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Knot traces and the slice genus

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Knot traces and the slice genus

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DISSERTATION

Presented to the Faculty of the Graduate School of

The University of Texas at Austin

in Partial Fulfillment

of the Requirements

for the Degree of

DOCTOR OF PHILOSOPHY

THE UNIVERSITY OF TEXAS AT AUSTIN

May 2019

Acknowledgments

I cannot imagine a more perfect environment to complete my PhD than the topology group at the UT Department of Mathematics. I would like to thank everyone who has contributed to the development of this community, of which it has been an honor to be a member.

In particular, this thesis and person grew from many insightful conversations with Allison N. Miller. It has been a privilege and a delight to talk with and learn from Cameron Gordon. And I am endlessly grateful to my adviser John Luecke, from whom I have learned how to ask questions, how to absorb an argument into the fiber of my being, how to read, speak and write carefully, and even some manifold topology along the way.

Knot traces and the slice genus

Publication No. ____________________

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The University of Texas at Austin, 2019

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Knot traces are elementary 4-manifolds built by attaching a single 2-handle to the 4-ball; these are the canonical examples 4-manifolds with non-trivial middle dimensional homology. In this thesis, we give a flexible technique for constructing pairs of distinct knots with diffeomorphic traces. Using this construction, we show that there are knot traces where the minimal genus smooth surface generating homology is not the canonical surface, resolving a question on the 1978 Kirby problem list. We also use knot traces to give a new technique for showing a knot does not bound a smooth disk in the 4-ball, and we show that the Conway knot does not bound a smooth disk in the 4-ball. This resolves a question from the 1960s, completes the classification of slice knots under 13 crossings, and gives the first example of a non-slice knot which is both topologically slice and a positive mutant of a slice knot.

Table of Contents

Acknowledgments **iv**

Abstract **v**

List of Tables **vii**

List of Figures **viii**

Chapter 1. Introduction **1**

Chapter 2. Constructing knots with diffeomorphic traces **8**

Chapter 3. Shake genus **14**

Chapter 4. The Conway knot **18**

Bibliography **23**

List of Tables

3.1 17

4.1 20

List of Figures

1.1 Positive mutation from the Conway knot C to the Kinoshita-Teresaka knot 6

2.1 Links L and L' 9

2.2 Banding near a crossing 9

2.3 A handle diagram for X_m and diffeomorphism to $X_0(K_m)$ . . . 10

2.4 The same handle diagram for X_m and diffeomorphism to $X_0(K'_m)$ 11

2.5 Non-slice K and K' can arise when the 'split' hypothesis is omitted 12

2.6 Constructing a link L associated to an unknotting number 1 knot K. Here D denotes a (blue) diagram of K with a positive unknotting crossing c, and $w(D)$ denotes the writhe of D. . . . 13

3.1 A somewhat reduced diagram of K'_0 16

4.1 The knot K' shares a trace with the Conway knot 19

4.2 Handle calculus exhibiting a diffeomorphism from $X(C)$ to $X(K')$ where K' is the knot defined in Figure 4.1. Handle slides are denoted with arrows, the transition from (L) to (M) includes canceling a 1-2 pair, and all other changes are isotopies. 21

Chapter 1

Introduction

The classical study of knots in S^3 is 3-dimensional; a knot is defined to be trivial if it bounds an embedded disk in S^3. Concordance, first defined by Fox in [8], is a 4-dimensional extension; a knot in S^3 is trivial in concordance if it bounds a smooth embedded disk in B^4, and a pair of knots $K \in S^3 \times \{0\}$ and $K' \in S^3 \times \{1\}$ are said to be concordant if they cobound a smooth embedded annulus in $S^3 \times I$.

Knot concordance is intimately tied to the study of 4-manifolds. Knot concordances are widely used in constructions of interesting 4-manifolds, and the absence of concordances is often leveraged as 4-manifold invariant. 4-manifolds also manifestly inform the study of knot concordance; the literature abounds with work using associated 4-manifolds to demonstrate that a pair of knots are not concordant, and the existence of particular 4-manifolds can be used to build concordances, especially in the topological category.

This thesis concerns the interplay between a particular facet of knot concordance, the slice genus, and a particular class of 4-manifolds, knot traces. The slice genus of a knot, denoted $g_4(K)$, is the minimum genus of any smooth surface properly embedded in B^4 with boundary K. Observe that a knot is slice if and only if it has slice genus zero, and that concordant knots have the same slice genus. A knot trace $X(K)$ is a four manifold obtained by attaching a 0-framed 2-handle to B^4 with attaching sphere K. The following observation about the relationship between knot traces and sliceness is folklore, for an early use see [14].

Lemma 1.0.1. *$g_4(K) = 0$ if and only if $X(K)$ smoothly embeds in S^4.*

Proof. For the 'only if' direction: Consider S^4 and a smooth S^3 therein which decomposes S^4 into the union of two 4-balls B_1 and B_2. Consider K sitting in this S^3. Since K is slice, we can find a smoothly embedded disk D_K which K bounds in B_1. Observe now that $B_2 \cup \overline{\nu(D_K)} \cong X(K)$ is smoothly embedded in S^4.

