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Optimal Multi-unit Auctions

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1. Introduction

Recently, a large literature has examined alternative methods for auctioning off an indivisible good. (See McAfee and McMillan 1987; Milgrom 1986; and Wilson 1987 for surveys.) Particular attention has been paid to two auctions used frequently in practice: the open, ascending-bid auction (also called the English auction), and the sealed, high-bid auction. A theoretical benchmark is provided by the Revenue Equivalence Theorem (Vickrey 1961b; Myerson 1981, and Riley and Samuelson 1981). This theorem asserts that, when each bidder's reservation price for the good is an independent draw from the same distribution and bidders are risk-neutral, the two common auctions give rise to exactly the same expected revenue for the seller.[1]

A good deal of research has considered the implications of relaxing one or more of the underlying hypotheses. Thus, Holt (1980) substitutes risk-averse for risk-neutral buyers and shows that, in this case, the sealed-bid auction generates greater expected revenue than its open counterpart.

In contrast, Milgrom and Weber (1982) show that, when reservation prices are not independent but are positively correlated, the additional informational about other buyers emerging in the open auction raises revenue on average relative to that in the sealed-bid auction.

A third strand of this research (Maskin and Riley 1986) relaxes symmetry. That is, buyers' reservation values are no longer postulated to be identically distributed. In this case, the ranking of the two auctions depends on how the distributions vary across buyers.

Rather than simply compare the expected revenue from specific auction schemes, one may wish to characterize *optimal* selling procedures, that is, selling procedures that maximize the seller's expected revenue. Under the hypotheses of the Revenue Equivalence Theorem, and provided that the distribution of reservation prices is sufficiently

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[1] For a formal statement of this result, generalized to the case of multiple units, see Section 2.

regular (see Section 3 for a precise definition of regularity), the open- and sealed-bid auctions are both optimal if the seller sets an appropriate minimum allowable bid (called a reserve price). Myerson (1981) characterizes optimal auctions when regularity fails and also when the symmetry assumption is dropped. Matthews (1983), Maskin and Riley (1984b), and Moore (1984) study the case of risk-averse buyers, whereas Myerson (1981), Maskin and Riley (1981), and Cremer and McLean (1985) consider correlated reservation prices. Finally, Harris and Raviv (1981) relax the assumption that only a single good is to be sold.

This last paper is the starting point of our analysis here. For the special case of a uniform distribution of reservation prices, Harris and Raviv show that the Revenue Equivalence Theorem continues to hold if there are multiple units for sale and each buyer wishes to purchase at most a single unit. Here we establish equivalence for all distributions, and also show that, as long as the regularity assumption mentioned above is satisfied, the standard auctions with appropriate reserve prices are optimal for the seller. In addition, we characterize the optimal auction when this restriction is violated.

We then relax the restriction to unit demand and instead assume simply that each buyer has a downward-sloping demand curve. We observe that, in general, the standard auctions are no longer optimal. Instead, an optimal procedure is to set a payment schedule $T(q)$ and ask each buyer to submit an order q; a buyer who demands q pays $T(q)$. If aggregate demand is less than supply, the auctioneer fills each order. If, however, orders exceed supply, the auctioneer scales down each buyer's demand, in a predetermined way, until the capacity constraint is met.

The optimal procedure is thus a nonlinear pricing scheme modified to take account of the supply constraint. Not surprisingly, therefore, the methods of analysis build on earlier work on nonlinear pricing, in particular that of Mussa and Rosen (1978) and Maskin and Riley (1984a).

2. Formulation of the Seller's Optimization Problem

The seller has q_0 units of a good for sale. There are n buyers, each of whose 'type' v is drawn independently from the same distribution $F(v)$. A buyer of type v has preferences represented by the utility function

$$U(q, R, v) \equiv \int_0^q p(x, v)\, dx - R \equiv N(q, v) - R \tag{1}$$

where q is the number of units purchased from the seller and R is total spending on these units. The seller and other buyers do not observe a buyer's v but know that it is drawn from $F(v)$. Throughout, we shall assume that higher levels of v are associated with higher demand.

Moreover, we suppose that buyers' demand curves are never positively sloped and that demand is finite for all $p \geq 0$.

To be precise, we impose the following restrictions.

ASSUMPTION A1. For all v, the demand price function $p(q, v)$ is finite, twice continuously differentiable, strictly decreasing in q, and strictly increasing in v whenever p is greater than zero.

Since it is of independent interest, we shall sometimes make the alternative assumption of unit demand, as follows.

ASSUMPTION A1*. *Unit Demand.* Preferences are given by the demand price function

$$p(q, v) = \begin{cases} v, & q \leq 1 \\ 0, & q > 1, \end{cases}$$

so that v is the buyer's reservation price.

We also assume that the unobservable parameter v is continuously distributed and that the cumulative distribution function $F(\cdot)$ satisfies the following assumption.

ASSUMPTION A2. The cumulative distribution function $F(v)$ is strictly increasing and continuously differentiable on the interval $[0, \bar{v}]$, with $F(0) = 0$ and $F(\bar{v}) = 1$.

Although, in general, it is not possible to rule out gains to randomized selling procedures, we show in Section 5 that, under a fairly weak restriction on the distribution of types, the following assumption is sufficient for the optimal selling scheme to be deterministic (if we interpret (1) to be a buyer's von Neumann–Morgerstern utility function).

ASSUMPTION B1. *Non-decreasing Price Elasticity.* Demand elasticity is non-decreasing in the demand price. That is,

$$\frac{\partial}{\partial v}\left(-\frac{q}{p}\frac{\partial p}{\partial q}\right) \leq 0.$$

For a buyer of type v, formula (1) gives us

$$\frac{\partial}{\partial v}\left(-\frac{\partial^2 U}{\partial q^2}\Big/\frac{\partial U}{\partial q}\right) = \frac{1}{q}\frac{\partial}{\partial v}\left(-\frac{q}{p}\frac{\partial p}{\partial q}\right).$$

Assumption B1 implies, therefore, that absolute risk aversion with respect to consumption is non-decreasing in v.

Readers should note that we have a great deal of flexibility of our choice of a parameterization. In particular, if $p(q, v)$ represents a family of inverse demand curves satisfying Assumptions A1, A2, and B1, then $p[q, \omega(v)]$ represents the same family and also satisfies these three assumptions if $\omega(\cdot)$ is strictly increasing and twice continuously differentiable. For convenience, we shall henceforth choose, without loss of generality, a parametrization for which the increases in demand price are non-decreasing as v rises.

ASSUMPTION B2. $p_{22}(q, v) \leq 0$.

Elsewhere (Maskin and Riley 1984a) we have shown that there are large classes of preferences satisfying these four assumptions.

The seller cannot force buyers to purchase his goods; his sales depend on their behaviour. Hence, a *selling procedure* is a rule that assigns buyers quantities and charges them prices on the basis of their actions. Depending on the procedure, an action might consist of making a bid, submitting a demand function, or, in principle, anything else that a buyer might do to signal his demand. Formally, a procedure is a schedule of pairs, one for each buyer:

$$[\tilde{q}_i(s_1, \ldots, s_n), \tilde{R}_i(s_1, \ldots, s_n)], \qquad i = 1, \ldots, n \tag{2}$$

where s_i is buyer i's strategy, lying in strategy space S_i, and $\tilde{R}_i$ and $\tilde{q}_i$ are, respectively, his payment and allocation of the good. Allocations must satisfy the aggregate supply constraint

$$\sum_{i=1}^{n} \tilde{q}_i(s) \leq q_0 \tag{3}$$

where $s \equiv (s_1, \ldots, s_n)$. The tildes reflect the possibility that payments and allocations may be random. Initially, however, we restrict attention to deterministic procedures, so that (2) can be rewritten in deterministic form as

$$[\hat{q}_i(s_i, s_{-i}), \hat{R}_i(s_i, s_{-i})] \tag{4}$$

where $s_{-i} \equiv (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$.

We suppose that the seller can select any selling procedure it desires and makes its selection to maximize expected revenue. The key to solving its optimization problem is the incorporation of the constraints implied by the buyers' choice of strategies. We assume that buyers choose their strategies without collusion. Thus, they play a game of incomplete information. A natural non-cooperative solution concept for such a game is the Bayesian equilibrium of Harsanyi (1967–8), an extension of ordinary Nash equilibrium.

