 3 \\ \frac{3}{2} & \frac{15}{8} & \frac{5}{4} & \frac{5}{4} \\ \frac{5}{2} & \frac{1}{2} & \frac{3}{8} & \frac{3}{4} \\ 2 & 0 & \frac{7}{4} & 2 \end{pmatrix} \end{array}.$$

Solving $G$ as explained in the last section, gives the value of the matrix $\frac{97}{75}$. □

*Remark* 17. Let $p^*$ denote the optimal mixed hider strategy in vector form and $q^*$ denote the $\varepsilon$-optimal mixed searcher strategy in vector form of the matrix game of Lemma 16. Then $p^* = (0.4133, 0.32, 0.2667, 0)^T$ and $q^* = (0.8, 0, 0, 0.1867, 0.0133)^T$ which means that the Princess doesn't use strategy $\pm Y$ at all because of the frequent use of the sweeper strategy and the Monster only uses $\pm A$, $\pm D$ and rarely $\pm M$.

To get more accurate bounds on $v$ it would theoretically be possible to proceed as before and consistently add ($\varepsilon$-)optimal responses to both players. Unfortunately according to [1] the increase of the number of the rather inefficient computable pure strategies is exponential and the convergence to the value of the game appears to be very slow. But there is another approach that might do better and suggests itself the most from now on. The Monster and the Princess both can use specific pure strategies that let them actually do the same thing, but let them always start at a different location. In particular, let them give strategies with continuous initial distributions. With their help it is possible to set further limits on the value of the game and obtain $\frac{15}{11} < v < \frac{13}{9}$.

## 3.2 A searcher strategy with continuous initial distribution

Define a new mixed searcher strategy $s_\Phi$ whose user starts in a point $x$ on the interval according to a continuous distribution function $\Phi(x)$. Then like in strategy $-B$ he runs right until meeting the sweeper $A$ which will be joined. Of course, by reaching $-1$ he has to turn once more and run to 1. Therefore the symmetric strategy $-s_\Phi$ let the Monster start in $x$ with probability $\Phi(-x)$ , just to let it run left until meeting $-A$, etc. In the rest of this section let $y = y(H) \leq 1$ be the first time a Princess using the pure strategy $H$ meets a sweeper. Then it can be claimed:

**Lemma 18.** *A Searcher using $\pm s_\Phi$ finds a Hider with pure strategy $H$ before time $y$ if and only if he starts in $(1-2y, H(0)]$ and runs to the right, or if he starts in $[H(0), -1+2y)$ and runs to the left.*

*Proof.* First of all the two intervals $(1-2y, H(0)]$ and $[H(0), -1+2y)$ are legitimate. According to Proposition 14, it yields either $H(t) \geq 0\ \forall\, t \geq 0$ (short: $H \geq 0$) or $H \leq 0$ and therefore either $H(y) = 1-y$ or $H(y) = -1+y$ due to the definition of $y$ and of the sweeper strategy. Because of $|H(y) - H(0)| \leq y$ (Proposition 7), $1-2y \leq H(0)$ and $-1+2y \geq H(0)$ are valid. Let $S$ be now a pure searcher strategy in $+s_\Phi$. The proof for $-s_\Phi$ is the same. Due to Proposition 14 assume moreover that $H \geq 0$. To show " $\Rightarrow$ " assume that a Searcher with $S$ does not start in the interval $(1-2y, H(0)]$, so either $S(0) > H(0)$ or $S(0) \leq 1-2y$. In the first case the Searcher runs to the right and meets the sweeper with strategy $A$ before the Hider such that the capture time is $y$ and not below. In the second case it is not possible for the Searcher to catch a Hider with $H(y) = 1-y$ before time $y$. " $\Leftarrow$ ": Now assume $S(0) \in (1-2y, H(0)]$. As the Searchers with $S$ and $A$ run toward each other, they meet on half way, that is $t_0 = \frac{1-S(0)}{2}$. Since $1-2y < S(0) \leq H(0)$ it yields $t_0 < y$. Therefore the Searcher meets the sweeper before the sweeper meets the Hider which means that the Searcher has to meet the Hider in the mean time. $\square$

Let in the following be $f = \Phi'$ the probability density of $\Phi$.

**Lemma 19.** *Searchers with $s_\Phi$ that start in $(1-2y, H(0)]$ catch any Hider using $H$ with expected capture time*

$$\int_0^y t \cdot f(H(t)-t)(1-H'(t))dt \tag{10}$$

*Searchers with $-s_\Phi$ that start in $[H(0), -1+2y)$ catch any Hider using $H$ with expected* *capture time*

$$\int_0^y t \cdot f(-H(t)-t)(1+H'(t))dt \tag{11}$$

*Consequently, the optimal hider path from $H(0)$ to $H(y)$ with a fixed $y$ against both type of Searchers categorized above, maximizes*