For the 'if' direction: Let Z denote the handle cobordism from S^3 to $\partial(X(K))$ given by attaching a 0-framed 2-handle to $S^3 \times I$ along K, and observe that $K \in S^3$ bounds a smooth disk in Z, namely the core of the 2-handle. Further, there is a natural smooth embedding $F : B^4 \to X(K)$ such that $\overline{X(K) \setminus F(B^4)} \cong Z$. Let $i : X(K) \to S^4$ be a smooth embedding. Since $(i \circ F)$ is a smooth embedding of B^4 in S^4, we have we have $W := \overline{S^4 \setminus i \circ F(B^4))} \cong B^4$. We also get a natural smooth embedding $Z \in W$, in which we have seen K bounds a smoothly embedded disk. □

This lemma has a natural corollary; if knots K and K' have diffeomorphic traces and K is slice, then K' is also slice. It is natural to ask whether this extends to higher slice genus; if K and K' have diffeomorphic traces, does $g_4(K) = g_4(K')$? Our first main result is to answer this is the negative.

Theorem 1.0.2. *There exist infinitely many pairs of knots K and K' such that $X(K)$ is diffeomorphic to $X(K')$ and $g_4(K) \neq g_4(K')$.*

Theorem 1.0.2 has a notable corollary, which we discuss now.

One of the key differences between smooth 4-manifolds and higher dimensional smooth manifolds is the ability to represent any middle-dimensional homology class of a simply connected manifold with a smoothly embedded sphere. For 4-manifolds this is not always possible; indeed not even among knot traces which are the simplest 4-manifolds with non-trivial H_2. The shake genus of K, denoted $g_{sh}(K)$, measures this failure to find a sphere representative by recording the minimal genus among smooth embedded generators of the second homology of $X(K)$.

There is a natural relationship between the shake genus and the slice genus. Let $\Sigma \hookrightarrow B^4$ a smooth properly embedded surface with boundary K. When we attach a 2-handle to B^4 along K, Σ can be capped off to a closed surface $\widehat{\Sigma} \hookrightarrow X(K)$ of the same genus. So we see that for all knots K the shake genus is bounded above by the slice genus. However, since $\widehat{\Sigma}$ is embedded in a restrictive manner ($\widehat{\Sigma}$ intersects the cocore of the handle in one point) one might expect that the shake genus can be strictly less than the slice genus; indeed this conjecture contitutes problem 1.41 of [2]. Since knots with diffeomorphic traces have the same shake genus by definition, we solve this as a corollary of Theorem 1.0.2.

Corollary 1.0.3. *There exist infinitely many knots K with $g_{sh}(K) < g_4(K)$.*

If one was hoping (perhaps motivated by Lemma 1.0.1) for a strong relationship between the 4-manifold trace of a knot and the concordance class of that knot, one might regard Theorem 1.0.2 as a disappointment; in particular Theorem 1.0.2 gives a strong proof that there are infinitely many pairs of knots with diffeomorphic traces which are not concordant. (The existence of knots with diffeomorphic traces which are not concordant was originally proven by the author and A.N. Miller in [18]). The second main result of this dissertation is the observation that the failure of Lemma 1.0.1 to extend to other concordance classes can provide a powerful new sliceness obstruction. We discuss the principle of our obstruction now, followed by an application.

There is plentiful literature constructing tools for obstructing the sliceness of a given knot, primarily through the computation of a concordance invariant that is known to "vanish" on the slice class. Concordance invariants have been derived from all fields of modern topology, and taken together as a suite they provide a robust tool for obstructing sliceness. Particular invariants are sometimes blinded by certain properties of a non-slice knot, but this is generally compensated for by some other invariant. However, it is possible that a knot may have many properties which together blind all known concordance invariants, and there are small examples of (presumably non-slice) knots for which all known concordance invariants vanish. Our new method can obstruct sliceness in these pathological cases.

Let K be a (presumably non-slice) knot, and let K' be some other knot with $X(K) \cong X(K')$. Then, by Lemma 1.0.1, K is slice if and only if K' is slice. If there were a strong relationship between the 4-manifold trace of a knot and the concordance class of that knot, then one would expect K' to be similar to K in concordance, and thus perhaps equally difficult to obstruct the sliceness of. But there is not, so it is reasonable to expect that known invariants may be able to detect the non-sliceness of K', thereby proving K is not slice.

Our second main result is an application of this outline; we begin by discussing two properties that each blind many concordance invariants and introducing a small knot which satisfies both.

A Conway sphere for an oriented knot K is an embedded S^2 in S^3 that meets the knot transversely in four points. The Conway sphere splits S^3 into two 3-balls, B_1 and B_2, and K into two tangles K_{B_1} and K_{B_2}. Any knot K^* obtained from K_{B_1} and K_{B_2} after regluing B_1 to B_2 via an involution of the Conway sphere is called a mutant of K. If K^* inherits a well-defined orientation from that of K_{B_1} and K_{B_2} then K^* is a positive mutant of K. The smallest pair of positive mutant knots, the 11 crossing Conway knot C and Kinoshita-Teresaka knot, were discovered by Conway in [7]; see Figure 1.1. We remark that the Kinoshita Teresaka knot is slice; we will be interested in obstructing the sliceness of the Conway knot.

