Classical and quantum wormholes in a flat Λ-decaying cosmology

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Abstract

We study the classical and quantum Euclidean wormholes for a flat Friedmann-Robertson-Walker universe with a perfect fluid including an ordinary matter source plus a source playing the role of a decaying cosmological term. It is shown that classical Euclidean wormholes exist for this model provided the strong energy condition is satisfied. Moreover, unlike the model adopted by Kim and Page in which quantum wormholes are incompatible with a cosmological constant, we show in the present model that quantum wormholes are compatible with a perfect fluid source including a decaying cosmological term.

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1 Introduction

Wormholes are usually considered as Euclidean metrics that consist of two large regions connected by a narrow throat. On the other hand, two kinds of wormholes, namely macroscopic and microscopic, are known. Macroscopic wormholes may provide the mechanism for black hole evaporation [1], while microscopic wormholes seem to play an important role in vanishing cosmological constant Λ [2]-[5]. One possible solution to the cosmological constant problem is that wormhole solutions can lead Λ to become a dynamical variable with a distribution function $P(\Lambda)$ [3]. This function is peaked with the Baum-Coleman-Hawking factor [3], [6], [7]

$$P(\Lambda) \sim \exp(1/\Lambda),$$

so predicting $\Lambda \to 0$. Wormholes have been studied mainly as instantons, namely solutions of the classical Euclidean field equations. In general, Euclidean wormholes can represent quantum tunneling between different topologies. These are saddle points in the path integral and form the basis in a semiclassical treatment, if one makes the dilute wormhole approximation of neglecting the interaction between the ends of different wormholes joining on the same large region [8].

It is well-known that real wormhole like solutions occur only for certain special kinds of matter that allow the Ricci tensor to have negative eigenvalues. These do not include pure Einstein gravity, or minimally coupled scalar fields (unless they are pure imaginary). But they include an antisymmetric tensor field whose field equations in four dimensions are equivalent to those of a scalar field [2]. There are also Yang-Mills solutions, however, in general do not seem to be local minima of the action [9]. Therefore, it is not clear that they really contribute to the semiclassical approximation. On the other hand, there are Yang-Mills solutions which are local minima of the action, but they exist only when the Yang-Mills field is not coupled to any fields in the fundamental representation [10]. Some solutions for matter content corresponding to $\mathcal{N} = 4$ SU(N) super Yang-Mills theory have also been obtained [11]. Recently, traversable wormholes coupled to nonlinear electrodynamics are also obtained [12].

The non existence of instantons, for general matter sources, not only makes it difficult to believe that wormholes are the mechanism for black hole evaporation but also casts doubt on whether wormholes are the reason why the cosmological constant is zero. Due to limited known classical wormhole solutions, Hawking and page advocated a different approach in which wormholes were regarded, not as solutions of the classical Euclidean field equations, but as solutions of the quantum mechanical Wheeler-DeWitt equation [8]. These wave functions have to obey certain boundary conditions in order that they represent wormholes. The boundary conditions seem to be that the wave function is exponentially damped for large tree geometries, and is regular in some suitable way when the tree-geometry collapses to zero.

Therefore, an open and interesting problem is whether Euclidean classical and quantum wormholes can occur for fairly general matter sources. Classical and quantum wormholes with perfect fluids and scalar fields have already been studied [13]. To the author's knowledge, the study of Λ-decaying cosmology in this framework has not received attention. In the present work, we shall consider a Λ-decaying cosmology and study its classical and quantum wormhole solutions.

