World Academy of Science, Engineering and Technology
International Journal of Mathematical and Computational Sciences
Vol:4, No:1, 2010

A neighborhood condition for fractional k-deleted graphs

Sizhong Zhou, Hongxia Liu

Abstract—Let $k \geq 3$ be an integer, and let G be a graph of order n with $n \geq 9k+3-4\sqrt{2(k-1)^2+2}$. Then a spanning subgraph F of G is called a k-factor if $d_F(x)=k$ for each $x \in V(G)$. A fractional k-factor is a way of assigning weights to the edges of a graph G (with all weights between 0 and 1) such that for each vertex the sum of the weights of the edges incident with that vertex is k. A graph G is a fractional k-deleted graph if there exists a fractional k-factor after deleting any edge of G. In this paper, it is proved that G is a fractional k-deleted graph if G satisfies $\delta(G) \geq k+1$ and $|N_G(x) \cup N_G(y)| \geq \frac{1}{2}(n+k-2)$ for each pair of nonadjacent vertices x, y of G.

Keywords—graph, minimum degree, neighborhood union, fractional k-factor, fractional k-deleted graph.

I. Introduction

In this paper, we consider only finite undirected graphs without loops or multiple edges. Let G be a graph. We use $V(G)$ and $E(G)$ to denote its vertex set and edge set, respectively. For $x \in V(G)$, we denote by $d_G(x)$ the degree of x in G and by $N_G(x)$ the set of vertices adjacent to x in G, and $N_G[x]$ for $N_G(x) \cup \{x\}$. For any $S \subseteq V(G)$, $N_G(S) = \cup_{x\in S} N_G(x)$ and we denote by $G[S]$ the subgraph of G induced by S, and $G-S = G[V(G) \setminus S]$. We say that S is independent if $N_G(S) \cap S = \emptyset$. Let S and T be disjoint subsets of $V(G)$. We use $e_G(S,T)$ to denote the number of edges joining S and T in G. The minimum vertex degree of G is denoted by $\delta(G)$.

Let k be a positive integer. Then a spanning subgraph F of G is called a k-factor if $d_F(x)=k$ for each $x \in V(G)$. If $k=1$, then a k-factor is simply called a 1-factor. A fractional k-factor is a way of assigning weights to the edges of a graph G (with all weights between 0 and 1) such that for each vertex the sum of the weights of the edges incident with that vertex is k. If $k=1$, then a fractional k-factor is a fractional 1-factor. A graph G is a fractional k-deleted graph if there exists a fractional k-factor after deleting any edge of G. If $k=1$, then a fractional k-deleted graph is a fractional 1-deleted graph. Some other terminologies and notations can be found in [1,2].

Many authors have studied graph factors [3-8]. Many authors have investigated fractional k-factors [9–12] and fractional k-deleted graphs [13,14]. The following results on k-factors, fractional k-factors and fractional k-deleted graphs are known.

Sizhong Zhou is with the School of Mathematics and Physics, Jiangsu University of Science and Technology, Mengxi Road 2, Zhenjiang, Jiangsu 212003, People's Republic of China, e-mail: zsz_cumt@163.com.

Hongxia Liu is with the School of Mathematics and Informational Science, Yantai University, Yantai, Shandong 264005, People's Republic of China, e-mail: mqy7174@sina.com

Manuscript received December 16, 2010.

Theorem 1[15] Let k be an integer such that $k \geq 2$, and let G be a connected graph of order n such that $n \geq 9k-1-4\sqrt{2(k-1)^2+2}$, kn is even, and the minimum degree is at least k. If G satisfies $|N_G(x) \cup N_G(y)| \geq \frac{1}{2}(n+k-2)$ for each pair of nonadjacent vertices $x, y \in V(G)$, then G has a k-factor.

Theorem 2[11] Let k be an integer such that $k \geq 2$, and let G be a connected graph of order n such that $n \geq 9k-1-4\sqrt{2(k-1)^2+2}$, and the minimum degree $\delta(G) \geq k$. If $|N_G(x) \cup N_G(y)| \geq \frac{1}{2}(n+k-2)$ for each pair of nonadjacent vertices $x, y \in V(G)$, then G has a fractional k-factor.

Theorem 3[16] Let $k \geq 2$ be an integer. Let G be a connected graph of order n with $n \geq 13k+1-4\sqrt{2(k-1)^2+2}$, $\delta(G) \geq k+2$. If $|N_G(x) \cup N_G(y)| \geq \frac{1}{2}(n+k-2)$ for each pair of nonadjacent vertices x, y of G, then G is a fractional k-deleted graph.

