

## Controllability of fractional noninstantaneous impulsive integrodifferential stochastic delay system

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The paper concerned with the controllability of nonlinear fractional noninstantaneous (NI) impulsive integrodifferential stochastic delay system (ISDS). Some sufficient conditions for the controllability of fractional NI impulsive ISDS have been derived by the new approach of measure of noncompactness in finite dimensional space. This NI impulsive ISDS is more reliable for the evolution process in pharmacotherapy. By using Mönch fixed point theorem, existence results have been proved. The result is new in the finite dimensional setting with NI impulse.

**Keywords:** delay differential system; fractional calculus; Mönch fixed point theorem; noninstantaneous impulse; stochastic differential calculus..

### 1. Introduction

In the real-world phenomena, stochastic models explain the functions of intrinsic noise, uncertainty, extrinsic noise and variability in the system and have attracted the research communities in recent days. Stochastic model is more realistic and challenging one when compared with deterministic model. Due to random environment noises, deterministic models often fluctuate, so it is apt to move to stochastic systems. Stochastic control is a developing field of control hypothesis that deals with the existence of uncertainty of the integrodifferential stochastic system described in Ahmed & El-Borai (2018); Balasubramaniam *et al.* (2018).

Generally, in the real-life modelling systems the fleeting changes in the state can be characterized through impulsive equations. The theory of impulsive differential equations (DEs) have been established by Lakshmikanthan *et al.* (1989). The fractional impulsive equations consist of two components of solutions via continuous and discontinuous, which describe the fleeting/sudden changes in the system.

The instantaneous impulse reaction may not describe the dynamics of evolution processes in pharmacotherapy, for instance taking into account the process the haemodynamic equilibrium of an individual. In case of sublimation of low or high levels of glucose, one can prescribe some intravenous drugs injecting into the body (insulin). The introduction of the drugs within the blood-stream and therefore the consequent absorption of the body are gradual and continuous processes. Thus, we don't expect to use the quality of the impulsive conditions to write down this process. In this situation, a new impulsive action occurs, which starts at an arbitrary fixed point and it will be active on a finite time interval. Concurrently, in many engineering and other disciplines

noninteger order DEs play a major role to develop the noninstantaneous (NI) models in the fields, including biological sciences, chemical sciences, physical sciences, medical sciences, engineering, industries etc. Many of the researchers have studied the fractional order instantaneous impulsive differential system, see Wang *et al.* (2011, 2014); Zhang *et al.* (2010). Recently NI impulsive differential systems have more applications and become popular amongst the researchers, see Abbas & Benchohra (2015); Abbas *et al.* (2015); Agarwal *et al.* (2016, 2017, 2017a); Gautam & Dabas (2015); Pandey *et al.* (2014); Saravananumar & Balasubramaniam (2020); Wang *et al.* (2016); Wang & Lin (2015).

Controllability plays a vital role in qualitative behaviour of the dynamical system. Sufficient conditions for the controllability of nonlinear DEs with NI impulses have been studied by Kumar *et al.* (2018). The qualitative behaviours have been extracted and extended to the fractional dynamical systems defined by Caputo, Riemann–Liouville (R-L) and Hilfer sense, for more details one can see the literature (Xexue & Jigen, 2011; Kilbas *et al.*, 2006; Kolmanovskii & Mysyshkis, 1992; Samko *et al.*, 1993).

Motivated by the above, we study the controllability of fractional order NI impulsive ISDS in finite dimensional setting. Existence of NI impulse with or without stochastic results have been derived using semigroup theory approach, see Feckan *et al.* (2014); Pandey *et al.* (2014). But there is no literature available to focus on NI impulsive ISDS in finite dimensional space. Controllability result is obtained from the solution represented in Mittag–Leffler (M-L) functions for different intervals by using fractional calculus, Grammian matrix and Mönch fixed point theorem.

In this paper, we derive the sufficient conditions for the controllability of the following nonlinear fractional NI impulsive ISDS described by

$$\begin{aligned} {}^C\mathcal{D}^\alpha x(t) &= \mathcal{A}x(t) + \mathcal{B}u(t) + f(t, x(t), x(t-h)) \\ &\quad + \int_0^t \Delta(s, x(s), x(s-h))dw(s), \quad t \in (s_j, t_{j+1}], \quad j = 0, 1, 2, \dots, m \\ x(t) &= I_j(t, x(t)), \quad t \in (t_j, s_j] \quad j = 1, 2, \dots, m \\ x(t) &= \phi(t) \in \mathcal{B}, \quad t \in [-h, 0] \end{aligned} \tag{1}$$

where  $\alpha \in (0, 1)$ ,  $x \in \mathbb{R}^n$  is the vector describing the NI state of the stochastic system,  $u(t) \in \mathbb{R}^m$  is a control input to the stochastic dynamical system,  $\mathcal{A}$  and  $\mathcal{B}$  are  $n \times n$ ,  $n \times m$  matrices. Also,  $u \in \mathcal{U}_{ad}$  an admissible control function, which is quadratically integrable and  $\mathcal{A}_t$ -measurable processed and  $w$  is an  $n$ -dimensional Wiener process.

Let  $\phi$  is an  $\mathcal{A}_0$  measurable  $\mathbb{R}^n$ -valued stochastic variable and let  $\mathcal{B} = \{\phi : [-h, 0] \rightarrow L_2(\Omega, \mathbb{R}^n), \phi$  is continuous everywhere expect points  $s$  where  $\phi(s^-), \phi(s^+)$  satisfy  $\phi(s^-) = \phi(s^+)\}$ . The nonlinear functions  $f : J := [0, b] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,  $\Delta : J \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$  are measurable and bounded functions and  $I_j : (t_j, s_j] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  are measurable and bounded functions for  $j = 1, 2, \dots, m$  and denote NI impulses and let  $J_j = (t_j, t_{j+1}]$ ,  $j = 1, 2, \dots, m$ , such that the impulse times satisfy  $0 = t_0 = s_0 < t_1 \leq s_1 < t_2 \dots < t_m \leq s_m < t_{m+1} = b$  be prefixed numbers. Also,  $x(t_j^-)$  and  $x(t_j^+)$  denote right and left hand limits, respectively, for  $x(t)$  at  $t = t_j$ ,  $x_t \in \mathcal{B}$  satisfies  $x_t(s) = x(t+s)$ ,  $s \in [-h, 0]$  is a state history.

The above model can describe the circumstance concerning the hemodynamical harmony of an individual, see Yan & Yang (2020). On account of a decompensation (for instance, high or low degrees of glucose) physician recommends some intravenous medications (insulin). Since the presentation of the medications in the circulatory system and the ensuing assimilation in the body are progressive and persistent cycles, one need to decipher the above circumstance as an incautious activity, which begins unexpectedly and remains dynamic on a limited time stretch.

The results are innovative in the following directions

- (i) In the finite dimensional space, no one derived the controllability results for fractional NI impulsive ISDS.
- (ii) The system (1) is an abstract model for studying the dynamics of periodic evolution processes in pharmacotherapy.
- (iii) Sufficient conditions are derived for controllability result by using Mönch fixed point theorem by constructing solutions in terms of M-L function.

The paper is exhibited as follows: In Section 2, some basic definitions on fractional calculus and few lemmas are provided. In Section 3, solution representation of fractional NI impulsive ISDS and the controllability results are demonstrated with the aid of a controllability Grammian matrix defined in the sense of M-L matrix function. Some sufficient conditions are derived in Section 4, for the system (1) to be controllable by utilizing the Mönch fixed point theorem. A computational example demonstrates for the validity of theoretical results in Section 5. In Section 6, conclusion is drawn.

#### Notations:

Let  $(\Omega, \mathcal{A}, \mathcal{P})$  be the complete probability space with a probability measure  $\mathcal{P}$  on  $\Omega$ . Let  $\{\mathcal{A}_t | t \in J\}$  is the filtration generated by  $\{w(s) : 0 \leq s \leq t\}$  defined on the probability space  $(\Omega, \mathcal{A}, \mathcal{P})$ . Let  $L_2(\Omega, \mathcal{A}_t, \mathcal{P}, \mathbb{R}^n) = L_2(\Omega, \mathbb{R}^n)$  – be the space of all  $\mathcal{A}_t$ - measurable square integrable random variables with values in  $\mathbb{R}^n$ .

Let  $\mathbb{C}(J, L_2(\Omega, \mathbb{R}^n))$  simply denoted by  $\mathbb{C}_n(J)$ , the Banach space of continuous maps from  $J$  into  $L_2(\Omega, \mathbb{R}^n)$ . Define norm of  $\phi$  on  $\mathcal{B}$  as  $\|\phi\|_{\mathcal{B}} = \sup_{s \in [-h, 0]} \{\mathbb{E}\|\phi(s)\|^2\}^{\frac{1}{2}}$

$$\begin{aligned} \mathcal{B}_b &= PC(J, L_2(\Omega, \mathbb{R}^n)) \\ &= \left\{ x : J \rightarrow L_2(\Omega, \mathbb{R}^n), x/J_j \in C(J_j, L_2(\Omega, \mathbb{R}^n)), \text{ and there exist } x(t_j^-) \text{ and } x(t_j^+) \right. \\ &\quad \left. \text{with } x(t_j) = x(t_j^-), x_0 = \phi(t) \in \mathcal{B}, j = 1, 2, \dots, m \right\}, \end{aligned}$$

where  $x/J_j$  is the restriction of  $x$  to  $J_j$ ,  $\mathcal{B}$  with  $\|\cdot\| = \|x\|_{\mathcal{B}_b} = \sup_{t \in [-h, b]} \{\mathbb{E}\|x(t)\|^2\}^{\frac{1}{2}}$  is a Banach space.

For  $t \in J$  and  $x \in \mathcal{B}_b$  we have  $x_t \in \mathcal{B}$ ,  $\mathbb{E}$  denotes the mathematical expectation operator of a stochastic process with respect to the given probability measure  $\mathcal{P}$ .

## 2. Preliminaries

**DEFINITION 1.** [Podlubny \(1998\)](#) The R-L integral of order  $\alpha$  with lower limit *zero* for a function  $f : [0, \infty) \rightarrow \mathbb{R}$  is defined as

$$(\mathcal{I}_{0+}^{\alpha} f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \quad t > 0, \quad \alpha > 0,$$

provided the right side is point-wise defined on  $[0, \infty)$ , where  $\Gamma(\cdot)$  is the Gamma function.

**DEFINITION 2.** [Podlubny \(1998\)](#) The R-L derivative of order  $\alpha$  with the lower limit *zero* for a function  $f : [0, \infty) \rightarrow \mathbb{R}$  is represented as

$$(\mathcal{D}_{0+}^{\alpha} f)(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\alpha+1-n}} ds, \quad t > 0, \quad n-1 < \alpha < n.$$

**DEFINITION 3.** [Podlubny \(1998\)](#) The Caputo fractional derivative of order  $\alpha$  with the lower limit *zero* for a function  $f : [0, \infty) \rightarrow \mathbb{R}$  is represented as,

$$\begin{aligned} {}^C \mathcal{D}_{0+}^{\alpha} f(t) &= \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds \\ &= \mathcal{I}_{0+}^{n-\alpha} f^{(n)}(t), \quad t > 0, \quad 0 < n-1 < \alpha \leq n. \end{aligned}$$

**DEFINITION 4.** [Miller & Ross \(1993\)](#) A two-parameter function of the M-L type is defined by the series expansion

$$M_{\alpha,\beta}(z) = \sum_{p=0}^{\infty} \frac{z^p}{\Gamma(p\alpha + \beta)}, \quad z \in \mathbb{C}.$$

**LEMMA 1.** [Mao \(1997\)](#) Let  $p \geq 2$  and  $\Delta \in L_p(J, \mathbb{R}^{n \times m})$  such that

$$\mathbb{E} \int_0^b |\Delta(s)|^p ds < \infty.$$

Then

$$\mathbb{E} \left| \int_0^b \Delta(s) dw(s) \right|^p \leq \left( \frac{p(p-1)}{2} \right)^{\frac{p}{2}} b^{\frac{p-2}{2}} \mathbb{E} \int_0^b |\Delta(s)|^p ds.$$

To state Mönch fixed point theorem, we provide some of the preliminary results.

First, we introduce the Hausdorff measure of noncompactness (MNC)  $\beta(\cdot)$  defined on each bounded subset  $\mathcal{B}$  of Banach space  $\chi$  by  $\beta(\mathcal{B}) = \inf\{\epsilon > 0 : \mathcal{B} \text{ has a finite } \epsilon - \text{net in } \chi\}$ .

