F. E. Udwadia
Professor of Aerospace and Mechanical Engineering, Mathematics, and Civil Engineering,
University of Southern California,
Los Angeles, CA 90089-1453

R. E. Kalaba
Professor of Biomedical Engineering, Economics, and Electrical Engineering,
University of Southern California,
Los Angeles, CA 90089

What is the General Form of the Explicit Equations of Motion for Constrained Mechanical Systems?

This paper presents the general form of the explicit equations of motion for mechanical systems. The systems may have holonomic and/or nonholonomic constraints, and the constraint forces may or may not satisfy D'Alembert's principle at each instant of time. The explicit equations lead to new fundamental principles of analytical mechanics.
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Introduction

Since its inception more than 200 years ago, analytical mechanics has been continually drawn to the determination of the equations of motion for constrained mechanical systems. Following the fundamental work of Lagrange [1] who bequeathed to us the so-called Lagrange multipliers in the process of determining these equations, numerous scientists and mathematicians have attempted this central problem of analytical dynamics. A comprehensive reference list would run into several hundreds; hence we shall provide here, by way of a thumbnail historical review of the subject, only some of the significant milestones and discoveries. In 1829, Gauss [2] introduced a general principle for handling constrained motion, which is commonly referred to today as Gauss's Principle; Gibbs [3] and Appell [4] independently obtained the so-called Gibbs-Appell equations of motion using the concept of (felicitously chosen) quasi-coordinates; Poincare [5], using group theoretic methods, generalized Lagrange's equations to include general quasi-coordinates; and Dirac [6], in a series of papers provided an algorithm to give the Lagrange multipliers for constrained, singular Hamiltonian systems. Udwadia and Kalaba [7] gave the explicit equations of motion for constrained mechanical systems using generalized inverses of matrices, a concept that was independently discovered by Moore [8] and Penrose [9]. The use of this powerful concept, which was further developed from the late 1950s to the 1980s, allows the generalized-inverse equations (Udwadia and Kalaba [7]) to go beyond, in a sense, those provided earlier; for, they are valid for sets of constraints that could be nonlinear in the generalized velocities, and that could be functionally dependent. Thus the problem of obtaining the equations of motion for constrained mechanical systems has a history that is indeed as long as that of analytical dynamics itself.

Yet, all these efforts have been solely targeted towards obtaining the equations of motion for holonomically and nonholonomically constrained systems that *all* obey D'Alembert's principle of virtual work at each instant of time. This principle, though introduced by D'Alembert, was precisely stated for the first time by Lagrange. The principle in effect makes *an assumption* about the nature of the forces of constraint that act on a mechanical system. It assumes that at each instant of time, t, during the motion of the mechanical system, the constraint forces do *no* work under virtual displacements.

This seemingly sweeping assumption is indeed a tribute to the genius of Lagrange, because: (1) it gives exactly the right amount of additional information regarding the nature of the constraint forces in a general constrained mechanical system so that the equations of motion are *uniquely* determined, and are thus in conformity with practical observation; (2) in the mathematical modeling of a mechanical system, it obviates the need for the mechanician to investigate each specific mechanical system at hand and to determine the nature of the constraint forces prevalent; and, (3) it yields equations of motion for constrained systems that seem to work well (or at least sufficiently well) in numerous practical situations.

However, there are many mechanical systems that are commonplace in Nature where D'Alembert's principle is not valid, such as when sliding friction becomes important. Such situations have so far been considered to lie beyond the compass of the Lagrangian formulation of mechanics. As stated by Goldstein [10], "This [total work done by forces of constraint under virtual displacements equal to zero] is no longer true if sliding friction is present, and we must exclude such systems from our [Lagrangian] formulation." And Pars [11] (p. 14) in his treatise on analytical dynamics writes, "There are in fact systems for which the principle enunciated [D'Alembert's Principle] ... does not hold. But such systems will not be considered in this book."

Constraint forces that *do* work under virtual displacements are called nonideal constraint forces, and such constraints themselves are often referred to as being nonideal. While it is possible, at times, to handle problems with holonomic, nonideal constraints (like sliding friction) by using a Newtonian approach, to date we do not have a general formulation for obtaining the equations of motion for systems where we have nonholonomic, nonideal constraints, i.e., nonholonomic constraints where the constraint forces do work under virtual displacements. The aim of this paper is to include such systems within the Lagrangian formulation of mechanics, and further to develop the general form of the explicit equations of motion for constrained systems that may or may not obey D'Alembert's principle at each instant of time. The approach we follow here is based on linear algebra, and it is different from that of Refs. [12], [13], and [14]. It leads us to the general structure of the equation of motion for constrained systems, and culminates in the statement of two fundamental principles of analytical dynamics.