$$\int_0^y \Phi(H(t)-t)+\Phi(-H(t)-t)dt \tag{12}$$

*Proof.* To proof (10) assume the Searcher to behave like stated above. Moreover let $\phi(t)$ be the probability density that describes the relative likelihood of the Hider with $H$ being catched by the Searcher. Due to Lemma 18 the Searcher needs less time than $y$ and hence the expected capture time is $\int_0^y t\cdot\phi(t)dt$. It remains to show $\phi(t)=f(H(t)-t)(1-H'(t))$. Therefore consider a small time interval $[t,t+\Delta t]$ in which the Searcher catches the Hider. At this time sequence the Hider can move from $H(t)$ to $H(t)+\Delta H$ with $|\Delta H|\leq\Delta t$ arbitrary due to Proposition 7. Because the Searcher starts to the left of the Hider and straightly runs to him, the Searcher starts in $[H(t)+\Delta H-(t+\Delta t), H(t)-t]$. The probability of the Searcher starting in this interval is equivalent to the probability of the Hider to get caught in $[t,t+\Delta t]$ and can be approximated by the length of $[H(t)+\Delta H-(t+\Delta t), H(t)-t]$ multiplicated with the density $f$ at $H(t)-t$. This approximation $f(H(t)-t)\cdot(\Delta t-\Delta H)$ in turn is equal to the approximation via $\phi(t)$ over $[t,t+\Delta t]$, that is $\phi(t)\cdot\Delta t$. By solving this equation for $\phi(t)$ while taking $\Delta t$ to 0, we obtain: $\phi(t)=\underset{\Delta t\longrightarrow 0}{lim}\frac{1}{\Delta t}\cdot f(H(t)-t)\cdot(\Delta t-\Delta H)=\underset{\Delta t\longrightarrow 0}{lim} f(H(t)-t)\cdot(1-\frac{H(t+\Delta t)-H(t)}{\Delta t})=f(H(t)-t)(1-H'(t))$. (11) is achieved in the same way.

It remains to show (12). Since the Monster uses $s_\Phi$ and $-s_\Phi$ with same probability, the Princess is caught with the expected capture time established by the arithmetic average of (10) and (11), that is $\frac{1}{2}\int_0^y t[f(H(t)-t)(1-H'(t))+f(-H(t)-t)(1+H'(t))]dt$. Maximizing this integral is equivalent to maximizing it without the constant $\frac{1}{2}$ and partial integration yields: $\int_0^y t[f(H(t)-t)(1-H'(t))+f(-H(t)-t)(1+H'(t))]dt = y[-\Phi(H(y)-y)-\Phi(-H(y)-y)]+\int_0^y \Phi(H(t)-t)+\Phi(-H(t)-t)dt$. The first term can be further simplified to the value $-y(\Phi(1-2y))$ independent of $H$ by $|H(y)|=1-y$. Though, the last term gives the sought-after integral (12). $\square$

**Lemma 20.** *Once a Hider with pure strategy $H$ would meet a sweeper, he should go to the end when playing against a Searcher that mixes the sweeper strategies with $\pm s_\Phi$.*

*Then the Searcher using only $\pm s_\Phi$ will catch the Hider ignoring $\varepsilon$ with expected time*

$$1-\Phi(-H(0))+2\Phi(1-2y)+\frac{y}{2}(1-\Phi(H(0)))-\frac{y}{2}\Phi(1-2y)+\frac{1}{2}\int_0^y \Phi(H(t)-t)+\Phi(-H(t)-t)dt \tag{13}$$

*Proof.* The first claim is clear, because once a Searcher with $\pm s_\Phi$ meets a sweeper, he will join him and run away from the end where the Hider optimally hides. To proof the second claim, consider all different locations for the Searcher to start in as well as both directions he can choose. Therefore it is not necessary to consider the symmetric counterpart of strategy $H$. Thus, assume as we may $H \geq 0$. A Searcher that chooses to use $-s_\Phi$ with probability $\frac{1}{2}$ and then starts out left from $H(0)$ with probability $1-\Phi(-H(0))$, runs to the left, joins the sweeper wih $-A$ and finds the Hider in 1 at time 2. Hence, the first term $\frac{1}{2}(1-\Phi(-H(0)))\cdot 2 = 1-\Phi(-H(0))$ is obtained. The second term is achieved by considering a Searcher that starts left from $1-2y$ and runs to the right, all in all with probability $\frac{1}{2}\Phi(1-2y)\cdot$. Due to Lemma 18 he meets the sweeper with $A$ first and after a reciprocating motion he finally reaches the Hider at time 4. Then a Searcher that starts right from $H(0)$ and runs to the right with probability $\frac{1}{2}(1-\Phi(H(0)))$, joins the sweeper with $A$, just to meet the Hider together at time $y$. This gives the third term. The last two terms are obtained by Searchers that behave like in Lemma 19. The expected values are therefore calculated by the same token as in the proof of that lemma. □

To tinker an upper bound for $v$, the payoff (13) has to become simpler. Therefore consider (12) which is a variational problem and its Euler-Lagrange equation is $f(H(t)-t) = f(-H(t)-t)$. If $f = \frac{1}{2}$ denoted the density of the uniform distribution, then this equation would be satisfied for every hider path $H$. Moreoever (12) would simplify itself to a term independent of $H$, that would be with $\Phi(x) = \frac{x+1}{2}$: $\int_0^y \frac{H(t)-t+1}{2} + \frac{-H(t)-t+1}{2}dt = -\frac{1}{2}y^2 + y$. So now we can give a explicit value for (13) if still adding just a few modifications here and there. The desired upper bound for $v$ can then be obtained out of it:

**Theorem 21.** $v \leq \frac{13}{9}$.

*Proof.* First let $\Phi(x) = \frac{x+1}{2}$ for the rest of (13) to obtain $1 - \frac{-H(0)-1}{2} + 2\cdot\frac{2-2y}{2} + \frac{y}{2} - \frac{y}{2}\cdot\frac{H(0)+1}{2} - \frac{y}{2}\cdot\frac{2-2y}{2} - \frac{1}{2}(-\frac{1}{2}y^2+y) = \frac{10+2H(0)-(7+H(0))y+y^2}{4}$. This term has to be maximized in order to get an upper bound on $v$. For $0 \leq y \leq 1$, it is obvious that $H(0) = 1$ does the job without fail and so we get $\frac{y^2}{4} - 2y + 3$. The Searcher now mixes $\{\pm A, \pm s_\Phi\}$ whereas he uses $\pm A$ with probability $\alpha$ and $\pm s_\Phi$ with probability $1-\alpha$. Denote this new mixed searcher strategy with $\sigma = \sigma(\alpha)$. As a Searcher using $\pm A$ meets a rational

Hider either at time $y$ or time 2 with equal probability, the expected capture time for $\sigma$ is $\alpha \cdot (\frac{y}{2}+1)+(1-\alpha)\cdot(\frac{y^2}{4}-2y+3) = \alpha(-2+\frac{5}{2}y-\frac{1}{4}y^2)+3-2y+\frac{1}{4}y^2$. This term has to be maximized in $y$ such that it is an upper bound for $v$, while the choice of $\alpha$ is up to the Searcher and can therefore be minimized for a sharper bound on $v$. Computing this maximin problem via Matlab gives $\alpha = \frac{7}{9}$ and $y$ is either 0 or 1. That means for the Princess either to hide at an end or to escape from a sweeper until time 1 and then run back to an end. Inserting the obtained values $\alpha$ and $y$ in the last term gives the desired upper bound $\frac{13}{9}$. □

Some might consider to vary $\Phi$ to even get a better upper bound on $v$. But to show that this bound can't be much more improved with the use of the mixed searcher strategy $\sigma$, let's construct a mixed hider strategy $\gamma$ such that the expected payoff $c(\sigma,\gamma) \leq v(\sigma)$ for both strategies is only marginally smaller than $\frac{13}{9}$. As the analysis of Theorem 21 indicates, it is maybe also here helpful for the Princess to choose $y=0$ and $y=1$. So let be $\gamma$ a mixed hider strategy that uses $\{\pm E, \pm H_0, H_{\pm\frac{1}{2}}, H_{\pm 1}\}$ whereas $H_x$ is defined as follows: The Princess using $H_x$ starts in $x$, runs to the nearest sweeper just to turn $\varepsilon$ in front of him and runs back to the middle where she turns once more to run back to the end. $-H_x$ and $H_{-x}$ denote the symmetric counterparts of $H_x$ whereas it is not necessary to consider them for the computation of (13). Moreover it suffices to define a path for $H_x$ that works until time $y$ because the integral (12) does not require anything beyond that. With $x \geq 0$ and ignoring $\varepsilon$ we have $H_x(t) = \begin{cases} H_x^1 = x+t, & \text{for } 0 \leq t \leq \frac{1-x}{2} \\ H_x^2 = 1-t, & \text{for } \frac{1-x}{2} \leq t \leq y \end{cases}$ and obtain therefore for (12) after substitution $\int_0^{\frac{1-x}{2}} \Phi(H_x^1(t)-t)+\Phi(-H_x^1(t)-t)dt + \int_{\frac{1-x}{2}}^{y} \Phi(H_x^2(t)-t)+\Phi(-H_x^2(t)-t)dt = \frac{1-x}{2}\cdot\Phi(x)+\frac{1}{2}\int_{-1}^{-x}\Phi(t)dt+\frac{1}{2}\int_{1-2y}^{x}\Phi(t)dt$. Putting that term together with (13) and $y=1$ yields

$$\frac{3}{2} - \Phi(-x) - \frac{(1+x)}{4}\cdot\Phi(x) + \frac{1}{4}\int\limits_{-1}^{-x}\Phi(t)dt + \frac{1}{4}\int\limits_{-1}^{x}\Phi(t)dt \tag{14}$$

Setting $x$ either to 0, $\frac{1}{2}$ or 1 and bounding the occurent integrals from below give a matrix game with a value that is a lower bound for $c(\sigma,\gamma)$ and thus also for the value of the strategy $\sigma$. As $\Phi$ being a distribution function, it is monotonically increasing and the integrals containing it, can for example be bounded from below by lower Riemann sums over even partitions of length $\frac{1}{2}$. That means e.g. $\int_{-1}^{1}\Phi(t)dt \geq \frac{1}{2}\underbrace{\Phi(-1)}_{=0}+\frac{1}{2}\Phi(-\frac{1}{2})+\frac{1}{2}\Phi(0)+\frac{1}{2}\Phi(\frac{1}{2})$. So we get for $\pm H_0$, $H_{\pm\frac{1}{2}}$ and $H_{\pm 1}$ against $\pm s_\Phi$ the set containing their payoffs in same order:

$$\left\{\frac{3}{2}+\frac{1}{4}\Phi(-\frac{1}{2})-\frac{5}{4}\Phi(0), \frac{3}{2}-\frac{7}{8}\Phi(-\frac{1}{2})+\frac{1}{8}\Phi(0)-\frac{3}{8}\Phi(\frac{1}{2}), 1+\frac{1}{8}\Phi(-\frac{1}{2})+\frac{1}{8}\Phi(0)+\frac{1}{8}\Phi(\frac{1}{2})\right\} \tag{15}$$