The properties of  $\beta(\cdot)$  are listed below

**LEMMA 2.** **Banas & Goebel (1980)** Let  $\chi$  be a Banach space and  $\mathcal{B}_1, \mathcal{B}_2 \subset \chi$  be bounded, then the following properties hold:

- (1)  $\mathcal{B}_1$  is precompact  $\iff \beta(\mathcal{B}_1) = 0$ ;
- (2)  $\beta(\mathcal{B}_1) = \beta(\bar{\mathcal{B}}_1) = \beta(\text{conv } \mathcal{B}_1)$  where  $\bar{\mathcal{B}}_1$  and  $\text{conv}(\mathcal{B}_1)$  are closure and the convex hull of  $\mathcal{B}_1$ , respectively;
- (3)  $\beta(\mathcal{B}_1) \leq \beta(\mathcal{B}_2)$  where  $\mathcal{B}_1 \subseteq \mathcal{B}_2$ ;
- (4)  $\beta(\mathcal{B}_1 + \mathcal{B}_2) \leq \beta(\mathcal{B}_1) + \beta(\mathcal{B}_2)$  where  $\mathcal{B}_1 + \mathcal{B}_2 = \{x + y : x \in \mathcal{B}_1, y \in \mathcal{B}_2\}$ ;
- (5)  $\beta(\mathcal{B}_1 \cup \mathcal{B}_2) = \max\{\beta(\mathcal{B}_1), \beta(\mathcal{B}_2)\}$ ;
- (6)  $\beta(\lambda \mathcal{B}_1) = |\lambda| \beta(\mathcal{B}_1)$  for any  $\lambda \in \mathbb{R}$ ;
- (7) If  $Z \subset C(J, \chi)$  is bounded then  $\beta(Z(t)) = \beta(Z)$  for all  $t \in J$ , where  $Z(t) = \{z(t) : z \in Z \subset \chi\}$ . Further, if  $Z$  is continuous on  $J$ , then  $t \rightarrow Z(t)$  is continuous on  $J$  and

$$\beta(Z) = \sup\{Z(t) : t \in J\};$$

- (8) If  $Z \subset C(J, \chi)$  is bounded and equicontinuous, then  $t \rightarrow \beta(Z(t))$  is continuous on  $J$  and  $\beta\left(\int_0^t Z(s)ds\right) \leq \int_0^t \beta(Z(s))ds$  for all  $t \in J$ , where  $\int_0^t Z(s)ds = \left\{ \int_0^t z(s)ds : z \in Z \right\}$ ;
- (9) Let  $\{z_n\}_{n=1}^\infty$  be a sequence of Bochner integrable functions from  $J$  into  $\chi$  with  $\|z_n(t)\| \leq \hat{m}(t)$  for almost all  $t \in J$  and every  $n \geq 1$ , where  $\hat{m}(t) \in L(J, \mathbb{R}^+)$ , then the function  $\psi(t) = \beta(\{y_n(s)\}_{n=1}^\infty) \in L(J, \mathbb{R}^+)$  and satisfies  $\beta\left(\left\{ \int_0^t y_n(s)ds : n \geq 1 \right\}\right) \leq 2 \int_0^t \psi(s)ds$ .

**LEMMA 3.** **Banas & Goebel (1980)** Let  $\chi$  be a Banach space, and let  $\mathcal{D} \subset PC(J, \chi)$  be bounded and equicontinuous. Then  $\beta(\mathcal{D}(t))$  is piecewise continuous on  $J$  and  $\beta(\mathcal{D}) = \sup_{t \in J} \beta(\mathcal{D}(t))$  where  $\mathcal{D}(t) = \{x(t) : x \in \mathcal{D}\}$ .

**LEMMA 4.** **Deng et al. (2018)** (Mönch fixed point theorem) Let  $\mathcal{D}$  is a closed convex subset of  $\chi$ ,  $0 \in \mathcal{D}$ . If the map  $\mathbb{H} : \mathcal{D} \rightarrow \chi$  is continuous and of Mönch type that is  $\mathbb{H}$  satisfies the property  $M \subset \mathcal{D}$ ,  $M$  is countable,  $M \subset \text{co}(\{0\} \cup \mathbb{H}(M)) \Rightarrow \bar{M}$  is compact, then  $\mathbb{H}$  has a fixed point in  $\mathcal{D}$ .

To prove the main part of the results, we consider the following assumptions for  $j = 1, 2, \dots, m$ .

For all  $x, y \in \mathbb{R}^n$  and  $t \in J$  the functions  $f, \Delta$  and  $I_j$  are measurable and bounded functions satisfy the following conditions

(A1): There exist some constants  $K_f, K_\Delta, K_j > 0$ , such that

- (i)  $E\|f(t, x(t), x(t-h)) - f(t, y(t), y(t-h))\|^2 \leq K_f \|x - y\|^2$
- (ii)  $E\|\Delta(t, x(t), x(t-h)) - \Delta(t, y(t), y(t-h))\|^2 \leq K_\Delta \|x - y\|^2$

$$(iii) \quad E\|I_j(t, x(t)) - I_j(t, y(t))\|^2 \leq K_j \|x - y\|^2.$$

(A2): There exist some constants  $\bar{K}_f, \bar{K}_\Delta, \bar{K}_j > 0$

$$(i) \quad E\|f(t, x(t), x(t-h))\|^2 \leq \bar{K}_f(1 + \|x\|^2)$$

$$(ii) \quad E\|\Delta(t, x(t), x(t-h))\|^2 \leq \bar{K}_\Delta(1 + \|x\|^2)$$

$$(iii) \quad E\|I_j(t, x(t))\|^2 \leq \bar{K}_j(1 + \|x\|^2).$$

(A3): There exist constants  $Q_f, Q_\Delta, Q_j > 0$ , such that for a countable subset  $\mathcal{D} = \{x^n\}_{n=1}^\infty \subset \mathcal{B}$ ,

$$(i) \quad \beta(f(t, x^n(t), x^n(t-h))) \leq Q_f \beta(\mathcal{D})$$

$$(i) \quad \beta(\Delta(t, x^n(t), x^n(t-h))) \leq Q_\Delta \beta(\mathcal{D})$$

$$(i) \quad \beta(I_j(t, x^n(t))) \leq Q_j \beta(\mathcal{D}).$$

(A4): The linear system of (1) is controllable on the interval  $J$ .

(A5):  $\Upsilon = \max_{1 \leq j \leq m} \{M^*, M_j^*, \bar{M}_j^*\} < 1$  where

$$\begin{aligned} M^* &= 4 \frac{t_1^\alpha}{\alpha} R_2 [Q_f + \sqrt{t_1} Q_\Delta] \\ M_j^* &= 2Q_j \\ \bar{M}_j^* &= 4 \left\{ \bar{R}_j Q_j + \frac{b^\alpha}{\alpha} R_j [Q_f + \sqrt{b} Q_\Delta] \right\}. \end{aligned}$$

For convenience, define the following constants, for  $j = 1, 2, \dots, m$

$$R_1 = \sup_{t \in t_1} \{E\|M_\alpha(\mathcal{A}t^\alpha)\|\}, \quad \bar{R}_j = \sup_{t \in (t_j, s_j]} \{E\|M_\alpha(\mathcal{A}(t-s_j)^\alpha)\|\},$$

$$R_2 = \sup_{t \in t_1} \{E\|M_{\alpha,\alpha}(\mathcal{A}(t-s)^\alpha)\|\}, \quad R_j = \sup_{t \in (s_j, t_{j+1}]} \{E\|M_{\alpha,\alpha}(\mathcal{A}(t-s)^\alpha)\|\},$$

$$\bar{l}_0 = \|\mathbb{W}_{[0,t_1]}^{-1}\|^2, \quad \bar{l}_j = \|\mathbb{W}_{(s_j, t_{j+1}]}^{-1}\|^2, \quad l = E\|x_{t_1}\|^2, \quad l_j = E\|x_{t_{j+1}}\|^2.$$

In addition, set

$$M_0 = 3 \frac{t_1^{2\alpha}}{\alpha} R_2^2 \left\{ \left( 1 + 2\|\mathcal{B}\|^2 \|\mathcal{B}^*\|^2 \bar{l}_0 R_2^4 \frac{t_1^{2\alpha}}{\alpha} \right) (K_f + t_1 K_\Delta) \right\},$$

$$M_j = 4 \left[ \bar{R}_j^2 K_j + R_j^2 \frac{b^{2\alpha}}{\alpha^2} \left\{ 1 + 3\|\mathcal{B}\|^2 \|\mathcal{B}^*\|^2 R_j^4 \bar{l}_j \frac{b^{2\alpha}}{\alpha^2} \right\} (K_f + b K_\Delta) \right], \quad j = 1, 2, \dots, m.$$

### 3. Solution representation

For  $t \in [0, t_1]$ , consider the following nonlinear fractional ISDS

$$\begin{aligned} {}^C\mathcal{D}^\alpha x(t) &= \mathcal{A}x(t) + \mathcal{B}u(t) + f(t, x(t), x(t-h)) + \int_0^t \Delta(s, x(s), x(s-h))dw(s), \quad t \in (0, t_1] \\ x(t) &= \phi(t), \quad t \in [-h, 0]. \end{aligned} \quad (2)$$

Taking the Laplace transform (LT) on (2), one can get

$$\begin{aligned} s^\alpha \hat{x}(s) - s^{\alpha-1}x(0) &= \mathcal{A}\hat{x}(s) + \mathcal{B}\hat{u}(s) + \hat{f}(s) + \hat{\Delta}(s) \\ (s^\alpha I - \mathcal{A})\hat{x}(s) &= s^{\alpha-1}\phi(0) + \mathcal{B}\hat{u}(s) + \hat{f}(s) + \hat{\Delta}(s) \\ \hat{x}(s) &= \frac{s^{\alpha-1}}{(s^\alpha I - \mathcal{A})}\phi(0) + \frac{1}{(s^\alpha I - \mathcal{A})}\mathcal{B}\hat{u}(s) + \frac{1}{(s^\alpha I - \mathcal{A})}\hat{f}(s) + \frac{1}{(s^\alpha I - \mathcal{A})}\hat{\Delta}(s) \end{aligned}$$

where  $I$  is an identity matrix.

Taking inverse LT and by M-L function, one can obtain the solution as

$$\begin{aligned} x(t) &= M_\alpha(\mathcal{A}t^\alpha)\phi(0) + \int_0^t (t-s)^{\alpha-1}M_{\alpha,\alpha}(\mathcal{A}(t-s)^\alpha) \left[ \mathcal{B}u(s) + f(s, x(s), x(s-h)) \right. \\ &\quad \left. + \int_0^s \Delta(\tau, x(\tau), x(\tau-h))dw(\tau) \right] ds, \end{aligned}$$

which is the solution for (2).

For  $t \in (t_1, s_1]$ ,  $x(t) = I_1(t, x(t_1))$ .

