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Hierarchical Three-Way Decisions With Intuitionistic Fuzzy Numbers in Multi-Granularity Spaces

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ABSTRACT Intuitionistic fuzzy sets, as extensions of fuzzy sets, are described by intuitionistic fuzzy numbers (IFNs) that describe the uncertain concepts with the membership and non-membership degrees together. Three-way decisions are effective classification methods, which are always utilized for solving uncertain problems with fixed cost parameter values and fixed attribute values in reality. However, under dynamic intuitionistic fuzzy environments, three-way decisions face a great challenge for processing uncertain problems with IFN cost parameters and IFN attribute values. In this paper, by considering the impact of IFNs, a hierarchical three-way decision model with IFN cost parameters (H3WDIF-I) is established in multi-granularity spaces for handling dynamic and uncertain problems first. The change rules of H3WDIF-I are discussed to analyze the relationships of decisions based on the granularity structures. In addition, when the IFN cost parameters and IFN attribute values arise together, a hierarchical three-way decision model with IFNs (H3WDIF-II) is proposed to explore three-way decisions in depth under dynamic intuitionistic fuzzy environments. Similarly, the change rules of H3WDIF-II are also discussed from the viewpoint of classification losses in multi-granularity spaces. Finally, the presented change rules are verified by many examples and experiments.

INDEX TERMS Intuitionistic fuzzy numbers, classification losses, uncertainty, multi-granularity spaces, hierarchical three-way decision models.

I. INTRODUCTION

Three-way decisions (3WDs), proposed by Yao [38], [41] have been developed greatly for dealing with uncertain knowledge discovering recently. Three-way decision model (3WDM) divides a universe into three disjoint regions [43], i.e., positive region, boundary region and negative region, which correspond to implement three actions, i.e., acceptance, deferment and rejection, respectively. Therefore, 3WDM supplies semantic interpretations to decision-theoretic rough sets, which conforms to human cognition.

As a classification model, 3WDM has been successfully applied to solve practical applications, such as engineering decisions [11], [48], information decisions [9], [24], [25], [35], [47], [55] and medical decisions [8], [37]. Cabitza *et al.* [3] presented two methods based on three-way decisions for collective knowledge extraction from questionnaires. By considering the misclassification cost and test cost, Li *et al.* [14] proposed a cost-sensitive sequential 3WD strategy to analyze image data. In addition, theoretic researches have also achieved many fruits [5], [6], [18], [22], [31], [46], [50]–[52]. Hu *et al.* [7] discussed the transformation from multiple three-way decision spaces to a single three-way decision space based on fuzzy lattices and partial ordered sets. Li *et al.* [16] proposed generalized 3WDMs to evaluate subsets of universe. Liu and Liang [21] presented some properties of a novel 3WDM by comparing ordered 3WDs with decision-theoretic rough sets. In Pythagorean fuzzy information systems, Liang *et al.* [20] proposed a new 3WDM as a natural extension of 3WDs.

In practice, granular computing simulates human thinking in processing complex problems by utilizing

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granularity structures. Yao [39] argued that granular computing is a web of interacting granules which could be used to establish multiple hierarchies for describing a universe. In addition, Yao and Yao [45] reformulated consistent classification problems with granular computing and proposed a more general framework of classification. Thus, by introducing granular computing, 3WDMs with sequential strategy were presented for achieving multi-division of boundary region [42], [44]. Li *et al.* [13] proposed a sequential three-way decision method based on granularity structures to balance conflicts between misclassification cost and insufficient high-quality facial image information. Yang *et al.* [34] proposed a unified sequential 3WDM by combining granularity for multilevel incremental processing. Chen *et al.* [4] established a sequential three-way decision model to deal with attribute updating in a multi-granularity universe. In summary, combining with granular computing, 3WDMs have been applied to deal with the complicated and uncertain problems in many aspects [10], [12], [27], [28], [30], [54].

Intuitionistic fuzzy set (IFS), proposed by Atanassov [1], [2], is also an important tool to deal with uncertain, imprecise and vague information [49], [53]. IFS describes a concept by using an IFN which is constituted by a membership degree and a non-membership degree together. Naturally, to extend 3WDMs, IFSs have been introduced [29], [33], [36]. Tan *et al.* [32] defined fuzzy information granules based on intuitionistic fuzzy relations and characterized the lower and upper approximations of intuitionistic fuzzy rough set with hierarchical structures. Liang *et al.* [19] introduced the intuitionistic fuzzy point operator into 3WDs to bring the variation of cost parameters and discussed the decision principles under intuitionistic fuzzy environments. Furthermore, combining intuitionistic fuzzy sets, a new three-way decision-theoretic model and a new rule induction algorithm were constructed by Liu *et al.* [23]. By considering new IFN cost parameters, a dynamic intuitionistic fuzzy decision-theoretic rough set model was proposed to analyze multi-period decisions by Liang and Liu [17]. Therefore, many successes were achieved by utilizing intuitionistic fuzzy information to aim at actual applications.

However, in reality, intuitionistic fuzzy environments always provide intuitionistic fuzzy information including IFN costs and IFN attribute values. Moreover, attribute increments always arise under intuitionistic fuzzy environments, which lead to different decision results. Therefore, 3WDM still exists several shortcomings for effective classifications currently under intuitionistic fuzzy environments:

1) how to present a 3WDM to apply intuitionistic fuzzy information effectively with constant increasing of attributes?
2) the relationships of different decisions in dynamic environments should be revealed.

To overcome the two shortcomings, a H3WDIF-I is derived in multi-granularity spaces, which targets for dealing with IFN cost parameters and fixed attribute values in this paper. Furthermore, relationships of decisions in different granularity levels are discussed from the viewpoint of classification losses. Then, by considering IFN attribute values and IFN cost parameter values together, a H3WDIF-II is established in multi-granularity spaces to deal with practical issues. As prerequisites for applications, the following relationships of decisions are discussed in detail in multi-granularity spaces:

1) the relationships of classification losses in successive granularity levels are locally analyzed from the membership and non-membership degree perspectives, respectively;
2) the changing trends of classification losses are discussed globally and clearly;
3) the changing trends of three regions in multi-granularity spaces are shown to achieve effective classification.

The rest of this paper is organized as follows. Section II provides many basic concepts and theories on three-way decisions and intuitionistic fuzzy sets. In Section III, a hierarchical three-way decision model with IFN cost parameters is defined and the relationships of decisions in successive granularity levels are discussed in multi-granularity spaces. In Section IV, a new H3WDIF-II is proposed and some change rules are analyzed in multi-granularity spaces. Many experiments are presented to verify proposed rules in Section V. Section VI concludes this paper.

II. PRELIMINARIES

In this section, some related theories and definitions on IFSs and 3WDs are reviewed briefly as prerequisites of this paper.

Definition 1 (Intuitionistic Fuzzy Set [1]): Given a fixed set U, an IFS E on U can be represented as follows,

$$E = \{\langle x, \mu_E(x), \nu_E(x)\rangle\},$$

where the functions $\mu_E(x) : U \to [0, 1]$ and $\nu_E(x) : U \to [0, 1]$ are the membership and non-membership degrees of object x, respectively. Moreover, for any $x \in U$, $0 \leq \mu_E(x) + \nu_E(x) \leq 1$, then the hesitation membership degree can be calculated as: $\pi_E(x) = 1 - \mu_E(x) - \nu_E(x)$, $0 \leq \pi_E(x) \leq 1$. An IFN can be compactly denoted as $E(x) = (\mu_E(x), \nu_E(x))$.

Given two IFNs $E(x_1) = (\mu_E(x_1), \nu_E(x_1))$, $E(x_2) = (\mu_E(x_2), \nu_E(x_2))$, their basic operations are:

1) $E(x_1) \oplus E(x_2) = (\mu_E(x_1) + \mu_E(x_2) - \mu_E(x_1)\mu_E(x_2), \nu_E(x_1)\nu_E(x_2))$,
2) $E(x_1) \otimes E(x_2) = (\mu_E(x_1)\mu_E(x_1), \nu_E(x_1) + \nu_E(x_2) - \nu_E(x_1)\nu_E(x_2))$,
3) $kE(x_1) = (1 - (1 - \mu_E(x_1))^k, (\nu_E(x_1))^k)$.