In order to minimize the value of the matrix game the Searcher should choose $\Phi$ so that the maximum of (15) is minimal. Solving this minimax problem with Matlab such that $0 \leq \Phi(-\frac{1}{2}) \leq \Phi(0) \leq \Phi(\frac{1}{2}) \leq 1$ and all values in (15) are positive, yields $\Phi(-\frac{1}{2}) = \frac{8}{25}$ and $\Phi(0) = \Phi(\frac{1}{2}) = \frac{9}{25}$. Then all payoffs in (15) are equal, that is the value 1.13. As the strategies $\pm H_0$, $H_{\pm\frac{1}{2}}$ and $H_{\pm 1}$ all possess the same expected capture times against $\sigma$, the game matrix can be simplified to $\begin{array}{c c} & \begin{array}{cc} \pm E & \pm H_x \end{array} \\ \begin{array}{c} \pm A \\ \pm s_\Phi \end{array} & \begin{pmatrix} 1 & \frac{3}{2} \\ 3 & 1.13 \end{pmatrix} \end{array}$. Its value is $\frac{337}{237} = 1.4219$ which is only marginally smaller than $\frac{13}{9}$.

## 3.3 A hider strategy with continuous initial distribution

Now it is time to improve the lower bound on $v$. Therefore we start with a definition of a mixed strategy with continuous initial distribution, this time for the Princess. As the strategies $\pm E$, $\pm F$ and $\pm X$ did a good job in section 3.1.3, it suggests itself to replace the similiar strategies $\pm F$ and $\pm G$ by a general version $\pm h_\Theta$ that works for arbitrary starting points on the interval. In particular, let $h_\Theta$ be similar to $H_x$ beside the fact that the Princess using $h_\Theta$ waits for the nearest sweeper instead of straightly running toward to him after she has picked a point $x \in (0, 1)$ according to a continuous distribution function $\Theta(x)$. Then she runs as with strategy $H_x$ in front of the sweeper that is $\varepsilon$ behind her and after reaching the middle, she turns and runs back to the end. In the symmetric strategy the Princess picks a point $x \in (-1, 0)$ according to $\Theta(-x)$.

The Princess now mixes $\{\pm E, \pm h_\Theta\}$. Denote this new mixed hider strategy with $\mu$. For the rest of this work redefine $y \leq 1$ as the first time that a Searcher meets a sweeper. Then by Proposition 9, any optimal response $S$ of the Searcher against $\mu$ has to let him start in $[-1, 0) \cup (0, 1]$. If the Searcher starts at an end, then $S$ should be a sweeper strategy. So assume $|S(0)| < 1$. Due to the usual arguments concerning symmetry of optimal strategies, assume moreover that $S(y) = 1 - y$. So the Searcher with $S$ meets the sweeper with $A$ first and approaches therefore from the right. The best thing he can do is to collect as many immobile hiders as possible that started in $x > S(0)$ and that either wait for the sweeper with $A$ to come or that run $\varepsilon$ in front of the sweeper with $A$. So similiar to section 3.1.3 by computing payoffs it emerges that turning until time $y$ would only lower the value of the searcher strategy. At time $y$ there are three types of hiders remaining: A lonely Hider with strategy $E$ is found in 1 and the hiders that started in $x < S(0)$, build two groups. One that stays in the positive half of the interval somewhen running into the Searcher and the other one in the negative half of the interval. The last group together with the Hider using $-E$ can be caught by a Searcher with $S$ at the earliest in $-1$. So the strategy $S$ splits by Proposition 9 at time $y$ into two strategies. $M_1$ denotes the strategy whose user goes for the lonely Hider with $E$ first and then catches the rest. $M_2$ denotes the strategy whose user similarly as in strategy $s_\Phi$ "joins" the sweeper with $A$ to catch all hiders with $x < S(0)$ and turns in $-1$ to get back to 1. Strictly speaking, such a Searcher using $M_2$ should not directly join the sweeper but run $\varepsilon$ in front of him like the hiders do. However, $\varepsilon$ is ignored as always.

Both strategies have simple payoffs against $\pm E$. A Monster with $M_1$ catches a Princess with $E$ at time $2y$ after meeting the sweeper on half way and a Princess with $-E$ at time $2y+2$ after crossing the whole interval in addition. That makes a expected payoff of $1+2y$ for $\pm M_1$ against $\pm E$. The expected payoff for $\pm M_2$ against $\pm E$ is known since dealing with $\pm s_\Phi$, and is therefore 3. The remaining payoffs are obtained in the following lemma, whereas $\theta = \Theta'$ denotes the probability density of $\Theta$:

**Lemma 22.** *Ignoring $\varepsilon$ the expected payoff for $\pm M_1$ against $\pm h_\Theta$ is*

$$\begin{gathered}(1+y)(1-\Theta(2y-1)) + \tfrac{1}{2}\int_0^{2y-1}(-t+2y-1)\cdot\theta(t)dt + \tfrac{1}{2}(1+y)\Theta(1-2y) \\ +\tfrac{1}{2}\int_{1-2y}^{1-y}(t-1+2y)\cdot\theta(t)dt + \tfrac{1}{2}y(1-\Theta(1-y))\end{gathered} \tag{16}$$