The purpose of this paper is to weaken the conditions on the order, minimum degree and connectivity of G in Theorem 3. The main result is the following theorem.

Theorem 4 Let $k \geq 3$ be an integer. Let G be a graph of order n with $n \geq 9k+3-4\sqrt{2(k-1)^2+2}$, $\delta(G) \geq k+1$. If

$$|N_G(x) \cup N_G(y)| \geq \frac{1}{2}(n+k-2)$$

for each pair of nonadjacent vertices x, y of G, then G is a fractional k-deleted graph.

II. The Proof of Theorem 4

The following result is essential to the proof of our main theorem.

Lemma 2.1[17] A graph G is a fractional k-deleted graph if and only if for any $S \subseteq V(G)$ and $T = \{x : x \in V(G) \setminus S, d_{G-S}(x) \leq k\}$

$$\delta_G(S,T) = k|S| + d_{G-S}(T) - k|T| \geq \varepsilon(S,T),$$

where $d_{G-S}(T) = \sum_{x\in T} d_{G-S}(x)$ and $\varepsilon(S,T)$ is defined as follows,

$$\varepsilon(S,T) = \begin{cases} 2, & if\ T\ is\ not\ independent, \\ 1, & if\ T\ is\ independent,\ and \\ & e_G(T, V(G) \setminus (S \cup T)) \geq 1, \\ 0, & otherwise. \end{cases}$$

Proof of Theorem 4. Let G be a graph satisfying the hypothesis of Theorem 4, we prove the theorem by contradiction.

Suppose that G is not a fractional k-deleted graph. Then by Lemma 2.1, there exists a subset S of $V(G)$ such that

$$\delta_G(S,T)=k|S|+d_{G-S}(T)-k|T|\leq \varepsilon(S,T)-1, \quad (1)$$

where $T=\{x: x\in V(G)\setminus S, d_{G-S}(x)\leq k\}$. Firstly, we prove the following claims.

Claim 1. $S\neq\emptyset$.

Proof. Note that $\varepsilon(S,T)\leq |T|$. If $S=\emptyset$, then by (1) we have

$$\begin{aligned}\varepsilon(S,T)-1 &\geq \delta_G(S,T)=k|S|+d_{G-S}(T)-k|T|\\ &= d_G(T)-k|T|\geq(\delta(G)-k)|T|\\ &\geq |T|\geq\varepsilon(S,T).\end{aligned}$$

It is a contradiction. This completes the proof of Claim 1.

Claim 2. $|T|\geq k+1$.

Proof. Assume that $|T|\leq k$. Then from (1) and $|S|+d_{G-S}(x)-k\geq d_G(x)-k\geq\delta(G)-k\geq 1$, we get

$$\begin{aligned}\varepsilon(S,T)-1 &\geq \delta_G(S,T)=k|S|+d_{G-S}(T)-k|T|\\ &\geq |T||S|+d_{G-S}(T)-k|T|\\ &= \sum_{x\in T}(|S|+d_{G-S}(x)-k)\\ &\geq |T|\geq\varepsilon(S,T).\end{aligned}$$

That is a contradiction. This completes the proof of Claim 2.

Claim 3. $|T|\geq|S|+1$.

Proof. Let $|T|\leq|S|$. Then by (1), we obtain

$$\varepsilon(S,T)-1\geq k|S|+d_{G-S}(T)-k|T|\geq d_{G-S}(T). \quad (2)$$

On the other hand, according to the definition of $\varepsilon(S,T)$, we have

$$d_{G-S}(T)\geq\varepsilon(S,T),$$

which contradicts (2). The proof of Claim 3 is complete.

Claim 4. $|S|\leq\frac{n-1}{2}$.

Proof. In terms of Claim 3 and $|S|+|T|\leq n$, we have

$$n\geq|S|+|T|\geq 2|S|+1,$$

that is,

$$|S|\leq\frac{n-1}{2}.$$

The proof of Claim 4 is complete.

In terms of Claim 2, $T\neq\emptyset$. Now we define

$$h_1=\min\{d_{G-S}(x): x\in T\}$$

and choose $x_1\in T$ such that $d_{G-S}(x_1)=h_1$. Clearly, we have $0\leq h_1\leq k$. In the following, we consider two cases.

Case 1. $T=N_T[x_1]$.

Using Claim 2, $T=N_T[x_1]$ and $0\leq h_1\leq k$, we obtain

$$k\geq h_1=d_{G-S}(x_1)\geq|T|-1\geq k,$$

which implies

$$h_1=k. \quad (3)$$

In terms of (3) and Claim 1, we get

$$\begin{aligned}\delta_G(S,T) &= k|S|+d_{G-S}(T)-k|T|\\ &\geq k|S|+h_1|T|-k|T|=k|S|\\ &\geq k>2\geq\varepsilon(S,T).\end{aligned}$$

That contradicts (1).

