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Journal of Engineering and Computer Sciences
Qassim University, Vol. 10, No. 1, pp. 53-88 (January 2017/Rabi' II 1438H)

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Abstract. This paper advocates and demonstrates the utility of the Karnaugh-map, as a pictorial manual tool, in the *Boolean analysis (BA)* of social and political problems, in general, and in problems of peace research, in particular, as exemplified herein by the problem of war termination. Analysis is performed for both the appearance and absence of a specified phenomenon for the cases where (a) the logical remainders (don’t cares) are ignored (actually nullified), and (b) the don’t cares are assigned certain deliberate but *independent* values, and (c) faithful representation is used via a partially-defined function whose asserted part constitutes the definite (certain) causes of the phenomenon while its don’t-care part is a disjunction of its potential (uncertain) causes. The paper also presents several novel extensions of BA in which the don’t-care entries in the Karnaugh-map are manipulated in an attempt to make the output function positive/negative in, independent of, or symmetric in some of its arguments, to assign an importance metric for each of these arguments, to obtain a threshold representation for it, or to fit a pre-specified hypothesis. An explanation is given for the relation between BA *per se*, and two variants thereof, namely *Crisp-set Qualitative Comparative Analysis (csQCA)* and *Coincidence Analysis (CNA).*

Key Words: Boolean analysis, Karnaugh map, Logical remainders (don’t cares), War termination, Minimization, Novel extensions, Crisp-set Qualitative Comparative Analysis (csQCA), Coincidence analysis (CNA).

1. Introduction

Boolean analysis (BA) was introduced by Flament [1, 2] for application to questionnaire data [3, 4]. This analysis is a partial-order generalization of scalogram analysis [5], and is particularly useful for hierarchical data [6], and generally any type of data that can be dichotomized, i.e., reduced to optimal binary data [7-9]. Boolean analysis is the core essence of Qualitative Comparative Analysis (QCA) first introduced in the seminal paper of Ragin *et al*. [10], well established via the celebrated text of Ragin [11], and further elaborated by Ragin, his associates and other scholars [12-24]. Qualitative Comparative Analysis has now branched into three variants, namely crisp-set Qualitative Comparative Analysis (csQCA), multi-value Qualitative Comparative Analysis (mvQCA), and fuzzy-set Qualitative Comparative Analysis (fsQCA) [20]. Qualitative Comparative Analysis bridges the gap between (and embodies some key strengths of) the qualitative and quantitative approaches in the social and political sciences [18]. Though csQCA has many subtle differences with rudimentary BA, we find it more convenient herein for the purposes of this paper to treat the terms BA and csQCA as synonymous. We will also include under the general umbrella of BA, all pertaining mathematical features and engineering applications [25-29]. A notable offshoot of BA is *Coincidence Analysis (CNA)* [30-33] which is formally more similar to the original work of Flament [1, 2]. This CNA offshoot differs slightly form the mainstream BA (or csQCA) studied herein. In CNA, a relation is studied between several variables without presuming which are causes and which are effects, while in mainstream BA a certain outcome (effect) is studied as a function of certain inputs (causes).

Perhaps the most important advantage of BA is that it addresses explicitly the concept that there can be multiple causal mechanisms producing the same output [34], a concept known as ‘multiple conjunctural causation’. Boolean analysis starts by identifying the outcome (*dependent* variable) as a *Boolean function* of n conditions (*independent* variables). This function is typically represented in the form of a truth table whose input domain consists of 2^n lines representing the 2^n possible combinations of the independent variables. These lines or combinations are usually called configurations. If the outcome has a specific value of 1 or 0 for each of these configurations, the Boolean function is said to be totally defined or completely specified. Other possible values for a configuration are

- The “indeterminate” value (-) among observed cases, a value that should be avoided since one is supposed to handle specific outcomes across well-selected cases. Rihoux & de Meur [19] call this value a don’t care, but we believe that this is a misnomer that might lead to confusion, and hence we will not adopt it herein. In fact, we reserve the term “don’t care” for use as an identical term to the term “logical remainder” to be discussed below.

- The “contradiction” value (C), for a configuration that has a “0” outcome for some observed cases and a “1” outcome for other observed cases. Such a value poses a logical contradiction that must be resolved before further processing *via* techniques discussed by Rihoux & de Meur [19] and Jordan et al. [14]. Fortunately, we do not encounter such a value in our forthcoming analysis based on that of Chan [34].

- The "Logical Remainder" value (L) or (R), which is what is called a don't-care (d) in digital design and electrical engineering circles [29, 35-39]. This value (which stands for either '1' or '0' but for nothing else) designates configurations that have not been observed among the empirical cases. We reiterate that we will consider Logical Remainders and don't-cares as synonymous herein.

The aim of this paper is to advocate and demonstrate the use of a pictorial manual tool (specifically, the Karnaugh-map) in general BA, with a stress on the Boolean Analysis of political problems pertaining to peace research such as the celebrated problem of war termination [34, 40-55]. At this point, we present several observations in order.

1.There is a long history of utilization of *pictorial tools* in logic, engineering, and mathematics [56]. These include the Venn diagram, the Carroll map, the Marquand-Veitch map, and the Karnaugh map [26, 57]. There are subtle differences among these tools; though sometimes these differences go unnoticed to the extent that one tool might be given the name of another. For example the 'Venn diagram' produced by the 'visualizer' tool of the Tosmana software [19] is in fact a Carroll map [57-58] though there is no significant fault (apart from a historical one pertaining to giving unfair credit) in naming it a Venn diagram. In Section 2, we will explain why the Karnaugh-map is our tool of choice and why it is more convenient to use than the other tools, a fact attested to by its widespread use (unopposed) in digital design circles [29,37,38,59], and by the existence of a variety of complaints concerning the use of other tools [58, 60-65].

2.Boolean Analysis or dichotomous *QCA (Crisp-set QCA)* is typically designed to address small-N or medium-N situations of less than 30-40 cases [66]. Since a configuration accommodates zero, one or more cases, the number of configurations is somewhat expected to be comparable to the number of cases, and hence the number of *independent* variables (which is the ceiling of (i.e., the smallest integer greater than or equal to) the natural logarithm of the number of configurations) is expected to be around six. In fact, in an overview of QCA applications in political science during the period 2003-2011, Marx, *et al.* [16] survey almost five hundred papers to find that the number n of conditions (*independent* variables) used in each of them ranged from 3 to 10. Table I summaries their findings concerning the possible values for the number m of papers employing the number of conditions n, $3 \leq n \leq 10$. We stress that the maximum value of n is 10, while its average value is $y/x = 5.287$ whose ceiling is 6. These maximum and average values seem as if deliberately set to suit the use of a Karnaugh map, whose conventional form is conveniently used up to 6 variables [29, 37], while its variable-entered form is conveniently used up to 12 variables [36, 37, 67-80].

3.Rihoux [66] characterizes QCA as a technique based on Boolean algebra and implemented by a set of computer programs. In fact, there are many such programs or packages including the Tosmana software [81] and QCA [22-24, 82]. We argue that implementation of QCA via computer programs is not really warranted for the dichotomous or crisp-set QCA (but could not probably be dispensed with for the multi-value QCA or fuzzy-set QCA). It is clear from the previous paragraph that a crisp-set QCA can be easily implemented by the conventional Karnaugh map (CKM), or if necessary, by its extension, the variable-entered Karnaugh map (VEKM) [36, 37, 67-80]. In fact, some *limited* use of the CKM has been already made by the QCA community [22, 82, 83].