Formulation of the Problem of Constrained Motion

Consider an "unconstrained" mechanical system described by the Lagrange equations

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$$M(q,t)\ddot{q}=Q(q,\dot{q},t), \quad q(0)=q_0, \quad \dot{q}(0)=\dot{q}_0 \tag{1}$$

where $q(t)$ is the n-vector (i.e., n by 1 vector) of generalized coordinates, M is an n by n symmetric, positive-definite matrix, Q is the ''known'' n-vector of impressed (also, called ''given'') forces, and the dots refer to differentiation with respect to time. By unconstrained, we mean that the components of the n-vector $\dot{q}_0$ can be arbitrarily specified. By ''known,'' we mean that the n-vector Q is a known function of its arguments. The acceleration, a, of the unconstrained system at any time t is then given by the relation $a(q,\dot{q},t)=M^{-1}(q,t)Q(q,\dot{q},t)$.

We next subject the system to a set of $m=h+s$ consistent, equality constraints of the form

$$\varphi(q,t)=0 \tag{2}$$

and

$$\psi(q,\dot{q},t)=0, \tag{3}$$

where φ is an h-vector and ψ an s-vector. Furthermore, we shall assume that the initial conditions q_0 and $\dot{q}_0$ satisfy these constraint equations at time $t=0$, i.e., $\varphi(q_0,0)=0$, $\dot{\varphi}(q_0,\dot{q}_0,0)=0$, and $\psi(q_0,\dot{q}_0,0)=0$.

Assuming that Eqs. (2) and (3) are sufficiently smooth,[1] we differentiate Eq. (2) twice with respect to time, and Eq. (3) once with respect to time, to obtain an equation of the form

$$A(q,\dot{q},t)\ddot{q}=b(q,\dot{q},t), \tag{4}$$

where the matrix A is m by n, and b is the m-vector that results from carrying out the differentiations. We place no restrictions on the rank of the matrix A.

This set of constraint equations includes, among others, the usual holonomic, nonholonomic, scleronomic, rheonomic, catastatic, and acatastatic varieties of constraints; combinations of such constraints may also be permitted in Eq. (4). Furthermore, the functions in (3) could be nonlinear in $\dot{q}$, and the m constraint equations need not be independent of one another.

It is important to note that Eq. (4), together with the initial conditions, is equivalent to Eqs. (2) and (3).

The equation of motion of the constrained mechanical system can then be expressed as

$$M(q,t)\ddot{q}=Q(q,\dot{q},t)+Q^c(q,\dot{q},t), \quad q(0)=q_0, \quad \dot{q}(0)=\dot{q}_0 \tag{5}$$

where the additional ''constraint force'' n-vector, $Q^c(q,\dot{q},t)$, arises by virtue of the constraints that are imposed on the unconstrained system, which we have described by Eq. (1). Since the n-vector Q is known, our aim is to determine a *general* explicit form for Q^c at any time t.

We shall see below that in any constrained mechanical system, the total constraint force n-vector, Q^c, at each instant of time t, can be thought of as made up of two components: $Q^c=Q_i^c+Q_{ni}^c$. The first component corresponds to the force of constraint, Q_i^c, *that would act were all the constraints ideal at that instant of time*; the second component, Q_{ni}^c, arises because of the nonideal nature of the constraints. This latter component is *situation specific* and needs to be specified by the mechanician entrusted with modeling the mechanical system. However, we shall show that this component too must always occur in the explicit equation of motion *in a specific form*.

In what follows, for brevity, we shall suppress the arguments of the various quantities, unless necessary for purposes of clarification.

[1]We assume throughout this paper that the presence of constraints does not change the rank of the matrix M. This is almost always true in mechanical systems.

The General Form of the Explicit Equation of Motion for any Constrained Mechanical Systems

We begin by stating our general result in the following three-part statement.