*Ignoring $\varepsilon$ the expected payoff for $\pm M_2$ against $\pm h_\Theta$ is*

$$\begin{gathered}1-\Theta(2y-1) + \tfrac{1}{2}\int_0^{2y-1}(-t+2y-1)\cdot\theta(t)dt + \tfrac{1}{2}\Theta(1-2y) \\ +\tfrac{1}{2}\int_{1-2y}^{1-y}(t-1+2y)\cdot\theta(t)dt + \tfrac{1}{2}y(1-\Theta(1-y))\end{gathered} \tag{17}$$

*Proof.* To compute the payoffs, consider due to symmetry only $M_1$ and $M_2$ against $\pm h_\Theta$. At first, we show (16). In exchange consider different cases. Thereof the first two cases concern the strategy $-h_\Phi$ whose user starts and stays in the negative half of the interval. In the first case the Hider is to the left of a Searcher using $M_1$ which means left to $1-2y$. As the Hider chooses his strategy with probability $\frac{1}{2}$ and his starting point $x$ with probability $\Theta(-x)$, the total probability of using $-h_\Theta$ with all $x \in [-1, 1-2y)$ is $\frac{1}{2}(1-\Theta(2y-1))$. The Searcher with $M_1$ finds such hiders not until the end of his journey, that is at time $2+2y$. This gives the first term. The second term is achieved by considering hiders that start in $x \in [1-2y, 0)$ with probability $\frac{1}{2}\int_{1-2y}^{0}\theta(-t)dt = \frac{1}{2}\int_0^{2y-1}\theta(t)dt$. Those hiders wait for the sweeper with strategy $-A$ and therefore get previously caught by a Searcher with $M_1$ at time $t(x) = x-(1-2y)\ \forall\, y \in [\frac{1}{2}, 1]$. The last three cases deal with Hiders in the positive half of the interval. In the third case the Hider is right to a Searcher with $y < \frac{1}{2}$, that gives the starting point $x \in [0, 1-2y)$ and the therefore corresponding starting probability $\frac{1}{2}\Theta(1-2y)$. The capture time is computed as follows: The Searcher starting in $1-2y$, runs to 1 and needs time $2y$ for it while the Hider stays immobile. Then the Hider runs to the middle while the Searcher is approaching about the distance of $1-2y$. Finally both Hider and Searcher meet on half way, that is the half of $2y$. Consequently the capture time sums itself to $2y + 1 - 2y + \frac{2y}{2} = 1 + y$. This gives the third term. The next case deals with hiders that start in $x \in [1-2y, 1-y)$ with corresponding probability $\frac{1}{2}\int_{1-2y}^{1-y} \theta(t)dt$. Those hiders wait for the sweeper using $A$ similiar to the second case and get caught by the Searcher at time $t(x) = x - (1-2y)\ \forall\, y \in [0, \frac{1}{2}]$. This gives the fourth term. The last case is about hiders that start right to the point where Searcher and sweeper meet. They start there with probability $\frac{1}{2}(1 - \Theta(1-y))$. However, they get pushed into the arms of the Searcher by the sweeper with $A$ at time $y$. This gives the final term. The second, the fourth and the fifth case are the same in (17). The only difference of $M_2$ is that its user catches hiders that start to the right of him, faster. So in the first case he catches respective hiders after joing the sweeper using $A$ at time 2 instead of $2+2y$ and in the third case he catches respective hiders after joining the sweeper using $A$ at time 1 instead of $1+y$. $\square$

*Remark* 23. Note that $\pm M_1$ and $\pm M_2$ do equal against $\pm h_\Theta$ apart from the double occurence of the factor $(1+y)$ for $y \geq 0$. So against $\pm h_\Theta$, $\pm M_2$ dominates $\pm M_1$. The only possible eligibility that $\pm M_1$ has and that it is actually designed for, arises from its better payoff against $\pm E$ because of $y \leq 1$.

As both terms (16) and (17) are only dependent on $y$, it is now possibile to form a lower bound on $v$.

**Theorem 24.** $v \geq \frac{15}{11}$.

*Proof.* To calculate concrete values for a matrix game, let the distribution be once more the uniform distribution like in the last section. So let $\theta(x) = \begin{cases} 1, & \text{for } 0 \leq x \leq 1 \\ 0, & \text{else} \end{cases}$ and

$\Theta(x) = \begin{cases} x, & \text{for } 0 \leq x \leq 1 \\ 0 & \text{for } x < 0 \\ 1 & \text{for } x > 1 \end{cases}$. Inserting that into (16) and (17) give the expected payoffs

$$t_1(y) = \begin{cases} -\frac{1}{4}y^2 + \frac{1}{2}y + \frac{3}{2}, & \text{for } 0 \leq y < \frac{1}{2} \\ -\frac{5}{4}y^2 + 2, & \text{for } \frac{1}{2} \leq y \leq 1 \end{cases} \text{ and } t_2(y) = \begin{cases} \frac{3}{4}y^2 - y + \frac{3}{2}, & \text{for } 0 \leq y < \frac{1}{2} \\ \frac{3}{4}y^2 - 2y + 2, & \text{for } \frac{1}{2} \leq y \leq 1 \end{cases}.$$