Table (1). Number of conditions (independent variables) in papers overviewed by Marx et al. (2013)

n=Number of conditions	3	4	5	6	7	8	9	10	Total	
m=Number of papers m using n	7	29	25	13	9	6	4	1	94	x
n*m	21	116	125	78	63	48	36	10	497	y

4.Rihoux [66] also characterizes QCA as a labor-intensive, interactive, and creative process. This means that QCA demands a lot of human work and intervention. Implementing Boolean minimization via a computer program, definitely saves some time, but this time is negligible compared to the total time of the whole process. This *not-so-important time saving* is achieved at the expense of the *loss of* the *ready insight* and *full control* gained via map use. In fact, with the iterative nature of QCA, a researcher must make sense out of the solution, interpret it by re-interrogating the cases, and go back to them to examine each one as a whole [66]. Working manually all throughout on a map might be preferable to a mixed type of work jumping between manual and automated work.

5.The seminal idea of Boolean Analysis has now outgrown to several distinguished variants or threads as shown in Fig. 1. One might go to one extreme of viewing each of these threads as essentially a *replica* of any of the others in *disguise*, or go to the other extreme of claiming each of these threads to be a *stand-alone* discipline marginally or superficially related to the others. Viewing these threads from a Karnaugh-map perspective might yield fruitful insights about certain striking similarities among these threads as well as some subtle differences among them. Such a perspective might remedy the problem that these threads currently exist in considerable *isolation* and at best have some *minimal interactions* among them.

Inspired by the aforementioned observations, we present this paper as a tutorial exposition of utilization of the Karnaugh map in political science in general, and in peace research in particular. In some sense, our paper can be thought of as a computer-science sequel to the seminal paper by Chan [34] that appeared earlier in a political-science journal. Our paper amplifies and clarifies the seminal work in Chan's paper and opens wide avenues allowing a variety of useful extensions for it.

The organization of the remainder of this paper is as follows. Section 2 discusses the characteristics of the Karnaugh map and offers a mini tutorial on its construction and its use in minimization. Section 3 presents a Boolean analysis, from a Karnaugh-map perspective, of the causes of war termination as reported by Chan [34]. The analysis is performed for both a Boolean outcome variable and its complement, and for three cases, namely: (a) when the logical remainders (don't cares) are ignored (actually deliberately nullified for an arbitrarily-selected form or literal of the outcome variable rather than for its complementary form or literal), (b) when the don't cares are assigned *independent* values so as to achieve a certain objective, typically minimization, and (c) when there is a need for faithful representation for the outcome via a partially-defined function whose asserted part constitutes the definite (certain) causes of the phenomenon, while its don't-care part is a disjunction of its potential (uncertain) causes. Section 4 demonstrates several novel extensions of the aforementioned analysis that make the most of the Karnaugh map by utilizing its don't-care entries in studying the possibility of making the outcome function *positive* or *negative* in, or *independent* of, some of its arguments, assigning an important metric for each of these arguments, rendering the outcome function partially or totally *symmetric* in these arguments, finding a threshold formulation of the outcome function, or making it fit a pre-specified hypothesis. Section 5 concludes the paper.

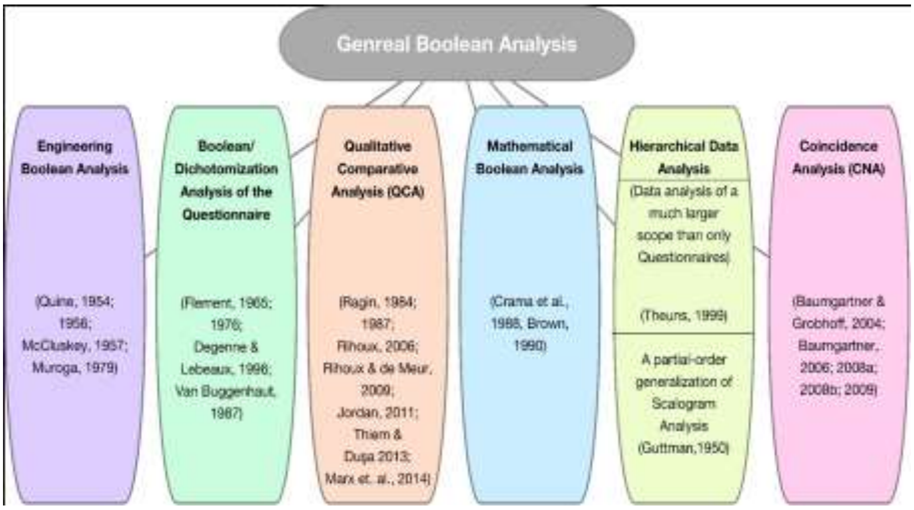


Fig. (1). Six distinguished threads or variants of General Boolean Analysis

2. On the Karnaugh-map

A classical or conventional Karnaugh map of n variables is in essence, a truth-table representation of a switching or Boolean function

$$f : B_2^n \to B_2 \qquad (1)$$

where B_2 is the bivalent Boolean carrier {0,1}. However, the Karnaugh map differs from an ordinary truth table in two major aspects:

1- The Karnaugh map is a two-dimensional display rather than a one-dimensional arrangement. Hence, the Karnaugh map makes the most of the human capability to view two independent dimensions simultaneously.

2- The combinations of the input variables (configurations) are ordered according to the Gray code (reflected binary code) [26, 37]. This code differs from (and has a spatial advantage over) the ordinary binary code shown in Fig. 2(a). For comparison, we display in Fig. 2(b) a 3-bit Gray-code representation for the integers from 0 to 7. Note that the leftmost bit X_1 is reflected w.r.t. the continuous lines, the middle bit X_2 is reflected w.r.t. the dashed lines, and the rightmost bit X_3 is reflected w.r.t. the dotted lines. The Gray ordering gives the map the visual advantage that *neighboring* cells are represented by *adjacent* input variable combinations, i.e., by binary numbers that differ in only one bit position [26, 29, 37]. The top and bottom rows of the map are viewed as *contiguous*. Similarly, the leftmost and rightmost columns are considered adjacent. In that sense, a Karnaugh map can be imagined to exist on the three-dimensional surface of a *torus*, albeit conveniently drawn on a two-dimensional plane [29].

Thanks to these two differences between a Karnaugh map and an ordinary truth table, the Karnaugh map is a more concise and time- and space-saving tool. It also provides pictorial insight to many switching-theoretic concepts such as duality, prime implicants, prime implicates, complementation, differentiation, and utilization of don't-care values. Rushdi [37] discusses advantages of the Karnaugh-map over other graphical representations of sets, events and propositions, including the ubiquitous Venn diagram, Euler diagram, Carroll diagram, and the Marquand-Veitch chart. Wheeler [65] laments that the Karnaugh map had not been much explored by teachers (of mathematics), a regrettable fact that seems to be still valid nowadays. In fact, a recent article by Mahoney and Vanderpoel [15] explored, admirably how visual tools (which they called set diagrams) can facilitate the application of qualitative methods and improve the presentation of qualitative findings. However, they mainly relied on Venn and Euler diagrams rather than Karnaugh maps.

The straightforward and mechanical conversion of a truth table to a Karnaugh map is discussed in numerous texts on logic design [29, 38, 84]. As an aid to the reader, we write in the right upper corner of each cell of the Karnaugh map in Fig. 3(a) the numerical value of the 4-bit binary digit.

$$(ABCD)_2 = 8A + 4B + 2C + D\ , \tag{2}$$

which represents the corresponding line number in the original truth table (numbered from 0 to 15).