For  $t \in (s_1, t_2]$ , consider the system (1) as

$$\begin{aligned} {}^C\mathcal{D}^\alpha x(t) &= \mathcal{A}x(t) + \mathcal{B}u(t) + f(t, x(t), x(t-h)) + \int_0^t \Delta(s, x(s), x(s-h))dw(s), \quad t \in (s_1, t_2] \\ x(t) &= I_1(t, x(t)), \quad t \in (t_1, s_1]. \end{aligned} \quad (3)$$

Using the similar process as above by taking LT, then inverse LT by convolution, one can get the solution for (3) as

$$\begin{aligned} x(t) = & I_2(s_1, x(s_1))M_\alpha(\mathcal{A}(t-s_1)^\alpha) + \int_{s_1}^t (t-s)^{\alpha-1} M_{\alpha,\alpha}(\mathcal{A}(t-s)^\alpha) \left[ \mathcal{B}u(s) + f(s, x(s), x(s-h)) \right. \\ & \left. + \int_0^s \Delta(\tau, x(\tau), x(\tau-h)) dw(\tau) \right] ds. \end{aligned}$$

Proceeding like this, the system (1) for general  $t \in (s_j, t_{j+1}]$ ,  $j = 1, 2, \dots, m$  is described as

$$\begin{aligned} {}^C\mathcal{D}^\alpha x(t) = & \mathcal{A}x(t) + \mathcal{B}u(t) + f(t, x(t), x(t-h)) \\ & + \int_0^t \Delta(s, x(s), x(s-h)) dw(s), \quad t \in (s_j, t_{j+1}], \quad j = 1, 2, \dots, m \\ x(t) = & I_j(t, x(t)), \quad t \in (t_j, s_j], \quad j = 1, 2, \dots, m. \end{aligned} \tag{4}$$

Taking LT, inverse LT by convolution one can get the solution for (4) as

$$\begin{aligned} x(t) = & I_j(s_j, x(s_j))M_\alpha(\mathcal{A}(t-s_j)^\alpha) + \int_{s_j}^t (t-s)^{\alpha-1} M_{\alpha,\alpha}(\mathcal{A}(t-s)^\alpha) \left[ \mathcal{B}u(s) + f(s, x(s), x(s-h)) \right. \\ & \left. + \int_0^s \Delta(\tau, x(\tau), x(\tau-h)) dw(\tau) \right] ds. \end{aligned}$$

**DEFINITION 5.** **Mao (1997)** A  $\mathbb{R}^n$ -valued  $\mathcal{A}_t$ -adapted stochastic process  $\{x(t)\}_{t \in J}$  is called a solution of equation (1) for  $\{f(t, x(t), x(t-h))\} \in L^1(J; \mathbb{R}^n)$ ,  $\Delta(s, x(s), x(s-h)) \in L^2(J; \mathbb{R}^{n \times n})$  and  $I_j(t, x(t)) \in L^1((t_j, s_j]; \mathbb{R}^n)$  provided the following integral equation holds

$$x(t) = \begin{cases} \phi(t), & t \in [-h, 0] \\ M_\alpha(\mathcal{A}t^\alpha)\phi(0) + \int_0^t (t-s)^{\alpha-1} M_{\alpha,\alpha}(\mathcal{A}(t-s)^\alpha) [\mathcal{B}u(s) + f(s, x(s), x(s-h)) \\ \quad + \int_0^s \Delta(\tau, x(\tau), x(\tau-h)) dw(\tau)] ds, & t \in [0, t_1] \\ I_j(t, x(t)), & t \in (t_j, s_j], j = 1, 2, \dots, m \\ I_j(s_j, x(s_j)) M_\alpha(\mathcal{A}(t-s_j)^\alpha) + \int_{s_j}^t (t-s)^{\alpha-1} M_{\alpha,\alpha}(\mathcal{A}(t-s)^\alpha) [\mathcal{B}u(s) + f(s, x(s), x(s-h)) \\ \quad + \int_0^s \Delta(\tau, x(\tau), x(\tau-h)) dw(\tau)] ds, & t \in (s_j, t_{j+1}], j = 1, 2, \dots, m, \end{cases} \quad (5)$$

where the control function is given by

$$u_x(t) = \begin{cases} \mathcal{B}^* M_{\alpha,\alpha}(\mathcal{A}^*(t_1 - t)^\alpha) W_{[0,t_1]}^{-1} \left\{ x_{t_1} - M_\alpha(\mathcal{A}t_1^\alpha)\phi(0) - \int_0^{t_1} (t_1 - s)^{\alpha-1} M_{\alpha,\alpha}(\mathcal{A}(t_1 - s)^\alpha) \right. \\ \times \left. [f(s, x(s), x(s-h)) + \int_0^s \Delta(\tau, x(\tau), x(\tau-h)) dw(\tau)] ds \right\}, & t \in [0, t_1] \\ 0, & t \in (t_j, s_j], j = 1, 2, \dots, m \\ \mathcal{B}^* M_{\alpha,\alpha}(\mathcal{A}^*(t_{j+1} - t)^\alpha) W_{[s_j, t_{j+1}]}^{-1} \left\{ x_{t_{j+1}} - I_j(s_j, x(s_j)) M_\alpha(\mathcal{A}(t-s_j)^\alpha) - \int_{s_j}^{t_{j+1}} (t_{j+1} - s)^{\alpha-1} \right. \\ \times \left. M_{\alpha,\alpha}(\mathcal{A}(t_{j+1} - s)^\alpha) [f(s, x(s), x(s-h)) + \int_0^s \Delta(\tau, x(\tau), x(\tau-h)) dw(\tau)] ds \right\}, & t \in (s_j, t_{j+1}], j = 1, 2, \dots, m. \end{cases} \quad (6)$$

In order to show that the system (1) is controllable, one can verify blue that  $x(t) = \phi(t)$  on  $t \in [-h, 0]$  and  $x(b) = x_b$  in the remaining interval. By using the control function defined in (6) into (5) one can easily see that  $x(0) = \phi(0)$  and  $x(b) = x_b$  on time  $b$ , which means (1) is controllable.

**THEOREM 3.1. Mahmudov & Denker (2000)** The linear control system is controllable on  $J$  if the controllability Grammian matrices

$$W_{[s_j, t_{j+1}]} = \int_{s_j}^{t_{j+1}} (t_{j+1} - s)^{\alpha-1} M_{\alpha,\alpha}(\mathcal{A}(b-s)^\alpha) \mathcal{B} \mathcal{B}^* M_{\alpha,\alpha}(\mathcal{A}^*(b-s)^\alpha) ds \Big|_{\mathcal{A}} \text{ a.e., } j = 0, 1, 2, \dots, m,$$

are positive definite for some  $t_{m+1} = b > 0$ .

Proof: Since  $W$  are positive definite, then they are nonsingular, and hence,  $W^{-1}$  are well defined. Take the control function defined in (6).

For  $t \in [0, t_1]$ , we have

$$\begin{aligned}
x(t_1) &= M_\alpha(\mathcal{A}t_1^\alpha)\phi(0) + \int_0^{t_1} (t_1 - s)^{\alpha-1} M_{\alpha,\alpha}(\mathcal{A}(t_1 - s)^\alpha) \mathcal{B}\mathcal{B}^* M_{\alpha,\alpha}(\mathcal{A}^*(t_1 - s)^\alpha) W_{[0,t_1]}^{-1} \\
&\quad \times \left\{ x_{t_1} - M_\alpha(\mathcal{A}t_1^\alpha)\phi(0) - \int_0^{t_1} (t_1 - s)^{\alpha-1} M_{\alpha,\alpha}(\mathcal{A}(t_1 - s)^\alpha) \left[ f(s, x(s), x(s-h)) \right. \right. \\
&\quad \left. \left. + \int_0^s \Delta(\tau, x(\tau), x(\tau-h)) dw(\tau) \right] ds \right\} + \int_0^{t_1} (t_1 - s)^{\alpha-1} M_{\alpha,\alpha}(\mathcal{A}(t_1 - s)^\alpha) f(s, x(s), x(s-h)) ds \\
&\quad + \int_0^{t_1} (t_1 - s)^{\alpha-1} M_{\alpha,\alpha}(\mathcal{A}(t_1 - s)^\alpha) \left( \int_0^s \Delta(\tau, x(\tau), x(\tau-h)) dw(\tau) \right) ds \\
x(t_1) &= x_{t_1}.
\end{aligned}$$

For  $t \in (s_j, t_{j+1}], j = 1, 2, \dots, m$  we have

$$\begin{aligned}
x(t_{j+1}) &= I_j(s_j, x(s_j)) M_\alpha(\mathcal{A}(t - s_j)^\alpha) + \int_{s_j}^{t_{j+1}} (t_{j+1} - s)^{\alpha-1} M_{\alpha,\alpha}(\mathcal{A}(t_{j+1} - s)^\alpha) \mathcal{B}\mathcal{B}^* M_{\alpha,\alpha} \\
&\quad \times (\mathcal{A}^*(t_{j+1} - s)^\alpha) W_{[s_j, t_{j+1}]}^{-1} \left\{ x_{t_{j+1}} - I_j(t_j, x(s_j)) M_\alpha(\mathcal{A}(t_{j+1} - s)^\alpha) \right. \\
&\quad \left. - \int_{s_j}^{t_{j+1}} (t_{j+1} - s)^{\alpha-1} M_{\alpha,\alpha}(\mathcal{A}(t_{j+1} - s)^\alpha) \left[ f(s, x(s), x(s-h)) \right. \right. \\
&\quad \left. \left. + \int_0^s \Delta(\tau, x(\tau), x(\tau-h)) dw(\tau) \right] ds \right\} \\
&\quad + \int_{s_j}^{t_{j+1}} (t_{j+1} - s)^{\alpha-1} M_{\alpha,\alpha}(\mathcal{A}(t_{j+1} - s)^\alpha) f(s, x(s), x(s-h)) ds \\
&\quad + \int_{s_j}^{t_{j+1}} (t_{j+1} - s)^{\alpha-1} M_{\alpha,\alpha}(\mathcal{A}(t_{j+1} - s)^\alpha) \left( \int_0^s \Delta(\tau, x(\tau), x(\tau-h)) dw(\tau) \right) ds \\
x(t_{j+1}) &= x_{t_{j+1}}.
\end{aligned}$$

Thus, system (1) is controllable for  $t \in [0, b]$ .

On the other way, if it is not positive definite, for  $t \in [0, b]$ ,  $\exists$  a nonzero  $\mathcal{Y}$ , such that  $\mathcal{Y}^* W_{[s_j, b]} \mathcal{Y} = 0$  that is

$$\int_0^b \mathcal{Y}^* (b-s)^{\alpha-1} M_{\alpha,\alpha} (\mathcal{A}(b-s)^\alpha) \mathcal{B} \mathcal{B}^* M_{\alpha,\alpha} (\mathcal{A}^*(b-s)^\alpha) \mathcal{Y} ds = 0.$$

Then

$$\mathcal{Y}^* M_{\alpha,\alpha} (\mathcal{A}(b-s)^\alpha) \mathcal{B} = 0. \quad (7)$$

Since,  $\exists$  control inputs  $u_1(t)$  and  $u_2(t)$ , such that

$$\begin{aligned} x(b) &= I_j(s_j, x(s_j)) M_\alpha (\mathcal{A}(b-s_j)^\alpha) + \int_0^b (b-s)^{\alpha-1} M_{\alpha,\alpha} (\mathcal{A}(b-s)^\alpha) \left[ f(s, x(s), x(s-h)) \right. \\ &\quad \left. + \int_0^s \Delta(\tau, x(\tau), x(\tau-h)) dw(\tau) + \mathcal{B} u_1(s) \right] ds = 0 \end{aligned} \quad (8)$$

$$\begin{aligned} \mathcal{Y} &= I_j(s_j, x(s_j)) M_\alpha (\mathcal{A}(b-s_j)^\alpha) + \int_0^b (b-s)^{\alpha-1} M_{\alpha,\alpha} (\mathcal{A}(b-s)^\alpha) \left[ f(s, x(s), x(s-h)) \right. \\ &\quad \left. + \int_0^s \Delta(\tau, x(\tau), x(\tau-h)) dw(\tau) + \mathcal{B} u_2(s) \right] ds = 0. \end{aligned} \quad (9)$$

Combining (8) and (9), we get

$$\mathcal{Y} = \int_0^b (b-s)^{\alpha-1} M_{\alpha,\alpha} (\mathcal{A}(b-s)^\alpha) \mathcal{B} [u_1(s) - u_2(s)] ds = 0. \quad (10)$$

Multiplying  $\mathcal{Y}^*$  on both sides of (9), we have

$$\mathcal{Y}^* \mathcal{Y} = \int_0^b (b-s)^{\alpha-1} M_{\alpha,\alpha} (\mathcal{A}(b-s)^\alpha) \mathcal{Y}^* \mathcal{B} [u_1(s) - u_2(s)] ds = 0.$$

By (7),  $\mathcal{Y}^* M_{\alpha,\alpha} (\mathcal{A}(t_{j+1} - s)^\alpha) \mathcal{B} = 0$ , we obtain  $\mathcal{Y}^* \mathcal{Y} = 0$ . Thus,  $\mathcal{Y} = 0$ , which is a contradiction.

#### 4. Controllability of NI impulsive stochastic system

This section deals with the controllability results for the proposed system (1).

**THEOREM 4.1.** Assume that hypotheses (A1)–(A5) hold and the corresponding linear system of (1) is controllable on  $J$ . Then, the nonlinear fractional NI impulsive ISDS is controllable on  $J$  provided that

$$\Gamma = \max_{1 \leq j \leq m} \{M_0, K_j, M_j\} < 1.$$

#### Proof:

Define  $\mathcal{B}_r = \{x : x \in \mathcal{B}_b, \|x\|_{\mathcal{B}_b}^2 \leq r\}$  be a closed, bounded and convex subset of  $\mathcal{B}$ .