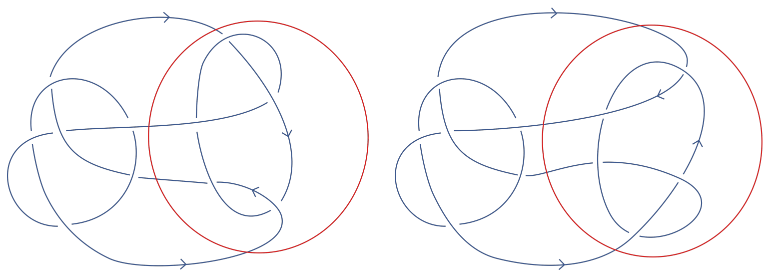

Figure 1.1: Positive mutation from the Conway knot C to the Kinoshita-Teresaka knot

Arising as a positive mutant of a slice knot is our first property that renders obstructing sliceness tricky; for such a knot all abelian and all but the subtlest metabelian sliceness obstructions vanish, Rasmussen's s-invariant is conjectured to vanish [5], and it is unknown whether any Heegaard Floer sliceness obstructions can detect such a knot. In 2001 Kirk and Livingston gave the first examples of non slice knots which are positive mutants of slice knots [15], and other examples have appeared since [13] [11] [17]. All of these works rely on careful analysis of metabelian sliceness obstructions.

Our second property that renders obstructing sliceness tricky is topological sliceness. A knot is topologically slice if it bounds a locally flat disk in B^4. All abelian and metabelian invariants vanish for topologically slice knots. It has been known since the early 1980's that there exist non-slice knots which are topologiclly slice; modern proofs use for example Heegaard Floer concordance invariants. By work of Freedman [9], the Conway knot is topologically slice.

Since its definition in 1969, the Conway knot has been a testing ground for all new concordance invariants, but it has remained open whether the Conway knot is slice. We resolve this as an application of our new method; this completes the classification of slice knots of under 13 crossings [6] and gives the first example of a non-slice knot which is both a positive mutant of a slice knot and topologically slice.

Theorem 1.0.4. *The Conway knot is not slice*

This dissertation is organized as follows: In chapter two, we give a construction for building pairs of knots with diffeomorphic traces, which is the primary constructive tool that powers this work. In chapter 3, we prove Theorem 1.0.2. In chapter 4, we prove Theorem 1.0.4.

Chapters 2 and 3 (with the exception of Lemma 2.0.4) are in the publication queue in Geometry & Topology, published by Mathematical Sciences Publishers, and are reproduced here with permission.

All manifolds, submanifolds, maps of manifolds and concordances are smooth throughout this work, all homology has integer coefficients and all knots and manifolds are taken to be oriented. We will use $\cong$ to denote diffeomorphic manifolds, $\simeq$ to denote isotopic links, and $\sim$ to denote concordant knots. All twist boxes in figures denote full twists, and positive twists are right handed. We will assume familiarity with handle calculus, for the details see [10].

Chapter 2

Constructing knots with diffeomorphic traces

We begin by constructing pairs of knots with diffeomorphic traces. Theorem 2.0.1 was motivated by an inside-out take on the well-known dualizable patterns construction.

Let L be a three component link with (blue, green, and red) components B, G, and R such that the following hold: the sublink $B \cup R$ is isotopic in S^3 to the link $B \cup \mu_B$ where μ_B denotes a meridian of B, the sublink $G \cup R$ is isotopic to the link $G \cup \mu_G$, and $\mathrm{lk}(B, G) = 0$. From L we can define an associated 4-manifold X by thinking of R as a 1-handle, in dotted circle notation, and B and G as attaching spheres of 0-framed 2-handles. See Figure 2.3 for an example of such a handle description. In a moment we will also define a pair of knots K and K' associated to L.

Theorem 2.0.1. *$X \cong X_0(K) \cong X_0(K')$.*

Proof. Isotope L to a diagram in which R has no self crossings (hence such that R bounds a disk D_R in the diagram) and in which $B \cap D_R$ is a single point. Slide G over B as needed to remove the intersections of G with D_R. After the slides we can cancel the 2-handle with attaching circle B with the

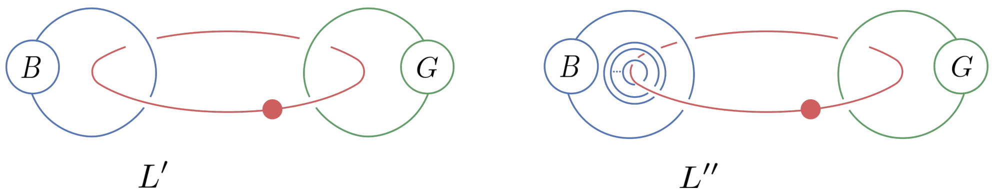

Figure 2.1: Links L and L'

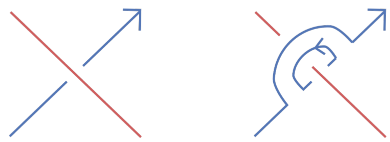

Figure 2.2: Banding near a crossing

1-handle and we are left with a handle description for a 0-framed knot trace; this knot is K'.