0	1	2	3	4	5	6	7
000	001	010	011	100	101	110	111

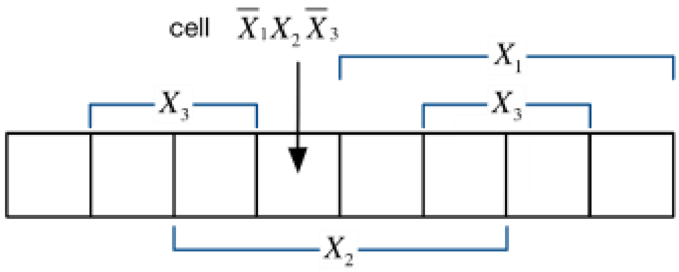
(a)

Fig. 2 (a). The 3-bit integers $(X_1X_2X_3)$, which represent 0 to 7 in the ordinary binary code. Here $(X_1X_2X_3) = 4X_1 + 2X_2 + X_3$

0	1	2	3	4	5	6	7
000	001	011	010	110	111	101	101

(b)

Fig. 2 (b). The 3-bit integers $(X_1X_2X_3)$ representing 0 to 7 in Gray code. The continuous lines is a mirror for the variable X_1, the dashed lines is a mirror for X_2, while the dotted lines is a mirror for X_3



(c)

Fig. 2 (c) A one-dimensional Karnaugh map utilizing the Gray ordering in Fig. 1b. Each map cell differs form an adjacent cell in a single bit. The leftmost and the rightmost cells are adjacent

The most famous application of the Karnaugh map is its use in minimization, i.e., obtaining a formula for the map function in sum-of-products (disjunction-of-conjunctions) form that has a minimum number of prime implicants as the primary objective and that has a minimum number of literals as the secondary objective. Dually, the map can also be used for expressing the map function in a product-of-sums (conjunction of disjunctions) form that has a minimum number of prime implicates as the primary objective, and again a minimum number of literals as the secondary objective [29].

The primal or sum-of-products map application heavily depends on the map capability to visually identify prime implicants. A prime implicant P of a given function f is:

(a) An implicant of f (i.e. $P \le f, P \to f$ or $f = g \vee P$),

(b) No other term subsumed by P (whose set of literals is a subset of the set of literals of P) is an implicant of f .

A prime implicant is represented by a (continuous or split) rectangular loop consisting of 2^k adjacent asserted cells (called 1-enterd cells or 1-cells) or don't-care cells (called d-cells), where k is a natural number chosen as large as possible. Generally, the Karnaugh map is used to represent an incompletely-specified (partially-defined) function. To express the map function one (a) must use prime-implicant loops to cover each 1-cell (cell entered with 1) *at least* once, (b) can cover each of d-cells (cells entered with don't cares) once or more (independently of the other d-cells), and (c) never cover any of the 0-cells (cells entered with 0). This means that the coverage of 1-cells, d-cells, and 0-cells is *obligatory*, optional (permissible) and *prohibited*, respectively. Once a 1-cell is covered, it is changed to a d-cell, i.e., its further covering is still allowed, i.e., such coverage becomes only permissible and ceases to be mandatory.

The Karnaugh map procedure for minimization consists of two major steps:

Step 1: Algorithmic covering of essential prime implicants:

Here, one checks the most isolated 1-cells first, so as to locate a 1-cell that can be covered uniquely, i.e., that is covered by a single prime implicant P_i and no more, called an *essential* prime implicant. The prime implicant is added to the disjunctive expression for the function f and entries of all cells within its loop are turned into don't cares. This process is repeated until all uniquely-covered 1-cells, if any, are exhausted.

Step 2: Heuristic (trial-and-error) covering of the remaining 1-cells:

Such coverage is achieved by as *few* and *large* prime implicants as possible. These *non-essential* prime implicants are added to the disjunctive expression of f. When f is totally covered, the final expression of f becomes an irredundant disjunctive from which might be minimal or not.

Detailed expositions of the Karnaugh map and its minimization procedure are available in classical textbooks on logic design such as those of Lee [84], Muroga [29], Hill and Peterson [85], and Roth & Kinney [37]. Some advanced aspects of map minimization are available in Rushdi [36, 69, 86], Rushdi & Ba-Rukab [87-89], Rushdi & Ghaleb [90] and Rushdi & Alturki [91]. Moreover, the examples on using the Karnaugh map in this paper will be explained in ample detail to allow readers with minimal map knowledge (or even with no previous such knowledge) to follow and understand them.

Rihoux & *de* Meur ([19, pp. 59-65]) devote a special section to explain why Logical Remainders are useful and how they can be utilized in obtaining more parsimonious minimal formulas. They raise the concern *"Isn't it altogether audacious to make assumptions about non-observed cases?"* and they point out that this concern is among the critiques targeted at QCA. It might sound strange enough that the above concern seems to have never been raised in electrical engineering circles. Electrical engineers used to obtain parsimonious minimal formulas that fill up the Boolean space well beyond the specified configurations. Their reasoning is that unspecified configurations are guaranteed to never happen, so though their formulas specify or assign outcomes for such configurations, practically such assigned outcomes will never be used or required. Of course, the situation in social and political sciences might be different, since an unobserved configuration might not be guaranteed to never happen, as it might be observed later. However, a strong rebuttal to the above concern can be stated as follows. If one refrains from utilizing Logical Remainders or don't-cares for minimization or for any other purpose, one is still setting certain unknown values to zero, and hence *is still making unwarranted* *or unjustified assumptions about non-observed data.* There is no reason to prefer these assumptions to the ones leading to minimization. In fact, the choice of nullifying k logical remainders can be thought to be equally likely to all other $(2^k - 1)$ possible choices for the logical remainders, and hence it can be equally accepted or rejected. We consider all these choices as equally interesting, and demonstrate how to utilize some of them to achieve certain useful purposes not excluded to minimization. We will also discuss herein how to obtain a faithful algebraic reproduction of the observed data that does not arbitrarily assign values to unobserved data. This faithful representation is in terms of partially-defined (incompletely-specified) functions. Again, the Karnaugh map will come to our aid in making such a representation. The spirit of such an incomplete representation is that "*When only partial observations are available, no method can provide definite answers,"* [27]. One should treat the observed data as *temporary* or *tentative*, and should seek more information to update one's findings. This process should continue as long as the observed data does not seem to be final. A stopping criterion need to be found so as to terminate the updating process whenever the data is deemed final in the sense that further observations are impossible or too costly.

3. Karnaugh-map Analysis of War Termination

We base our analysis in this section on Table II of Chan [34], which is a Boolean truth table for short wars. This table summarizes observations of important violent conflicts in modern history all over the world. For simplicity, we designate the outcome (dependent variable) of short wars by the single literal S, and use the variables A, B, C and D (as in Chan [34]) to denote Chan's hypothesized dichotomous determinants of war duration, namely:

$A =$ 'The war has a 'big and fast start' rather than a 'small and slow one' (Big start), $\overline{A} =$ Small start,

$B =$ 'The war involves major-minor dyads rather than more evenly-matched contestants' (Major-minor), $\overline{B} =$ 'Matched contestants'

$C =$ 'The war is multilateral rather than bilateral' (Multilateral), $\overline{C} =$ 'Bilateral'

$D =$ 'The war involves moderately repressive and exclusionary regimes rather than goes without such involvement' (Semi-rep.), $\overline{D} =$ 'Not Semi-rep.'.