How they are put together can be found in Table 1. Since $\pm M_2$ dominates $\pm M_1$ against

| | | term 1 | term 2 | term 3 | term 4 | term 5 |
|---|---|---|---|---|---|---|
| $t_1$: | $0 \leq y < \frac{1}{2}$: | $1+y$ | $0$ | $\frac{1}{2}(1+y)(1-2y)$ | $\frac{1}{2}\int_{1-2y}^{1-y}(t-1+2y)dt = \frac{1}{4}y^2$ | $\frac{1}{2}y^2$ |
| | $\frac{1}{2} \leq y \leq 1$: | $(1+y)(2-2y)$ | $\frac{1}{2}\int_0^{2y-1}(-t+2y-1)dt = y^2-y+\frac{1}{4}$ | $0$ | $\frac{1}{2}\int_0^{1-y}(t-1+2y)dt = -\frac{3}{4}y^2+y-\frac{1}{4}$ | $\frac{1}{2}y^2$ |
| $t_2$: | $0 \leq y < \frac{1}{2}$: | $1$ | $0$ | $\frac{1}{2}(1-2y)$ | $\frac{1}{2}\int_{1-2y}^{1-y}(t-1+2y)dt = \frac{1}{4}y^2$ | $\frac{1}{2}y^2$ |
| | $\frac{1}{2} \leq y \leq 1$: | $2-2y$ | $\frac{1}{2}\int_0^{2y-1}(-t+2y-1)dt = y^2-y+\frac{1}{4}$ | $0$ | $\frac{1}{2}\int_0^{1-y}(t-1+2y)dt = -\frac{3}{4}y^2+y-\frac{1}{4}$ | $\frac{1}{2}y^2$ |

Table 1: The individual payoffs for $t_1$ and $t_2$ whereas each term belongs to the corresponding term of (16), respectively (17). Adding those terms yields $t_1$ and $t_2$.

$\pm h_\Theta$, we pretend that $\pm M_2$ dominates $\pm M_1$ against $\mu$ as well, as if the better payoff of $\pm M_1$ against $\pm E$ does not justify the use of $\pm M_1$ at all. So the Monster chooses strategies only in $\{\pm A, \pm M_2\}$. Because the payoff of $\pm M_2$ against $\pm E$ is independent of $y$, it is sufficient that the Monster picks a $y$ with $0 \leq y \leq 1$ such that $t_2(y)$ is minimal. This happens for $y = 1$ and the value $t_2(1) = \frac{3}{4}$ is put into the game matrix $\begin{array}{c|c} & \begin{array}{cc} \pm E & \pm h_\Theta \end{array} \\ \begin{array}{c} \pm A \\ \pm M_2 \end{array} & \begin{pmatrix} 1 & \frac{3}{2} \\ 3 & \frac{3}{4} \end{pmatrix} \end{array}$. Solving the matrix game with Matlab yields the value $\frac{15}{11}$ whereas the Princess uses the strategy $\pm E$ with probability $\frac{3}{11}$ and all strategies in $\pm h_\Theta$ together with probability $\frac{8}{11}$. To show that $\frac{15}{11}$ is a lower bound on $v$, it remains to show that the Searcher can't improve by including the strategy $\pm M_1$. And this is true, because if the Hider keeps using his strategy with the above specified probabilities, then $\frac{3}{11} \cdot (1+2y) + \frac{8}{11}t_1(y)$ is minimal for $y = 0$ and $y = 1$ exactly with value $\frac{15}{11}$. So $\frac{3}{11} \cdot (1+2y) + \frac{8}{11} \cdot t_1(y) \geq \frac{15}{11}\ \forall\, y \in [0,1]$ yields.

□

To make further statements about the lower bound of $v$, consider this time another approach which is dealt with in detail in the next section.

# 4 Numerical approach of the interval game

Let's restrict the 'Princess and Monster' game on an interval and only permit the strategies used in the last section. The value of this restricted game is denoted by $v_r$ and has already been estimated for the case that the Princess chooses every starting point in $(-1,1)$ with same probability. In contrast to section 3.2 where only the strategy of the Monster could have been specified directly to give a bound on $v$, the strategies of both players as well as their corresponding payoffs of section 3.3 are known. This enables us to re-pick up and realize the thoughts in the proof of Proposition 8: A discretization of the interval game might sharpen the lower bound on $v$, since the value of a discretized game would converge to $v_r \leq v$ for an increasing number of grid points due to the same reasons given in the proof of Proposition 8. In particular, the discretization of this section looks like this: The interval splits into $2n$ equidistant grid points with $n \in \mathbb{N}$ whereas the first and the last grid point represent the end points $\pm 1$ and the middle of the interval, respectively 0 gets only approximated by the two surrounding grid points. The corresponding mesh size of this discretization is therefore $\triangle = \frac{2}{2n-1}$. Now the Monster and the Princess behave as before, but they have to choose a grid point as starting point and they can move in a time step equal to $\triangle$ from one grid point to another. Capture occurs when both players occupy the same grid point at the same time. Hence, if the players start at the same grid point for example, the game is immediately over. In order to solve this approximation of the restricted game, we can compute a capture time due to the finite number of grid points for every strategy and every starting point of the player and put it then into a game matrix $G = (g_{i,j})_{i=1,\dots,8n, j=1,\dots,2n}$. Denote the value of this matrix game with $v_n$ and let $k, l \in [2n]$ denote the grid points where the Monster, respectively the Princess starts in at which $k = 1$ or $l = 1$ relates to the end point $-1$, $k = 2$ or $l = 2$ relates to $-1 + \triangle$, etc. Then the game matrix yields