To construct K and see $X \cong X_0(K)$, perform the above again with the roles of B and G reversed. □

Remark 2.0.1. By modifying the framing hypotheses in Theorem 2.0.1 this technique can be easily modified to produce knots J and J' with $X_n(J) \cong X_n(J')$ for any integer n.

For a link L in S^3, define $-L$ to be the mirror of L with its orientation reversed. Two n-component links L_0 and L_1 are said to be *strongly concordant* if they cobound a smoothly embedded surface Σ in $S^3 \times [0,1]$ such that Σ is a disjoint union of n annuli and $\Sigma \cap (S^3 \times \{0\}) = -L_0$ and $\Sigma \cap (S^3 \times \{1\}) = L_1$. When $n = 1$ we omit the word strongly.

Theorem 2.0.2. *Let X be a 4-manifold with a handle description $L := R \cup$*

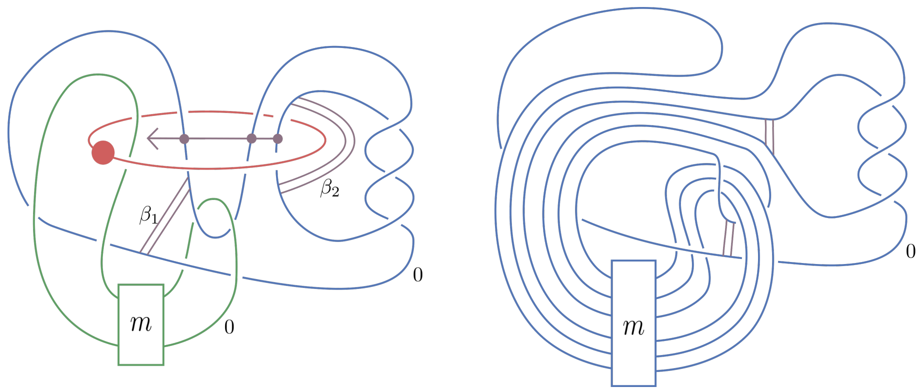

Figure 2.3: A handle diagram for X_m and diffeomorphism to $X_0(K_m)$

$B \cup G$ as in Theorem 2.0.1. Further, suppose that $G \sim U$ and the link $B \cup G$ is split. If K arises from L as in Theorem 2.0.1 then $K \sim B$.

Proof. By definition, K arises from L by banding B to several 0-framed parallel copies of G. Since $B \cup G$ is split and G is slice, the bands from the construction of K together with several parallel copies of a slice disk for G give a slice disk for K. □

Example 2.0.3. *Let m be an integer and L_m be the decorated link on the left hand side of Figure 2.3, which describes a 4-manifold X_m, and observe that L_m satisfies the hypotheses of Theorem 2.0.2. After the indicated slides we obtain a diagram of X_m as the 0-trace of a knot we call K_m. By Theorem 2.0.2, K_m is concordant to B which we see is isotopic to the right-hand trefoil for all m. We then isotope L_m to get a handle diagram for X_m as the 0-trace of a knot we call K'_m. See Figure 2.4.*

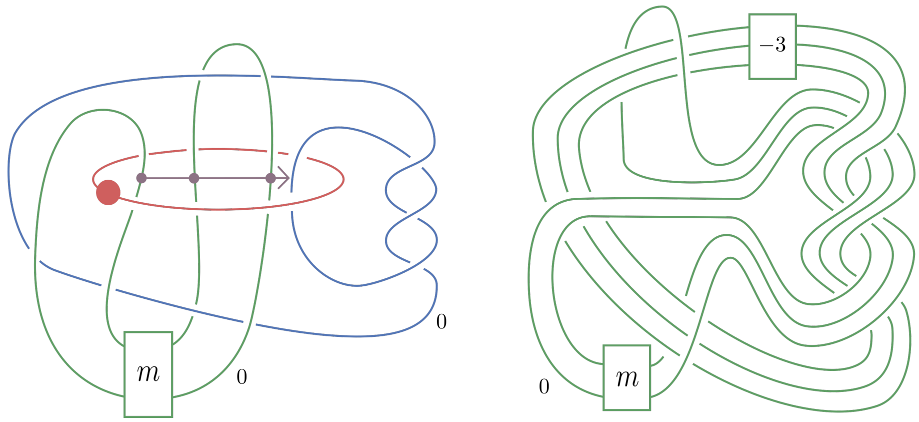

Figure 2.4: The same handle diagram for X_m and diffeomorphism to $X_0(K'_m)$

Remark 2.0.2. In Figure 2.3 we have illustrated bands $\{\beta_i\}$ such that banding along $\{\beta_i\}$ in the left hand diagram changes B into a three component link split from G, where two components are isotopic to μ_R, as in the proof of Theorem 2.0.2. We also kept track of $\{\beta_i\}$ through the diffeomorphism. In practice neither exhibiting nor keeping track of the bands is necessary; we have included it here to build intuition for the proof of Theorem 2.0.2 and demonstrate how Theorem 2.0.2 can be used to give an explicit description of the implied concordance.