Define a nonlinear operator  $\mathbb{H} : \mathcal{B}_b \rightarrow \mathcal{B}_b$  as

$$(\mathbb{H}x)(t) = \begin{cases} \phi(t), & t \in [-h, 0] \\ M_\alpha(A t^\alpha) \phi(0) + \int_0^{t_1} (t_1 - s)^{\alpha-1} M_{\alpha,\alpha}(A(t_1 - s)^\alpha) \left[ \mathcal{B}u(s) + f(s, x(s), x(s-h)) \right. \\ \left. + \int_0^{t_1} \Delta(\tau, x(\tau), x(\tau-h)) dw(\tau) \right] ds, & t \in [0, t_1] \\ I_j(t, x(t)), & t \in (t_j, s_j], j = 1, 2, \dots, m \\ I_j(s_j, x(s_j)) M_\alpha(A(t - s_j)^\alpha) + \int_{s_j}^t (t - s)^{\alpha-1} M_{\alpha,\alpha}(A(t - s)^\alpha) \left[ \mathcal{B}u(s) + f(s, x(s), x(s-h)) \right. \\ \left. + \int_0^s \Delta(\tau, x(\tau), x(\tau-h)) dw(\tau) \right] ds, & t \in (s_j, t_{j+1}], j = 1, 2, \dots, m. \end{cases}$$

**Step 1:**  $\mathbb{H}$  maps bounded sets into bounded sets in  $\mathcal{B}_r$ .

We need to prove  $\|\mathbb{H}x\|^2 \leq N$ ,  $N > 0$  for  $t \in [0, t_1]$ ,  $t \in (t_j, s_j]$  and  $t \in (s_j, t_{j+1}]$ ,  $j = 1, 2, \dots, m$ .

For  $t \in [0, t_1]$

$$\begin{aligned} \mathbb{E}\|(\mathbb{H}x)(t)\|^2 &\leq 4 \left\{ \|M_\alpha(A t^\alpha) \mathbb{E}\|\phi(0)\|^2 + \left( \int_0^t (t-s)^{\alpha-1} ds \right)^2 \|M_{\alpha,\alpha}(A(t-s)^\alpha)\|^2 \right. \\ &\quad \times \left[ \mathbb{E}\|u(t)\|^2 \|\mathcal{B}\|^2 + \mathbb{E}\|f(t, x(t), x(t-h))\|^2 \right. \\ &\quad \left. \left. + \mathbb{E}\left\| \int_0^s \Delta(\tau, x(\tau), x(\tau-h)) dw(\tau) \right\|^2 \right] \right\}. \end{aligned}$$

Using (A2), Lemma 1 and other estimates one can compute for  $t \in [0, t_1]$

$$\begin{aligned}
E\|u(t)\|^2 &\leq 4\|\mathcal{B}^*\|^2 R_2^2 \bar{l}_0 \left\{ E\|x_{t_1}\|^2 + R_1^2 E\|\phi(0)\|^2 + \left( \int_0^{t_1} (t_1 - s)^{\alpha-1} ds \right)^2 R_2^2 \right. \\
&\quad \times \left. \left[ E\|f(t, x(t), x(t-h))\|^2 + E \left\| \int_0^s \Delta(\tau, x(\tau), x(\tau-h)) dw(\tau) \right\|^2 \right] \right\} \\
&\leq 4\|\mathcal{B}^*\|^2 R_2^2 \bar{l}_0 \left\{ l + R_1^2 \|\phi(0)\|^2 + \frac{t_1^{2\alpha}}{\alpha^2} R_2^2 \bar{K}_f (1 + \|x\|^2) + \frac{t_1^{2\alpha+1}}{\alpha^2} R_2^2 \bar{K}_\Delta (1 + \|x\|^2) \right\} \\
&\leq 4\|\mathcal{B}^*\|^2 R_2^2 \bar{l}_0 \left\{ l + R_1^2 \|\phi(0)\|^2 + \frac{t_1^{2\alpha}}{\alpha^2} R_2^2 (\bar{K}_f + t_1 \bar{K}_\Delta) (1 + \|x\|^2) \right\}.
\end{aligned}$$

Also, for  $t \in (s_j, t_{j+1}]$ ,  $j = 1, 2, \dots, m$

$$\begin{aligned}
E\|u(t)\|^2 &\leq 4\|\mathcal{B}^*\|^2 R_j^2 \bar{l}_j \left\{ E\|x_{t_{j+1}}\|^2 + \bar{R}_j^2 E\|I_j(t_j, x(s_j))\|^2 \right. \\
&\quad + \left( \int_{s_j}^{t_{j+1}} (t_{j+1} - s)^{\alpha-1} ds \right)^2 R_j^2 \left[ E\|f(t, x(t), x(t-h))\|^2 \right. \\
&\quad \left. \left. + E \left\| \int_0^s \Delta(\tau, x(\tau), x(\tau-h)) dw(\tau) \right\|^2 \right] \right\} \\
&\leq 4\|\mathcal{B}^*\|^2 R_j^2 \bar{l}_j \left\{ l_j + \bar{R}_j^2 \bar{K}_j (1 + \|x\|^2) + \frac{(t_{j+1} - s_j)^{2\alpha}}{\alpha^2} R_j^2 \bar{K}_f (1 + \|x\|^2) \right. \\
&\quad \left. + \frac{(t_{j+1} - s_j)^{2\alpha+1}}{\alpha^2} R_j^2 \bar{K}_\Delta (1 + \|x\|^2) \right\} \\
&\leq 4\|\mathcal{B}^*\|^2 R_j^2 \bar{l}_j \left\{ l_j + \left[ \bar{R}_j^2 \|\bar{K}_j\|^2 + \frac{(t_{j+1} - s_j)^{2\alpha}}{\alpha^2} R_j^2 (\bar{K}_f + (t_{j+1} - s_j) \bar{K}_\Delta) \right] \right. \\
&\quad \left. \times (1 + \|x\|^2) \right\}.
\end{aligned}$$

Therefore, for  $t \in [0, t_1]$

$$\begin{aligned}
E\|(\mathbb{H}x)\|^2 &\leq 4 \left\{ R_1^2 \|\phi(0)\|^2 + 4 \frac{t_1^{2\alpha}}{\alpha^2} R_2^4 \|\mathcal{B}\|^2 \|\mathcal{B}^*\|^2 \bar{l}_0 \left\{ l + R_1^2 \|\phi(0)\|^2 + \frac{t_1^{2\alpha}}{\alpha^2} R_2^2 \right. \right. \\
&\quad \times (\bar{K}_f + t_1 \bar{K}_\Delta) (1 + \|x\|^2) \left. \right\} + \frac{t_1^{2\alpha}}{\alpha^2} R_2^2 (\bar{K}_f + t_1 \bar{K}_\Delta) (1 + \|x\|^2) \left. \right\}
\end{aligned}$$

$$\begin{aligned}
\|(Hx)\|^2 &\leq \left(1 + 4\frac{t_1^{2\alpha}}{\alpha^2}R_2^4\|\mathcal{B}\|^2\|\mathcal{B}^*\|^2\bar{l}_0\right)R_1^2\|\phi(0)\|^2 + 16\frac{t_1^{2\alpha}}{\alpha^2}R_2^4\|\mathcal{B}\|^2\|\mathcal{B}^*\|^2\bar{l}_0l \\
&\quad \times \left(1 + 4\frac{t_1^{2\alpha}}{\alpha^2}R_2^4\|\mathcal{B}\|^2\|\mathcal{B}^*\|^2\bar{l}_0\right)R_2^2\frac{t_1^{2\alpha}}{\alpha^2}(\bar{K}_f + t_1\bar{K}_\Delta)(1+r) \\
&\cong N_0(\text{say}).
\end{aligned}$$

For  $t \in (t_j, s_j]$ ,  $j = 1, 2, \dots, m$

$$\begin{aligned}
E\|(\mathbb{H}x)\|^2 &\leq \|I_j(t, x(t))\|^2 \\
&\leq \bar{K}_j(1 + \|x\|^2). \\
\|(\mathbb{H}x)\|^2 &\leq \bar{K}_j(1 + r) \\
&\cong N_j(\text{say}).
\end{aligned}$$

For  $t \in (s_j, t_{j+1}]$ ,  $j = 1, 2, \dots, m$ , we have

$$\begin{aligned}
E\|(\mathbb{H}x)\|^2 &\leq 4\left\{\bar{R}_j^2\bar{K}_j(1 + \|x\|^2) + 4\frac{(t_{j+1} - s_j)^{2\alpha}}{\alpha^2}R_j^4\|\mathcal{B}\|^2\|\mathcal{B}^*\|^2\bar{l}_j\left\{l_j + \bar{R}_j^2\bar{K}_j(1 + \|x\|^2)\right.\right. \\
&\quad \left.\left.+ \frac{(t_{j+1} - s_j)^{2\alpha}}{\alpha^2}R_j^2(\bar{K}_f + (t_{j+1} - s_j)\bar{K}_\Delta)(1 + \|x\|^2)\right\}\right. \\
&\quad \left.\left.+ \frac{(t_{j+1} - s_j)^{2\alpha}}{\alpha^2}R_j^2(\bar{K}_f + (t_{j+1} - s_j)\bar{K}_\Delta)(1 + \|x\|^2)\right\}\right. \\
&\quad \left.\left.\|(\mathbb{H}x)\|^2 \leq 16\frac{(t_{j+1} - s_j)^{2\alpha}}{\alpha^2}R_j^4\|\mathcal{B}\|^2\|\mathcal{B}^*\|^2\bar{l}_j l_j\right.\right. \\
&\quad \left.\left.+ \left(1 + 4\frac{(t_{j+1} - s_j)^{2\alpha}}{\alpha^2}R_j^4\|\mathcal{B}\|^2\|\mathcal{B}^*\|^2\bar{l}_j\right)\left[\bar{R}_j^2\bar{K}_j\right.\right.\right. \\
&\quad \left.\left.\left.+ \frac{(t_{j+1} - s_j)^{2\alpha}}{\alpha^2}R_j^2(\bar{K}_f + (t_{j+1} - s_j)\bar{K}_\Delta)\right](1+r)\right.\right. \\
&\leq 16\frac{b^{2\alpha}}{\alpha^2}R_j^4\|\mathcal{B}\|^2\|\mathcal{B}^*\|^2\bar{l}_j l_j + \left(1 + 4\frac{b^{2\alpha}}{\alpha^2}R_j^4\|\mathcal{B}\|^2\|\mathcal{B}^*\|^2\bar{l}_j\right) \\
&\quad \times \left[\bar{R}_j\bar{K}_j + \frac{b^{2\alpha}}{\alpha^2}R_j^2(\bar{K}_f + b\bar{K}_\Delta)\right](1+r) \\
&\cong \bar{N}_j(\text{say}).
\end{aligned}$$

Take  $N = \max_{1 \leq j \leq m} \{N_0, N_j, \bar{N}_j\}$ ,  $j = 1, 2, \dots, m$ , then for each  $x \in \mathcal{B}_r$ , we have  $\|\mathbb{H}x\|^2 \leq N$ .

**Step 2:**  $\mathbb{H}$  is continuous on  $\mathcal{B}_r$ .

Let the sequence  $\{x^n(t)\}_{n=1}^{\infty} \subset \mathcal{B}_r$  with  $x^n \rightarrow x$  as  $n \rightarrow \infty$  in  $\mathcal{B}_r$ . Then, the following are true due to the continuity of  $f$ ,  $\Delta$  and  $I_j$ ,  $j = 1, 2, \dots, m$  as  $n \rightarrow \infty$