Definition 2 (Equivalence Class and Equivalence Relation) [26]: Given an information table $S = (U, AT, V, f)$, a nonempty finite set U, an attribute set AT and an attribute value set V. Let $f : U \to V$ be an information function, for any attribute subset $C \subseteq AT$,

$$IND(C) = \{(x, y) | (x, y) \in U^2, \forall_{c \in C}(c(x) = c(y))\},$$

$U/IND(C) = \{[x]_{IND(C)} | x \in U\}$ is a partition, where $[x]_{IND(C)}$ and $IND(C)$ are called an equivalence class and an

TABLE 1. Intuitionistic fuzzy cost parameter matrix.

	X	$\neg X$
a_P	$E(\lambda_{PP}) = (\mu_E(\lambda_{PP}), \nu_E(\lambda_{PP}))$	$E(\lambda_{PN}) = (\mu_E(\lambda_{PN}), \nu_E(\lambda_{PN}))$
a_B	$E(\lambda_{BP}) = (\mu_E(\lambda_{BP}), \nu_E(\lambda_{BP}))$	$E(\lambda_{BN}) = (\mu_E(\lambda_{BN}), \nu_E(\lambda_{BN}))$
a_N	$E(\lambda_{NP}) = (\mu_E(\lambda_{NP}), \nu_E(\lambda_{NP}))$	$E(\lambda_{NN}) = (\mu_E(\lambda_{NN}), \nu_E(\lambda_{NN}))$

equivalence relation, respectively. In the following sections, equivalence class of x can be simplified as $[x]$.

Definition 3 (Conditional Probability of Classification) [26]: Given an information table $S = (U, AT, V, f)$ and a subset $X \subseteq U$. $\forall x \in U$, the conditional probability of classification $P(X|[x])$ can be defined as follows,

$$P(X|[x]) = \frac{|X \cap [x]|}{|[x]|},$$

where $|\bullet|$ denotes the cardinality of a finite set.

Definition 4 (Decision Information Table) [26]: Given an information table $S = (U, AT, V, f)$, where U is an object set, AT is an attribute set, V is an attribute value set and $f : U \to V$ is an information function. If $AT = C \cup D$, where C is a set of conditional attributes and D is a set of decision attributes, then S is called a decision information table and denoted as $S = (U, C \cup D, V, f)$.

Definition 5 (Consistent Decision Table) [15]: Given a decision information table $S = (U, C \cup D, V, f)$. Given an object $x \in U$ and a subset $X \subseteq U$. S is called a consistent decision table if and only if $POS_C(D) = U$, where $POS_C(D) = \bigcup_{X \subseteq C/D} \underline{apr}(X)$, and $\underline{apr}(X) = \{x|[x]_{IND(C)} \subseteq X\}$.

Definition 6 (Three-Way Decisions) [40], [41]: Given a set of states $\Omega = (X, \neg X)$ and a set of actions $A = \{a_P, a_B, a_N\}$ where a_P, a_B and a_N represent three actions, i.e., deciding positive region $POS(X)$, deciding boundary region $BND(X)$, and deciding negative region $NEG(X)$, corresponding to acceptance, deferment and rejection decisions, respectively. Given a pair of thresholds (α, β) $(0 \leq \beta < \alpha \leq 1)$, then decision rules of 3WDs are shown,

1) if $P(X|[x]) \geq \alpha$, then $x \in POS(X)$,
2) if $\beta < P(X|[x]) < \alpha$, then $x \in BND(X)$,
3) if $P(X|[x]) \leq \beta$, then $x \in NEG(X)$.

Under intuitionistic fuzzy environments, a 3WDM with IFN cost parameters is reviewed by considering an intuitionistic fuzzy cost parameter matrix $IM = \{E(\lambda_{\cdot k}) = (\mu_E(\lambda_{\cdot k}), \nu_E(\lambda_{\cdot k}))\}_{3\times 2}$ $(\cdot = P, B, N$ and $k = P, N)$ in Table 1 [17]. In Table 1, X and $\neg X$ are complementary state sets. IFNs $E(\lambda_{PP})$, $E(\lambda_{BP})$ and $E(\lambda_{NP})$ are cost parameter values where $\mu_E(\lambda_{PP})$, $\mu_E(\lambda_{BP})$ and $\mu_E(\lambda_{NP})$ are the membership degrees of cost, and $\nu_E(\lambda_{PP})$, $\nu_E(\lambda_{BP})$ and $\nu_E(\lambda_{NP})$ are the non-membership degrees for objects in X taking a_P, a_B and a_N, however, IFNs $E(\lambda_{PN})$, $E(\lambda_{BN})$ and $E(\lambda_{NN})$ are cost parameter values where $\mu_E(\lambda_{PN})$, $\mu_E(\lambda_{BN})$ and $\mu_E(\lambda_{NN})$ are the membership degrees, and $\nu_E(\lambda_{PN})$, $\nu_E(\lambda_{BN})$ and $\nu_E(\lambda_{NN})$ are the non-membership degrees for objects not in X taking a_P, a_B and a_N. Therefore, the classification losses with intuitionistic fuzzy cost parameters can be expressed as:

$$\begin{aligned}
R(a_P|[x]) &= [1-(1-\mu_E(\lambda_{PP}))^{P(X|[x])}(1-\mu_E(\lambda_{PN}))^{1-P(X|[x])},\\
&\quad \nu_E(\lambda_{PP})^{P(X|[x])}\nu_E(\lambda_{PN})^{1-P(X|[x])}],\\
R(a_B|[x]) &= [1-(1-\mu_E(\lambda_{BP}))^{P(X|[x])}(1-\mu_E(\lambda_{BN}))^{1-P(X|[x])},\\
&\quad \nu_E(\lambda_{BP})^{P(X|[x])}\nu_E(\lambda_{BN})^{1-P(X|[x])}],\\
R(a_N|[x]) &= [1-(1-\mu_E(\lambda_{NP}))^{P(X|[x])}(1-\mu_E(\lambda_{NN}))^{1-P(X|[x])},\\
&\quad \nu_E(\lambda_{NP})^{P(X|[x])}\nu_E(\lambda_{NN})^{1-P(X|[x])}].
\end{aligned}$$

The decision rules of 3WDM with IFN cost parameters (3WDIF-I) are provided by Bayesian decision theory for minimum loss with a pair of thresholds (α_1, β_1) $(0 \leq \beta_1 < \alpha_1 \leq 1)$ from the membership degree perspective:

1) if $P(X|[x]) \geq \alpha_1$, then $x \in POS(X)$,
2) if $\beta_1 < P(X|[x]) < \alpha_1$, then $x \in BND(X)$,
3) if $P(X|[x]) \leq \beta_1$, then $x \in NEG(X)$,

where

$$\alpha_1 = \frac{\ln \frac{1-\mu_E(\lambda_{BN})}{1-\mu_E(\lambda_{PN})}}{\ln\left(\frac{1-\mu_E(\lambda_{PP})}{1-\mu_E(\lambda_{BP})} \times \frac{1-\mu_E(\lambda_{BN})}{1-\mu_E(\lambda_{PN})}\right)},$$

$$\beta_1 = \frac{\ln \frac{1-\mu_E(\lambda_{NN})}{1-\mu_E(\lambda_{BN})}}{\ln\left(\frac{1-\mu_E(\lambda_{BP})}{1-\mu_E(\lambda_{NP})} \times \frac{1-\mu_E(\lambda_{NN})}{1-\mu_E(\lambda_{BN})}\right)}.$$