(1) The general ''explicit'' equation of motion at time t for *any* constrained mechanical system, *whether or not the constraint forces satisfy D'Alembert's Principle at that time t*, is given by

$$\begin{aligned} M\ddot{q}&=Q+Q^c=Q+Q_i^c+Q_{ni}^c \\ &=Q+M^{1/2}B^+(b-AM^{-1}Q)+M^{1/2}(I-B^+B)z \end{aligned} \tag{6}$$

where the matrix $B=AM^{-1/2}$, B^+ is the generalized inverse[2] of the matrix B, and $z(q(t),\dot{q}(t),t)$ is some suitable n-vector. (When z is C^1, Eq. (6) yields a unique solution.) The matrix A is defined in relation (4), as is the m-vector b. The n-vector Q is the impressed force. By ''explicit'' we mean here that the acceleration n-vector, $\ddot{q}$, on the left-hand side of Eq. (6) is explicitly expressed in terms of quantities that are functions of q, $\dot{q}$, and t on the right-hand side.

Alternately stated, the total constraint force n-vector, Q^c, at any instant of time t is made up of the sum of two components Q_i^c and Q_{ni}^c that can be explicitly written as

$$Q_i^c=M^{1/2}B^+(b-AM^{-1}Q), \tag{7}$$

and,

$$Q_{ni}^c=M^{1/2}(I-B^+B)z. \tag{8}$$

(2) To mathematically model a *given* constrained mechanical system adequately, the mechanician must *specify* the vector $z(q,\dot{q},t)$ in the third member on the right-hand side of Eq. (6) at each instant of time. This may be done by inspection of the specific system at hand, by analogy with other systems that the mechanician may have dealt with in the past, by experimentation with the specific system or similar systems, or otherwise.

(3) However, no matter how the mechanician comes up with the prescription of the n-vector z for adequately modeling a *given* constrained mechanical system under consideration, specification of this n-vector at each time t uniquely determines Q_{ni}^c, and hence the acceleration n-vector, $\ddot{q}(t)$, of the constrained system. Such a prescription of $z(t)$ is *equivalent* to prescribing the work done by all the constraint forces under virtual displacements at that time t, in the following sense.

(a) When the vector $z(t)$ is prescribed, it can always be expressed as

$$z(t)=M^{-1/2}(q,t)C(q,\dot{q},t) \tag{9}$$

since, M is a positive definite matrix. The total work done, $W:=v^TQ^c$, by all the forces of constraint under (nonzero) virtual displacements v at time t, is then given by

$$W(t):=v(t)^TQ^c=v(t)^TC(q,\dot{q},t). \tag{10}$$

(b) When, for a given specific constrained mechanical system, the work done, W, at time t by the forces of constraint under virtual displacements v is prescribed through specification of the n-vector $C(q,\dot{q},t)$ such that

$$W(t)=v(t)^TC(q,\dot{q},t), \tag{11}$$

this determines the equation of motion of the constrained system *uniquely* at time t. This equation of motion is obtained by setting $z(t)=M^{-1/2}(q,t)C(q,\dot{q},t)$, in Eq. (6). The work done, $W(t)$, may be positive, zero, or negative, at the instant of time t. □

[2]Some of the basic properties of the Moore-Penrose generalized inverse that are used throughout this paper may be found in Chapter 2 of Ref. [15].

We note from Eq. (9) above, that prescribing z to be the zero n-vector at any time t, is equivalent to specifying $C=0$ at that specific time t, and then by (10), the constraint forces do *no* work under virtual displacements and therefore they satisfy D'Alembert's principle at that instant of time t. In what follows we shall also show that when the constraints do no work under virtual displacements at time t, because of Eq. (10), the n-vector C must belong to the range space of A^T; the third member on the right in Eq. (6) then becomes zero at that time. Further, if throughout the motion of the constrained system the work done by the constraint forces under virtual displacements is zero, then the third member on the right-hand side in Eq. (6) disappears for all time. The equation of motion (6) then becomes

$$M\ddot{q}=Q+Q^c=Q+Q_i^c=Q+M^{1/2}B^+(b-AM^{-1}Q), \quad (12)$$

which is identical to that obtained by Udwadia and Kalaba [7] for systems that obey D'Alembert's principle. Equation (12) is equivalent to the Gibbs-Appell equations (see Ref. [15]). We then see that the component Q_i^c in Eq. (7) therefore gives the constraint force at time t that *would be generated were all the constraints ideal at that time*. And Q_{ni}^c explicitly gives the contribution to the total constraint force, Q^c, made by the nonideal nature of the constraints.