$$(A5): f(t, x^n(t), x^n(t-h)) \rightarrow f(t, x(t), x(t-h))$$

$$\Delta(t, x^n(t), x^n(t-h)) \rightarrow \Delta(t, x(t), x(t-h))$$

$$I_j(t, x^n(t)) \rightarrow I_j(t, x(t)).$$

Using Lebesgue dominated convergence theorem, one can have

$$\begin{aligned} & \left\{ 3 \left\{ \mathbb{E} \left\| \int_0^t (t-s)^{\alpha-1} M_{\alpha,\alpha}(\mathcal{A}(t-s)^\alpha) \mathcal{B}[u_{x^n}(s) - u_x(s)] ds \right\|^2 \right. \right. \\ & \quad + \mathbb{E} \left\| \int_0^t (t-s)^{\alpha-1} M_{\alpha,\alpha}(\mathcal{A}(t-s)^\alpha) \right. \\ & \quad \times [f(s, x^n(s), x^n(s-h)) - f(s, x(s), x(s-h))] ds \left. \right\|^2 \\ & \quad + \mathbb{E} \left\| \int_0^t (t-s)^{\alpha-1} M_{\alpha,\alpha}(\mathcal{A}(t-s)^\alpha) \left[ \int_0^s (\Delta(\tau, x^n(\tau), x^n(\tau-h)) \right. \right. \\ & \quad \left. \left. - \Delta(\tau, x(\tau), x(\tau-h)) \right] dw(\tau) \right] ds \Big\}, \quad t \in [0, t_1] \\ & \mathbb{E} \|(\mathbb{H}x^n)(t) - (\mathbb{H}x)(t)\|^2 = \begin{cases} \mathbb{E} \|I_j(t, x^n(t)) - I_j(t, x(t))\|^2, & t \in (t_j, s_j], j = 1, 2, \dots, m \\ 4 \left\{ \mathbb{E} \|I_j(s_j, x^n(s_j)) - I_j(s_j, x(s_j))\|^2 \|M_\alpha(\mathcal{A}(t-s_j)^\alpha)\|^2 \right. \\ \left. \mathbb{E} \left\| \int_{s_j}^t (t-s)^{\alpha-1} M_{\alpha,\alpha}(\mathcal{A}(t-s)^\alpha) \mathcal{B}[u_{x^n}(s) - u_x(s)] ds \right\|^2 \right. \\ \left. + \mathbb{E} \left\| \int_{s_j}^t (t-s)^{\alpha-1} M_{\alpha,\alpha}(\mathcal{A}(t-s)^\alpha) \right. \right. \\ \left. \left. \times [f(s, x^n(s), x^n(s-h)) - f(s, x(s), x(s-h))] ds \right\|^2 \right. \\ \left. + \mathbb{E} \left\| \int_{s_j}^t (t-s)^{\alpha-1} M_{\alpha,\alpha}(\mathcal{A}(t-s)^\alpha) \left[ \int_0^s (\Delta(\tau, x^n(\tau), x^n(\tau-h)) \right. \right. \right. \\ \left. \left. \left. - \Delta(\tau, x(\tau), x(\tau-h)) \right] dw(\tau) \right] ds \right\}, \quad t \in (s_j, t_{j+1}], j = 1, 2, \dots, m, \end{cases} \end{aligned}$$

which tends to zero as  $n \rightarrow \infty$ . Thus,  $\mathbb{H}$  is a continuous operator on  $\mathcal{B}_r$ .

**Step 3:**  $\mathbb{H}$  is equicontinuous.

Let  $\rho_1, \rho_2 \in [0, t_1]$  with  $0 < \rho_1 < \rho_2 \leq t_1$ . Then,

$$\mathbb{E} \|(\mathbb{H}x)(\rho_2) - (\mathbb{H}x)(\rho_1)\|^2$$

$$\begin{aligned} & \leq 7 \left\{ \mathbb{E} \|M_\alpha(\mathcal{A}\rho_2^\alpha) - M_\alpha(\mathcal{A}\rho_1^\alpha)\|^2 \|\phi(0)\|^2 + \mathbb{E} \left\| \int_{\rho_1}^{\rho_2} (\rho_2-s)^{\alpha-1} M_{\alpha,\alpha}(\mathcal{A}(\rho_2-s)^\alpha) \mathcal{B}u_x(s) ds \right\|^2 \right. \\ & \quad + \mathbb{E} \left\| \int_0^{\rho_1} [(\rho_2-s)^{\alpha-1} M_{\alpha,\alpha}(\mathcal{A}(\rho_2-s)^\alpha) - (\rho_1-s)^{\alpha-1} M_{\alpha,\alpha}(\mathcal{A}(\rho_1-s)^\alpha)] \mathcal{B}u_x(s) ds \right\|^2 \end{aligned}$$

$$\begin{aligned}
& + \mathbb{E} \left\| \int_{\rho_1}^{\rho_2} (\rho_2 - s)^{\alpha-1} M_{\alpha,\alpha}(\mathcal{A}(\rho_2 - s)^\alpha) f(s, x(s), x(s-h)) ds \right\|^2 \\
& + \mathbb{E} \left\| \int_0^{\rho_1} [(\rho_2 - s)^{\alpha-1} M_{\alpha,\alpha}(\mathcal{A}(\rho_2 - s)^\alpha) - (\rho_1 - s)^{\alpha-1} M_{\alpha,\alpha}(\mathcal{A}(\rho_1 - s)^\alpha)] f(s, x(s), x(s-h)) ds \right\|^2 \\
& + \mathbb{E} \left\| \int_{\rho_1}^{\rho_2} (\rho_2 - s)^{\alpha-1} M_{\alpha,\alpha}(\mathcal{A}(\rho_2 - s)^\alpha) \left( \int_0^s \Delta(\tau, x(\tau), x(\tau-h)) dw(\tau) \right) ds \right\|^2 \\
& + \mathbb{E} \left\| \int_0^{\rho_1} [(\rho_2 - s)^{\alpha-1} M_{\alpha,\alpha}(\mathcal{A}(\rho_2 - s)^\alpha) - (\rho_1 - s)^{\alpha-1} M_{\alpha,\alpha}(\mathcal{A}(\rho_1 - s)^\alpha)] \right. \\
& \times \left. \left( \int_0^s \Delta(\tau, x(\tau), x(\tau-h)) dw(\tau) \right) ds \right\|^2.
\end{aligned}$$

Thus,

$$\begin{aligned}
& \|(\mathbb{H}x)(\rho_2) - (\mathbb{H}x)(\rho_1)\|^2 \\
& \leq 7 \left\{ \|M_\alpha(\mathcal{A}\rho_2^\alpha) - M_\alpha(\mathcal{A}\rho_1^\alpha)\|^2 \|\phi(0)\|^2 + \frac{(\rho_2 - \rho_1)^\alpha}{\alpha} \int_{\rho_1}^{\rho_2} \|(\rho_2 - s)^{\alpha-1} M_{\alpha,\alpha}(\mathcal{A}(\rho_2 - s)^\alpha)\|^2 \right. \\
& \times \|\mathcal{B}\|^2 \|u_x(s)\|^2 ds + \rho_1 \int_0^{\rho_1} \|[(\rho_2 - s)^{\alpha-1} M_{\alpha,\alpha}(\mathcal{A}(\rho_2 - s)^\alpha) - (\rho_1 - s)^{\alpha-1} M_{\alpha,\alpha}(\mathcal{A}(\rho_1 - s)^\alpha)]\|^2 \\
& \times \|\mathcal{B}\|^2 \|u_x(s)\|^2 ds + \frac{(\rho_2 - \rho_1)^\alpha}{\alpha} \int_{\rho_1}^{\rho_2} \|(\rho_2 - s)^{\alpha-1} M_{\alpha,\alpha}(\mathcal{A}(\rho_2 - s)^\alpha)\|^2 \|f(s, x(s), x(s-h))\|^2 ds \\
& + \rho_1 \int_0^{\rho_1} \|[(\rho_2 - s)^{\alpha-1} M_{\alpha,\alpha}(\mathcal{A}(\rho_2 - s)^\alpha) - (\rho_1 - s)^{\alpha-1} M_{\alpha,\alpha}(\mathcal{A}(\rho_1 - s)^\alpha)]\|^2 \\
& \times \|f(s, x(s), x(s-h))\|^2 ds + \frac{(\rho_2 - \rho_1)^\alpha}{\alpha} \int_{\rho_1}^{\rho_2} \|(\rho_2 - s)^{\alpha-1} M_{\alpha,\alpha}(\mathcal{A}(\rho_2 - s)^\alpha)\|^2 \\
& \times \left\| \int_0^s \Delta(\tau, x(\tau), x(\tau-h)) dw(\tau) \right\|^2 ds
\end{aligned}$$

$$\begin{aligned}
& + \rho_1 \int_0^{\rho_1} \| [(\rho_2 - s)^{\alpha-1} M_{\alpha,\alpha}(\mathcal{A}(\rho_2 - s)^\alpha) - (\rho_1 - s)^{\alpha-1} M_{\alpha,\alpha}(\mathcal{A}(\rho_1 - s)^\alpha)] \|^2 \\
& \times \left\| \int_0^s \Delta(\tau, x(\tau), x(\tau - h)) dw(\tau) \right\|^2 ds \}
\end{aligned}$$

as  $\rho_2 \rightarrow \rho_1$ , the RHS of the above inequality tends to zero.

For  $\rho_1, \rho_2 \in (t_j, s_j], j = 1, 2, \dots, m, t_j < \rho_1 < \rho_2 \leq s_j$ , one can estimate

$$\begin{aligned}
E\|(\mathbb{H}x)(\rho_2) - (\mathbb{H}x)(\rho_1)\|^2 & \leq E\|I_j(\rho_2, x(t)) - I_j(\rho_1, x(t))\|^2 \\
\text{Thus, } \|(\mathbb{H}x)(\rho_2) - (\mathbb{H}x)(\rho_1)\|^2 & \leq \|I_j(\rho_2, x(t)) - I_j(\rho_1, x(t))\|^2
\end{aligned}$$

as  $\rho_2 \rightarrow \rho_1$ , the RHS  $\rightarrow 0$ .

Similarly for  $\rho_1, \rho_2 \in (s_j, t_{j+1}], j = 1, 2, \dots, m, s_j < \rho_1 < \rho_2 \leq t_{j+1}$ , one can compute the following estimate

$$E\|(\mathbb{H}x)(\rho_2) - (\mathbb{H}x)(\rho_1)\|^2$$

$$\begin{aligned}
& \leq 7 \left\{ E\|I_j(s_j, x(s_j))M_\alpha(\mathcal{A}(\rho_2 - s)^\alpha) - I_j(s_j, x(s_j))M_\alpha(\mathcal{A}(\rho_1 - s)^\alpha)\|^2 \right. \\
& \quad \left. + E\left\| \int_{\rho_1}^{\rho_2} (\rho_2 - s)^{\alpha-1} M_{\alpha,\alpha}(\mathcal{A}(\rho_2 - s)^\alpha) Bu_x(s) ds \right\|^2 \right. \\
& \quad \left. + E\left\| \int_{s_j}^{\rho_1} [(\rho_2 - s)^{\alpha-1} M_{\alpha,\alpha}(\mathcal{A}(\rho_2 - s)^\alpha) - (\rho_1 - s)^{\alpha-1} M_{\alpha,\alpha}(\mathcal{A}(\rho_1 - s)^\alpha)] Bu_x(s) ds \right\|^2 \right. \\
& \quad \left. + E\left\| \int_{\rho_1}^{\rho_2} (\rho_2 - s)^{\alpha-1} M_{\alpha,\alpha}(\mathcal{A}(\rho_2 - s)^\alpha) f(s, x(s), x(s-h)) ds \right\|^2 \right. \\
& \quad \left. + E\left\| \int_{s_j}^{\rho_1} [(\rho_2 - s)^{\alpha-1} M_{\alpha,\alpha}(\mathcal{A}(\rho_2 - s)^\alpha) - (\rho_1 - s)^{\alpha-1} \right. \right. \\
& \quad \left. \left. \times M_{\alpha,\alpha}(\mathcal{A}(\rho_1 - s)^\alpha)] f(s, x(s), x(s-h)) ds \right\|^2 \right\}
\end{aligned}$$

$$\begin{aligned}
& + \mathbb{E} \left\| \int_{\rho_1}^{\rho_2} (\rho_2 - s)^{\alpha-1} M_{\alpha,\alpha}(\mathcal{A}(\rho_2 - s)^\alpha) \left( \int_0^s \Delta(\tau, x(\tau), x(\tau - h)) dw(\tau) \right) ds \right\|^2 \\
& + \mathbb{E} \left\| \int_{s_j}^{\rho_1} [(\rho_2 - s)^{\alpha-1} M_{\alpha,\alpha}(\mathcal{A}(\rho_2 - s)^\alpha) - (\rho_1 - s)^{\alpha-1} M_{\alpha,\alpha}(\mathcal{A}(\rho_1 - s)^\alpha)] \right. \\
& \times \left. \left( \int_0^s \Delta(\tau, x(\tau), x(\tau - h)) dw(\tau) \right) ds \right\|^2.
\end{aligned}$$

