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Note

A Note on an Evaluation of Abel Sums

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Following [1, p. 18], the Abel Sums are taken as defined by

$$A_n(x, y; p, q) = \sum_{k=0}^{n} \binom{n}{k} (k + x)^{k+p} (n - k + y)^{n-k+q}. \tag{1}$$

Here x, y are indeterminates, n is a nonnegative integer, and p, q are arbitrary integers.

The purpose of this note is to show the utility in their evaluation of auxiliary sums, here called Associated Abel Sums, and defined by

$$F_n(x, y; p, q) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} (k + x)^{p} (n - k + y)^{q}. \tag{2}$$

Indeed, the two are interrelated by

$$A_n(x, y; p, q) = \sum_{k=0}^{n} \binom{n}{k} (n + x + y)^{k} F_{n-k}(x, y + k; p + n - k, q), \tag{3}$$

$$F_n(x, y; p, q) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{k} (n + x + y)^{k} A_{n-k}(x, y + k; p - n, q). \tag{4}$$

(3) is verified as follows:

$A_n(x, y; p, q)$

$$= \sum_{k=0}^{n} \binom{n}{k} (k + x)^{k+p} (n - k + y)^{n-k+q}$$

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$$= \sum_{k=0}^{n} \binom{n}{k} (k + x)^{k+p} (n - k + y)^{q}$$

$$\times \sum_{j=0}^{n-k} \binom{n-k}{j} (-1)^{n-k-j} (k + x)^{n-k-j} (n + x + y)^{j}$$

$$= \sum_{k=0}^{n} \sum_{j=0}^{n-k} \binom{n}{k}\binom{n-k}{j} (-1)^{n-k-j} (k + x)^{p+n-j} (n - k + y)^{q} (n + x + y)^{j}$$

$$= \sum_{j=0}^{n} \sum_{k=0}^{n-j} \binom{n}{j}\binom{n-j}{k} (-1)^{n-k-j} (k + x)^{p+n-j} (n - k + y)^{q} (n + x + y)^{j}$$

$$= \sum_{j=0}^{n} \binom{n}{j} (n + x + y)^{j} F_{n-j}(x, y + j; p + n - j, q)$$

$$= \sum_{j=0}^{n} \binom{n}{j} (n + x + y)^{n-j} F_{j}(x, n - j + y; p + j, q).$$

(4) is verified by

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^{k} (n + x + y)^{k} A_{n-k}(x, y + k; p - n, q)$$

$$= \sum_{k=0}^{n} \binom{n}{k} (-1)^{k} (n + x + y)^{k}$$

$$\times \sum_{j=0}^{n-k} \binom{n-k}{j} (j + x)^{p-n+j} (n - j + y)^{q+n-j-k}$$

$$= \sum_{j=0}^{n} \sum_{k=0}^{n-j} \binom{n}{j}\binom{n-j}{k} (-1)^{k} (n + x + y)^{k} (j + x)^{p-n+j} (n - j + y)^{q+n-j-k}$$

$$= \sum_{j=0}^{n} \binom{n}{j} (j + x)^{p-n+j} (n - j + y)^{q}$$

$$\times \sum_{k=0}^{n-j} \binom{n-j}{k} (-1)^{k} (n + x + y)^{k} (n - j + y)^{n-k-j}$$

$$= \sum_{j=0}^{n} \binom{n}{j} (j + x)^{p-n+j} (n - j + y)^{q} (-1)^{n-j} (j + x)^{n-j}.$$

$$= \sum_{j=0}^{n} \binom{n}{j}(-1)^{n-j} (j + x)^{p} (n - j + y)^{q}.$$

Formula (3) is an evaluation of Abel Sums as a linear combination of

Associated Abel Sums, whose coefficients are monomials in $(n + x + y)$.

The next formula gives a simple expression for $F_n(x, y; p, q)$ in terms of difference operators.

$$F_n(x, y; p, q) = \sum_{k=0}^{n} \binom{n}{k}(-1)^{n-k} (k + x)^p (n - k + y)^q \tag{5}$$

$$= \sum_{k=0}^{n} \binom{n}{k}(-1)^{n-k} E_x^k x^p E_y^{n-k} y^q$$

$$= (E_x - E_y)^n x^p y^q = (\Delta_x - \Delta_y)^n x^p y^q$$

where $E_x(E_y)$ is the shift operator operating on $x(y)$, and $\Delta_x(\Delta_y)$ is the corresponding difference operator, i.e.

$$E_x f(x, y) = f(x + 1, y), \qquad \Delta_x f(x, y) = f(x + 1, y) - f(x, y).$$

This implies the instances:

$$F_n(x, y; p, 0) = \Delta^n x^p \tag{6}$$

$$F_n(x, y; p, 1) = y\Delta^n x^p - n\Delta^{n-1} x^p \tag{7}$$

$$F_n(x, y; p, 2) = y^2 \Delta^n x^p - n(2y + 1) \Delta^{n-1} x^p + 2 \binom{n}{2} \Delta^{n-2} x^p. \tag{8}$$