2 Classical wormholes

We consider a Friedmann-Robertson-Walker (FRW) universe

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right], \tag{1}$$

where $a(t)$ is the scale factor and $k = 0, +1$ and -1 accounts for flat, closed and open universes, respectively. This metric evolves according to the Einstein equation [1]

$$\mathcal{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R} = T_{\mu\nu}, \tag{2}$$

where we take the energy momentum tensor $T_{\mu\nu}$ to be perfect fluid

$$T_{\mu\nu} = diag(-\rho, p, p, p). \tag{3}$$

The time-time and space-space components of the Einstein equation (2) leads respectively to

$$\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \frac{\rho}{3}, \tag{4}$$

$$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = -p. \tag{5}$$

where Eq.(4) is the Friedmann equation. Combining Eqs.(4) and (5) we obtain the acceleration equation

$$\frac{\ddot{a}}{a} = -\frac{1}{6}(\rho + 3p). \tag{6}$$

There is also a conservation equation $\nabla_\mu T^{\mu\nu} = 0$ whose time component gives the fluid equation

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0. \tag{7}$$

By analytic continuation $t \to it$, the Friedmann equation can be written in Euclidean form

$$\frac{\dot{a}^2}{a^2} = \frac{k}{a^2} - \frac{\rho}{3}, \tag{8}$$

whereas the Euclidean form of the fluid equation (7) has the same Lorentzian form.

In FRW models, wormholes are typically described by a constraint equation of the form

$$\frac{\dot{a}^2}{a^2} = \frac{1}{a^2} - \frac{const}{a^n}, \tag{9}$$

where the derivative is with respect to the Euclidean time. In order to have an asymptotically Euclidean wormhole it is necessary that $\dot{a}^2$ remains positive at large a and this requires $n > 2$. This wormhole represents two separate asymptotically Euclidean regions joined together by a

[1] We have used the units where $8\pi G = c = 1$.

throat with the finite size a_0 at which $\dot{a} = 0$. For example, $n = 4$ and $n = 6$ account for the wormholes corresponding to the conformal scalar field and axion, respectively[2]

Carlini and Mijic have introduced some possible wormhole solutions by analytic continuation of closed ($k = 1$) FRW universes [13]. In fact, for a perfect fluid equation of state $p = (\gamma - 1)\rho$, recollapsing closed universes require that the strong energy condition be satisfied, namely $\gamma > 2/3$. The continuation of this theory to the Euclidean domain then gives rise to wormhole solutions satisfying the strong energy condition.

Contrary to the model of Carlini and Mijic for *closed* FRW universes with the usual equation of state and ordinary matter source, we assume a *flat* FRW universe with a perfect fluid source combined of ordinary matter and a source evolving with the scale factor

$$\rho = \rho_m + \rho_v = \left(\frac{\rho_0}{a^{3\gamma}} - \frac{\Lambda_0}{a^2} \right). \tag{11}$$

The first term is the ordinary matter density (with the constant $\rho_0 > 0$) and the second term is a density playing the role of a decaying cosmological term (with the constant $\Lambda_0 > 0$). In fact, there are strong observational motivation for considering models in which $k = 0$ and Λ decreases as $\Lambda \sim R^{-m}$ (m is a parameter). For $0 \leq m < 3$ [14], the effect of decaying cosmological term on the cosmic microwave background anisotropy is studied and the angular power spectrum for different values of m and density parameter Ω_{m0} is computed. Models with $\Omega_{m0} \geq 0.2$ and $m \geq 1.6$ are shown to be in good agreement with data. For $m = 2$ [15], it is shown that in the early universe Λ could be several tens of orders bigger than its present value, but not big enough disturbing the physics in the radiation-dominant epoch in the standard cosmology. In the matter-dominant epoch such a varying Λ shifts the three space curvature parameter k by a constant which changes the standard cosmology predictions reconciling observations with the inflationary scenario. Such a vanishing cosmological term also leads to present creation of matter with a rate comparable to that in the steady-state cosmology [15].