Thus,

$$\begin{aligned}
\|(\mathbb{H}x)(\rho_2) - (\mathbb{H}x)(\rho_1)\|^2 & \leq 7 \left\{ I_j(s_j, x(s_j)) M_\alpha(\mathcal{A}(\rho_2 - s)^\alpha) - I_j(s_j, x(s_j)) M_\alpha(\mathcal{A}(\rho_1 - s)^\alpha) \right\|^2 \\
& + \frac{(\rho_2 - \rho_1)^\alpha}{\alpha} \int_{\rho_1}^{\rho_2} \|(\rho_2 - s)^{\alpha-1} M_{\alpha,\alpha}(\mathcal{A}(\rho_2 - s)^\alpha)\|^2 \|\mathcal{B}\|^2 \|u_x(s)\|^2 ds \\
& + (\rho_1 - s_j) \int_{s_j}^{\rho_1} \|[(\rho_2 - s)^{\alpha-1} M_{\alpha,\alpha}(\mathcal{A}(\rho_2 - s)^\alpha) - (\rho_1 - s)^{\alpha-1} \right. \\
& \times \left. M_{\alpha,\alpha}(\mathcal{A}(\rho_1 - s)^\alpha)]\|^2 \|\mathcal{B}\|^2 \|u_x(s)\|^2 ds \\
& + \frac{(\rho_2 - \rho_1)^\alpha}{\alpha} \int_{\rho_1}^{\rho_2} \|(\rho_2 - s)^{\alpha-1} M_{\alpha,\alpha}(\mathcal{A}(\rho_2 - s)^\alpha)\|^2 \|f(s, x(s), x(s - h))\|^2 ds \\
& + (\rho_1 - s_j) \int_{s_j}^{\rho_1} \|[(\rho_2 - s)^{\alpha-1} M_{\alpha,\alpha}(\mathcal{A}(\rho_2 - s)^\alpha) - (\rho_1 - s)^{\alpha-1} \right. \\
& \times \left. M_{\alpha,\alpha}(\mathcal{A}(\rho_1 - s)^\alpha)]\|^2 \|f(s, x(s), x(s - h))\|^2 ds + \frac{(\rho_2 - \rho_1)^\alpha}{\alpha} \\
& \times \int_{\rho_1}^{\rho_2} \|(\rho_2 - s)^{\alpha-1} M_{\alpha,\alpha}(\mathcal{A}(\rho_2 - s)^\alpha)\|^2 \left\| \int_0^s \Delta(\tau, x(\tau), x(\tau - h)) dw(\tau) \right\|^2 ds \\
& + (\rho_1 - s_j) \int_{s_j}^{\rho_1} \|[(\rho_2 - s)^{\alpha-1} M_{\alpha,\alpha}(\mathcal{A}(\rho_2 - s)^\alpha) - (\rho_1 - s)^{\alpha-1} \right. \\
& \times \left. M_{\alpha,\alpha}(\mathcal{A}(\rho_1 - s)^\alpha)]\|^2 \left\| \int_0^s \Delta(\tau, x(\tau), x(\tau - h)) dw(\tau) \right\|^2 ds \}.
\end{aligned}$$

as  $\rho_2 \rightarrow \rho_1$ , the RHS of the above inequality tends to zero. Thus,  $\mathbb{H}$  maps  $\mathcal{B}_r$  into an equicontinuous family of functions.

**Step 4:** The following conditions are proved to apply Mönch fixed point theorem (Lemma 4).

**Claim 1:**  $\mathbb{H}$  is Lipschitz continuous.

Let  $x, y \in \mathcal{B}_r$ , for  $t \in [0, t_1]$ . Using (A1), Lemma 1 and other estimates, we have

$$\begin{aligned}
E\|(\mathbb{H}x)(t) - (\mathbb{H}y)(t)\|^2 &\leq 3 \left\{ \frac{t_1^\alpha}{\alpha} \int_0^{t_1} (t_1 - s)^{\alpha-1} \|M_{\alpha,\alpha}(\mathcal{A}(t_1 - s)^\alpha)\|^2 \left[ \|B\|^2 E\|u_x(s) - u_y(s)\|^2 \right. \right. \\
&\quad + E\|f(s, x(s), x(s-h)) - f(s, y(s), y(s-h))\|^2 \\
&\quad \left. \left. + E\left\| \int_0^s [\Delta(\tau, x(\tau), x(\tau-h)) - \Delta(\tau, y(\tau), y(\tau-h))] dw(\tau) \right\|^2 ds \right] \right\} \\
&\leq 3 \frac{t_1^{2\alpha}}{\alpha^2} R_2^2 \left\{ 2\|B\|^2 \|B^*\|^2 \bar{l}_0 R_2^4 \frac{t_1^{2\alpha}}{\alpha^2} (K_f + t_1 K_\Delta) \|x - y\|^2 \right. \\
&\quad \left. + (K_f + t_1 K_\Delta) \|x - y\|^2 \right\} \\
&\leq 3 \frac{t_1^{2\alpha}}{\alpha^2} R_2^2 \left\{ \left( 1 + 2\|B\|^2 \|B^*\|^2 \bar{l}_0 R_2^4 \frac{t_1^{2\alpha}}{\alpha} \right) (K_f + t_1 K_\Delta) \right\} \|x - y\|^2 \\
&\cong M_0 \|x - y\|^2 \\
\text{where } M_0 &= 3 \frac{t_1^{2\alpha}}{\alpha^2} R_2^2 \left\{ \left( 1 + 2\|B\|^2 \|B^*\|^2 \bar{l}_0 R_2^4 \frac{t_1^{2\alpha}}{\alpha} \right) (K_f + t_1 K_\Delta) \right\}.
\end{aligned}$$

For  $t \in (t_j, s_j]$

$$\begin{aligned}
E\|(\mathbb{H}x)(t) - (\mathbb{H}y)(t)\|^2 &\leq E\|I_j(t, x(t)) - I_j(t, y(t))\|^2 \\
&\leq K_j \|x - y\|^2, \quad j = 1, 2, \dots, m.
\end{aligned}$$

For  $t \in (s_j, t_{j+1}]$ , one can compute

$$\begin{aligned}
E\|(\mathbb{H}x)(t) - (\mathbb{H}y)(t)\|^2 &\leq 4 \left\{ \bar{R}_j^2 K_j \|x - y\|^2 + R_j^2 \frac{(t_{j+1} - s_j)^{2\alpha}}{\alpha^2} \|B\|^2 \|u_x(t) - u_y(t)\|^2 \right. \\
&\quad \left. + \frac{(t_{j+1} - s_j)^{2\alpha}}{\alpha^2} R_j^2 [K_f + (t_{j+1} - s_j) K_\Delta] \|x - y\|^2 \right\}
\end{aligned}$$

$$\begin{aligned}
&\leq 4 \left[ \bar{R}_j^2 K_j \|x - y\|^2 + R_j^2 \frac{(t_{j+1} - s_j)^{2\alpha}}{\alpha^2} \left\{ 3\|\mathcal{B}\|^2 \|\mathcal{B}^*\|^2 R_j^4 \bar{l}_j \frac{(t_{j+1} - s_j)^{2\alpha}}{\alpha^2} \right. \right. \\
&\quad \times [K_f + (t_{j+1} - s_j)K_\Delta] \|x - y\|^2 \Big\} + \frac{(t_{j+1} - s_j)^{2\alpha}}{\alpha^2} [K_f + (t_{j+1} - s_j)K_\Delta] \|x - y\|^2 \Big] \\
&\leq 4 \left[ \bar{R}_j^2 K_j + R_j^2 \frac{(t_{j+1} - s_j)^{2\alpha}}{\alpha^2} \left\{ 1 + 3\|\mathcal{B}\|^2 \|\mathcal{B}^*\|^2 R_2^4 l \frac{(t_{j+1} - s_j)^{2\alpha}}{\alpha^2} \right\} \right. \\
&\quad \times [K_f + (t_{j+1} - s_j)K_\Delta] \Big] \|x - y\|^2 \\
&\leq 4 \left[ \bar{R}_j^2 K_j + R_j^2 \frac{b^{2\alpha}}{\alpha^2} \left\{ 1 + 3\|\mathcal{B}\|^2 \|\mathcal{B}^*\|^2 R_2^4 \bar{l}_j \frac{b^{2\alpha}}{\alpha^2} \right\} (K_f + bK_\Delta) \right] \|x - y\|^2 \\
&\equiv M_j \|x - y\|^2
\end{aligned}$$

where  $M_j = 4 \left[ \bar{R}_j^2 K_j + R_j^2 \frac{b^{2\alpha}}{\alpha^2} \left\{ 1 + 3\|\mathcal{B}\|^2 \|\mathcal{B}^*\|^2 R_2^4 \bar{l}_j \frac{b^{2\alpha}}{\alpha^2} \right\} (K_f + bK_\Delta) \right]$ .

Take  $\Gamma = \max_{1 \leq j \leq m} \{M_0, K_j, M_j\}$ , since  $\Gamma < 1$ , for all  $t \in J$ ,  $\mathbb{H}$  is Lipschitz continuous.

**Claim 2:**

Suppose that  $\mathcal{D} \subseteq \mathcal{B}_r$  is countable and  $\mathcal{D} \subseteq \bar{co}(\{0\} \cup \mathbb{H}(\mathcal{D}))$ , we show that  $\mathcal{D}$  is relatively compact. Without loss of generality, we assume that  $\mathcal{D} = \{x^n\}_{n=1}^\infty$ . By Step 3, it is easy to verify that  $\mathcal{D} \subseteq \bar{co}(\{0\} \cup \mathbb{H}(\mathcal{D}))$  is equicontinuous on  $J$ . We need to estimate  $\beta(\mathcal{D}) = 0$ , where  $\beta$  is the Hausdorff MNC.

By using Lemma 2, we have  $\beta(\mathbb{H}x(t)) = \beta(\hat{\mathbb{H}}x(t))$ .

As  $M_\alpha(\mathcal{A}t^\alpha)$  is compact,  $\beta[M_\alpha(\mathcal{A}t^\alpha)\phi(0)] = 0$  remaining terms for  $t \in [0, t_1]$

$$\begin{aligned}
(\mathbb{H}x)(t) &= \int_0^t (t-s)^{\alpha-1} M_{\alpha,\alpha}(\mathcal{A}(t-s)^\alpha) \mathcal{B}u(s) ds \\
&\quad + \int_0^t (t-s)^{\alpha-1} M_{\alpha,\alpha}(\mathcal{A}(t-s)^\alpha) f(s, x(s), x(s-h)) \\
&\quad + \int_0^t (t-s)^{\alpha-1} M_{\alpha,\alpha}(\mathcal{A}(t-s)^\alpha) \int_0^s \Delta(\tau, x(\tau), x(\tau-h)) dw(\tau) ds \\
&:= \hat{\mathbb{H}}_1 + \hat{\mathbb{H}}_2 + \hat{\mathbb{H}}_3
\end{aligned}$$

$$\begin{aligned}
\beta\left(\{\hat{\mathbb{H}}_1 x^n(t)\}_{n=1}^{\infty}\right) &= \beta\left(\int_0^t (t-s)^{\alpha-1} M_{\alpha,\alpha}(\mathcal{A}(t-s)^\alpha) \mathcal{B} u_{x^n}(s) ds\right) \\
&= \beta\left(\int_0^t (t-s)^{\alpha-1} M_{\alpha,\alpha}(\mathcal{A}(t-s)^\alpha) \mathcal{B} \mathcal{B}^* M_{\alpha,\alpha}(\mathcal{A}^*(t-s)^\alpha) W_{[0,t]}^{-1} \right. \\
&\quad \times \left. \left\{ - \int_0^{t_1} (t_1-s)^{\alpha-1} M_{\alpha,\alpha}(\mathcal{A}(t_1-s)^\alpha) f(s, x^n(s), x^n(s-h)) ds \right. \right. \\
&\quad \left. \left. - \int_0^{t_1} (t_1-s)^{\alpha-1} M_{\alpha,\alpha}(\mathcal{A}(t_1-s)^\alpha) \right. \right. \\
&\quad \left. \left. \times \left( \int_0^s \Delta(\tau, x^n(\tau), x^n(\tau-h)) dw(\tau) \right) ds \right\}_{n=1}^{\infty} \right).
\end{aligned}$$