The diagram we give of K_m in Figure 2.3 can certainly be simplified, but since we will only be concerned with K_m up to concordance and we understand $[K_m]$ by Theorem 2.0.2, we don't pursue this. This illustrates the usefulness of Theorem 2.0.2; if one wants to compare the concordance properties of knots with diffeomorphic traces one can get a tractable pair by choosing L so that K' remains relatively simple (in crossing number perhaps, or what-

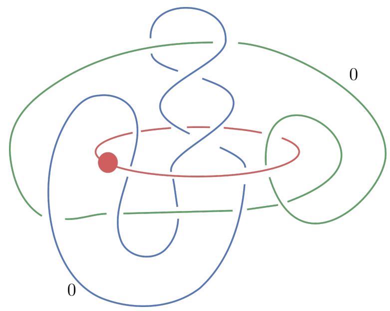

Figure 2.5: Non-slice K and K' can arise when the 'split' hypothesis is omitted

ever is convenient) and since we understand $[K]$ it does not matter if the knot K is complicated.

Remark 2.0.3. The split hypothesis of Theorem 2.0.2 is essential. For example, consider the handle diagram L in Figure 2.5 and let K be the knot obtained from L as in Theorem 2.0.1. K is isotopic to the pretzel knot $P(5, -3, -3)$, which is not slice.

The following proposition gives a class of knots which can arise as K or K' in Theorem 2.0.1, and is useful for applications. A related statement appears in [4].

Proposition 2.0.4. *For any unknotting number 1 knot K, there exists a link L as above and 4-manifold X associated to L as above so that $X \cong X(K)$.*

Proof. Choose a (blue) diagram D of K with an unknotting crossing c. We will prove the claim for c positive, the proof for c negative is similar. Define knots R and G in $S^3 \setminus \nu(K)$ as in the left frame of Figure 2.6, where R is

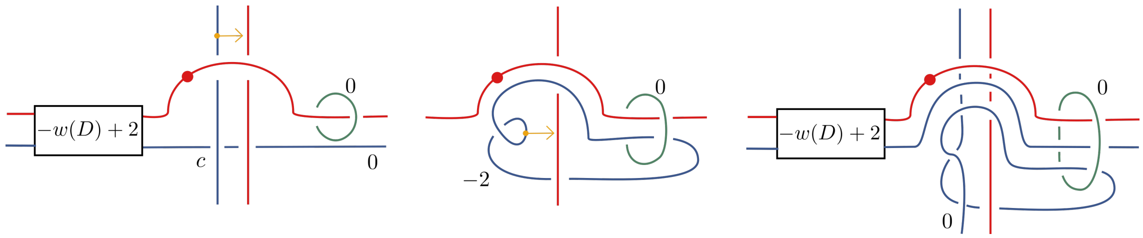

Figure 2.6: Constructing a link L associated to an unknotting number 1 knot K. Here D denotes a (blue) diagram of K with a positive unknotting crossing c, and $w(D)$ denotes the writhe of D.

a blackboard parallel of D outside of the diagram. Define X to be the four manifold obtained by thinking of R as a one handle in dotted circle notation, and attaching 0-framed 2-handles along K and G. Since G and R are a canceling 1-2 pair, we see that $X \cong X(K)$. It remains to construct a link L presenting X, where L satisfies the construction preceding Theorem 2.0.1.

To this end, slide K over R as indicated in Figure 2.6 to get a handle description for X as in the center frame. Observe that the blue attaching sphere is isotopic to a meridian of R. As such, performing the indicated slide to get the right frame will yield a link L with 0-framed blue attaching sphere B which can be isotoped so that $B \cup R$ is isotopic to $B \cup \mu_B$, and one observes that $\text{lk}(B, G) = 0$. □

Thus for any unknotting number one knot K, one can produce a link L as in Theorem 2.0.1, and use L to produce a knot K' with $X(K') \cong X(K)$. We remark that the unknotting number one knot K is in fact isotopic to the knot K produced from L as in the proof of Theorem 2.0.1, though we will not rely on that here.

Chapter 3

Shake genus

Armed with Theorem 2.0.1, we can now produce pairs of knots with the same shake genus by producing pairs of knots K, K' with diffeomorphic 0-traces. Our plan to prove Theorem 1.0.2 is to build such a pair where we loosely expect the shake genus and slice genus of both knots to be large. However, we will do this while satisfying the hypothesis of Theorem 2.0.2 so that K, surprisingly, has some prescribed small slice genus. Since Theorem 2.0.2 is non-symmetric in K and K', one might still expect that $g_4(K')$ is large. In this chapter we give an infinite family of pairs where this occurs. The main technical work lies in giving lower bounds on $g_4(K')$, for which we use Rasmussen's s invariant. We recall the relevant properties of Rasmussen's s invariant now.