Case 2. $T\setminus N_T[x_1]\neq\emptyset$.

Define

$$h_2=\min\{d_{G-S}(x): x\in T\setminus N_T[x_1]\}.$$

We choose $x_2\in T\setminus N_T[x_1]$ such that $d_{G-S}(x_2)=h_2$. Obviously, $0\leq h_1\leq h_2\leq k$ and $x_1x_2\notin E(G)$. According to the hypothesis of Theorem 4, we have

$$\begin{aligned}\frac{n+k-2}{2} &\leq |N_G(x_1)\cup N_G(x_2)|\\ &\leq d_{G-S}(x_1)+d_{G-S}(x_2)+|S|\\ &= h_1+h_2+|S|,\end{aligned}$$

which implies

$$|S|\geq\frac{n+k-2}{2}-h_1-h_2. \quad (4)$$

By (4) and Claim 4, we obtain

$$\frac{n-1}{2}\geq\frac{n+k-2}{2}-h_1-h_2,$$

that is,

$$h_1+h_2\geq\frac{k-1}{2}. \quad (5)$$

In terms of (5), $k\geq 3$, $0\leq h_1\leq h_2\leq k$ and the integrity of h_2, we get

$$h_2\geq 1. \quad (6)$$

Claim 5. $0\leq h_1\leq k-1$.

Proof. If $h_1=k$, then by (1) and Claim 1 we get

$$\begin{aligned}\varepsilon(S,T)-1 &\geq \delta_G(S,T)=k|S|+d_{G-S}(T)-k|T|\\ &\geq k|S|+h_1|T|-k|T|=k|S|\geq k\\ &> 2\geq\varepsilon(S,T),\end{aligned}$$

which is a contradiction. This completes the proof of Claim 5.

Note that

$$|N_T[x_1]|\leq d_{G-S}(x_1)+1=h_1+1. \quad (7)$$

From (4), (7), $0\leq h_1\leq h_2\leq k$ and $|S|+|T|\leq n$, we have

$$\begin{aligned}\delta_G(S,T) &= k|S|+d_{G-S}(T)-k|T|\\ &\geq k|S|+h_1|N_T[x_1]|+h_2(|T|-|N_T[x_1]|)\\ &\quad -k|T|\\ &= k|S|-(h_2-h_1)|N_T[x_1]|-(k-h_2)|T|\\ &\geq k|S|-(h_2-h_1)(h_1+1)\\ &\quad -(k-h_2)(n-|S|)\\ &= (2k-h_2)|S|-(h_2-h_1)(h_1+1)\\ &\quad -(k-h_2)n\\ &\geq (2k-h_2)(\frac{n+k-2}{2}-h_1-h_2)\\ &\quad -(h_2-h_1)(h_1+1)-(k-h_2)n,\end{aligned}$$

that is,

$$\delta_G(S,T) \geq (2k-h_2)(\frac{n+k-2}{2}-h_1-h_2) -(h_2-h_1)(h_1+1)-(k-h_2)n. \quad (8)$$

Let $F(h_1,h_2) = (2k-h_2)(\frac{n+k-2}{2}-h_1-h_2)-(h_2-h_1)(h_1+1)-(k-h_2)n$. Then by Claim 5, we have

$$\begin{aligned} F'_{h_1}(h_1,h_2) &= -(2k-h_2)+(h_1+1)-(h_2-h_1) \\ &= 2h_1-2k+1 \leq 2(k-1)-2k+1 \\ &= -1 < 0. \end{aligned}$$

Combining this with $h_1 \leq h_2$, we obtain

$$F(h_1,h_2) \geq F(h_2,h_2). \quad (9)$$

Using (8) and (9), we get

$$\delta_G(S,T) \geq (2k-h_2)(\frac{n+k-2}{2}-2h_2)-(k-h_2)n. \quad (10)$$

According to (1), (10) and $\varepsilon(S,T) \leq 2$, we get

$$\begin{aligned} 1 &\geq \varepsilon(S,T)-1 \geq \delta_G(S,T) \\ &\geq (2k-h_2)(\frac{n+k-2}{2}-2h_2)-(k-h_2)n \\ &= \frac{1}{2}(4h_2^2+(n-9k+2)h_2+2k^2-4k), \end{aligned}$$

which implies

$$4h_2^2+(n-9k+2)h_2+2k^2-4k-2 \leq 0. \quad (11)$$

Claim 6. For $k \geq 3$, we have $\sqrt{\frac{(k-1)^2+1}{2}}-1 > \frac{1}{2}$.