By Lemma 2 and (A3), we have

$$\begin{aligned}
&\beta\left(\{\hat{\mathbb{H}}_1 x^n(t)\}_{n=1}^{\infty}\right) \\
&\leq 2\left(\beta\left(\left\{ \int_0^{t_1} (t_1-s)^{\alpha-1} M_{\alpha,\alpha}(\mathcal{A}(t_1-s)^\alpha) f(s, x^n(s), x^n(s-h)) ds \right\}_{n=1}^{\infty} \right)\right. \\
&\quad \left. + \beta\left(\left\{ \int_0^{t_1} (t_1-s)^{\alpha-1} M_{\alpha,\alpha}(\mathcal{A}(t_1-s)^\alpha) \left[ \left( \int_0^s \Delta(\tau, x^n(\tau), x^n(\tau-h)) dw(\tau) \right)^2 \right]^{\frac{1}{2}} ds \right\}_{n=1}^{\infty} \right) \right) \\
&\leq 2\left(\left\{ \int_0^{t_1} (t_1-s)^{\alpha-1} M_{\alpha,\alpha}(\mathcal{A}(t_1-s)^\alpha) \beta(f(s, x^n(s), x^n(s-h))) ds \right.\right. \\
&\quad \left. \left. + \int_0^{t_1} (t_1-s)^{\alpha-1} M_{\alpha,\alpha}(\mathcal{A}(t-s)^\alpha) \beta\left(\left[ \left( \int_0^s \Delta(\tau, x^n(\tau), x^n(\tau-h)) dw(\tau) \right)^2 \right]^{\frac{1}{2}} ds \right\}_{n=1}^{\infty} \right) \right) \\
&\leq 2\left(\frac{t_1^\alpha}{\alpha} R_2 Q_f \sup_{-h < \theta \leq 0} \beta\left(\{x^n(\theta+s)\}_{n=1}^{\infty}\right) + \frac{t_1^\alpha}{\alpha} R_2 \left( \int_0^s [\Delta(\tau, x^n(\tau), x^n(\tau-h))]^2 d(\tau) \right)^{\frac{1}{2}} \right. \\
&\quad \left. \times \sup_{-h < \theta \leq 0} \beta\left(\{x^n(\theta+s)\}_{n=1}^{\infty}\right) \right) \\
&\leq 2\frac{t_1^\alpha}{\alpha} R_2 [Q_f + \sqrt{t_1} Q_\Delta] \sup_{-h < \theta \leq 0} \beta\left(\{x^n(\theta+s)\}_{n=1}^{\infty}\right) \\
&\leq 2\frac{t_1^\alpha}{\alpha} R_2 [Q_f + \sqrt{t_1} Q_\Delta] \sup_{t \in [0, t_1]} \beta\left(\{x^n(t)\}_{n=1}^{\infty}\right).
\end{aligned}$$

In a similar fashion, we have

$$\begin{aligned}
\beta\left(\{\hat{\mathbb{H}}_2x^n(t)\}_{n=1}^{\infty}\right) &\leq \beta\left(\left\{\int_0^t(t-s)^{\alpha-1}M_{\alpha,\alpha}(\mathcal{A}(t-s)^{\alpha})f(s,x^n(s),x^n(s-h))ds\right\}_{n=1}^{\infty}\right) \\
&\leq 2\frac{t^{\alpha}}{\alpha}R_2Q_f\sup_{-h<\theta\leq 0}\beta\left(\{x^n(\theta+s)\}_{n=1}^{\infty}\right) \\
&\leq 2\frac{t_1^{\alpha}}{\alpha}R_2Q_f\sup_{t\in[0,t_1]}\beta\left(\{x^n(t)\}_{n=1}^{\infty}\right) \\
\beta\left(\{\hat{\mathbb{H}}_3x^n(t)\}_{n=1}^{\infty}\right) &\leq \beta\left(\left\{\int_0^t(t-s)^{\alpha-1}M_{\alpha,\alpha}(\mathcal{A}(t-s)^{\alpha})\right.\right. \\
&\quad \times\left[\left(\int_0^s\Delta(\tau,x^n(\tau),x^n(\tau-h))dw(\tau)\right)^2\right]^{\frac{1}{2}}ds\left\}_{n=1}^{\infty}\right) \\
&\leq 2\frac{t^{\alpha}}{\alpha}R_2\left(\int_0^s\left[\Delta(\tau,x^n(\tau),x^n(\tau-h))\right]^2d(\tau)\right)^{\frac{1}{2}}\sup_{-h<\theta\leq 0}\beta\left(\{x^n(\theta+s)\}_{n=1}^{\infty}\right) \\
&\leq 2\frac{t_1^{\alpha}}{\alpha}R_2\sqrt{t_1}Q_{\Delta}\sup_{t\in[0,t_1]}\beta\left(\{x^n(t)\}_{n=1}^{\infty}\right).
\end{aligned}$$

Combining the above inequalities, we obtain that

$$\begin{aligned}
\beta\left(\{\hat{\mathbb{H}}x^n(t)\}_{n=1}^{\infty}\right) &= \beta\left(\{\hat{\mathbb{H}}_1x^n(t)\}_{n=1}^{\infty}\right) + \beta\left(\{\hat{\mathbb{H}}_2x^n(t)\}_{n=1}^{\infty}\right) + \beta\left(\{\hat{\mathbb{H}}_3x^n(t)\}_{n=1}^{\infty}\right) \\
&\leq 2\frac{t_1^{\alpha}}{\alpha}R_2[Q_f + \sqrt{t_1}Q_{\Delta}]\sup_{t\in[0,t_1]}\beta\left(\{x^n(t)\}_{n=1}^{\infty}\right) \\
&\quad + 2\frac{t_1^{\alpha}}{\alpha}R_2Q_f\sup_{t\in[0,t_1]}\beta\left(\{x^n(t)\}_{n=1}^{\infty}\right) \\
&\quad + 2\frac{t_1^{\alpha}}{\alpha}R_2\sqrt{t_1}Q_{\Delta}\sup_{t\in[0,t_1]}\beta\left(\{x^n(t)\}_{n=1}^{\infty}\right) \\
&\leq 4\frac{t_1^{\alpha}}{\alpha}R_2[Q_f + \sqrt{t_1}Q_{\Delta}]\sup_{t\in[0,t_1]}\beta\left(\{x^n(t)\}_{n=1}^{\infty}\right).
\end{aligned}$$

Using Lemma 2.7, we have

$$\beta\left(\{\hat{\mathbb{H}}x^n(t)\}_{n=1}^{\infty}\right) = M^*\beta(\mathcal{D}).$$

For  $t \in (t_j, s_j]$ , compute

$$\begin{aligned}\beta\left(\{\hat{\mathbb{H}}x^n(t)\}_{n=1}^{\infty}\right) &= \beta\left(\{I_j(t, x^n(t))\}_{n=1}^{\infty}\right) \\ &\leq 2Q_j \sup_{-h < \theta \leq 0} \beta\left(\{x^n(\theta + s)\}_{n=1}^{\infty}\right) \\ &\leq 2Q_j \sup_{t \in (t_j, s_j]} \beta\left(\{x^n(t)\}_{n=1}^{\infty}\right).\end{aligned}$$

Using Lemma 2.7, we have

$$\beta\left(\{\hat{\mathbb{H}}x^n(t)\}_{n=1}^{\infty}\right) = M_j^* \beta(\mathcal{D}).$$

As  $M_{\alpha, \alpha}(\mathcal{A}(t - s_j)^\alpha)$  is compact,  $\beta[M_\alpha(\mathcal{A}(t - s_j)^\alpha)\phi(0)] = 0$  remaining terms for  $t \in (s_j, t_{j+1}]$

$$\begin{aligned}(\mathbb{H}x)(t) &= I_j(s_j, x(s_j))M_\alpha(\mathcal{A}(t - s_j)^\alpha) + \int_{s_j}^t (t - s)^{\alpha-1} M_{\alpha, \alpha}(\mathcal{A}(t_{j+1} - s)^\alpha) \mathcal{B}u(s) ds \\ &\quad + \int_{s_j}^t (t - s)^{\alpha-1} M_{\alpha, \alpha}(\mathcal{A}(t - s)^\alpha) f(s, x(s), x(s - h)) \\ &\quad + \int_{s_j}^t (t - s)^{\alpha-1} M_{\alpha, \alpha}(\mathcal{A}(t_{j+1} - s)^\alpha) \int_0^s \Delta(\tau, x(\tau), x(\tau - h)) dw(\tau) ds \\ &:= \hat{\mathbb{H}}_1 + \hat{\mathbb{H}}_2 + \hat{\mathbb{H}}_3 + \hat{\mathbb{H}}_4.\end{aligned}$$

Now, we estimate the following

$$\begin{aligned}\beta\left(\{\hat{\mathbb{H}}_1 x^n(t)\}_{n=1}^{\infty}\right) &= \beta\left(\left\{I_j(s_j, x^n(s_j))M_\alpha(\mathcal{A}(t - s_j)^\alpha)\right\}_{n=1}^{\infty}\right) \\ &\leq \bar{R}_j \beta(\{I_j(s_j, x^n(s_j))\}_{n=1}^{\infty}) \\ &\leq \bar{R}_j Q_j \sup_{-h < \theta \leq 0} \beta\left(\{x^n(\theta + s)\}_{n=1}^{\infty}\right) \\ &\leq \bar{R}_j Q_j \sup_{t \in [s_j, t_{j+1}]} \beta\left(\{x^n(t)\}_{n=1}^{\infty}\right)\end{aligned}$$

$$\begin{aligned}
\beta\left(\{\hat{\mathbb{H}}_2 x^n(t)\}_{n=1}^{\infty}\right) &= \beta\left(\int_{s_j}^t (t-s)^{\alpha-1} M_{\alpha,\alpha}(\mathcal{A}(t-s)^\alpha) \mathcal{B} u_{x^n}(s) ds\right) \\
&= \beta\left(\int_{s_j}^t (t-s)^{\alpha-1} M_{\alpha,\alpha}(A(t-s)^\alpha) \mathcal{B} \mathcal{B}^* M_{\alpha,\alpha}(\mathcal{A}^*(t-s)^\alpha) W_{(s_j,t_{j+1})}^{-1}\right. \\
&\quad \times \left\{ -I_j(s_j, x^n(s_j)) M_\alpha(\mathcal{A}(t_{j+1}-s_j)^\alpha) - \int_{s_j}^{t_{j+1}} (t_{j+1}-s)^{\alpha-1} M_{\alpha,\alpha}(\mathcal{A}(t_{j+1}-s)^\alpha) \right. \\
&\quad \times f(s, x^n(s), x^n(s-h)) ds - \int_{s_j}^{t_{j+1}} (t_{j+1}-s)^{\alpha-1} M_{\alpha,\alpha}(\mathcal{A}(t_{j+1}-s)^\alpha) \\
&\quad \times \left. \left. \left. \Delta(\tau, x^n(\tau), x^n(\tau-h)) dw(\tau) \right) ds \right\}_{n=1}^{\infty}\right).
\end{aligned}$$

By Lemma 2 and (A3), we have

$$\begin{aligned}
\beta\left(\{\hat{\mathbb{H}}_2 x^n(t)\}_{n=1}^{\infty}\right) &\leq 2\beta\left(\left\{\bar{R}_j Q_j \beta\left(\{x^n(\theta+s)\}_{n=1}^{\infty}\right) + \int_{s_j}^{t_{j+1}} (t_{j+1}-s)^{\alpha-1} M_{\alpha,\alpha}(\mathcal{A}(t_{j+1}-s)^\alpha) \right.\right. \\
&\quad \times \left. \left. \beta\left(f(s, x^n(s), x^n(s-h))\right) ds + \int_{s_j}^{t_{j+1}} (t_{j+1}-s)^{\alpha-1} M_{\alpha,\alpha}(\mathcal{A}(t-s)^\alpha) \right. \right. \\
&\quad \times \left. \left. \beta\left(\left[\left(\int_0^s \Delta(\tau, x^n(\tau), x^n(\tau-h)) dw(\tau)\right)^2\right]^{\frac{1}{2}} ds\right\}_{n=1}^{\infty}\right)\right) \\
&\leq 2\left(\bar{R}_j Q_j + \frac{(t_{j+1}-s_j)^\alpha}{\alpha} R_j \left[Q_f + \sqrt{(t_{j+1}-s_j) Q_\Delta}\right]\right) \sup_{-h < \theta \leq 0} \beta\left(\{x^n(\theta+s)\}_{n=1}^{\infty}\right) \\
&\leq 2\left(\bar{R}_j Q_j + \frac{(t_{j+1}-s_j)^\alpha}{\alpha} R_j \left[Q_f + \sqrt{(t_{j+1}-s_j) Q_\Delta}\right]\right) \sup_{t \in (s_j, t_{j+1}]} \beta\left(\{x^n(t)\}_{n=1}^{\infty}\right) \\
\beta\left(\{\hat{\mathbb{H}}_3 x^n(t)\}_{n=1}^{\infty}\right) &\leq \beta\left(\left\{\int_{s_j}^t (t-s)^{\alpha-1} M_{\alpha,\alpha}(\mathcal{A}(t-s)^\alpha) f(s, x^n(s), x^n(s-h)) ds\right\}_{n=1}^{\infty}\right) \\
&\leq 2 \frac{(t-s_j)^\alpha}{\alpha} R_j Q_f \sup_{-h < \theta \leq 0} \beta\left(\{x^n(\theta+s)\}_{n=1}^{\infty}\right) \\
&\leq 2 \frac{(t_{j+1}-s_j)^\alpha}{\alpha} R_j Q_f \sup_{t \in (s_j, t_{j+1}]} \beta\left(\{x^n(t)\}_{n=1}^{\infty}\right)
\end{aligned}$$

$$\begin{aligned}
\beta\left(\{\hat{\mathbb{H}}_4x^n(t)\}_{n=1}^{\infty}\right) &\leq \beta\left(\left\{\int_{s_j}^t(t-s)^{\alpha-1}M_{\alpha,\alpha}(\mathcal{A}(t-s)^{\alpha})\right.\right. \\
&\quad \times \left[\left(\int_0^s\Delta(\tau,x^n(\tau),x^n(\tau-h))dw(\tau)\right)^2\right]^{\frac{1}{2}}ds\left.\right\}_{n=1}^{\infty}\Big) \\
&\leq 2\frac{(t-s_j)^{\alpha}}{\alpha}R_j\left(\int_0^s\left[\Delta(\tau,x^n(\tau),x^n(\tau-h))\right]^2d(\tau)\right)^{\frac{1}{2}} \\
&\quad \times \sup_{-h<\theta\leq 0}\beta\left(\{x^n(\theta+s)\}_{n=1}^{\infty}\right) \\
&\leq 2\frac{(t_{j+1}-s_j)^{\alpha}}{\alpha}R_j\sqrt{(t_{j+1}-s_j)}Q_{\Delta}\sup_{t\in(s_j,t_{j+1}]}\beta\left(\{x^n(t)\}_{n=1}^{\infty}\right).
\end{aligned}$$