To define a Bayesian equilibrium, we introduce the notion of a strategy rule for buyer i, a function $s_i^*(\cdot)$ that, for each possible type v_i, assigns a strategy $s_i^*(v_i)$. With buyers behaving non-cooperatively, the vector of strategy rules $(s_1^*(\cdot), \ldots, s_n^*(\cdot))$ is an equilibrium if, when conformed to by all others, each buyer's best option (in the sense of maximizing his expected utility) is to conform to it also. Of course, there is no reason, in general, why equilibrium should be unique. (But see Maskin and Riley 1982 for a treatment of uniqueness in sealed-bid and open auctions.) Thus, when we speak of a procedure that maximizes expected revenue, we really mean that there is *an* equilibrium of that procedure that maximizes expected revenue.

Consider the expected surplus of buyer i if, when his parameter value is v_i, he chooses the strategy $s_i = s_i^*(x)$ instead of $s_i^*(v_i)$. With other buyers adopting the strategy rules

$$s_{-i}^*(\cdot) \equiv [s_1^*(\cdot), \ldots, s_{i-1}^*(\cdot), s_{i+1}^*(\cdot), \ldots, s_n^*(\cdot)],$$

buyer i has an expected surplus of

$$\Pi_i(x, v_i) = E_{v_{-i}}[N(\hat{q}_i(s_i^*(x), s_{-i}^*(v_{-i})), v_i) - \hat{R}_i(s_i^*(x), s_{-i}^*(v_{-i}))] \quad (5)$$

where $N(q, v_i) \equiv \int_0^q p(z, v_i)\,dz$ and where the expectation is taken with respect to the distribution $F(\cdot)$. Let us suppress the functions $s_i^*(\cdot)$ and define

$$\left.\begin{aligned} q_i(x, v_{-i}) &= \hat{q}_i(s_i^*(x), s_{-i}^*(v_{-i})) \\ R_i(x) &= E_{v_{-i}}\hat{R}_i(s_i^*(x), s_{-i}^*(v_{-i})). \end{aligned}\right\} \quad (6)$$

Then (5) can be rewritten as

$$\Pi_i(x, v_i) = E_{v_{-i}}N(q_i(x, v_{-i}), v_i) - R_i(x). \quad (7)$$

Since we have defined $s_i^*(v_i)$ to be buyer i's optimal strategy, if his parameter value is v_i, it follows that $\Pi_i(x, v_i)$ must take its maximum at $x = v_i$; that is, for all i and v_i,

$$\Pi_i(v_i, v_i) = \max_x \Pi_i(x, v_i). \quad (8)$$

We now show that, given (6)–(8), we can express maximized surplus $\Pi_i(v_i, v_i)$ solely in terms of the allocation rule $q_i(v_i, v_{-i})$.

Proposition 1. *Necessary Condition for Self-selection, i.e., Incentive Compatibility.* Under Assumptions A1 (or A1*) and A2, the maximized expected surplus of buyer i with parameter value v_i can be written as

$$\Pi_i(v_i, v_i) = \Pi_i(0, 0) + E_{v_{-i}}\int_0^{v_i} N_2(q_i(z, v_{-i}), z)\,dz. \quad (9)$$

Remark. We have derived $[q_i(\cdot), R_i(\cdot)]$ from $[\hat{q}_i(\cdot), \hat{R}_i(\cdot)]$ through (6). However we can think of $[q_i(\cdot), R_i(\cdot)]$, $i = 1, \ldots, n$, as a selling procedure itself in which buyers announce parameter values as strategies; in other words, it is a *direct revelation mechanism*. Condition (8), moreover, ensures that buyers announce their *true* values in equilibrium. Of course, if instead we confront a buyer with *arbitrary* functions $q_i(\cdot)$ and $R_i(\cdot)$, it may not be in his interest to reveal truthfully. Proposition 1 implies that, if $\Pi_i(x, v_i)$ is defined by (7), a necessary condition for truthful revelation

is (9). That is why we have attached the label 'incentive compatibility' to the proposition.

Proof.[2] Because $\Pi_i(v_i, v_i) \geq \Pi_i(x, v_i)$ (from (8)), and if $(v_i > x)$ $\Pi_i(x, v_i) \geq \Pi_i(x, x)$,

$$\Pi_i(v_i, v_i) - \Pi_i(x, x) \geq 0, \quad \text{for} \quad v_i > x. \quad (10)$$

Also, from (7),

$$\begin{aligned} \Pi_i(v_i, v_i) - \Pi_i(v_i, x) &= E_{v_{-i}}(N(q_i(v_i, v_{-i}), v_i) - N(q_i(v_i, v_{-i}), x)) \\ &= E_{v_{-i}}\int_0^{q_i(v_i, v_{-i})}\int_x^{v_i} p_2(z, y)\,dy\,dz \\ &\leq E_{v_{-i}}\int_0^{q_i(v_i, v_{-i})}\int_x^{v_i} p_2(z, x)\,dy\,dz, \end{aligned} \quad (11)$$

where the inequality follows from Assumption B2 (which, without loss of generality, we can assume holds). Hence, for all $v_i \geq x$,

$$0 \leq \Pi_i(v_i, v_i) - \Pi_i(x, x) \leq (v_i - x)E_{v_{-i}}\int_0^{q_i(v_i, v_{-i})} p_2(z, x)\,dz. \quad (12)$$

Therefore, $\Pi_i(v_i, v_i)$ is continuous. In fact, since $0 \leq q_i \leq q_0$, the expectation in (12) is bounded and so $\Pi_i(v_i, v_i)$ is an absolutely continuous function.

From (8), for all x and v_i,

$$v_i \in \arg\min_x [\Pi_i(x, x) - \Pi_i(v_i, x)]. \quad (13)$$

From (7), $\Pi_i(v_i, x)$ is a differentiable function of x. Moreover, as we have just argued, $\Pi_i(x, x)$ is continuous and non-decreasing, hence differentiable almost everywhere. Thus, almost everywhere we can write the first-order condition for (13) as

$$\frac{d\Pi_i}{dx}(x, x) - \frac{\partial \Pi_i}{\partial x}(v_i, x) = 0 \quad \text{at} \quad x = v_i.$$

From (7),

$$\left.\frac{\partial \Pi_i}{\partial x}(v_i, x)\right|_{x = v_i} = E_{v_{-i}}N_2[q_i(v_i, v_{-i}), v_i].$$

A necessary condition for (8) to hold, therefore, is

$$\frac{d}{dv_i}\Pi_i(v_i, v_i) = E_{v_{-i}}N_2(q_i(v_i, v_{-i}), v_i) \text{ almost everywhere.} \quad (14)$$

Moreover, since $\Pi_i(v_i, v_i)$ is absolutely continuous, we can rewrite (14) in the more convenient integral form (9). Q.E.D

[2] For the proof presented here we acknowledge the helpful suggestions of Steven Matthews.

Because the seller cannot force any buyer to participate, the expected surplus of every buyer must be non-negative. Since $\Pi_i(v_i, v_i)$ is a non-decreasing function, this 'voluntary participation' constraint can be expressed simply as

$$\Pi_i(0, 0) \geq 0. \tag{15}$$

Propositions 1 showed that (9) is a necessary condition for arbitrary functions $[q_i(\cdot), R_i(\cdot)]$, $i = 1, \ldots, n$, to constitute a direct revelation mechanism in which truth-telling is an equilibrium. We next show that, if the function $q_i(\cdot)$ is suitably monotonic, then conditions (9) and (15) are sufficient conditions.

LEMMA 1. Suppose that preferences satisfy Assumption A1 or A1*. Assume that $q_i(v_i, v_{-i})$ is a non-decreasing function of v_i, and *define* Π_i by (9). Then if Π_i satisfies (15), we have

(i) $\Pi_i(v_i, v_i) \geq \Pi_i(x, v_i)$

and

(ii) $\Pi_i(v_i, v_i) \geq 0$

for all x and v_i.