We reproduce the aforementioned Table II of Chan [34] in the form of the Karnaugh map of Fig. 3(a). The interested reader might wish to consult typical texts on digital design [29, 38, 84] about the mechanical procedure of conversion from a truth table to a Karnaugh map. We remind the reader that the integer in the right upper corner of each map cell corresponds to the line number in Table II of Chan [34], where the line numbers range from 0 to 15.

As stated earlier in the introduction, we analyze this problem for three cases, namely: (a) when the logical remainders (don't cares) are ignored (actually deliberately nullified for an arbitrarily-selected form or literal of the outcome variable rather than for its complementary form or literal), (b) when the don't cares are assigned *independent* values so as to achieve a certain objective, typically minimization, and (c) when there is a need for faithful representation for the outcome via a partially-defined function whose asserted part constitutes the definite (certain). We devote separate subsections to the aforementioned three cases.

3.1 The Case of Ignoring the Logical Remainders

A first possible expression for the proposition of short wars (S) can be obtained by identifying the six combinations in which the outcome code in Table I of Chan [34] is asserted namely:

$$S_1 = \bar{A}\bar{B}CD \vee A\bar{B}\bar{C}D \vee A\bar{B}CD \vee AB\bar{C}\,\bar{D} \vee AB\bar{C}D \vee ABCD. \tag{3}$$

Our Equation (3) is identical to Equation (1) in Chan [34] with the single exception that we use a bar to designate a complemented literal, while such a literal is depicted therein by a lower–case letter. Figure 3(a) demonstrates Equation (3) simply as coverage of single cells in the corresponding Karnaugh map. This is called a *minterm* expression of (the asserted part of) S in digital design texts [29], or a listing of (asserted) discriminates of S in Boolean reasoning texts [26]. Equation (3) is simplified algebraically in Chan [34] via

Table (2). Interpretation of Results via Verbal Statements.

Class of Results	**Equation**	**No.**	**Verbal Statement**
Nullifying don't-cares for S	$S_1 = \bar{A}\bar{B}CD \vee A\bar{B}\bar{C}D \vee$ $A\bar{B}CD \vee AB\bar{C}\bar{D} \vee$ $AB\bar{C}D \vee ABCD$ Same as (1) of Chan (2003)	(3)	Short War = Small Start AND Matched contestants AND Multilateral AND Semi-rep. OR Big Start AND Matched contestants AND Bilateral AND Semi-rep. OR Big Start AND Matched contestants AND Multilateral AND Semi-rep. OR Big Start AND Major-minor AND Bilateral AND No Semi-rep. OR Big Start AND Major-minor AND Bilateral AND Semi-rep. OR Big Start AND Major-minor

Class of Results	Equation	No.	Verbal Statement
			AND Multilateral AND Semi-rep.
	$S_2 = AB\overline{C} \vee AD \vee \overline{B}CD$ Same as (2) of Chan (2003)	(11)	Short War = Big Start AND Major-minor AND Bilateral OR Big Start AND Semi-rep. OR Matched contestants AND Multilateral AND Semi-rep.
	$\overline{S}_4 = \overline{A}B \vee \overline{B}\,\overline{D} \vee \overline{A}\,\overline{C} \vee C\overline{D}$ Same as (3) of Chan (2003)	(12)	No Short War = Small Start AND Major-minor OR Matched contestants AND No Semi-rep. OR Small Start AND Bilateral OR Multilateral AND No Semi-rep.
Utilizing don't cares for minimization	$S_3 = AB \vee AD \vee \overline{B}C$	(13)	Short War = Big Start AND Major-minor OR Big Start AND Semi-rep. OR Matched contestants AND Multilateral
	$\overline{S}_5 = \overline{A}B \vee \overline{B}\,\overline{D} \vee \overline{A}\,\overline{C}$	(14)	No Short War = Small Start AND Major-minor OR Matched contestants AND No Semi-rep. OR Small Start AND Bilateral
Faithful Representation	$S = AD \vee AB\overline{C} \vee \overline{B}CD \vee$ $d(C\overline{D} \vee B\overline{C}D \vee \overline{A}$ $\overline{B}\,\overline{D})$	(17)	Short War = Big Start AND Semi-rep. OR Big Start AND Major-minor AND Bilateral OR Matched contestants AND Multilateral AND Semi-rep. OR d(Multilateral AND No Semi-rep. OR Major-minor AND Bilateral AND Semi-rep. OR Small Start AND Matched contestants AND No Semi-rep.)
			No Short War = Big Start AND Matched

Class of Results	Equation	No.	Verbal Statement
	$\overline{S} = A\overline{B}\,\overline{C} \quad \overline{D} \vee \overline{A}B\overline{C} \quad \overline{D} \vee \overline{A}\cdot$ $\overline{B}\,\overline{C}D$ $\vee \overline{A}BCD \vee d(C\overline{D} \vee \overline{A}\ \ \overline{C})$	(18)	contestants AND Bilateral AND No Semi-rep. OR Small Start AND Major-minor AND Bilateral AND No Semi-rep. OR Small Start AND Matched contestants AND Bilateral AND Semi-rep. OR Small Start AND Major-minor AND Multilateral AND Semi-rep. OR d(Multilateral AND No Semi-rep. OR Small Start AND Bilateral.

reasoning texts [26]. Equation (3) is simplified algebraically in Chan [34] via painstaking and elaborate efforts to find ways for combining minterms. For example, the fact that the 'AND' operation is distributive over the 'OR' operation namely:

$$X \wedge (Y \vee Z) = (X \wedge Y) \vee (X \wedge Z), \tag{4}$$

and that 'OR' is idempotent, namely:

$$X \vee X = X, \tag{5}$$

can be used to write

$$AB\overline{C}\,\overline{D} \vee \underline{\underline{AB\overline{C}D}} = AB\overline{C}(\overline{D} \vee D) = AB\overline{C}, \tag{6}$$

$$A\overline{B}\,\overline{C}D \vee \underline{A\overline{B}CD} \vee \underline{\underline{AB\overline{C}D}} \vee ABCD \tag{7}$$

$$= A(\overline{B} \vee B)(\overline{C} \vee C)D = AD,$$

$$\overline{A}\,\overline{B}CD \vee \underline{A\overline{B}CD} = (\overline{A} \vee A)\overline{B}CD = \overline{B}CD. \tag{8}$$

For clarity, we underlined the minterms that are used repeatedly since duplicate instances of them are available due to the idempotency of OR in (5), namely

$$A\overline{B}CD = \underline{A\overline{B}CD} \vee \underline{A\overline{B}CD} \tag{9}$$

$$AB\overline{C}D = \underline{\underline{AB\overline{C}D}} \vee \underline{\underline{AB\overline{C}D}}\,. \tag{10}$$

The result is Equation (2) in Chan (2003), namely

$$S_2 = AB\overline{C} \vee AD \vee \overline{B}CD. \tag{11}$$

3.2 The Case of Assigning Independent Values to the Logical Remainders

The great effort exerted in Chan [34] to go from (3) to (11) via unguided verbal arguments (and not even via Equations (6)–(8)) has surprisingly reached the correct result (11). Such an effort can be substantially reduced via the pictorial insight offered by the Karnaugh map in Fig. 3(b) in which the distributive law is manifest in combining $2^m (m \geq 1)$ adjacent cells, and the idempotency law is exhibited in permitting the resulting loops to be overlapping, i.e., allowing certain cells to be covered more than once. Now, we come to the most important issue, which is well known in electrical-engineering circles and seems to be somewhat unknown, occasionally ignored, or deliberately avoided in some other fields. This issue pertains to the lines left blank in Table II of Chan [34] and in Figs. 1(a) and 1(b) because they correspond to cases that were never observed. These cases actually never happened so far, and presumably expected to never happen if observations are deemed final. The function $S(A,B,C,D)$ should be identified as an incompletely specified switching function [29], or equivalently as a partially defined two-valued Boolean function [27]. The blank lines or cells are now assigned the values of don't-cares (d), which can be either 0 or 1, but nothing else. These values are called logical remainders in the QCA literature [19].