Combining the above inequalities, we have

$$\begin{aligned}
\beta\left(\{\hat{\mathbb{H}}x^n(t)\}_{n=1}^{\infty}\right) &= \beta\left(\{\hat{\mathbb{H}}_1x^n(t)\}_{n=1}^{\infty}\right) + \beta\left(\{\hat{\mathbb{H}}_2x^n(t)\}_{n=1}^{\infty}\right) + \beta\left(\{\hat{\mathbb{H}}_3x^n(t)\}_{n=1}^{\infty}\right) \\
&\quad + \beta\left(\{\hat{\mathbb{H}}_4x^n(t)\}_{n=1}^{\infty}\right) \\
&= 2\left(\bar{R}_jQ_j + \frac{(t_{j+1}-s_j)^{\alpha}}{\alpha}R_j\left[Q_f + \sqrt{(t_{j+1}-s_j)}Q_{\Delta}\right]\right) \\
&\quad \times \sup_{t\in(s_j,t_{j+1}]} \beta\left(\{x^n(t)\}_{n=1}^{\infty}\right) + 2\bar{R}_jQ_j\sup_{t\in(s_j,t_{j+1}]} \beta\left(\{x^n(t)\}_{n=1}^{\infty}\right) \\
&\quad + 2\frac{(t_{j+1}-s_j)^{\alpha}}{\alpha}R_jQ_f\sup_{t\in(s_j,t_{j+1}]} \beta\left(\{x^n(t)\}_{n=1}^{\infty}\right) \\
&\quad + 2\frac{(t_{j+1}-s_j)^{\alpha+1}}{\alpha}R_jQ_{\Delta}\sup_{t\in(s_j,t_{j+1}]} \beta\left(\{x^n(t)\}_{n=1}^{\infty}\right) \\
&\leq 4\left(\bar{R}_jQ_j + \frac{(t_{j+1}-s_j)^{\alpha}}{\alpha}R_j\left[Q_f + \sqrt{(t_{j+1}-s_j)}Q_{\Delta}\right]\right) \\
&\quad \times \sup_{t\in(s_j,t_{j+1}]} \beta\left(\{x^n(t)\}_{n=1}^{\infty}\right) \\
&\leq 4\left(\bar{R}_jQ_j + \frac{b^{\alpha}}{\alpha}R_j\left[Q_f + \sqrt{b}Q_{\Delta}\right]\right)\sup_{t\in J}\beta\left(\{x^n(t)\}_{n=1}^{\infty}\right).
\end{aligned}$$

Using Lemma 2.7, we have

$$\beta\left(\{\hat{\mathbb{H}}x^n(t)\}_{n=1}^{\infty}\right) = \bar{M}_j^*\beta(\mathcal{D}).$$

Take  $\Upsilon = \max_{1 \leq j \leq m} \{M^*, M_j^*, \bar{M}_j^*\}$ , by (A5)  $\Upsilon < 1$ , for all  $t \in J$ ; therefore, Mönch condition to be verified.

By Lemma 2, we obtain that  $\beta(\mathcal{D}) \subseteq \beta(\bar{co}(\{0\} \cup \mathbb{H}(\mathcal{D})))$ . This implies  $\beta(\mathcal{D}) = 0$ ,  $\mathcal{D}$  is relatively compact set. Hence, we derive that  $\mathbb{H}$  has a fixed point in  $\mathcal{D}$ , which is a solution of system (1). Hence, the system (1) is controllable.

## 5. Computational example

Consider the following NI impulsive ISDS as

$$\begin{aligned} D^{0.5}x_1(t) &= x_2(t) + u_1(t) + \frac{1}{5} \left( \frac{x_1(t-0.2)}{1+x_1^2(t-0.2)+x_2^2(t-0.2)} \right) + \int_0^t \frac{x_1(s-0.2)e^{-s}}{3} dw_1(s) \\ D^{0.5}x_2(t) &= x_1(t) - u_2(t) + \int_0^t \frac{x_2(s-0.2)e^{-s}}{3} dw_2(s), \quad t \in (t_j, s_j], \quad j = 0, 1, \dots, m \\ x(t) &= \frac{1}{4}x(t, t_{j+1}], \quad j = 1, 2, \dots, m \\ x(t) &= \phi(t), \quad t \in [-0.2, 0], \end{aligned} \tag{11}$$

where  $x \in \mathbb{R}^2$ ,  $\alpha = 0.5$ ,  $0 = t_0 = s_0 < t_1 = 0.5 \leq s_1 < t_2 \dots < t_m \leq s_m < t_{m+1} = 1$  are prefixed numbers,  $J = [0, 1]$ . Here

$$\mathcal{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$f(t, x(t)) = \begin{pmatrix} \frac{x_1(t-0.2)}{5(1+x_1^2(t-0.2)+x_2^2(t-0.2))} \\ 0 \end{pmatrix}, \quad \Delta(t, x(t)) = \begin{pmatrix} \frac{x_1(s-0.2)e^{-t}}{3} & 0 \\ 0 & \frac{x_2(s-0.2)e^{-t}}{3} \end{pmatrix}$$

The Mittag-Leffler matrix function of the systems is given by

$$M_\alpha(\mathcal{A}b^\alpha) = \begin{pmatrix} S_1 & S_2 \\ -S_3 & S_4 \end{pmatrix}$$

where

$$\begin{aligned} S_1 = S_4 &= \sum_{j=0}^{\infty} \frac{(-1)^j b^{2j\alpha}}{\Gamma(2j\alpha + \alpha)} = -0.33777 \\ S_2 = -S_3 &= -\sum_{j=0}^{\infty} \frac{(-1)^j b^{(2j+1)\alpha}}{\Gamma[(2j+1)\alpha]} = -0.1666. \end{aligned}$$

Now the controllability Grammian matrix is described as below:

$$\begin{aligned} W &= \int_0^b (b-s)^{\alpha-1} [M_{\alpha,\alpha}(\mathcal{A}(b-s)^\alpha)\mathcal{B}] [M_{\alpha,\alpha}(\mathcal{A}(b-s)^\alpha)\mathcal{B}]^* ds \\ &= \int_0^b (b-s)^{q-1} \begin{pmatrix} P_1^2 + P_2^2 & P_1 P_3 + P_2 P_4 \\ P_1 P_3 + P_2 P_4 & P_1^2 + P_2^2 \end{pmatrix} ds. \end{aligned}$$

Here

$$M_{\alpha,\alpha}(\mathcal{A}(b-s)^\alpha) = \begin{pmatrix} P_1 & P_2 \\ -P_3 & P_4 \end{pmatrix},$$

$$\begin{aligned} P_1 &= P_4 = \sum_{j=0}^{\infty} \frac{(-1)^j (b-s)^{2j\alpha}}{\Gamma(2j\alpha + \alpha)} = -0.7477 \\ P_2 &= -P_3 = -\sum_{j=0}^{\infty} \frac{(-1)^j (b-s)^{(2j+1)\alpha}}{\Gamma[(2j+1)\alpha]} = -0.4203. \end{aligned}$$

Therefore,

$$\begin{aligned} W &= \int_0^1 (1-0.6)^{0.5-1} \begin{pmatrix} 0.5590 & 0 \\ 0 & 0.5990 \end{pmatrix} ds \\ &= 0.2166, \end{aligned}$$

which is positive definite for any  $b > 0$ . Therefore, the corresponding linear system of (11) is controllable on  $[0, 1]$ . It is easy to compute that the nonlinear functions satisfy the assumptions stated in the Theorem 4.1. Further

$$\begin{aligned} \mathbb{E}\|f(t, x_1(t), x_1(t-h)) - f(t, x_2(t), x_2(t-h))\|^2 &\leq \frac{1}{25} \mathbb{E}\|x_1 - x_2\|^2 \\ \mathbb{E}\|\Delta(t, x_1(t), x_1(t-h)) - \Delta(t, x_2(t), x_2(t-h))\|^2 &\leq \frac{1}{9} e^{-2t} \mathbb{E}\|x_1 - x_2\|^2 \\ \mathbb{E}\|I_j(t, x_1(s_{j-1})) - I_j(t, x_2(s_{j-1}))\|^2 &\leq \frac{1}{16} \mathbb{E}\|x_1 - x_2\|^2, \quad j = 1, 2, \dots, m. \end{aligned}$$

Here

$$\begin{aligned}
 M_0 &= 3 \times \frac{0.25^{2 \times 0.5}}{0.5^2} \times 0.7357 \left\{ 1 + 2 \times 0.2929 \times 4.6 \times 0.7357 \times \frac{0.25^{2 \times 0.5}}{0.5} \right\} \left( \frac{1}{25} + 0.067 \right) \\
 &= 2.2 \times 1.29 \times 0.107 \\
 &= 0.2761. \\
 K_j &= 0.0625. \\
 M_j &= 0.486.
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 \Gamma &= \max\{M_0, K_j, M_j\} \\
 &= \max\{0.2761, 0.0625, 0.486\} < 1;
 \end{aligned}$$

thus, the system (11) is controllable on  $[0, 1]$ .

**REMARK 1.** Some sufficient conditions for the controllability of nonlinear stochastic impulsive system have been obtained by [Karthikeyan & Balachandran \(2011\)](#). Controllability of nonlinear DEs with NI impulses in second-order have been studied by [Kumar et al. \(2018\)](#). Controllability of fractional stochastic with Poisson jumps have been derived by [Sathiyaraj & Balasubramaniam \(2015\)](#). A series of sufficient conditions have been derived in this manuscript, which differ from the above existing literature in terms of NI impulse conditions and Mönch fixed point theorem in finite dimensional settings.

**REMARK 2.** Here a special type of impulse is used in the sense of Caputo fractional DEs in finite dimensional stochastic settings. This type of DEs are used to model the dynamics of evolution process in pharmacotherapy. This process starts abruptly at some points and their actions continues in a finite interval. Hence, we derived the controllability results on NI impulsive ISDS in finite dimensional  $\mathbb{R}^n$ .

**REMARK 3.** The model in (11) is associated with a typical cycle for the streamlining control of advancement measures in pharmacotherapy. In particular, we think about the accompanying disentangled circumstance concerning the haemodynamical harmony of an individual. For the situation of a decompensation, one can recommend some intravenous medications.

Since the presentation of the medications in the circulatory system and the ensuing ingestion of the body is gradual, hence it is a continuous processes; we can decipher the above circumstance as an imprudent activity, which begins unexpectedly and remains dynamic on a limited time stretch.

From a practical view, the incredible abilities of partial subsidiaries in displaying elements of complex stochastic systems and processes, it appears to be encouraging to utilize fragmentary analytics to abnormal body balance measures all the more precisely in the future. We comment that the applications inspiration for the investigation of the system (11) is identified with the fractional stochastic differential control system administering the dynamic system with fractional order and NI impulses one can refer ([Wang et al. 2016](#); [Yan & Yang 2020](#)).

## 6. CONCLUSION

This paper provides sufficient conditions for controllability of NI impulsive ISDS by using MNC based on Mönch fixed point theorem. The controllability Grammian matrix is employed by means of M-L function. Finally, an example is provided to show the effectiveness of the proposed results.

## CONFLICT OF INTEREST

There is no conflict of interest.

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