Proof. If $\Pi_i(v_i, v_i)$ satisfies (9), then for any $y \geq x$

$$\Pi_i(y, y) - \Pi_i(x, x) = E_{v_{-i}} \int_x^y N_2(q_i(z, v_{-i}), z)\, dz$$

$$\geq E_{v_{-i}} \int_x^y N_2(q_i(x, v_{-i}), z)\, dz \tag{16}$$

since, by hypothesis, $q_i(z, v_{-i})$ is non-decreasing in z, and, by Assumption A1 (or A1*), $p_2 = N_{12}$ is non-negative and N_2 is positive. Hence, from (15), (ii) holds. But, from (7),

$$\Pi_i(x, y) - \Pi_i(x, x) = E_{v_{-i}} \int_x^y N_2(q_i(x, v_{-i}), z)\, dz. \tag{17}$$

Thus, combining (16) and (17), we obtain

$$\Pi_i(y, y) \geq \Pi_i(x, y),\ y > x.$$

Hence, (i) holds for all $x < v_i$. An almost identical argument establishes that it holds for all $x > v_i$ as well. Q.E.D

A selling procedure can be extremely complicated, and therefore, in principle, so can be maximizing expected revenue over the class of all procedures. As we have seen, however, any selling procedure is equivalent (in allocation and expected payments) to a direct revelation mechanism. (This equivalence is sometimes called the Revelation Principle.) Thus, the optimization can be restricted to the much smaller class of such mechanisms. Lemma 1 helps simplify this optimization by establishing that, if the functions $[q_i(v_i, v_{-i})]$ are non-decreasing, then there exist corresponding payment functions $[R_i(\cdot)]$ such that $[q_i(\cdot), R_i(\cdot)]$, $i = 1, \ldots, n$ is a direct revelation mechanism in which truth-telling is an equilibrium. Indeed, from (7) and (9), the expected payment by buyer i with valuation v_{-i} is

$$R_i(v_i) = E_{v_{-i}}\left[N(q_i(v_i, v_{-i}), v_i) - \int_0^{v_i} N_2(q_i(z, v_{-i}), z)\, dz\right] - \Pi_i(0, 0). \tag{18}$$

Thus, the seller's problem boils down to maximizing over functions $q_i(\cdot)$. Specifically, the expected revenue from buyer i can be written as

$$\bar{R}_i = E_{v_i, v_{-i}}\left[N(q_i(v_i, v_{-i}), v_i) - \int_0^{v_i} N_2(q_i(z, v_{-i}), z)\, dz\right] - \Pi_i(0, 0).$$

Integrating the second term on the right by parts, we obtain

$$\bar{R}_i = E_{v_i, v_{-i}}[N(q_i(v_i, v_{-i}), v_i) - N_2(q_i(v_i, v_{-i}), v_i)/\rho(v_i)] - \Pi_i(0, 0) \tag{19}$$

where $\rho(v_i) = (dF(v_i)/dv)/[1 - F(v_i)]$ is the hazard rate for F.

Because the expression in (19) enclosed in braces is independent of $\Pi_i(0, 0)$, and because the latter must satisfy (15), maximization of expected revenue clearly implies setting $\Pi_i(0, 0) = 0$. Summing over n, we obtain the following proposition.

PROPOSITION 2. *Expected Seller Revenue.* Consider a selling procedure $[\hat{q}_i(s), \hat{R}_i(s)]_{s_i \in S_i}$ in which a buyer with parameter value zero has zero expected surplus in equilibrium. Under Assumptions A1 or A1* and A2, expected revenue equals

$$E_{v_i, v_{-i}}\left[\sum_{i=1}^n I(q_i(v_i, v_{-i}), v_i) \,\middle|\, \sum_{i=1}^n q_i(v_i, v_{-i}) \leq q_0\right] \tag{20}$$

where

$$I(q_i, v_i) \equiv N(q_i, v_i) - N_2(q_i, v_i)/\rho(v_i) \tag{21}$$

and $[q_i(\cdot), R_i(\cdot)]$ is the direct revelation mechanism corresponding to $[\hat{q}_i(\cdot), \hat{R}_i(\cdot)]$.

As a direct implication of Proposition 2, we can demonstrate that two standard selling procedures—the open and sealed-bid auctions—generate the same expected revenue when buyers have unit demand. In the open auction, the auctioneer raises the asking price continuously until all but q_0 bidders have dropped out (assuming that there are q_0 units for sale). Each remaining bidder receives one item and pays the final price. In the sealed-bid auction, buyers all submit secret bids. The winners are the q_0 highest bidders, and they pay their bids.

PROPOSITION 3. *Revenue Equivalence.* If Assumptions A1* and A2 hold, expected seller revenue is the same under the sealed-bid and the open auctions.

Remark. Harris and Raviv (1981) and Vickrey (1961a) establish this result for the case of a uniform distribution.

Proof. It is clear, first, that in both types of auctions $\Pi_i(0, 0) = 0$. In the open auction, each buyer's dominant strategy is to remain in the auction until the asking price equals his parameter value. The items for sale are thus sold to those with the q_0 highest values; that is,

$$q_i(v_i, v_{-i}) = \begin{cases} 1, & \text{if } v_i \text{ is among the } q_0 \text{ highest values} \\ 0, & \text{otherwise.} \end{cases} \tag{22}$$

In view of Proposition 2, it remains only to show that (22) holds as well in the sealed-bid auction. To do so, it suffices to show that there exists an equilibrium in which buyers all use the same, strictly increasing, bidding strategy $b_i = B(v_i)$. In such an equilibrium, the goods are clearly sold to those with the highest values. The methods of Maskin and Riley (1982) can, moreover, be applied to establish that this is the unique equilibrium. Define

$$P(x) = \Pr\{\text{fewer than } q_0 \text{ of } n-1 \text{ buyers have valuations greater than } x\}$$
$$= \sum_{k=0}^{q_0-1} \binom{n-1}{k} F(x)^{n-1-k}[1 - F(x)]^k.$$

Suppose $B(x)$ is the solution to the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}x}[P(x)B(x)] = x\frac{dP(x)}{\mathrm{d}x}, \qquad B(0) = 0. \tag{23}$$

Rewriting (23) in integral form, we obtain

$$P(v_i)B(v_i) = \int_0^{v_i} x\frac{\mathrm{d}P}{\mathrm{d}x}(x)\,\mathrm{d}x. \tag{24}$$

Because $\mathrm{d}P(x)/\mathrm{d}x > 0$,

$$\int_0^{v_i} x\frac{\mathrm{d}P}{\mathrm{d}x}(x)\,\mathrm{d}x < \int_0^{v_i} v_i\frac{dP}{\mathrm{d}x}(x)\,\mathrm{d}x = v_iP(v_i), \qquad v_i > 0.$$

From (24) it thus follows that, for $v_i > 0$, $B(v_i) < v_i$. From (23),

$$P(v_i)\frac{dB(v_i)}{\mathrm{d}v} = \frac{\mathrm{d}P}{\mathrm{d}v}(v_i)[v_i - B(v_i)].$$

Because $B(v_i) < v_i$, it follows that $B(v_i)$ is strictly increasing.

Suppose that all buyers but i bid according to $B(\cdot)$ and that buyer i bids $b_i = B(x)$ for some x not necessarily equal to v_i. His expected surplus is

then

$$\Pi_i(x, v_i) = \Pr\{x \text{ is among the } q_0 \text{ highest valuations}\}(v_i - b_i)$$
$$= P(x)[v_i - B(x)].$$

Differentiating by x, we obtain

$$\frac{\partial \Pi_i}{\partial x}(x, v_i) = v_i\frac{\mathrm{d}P(x)}{\mathrm{d}x} - \frac{\mathrm{d}}{\mathrm{d}x}[P(x)B(x)].$$

Substituting from (23), we can rewrite this as

$$\frac{\partial \Pi_i}{\partial x}(x, v_i) = (v_i - x)\frac{dP(x)}{\mathrm{d}x}.$$

Because $P(x)$ is strictly increasing, it follows immediately that buyer i's optimal choice is $x = v_i$, that is, to bid $B(v_i)$. Thus, $B(\cdot)$ is indeed an equilibrium bidding strategy. Because it is strictly increasing, we conclude that (22) holds for the sealed-bid auction. Q.E.D

Remark 1. An almost identical argument can be used to establish the equivalence of the open and sealed-bid auctions when the seller sets a non-zero reserve price.

Remark 2. We have concentrated in Proposition 3 on the open and sealed-bid auctions, but it is clear that there are many other auctions as well that satisfy $\Pi_i(0, 0) = 0$ and (22) and so generate the same expected revenue. For example, the (admittedly peculiar) auction in which buyers submit sealed bids and the q_0 highest are winners, but only *losers* pay their bids satisfies these conditions.