Figure 1(c) shows the blank cells entered with don't-care values d_i, $1 \leq i \leq 6$. Contrary to the common practice of assigning the same symbol $'d'$ to every don't-care cell (albeit with an implicit assumption that the d values in different cells are different), we have assigned explicitly different don't-care values to different cells to stress that these values are *independent*, and to deliberately avoid a potential misconception that they are necessarily the same. The solution obtained via conventional map minimization techniques [29, 38] is:

$$S_3 = AB \vee AD \vee \overline{B}C \, , \tag{12}$$

and represents a true minimal representation of S, typically called a 'minimal sum' [29], or a most compact or parsimonious formula [66]. The representation in (12) happens to be *unique*, since each of the three loops in Fig. 1(c) is an *essential* prime implicant, due to the fact that it is the only prime implicant covering a particular cell. The particular cells uniquely covered by essential prime implicants are each distinguished in Fig. 3(c) with a star in it. Note that the minimal solution (12) is obtained for the choice $d_1 = d_2 = d_4 = 0$ and $d_3 = d_5 = d_6 = 1$, while the solutions in Fig. 1(a) and Fig. 1(b) *implicitly assume* that all these $d's$ are zeros, i.e. that all blank cells in Figs. 1(a) and 1(b) are entered by 0's. This implicit assumption in Chan [34] is made explicit when expressing the non-short war proposition $(\overline{S})$ via Equation (3) in Chan [34], namely

$$\overline{S}_4 = \overline{A}B \vee \overline{B}\ \overline{D} \vee \overline{A}\ \overline{C} \vee C\overline{D}. \tag{13}$$

Equation (13) can be visually obtained via Fig. 4(a), which represents $\overline{S}_4$ as the complement of S_2 in Fig. 3(b) with all blank cells therein being assumed 0-entered and hence with the corresponding cells in Fig. 4(a) being explicitly 1-entered. The actual minimal solution in this case is shown in Fig. 4(b) (in which $\overline{S}$ is obtained by complementing the entries in Fig.3(c)) namely,

$$\overline{S}_5 = \overline{A}B \vee \overline{B}\overline{D} \vee \overline{A}\overline{C}. \tag{14}$$

Again, the expression in (14) happens to be unique, since it consists of prime implicants that are all essential, thanks to the uniquely-covered starred cells in Fig. 4(b). The only difference between (13) and (14) is the appearance of the extraneous or the all-d implicant $C\overline{D}$ in (13), which makes (13) less parsimonious than (14). This prime implicant is called *absolutely eliminable* [29], since it is possible (but always unnecessary) to cover it. Hence, this prime implicant is definitely useless [29, 92], as it is never needed in a minimal representation of its implied function.

In passing, we comment on the values of the logical remainders or don't-cares in Fig. 3(c). There are six don't-cares $(d_1 \ to \ d_6)$ that could each be assigned one of the values 0 or 1 *independently*. This means that there are 2^6 possibilities for selecting the d_i values. These values are selected to be the same as

$$d_i = 0, \ 1 \ \leq \ i \ \leq \ 6, \tag{15}$$

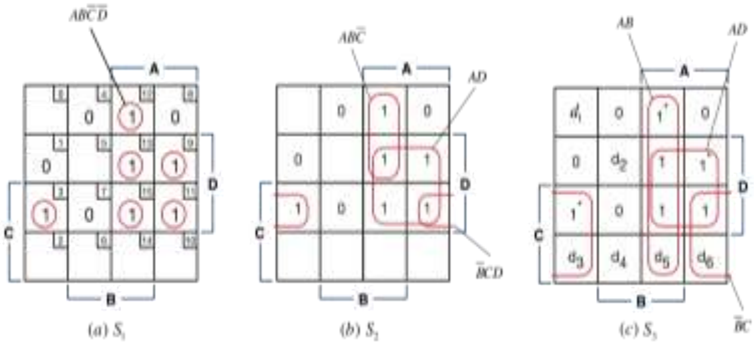


Fig. (3). The truth table for short wars (Table 1 in Chan (2003)) redrawn as a Karnaugh map with (a) expressing a minterm expansion, 1(b) expressing further simplification, and (c) expressing true minimization involving don't cares.

in each of Fig. 3(a) and Fig. 3(b). With such an implicit choice, *one is making certain assumptions about unobserved data.* The only reason to think of this choice as the appropriate one is that it entails apparently no action, though, in fact, it demands the particular action of nullifying specific d values. The alternative choices made in Fig. 3(c) or Fig. 4(b) are equally likely (or equally objectionable to) compared with the choice (15). Somehow, some people might think that the choice of (supposedly) no action is more appealing or more acceptable than other choices. An intriguing question is why such a (supposedly) no-action choice is equated to a no-assumption one for a function S and not for its complement $\overline{S}$. Note that one can equally well apply the (supposedly) no-action choice to $\overline{S}$ rather than to S, and obtain a possibly different result

3.3 The Case of Faithfully Representing the Boolean Function as a Partially Defined One

A faithful representation of S is to express it as a partially-defined function [35, 36, 39], namely,

$$S = g \vee d(h), \tag{16}$$

where g and h are called the *asserted* and *don't-care* parts of S. Equation (16) means that

$$\{g = 1\} \rightarrow \{S = 1\}, \tag{16a}$$

$$\{g = h = 0\} \rightarrow \{S = 0\}. \tag{16b}$$

Note that we keep silent about specifying the value of S when $\{g = 0, h = 1\}$, i.e. we think of this value as a don't-care. The representation (16) has the pictorial interpretation of Fig. 5. We have already obtained the minimal asserted part of the function S displayed in Fig. 3(b) and given by Equation (11). The minimal don't-care part of S is obtained in Fig. 6 in which we are obliged to cover every d-cell at least once and are allowed to cover any 1-cell, possibly repeatedly. The minimal asserted part of $\overline{S}$ was not obtained in Fig. 4(a) since the d-cells were not replaced by 0's but were replaced by 1's (in an unjustified bias for S against $\overline{S}$). Figure 7 presents Karnaugh-map minimization for the asserted and don't-care parts of $\overline{S}$. Now, we obtain the following expressions for S and $\overline{S}$:

$$S = AD \vee AB\overline{C} \vee \overline{B}CD \vee d(C\overline{D} \vee B\overline{C}D \vee \overline{A}\overline{B}\overline{D}), \tag{17}$$

$$\overline{S} = A\overline{B}\overline{C}\overline{D} \vee \overline{A}B\overline{C}\overline{D} \vee \overline{A}\overline{B}\overline{C}D \vee \overline{A}BCD \vee d(C\overline{D} \vee \overline{A}\overline{C}). \tag{18}$$

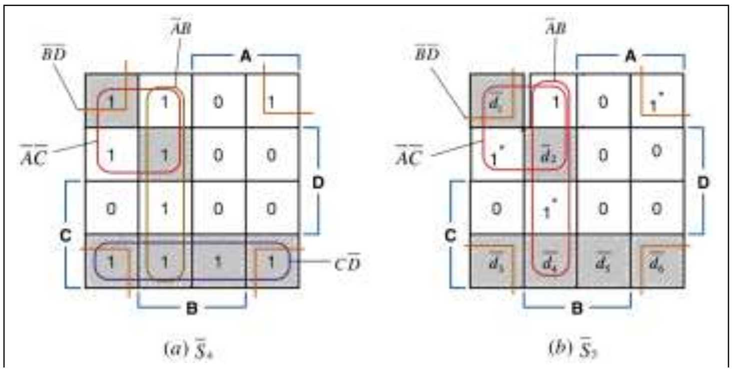


Fig. (4). The complementary function for the function in Fig. 2(a), (a) with the blank cells 1-entered, and (b) with a true minimal coverage

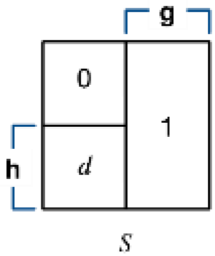


Fig. (5). A Karnaugh-map-like structure expressing   in terms of its asserted part   and don't-care part . Here d means either 0 or 1 (but nothing else).