In [12] Khovanov introduced a link invariant $Kh^{i,j}(L)$ which is the (co)homology of a finitely generated bigraded chain complex $(C^{i,j}(D_L), d)$. In our notation, D_L denotes a diagram of L and i is refered to as the homological grading and j the quantum grading. Later Lee [16] introduced a modification of the Khovanov differential: she considered instead a graded filtered complex $(C^{i,j}(D_L), d')$, such that d' raises homological grading by 1 and for any homogenous $v \in C^{i,j}(D_L)$ the quantum grading of every monomial in $d'(v)$ is greater than or equal to the quantum grading of v. As a consequence of her construction, there exists a spectral sequence with $(E_1^{i,j}(D_L), d_1) = (C^{i,j}(D_L), d)$ and $E_2^{i,j} = Kh^{i,j}(L)$ which converges to the homology of the Lee complex for L. We will denote this homology group $KhL^{i,j}(L)$. It will be relevant for us that the differentials d_n of the spectral sequence have bidegree $(1, 4(n-1))$ (see [19] or [16]). Lee proves that for any knot K, $KhL(K) = \mathbb{Q} \oplus \mathbb{Q}$ where both generators are located in grading $i = 0$. Rasmussen used this to define an integer valued knot invariant $s(K)$ as follows.

Theorem 3.0.1 ([19]). *For any knot K the generators of Lee homology are located in gradings $(i, j) = (0, s(K) \pm 1)$.*

It will suffice for this work to recall the following properties of $s(K)$.

Theorem 3.0.2 ([19]). *For any knot K in S^3, the following hold:*

1. $|s(K)| \leq 2g_4(K)$
2. *rank*$(Kh(K)^{0,s(K)\pm1}) \neq 0$
3. *Suppose K_+ and K_- are knots that differ by a single crossing change, from a positive crossing in K_+ to a negative one in K_-. Then $s(K_-) \leq s(K_+) \leq s(K_-) + 2$.*

Proof of Theorem 1.0.2. For a fixed $m \leq 0$ let K_m and K'_m be the knots from Example 2.0.3. By Theorem 2.0.1 $X_0(K_m) \cong X_0(K'_m)$, and as remarked in

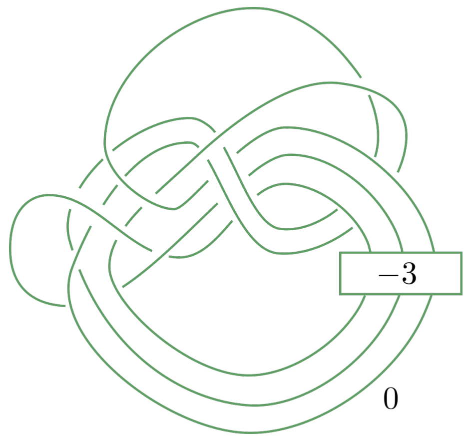

Figure 3.1: A somewhat reduced diagram of K'_0

Example 2.0.3, $g_4(K_m) = 1$ for all m. For $m \leq 0$ we will bound the slice genus of K'_m from below by bounding $s(K'_m)$ from below. See Figure 3.1 for a somewhat reduced diagram of K'_0 with approximately 40 crossings. We make use of the JavaKh routines, available at [1] to compute $Kh^{i,j}(K'_0)$. We plot the values $\text{rank}(Kh^{i,j}(K') \otimes \mathbb{Q})$ in Table 4.1.

By item 2 of Theorem 3.0.2 we have $s(K'_0) = 4$, and by item 3 we have $s(K'_m) \geq 4$ for all $m \leq 0$. We conclude by appealing to item 1 of Theorem 3.0.2. □

Corollary 3.0.3. *Rasmussen's s invariant is not a 0-trace invariant*

This addresses problem 12 of [3] which was given by Tetsuya Abe. It is still unknown whether Ozsváth-Szabó's τ invariant is an invariant of the 0-trace of K.

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Table 3.1

Remark 3.0.1. It is not hard to check that $g_4(K'_m) \leq 2$ for all $m \in \mathbb{Z}$. Hence all bounds in the above proof are sharp.

Chapter 4

The Conway knot

As outlined in the introduction, we will obstruct the sliceness of the Conway knot, which we call C, by constructing a knot K' such that $X(C) \cong X(K')$, hence such that the Conway knot is slice if and only if K' is slice. The advantage of this approach is that we do not have any reason to expect K' is a mutant of a slice knot, so we hope that not all sliceness obstructions for K' will vanish. In this chapter we build such a K', and use Rasmussen's s invariant to show that our K' is not slice. It is perhaps worth remarking then that for any K' with $X(C) \cong X(K')$ one can show $\tau(K') = 0$.