3. Solving For the Revenue-maximizing Selling Procedure: the Regular Case

To solve for the optimal (deterministic) selling procedure, we begin by choosing $q(\cdot) = [q_1(\cdot), \ldots, q_n(\cdot)]$ to maximize (20). We then show that the solution to this problem $q^*(\cdot) = [q_1^*(\cdot), \ldots, q_n^*(\cdot)]$ is monotonic as required by Lemma 1 if the distribution is 'regular' in the sense defined below. Thus, $q^*(\cdot)$ solves the seller's optimization problem. The regularity assumption, which we will invoke throughout this section, is as follows.

ASSUMPTION C. *Regularity of the Distribution Function*[3]

$$J(v) \equiv v - \frac{1}{\rho(v)} \tag{25a}$$

[3] See Maskin and Riley (1984a) for a discussion of this assumption. Clearly, it is satisfied if the hazard rate $\rho(v)$ either increases or does not decline too rapidly with v. We noted earlier that we can always choose our parametrization so that Assumption B2 is satisfied. The choice, however, may affect whether or not Assumption C holds.

is increasing or

$$\frac{1}{p_2(\bar{q}(v), v)} \frac{\partial}{\partial v}\left[\frac{p_2(\bar{q}(v), v)}{\rho(v)}\right] < 1 \tag{25b}$$

where $\bar{q}(v)$ solves $(\partial I/\partial q)(q, v) = 0$.

We first consider the case of unit demand, Assumption A1*. Although units are themselves indivisible, the optimization problem is not so constrained, since we can give q_i a probabilistic interpretation. That is, q_i between zero and unity should be thought of as the probability that buyer i receives a unit. Given Assumption A1*,

$$I(q_i, v_i) = J(v_i) \min\{q_i, 1\} \tag{26}$$

where $J(\cdot)$ is given by (25a). Substituting (26) into (20), we seek the solution to

$$\max_{q(v)} \left[E_{v_i, v_{-i}} \sum_{i=1}^{n} J(v_i) q_i \,\middle|\, 0 \leq q_i \leq 1, \sum_{i=1}^{n} q_i \leq q_0\right]. \tag{27}$$

Define

$$v^0 = \max\{v \mid J(v) = 0\}. \tag{28}$$

If Assumption C holds, so that $J(v)$ is increasing,[4] then $J(v)$ is positive if and only if $v > v^0$. It follows immediately that the solution to (27), $q^*(\cdot)$, satisfies

$$q_i^*(v_i, v_{-i}) = 0, \; v_i < v^0.$$

We now establish the following proposition.

Proposition 4. *Optimal Selling Procedure for Unit Demand: the Regular Case.* If buyers' preferences satisfy A1* and $F(v)$ satisfies Assumption C (so that $J(v)$ is increasing), expected seller revenue is maximized by selling up to q_0 units to those buyers with the highest reservation prices in excess of v^0 (defined by (28)).

Remark. There are clearly many selling procedures that satisfy the conditions of Proposition 4 and are therefore optimal. Indeed, the open and sealed-bid auctions described above are optimal as long as the auctioneer sets a minimum price of v^0.

Proof. Suppose there are m buyers for whom $J(v_i) > 0$, that is, m buyers with reservation values exceeding v^0. If $m \leq q_0$, the term in braces in (27) is maximized by setting $q_i = 1$ if $v_i > v^0$ and $q_i = 0$ otherwise. If $m > q^0$, the term in braces is maximized by setting $q_i = 1$ for those q_0 buyers with the highest values of J, that is, with the q_0 highest reservation values. To summarize, (27) is solved by choosing

$$q_i^*(v_i, v_{-i}) = \begin{cases} 1, & \text{if } v_i \geq v^0 \text{ and } v_i \text{ is among the } q_0 \text{ highest reservation values} \\ 0, & \text{otherwise.} \end{cases} \tag{28a}$$

Since $q_i^*(v_i, v_{-i})$ is a non-decreasing function of v_i, it satisfies the hypotheses of Lemma 1. Thus, $q_i^*(v_i, v_{-i})$, $i = 1, \ldots, n$, is the expected revenue-maximizing allocation rule. Q.E.D

We now consider the problem of general downward-sloping demand curves. Perhaps the best-known selling procedure for demand curves of this type is the US Treasury bill auction. Buyers may submit orders at one or more prices. Thus, in principle, a buyer can approximate any demand curve arbitrarily closely. Current practice is for the Treasury to fill orders at the prices submitted until orders filled equal the size of the offering. However, the Treasury has also experimented with a sealed bid auction in which all buyers pay the price of the highest unsuccessful bidder.[5]

As we will see, neither of these auctions is optimal even with a reserve price. Moreover, expected revenue from the two auctions is not in general the same.

Suppose that Assumptions A1, A2, B1, B2, and C hold. Consider the problem of maximizing (20). If, for all i, the solution $q_i^*(v_i, v_{-i})$ is non-decreasing in v_i, then once again the hypotheses of Lemma 1 are satisfied. Thus, if $q_i^*(v_i, v_{-i})$ satisfies this monotonicity property, it is the solution to the seller's optimization problem. The following lemma is helpful.

Lemma 2. If Assumptions A1, A2, B1, B2, and C hold, then

(a) $I(q, v)$ is a strictly quasi-concave function of q (that is, its second derivative with respect to q is negative whenever its first derivative is non-negative); and
(b) $\partial I/\partial q$ is strictly increasing in v.

Proof. We first establish that Assumptions A1, A2, and B1 together imply (a). From (21),

$$\frac{\partial I}{\partial q} = p(q, v) - p_2(q, v)/\rho(v). \tag{29}$$

[4] For the unit demand case, condition (25b) reduces to (25a).

[5] The Treasury has not yet announced the results of its experiment with the one-price auction. In future work we plan to use the results of this paper to compare the two forms of Treasury bill auctions with the theoretical optimum. For a discussion of the one-price auction when buyers bid for a share of a divisible good, see Wilson (1979) and Maxwell (1982).

Hence,

$$\frac{\partial I}{\partial q} > 0 \text{ if and only if } 1/\rho < p/p_2. \tag{30}$$

Note that

$$\frac{\partial^2 I}{\partial q^2} = p_1 - p_{12}/\rho.$$

If p_{12} is non-negative, $\partial^2 I/\partial q^2$ is negative, since, by Assumption A1, $p_1 < 0$ for $p > 0$. If p_{12} is negative, then by (30),

$$\frac{\partial I}{\partial q} > 0 \rightarrow \frac{\partial^2 I}{\partial q^2} < p_1 - \frac{p_{12} p}{p_2} = \frac{p_1^2}{q p_2} \frac{\partial}{\partial v}\left(\frac{-q p_1}{p}\right).$$

By Assumption B1, the final expression is non-positive. Thus, $I(q, v)$ is indeed strictly quasi-concave. Furthermore,

$$\begin{aligned}\frac{\partial^2 I}{\partial q \partial v} &= p_2\left[1 - \frac{1}{p_2}\frac{\partial}{\partial v}\left(\frac{p_2}{\rho}\right)\right] \\ &= p_2\left(1 + \frac{\mathrm{d}\rho/\mathrm{d}v}{\rho^2}\right) - \frac{p_{22}}{\rho} \\ &= p_2 \frac{\mathrm{d}J}{\mathrm{d}v}(v) - \frac{p_{22}}{\rho},\end{aligned}$$

since $\mathrm{d}J(v)/\mathrm{d}v = 1 + (\mathrm{d}\rho/\mathrm{d}y)/\rho^2$. Thus, by Assumptions B2 and C, $\partial^2 I/\partial q\, \partial v$ is strictly positive Q.E.D

Form the Lagrangean for the maximization of (27); that is,

$$L = E_{v_i, v_{-i}}\left[\sum_{i=1}^{n} I(q_i, v_i) + \mu(v_i, v_{-i})\left(q_0 - \sum_{i=1}^{n} q_i\right)\right].$$

The solution $q_1^*(\cdot), \ldots, q_n^*(\cdot)$ satisfies

$q_i^*(v_i, v_{-i}) \geq 0$ and $\mu(v_i, v_{-i})$ non-negative

$$\left.\begin{aligned}&\mu(v_i, v_{-i})\left[\sum_{i=1}^{n} q_i^*(v_i, v_{-i}) - q_0\right] = 0 \\ &q_i^*(v_i, v_{-i})\left[\frac{\partial I}{\partial q}(q_i^*(v_i, v_{-i}), v_i) - \mu(v_i, v_{-i})\right] = 0 \\ &q_i^*(v_i, v_{-i}) = 0 \rightarrow \frac{\partial I}{\partial q}(0, v_i) \leq \mu(v_i, v_{-i}).\end{aligned}\right\} \tag{31}$$