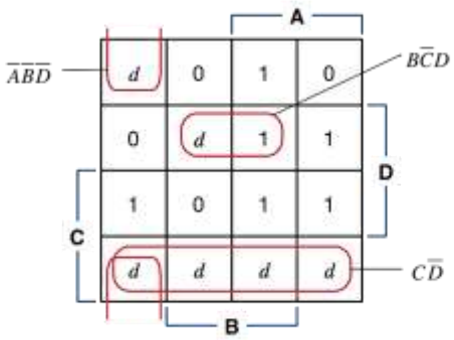


Fig. (6). The don't-care part of the function in Fig. 1(c).

Note that we have not subscripted the symbols S and $\overline{S}$ in (17) and (18), since each is a faithful representation of its respective function, and since $\overline{S}$ is now indeed the complement of S. This faithful representation defers the question of assigning values of non-observed configurations, maybe until they are observed. The terms in the asserted part represent *definite* or *certain* causes, while those in the don't-care part represent *potential* or *uncertain* causes. For example the term AD is a definite (certain) cause of S while the term $C\overline{D}$ is a potential (uncertain) cause for it.

We seem to have been too much involved in mathematical manipulations to the extent that we might have forgotten our original problem. To remedy this potential shortcoming, we use Table II to restate our results verbally. We express each of the outcomes S (short war) and $\overline{S}$ (no short war) in terms of the four determinants big start/small start, major-minor/matched contestants, multilateral/ bilateral, and finally semi-rep/no semi-rep. Table II interprets various results obtained when (a) nullifying the don't-cares of S while asserting those of $\overline{S}$, (b) utilizing the don't cares for independent minimizations of S and $\overline{S}$, and (c) using a faithful representation for each of S and $\overline{S}$.

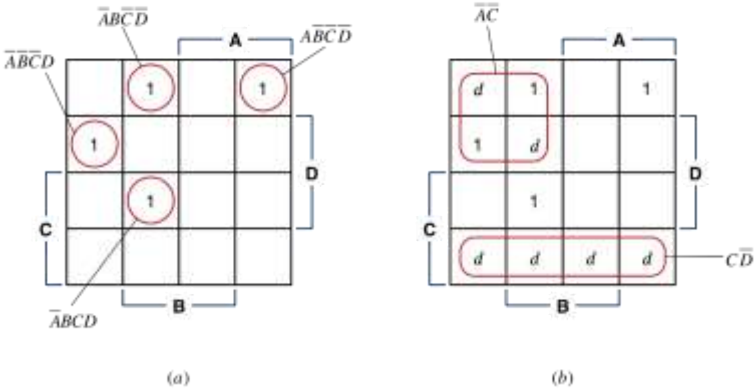


Fig. (7). The asserted part (a) and the don't-care part (b) for the complement of the function in Fig. 1(c).

4. Novel Extensions for Boolean Analysis

The Availability of the Karnaugh-map representations of Fig. 3(c) and fig. 4(b) allow several useful extensions of the Boolean Analysis discussed in Sec. 3. These include (a) deciding whether the pertinent function S is *independent* of some of its (supposed) arguments, (b) assigning an importance metric for each variable, (c) investigating whether S is positive or negative in its arguments, (d) checking whether S is partially symmetric in some of its arguments, and hence whether it is totally symmetric in all of them, and (e) deciding if S is a threshold function, and if yes, obtaining its threshold-function representation. The aforementioned extensions are discussed in the following subsections.

4.1 Possible Outcome Independence of Certain Arguments

To investigate the possibility of independence of the pertinent outcome function $S(A,B,C,D)$ of each of its arguments, we construct the Boolean difference (derivative) [29, 84, 86, 93].

$$\partial S / \partial X = S(A,B,C,D \,|\, 1_X) \oplus S(A,B,C,D) \,|\, 0_X), \tag{19}$$

where X stands for $A, B, C,$ or D, and $\oplus$ represents the XOR operator defined by the function table of Fig. 8 [94]. Note that, strictly speaking $1 \oplus d = d \oplus 1 = \overline{d}$ but $\overline{d}$ can be simply renamed as d, and $S(\ldots|\, j_X)$ stands for the subfunction or cofactor of S obtained by restricting the argument X to the value j, with j = 0 or 1. For example

$$\partial S / \partial A = S(1,B,C,D) \oplus S(0,B,C,D). \tag{20}$$

Note that $S(\ldots|1_x)$ can be written as S/X and $S(\ldots|0_x)$ can be written as $S/\overline{X}$. The function S is *independent* of X if $\partial S / \partial X$ is identically equal to 0. Figure 9(a) illustrates a pictorial illustration for computing (19) via map rotation or folding [78, 84, 86, 93]. The maps of the various Boolean derivatives

$\oplus$	0	1	d
0	0	1	d
1	1	0	d
d	d	d	d

Fig. (8). Function table for the XOR operation. Note that $d \oplus d$ is d and not 0 since the input d's are independatnt and not necessarily identical.

$\partial S/\partial A, \partial S/\partial B, \partial S/\partial C$ and $\partial S/\partial D$ are shown in Figs. 9(b)-9(e). None of these derivatives can be made identically zero by any choice of the don't cares (in fact, each of the maps in Figs. 9(b)-9(e) has at least one cell that is entered with 1). Hence, the function S is not vacuous in (independent of) any of its arguments. The arguments (independent variables) for S chosen in Chan [34] are definitely justified from a mathematical point of view, since the outcome function cannot be made independent of any of them. However, the possibility of existence of other arguments affecting S is not mathematically ruled out, but could be excluded via the purely historical reasoning given by Chan [34].

4.2 Importance Metrics for the Arguments

There are many ways to devise importance metrics for the arguments (independent variables) that measure the relative importance of each argument or its relative power in influencing the outcome. We will adopt herein a simple metric given by the Banzhaf index [95]

I_x = The weight of the partial derivative of S *w.r.t.* X

= The number of asserted minterms of $\partial S/\partial X$. (21)

Since the function S and hence the functions $(\partial S/\partial X)$, where $X \in \{A, B, C, D\}$ are incompletely specified, we do not obtain definite values for the importance metric I_x. However, we can obtain an upper or a lower bound (i.e., a maximum or a minimum value) for I_x when all the don't cares in the map for $(\partial S/\partial X)$ are set to 1 and 0, respectively: These bounds are

$$3 \le I_A \le 7, \quad (22a)$$

$$2 \le I_B \le 6, \quad (22b)$$

$$1 \le I_C \le 6, \quad (22c)$$

$$1 \le I_D \le 7. \quad (22d)$$

We cannot use (22) to rank the variables A, B, C and D according to their importance (influence on the outcome). There are 2^6 equally likely ways for assigning the $d's$ in Fig. 9(a) and hence in Figs. 9(b)-9(e). For each of these assignments, one can get specific values of I_A, I_B, I_C, and I_D, and hence a definite ranking of $A, B, C,$ and D. For example, with the d's assignment in Fig.