From the non-membership degree perspective with a pair of thresholds (α_2, β_2) $(0 \leq \beta_2 < \alpha_2 \leq 1)$:

1) if $P(X|[x]) \geq \alpha_2$, then $x \in POS(X)$,
2) if $\beta_2 < P(X|[x]) < \alpha_2$, then $x \in BND(X)$,
3) if $P(X|[x]) \leq \beta_2$, then $x \in NEG(X)$,

where

$$\alpha_2 = \frac{\ln \frac{\nu_E(\lambda_{BN})}{\nu_E(\lambda_{PN})}}{\ln\left(\frac{\nu_E(\lambda_{PP})}{\nu_E(\lambda_{BP})} \times \frac{\nu_E(\lambda_{BN})}{\nu_E(\lambda_{PN})}\right)},$$

$$\beta_2 = \frac{\ln \frac{\nu_E(\lambda_{NN})}{\nu_E(\lambda_{BN})}}{\ln\left(\frac{\nu_E(\lambda_{BP})}{\nu_E(\lambda_{NP})} \times \frac{\nu_E(\lambda_{NN})}{\nu_E(\lambda_{BN})}\right)}.$$

III. HIERARCHICAL THREE-WAY DECISION MODEL WITH IFN COST PARAMETERS

3WDIF-I only provides a method to make three-way decisions in a granularity level under intuitionistic fuzzy environments. Therefore, under dynamic intuitionistic fuzzy environments, it may be only possible to make decisions with coarse granules. However, with coarse-decreased granules, more detailed information is supplied, and hence establishing multi-granularity spaces is an important method to fully use information in decision-making. In this paper, inspired by 3WDIF-I [17], a hierarchical three-way decision model is established from two perspectives in multi-granularity spaces for solving attribute increment problems by Definition 7.

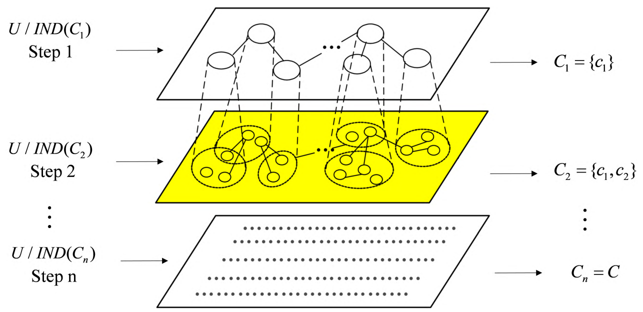

FIGURE 1. **Partitions of objects in multi-granularity spaces.**

Definition 7 (H3WDIF-I): Given a decision information table $S=(U,C\cup D,V,f)$. Let attribute subset $C_i\subseteq C$, $C_i=\{c_1,c_2,\cdots,c_i\}$ ($|C|=n$ and $i=1,2,\cdots,n$). If $IM=\{E(\lambda_{\cdot k})=(\mu_E(\lambda_{\cdot k}),\nu_E(\lambda_{\cdot k}))\}_{3\times 2}$ ($\cdot=P,B,N$ and $k=P,N$) is an intuitionistic fuzzy cost parameter matrix, for any object $x\in U$, a hierarchical three-way decision model with IFN cost parameters (H3WDIF-I) can be defined as follows,

$$HIFD(I)=(HIFD_1(I),HIFD_2(I),\cdots,HIFD_i(I),\\ \cdots,HIFD_n(I)),$$

where $HIFD_i(I)$ denotes a decision based on $IND(C_i)$.

With the refinement of granules, a multi-granularity space is always established and granularity levels are formed. Fig.1 shows the partitions of objects in granularity levels of a multi-granularity space. Observably, the more refined granules are, the more attributes are supplied. For convenience, decisions with the coarsest granule are written as the first step of hierarchical models. According to Definition 7, the classification loss in $HIFD_i(I)$ is expressed as

$$R([x],HIFD_i(I))=\min_{\cdot=P,B,N}(R_i(a_{\cdot}|[x])),$$

where

$$\begin{aligned}&R_i(a_{\cdot}|[x])_{\cdot=P,B,N}\\&=(1-(1-\mu_E(\lambda_{\cdot P}))^{P_i(X|[x])}(1-\mu_E(\lambda_{\cdot N}))^{P_i(\neg X|[x])},\\&\quad(\nu_E(\lambda_{\cdot P}))^{P_i(X|[x])}(\nu_E(\lambda_{\cdot N}))^{P_i(\neg X|[x])}),\end{aligned}$$

and $P_i(X|[x])$ is the conditional probability of classification at the i-th step.

Let cost parameters $\mu_E(\lambda_{PP})=0$, $\mu_E(\lambda_{NN})=0$, $\nu_E(\lambda_{PP})=1$, and $\nu_E(\lambda_{NN})=1$, then $R([x],HIFD_i(I))$ can be simplified as follows,

$$\begin{aligned}&R([x],HISD_i(I))\\&=\begin{cases}(\mu_i([x],R)_P,\nu_i([x],R)_P), & \text{if } P_i(X|[x])\geq\alpha_I\\(\mu_i([x],R)_N,\nu_i([x],R)_N), & \text{if } P_i(X|[x])\leq\beta_I\\(\mu_i([x],R)_B,\nu_i([x],R)_B), & \text{if } \beta_I<P_i(X|[x])<\alpha_I,\end{cases}\end{aligned}$$

where

$$\begin{aligned}\mu_i([x],R)_P&=1-(1-\mu_E(\lambda_{PN}))^{P_i(\neg X|[x])},\\\nu_i([x],R)_P&=(\nu_E(\lambda_{PN}))^{P_i(\neg X|[x])},\\\mu_i([x],R)_N&=1-(1-\mu_E(\lambda_{NP}))^{P_i(X|[x])},\\\nu_i([x],R)_N&=(\nu_E(\lambda_{NP}))^{P_i(X|[x])},\\\nu_i([x],R)_B&=(\nu_E(\lambda_{BP}))^{P_i(X|[x])}(\nu_E(\lambda_{BN}))^{P_i(\neg X|[x])},\\\mu_i([x],R)_B&=1-(1-\mu_E(\lambda_{BP}))^{P_i(X|[x])}\\&\quad(1-\mu_E(\lambda_{BN}))^{P_i(\neg X|[x])}.\end{aligned}$$

In addition, if $R([x],HISD_i(I))$ is discussed from the membership degree perspective, $\alpha_I=\alpha_1$ and $\beta_I=\beta_1$, however, from the non-membership degree perspective, $\alpha_I=\alpha_2$ and $\beta_I=\beta_2$.

In reality, more and more information is supplied by attribute increment at continuous steps for different decision results in multi-granularity spaces. Therefore, it is significant to find the relationships of decisions in granularity levels. Theorems 1 and 2 show the relative change rules of classification loss for x in the positive region from the membership degree and non-membership degree perspectives, respectively.

Theorem 1: Given a successive decision $HIFD_j(I)$ of $HIFD_i(I)$ ($i<j$). For any x satisfying $P_i(X|[x])\geq\alpha_1$,

1) if $P_i(X|[x])\leq P_j(X|[x])$, then $\mu_j([x],R)_P\leq\mu_i([x],R)_P$,
2) if $\alpha_1\leq P_j(X|[x])\leq P_i(X|[x])$, then $\mu_j([x],R)_P\geq\mu_i([x],R)_P$.