Given the hypotheses of Lemma 2, $I(q_i, v_i)$ is strictly quasi-concave in q; hence the necessary conditions (31) are also sufficient. To show that $q_1^*(\cdot), \ldots, q_n^*(\cdot)$ solves the seller's maximization problem, it remains to argue that $q_i^*(v_i, v_{-i})$ is non-decreasing in v_i, so that we can apply

Lemma 1. For given (v_i, v_{-i}), either $q_i^*(v_i, v_{-i}) = 0$, in which case $(\partial q_i^*/\partial v_i)(v_i, v_{-i})$ is trivially non-negative, or else $q_i^*(v_i, v_{-i}) > 0$. In the latter case, (31) implies that

$$\frac{\partial I}{\partial q}(q_i^*(v_i, v_{-i}), v_i) = \mu(v_i, v_{-i}).$$

Moreover, the equality holds in a neighbourhood of (v_i, v_{-i}). Differentiating this last equation with respect to v_i, we obtain

$$\frac{\partial^2 I}{\partial q^2}\frac{\partial q_i^*}{\partial v_i} + \frac{\partial^2 I}{\partial q\, \mathrm{d}v} = \frac{\partial \mu}{\partial v_i}. \tag{31a}$$

Suppose that $\partial\mu/\partial v_i$ is non-positive. Because $\partial^2 I/\partial q\, \partial v$ is positive and (thanks to strict quasi-concavity) $\partial^2 I/\partial q^2$ is negative. (31a) implies that $\partial q_i^*/\partial v_i$ is positive. Assume, therefore, that $\partial\mu/\partial v_i$ is positive; this implies, in particular, that $\mu > 0$. If, for $j \neq i$, $\partial I(q_j^*(v_i, v_{-i}), v_j)/\partial q < \mu$, then $\partial q_j^*/\partial v_i = 0$. Moreover, if $\partial I(q_j^*(v_i, v_{-i}), v_j)/\partial q = \mu$, then $(\partial^2 I/\partial q^2)(\partial q_j^*/\partial v_i) = \partial\mu/\partial v_i$, implying that $\partial q_j^*/\partial v_i \leq 0$. In either case, therefore $\partial q_j^*/\partial v_i$ is non-positive for $j \neq i$. But because

$$\sum_{k=1}^{n} \frac{\partial q_k^*}{\partial v_i}(v_i, v_{-i}) = 0$$

(since $\mu > 0$), we can deduce again that $\partial q_i^*/\partial v_i$ is non-negative. Hence $q_1^*(\cdot), \ldots, q_n^*(\cdot)$ solves the seller's problem.

Define $\bar{q}(v_i)$ so that $(\partial I/\partial q)(\bar{q}(v_i), v_i) = 0$ for all v_i, and let $\phi(\cdot)$ be the inverse of $\bar{q}(v_i)$. Take

$$\bar{R}(q_i) = R_i^*(\phi(q_i)) \tag{32}$$

where $R_i^*(v_i)$ satisfies (18) with $q_i(\cdot) = q_i^*(\cdot)$ and $\Pi_i(0, 0) = 0$. Given the preceding analysis, the following result describes an optimal selling procedure.

Proposition 5. *Optimal Selling Procedure for General Demand: the Regular Case.* If Assumptions A1, A2, B1, B2, and C hold, expected revenue from the sale of q_0 units is maximized if the seller sets the payment schedule $\bar{R}(q_i)$ defined by (32). Each buyer i submits an order q_i and pays $\bar{R}(q_i)$. If total orders exceed supply, final allocations are reduced according to the rationing scheme:

$$\begin{aligned}&q_i^*\left[\frac{\partial I}{\partial q_i}(q_i^*, \phi(q_i)) - \mu\right] = 0 \\ &\sum_{i=1}^{n} q_i^* \leq q_0 \\ &q_i^* = 0 \rightarrow \frac{\partial I}{\partial q_i}(0, \phi(q_i)) \leq \mu.\end{aligned} \tag{33}$$

For the special case in which demand curves have the simple form

$$p(q_i, v_i) = v_i - \gamma q_i,$$

the allocation rule is especially straightforward. From (33), we obtain

$$q_i^* > 0 \rightarrow \gamma(q_i - q_i^*) - \mu = 0 \rightarrow q_i^* = q_i - \mu/\gamma.$$

Thus, the seller simply reduces each buyer's order by the same amount (subject to its remaining non-negative) if demand exceeds supply.

It is easy to see that the open and sealed-bid auctions,[6] as well as the two Treasury bill procedures, cannot be optimal in general. Suppose, for example, that q_0 is so large that the supply constraint is never binding. Then all these auctions have the property that the equilibrium price is just the seller reserve price, at which buyers can buy all they want. In the optimal selling procedure, however, pricing is nonlinear: a buyer with value v_i buys $\bar{q}(v)$, solving $I(\bar{q}(v), v) = 0$, and pays

$$R(v) = N(\bar{q}(v), v) - N_2(\bar{q}(v), v)/\rho(v).$$

4. Optimal Selling Procedures: the General Case

We next study revenue-maximizing procedures when Assumption C is not imposed. To simplify matters, we consider only the case of unit demand.

In addition to the necessary conditions (9) and (15) derived in Section 2, we first note that the allocation rule must satisfy a monotonicity conditoon. (Earlier we noted that monotonicity was a *sufficient* hypothesis for Lemma 1.) For unit demand, (7) becomes

$$\Pi_i(x, v_i) = E_{v_{-i}} v_i q_i(x, v_{-i}) - R_i(x).$$

Thus,

$$\Pi_i(x, x) - \Pi_i(x, v_i) = (x - v_i) E_{v_{-i}} q_i(x, v_{-i}). \tag{34}$$

From (8),

$$\Pi_i(x, x) - \Pi_i(v_i, x) \geq 0 \text{ and } \Pi_i(v_i, v_i) - \Pi_i(x, v_i) \geq 0.$$

Adding these two inequalities and substituting from (34), we obtain

$$(v_i - x) E_{v_{-i}}[q_i(v_i, v_{-i}) - q_i(x, v_{-i})] \geq 0.$$

Thus, the allocation rule $q_i(v_i, v_{-i})$ must satisfy the condition that

$$E_{v_{-i}} q_i(v_i, v_{-i}) \text{ is non-decreasing in } v_i. \tag{35}$$

[6] When buyers may want more than one unit, these auctions must be modified slightly. In the open auction the auctioneer continuously raises the price, and at each level buyers indicate how many units they would want to buy. The actual price is determined when the level is raised high enough so that supply equals demand. In the sealed-bid auction, buyers submit demand curves, and the auctioneer uses these to compute the market-clearing price and allocations.

Adding this constraint to programme (27), we obtain the maximization problem

$$\max_{\{q_i(\cdot)\}} \left\{ E_{v_i, v_{-i}} \sum_{i=1} J(v_i) q_i \;\middle|\; E_{v_{-i}} q_i(v_i, v_{-i}) \text{ is non-decreasing}, \; 0 \leq q_i \leq 1, \sum_{i=1}^{n} q_i \leq q_0 \right\}. \tag{36}$$

In general, the function $J(v) = v - 1/\rho(v)$ is non-monotonic, and so the earlier argument does not generalize immediately. Instead we begin by defining a modified function $J^*(v)$ that *is* monotonic, and solve for the optimal $\{q_i^*(\cdot)\}$ with J^* replacing J. We then show that this allocation rule also solves the original problem. Finally, we interpret the optimal selling procedure as an auction.

Modified J Function. Let $\{[x^\omega, y^\omega] \mid y^\omega < x^{\omega+1}\}_{\omega \in \Omega}$ be a collection of subintervals of $[0, \bar{v}]$ such that (a) the function

$$J^*(v) = \begin{cases} J(v), & \text{if } v \in \bigcup_{\omega \in \Omega} [x^\omega, y^\omega] \\ J(y^\omega), & \text{if } v \in [x^\omega, y^\omega] \quad \text{for some } \omega \end{cases}$$

is non-decreasing, and (b) the function

$$K^\omega(v) = \int_v^{y^\omega} [J(z) - J(y^\omega)] \, dF(z)$$

satisfies

$$K^\omega(v) \begin{cases} \leq 0, & \text{for all } v \leq y^\omega \\ = 0, & v = x^\omega. \end{cases}$$

A proof that the collection $\{[x^\omega, y^\omega]\}$ exists can be constructed along the following geometrical lines. Consider Figure 14.1. Starting at $v = \bar{v}$ and moving to the left, we define $J^*(v) = J(v)$ until a point y^1 is reached at which, for some $x < y^1$,

$$\int_x^{y^1} [J(z) - J(y^1)] \, dF(z) \geq 0. \tag{**}$$

Since $J(v) \leq v_i \leq \bar{v} = J(\bar{v})$, y^1, if it exists, is less than $\bar{v}$. Define x^1 to be the smallest such x satisfying inequality (**) and define $J^*(v) = J(y^1)$ over $[x^1, y^1]$. This process is continued until $v = 0$.