Substitution for ρ and k in eq.(8) leads to

$$\frac{\dot{a}^2}{a^2} = \frac{1}{3} \left(\frac{\Lambda_0}{a^2} - \frac{\rho_0}{a^{3\gamma}} \right). \tag{12}$$

By defining $R = \sqrt{\frac{3}{\Lambda_0}} a$ we obtain

$$\frac{\dot{R}^2}{R^2} = \frac{1}{R^2} - \frac{\alpha_0}{R^{3\gamma}}, \tag{13}$$

[2]The typical solution of Eq.(9) is obtained, using the conformal time $dt = a d\tau$, as [16]

$$a^{n/2-1} \sim (Const)^{1/2} \cosh \left[\frac{n-2}{2} \tau \right]. \tag{10}$$

For example, the conformal scalar field ($n = 4$) leads to the typical wormhole shape

$$a \sim \cosh(\tau).$$

with the constant

$$\alpha_0 = \frac{\rho_0}{(\frac{3}{\Lambda_0})^{3\gamma/2}}. \tag{14}$$

This equation has the form of the constraint (9) describing an Euclidean wormhole with the correspondence $3\gamma = n$. Therefore, classical Euclidean wormholes are possible for the combined source (11) with any $\gamma > \frac{2}{3}$.

Now, we have to determine the equation of state. To this end, we substitute for ρ from Eq.(11) in the conservation equation (7) and obtain the following equation of state

$$\begin{aligned} p &= p_m + p_v \\ &= \rho_m(\gamma - 1) - \frac{1}{3}\rho_v, \end{aligned} \tag{15}$$

where the first term describes the standard equation of state for ordinary matter and the second term accounts for equation of state for the Λ-decaying source. It is easy to show that $\gamma > \frac{2}{3}$ leads the strong energy condition to hold for the total pressure p and total density ρ.

3 Quantum Wormholes

The number of known classical wormholes is so limited. It casts doubt on whether wormholes are important, only in the very restricted class of theories, in which the matter content allows wormhole instantons? To resolve this problem, Hawking and Page advocated a different approach and considered that solutions of the Wheeler-DeWitt equation could more generally represent the wormholes [8]. They realized that for the mini superspace models one may consider metrics of the Euclidean Friedmann form

$$ds^2 = N^2(t)dt^2 + a^2(t)d\Omega_3^2. \tag{16}$$

If N is imaginary, this is the Lorentzian metric, and if N is real, it is the metric of an Euclidean wormhole. However, solutions of the Wheeler-DeWitt equation are independent of the lapse function N and t. So, they can be interpreted either as Friedmann universe, or as wormholes according to the appropriate boundary conditions. The boundary conditions for wormholes seems to be that the wave functions should decay exponentially for large scale factor a, so as to represent Euclidean space, and that they be regular in some suitable way as $a \to 0$, so that no singularities are present.

The quantum mechanical version of Eq.(13) is given by [17]

$$\left(R^2\frac{d^2}{dR^2} + qR\frac{d}{da} + \alpha_0 R^{6-3\gamma} - R^4\right)\Psi(R) = 0, \tag{17}$$

where q represents part of the factor ordering ambiguities. We set $q = 0$ and study the potential to get some idea as to when a Euclidean domain occurs at large a by considering the sign of the potential

$$U(R) = \alpha_0 R^{4-3\gamma} - R^2, \tag{18}$$

in the equation

$$\left[\frac{d^2}{dR^2} + U(R)\right]\Psi(R) = 0. \tag{19}$$

For positive potential $U(R) > 0$, oscillating solutions occur which represent Lotentzian metrics. On the contrary, for negative potential $U(R) < 0$, wormhole solutions can occur which are asymptotically Euclidean at large R. The potential is negative for $\gamma > 2/3$ (in the case of positive energy density ρ_0) which is the same strong energy condition that is obtained by Carlini and Mijic for the occurrence of classical wormhole solutions. Therefore, wormholes obeying the Hawking-Page boundary condition at large R, occur when the strong energy condition $\gamma > 2/3$ is valid for the source (11). In other words, perfect fluid sources violating the strong energy condition are incompatible with wormholes obeying the Hawking-Page boundary condition. The presence of any matter source with $\gamma < 2/3$ will eventually dominate for large R and prevent the asymptotically Euclidean wormholes to occur.