3(c), one obtains $I_A = 5, I_B = I_C = 3, I_D = 1$, which amounts to saying that A is the most important (influential) input variable, D is the least important one, while B and C are of intermediate and same importance.

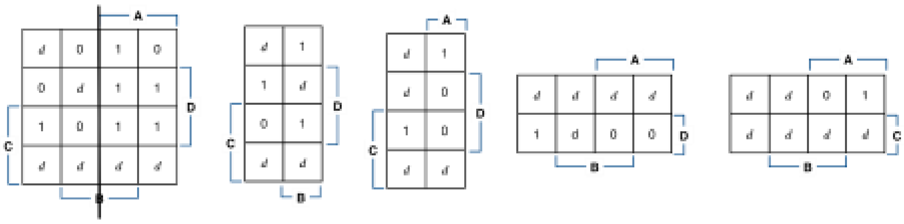
4.3. Whether the Function is Positive/Negative in Its Arguments

The function S is said to be positive in its argument X when

$$S(1_X) \geq S(0_X), \tag{23}$$

and if so, the dependence of S on the variable X can be expressed solely in terms of the uncomplemented literal X . The function S is said to be negative in X when

$$S(0_X) \geq S(1_X), \tag{24}$$



(a) S (b) $\partial S/\partial A$ (c) $\partial S/\partial B$ (d) $\partial S/\partial C$ (e) $\partial S/\partial D$

Fig. (9). (a) Map rotation or folding to obtain a map of the derivative w.r.t. A, (b-e) Maps for Boolean differences (derivatives) of S w.r.t. each of its four arguments.

and if so, the dependence of S on the variable S can be made in terms of the complemented literal $\overline{X}$ alone. For either case, the function S is a *unate* function and the variable X is a mono-form or a pure variable. Otherwise, the variable X is necessarily biform (appears both complemented and uncomplemented) in any formula for S and function S is *binate*. It is clear from Fig. 3(c) that S could be made positive in $A, C,$ and D because we could obtain $S = S_3$ in (12) to involve only the uncomplemented literals $A, C,$ and D, respectively. Equation (12) involves a mixed-polarity in the variable B , where both the complemented literal $\overline{B}$ and the uncomplemented one B appear. Note that we can neither

a. make S positive in B (because there is a 0 in cell 7 within the $B-$ domain, and a 1 in cell 3, the adjacent cell in the $\overline{B}-$domain).

b. make S negative in B (because there is a 0 in cell 8 within the $\overline{B}-$ domain, and a 1 in cell 12, the adjacent cell in the $B-$domain).

The failure to make S of a single-polarity in B means that S cannot become a unate function (one of fixed-polarity in each of its arguments). Unateness of the function is a sufficient but not necessary condition for it to be a threshold function [84, 96]. This means that there is still some possibility but no guarantee to express S as a threshold function, which we are going to explore in subsection 4.5.

4.4. Symmetry Considerations

The function S is partially symmetric in its arguments A and B if and only if

$$S / A\overline{B} = S / \overline{A}B, \quad (25)$$

where each of $S / A\overline{B}$ or $S / \overline{A}B$ is called a Boolean quotient or ratio [26], a context [82], a subfunction [35, 36], or a restriction[25], namely

$$S / A\overline{B} = S]_{A=1,B=0}, \quad (26)$$

$$S / \overline{A}B = S]_{A=0,B=1}. \quad (27)$$

Figure 10(a) shows that each of $S / A\overline{B}$ and $S / \overline{A}B$ is represented by one quarter of the original Karnaugh map in Fig. 3(c). The cell-wise comparison between these two quarter maps indicates the impossibility of making them equal since the ‘1’ in cell 11 is not equal to the ‘0’ in cell 7. Hence, S cannot be made partially symmetric in A and B. However, S can be made partially symmetric in B and C by selecting $d_2 = d_6 = 1$ and $d_3 = 0$. Likewise, S can be made partially symmetric in B and D by choosing $d_4 = d_5 = 1$. Similarly, S can be made symmetric in C and D by making $d_2 = d_4$, $d_3 = 0$, and $d_5 = d_6 = 1$. The net result is that S can be made symmetric in the three arguments $B, C,$ and D by choosing $d_2 = d_4 = d_5 = d_6 = 1$ and $d_3 = 0$. Figure 11 demonstrates a unique minimal coverage of S under these conditions as

$$S = A(B \vee C \vee D) \vee (\overline{B}CD \vee B\overline{C}D \vee BC\overline{D})\,, \tag{28}$$

The above expression (28) for S is symmetric in $B, C,$ and D in the sense that any two of these three variables can be interchanged without affecting the function.

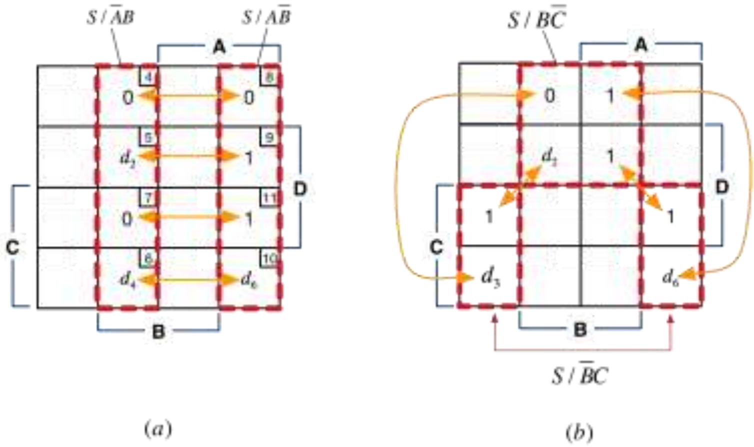


Fig. (10). (a) Comparisson of $S / A\overline{B}$ and $S / \overline{A}B$ showing that they cannot be made equal. (b) Comparison of $S / B\overline{C}$ and $S / \overline{B}C$ showing that they can be made equal

4.5. Possibility of Expressing the Function as a Threshold One

The function S can be made a threshold function if we find weights W_A, W_B, W_C, and W_D together with a threshold T such that [29, 91, 95, 96]

$$\{S = 1\} \Leftrightarrow \{W_A A + W_B B + W_C C + W_D D \geq T\}\,, \tag{29}$$

or equivalently

$$\{S = 0\} \Leftrightarrow \{W_A A + W_B B + W_C C + W_D D < T\}\,, \tag{30}$$

where the plus sign $(+)$ holds its converntional meaning of arithmatic addition. Figure 11 gives detailed inequalities needed to achive (29) or (30). The inequalities in cells (7) and (3) can be combined to give

$$W_B + W_C + W_D < T \leq W_C + W_D\,, \tag{31}$$

or

$$W_B < 0\,, \tag{32}$$

while those in cells 8 and 12 yield

$$W_A < T \leq W_A + W_B\,, \tag{33}$$

$$W_B \geq 0\,, \tag{34}$$

Note that (32) and (34) are in obvious contradiction. This means that S cannot be made threshold. However, many of its subfucnctions or contexts are threshold. For example, the subfunction S/B is preresented by the middle two columns in Fig. 12 and can be given by the non-unique threshold representation

$$\{S/B = 1\} \Leftrightarrow \{4W_A - W_C - W_D \geq 2\}\,, \tag{35}$$

and its sister subfunction $S/\overline{B}$ is represented by the first and last columns in Fig. 12, and can be given by the non-unique threshold representation.