Proposition 6. *Optimal Allocation Rule.* For any (v_i, v_{-i}), choose $\hat{J}$ so that the number, M, of buyers with parameter values v_i for which $J^*(v_i) \geq \hat{J}$ is at least q_0, and the number, m, for which $J^*(v_i) > \hat{J}$ is at most $q_0 - 1$. Then expected seller revenue is maximized by

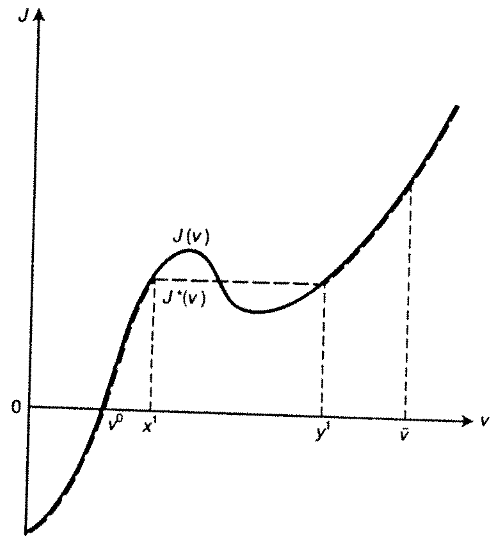

Fig. 14.1 *Derivation of J^*.*

$q_1^*(\cdot), \ldots, q_n^*(\cdot)$ satisfying

$$q_i^*(v_i, v_i) = \begin{cases} 1, & \text{if } J^*(v_i) > \hat{J} > 0 \\ \dfrac{q_0 - m}{M - m}, & \text{if } J^*(v_i) = \hat{J} \geq 0 \\ 0, & \text{otherwise.} \end{cases} \quad (37)$$

Proof. Because $J^*(\cdot)$ is non-decreasing, $q_i^*(v_i, v_{-i})$ is non-decreasing in v_i. Hence, (35) is satisfied. The proof is completed in three steps. First, we show that, for any $q_1(\cdot), \ldots, q_n(\cdot)$ satisfying the constraints of (36),

$$E_{v_i, v_{-i}} \sum_{i=1}^{n} J(v_i) q_i(v_i, v_{-i}) \leq E_{v_i, v_{-i}} \sum_{i=1}^{n} J^*(v_i) q_i(v_i, v_{-i}). \quad (38)$$

Next, we show that $q_1^*(\cdot), \ldots, q_n^*(\cdot)$ defined by (37) solves the modified optimization problem in which $J(\cdot)$ in (36) is replaced by $J^*(\cdot)$. Finally, we confirm that, for $q_i(\cdot) = q_i^*(\cdot)$, (38) holds with equality.

To prove the first step, we define

$$\mathring{q}_i(v_i) = E_{v_{-i}} q_i(v_i, v_{-i})$$

for any allocation $\{q_i(\cdot)\}$ satisfying the constraints of (36). Then,

$$\begin{aligned} E_{v_i, v_{-i}}[J(v_i) - J^*(v_i)] q_i(v_i, v_{-i}) &= \int_0^{\bar{v}} [J(v) - J^*(v)] \mathring{q}_i(v) \, dF(v) \\ &= \sum_{\omega \in \Omega} \int_{x^\omega}^{y^\omega} [J(v) - J^*(v)] \frac{dF}{dv}(v) \mathring{q}_i(v) \, dv \\ &= \sum_{\omega \in \Omega} - \int_{x^\omega}^{y^\omega} \frac{dK^\omega}{dv}(v) \mathring{q}_i(v) \, dv \end{aligned} \quad (39)$$

where the last two equations follow directly from the definitions of J^* and K^ω. By construction, $K^\omega(v)$ is non-positive and $\mathring{q}_i(v)$ is non-decreasing. Thus, integrating by parts, we obtain

$$- \int_{x^\omega}^{y^\omega} \frac{dK^\omega}{dv}(v) \mathring{q}_i(v) \, dv \leq K^\omega(x^\omega) \mathring{q}_i(x^\omega) - K^\omega(y^\omega) \mathring{q}_i(y^\omega) \leq 0 \quad (40)$$

from the definition of K^ω. Inequality (38) follows from (39) and (40).

Next, consider the maximization problem

$$\max_{\{q_i(\cdot)\}} \left\{ E_{v_i, v_{-i}} \sum_{i=1}^{n} J^*(v_i) q_i(v_i, v_{-i}) \,\middle|\, 0 \leq q_i \leq 1, \sum_{i=1}^{n} q_i \leq q_0 \right\}. \quad (41)$$

Because J^* is non-decreasing, the solution, from the argument in Section 3, is to set $q_i^* = 1$ for the (up to) q_0 buyers with the highest non-negative values of J^*. Since J^* is not strictly increasing, ties occur with positive probability. These can be broken by randomizing—that is, by giving all buyers with $J^* = \hat{J}$ a chance $(q_0 - M)/(M - m)$ of winning—thereby obtaining exactly q_0 'winners'. Thus, $\{q_i^*(\cdot)\}$ given by (37) solves the maximization problem (41).

By definition, $q_i^*(v_i, v_{-i})$ is a constant as a function of v_i on any interval $[x^\omega, y^\omega]$, $\omega \in \Omega$. Hence,

$$\int_{x^\omega}^{y^\omega} \frac{dK^\omega}{dv}(v) \mathring{q}_i^*(v) \, dv = [K^\omega(y^\omega) - K^\omega(x^\omega)] \mathring{q}_i^*(y^\omega) = 0$$

where

$$\mathring{q}_i^*(v) = E_{v_{-i}} q_i^*(v, v_{-i}).$$

Thus, (39) implies that (38) holds with equality.

Finally, note that, because $q_i^*(v_i, v_{-i})$ is non-decreasing in v_i, Lemma 1 implies that, because it solves (36), it solves the seller's optimization problem. Q.E.D

Combining (18) and (37), we can readily compute the expected payment $R_i^*(v_i)$ made by buyer i with parameter value v_i in the optimal selling procedure. Thus, the seller can maximize expected revenue through a direct revelation mechanism in which, if the n buyers 'announce' parameter values $(x_1, \ldots, x_n)$, the allocation and expected payments are

$$[q_i^*(x_1, \ldots, x_n), R_i^*(x_i)], \qquad i = 1, \ldots, n.$$

We next show that, alternatively, the seller can use a modification of the open-bid auction.

PROPOSITION 7. *Optimal Multi-unit Auctions with Unit Demand.* Let $\{[x^\omega, y^\omega]\}_{\omega\in\Omega}$ be the collection of intervals in the definition of J^*. For each ω, there exists $z^\omega \in (x^\omega, y^\omega)$ such that if, in an open auction, the asking price is started at $v^0 = \max[v \mid J^*(v) = 0]$, and is raised discontinuously from x^ω to z^ω whenever it reaches x^ω, then that auction is optimal.

Remark. When the price rises from x^ω to z^ω, buyers' decisions about whether to continue bidding must be revealed simultaneously (since, with positive probability, several will drop out at the same time). One way of achieving this is for the auctioneer to confer (privately) with each buyer to determine whether more than q_0 wish to continue bidding. If not, those remaining in the auction pay z^ω and receive one unit. The winners among those dropping out are selected at random and pay x^ω.