$$
\begin{array}{cc}
\begin{array}{ccc}
 & & \\
i=1 & \rightarrow & -A \\
 & \vdots & \\
i=k & \rightarrow & M_1 \\
 & \vdots & \\
i=2n & \rightarrow & A \\
i=1+2n & \rightarrow & -A \\
 & \vdots & \\
i=k+2n & \rightarrow & -M_1 \\
 & \vdots & \\
i=4n & \rightarrow & A \\
i=1+4n & \rightarrow & -A \\
 & \vdots & \\
i=k+4n & \rightarrow & M_2 \\
 & \vdots & \\
i=6n & \rightarrow & A \\
i=1+6n & \rightarrow & -A \\
 & \vdots & \\
i=k+6n & \rightarrow & -M_2 \\
 & \vdots & \\
i=8n & \rightarrow & A
\end{array}
&
\begin{array}{c}
\begin{array}{ccccccc}
\begin{smallmatrix} j=1 \\ \downarrow \\ -E \end{smallmatrix} & \cdots & \begin{smallmatrix} j=l\leq n \\ \downarrow \\ -h_\ominus \end{smallmatrix} & \cdots & \begin{smallmatrix} j=l>n \\ \downarrow \\ h_\ominus \end{smallmatrix} & \cdots & \begin{smallmatrix} j=2n \\ \downarrow \\ E \end{smallmatrix}
\end{array}
\\
\begin{pmatrix}
0 & \cdots & 1 & \cdots & 2 & \cdots & 2 \\
\vdots & \ddots & \vdots & & \vdots & & \vdots \\
4-(k-1)\triangle & \cdots & = \begin{cases} 4-(k-1)\triangle, & l<k \\ (l-k)\triangle, & \text{else} \end{cases} & \cdots & = \begin{cases} (n-1+\lceil \frac{2n-k}{2} \rceil)\triangle, & l<k \\ (2n-l+\lceil \frac{2l-2n-k}{2} \rceil)\triangle, & l>n+\frac{k}{2} \\ (l-k)\triangle, & \text{else} \end{cases} & \cdots & 2-(k-1)\triangle \\
\vdots & & \vdots & & \vdots & \ddots & \vdots \\
2 & \cdots & 2 & \cdots & 1 & \cdots & 0 \\
0 & \cdots & 1 & \cdots & 2 & \cdots & 2 \\
\vdots & \ddots & \vdots & & \vdots & & \vdots \\
(k-1)\triangle & \cdots & g_{2n+1-k,2n+1-l} & \cdots & g_{2n+1-k,2n+1-l} & \cdots & 2+(k-1)\triangle \\
\vdots & & \vdots & & \vdots & \ddots & \vdots \\
2 & \cdots & 2 & \cdots & 1 & \cdots & 0 \\
0 & \cdots & 1 & \cdots & 2 & \cdots & 2 \\
\vdots & \ddots & \vdots & & \vdots & & \vdots \\
2 & \cdots & = \begin{cases} 2, & l<k \\ (l-k)\triangle, & \text{else} \end{cases} & \cdots & = \begin{cases} (n-1)\triangle, & l<k \\ (2n-l+\lceil \frac{2l-2n-k}{2} \rceil)\triangle, & l>n+\frac{k}{2} \\ (l-k)\triangle, & \text{else} \end{cases} & \cdots & 4 \\
\vdots & & \vdots & & \vdots & \ddots & \vdots \\
2 & \cdots & 2 & \cdots & 1 & \cdots & 0 \\
0 & \cdots & 1 & \cdots & 2 & \cdots & 2 \\
\vdots & \ddots & \vdots & & \vdots & & \vdots \\
4 & \cdots & g_{6n+1-k,2n+1-l} & \cdots & g_{6n+1-k,2n+1-l} & \cdots & 2 \\
\vdots & & \vdots & & \vdots & \ddots & \vdots \\
2 & \cdots & 2 & \cdots & 1 & \cdots & 0
\end{pmatrix}
\end{array}
\end{array}
\tag{18}
$$

*Remark* 25. The computation of the capture times of $\pm h_\Theta$ against $M_1$ and $M_2$ and their associated distinction of cases in (18) relates to the computations and distinction of cases in the proof of Lemma 22. Then those payoffs also offer the capture times of $\pm h_\Theta$ against $-M_1$ and $-M_2$ which can be computed because of symmetry by a clever shifting of indices. The only difference is that due to the rules of this discretized game the starting of both players and the capture takes place on grid points which has thereby to be considered when computing capture times of this discretized game in general. This leads for example to terms with ceiling functions to ensure that if the Monster and the Princess locate on adjacent grid points at the same time and then run toward to each other, they don't meet in the mid of those grid points. Since this sort of modeling means a disadvantage for a Monster starting in a grid point with odd index $k$, a resultant 'zig-zag-distribution' of starting points can be avoided by omitting these ceiling functions.