Proposition 4.0.1. *The knot K' in Figure 4.1 has $X(C) \cong X(K')$.*

Proof. We proceed as in the proofs of Proposition 2.0.4 and Theorem 2.0.1; in order to produce a diagram of K' with small crossing number we will perform additional isotopies throughout. See Figure 4.2. □

As discussed, Theorem 1.0.4 follows as a corollary of the following:

Theorem 4.0.2. *K' is not slice.*

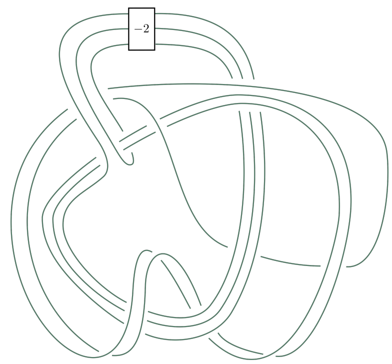

Figure 4.1: The knot K' shares a trace with the Conway knot

Proof. Let K' be the knot from Proposition 4.0.1; to show K' is not slice we will calculate $s(K')$. To begin, we compute the Khovanov homology of K', using Bar-Natan's Fast-Kh routines available at [1]. These routines produce the polynomial $Kh(K)(t,q) := \Sigma_{i,j} t^i q^j \text{rank}(Kh^{i,j}(K) \otimes \mathbb{Q})$. We plot the values $\text{rank}(Kh^{i,j}(K') \otimes \mathbb{Q})$ in Table 4.1.

Since the Lee homology is supported in grading $i = 0$, we see that $s(K') \in \{0, 2\}$. To demonstrate that in fact $s(K') = 2$ we will use the fact that all higher differentials in the spectral sequence to the Lee homology have bidegree $(1, 4(n-1))$. Consider a generator x of $Kh^{0,3}(K')$. If x were to die on the n^{th} page of the spectral sequence ($n \geq 2$) we would have to have that either $d_n(x) \neq 0$ or there exists a y with $d_n(y) = x$. Since $Kh^{i,j}(K')$ has no generators in gradings $\{1, 4(n-1)+3\}$ or $\{-1, -4(n-1)+3\}$ for any $n \geq 2$, neither of these can happen. As such, x survives to the E^∞ page, and $s(K') = 2$. □

	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
49																																				1
47																																				
45																																	1	1	1	
43																																1	1			
41																														1	1	1	1			
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37																											1	1	2	1						
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33																								2	3	2	1	1								
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29																					2	2	4	4	1											
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3				1	3	3																														
1			2	2	2																															
-1		1		1																																
-3		2																																		
-5	1																																			

Table 4.1

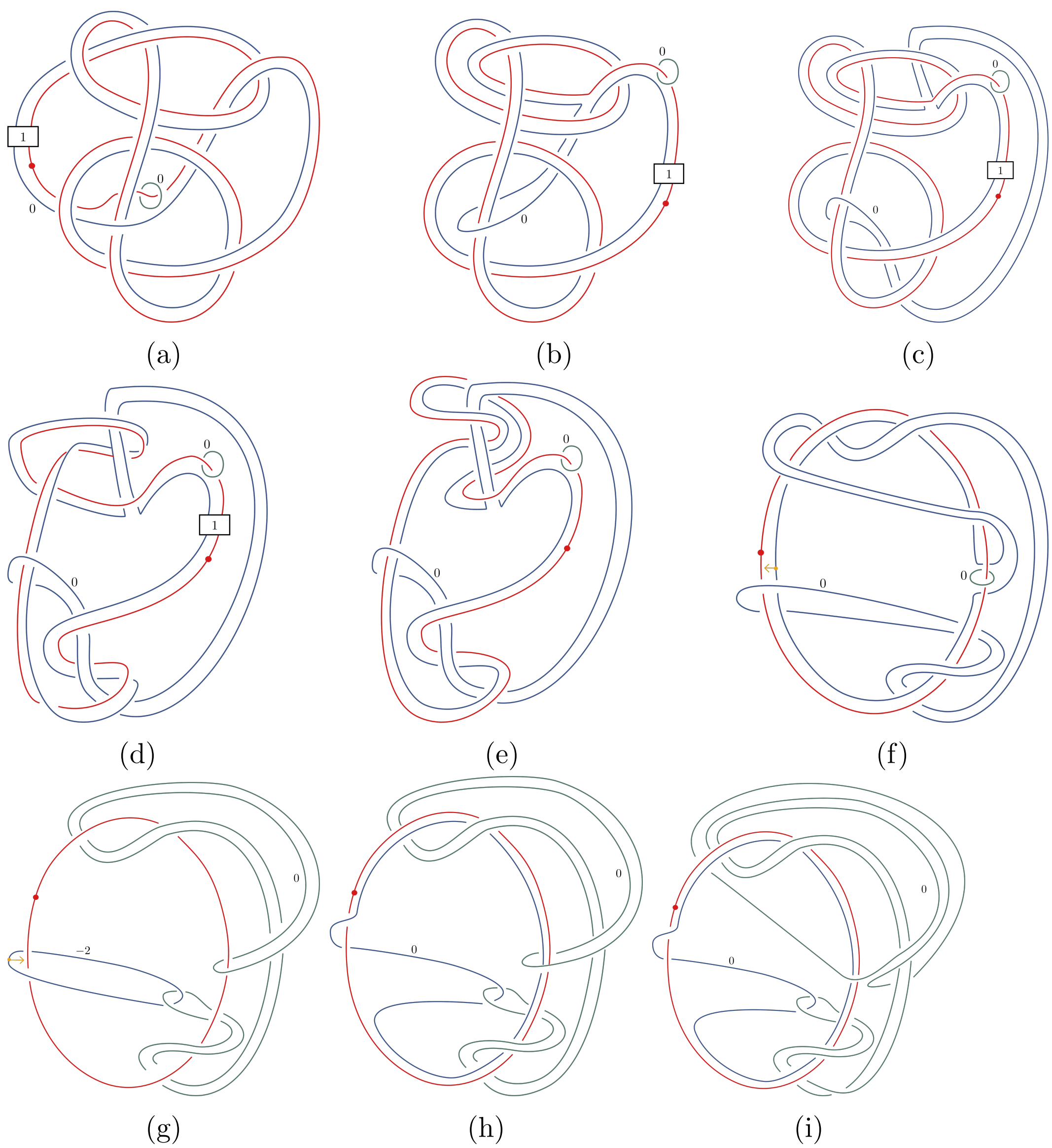
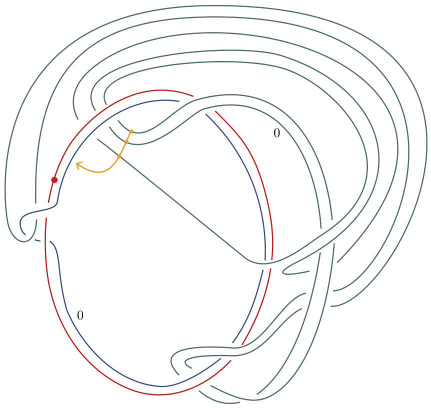
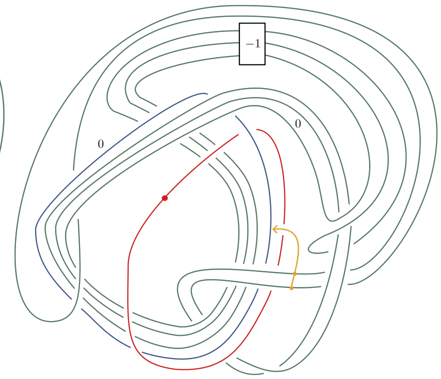