Were the acceleration, $a=M^{-1}Q$, of the unconstrained system at time t to be inserted into the equation of constraint (4), this equation would not, in general, be satisfied at that time. The extent to which the constraint (Eq. (4)) would not be satisfied by this acceleration, a, of the unconstrained system at time t would then be given by

$$e=b-Aa=b-AM^{-1}Q. \quad (13)$$

The force of constraint can now be rewritten as

$$Q^c=Q_i^c+Q_{ni}^c=M^{1/2}B^+e+M^{1/2}(I-B^+B)z. \quad (14)$$

Also, the effect of this constraint force in altering the acceleration of the unconstrained system can be explicitly determined. For, the deviation, $\Delta\ddot{q}$, at time t of the acceleration of the constrained system from that of the unconstrained system becomes, by Eq. (6),

$$\Delta\ddot{q}=\ddot{q}-a=M^{-1/2}B^+e+M^{-1/2}(I-B^+B)z. \quad (15)$$

Equations (14) and (15) lead us to a new fundamental principle of Lagrangian mechanics which we now state in two equivalent forms.

1 A constrained mechanical system evolves in such a way that, at each instant of time, the deviation, $\Delta\ddot{q}$, of its acceleration from what it would have been at that instant had there been no constraints on it, is given by a sum of two components: the first component is proportional to the extent, e, to which the unconstrained acceleration does not satisfy the constraints at that instant of time, the matrix of proportionality being the matrix $M^{-1/2}B^+$; the second is proportional to an n-vector z that needs, in general, to be specified at each instant of time, the matrix of proportionality being $M^{-1/2}(I-B^+B)$, where $B=AM^{-1/2}$. The specification of z at any time, t, is dependent on the nature of the forces of constraint that are generated. Its specification for a given system at hand is tantamount to the specification of the total work done under virtual displacements by all the forces of constraint at that time. Such a specification of the work done at each instant of time uniquely determines the equation of motion of the constrained system.

2 At each instant of time t, the force of constraint acting on a constrained mechanical system is made up of two components: the first component is proportional to the extent, e, to which the unconstrained acceleration of the system does not satisfy the constraints at that instant of time, and the matrix of proportionality is $M^{1/2}B^+$; the second is proportional to an n-vector z that, in general, needs specification at each instant of time, the matrix of proportionality being $M^{1/2}(I-B^+B)$, where $B=AM^{-1/2}$. This vector z is specific to a given mechanical system and needs to be prescribed by the mechanician who is modeling the system. Whether or not the constraints are ideal, the first component is always present and constitutes the constraint force at the instant of time t that would have been generated were all the constraints ideal at that time. The second component depends on the nature of the constraint forces generated in the specific mechanical system that is being modeled; it prevails only when the total work done by the constraint forces under virtual displacements differs from zero.

Proof of the General Form of the Equations of Motion for Constrained Systems

We begin by considering the "scaled accelerations" defined by the relations

$$\ddot{q}_s=M^{1/2}\ddot{q}; \quad (16)$$

$$a_s=M^{-1/2}Q=M^{1/2}a; \quad (17)$$

and,

$$\ddot{q}_s^c=M^{-1/2}Q^c=M^{1/2}\ddot{q}^c. \quad (18)$$

By Eq. (5), we then have

$$\ddot{q}_s=a_s+\ddot{q}_s^c. \quad (19)$$

Furthermore, Eq. (4) can be expressed as

$$B\ddot{q}_s=b, \quad (20)$$

where

$$B=AM^{-1/2}. \quad (21)$$

Consider the matrices $T=B^+B$ and $N=(I-B^+B)$, where the matrix B^+ is the Moore-Penrose (MP) inverse of the matrix B. The matrix T is an orthogonal projection operator since $(B^+B)^T=B^+B$, and $T^2=(B^+B)(B^+B)=B^+B=T$. Also, N is an orthogonal projection operator since $(I-B^+B)^T=I-(B^+B)^T=I-B^+B$, and $N^2=N$. Since $R^n=\mathcal{R}(B^T)\oplus\mathcal{N}(B)$, any n-vector w has a unique orthogonal decomposition $w=B^+Bw+(I-B^+B)w$; and so also our n-vector $\ddot{q}_s$. This yields the identity