Proof. It suffices to show that we can choose z^ω such that the corresponding allocation rule is defined by (37). For each ω, choose z^ω so that

$$E_{v_{-i}}[q_i^*(x^\omega, v_{-i}) \mid \#_i(x^\omega, v_{-i}) > q_0 - 1](y^\omega - x^\omega)$$
$$= E_{v_{-i}}[q_i^*(y^\omega, v_{-i}) \mid \#_i(x^\omega, v_{-i}) > q_0 - 1](y^\omega - z^\omega), \quad (42)$$

where $\#_i(v_i, v_{-i})$ is defined to be the number of buyers (other than i) whose parameter value is at least v_i. In (42), z^ω is chosen so that a buyer with reservation value y^ω is indifferent between staying in and dropping out when the price reaches x^ω. Hence all buyers with values less than y^ω drop out when (or before) the price rises to x^ω but stay in if their reservation values exceed y^ω. The induced allocation rule of this modified open auction thus equals that of (37). Q.E.D

We should point out that in the auction of Proposition 7 all buyers with reservation values in an interval $[x^\omega, y^\omega]$ have an equal chance of winning. This means that there is a positive probability that a buyer who does not have one of the q_0 highest reservation values will be assigned a unit. The proposition therefore assumes implicitly that the seller can enforce a no-resale provision. In the absence of such a provision, the prospect of resale changes buyers' behaviour, and expected seller revenue declines. None the less, we show in Maskin and Riley (1980) that the conditions under which it is optimal for the seller to raise the asking price discontinuously are the same with and without resale.

We conclude this section by illustrating the seller's potential gain from using the optimal rather than the sealed-bid or open auctions. Suppose that the distribution $F(\cdot)$ can be approximated by the following two-point distribution:

$$F(v) \approx \begin{cases} 0, & v < 32 \\ 3/4, & 32 \leq v < 80 \\ 1, & v \geq 80. \end{cases}$$

Suppose there are two buyers and one unit for sale. Clearly, if the ordinary open auction with reserve price is to be used, the seller is best off setting the reserve price equal to 32 or 80. If the former, the item sells for 32 unless both buyers have a valuation of 80. Since the latter occurs with probability 1/16, expected seller revenue is

$$\left(\frac{15}{16}\right)32 + \left(\frac{1}{16}\right)80 = 35.$$

If the reserve price is 80, there are no bids with probability (3/4)(3/4) = 9/16. Expected seller revenue is therefore

$$\left(\frac{7}{16}\right)80 = 35.$$

Thus, in this example a reserve price of either 32 or 80 is optimal for the seller in an open auction.

Alternatively, suppose that the seller uses an auction like that of Proposition 7 and opens the bidding at 32 but then jumps the bid to (just less than) 56. Suppose that buyer 2 stays in the auction only if his reservation value is 80. Then buyer 1 gains from staying in himself only if buyer 2 has a low reservation value. (If both stay in, all consumer surplus is bid away.) His expected gain is therefore slightly greater than

$$\frac{3}{4}(80 - 56) = 18.$$

If buyer 1 chooses not to stay in the auction, he wins (with probability 1/2) only if buyer 2 has a low valuation. Thus his expected gain is

$$\frac{3}{8}(80 - 32) = 18.$$

Buyer 1 therefore has an incentive to use the same strategy as buyer 2, so that this is the equilibrium bidding strategy. Expected seller revenue is therefore

$$\left(\frac{9}{16}\right)32 + \left(\frac{1}{16}\right)80 + \left(\frac{6}{16}\right)56 = 44.$$

By jumping the bid, the seller can thus increase its expected revenue by 9, a gain over 25 per cent.

5. Randomized Selling Schemes

At the outset, we noted that the seller could in principle use a selling procedure in which the outcome is a random schedule

$$[\tilde{q}_i(v_i, v_{-i}), \tilde{R}_i(v_i, v_{-i})]_{i=1,\ldots,n}.^7$$

We restricted attention however, to deterministic selling procedures. In this section we investigate the desirability of randomness.

Because preferences take the form

$$U_i(q, R, v) = \int_0^q p(x, v)\,dy - R,$$

so that buyers are neutral towards income risk, i.e., they are indifferent between the random payment $\tilde{R}_i(v_i, v_{-i})$ and its mean (given v_i). Thus, we may assume that the optimal selling procedure is of the form

$$[\tilde{q}_i(v_i, v_{-i}), R_i(v_i)].$$

Moreover, for the special case of unit demand, it is clear that there is no loss in generality in assuming that each realization of $\tilde{q}_i(v_i, v_{-i})$ is no greater than unity. Thus,

$$U_i(\tilde{q}_i, R_i) = v_i\tilde{q}_i - R_i$$

is linear in $\tilde{q}_i$, and so buyers are again indifferent to risk. Hence, in the case of unit demand, the seller gains nothing by using random selling procedures. We next show that the same principle applies to a broad class of smooth demand curves.

Proposition 8. Under Assumptions A1, A2, B1, B2, and C, the optimal deterministic selling procedure generates at least as much expected revenue as any random one.

Proof. Let $[\tilde{q}_i(\cdot), R_i(\cdot)]$ be a random selling procedure. Although $\tilde{q}_i(v_i, v_{-i})$ is now a random variable, we can still argue as in Section 2 to establish the following counterpart of (9):

$$\Pi_i(v_i, v_{-i}) - \Pi_i(0, 0) = E_{v_{-i}}\left[E\int_0^{v_i} N_2(\tilde{q}_i(z, v_{-i}), z)\,dz\right] \qquad (43)$$

where the inner expectation is over the possible realizations of $\tilde{q}_i(z, v_{-i})$. It follows immediately that the counterpart of (18) holds, namely,

$$R_i(v_i) = E_{v_{-i}}\left\{E\left[N(\tilde{q}_i(v_i, v_{-i})) - \int_0^{v_i} \tilde{N}_2(\tilde{q}_i(z, v_{-i}), z)\,dz\right]\right\} - \Pi_i(0, 0). \qquad (44)$$

[7] We are expressing all selling procedures in this section as direct revelation mechanisms, which, by the Revelation Principle, we are entitled to do.

Since there can be no gain to supplying the buyer with more than $q^0(v_i)$, the amount he would purchase at a zero price, we can assume that each realization of $\tilde{q}_i$ satisfies $\tilde{q}_i \leq q^0(v_i)$. Thus, $N(q_i, v_i)$ is strictly increasing over the domain of q_i, and we can define the inverse function

$$q_i = N^{-1}(n, v_i). \qquad (45)$$

For any random variable $\tilde{q}_i$ and $\tilde{n} = N(\tilde{q}_i, v_i)$, we can then choose $\bar{q}_i = N^{-1}(\bar{n}, v_i)$ where $\bar{n} \equiv E(\tilde{n})$. That is,

$$EN(\tilde{q}_i, v_i) = N(\bar{q}_i, v_i) = \bar{n}. \qquad (46)$$

Consider the function

$$G(n) = N_2(N^{-1}(n, v_i), v_i). \qquad (47)$$

We shall suppose that $G(\cdot)$ is convex. (We will later confirm that this is the case.) Thus

$$EG(\tilde{n}) = EN_2(\tilde{q}_i, v_i) \geq G(\bar{n}) = N_2(\bar{q}_i, v_i). \qquad (48)$$

Next, define

$$\bar{\Pi}_i(v_i, v_i) = E_{v_{-i}}\int_0^{v_i} N_2(\bar{q}_i(z, v_{-i}), z)\,dz, \qquad (49)$$

where $N(\bar{q}_i(v_i, v_{-i}), v_i) = EN(\tilde{q}_i(v_i, v_{-i}), v_i)$. Then $\bar{q}_i$ satisfies the necessary condition (9). Arguing exactly as in Section 2, we deduce that the expected payment schedule for buyer i is

$$\bar{R}_i(v_i) = E_{v_{-i}}\left[N(\bar{q}_i(v_i, v_{-i}), v_i) - \int_0^{v_i} N_2(\bar{q}_i(z, v_{-i}), z)\,dz\right] - \Pi_i(0, 0). \qquad (50)$$

From (46), the first terms on the right-hand side of (44) and (50) are equal. From (48), the second term in (44) is no greater than the second term in (50). Thus

$$\bar{R}_i(v_i) \geq R_i(v_i). \qquad (51)$$

We now show that the procedure $[\bar{q}_i(\cdot), \bar{R}_i(\cdot)]$ satisfies the aggregate feasibility condition

$$\sum_{i=1}^n \bar{q}_i(v_i, v_{-i}) \leq q_0.$$

By Assumption A1, N is an increasing, concave function of q. Therefore, from Jensen's Inequality,

$$EN(\tilde{q}_i(v_i, v_{-i}), v_i) \leq N(E\tilde{q}_i(v_i, v_{-i}), v_i),$$

and so

$$\bar{q}_i(v_i, v_{-i}) \leq E\tilde{q}_i(v_i, v_{-i}).$$

But

$$\sum_{i=1}^n \tilde{q}_i(v_i, v_{-i}) \leq q_0,$$

establishing feasibility.