From Riordan [1, p. 203] we have for $p \geqslant n$:

$$\Delta^n x^p = n! \sum_{j=n}^{p} \binom{p}{j} x^{p-j} S(j, n) \tag{9}$$

where $S(j, n)$ are Stirling numbers of the second kind.
Thus we have

$$A_n(x, y; p, 0) = \sum_{k=0}^{n} \binom{n}{k} (n + x + y)^{n-k} \Delta^k x^{p+k}. \tag{10}$$

This implies the instances: (for $0 \leqslant p < n$, $\Delta^n x^p = 0$)

$$A_n(x, y; -1, 0) = \sum_{k=0}^{n} \binom{n}{k} (k + x)^{k-1} (n - k + y)^{n-k}$$

$$= (n + x + y)^n x^{-1}. \tag{11}$$

which is the original identity of Abel, and

$$A_n(x, y; -2, 0) = \binom{n}{0} (n + x + y)^n \Delta^0 x^{-2} + \binom{n}{1} (n + x + y)^{n-1} \Delta^1 x^{-1}$$

$$= \frac{-n(n + x + y)^{n-1}}{x(x + 1)} + \frac{(n + x + y)^n}{x^2}. \tag{12}$$

In general for $p > 0$

$$\begin{aligned} A_n(x, y; -p, 0) &= \sum_{k=0}^{n} \binom{n}{k} (n + x + y)^{n-k} \Delta^k x^{k-p} \\ &= \sum_{k=0}^{p-1} \binom{n}{k} (n + x + y)^{n-k} \Delta^k x^{k-p}. \end{aligned} \tag{13}$$

Thus the number of terms in the sum is bounded for fixed p, (even if $\Delta^k x^{k-p}$ is written out as a polynomial in x).
For $p = 0$ and $p = 1$ we obtain

$$A_n(x, y; 0, 0) = \sum_{k=0}^{n} \binom{n}{k} (n + x + y)^k (n - k)! = n! \sum_{k=0}^{n} \frac{(n + x + y)^k}{k!}, \tag{14}$$

$$\begin{aligned} A_n(x, y; 1, 0) &= \sum_{k=0}^{n} \binom{n}{k} (n + x + y)^k \Delta^{n-k} x^{n-k+1} \\ &= \sum_{k=0}^{n} \binom{n}{k} (n + x + y)^k (n - k + 1)! \left(x + \frac{n - k}{2}\right). \end{aligned} \tag{15}$$

Another evaluation of $F_n(x, y; p, q)$ for $q \geqslant 0$, similar to (5), is:

$$\begin{aligned} F_n(x, y; p, q) &= \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (k + x)^p \sum_{j=0}^{q} \binom{q}{j} (-k - x)^{q-j} (n + x + y)^j \\ &= \sum_{j=0}^{q} \binom{q}{j} (-1)^{q-j} (n + x + y)^j \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} (k + x)^{p+q-j} \\ &= \sum_{j=0}^{q} \binom{q}{j} (-1)^{q-j} (n + x + y)^j F_n(x, 0; p + q - j, 0) \\ &= \sum_{j=0}^{q} \binom{q}{j} (-1)^{q-j} (n + x + y)^j \Delta^n x^{p+q-j}. \end{aligned} \tag{16}$$

Similarly, (for $q \geqslant 0$):

$$A_n(x, y; p, q) = \sum_{j=0}^{q} \binom{q}{j} (-1)^{q-j} (n + x + y)^j A_n(x, y; p + q - j, 0)$$

$$= \sum_{j=0}^{q} \binom{q}{j} (-1)^{q-j} (n + x + y)^j$$

$$\times \sum_{k=0}^{n} \binom{n}{k} (n + x + y)^{n-k} \Delta^k x^{p+q-j+k}. \tag{17}$$

In particular we have for $p, q \geqslant 0, p + q < n$:

$$F_n(x, y; p, q) = 0. \tag{18}$$

And therefore our evaluation (17) of $A_n(x, y; -p, q)$ for $p, q \geqslant 0, p \geqslant q$ is a sum in which the number of terms is bounded for fixed p and q.

Finally we present an example of (4), the inverse of our main identity. For $0 \leqslant p < n$

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k (n + x + y)^k A_{n-k}(x, k + y; p - n, 0)$$

$$= F_n(x, y; p, 0) = 0. \tag{19}$$

Particularly, for $p = n - 1$

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k (n + x + y)^k A_{n-k}(x, y + k; -1, 0)$$

$$= \sum_{k=0}^{n} \binom{n}{k} (-1)^k (n + x + y)^k (n + x + y)^{n-k} x^{-1} = 0. \tag{20}$$

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References

1. J. Riordan, "Combinatorial Identities," Wiley, New York, 1968.