Solving this matrix game yields slight different values and different optimal distributions of strategies of both players than in [2]. The values of the matrix game can be found in Table 2 while the Princess and the Monster use their strategies in this following manner: Both utilize two types of discrete strategies at a time and one continuous strategy at a time. For the Princess this means that she either hides at each end point with probability $\approx 0.127$, she uses $\pm h_\Theta$ with starting point $\pm\varepsilon$ with total probability $\approx 0.236$, or she uses $\pm h_\Theta$ when starting in the rest of the interval according to a initial continuous distribution described by the density function in Fig. 3 on the left side. In contrast the Monster either uses the sweeper strategy with total probability $\approx 0.814$, strategy $\pm M_1$ when starting near the end in $\pm(1-\varepsilon)$ with probability $\approx 0.059$ at each point, or strategy $\pm M_2$ when starting in the rest of the interval according to a initial continuous distribution described by the density function in Fig. 3 on the right side. Note that the Monster uses $M_2$ only when starting in the negative half of the interval and $-M_2$ only when starting in the positive half of the interval. In fact it additionally attracts attention that for a not large enough $n$ the Princess as well as the Monster very rarely use further discrete strategies near the end points and near the middle as a consequence to numerical side-effects, but they disappear with an increasing number of grid points.

| $n$ | $v_n$ |
|---|---|
| 1 | 1 |
| 2 | 1.2667 |
| 4 | 1.3303 |
| 8 | 1.3547 |
| 16 | 1.3647 |
| 32 | 1.3689 |
| 64 | 1.3709 |
| 128 | 1.3719 |
| 256 | 1.3724 |
| 512 | 1.3726 |
| 1024 | 1.3727 |

Table 2: The values of the matrix game which approximates the restricted game with $v_r$ dependent on $n$

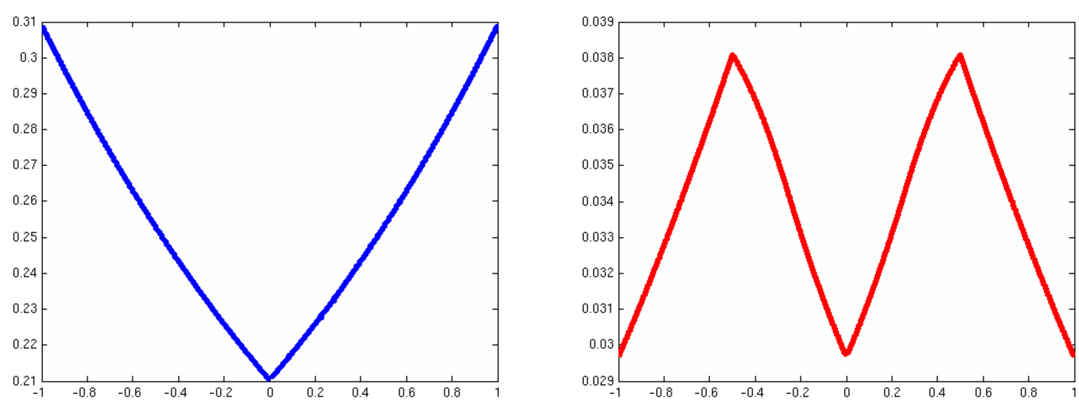


Figure 3: Left: Probability density of the continuous Princess strategy; Right: Probability density of the continuous Monster strategy

# 5 Conclusion

Summing up it can be said that the solution of this game is still unknown, but at least there exist good estimates of $v$ as well as some helpful evidence about properties of optimal strategies. The latter can be used to test strategies on optimality as it has already been demonstrated in this work. Due to the mixed strategies of both players being very complex – and this can be seen once again in the evaluation of the restricted game of the last section – it is conjectured that this game won't ever get solved analytically after all. But since the mixture of the strategies $\{\pm E, \pm h_\Theta\}$ might seemingly be optimal for the Princess, the value $v_r$ could be the value of the game. Unfortunately the $(\varepsilon-)$optimality of $\mu$ could not have been established yet. Otherwise the results of section 4 would then at least approximatively solve the game. Here the Monster uses as expected his sweeper strategy the most which forces the Princess to use $\pm E$ not too often. Moreover the strategy $\pm M_1$ gets almost completely dominated by $\pm M_2$ as indicated in section 3.3: Going for one lonely Hider at the nearest end isn't profitable for the Monster. It only uses $\pm M_1$ when starting in $\pm(1-\varepsilon)$ because there is no Princess with inner strategy $\pm h_\Theta$ left to catch at time $y$ and therefore its better payoff against the end point strategies $\pm E$ let $\pm M_1$ become preferred in this special case. The relative high probability that goes along with this discrete strategy is a result of the fact that the Monster starts near one end here and – as said above – is able to catch therefore all hiders using $\pm h_\Theta$ until meeting the sweeper. Such hiders with $\pm h_\Theta$ do best in average against this if starting in the middle and this maybe explains the relative common appearence of the Princess in the mid, although the density in Fig. 3 otherwise suggests a minimum there. However note that the Monster doesn't often start in the rest of the interval, but if still doing so, it prefers $y \geq \frac{1}{2}$ such that it collects enough immobile hiders using strategy $\pm h_\Theta$. Thereby the Monster apparently starts the most time in $\pm\frac{1}{2}$ such that $y = \frac{3}{4}$. The reasons of that maximum are not obvious, but choosing $y = \frac{3}{4}$ maybe presents an equilibrium of two aims of the Monster, that is catching a lot of hiders with one move and catching all hiders as fast as possible. Hence it is also not bad for the Princess to start behind the Monster and this becomes the more likely the more she is near an end. This explains the density in Fig. 3. All in all, the results of section 4 seem reasonable. And even if $\mu$ was not quite optimal, I have gained the intuition that it would not be far away from that.

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