Because we are imposing regularity (Assumption C), the optimal deterministic selling procedure $[q_i^*(\cdot), R_i^*(\cdot)]_{i=1,\dots,n}$ solves the problem of maximizing expected revenue subject *only* to feasibility, (9), and (15). In particular, we need not impose monotonicity; thanks to the proof of Proposition 4, it is satisfied automatically. Now, $[\bar{q}_i(\cdot), \bar{R}_i(\cdot)]$ satisfies (9) and (15) by construction, and as we have seen it satisfies feasibility. Hence $E\Sigma R_i^*(v_i) \geqslant E\Sigma \bar{R}_i(v_i)$. Thus, in view of (51), $E\Sigma R_i^*(v_i) \geqslant E\Sigma R_i(v_i)$; that is, the optimal deterministic selling procedure generates at least as much expected revenue as the random one.

It remains to establish that, as hypothesized, $G(n)$ is convex. From (45) and (47),

$$G(N(q_i, v_i)) = N_2(q_i, v_i).$$

Thus, differentiating by q_i and rearranging, we obtain

$$\frac{dG}{dn}(N(q_i, v_i)) = N_{12}/N_1 = p_2/p.$$

Differentiating again by q_i, we obtain

$$\begin{aligned}\frac{d^2G}{dn^2}[N(q_i, v_i)]p(q_i, v_i) &= \frac{\partial}{\partial q}\left(\frac{p_2}{p}\right)\\ &= \frac{p_{12}p - p_1p_2}{p^2}\\ &= \frac{\partial}{\partial v}\left(\frac{p_1}{p}\right)\\ &\geqslant 0 \text{ by Assumption B1.}\end{aligned}$$

6. Concluding Remarks

In this paper we have shown how the earlier analysis of optimal auctions by Harris and Raviv (1981), Myerson (1981), and Riley and Samuelson (1981) can be generalized to multiple units. We conclude with some comments on the crucial assumptions.

First of all, we have assumed agents to be neutral towards income risk. With risk-averse buyers, the analysis is considerably more complicated. With only a single, indivisible unit for sale, it is relatively easy to show that the sealed high-bid auction generates greater expected revenue than the second-bid auction. However, the expected profit-maximizing selling scheme is no longer a simple auction. Instead, as Matthews (1983) and Maskin and Riley (1984a) establish, the seller can exploit buyer risk aversion still further by making losers as well as winners pay in a sealed-bid auction.

A second important assumption is that parameter values are drawn independently. This implies that any pair of buyers, with possibly very different parameter values, has the same beliefs about the parameter value of a third buyer. Although this is the natural first approximation, there are situations in which it is clearly deficient. For example, suppose that, as in the auctioning of mineral rights, the true value of the item is unknown. Each buyer has an estimate based on his research. In this case it is natural to assume that a buyer with a low estimate will have more conservative beliefs about the estimates of other buyers than a buyer with a high estimate. Milgrom and Weber (1982) apply the concept of 'affiliatedness' (implying positive correlation of parameter values) to formalize this idea to compare the sealed-bid and open auctions. A central result is that the information revealed as the open auction progresses raises the expected selling price. With risk-neutral buyers, there is no equivalent effect in the sealed-bid auction, and so the open auction dominates in terms of expected revenue.[8]

This conclusion suggests that the seller might be able to exploit the correlation of buyer's reservation values with a selling procedure very different from either of the usual auctions. Indeed, work by Myerson (1981), Cremer and McLean (1985), and Maskin and Riley (1981) shows that, when buyers are risk-neutral and their parameter values are correlated and discretely distributed, the seller can extract *all* surplus.

Finally, agents' parameter values are assumed to have been drawn from the same distribution. Although symmetry is a commonly invoked theoretical simplification, it is certainly a strong restriction. To illustrate, suppose that several contractors bid for the right to resurface a section of roadway. If one contractor is much busier than the others, he will have to hire workers overtime, reduce maintenance, and so on. If, moreover, the other bidders know about this, symmetry is violated. As we show in Maskin and Riley (1986), either the sealed-bid or the open auction can dominate the other (in terms of expected revenues), depending on the nature of the asymmetry.

[8] Because risk aversion has the effect of improving the sealed bid auction relative to the open auction, there is no simple ranking of the two except given risk-neutral buyers.

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Optimal Multi-unit Auctions

ERIC MASKIN AND JOHN RILEY

1. Introduction

Recently, a large literature has examined alternative methods for auctioning off an indivisible good. (See McAfee and McMillan 1987; Milgrom 1986; and Wilson 1987 for surveys.) Particular attention has been paid to two auctions used frequently in practice: the open, ascending-bid auction (also called the English auction), and the sealed, high-bid auction. A theoretical benchmark is provided by the Revenue Equivalence Theorem (Vickrey 1961b; Myerson 1981, and Riley and Samuelson 1981). This theorem asserts that, when each bidder's reservation price for the good is an independent draw from the same distribution and bidders are risk-neutral, the two common auctions give rise to exactly the same expected revenue for the seller.[1]

A good deal of research has considered the implications of relaxing one or more of the underlying hypotheses. Thus, Holt (1980) substitutes risk-averse for risk-neutral buyers and shows that, in this case, the sealed-bid auction generates greater expected revenue than its open counterpart.

In contrast, Milgrom and Weber (1982) show that, when reservation prices are not independent but are positively correlated, the additional informational about other buyers emerging in the open auction raises revenue on average relative to that in the sealed-bid auction.

A third strand of this research (Maskin and Riley 1986) relaxes symmetry. That is, buyers' reservation values are no longer postulated to be identically distributed. In this case, the ranking of the two auctions depends on how the distributions vary across buyers.

Rather than simply compare the expected revenue from specific auction schemes, one may wish to characterize *optimal* selling procedures, that is, selling procedures that maximize the seller's expected revenue. Under the hypotheses of the Revenue Equivalence Theorem, and provided that the distribution of reservation prices is sufficiently

regular (see Section 3 for a precise definition of regularity), the open- and sealed-bid auctions are both optimal if the seller sets an appropriate minimum allowable bid (called a reserve price). Myerson (1981) characterizes optimal auctions when regularity fails and also when the symmetry assumption is dropped. Matthews (1983), Maskin and Riley (1984b), and Moore (1984) study the case of risk-averse buyers, whereas Myerson (1981), Maskin and Riley (1981), and Cremer and McLean (1985) consider correlated reservation prices. Finally, Harris and Raviv (1981) relax the assumption that only a single good is to be sold.

This last paper is the starting point of our analysis here. For the special case of a uniform distribution of reservation prices, Harris and Raviv show that the Revenue Equivalence Theorem continues to hold if there are multiple units for sale and each buyer wishes to purchase at most a single unit. Here we establish equivalence for all distributions, and also show that, as long as the regularity assumption mentioned above is satisfied, the standard auctions with appropriate reserve prices are optimal for the seller. In addition, we characterize the optimal auction when this restriction is violated.

We then relax the restriction to unit demand and instead assume simply that each buyer has a downward-sloping demand curve. We observe that, in general, the standard auctions are no longer optimal. Instead, an optimal procedure is to set a payment schedule $T(q)$ and ask each buyer to submit an order q; a buyer who demands q pays $T(q)$. If aggregate demand is less than supply, the auctioneer fills each order. If, however, orders exceed supply, the auctioneer scales down each buyer's demand, in a predetermined way, until the capacity constraint is met.

The optimal procedure is thus a nonlinear pricing scheme modified to take account of the supply constraint. Not surprisingly, therefore, the methods of analysis build on earlier work on nonlinear pricing, in particular that of Mussa and Rosen (1978) and Maskin and Riley (1984a).

2. Formulation of the Seller's Optimization Problem

The seller has q_0 units of a good for sale. There are n buyers, each of whose 'type' v is drawn independently from the same distribution $F(v)$. A buyer of type v has preferences represented by the utility function

$$U(q, R, v) \equiv \int_0^q p(x, v)\,dx - R \equiv N(q, v) - R \qquad (1)$$

where q is the number of units purchased from the seller and R is total spending on these units. The seller and other buyers do not observe a buyer's v but know that it is drawn from $F(v)$. Throughout, we shall assume that higher levels of v are associated with higher demand.

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[1] For a formal statement of this result, generalized to the case of multiple units, see Section 2.