Asymptotically Euclidean property of the wave function is not sufficient to make it a wormhole. It also requires regularity for small R. In order to realize this, we can ignore R^4 term in Eq.(17) as $R \to 0$, when $\gamma > 2/3$. In this case, the Wheeler-DeWitt equation (17) (for $\gamma \neq 2$) simplifies to a Bessel differential equation with solution

$$\Psi(R) \simeq R^{(1-q)/2}\left[c_1 J_\nu\left(\frac{2\sqrt{\alpha_0}}{3(2-\gamma)}R^{3-3\gamma/2}\right) + c_2 Y_\nu\left(\frac{2\sqrt{\alpha_0}}{3(2-\gamma)}R^{3-3\gamma/2}\right)\right], \tag{20}$$

where use has been made of $\nu \equiv (1-q)/3(2-\gamma)$. The wormhole boundary condition at $R \to 0$ is satisfied for the Bessel function of the J kind.

In the particular case $\gamma = 2$, the solution of Eq.(17) with $q = 1$ is a linear combination of Bessel functions $J_{\pm i\sqrt{\alpha_0}/2(ia^2/2)}$ which oscillates an infinite number of times at $R \to 0$ and therefore can not satisfy the required regularity condition for a quantum wormhole.

On the other hand, in the case of $\gamma = 4/3$ which represents radiation (or equivalently, that of a conformally coupled scalar field) dominated FRW ansatz, Eq.(17) for $q = 0$ is written as

$$\left(\frac{d^2}{dR^2} + \alpha_0 - R^2\right)\Psi(R) = 0, \tag{21}$$

which is in the form of a parabolic equation with solution in terms of confluent hypergeometric functions [18]

$$\Psi(R) \simeq \exp(-R^2/2)[c_3\, {}_1F_1(\frac{1}{4}(1-\alpha_0); 1/2; R^2) + c_4\, {}_1F_1(\frac{1}{4}(3-\alpha_0); 3/2; R^2)]. \tag{22}$$

For example, for $c_3 = 0$ with $\alpha_0 = (35, 55)$, and $c_4 = 0$ with $\alpha_0 = (25, 37)$, we obtain regular oscillations at $R \to 0$, and Euclidean regimes for large R, see Figs.1, 2, 3, 4.

Therefore, Hawking-Page boundary conditions are satisfied for some special values of α_0 and so we have a spectrum of wormholes. Considering Eq.(14) it turns out that for a given equation of state γ, the existence of quantum wormholes depends on the special values of ρ_o and Λ_0. In other words the spectrum of wormholes depends on the spectrum of ρ_o and Λ_0. We notice that there is no such constraint on the values of ρ_o and Λ_0 for the occurrence of classical wormholes.

Conclusion

The classical and quantum Euclidean wormhole solutions have been studied for a flat Friedmann-Robertson-Walker metric coupled with a perfect fluid combined of an ordinary matter source and a source playing the role of a decaying cosmological term. Kim and Page had already found that the quantum wormholes are incompatible with a cosmological constant. We have shown here that classical and quantum Euclidean wormholes exist for the case of a perfect fluid source including a decaying cosmological term.

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Figure captions

Figure 1. A quantum wormhole solution for the case $\gamma = 4/3$ with $c_3 = 0$ and $\alpha_0 = 35$.

Figure 2. A quantum wormhole solution for the case $\gamma = 4/3$ with $c_3 = 0$ and $\alpha_0 = 55$.

Figure 3. A quantum wormhole solution for the case $\gamma = 4/3$ with $c_4 = 0$ and $\alpha_0 = 25$.

Figure 4. A quantum wormhole solution for the case $\gamma = 4/3$ with $c_4 = 0$ and $\alpha_0 = 37$.

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