$$\ddot{q}_s=B^+B\ddot{q}_s+(I-B^+B)\ddot{q}_s. \quad (22)$$

Using relation (20) in the first member on the right, and relation (19) in the second member, we obtain

$$\ddot{q}_s=a_s+B^+(b-Ba_s)+(I-B^+B)\ddot{q}_s^c. \quad (23)$$

Comparison of Eq. (19) with Eq. (23) then yields

$$B^+B\ddot{q}_s^c=B^+(b-Ba_s) \quad (24)$$

which can be solved for $\ddot{q}_s^c$ to yield

$$\ddot{q}_s^c=B^+BB^+(b-Ba_s)+\{I-(B^+B)^+(B^+B)\}z$$
$$=B^+(b-Ba_s)+(I-B^+B)z \quad (25)$$

for some n-vector z.

Equation (18), then gives

$$Q^c=M^{1/2}B^+(b-Aa)+M^{1/2}(I-B^+B)z \quad (26)$$

and the general equation of motion of the constrained system, by Eq. (5), becomes

$$M\ddot{q}=Q+Q^c=Q+M^{1/2}B^+(b-Aa)+M^{1/2}(I-B^+B)z \quad (27)$$

where z is some n-vector. q.e.d.

To obtain the unique equation of motion for a *specific* mechanical system, the mechanician needs to prescribe the vector $z(q(t),\dot{q}(t),t)$ at each instant of time. Specification of the vector

$z(t)$ yields explicitly and uniquely the component Q^c_{ni} of the constraint force, Q^c, at each instant of time t. In fact, given an n-vector z at a specific time t, we can form the n-vector $C = M^{1/2}z$ at time t. The vector C can *now be interpreted* as providing the work done, $W = v^T C$, by the constraint force n-vector Q^c under virtual displacements v at time t.

We now show that Q^c_{ni} can also be uniquely determined at each instant of time t by specifying the work done by the constraint force n-vector, Q^c, under virtual displacements at that time. *Proof*: A virtual displacement is any nonzero n-vector v such that $Av = 0$ (see Ref. [15]). Using Eq. (21) this relation can also be written as $Av = (AM^{-1/2})M^{1/2}v = B(M^{1/2}v) = B\mu = 0$, where we have denoted the n-vector $M^{1/2}v$ by μ. Thus a virtual displacement can also be considered as any (nonzero) n-vector μ such that $B\mu = 0$. Using Eq. (27), the work done by the force of constraint under all virtual displacements v is then given by

$$\begin{aligned} W &:= v^T Q^c = v^T(Q^c_i + Q^c_{ni}) \\ &= v^T M^{1/2} B^+(b - Aa) + v^T M^{1/2}(I - B^+B)z \\ &= \mu^T B^+(b - Aa) + \mu^T(I - B^+B)z. \end{aligned} \tag{28}$$

The first member in the last expression on the right of equation (28) is zero since $B\mu = 0$ implies $\mu^T B^+ = 0$. *Hence the component Q^c_i of the total force of constraint, Q^c, does no work under virtual displacements*. Equation (28) then becomes

$$W := v^T Q^c = v^T Q^c_{ni} = \mu^T z = v^T(M^{1/2}z). \tag{29}$$

Let $W(t)$ to be prescribed at time t by the mechanician through a specification of the n-vector $C(q,\dot{q},t)$ so that $W := v^T Q^c = v^T C$. Then by Eq. (29), we have

$$v^T(M^{1/2}z) = v^T C. \tag{30}$$

Since v is such that $Av = 0$, this requires that

$$z = M^{-1/2}(C + A^T w) = M^{-1/2}C + B^T w \tag{31}$$

where w is any arbitrary m-vector. Using this expression for z in Eq. (27) we obtain the *unique* equation of motion of the constrained system to be

$$\begin{aligned} M\ddot{q} &= Q + Q^c = Q + Q^c_i + Q^c_{ni} = Q + M^{1/2}B^+(b - Aa) \\ &\quad + M^{1/2}(I - B^+B)M^{-1/2}C, \end{aligned} \tag{32}$$

since $(I - B^+B)B^T = \{B(I - B^+B)\}^T = 0$.

We now see that Eq. (6) is identical to Eq. (32) with $z = M^{-1/2}C$! The component of z in the range space of B^T—the second member on the right in Eq. (31)—does not affect Q^c_{ni}, and therefore the equation of motion of the constrained system.