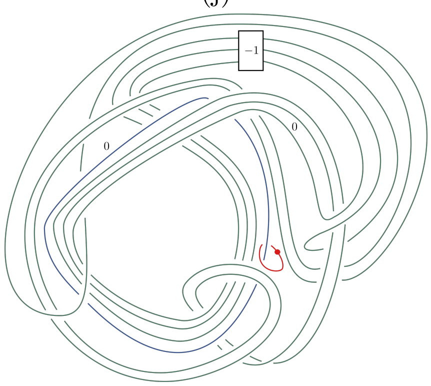
Figure 4.2: Handle calculus exhibiting a diffeomorphism from $X(C)$ to $X(K')$ where K' is the knot defined in Figure 4.1. Handle slides are denoted with arrows, the transition from (L) to (M) includes canceling a 1-2 pair, and all other changes are isotopies.



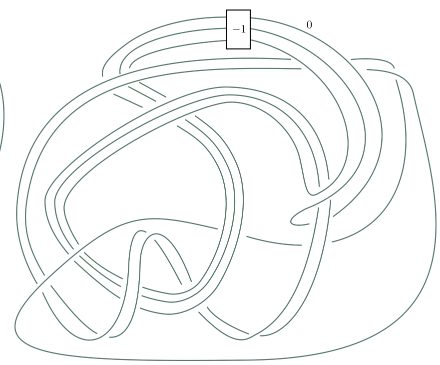
(j)



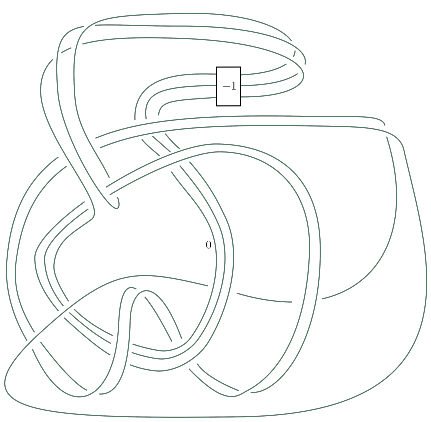
(k)



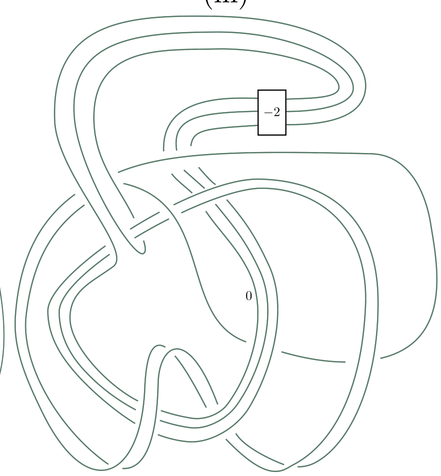
(l)



(m)



(n)



(o)

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