$$\{S/\overline{B} = 1\} \Leftrightarrow \{4W_A + W_B + W_C \geq 2\}\,. \tag{36}$$

5. Discussion

The Karnaugh map can be used to successfully verify or recover many results obtained earlier by available software packages. For example, the four results displayed in Figs. 3-2 to 3-5 of Rihoux & *de* Meur [19] as Carroll diagrams (or Venn diagrams, as they are called therein) have simple and insightful Karnaugh map representations that are easy to obtain manually in a very short time. We used a running example dealing with the problem of war termination to demonstrate the utility of the Karnaugh map in Boolean analysis. Our methods are easily applicable in many other QCA problems such as those in Berg-Schlosser, *et al.* [97], Breuer [98], Delreux [99], Ishida, *et al.* [100], Marx & Duşa [101], Ragin [102], Valtonen, *et al.* [103] and Zeng [104].

The BA implemented herein via the Karnaugh map corresponds to the csQCA variant, in which the Karnaugh map represents a single effect (map output) in terms of several causes (map variables, presumably independent). The initial work of Flament [1, 2] and the subsequent methodology of Coincidence Analysis [30-33] might similarly be achieved by a Karnaugh map representing a function relating all pertinent (and maybe non-pertinent) variables [105]. The map variables initially stand on equal footing, as the researcher cannot distinguish some of them as causes and others as effects. It is the job of the analysis to make a distinction between effects and causes as well as express the effects in terms of causes. An ambitious sequel to the current work is to make a detailed comparison of the BA variants in Fig. 1 from a Karnaugh-map perspective. A possible outcome is to obtain a unified BA analysis that encompasses all the variants in one common setting. A relatively simpler task to follow the current work, is to extend this work to the largest problems in social and political sciences that require Boolean Analysis. As stated earlier, such problems involve up to 10 independent variables and hence cannot be handled by the conventional Karnaugh map. A useful plan is to use the variable-entered Karnaugh map [36, 37, 67-80] to analyze some such problems for a number of *independent* variables ranging from 7 to 10. There are several candidate papers to utilize in this endeavor [106-121]. Insight provided by the map is remarkably beneficial in achieving better understanding of the underlying problems.

6. Conclusions

This paper advocated more utilization of the Karnaugh map in the Boolean Analysis used in social, political, economic, managerial and engineering sciences. The paper made its point by a detailed exposition of a prominent problem in peace research, namely that of the causes of war termination. As a bonus, the paper demonstrated the Karnaugh map utility for achieving important purposes other than minimization.

The paper strived to review all existing concepts in ample details, while occasionally contributing some novel concepts and methods. To help the readers pursue the topic further in its original sources, we not only cited most recent papers, but we also referenced older publications that we felt are of paramount importance, seminal contribution or everlasting impact.

Acknowledgement

This work was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah. Therefore, the authors acknowledge, with thanks, the DSR for their financial and technical support. The authors are also indebted to two anonymous reviewers for their helpful comments.

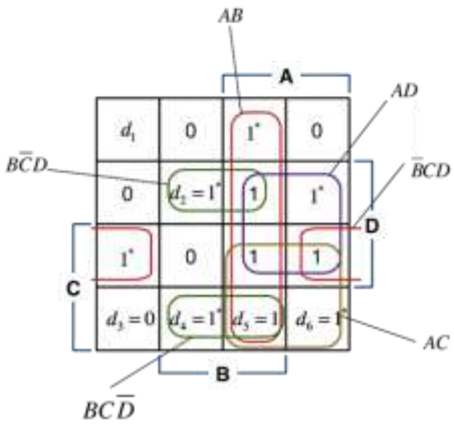


Fig. (11). The function made symmetric in B, C, and D

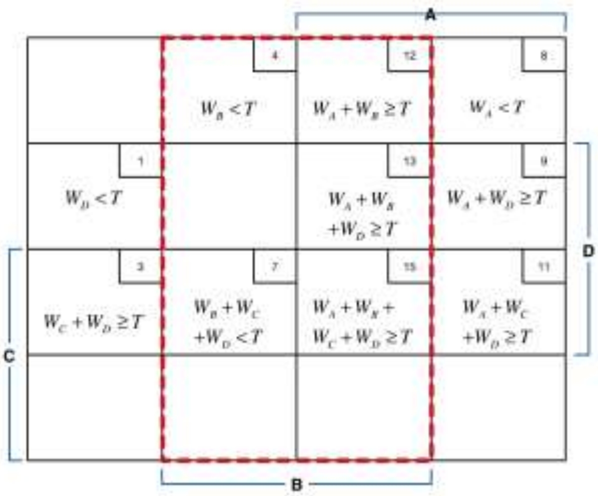


Fig. (12). Detailed inequalities needed to achieve (30) or (31).

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الانتفاع بخريطة كارنوه في التحليل البولاني: دراسة متى تضع الحرب أوزارها

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ملخص البحث. يعتبر التحليل البولاني (ح ب) تعميماً ذا ترتيب جزئي لتحليل رسوم القياس، وهو مفيد بصفة خاصة للبيانات التسلسلية وبصفة أعم لكافة البيانات التي يمكن وصفها كأرقام ثنائية، ومن ثم فهو خلاصة أو لب التحليل المقارن الوصفي (ح ق و). يدعو هذا البحث ويوضح كيفية الانتفاع بخريطة كارنوه بأنواعها المختلفة كوسيلة يدوية تصويرية في التحليل البولاني للمسائل الهندسية والاجتماعية والسياسية. تم التحليل لمؤشري الوجود والغياب لظاهرة معينة (أ) عند إهمال البواقي المنطقية (الحالات التي لا يُعبَأ بها) ، أو بمعنى أصح عند اعتبارها أصفاراً، (ب) عند تعيين قيم متعمدة مستقلة لهذه البواقي، و(ج) عند التمثيل الأمين لدالة الخرج المعينة كدالة محددة جزئياً يمثل جزؤها المؤكد الأسباب المتيقنة للظاهرة ويمثل جزؤها الذي لا يُعبأ به اتحاداً لأسبابها المحتملة غير المتيقنة. يقدم هذا البحث توسعات مختلفة مبتكرة للتحليل البولاني يتم فيها استعمال البواقي المنطقية في خريطة كارنوه في محاولات لجعل دالة الخرج موجبة أو سالبة في، أو مستقلة عن، أو متماثلة في بعض المدخلات، أو للحصول على تمثيل لها كدالة حدية، أو لجعلها تطابق فرضية مسبقة. يتم شرح العلاقة بين التحليل البولاني واثنتين من تفريعاته هما التحليل المقارن الوصفي ذو المجموعات الجاسئة (ح ق وج ج)، وتحليل التطابق (ح ط).

الكلمات المفتاحية: التحليل البولاني، خريطة كارنوه، البواقي المنطقية (الحالات التي لا يُعبَأ بها)، إنهاء الحروب، التصغير الأعظمي، توسعات مبتكرة، التحليل المقارن الوصفي ذو المجموعات الجاسئة (ح ق وج ج)، تحليل التطابق (ح ط).