Proof: If $P_i(X|[x])\leq P_j(X|[x])$, then $\alpha_1\leq P_i(X|[x])\leq P_j(X|[x])$.

$$\begin{aligned}\mu_i([x],R)_P&=1-(1-\mu_E(\lambda_{PN}))^{P_i(\neg X|[x])}\\&=1-(1-\mu_E(\lambda_{PN}))^{1-P_i(X|[x])},\\\mu_j([x],R)_P&=1-(1-\mu_E(\lambda_{PN}))^{1-P_j(X|[x])},\end{aligned}$$

then $\mu_j([x],R)_P\leq\mu_i([x],R)_P$. Similarly, if $\alpha_1\leq P_j(X|[x])\leq P_i(X|[x])$, thus $\mu_i([x],R)_P\leq\mu_j([x],R)_P$ holds. ■

Theorem 1 shows that the classification loss at the i-th step is larger than it at the $(i+1)$-th step when the conditional probability of classification is smaller, however, when the conditional probability of classification at the i-th step is larger, the classification loss at the i-th step is smaller. Therefore, from the membership degree perspective, the relationships of classification loss in successive granularity levels are revealed. Meaningfully, although there exist a non-monotonicity relationship of changes in multi-granularity spaces, the changes of classification loss still depend on conditional probability of classification.

Theorem 2: For any object x satisfying $P_i(X|[x])\geq\alpha_2$ in $HIFD_i(I)$,

1) if $P_j(X|[x])\geq P_i(X|[x])$, then $\nu_j([x],R)_P\geq\nu_i([x],R)_P$,
2) if $\alpha_2\leq P_j(X|[x])\leq P_i(X|[x])$, then $\nu_j([x],R)_P\leq\nu_i([x],R)_P$,

Proof: If $P_j(X|[x])\geq P_i(X|[x])$, then $P_j(X|[x])\geq P_i(X|[x])\geq\alpha_2$.

$$\begin{aligned}\nu_i([x],R)_P&=(\nu_E(\lambda_{PN}))^{1-P_i(X|[x])},\\\nu_j([x],R)_P&=(\nu_E(\lambda_{PN}))^{1-P_j(X|[x])},\end{aligned}$$

thus $\nu_j([x],R)_P\geq\nu_i([x],R)_P$.

TABLE 2. Influenza decision information based on H3WDIF-I.

U	c_1 (headache)	c_2 (myalgia)	c_3 (temperature)	d (influenza)
x_1	Yes	Yes	Normal	No
x_2	Yes	Yes	High	Yes
x_3	Yes	Yes	Higher	Yes
x_4	No	Yes	Normal	No
x_5	No	No	High	No
x_6	No	Yes	Higher	Yes
x_7	No	Yes	Higher	Yes
x_8	Yes	No	Higher	Yes

TABLE 3. Intuitionistic fuzzy cost parameters.

	X	$\neg X$
a_P	$E(\lambda_{PP}) = (0, 1)$	$E(\lambda_{PN}) = (0.8, 0.1)$
a_B	$E(\lambda_{BP}) = (0.3, 0.7)$	$E(\lambda_{BN}) = (0.6, 0.3)$
a_N	$E(\lambda_{NP}) = (0.9, 0.1)$	$E(\lambda_{NN}) = (0.05, 0.8)$

Similarly, if $\alpha_2 \leq P_j(X|[x]) \leq P_i(X|[x])$, thus $\nu_j([x], R)_P \leq \nu_i([x], R)_P$ holds. ■

Example 1: To verify Theorems 1 and 2, Table 2 shows some decision information for judging influenza, and Table 3 gives an intuitionistic fuzzy cost parameter matrix to calculate two pairs of thresholds based on 3WDIF-I.

It is easy to obtain that $\alpha_1 = 0.66$, $\beta_1 = 0.31$, $\alpha_2 = 0.75$ and $\beta_2 = 0.34$ from the membership and non-membership degree perspectives, respectively. The following partitions by the equivalence relation $IND(D)$ (decision attribute set $D = \{d\}$) are shown.

$$U/IND(D) = \{\{x_2, x_3, x_6, x_7, x_8\}, \{x_1, x_4, x_5\}\} = \{X, \neg X\}.$$

Suppose C_1, C_2 and C_3 are a series of attribute subsets, and let $C_1 = \{c_1\}$, $C_2 = \{c_1, c_2\}$ and $C_3 = \{c_1, c_2, c_3\}$. For a concept X, the conditional probabilities of classification based on $IND(C_1)$, $IND(C_2)$ and $IND(C_3)$ are as follows,

$$U/IND(C_1) = \{\{x_1, x_2, x_3, x_8\}, \{x_4, x_5, x_6, x_7\}\},$$

$P_1(X|[x_1]) = \frac{3}{4}$, $P_1(X|[x_4]) = \frac{1}{2}$.

$$U/IND(C_2) = \{\{x_1, x_2, x_3\}, \{x_4, x_6, x_7\}, \{x_5\}, \{x_8\}\},$$

$P_2(X|[x_1]) = \frac{2}{3}$, $P_2(X|[x_4]) = \frac{2}{3}$, $P_2(X|[x_5]) = 0$, $P_2(X|[x_8]) = 1$.

$$U/IND(C_3) = \{\{x_1\}, \{x_2\}, \{x_3\}, \{x_4\}, \{x_5\}, \{x_6, x_7\}, \{x_8\}\},$$

$P_3(X|[x_1]) = 0$, $P_3(X|[x_2]) = 1$, $P_3(X|[x_3]) = 1$, $P_3(X|[x_4]) = 0$, $P_3(X|[x_5]) = 0$, $P_3(X|[x_6]) = 1$, $P_3(X|[x_8]) = 1$.

The relation $P_3(X|[x_3]) \geq P_1(X|[x_1]) \geq P_2(X|[x_4]) \geq 0.66$ is provided in successive granularity levels. From the membership degree perspective, the classification losses are calculated based on H3WDIF-I.

$$\begin{aligned}\mu_1([x_1], R)_P &= 1 - (1 - \mu_E(\lambda_{PN}))^{P_1(\neg X|[x_1])}\\ &= 1 - (1 - 0.8)^{1-\frac{3}{4}} = 0.33,\\ \mu_2([x_4], R)_P &= 1 - (1 - \mu_E(\lambda_{PN}))^{P_2(\neg X|[x_4])}\\ &= 1 - (1 - 0.8)^{1-\frac{2}{3}} = 0.42,\\ \mu_3([x_3], R)_P &= 1 - (1 - \mu_E(\lambda_{PN}))^{P_3(\neg X|[x_3])}\\ &= 1 - (1 - 0.8)^{1-1} = 0,\end{aligned}$$

then $\mu_3([x_3], R)_P \leq \mu_1([x_1], R)_P \leq \mu_2([x_4], R)_P$.

Furthermore, $0.75 \leq P_1(X|[x_1]) \leq P_2(X|[x_8])$ in a multi-granularity space,

$$\begin{aligned}\nu_1([x_1], R)_P &= (\nu_E(\lambda_{PN}))^{P_1(\neg X|[x_1])}\\ &= (1 - 0.1)^{1-\frac{3}{4}} = 0.97,\\ \nu_2([x_8], R)_P &= (\nu_E(\lambda_{PN}))^{P_2(\neg X|[x_8])}\\ &= (1 - 0.1)^{1-1} = 1.\end{aligned}$$

Then $\nu_1([x_1], R)_P \leq \nu_2([x_8], R)_P$.

Theorems 1 and 2 analyze the relationships in successive granularity levels from the membership and non-membership degree perspectives, respectively. In the positive region, non-monotonicity changes of classification loss are found in a multi-granularity space, which are locally discussed based on H3WDIF-I. However, some properties are also presented for globally discussing the trends of classification losses by Theorems 3 and 4.