Though the n-vector $C(t)$ specifies the work done, $W := v^T Q^c = v^T Q^c_{ni} = v^T C$, by the constraint force under all virtual displacements v at time t, Eq. (32) states that, in general, $Q^c_{ni} \neq C$. At instants of time t when $W = (v^T M^{1/2})(M^{-1/2}C(q,\dot{q},t)) = 0$, $M^{-1/2}C$ belongs to the range space of B^T, and hence by Eq. (32), $Q^c_{ni} = 0$ since $(I - B^+B)B^T = 0$. If further, $W = 0$ for *all* time, then the force of constraint satisfies D'Alembert's principle, and $Q^c_{ni}(t) \equiv 0$; the equation of motion for the constrained system then reduces to that given in (12). At instants of time t when $M^{-1/2}C$ belongs to the null space of B, $Q^c_{ni} = C$. In general, the n-vector $M^{-1/2}C$ can have components in both the null space of B and the range space of B^T. We note that at each instant of time, it is only the component of $M^{-1/2}C$ in the null space of B that contributes to Q^c_{ni}, and hence to the equation of motion of the constrained system.

Conclusions

The equations of motion for constrained systems obtained to date have all been based upon D'Alembert's principle of virtual work. So far, no general equations of motion have been discovered within the Lagrangian formalism in situations where this central principle of analytical dynamics is not applicable.

This paper provides the general explicit form of the equation of motion for *any* holonomically and/or nonholonomically constrained mechanical system. The equation is

$$M\ddot{q} = Q + M^{1/2}B^+(b - AM^{-1}Q) + M^{1/2}(I - B^+B)z. \tag{33}$$

The n-vector Q is the given force, the m by n matrix A and the m-vector b are defined in Eq. (4), $B = AM^{-1/2}$, and B^+ is the generalized inverse of B. *The equation applies to all constrained mechanical systems whether or not they satisfy D'Alembert's principle*. The second member on the right in Eq. (33) explicitly gives the force of constraint, Q^c_i that *would have been* generated at time t were all the constraint forces ideal, and thus satisfy D'Alembert's principle. The third member on the right in Eq. (33) explicitly gives the contribution, Q^c_{ni}, to the total force of constraint because of the presence of nonideal constraints.

To obtain the equation of motion for a given, specific, mechanical system, the mechanician needs to provide the n-vector $z(q,\dot{q},t)$ suitably at each instant of time, thereby uniquely specifying the third member on the right in Eq. (33). The provision of this vector $z(t)$ depends on the judgement and discernment of the mechanician and may be determined by experiment, experience, intuition, inspection, or otherwise. However, no matter how this vector is arrived at, the total work done, $W(t) := v^T(t)Q^c(t)$, by the force of constraint under virtual displacements $v(t)$ at any instant of time t is always given by $v^T(t)C(t)$, where the n-vector $C(t) = M^{1/2}(q,t)z(q,\dot{q},t)$. This work, $W(t)$, may, in general, be positive, zero, or negative.

We show that to model a *given* constrained mechanical system adequately one needs, in general, to provide *more than* just the equations of constraint (Eqs. (2) and (3)), be they holonomic or nonholonomic. While at each instant of time the component Q^c_i of the total constraint force n-vector, Q^c, is determined solely from the kinematical description of the constraints (Eqs. (2) and (3)), to determine the component Q^c_{ni} one *always* needs to rely on the mechanician's discernment and judgement. However, as shown in the equation above, this component (see also Eq. (8)) *must* appear in a *specific form* in the explicit equation of motion of the constrained system. When the mechanical system satisfies D'Alembert's principle at every instant of time, $Q^c_{ni}(t) \equiv 0$, and the third member on the right in (33) becomes zero. Then our general equation yields the known equation of motion ([15]) for constrained systems that satisfy D'Alembert's principle.

It is perhaps noteworthy that though the equations of motion of even very simple mechanical systems are often highly nonlinear, the general form of the equation of motion obtained here relies on techniques from linear algebra. The fundamental principles of analytical dynamics obtained in this paper may have been impossible to state in such a simple form without the concept of the generalized inverse of a matrix, a concept first invented by Penrose [9].

The equation of motion obtained in this paper appears to be the simplest and most general so far discovered for mechanical systems within the framework of classical mechanics.

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