Proof. Since $k \geq 3$, we have

$$\frac{(k-1)^2+1}{2} \geq \frac{5}{2} > \frac{9}{4},$$

that is,

$$\sqrt{\frac{(k-1)^2+1}{2}} > \frac{3}{2}.$$

Thus, we obtain

$$\sqrt{\frac{(k-1)^2+1}{2}}-1 > \frac{1}{2}.$$

The proof of Claim 6 is complete.

According to (6), (11), $n \geq 9k+3-4\sqrt{2(k-1)^2+2}$, $k \geq 3$ and Claim 6, we obtain

$$\begin{aligned} 0 &\geq 4h_2^2+(n-9k+2)h_2+2k^2-4k-2 \\ &\geq 4h_2^2+(-4\sqrt{2(k-1)^2+2}+5)h_2+2k^2-4k-2 \\ &\geq 4h_2^2-8\sqrt{\frac{(k-1)^2+1}{2}}h_2+2(k-1)^2+2+5h_2-6 \\ &= 4(\sqrt{\frac{(k-1)^2+1}{2}}-h_2)^2+5h_2-6 \\ &\geq 4(\sqrt{\frac{(k-1)^2+1}{2}}-1)^2-1 \\ &> 4(\frac{1}{2})^2-1 \geq 0, \end{aligned}$$

which is a contradiction.

From all the cases above, we deduce the contradictions. Hence, G is a fractional k-deleted graph. This completes the proof of Theorem 4.

Acknowledgment

This research was supported by Natural Science Foundation of the Higher Education Institutions of Jiangsu Province (10KJB110003) and Jiangsu University of Science and Technology (2009SL148J, 2009SL154J) and Shandong Province Higher Educational Science and Technology Program(J10LA14), and was sponsored by Qing Lan Project of Jiangsu Province.

References

[1] J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications*, London, The Macmillan Press, 1976.
[2] R. Schirerman Edward and H. Ullman Daniel, *Fractional Graph Theory*, John Wiley and Son. Inc. New York, 1997.
[3] J. R. Correa and M. Matamala, *Some remarks about factors of graphs*, Journal of Graph Theory 57(2008), 265–274.
[4] H. Matsuda, *Fan-type results for the existence of $[a,b]$-factors*, Discrete Mathematics 306(2006), 688–693.
[5] S. Zhou, *Independence number, connectivity and (a,b,k)-critical graphs*, Discrete Mathematics 309(12)(2009), 4144–4148.
[6] S. Zhou, *A sufficient condition for a graph to be an (a,b,k)-critical graph*, International Journal of Computer Mathematics 87(10)(2010), 2202–2211.
[7] S. Zhou, *Remarks on (a,b,k)-critical graphs*, Journal of Combinatorial Mathematics and Combinatorial Computing 73(2010), 85–94.
[8] H. Liu and G. Liu, *Binding number and minimum degree for the existence of (g,f,n)-critical graphs*, Journal of Applied Mathematics and Computing 29(1-2)(2009), 207–216.
[9] G. Liu and L. Zhang, *Toughness and the existence of fractional k-factors of graphs*, Discrete Mathematics 308(2008), 1741–1748.
[10] J. Yu and G. Liu, *Fractional k-factors of graphs*, Chinese Journal of Engineering Mathematics 22(2)(2005), 377–380.
[11] S. Zhou and H. Liu, *Neighborhood conditions and fractional k-factors*, Bulletin of the Malaysian Mathematical Sciences Society 32(1)(2009), 37–45.
[12] S. Zhou, *Some results on fractional k-factors*, Indian Journal of Pure and Applied Mathematics 40(2)(2009), 113–121.
[13] S. Zhou, *A result on fractional k-deleted graphs*, Mathematica Scandinavica 106(1)(2010), 99–106.
[14] K. Kotani, *Binding numbers of fractional k-deleted graphs*, Proceedings of the Japan Academy, Ser. A, Mathematical Sciences 86(2010), 85–88.
[15] T. Iida and T. Nishimura, *Neighborhood conditions and k-factors*, Tokyo Journal of Mathematics 20(2)(1997), 411–418.
[16] S. Zhou, *A neighborhood condition for graphs to be fractional (k,m)-deleted graphs*, Glasgow Mathematical Journal 52(1)(2010), 33–40.
[17] Z. Li, G. Yan and X. Zhang, *On fractional (g,f)-deleted graphs*, Mathematica Applicata (Wuhan) 16(1)(2003), 148–154.