Theorem 3: Given a consistent decision table $S = (U, C \cup D, V, f)$, let $n = |C|$ and $1 \leq i \leq n$. For any $x \in U$, there must exist $HIFD_i(I)$ for classification loss $\mu_i([x], R)_\cdot = 0$ $(\cdot = P, N)$. *Proof:* Suppose S is a consistent decision table, $POS_C(D) = U$ and $U/IND(D) = \{X, \neg X\}$. Let $[x]_{IND(C)} = [x]$ where $x \in U$, thus $[x] \subseteq X$ or $[x] \subseteq \neg X$. If $[x] \subseteq X$, $P_n(X|[x]) = 1 \geq \alpha_1$ $(0 \leq \alpha_1 \leq 1)$, then

$$\mu_n([x], R)_P = 1 - (1 - \mu_E(\lambda_{PN}))^{1-P_n(X|[x])} = 0.$$

If $[x] \subseteq \neg X$, $P_n(X|[x]) = 0 \leq \beta_1$ $(0 \leq \beta_1 \leq 1)$, then

$$\mu_n([x], R)_N = 1 - (1 - \mu_E(\lambda_{NP}))^{P_n(X|[x])} = 0.$$

Therefore $\mu_i([x], R)_\cdot = 0$ when $i < n$ or $i = n$. ■

Theorem 3 reveals that classification losses would reach to 0 from the membership degree perspective in multi-granularity spaces based on H3WDIF-I. Meanwhile, the trend of classification losses is globally discussed from the non-membership degree perspective by Theorem 4.

Theorem 4: Given a consistent decision table $S = (U, C \cup D, V, f)$, let $n = |C|$ and $1 \leq i \leq n$. For any $x \in U$, there must exist $HIFD_i(I)$ for classification loss $\nu_i([x], R)_\cdot = 1$ $(\cdot = P, N)$.

Proof: Suppose S is a consistent decision table, $x \in U$, $POS_C(D) = U$ and $U/IND(D) = \{X, \neg X\}$. Let $[x]_{IND(C)} = [x]$, $[x] \subseteq X$ or $[x] \subseteq \neg X$. If $[x] \subseteq X$, $P_n(X|[x]) = 1 \geq \alpha_2$ $(0 \leq \alpha_2 \leq 1)$, then

$$\nu_n([x], R)_P = (\nu_E(\lambda_{PN}))^{1-P_n(X|[x])} = 1.$$

If $[x] \subseteq \neg X$, $P_n(X|[x]) = 0 \leq \beta_2$ $(0 \leq \beta_2 \leq 1)$, then

$$\nu_n([x], R)_N = (\nu_E(\lambda_{NP}))^{P_n(X|[x])} = 1.$$

Therefore $\nu_i([x], R)_\cdot = 1$ when $i < n$ or $i = n$. ■

Theorem 4 indicates that there must exist one decision for classification losses reaching to 1 from the non-membership degree perspective in multi-granularity spaces. What causes the differences between Theorem 3 and Theorem 4? Understandably, from the non-membership degree perspective, the larger loss is, the more expected decision is. For example, in a voting, when the loss is too large for con, the result of voting tends to pro. Conversely, when the loss is too large for pro, con is always chosen. Therefore the classification losses from the membership degree prespective are hoped to be small but to be large from the non-membership degree perspective. In reality, a better and more correct decision with less loss is eager by limited and effective information. By Theorems 3 and 4, the trends of classification losses are analyzed globally, which give the optimal decision of H3DWIF with consistent information.

Example 1: At the 3-th step, $P_3(X|[x_1]) = 0$, $P_3(X|[x_2]) = 1$, $P_3(X|[x_3]) = 1$, $P_3(X|[x_4]) = 0$, $P_3(X|[x_5]) = 0$, $P_3(X|[x_6]) = 1$ and $P_3(X|[x_8]) = 1$. Therefore, whatever from the membership and non-membership degree perspectives, $[x_1]$, $[x_4]$, and $[x_5]$ are separated towards the negative region. The classification loss of $[x_1]$ is shown from the membership degree perspective as examples.

$$\begin{aligned}\mu_3([x_1], R)_N &= 1-(1-\mu_E(\lambda_{NP}))^{P_3(X|[x_1])}\\ &= 1-(1-0.9)^0 = 0,\end{aligned}$$

Meanwhile, $[x_2]$, $[x_3]$, $[x_6]$ and $[x_8]$ are classified into positive region. Similarly, The classification loss of $[x_2]$ is only given as follows,

$$\begin{aligned}\mu_3([x_2], R)_P &= 1-(1-\mu_E(\lambda_{PN}))^{P_3(\neg X|[x_2])}\\ &= 1-(1-0.8)^{1-1} = 0.\end{aligned}$$

Observably, the classification losses equal to 0 at the last step for any objects in a multi-granularity space even though that of some objects tend to 0 in earlier. In addition, the classification losses are also calculated from the non-membership degree perspective as follows,

$$\begin{aligned}\nu_3([x_1], R)_N &= (\nu_E(\lambda_{NP}))^{P_3(X|[x_1])}\\ &= (1-0.1)^0 = 1,\\ \nu_3([x_2], R)_P &= (\nu_E(\lambda_{PN}))^{P_3(\neg X|[x_2])}\\ &= (1-0.1)^{1-1} = 1.\end{aligned}$$

Similarly, at the 3-th step, all objects are classified into the positive and negative regions, and the classification losses of all objects from the non-membership degree are equal to 1. Therefore, Theorems 3 and 4 are respectively validated from the membership and non-membership degree perspectives.

Ideally, the change rule of boundary region deserves discussing by considering a consistent decision table. Theorem 5 proves corresponding change rules of boundary region based on Theorems 3 and 4. Moreover, an example is presented to illustrate Theorem 5.

Theorem 5: Let $BND_i(X)$ be a boundary region of $HIFD_i(I)$. There must exist a step i for $BND_i(X) = \emptyset$.

Proof:

1) From the membership degree perspective, according to Theorem 3, $\mu_i([x], R)_P = 0$ or $\mu_i([x], R)_N = 0$ at the i-th step. And $[x]$ must be separated towards positive and negative regions when $i = n$. Thus, there at least exist a step $i = n$ for $BND_i(X) = \emptyset$.
2) Similarly, there must exist $BND_i(X) = \emptyset$ at step $i = n$ from the non-membership degree perspective according to Theorem 4. ∎

Example 1: From the membership degree perspective, the thresholds $\alpha_1 = 0.66$ and $\beta_1 = 0.31$.

1) At the first step, $P_1(X|[x_1]) = \frac{3}{4}$, then $[x_1]$ is in the positive region but $[x_4]$ is in $BND_1(X)$.
2) At the second step, $P_2(X|[x_1]) = \frac{2}{3}$, $P_2(X|[x_4]) = \frac{2}{3}$, $P_2(X|[x_5]) = 0$ and $P_2(X|[x_8]) = 1$, then all objects are separated towards the positive and negative regions. Thus, there is no objects in $BND_2(X)$ from the membership degree perspective.

From the perspective of the non-membership degree, $\alpha_2 = 0.75$ and $\beta_2 = 0.34$. Thus, the classification results of boundary regions in multi-granularity spaces are same as results from the membership degree perspective, and the decisions at the third step also can be as an example for discussing $BND_i(X) = \emptyset$.

Actually, according to Theorems 3 and 4, boundary region will lead to be an empty set by refinement of granules with consistent information. By Theorem 5, an optimal decision is also revealed in multi-granularity spaces with enough information.

IV. HIERARCHICAL THREE-WAY DECISION MODEL WITH IFNS

Let $IS = (U, AT, IV, f)$ be an information table, where U denotes a nonempty finite set, AT denotes an attribute set, and IV denotes an attribute value set. Given an object $x \in U$, if IFN $E(x) = (\mu(x), \nu(x))$ $(E(x) \in IV)$ and a function $f : U \to IFV$, then IS is called intuitionistic fuzzy information table.

3WDs with IFN cost parameters and IFN attribute values (3WDIF-II) are discussed shortly. As follows, the new classification losses are defined based on Table 1,

$$\begin{aligned}R(a_P|x) &= E(\lambda_{PP})\mu(x) \oplus E(\lambda_{PN})\nu(x),\\ R(a_B|x) &= E(\lambda_{BP})\mu(x) \oplus E(\lambda_{BN})\nu(x),\\ R(a_N|x) &= E(\lambda_{NP})\mu(x) \oplus E(\lambda_{NN})\nu(x).\end{aligned}$$

To detail discuss, they are rewritten as,

$$\begin{aligned}&R(a_P|x)\\ &\quad= [1-(1-\mu_E(\lambda_{PP}))^{\mu(x)}(1-\mu_E(\lambda_{PN}))^{1-\pi(x)-\mu(x)},\\ &\qquad \nu_E(\lambda_{PP})^{\mu(x)}\nu_E(\lambda_{PN})^{1-\pi(x)-\mu(x)}],\\ &R(a_B|x)\\ &\quad= [1-(1-\mu_E(\lambda_{BP}))^{\mu(x)}(1-\mu_E(\lambda_{BN}))^{1-\pi(x)-\mu(x)},\\ &\qquad \nu_E(\lambda_{BP})^{\mu(x)}\nu_E(\lambda_{BN})^{1-\pi(x)-\mu(x)}],\\ &R(a_N|x)\\ &\quad= [1-(1-\mu_E(\lambda_{NP}))^{\mu(x)}(1-\mu_E(\lambda_{NN}))^{1-\pi(x)-\mu(x)},\\ &\qquad \nu_E(\lambda_{NP})^{\mu(x)}\nu_E(\lambda_{NN})^{1-\pi(x)-\mu(x)}].\end{aligned}$$

Combining with Bayesian decision theory, decision rules of 3WDIF-II for x are defined by a pair of thresholds $(\alpha_1(x), \beta_1(x))$ $(0 \leq \beta_1(x) < \alpha_1(x) \leq 1)$ from the membership degree perspective,

1) if $\mu(x) \geq \alpha_1(x)$, then $x \in POS(X)$,
2) if $\beta_1(x) < \mu(x) < \alpha_1(x)$, then $x \in BND(X)$,
3) if $\mu(x) \leq \beta_1(x)$, then $x \in NEG(X)$,

where

$$\alpha_1(x) = (1-\pi(x))\frac{\ln\frac{1-\mu_E(\lambda_{BN})}{1-\mu_E(\lambda_{PN})}}{\ln\left(\frac{1-\mu_E(\lambda_{PP})}{1-\mu_E(\lambda_{BP})}\times\frac{1-\mu_E(\lambda_{BN})}{1-\mu_E(\lambda_{PN})}\right)} = (1-\pi(x))\alpha_1,$$

$$\beta_1(x) = (1-\pi(x))\frac{\ln\frac{1-\mu_E(\lambda_{NN})}{1-\mu_E(\lambda_{BN})}}{\ln\left(\frac{1-\mu_E(\lambda_{BP})}{1-\mu_E(\lambda_{NP})}\times\frac{1-\mu_E(\lambda_{NN})}{1-\mu_E(\lambda_{BN})}\right)} = (1-\pi(x))\beta_1.$$

Furthermore, by a pair of thresholds $(\alpha_2(x), \beta_2(x))$ $(0 \leq \beta_2(x) < \alpha_2(x) \leq 1)$, the decision rules are shown from the non-membership degree perspective:

1) if $\mu(x) \geq \alpha_2(x)$, then $x \in POS(X)$,
2) if $\beta_2(x) < \mu(x) < \alpha_2(x)$, then $x \in BND(X)$,
3) if $\mu(x) \leq \beta_2(x)$, then $x \in NEG(X)$,

where

$$\alpha_2(x) = (1-\pi(x))\frac{\ln\frac{\nu_E(\lambda_{BN})}{\nu_E(\lambda_{PN})}}{\ln\left(\frac{\nu_E(\lambda_{PP})}{\nu_E(\lambda_{BP})}\times\frac{\nu_E(\lambda_{BN})}{\nu_E(\lambda_{PN})}\right)} = (1-\pi(x))\alpha_2,$$

$$\beta_2(x) = (1-\pi(x))\frac{\ln\frac{\nu_E(\lambda_{NN})}{\nu_E(\lambda_{BN})}}{\ln\left(\frac{\nu_E(\lambda_{BP})}{\nu_E(\lambda_{NP})}\times\frac{\nu_E(\lambda_{NN})}{\nu_E(\lambda_{BN})}\right)} = (1-\pi(x))\beta_2.$$

Definition 8 (Decision IFN): Given an intuitionistic fuzzy information table $IS = (U, AT, IV, f)$, an attribute subset $C = \{c_1, c_2\}$ $(C \subseteq AT)$ and an object $x \in U$. Given two IFNs $E(c_1) = (\mu(c_1), \nu(c_1))$ and $E(c_2) = (\mu(c_2), \nu(c_2))$. Let δ_{c_1} and δ_{c_2} $(0 \leq \delta_{c_1}, \delta_{c_2} \leq 1)$ be two important degrees of c_1 and c_2. Then a decision IFN $DA(x)$ of x on C is defined as $DA(x) = \delta_{c_1}E(c_1) + \delta_{c_2}E(c_2)$.

Definition 9 (H3WDIF-II): Given an intuitionistic fuzzy information table $IS = (U, AT, IV, f)$, an attribute subset $C_i \subseteq AT$, and $C_i = \{c_1, c_2, \cdots, c_i\}$ $(|AT| = n$ and $i = 1, 2, \cdots, n)$. Let $IM = \{E(\lambda_{\cdot k}) = (\mu_E(\lambda_{\cdot k}), \nu_E(\lambda_{\cdot k}))\}_{3\times 2}$ $(\cdot = P, B, N$ and $k = P, N)$ be an intuitionistic fuzzy cost parameter matrix. For any $x \in U$, a hierarchical three-way decision with IFNs (H3WDIF-II) can be defined as follows,

$$HIFD(II) = (HIFD_1(II), HIFD_2(II), \cdots, HIFD_i(II), \cdots, HIFD_n(II)),$$

where $HIFD_i(II)$ is a decision at the i-th step on C_i in multi-granularity spaces.

Given the membership degree $\mu_i(x)$ and the non-membership degree $\nu_i(x)$ of decision IFNs at the i-th step. Similarly, with $\mu_E(\lambda_{PP}) = 0$, $\mu_E(\lambda_{NN}) = 0$, $\nu_E(\lambda_{PP}) = 1$ and $\nu_E(\lambda_{NN}) = 1$, the classification losses $R(x, HIFD_i(II))$ in $HIFD_i(II)$ can be simplified as follows,

$$R(x, HIFD_i(II)) = \begin{cases} (\mu_i(x,R)_P, \nu_i(x,R)_P), \text{if} \mu_i(x) \geq \alpha_{IIi}(x) \\ (\mu_i(x,R)_N, \nu_i(x,R)_N), \text{if} \mu_i(x) \leq \beta_{IIi}(x) \\ (\mu_i(x,R)_B, \nu_i(x,R)_B), \text{if} \beta_{IIi}(x) < \mu_i(x) < \alpha_{IIi}(x) \end{cases}, \text{ where}$$

$$\begin{aligned} \mu_i(x,R)_P &= 1-(1-\mu_E(\lambda_{PN}))^{1-\mu_i(x)-\pi_i(x)}, \\ \nu_i(x,R)_P &= (\nu_E(\lambda_{PN}))^{1-\mu_i(x)-\pi_i(x)}, \\ \mu_i(x,R)_N &= 1-(1-\mu_E(\lambda_{NP}))^{\mu_i(x)}, \\ \nu_i(x,R)_N &= (\nu_E(\lambda_{NP}))^{\mu_i(x)}, \\ \nu_i(x,R)_B &= \nu_E(\lambda_{BP}))^{\mu_i(x)}(\nu_E(\lambda_{BN}))^{1-\mu_i(x)-\pi_i(x)}, \\ \mu_i(x,R)_B &= 1-(1-\mu_E(\lambda_{BP}))^{\mu_i(x)}(1-\mu_E(\lambda_{BN}))^{1-\mu_i(x)-\pi_i(x)}. \end{aligned}$$

Let $\alpha_{IIi}(x) = \alpha_{1i}(x) = (1-\pi_i(x))\alpha_1$ and $\beta_{IIi}(x) = \beta_{1i}(x) = (1-\pi_i(x))\beta_1$ when $R(x, HIFD_i(II))$ is discussed from the membership degree perspective. In addition, $\alpha_{IIi}(x) = \alpha_{2i}(x) = (1-\pi_i(x))\alpha_2$ and $\beta_{IIi}(x) = \beta_{2i}(x) = (1-\pi_i(x))\beta_2$ from the non-membership degree perspective.

With respect to the definition of H3WDIF-II, three-way decisions are presented by the characteristic of separate object, i.e. IFN attribute values. However, in multi-granularity spaces, distinctions of decision IFNs that are calculated by IFN attribute values appear with the change of granules. To better analyze relationships of decisions based on H3WDIF-II in two granularity levels, Theorem 6 is proposed by considering the impact of INFs in multi-granularity spaces from the membership and non-membership perspectives, respectively.

Theorem 6: Let $HIFD_j(II)$ be a successive decision of $HIFD_i(II)$ $(i < j)$, for $\mu_i(x) \geq \alpha_{IIi}(x)$,

1) if $\mu_j(x) \geq \alpha_{1j}(x)$ and $\nu_i(x) \leq \nu_j(x)$, then $\mu_i(x,R)_P \leq \mu_j(x,R)_P$,
2) if $\mu_j(x) \geq \alpha_{1j}(x)$ and $\nu_i(x) \geq \nu_j(x)$, then $\mu_i(x,R) \geq \mu_j(x,R)_P$,
3) if $\mu_j(x) \geq \alpha_{2j}(x)$ and $\nu_i(x) \leq \nu_j(x)$, then $\nu_i(x,R)_P \geq \nu_j(x,R)_P$,
4) if $\mu_j(x) \geq \alpha_{2j}(x)$ and $\nu_i(x) \geq \nu_j(x)$, then $\nu_i(x,R)_P \leq \nu_j(x,R)_P$.

Proof: From the membership degree perspective, because $\mu_i(x) \geq \alpha_{IIi}(x)$, $\mu_i(x,R)_P = 1-(1-\mu_E(\lambda_{PN}))^{1-\mu_i(x)-\pi_i(x)}$.

If $\mu_j(x) \geq \alpha_{1j}(x)$, then $\mu_j(x) \geq (1-\pi_j(x))\alpha_1$ and

$$\mu_j(x,R)_P = 1-(1-\mu_E(\lambda_{PN}))^{1-\mu_j(x)-\pi_j(x)}.$$

And because $\nu_i(x) \leq \nu_j(x)$, $\mu_i((x),R)_P \leq \mu_j(x,R)_P$ holds. Similarly, 2), 3) and 4) could be also proven from the non-membership degree perspective. ∎

Compared with H3WDIF-I, the relationships of classification losses in successive granularity levels are shown by discussing a single object instead of an equivalence class, which is resulted from the characteristic of objects in H3WDIF-II. Meanwhile, Theorem 6 also reflects the change rules of classification losses effecting by decision IFNs. Similar to H3WDIF-I, the non-monotonicity of classification losses in

TABLE 4. Intuitionistic fuzzy information for decisions based on H3WDIF-II.

U	c_1	c_2
x_1	(0.68, 0.21)	(0.8872, 0.0685)
x_2	(0.78, 0.06)	(0.9852, 0.0003)
x_3	(0.79, 0.02)	(0.9244, 0.0001)
x_4	(0.72, 0.05)	(0.9289, 0.0600)
x_5	(0.7, 0.05)	(0.7857, 0.0001)
x_6	(0.1, 0.6)	(0.9535, 0.0064)
x_7	(0.2, 0.4)	(0.1287, 0.0392)

TABLE 5. The first decision thresholds from the membership degree.

U	DA_1	$\alpha_{11}(x)$	$\beta_{11}(x)$
x_1	(0.68, 0.21)	0.59	0.28
x_2	(0.78, 0.06)	0.55	0.26
x_3	(0.79, 0.02)	0.53	0.25
x_4	(0.72, 0.05)	0.71	0.21
x_5	(0.7, 0.05)	0.50	0.23
x_6	(0.1, 0.6)	0.46	0.22
x_7	(0.2, 0.4)	0.40	0.19

TABLE 6. The second decision thresholds from the membership degree.

U	DA_2	$\alpha_{12}(x)$	$\beta_{12}(x)$
x_1	(0.62, 0.29)	0.60	0.28
x_2	(0.81, 0.04)	0.56	0.26
x_3	(0.7, 0.01)	0.47	0.22
x_4	(0. 68, 0.18)	0.57	0.27
x_5	(0.55, 0.03)	0.38	0.18
x_6	(0.6, 0.2)	0.53	0.25
x_7	(0.1, 0.3)	0.26	0.12

the positive region still exists from the membership and non-membership degree perspectives. Therefore, the change rules of classification losses could be only discussed locally by Theorem 6, and Example 2 is given to illustrate the rules.

Example 2: Given an intuitionistic fuzzy information in Table 4 including 2 attributes and 7 objects. The cost parameter matrix is shown in Table 3. Then the first decision thresholds for 3WDIF-II from the membership degree perspective and decision IFNs DA_1 are shown in Table 5 where $DA_1 = c_1$.

Let δ_{c_1} be 0.29 and δ_{c_2} be 0.288, then the second decision IFNs DA_2 are calculated by Definition 8 in Table 6.

By Tables 5 and 6, at the first and the second steps, x_1 and x_4 are always in the positive region based on decision rules of 3WDIF-II, and x_1 is given for verifying 1) of Theorem 6 as an example.

$$\mu_1(x_1) = 0.68, \alpha_{11}(x_1) = (1 - \pi_1(x_1))\alpha_1 = 0.59,$$
$$\mu_2(x_1) = 0.62, \alpha_{12}(x_1) = (1 - \pi_2(x_1))\alpha_1 = 0.6,$$

then $\mu_1(x_1) \geq \alpha_{11}(x_1)$ and $\mu_2(x_1) \geq \alpha_{12}(x_1)$. In addition, with $\nu_1(x_1) = 0.21$ and $\nu_2(x_1) = 0.29$, then $\nu_2(x_1) \geq \nu_1(x_1)$.

$$\begin{aligned}\mu_1(x_1, R)_P &= 1 - (1 - \mu_E(\lambda_{PN}))^{1-\mu_1(x_1)-\pi_1(x_1)} \\ &= 1 - 0.2^{0.21} = 0.29, \\ \mu_2(x_1, R)_P &= 1 - (1 - \mu_E(\lambda_{PN}))^{1-\mu_2(x_1)-\pi_2(x_1)} \\ &= 1 - 0.2^{0.29} = 0.37,\end{aligned}$$

$\mu_1(x_1, R)_P \leq \mu_2(x_1, R)_P$ is verified.

x_2 and x_3 are classified into the positive region at the first and the second steps. And their non-membership degrees of DA_2 are smaller than that of DA_1. Therefore, 2) of Theorem 6 is shown by x_2 as follows,

$$\mu_1(x_2) = 0.78, \alpha_{11}(x_2) = (1 - \pi_1(x_2))\alpha_1 = 0.55,$$
$$\mu_2(x_2) = 0.81, \alpha_{12}(x_2) = (1 - \pi_2(x_2))\alpha_1 = 0.56,$$

then $\mu_1(x_2) \geq \alpha_{11}(x_2)$ and $\mu_2(x_2) \geq \alpha_{12}(x_2)$. Moreover, $\nu_1(x_2) = 0.06$ and $\nu_2(x_2) = 0.04$, then $\nu_2(x_2) \leq \nu_1(x_2)$

$$\begin{aligned}\mu_1(x_2, R)_P &= 1 - (1 - \mu_E(\lambda_{PN}))^{1-\mu_1(x_2)-\pi_1(x_2)} \\ &= 1 - 0.2^{0.06} = 0.09, \\ \mu_2(x_2, R)_P &= 1 - (1 - \mu_E(\lambda_{PN}))^{1-\mu_2(x_2)-\pi_2(x_2)} \\ &= 1 - 0.2^{0.04} = 0.06,\end{aligned}$$

$\mu_1(x_2, R)_P \geq \mu_2(x_2, R)_P$.

From the non-membership degree perspective, x_4, x_5 and x_2 are given to verify 3) and 4) of Theorem 6, respectively. With respect to Example 1, $\alpha_2 = 0.75$ and $\beta_2 = 0.34$. Tables 7 and 8 show both the first and the second decision thresholds from the non-membership degree perspective.

TABLE 7. The first decision thresholds from the non-membership degree.

U	DA_1	$\alpha_{21}(x)$	$\beta_{21}(x)$
x_1	(0.68, 0.21)	0.67	0.30
x_2	(0.78, 0.06)	0.63	0.29
x_3	(0.79, 0.02)	0.61	0.27
x_4	(0.72, 0.05)	0.65	0.29
x_5	(0.7, 0.05)	0.56	0.26
x_6	(0.1, 0.6)	0.53	0.24
x_7	(0.2, 0.4)	0.45	0.20

TABLE 8. The second decision thresholds from the non-membership degree.

U	DA_2	$\alpha_{22}(x)$	$\beta_{22}(x)$
x_1	(0.62, 0.29)	0.68	0.31
x_2	(0.81, 0.04)	0.64	0.29
x_3	(0.7, 0.01)	0.53	0.24
x_4	(0.68, 0.18)	0.58	0.26
x_5	(0.55, 0.03)	0.44	0.20
x_6	(0.6, 0.2)	0.6	0.27
x_7	(0.1, 0.3)	0.3	0.14

$$\mu_1(x_4) = 0.72, \alpha_{21}(x_4) = (1 - \pi_1(x_4))\alpha_2 = 0.58,$$
$$\mu_2(x_4) = 0.68, \alpha_{22}(x_4) = (1 - \pi_2(x_4))\alpha_2 = 0.65,$$

then $\mu_1(x_4) \geq \alpha_{21}(x_4)$ and $\mu_2(x_4) \geq \alpha_{22}(x_4)$. Furthermore $\nu_2(x_4) \leq \nu_1(x_4)$, then $\nu_1(x_4, R)_P \geq \nu_2(x_4, R)_P$, with

$$\nu_1(x_4, R)_P = (\nu_E(\lambda_{PN}))^{1-\mu_1(x_4)-\pi_1(x_4)} = 0.1^{0.05} = 0.89,$$
$$\nu_2(x_4, R)_P = (\nu_E(\lambda_{PN}))^{1-\mu_2(x_4)-\pi_2(x_4)} = 0.1^{0.18} = 0.66,$$

Thus, 3) of Theorem 6 has been illustrated by x_4 in detail, and 4) of Theorem 6 is also explained by x_2 as follows.

With $\mu_1(x_2) \geq \alpha_{21}(x_2)$, $\mu_2(x_2) \geq \alpha_{22}(x_2)$ and $\nu_2(x_2) \leq \nu_1(x_2)$, then $\nu_1(x_2, R)_P \leq \nu_2(x_2, R)_P$ where

$$\nu_1(x_2, R)_P = (\nu_E(\lambda_{PN}))^{1-\mu_1(x_2)-\pi_1(x_2)} = 0.1^{0.06} = 0.87,$$
$$\nu_2(x_2, R)_P = (\nu_E(\lambda_{PN}))^{1-\mu_2(x_2)-\pi_2(x_2)} = 0.1^{0.04} = 0.91.$$

Definition 10 (Consistent Intuitionistic Fuzzy Table): Given an intuitionistic fuzzy information table $IS = (U, AT, IV, f)$, a subset $X \subseteq U$ and an object $x \in U$. Let $POS(X)$ and $NEG(X)$ be the positive and negative regions

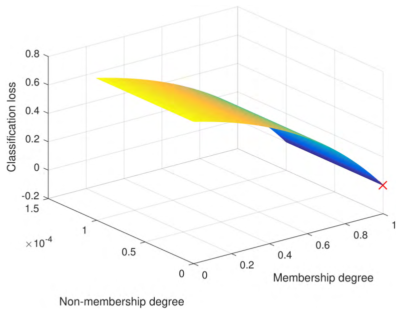

FIGURE 2. Trends of $\mu_i(x, R)_P$ for objects in the positive region.

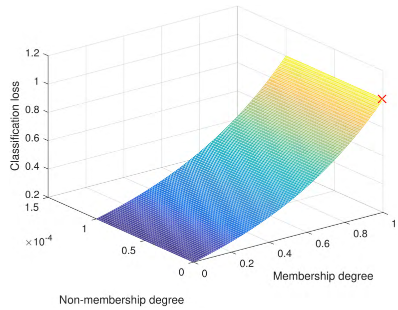

FIGURE 3. Trends of $\nu_i(x, R)_P$ for objects in the positive region.

of there-way decisions on attribute AT, if and only if $U = \{x|x \in (POS(X) \cup NEG(X))\}$, IS is called a consistent intuitionistic fuzzy table.

Theorem 7: Given a consistent intuitionistic fuzzy table $IS = (U, AT, IV, f)$ and let $POS_i(X)$ and $NEG_i(X)$ be the positive and negative regions of $HIFD_i(II)$,

1) $\forall x \in POS_i(X)$, if $\mu_i(x) \to 1$, then $\mu_i(x, R)_P \to 0$ and $\nu_i(x, R)_P \to 1$,
2) $\forall x \in NEG_i(X)$, if $\mu_i(x) \to 0$, then $\mu_i(x, R)_N \to 0$ and $\nu_i(x, R)_N \to 1$.

Proof: Suppose IS is a consistent intuitionistic fuzzy table and $U = \{X, \neg X\}$. For $x \in U$, $x \in X$ or $x \in \neg X$. If $x \in POS_i(X)$ and $\mu_i(x) \to 1$, then $\mu_i(x) \geq \alpha_{IIi}(x)$ $(0 \leq \alpha_{IIi}(x) \leq 1)$. For any x,

$$\lim_{\mu_i(x)\to 1} \nu_i(x, R)_P = \lim_{\mu_i(x)\to 1} (\nu_E(\lambda_{PN}))^{1-\mu_i(x)-\pi_i(x)} = 1,$$
$$\lim_{\mu_i(x)\to 1} \mu_i(x, R)_P = 0.$$

If $x \in NEG_i(X)$ and $\mu_i(x) \to 0$, $\mu_i(x) \leq \beta_{IIi}(x)$ $(0 \leq \beta_{IIi}(x) \leq 1)$. For any x,

$$\lim_{\mu_i(x)\to 0} \nu_i(x, R)_N = \lim_{\mu_i(x)\to 0} (\nu_E(\lambda_{NP}))^{\mu_i(x)} = 1,$$
$$\lim_{\mu_i(x)\to 0} \mu_i(x, R)_N = 0.$$

■

Theorem 7 has globally proven the trends of classification losses in the multi-granularity space. According to Theorem 7, when $\mu_i(x)$ closes to 1, the classification losses $\mu_i(x, R)_P$ and $\nu_i(x, R)_P$ of objects in the positive region tend to 0 and 1, respectively. Actually, $\mu_i(x, R)_P = 0$ and $\nu_i(x, R)_P = 1$ with a prerequisite of $\mu_i(x) = 1$. Correspondingly, classification losses of objects in the negative region satisfy the similar change rules when $\mu_i(x)$ closes to 0.

Figs.2 and 3 describe the trends of classification losses from the membership and non-membership degree perspectives, respectively. Observably, the lowest point of Fig.2 is $\mu_i(x) = 1$ and the highest point of Fig.3 is $\mu_i(x) = 1$ in the positive region. Similarly, the minimum value of classification losses is obtained when $\mu_i(x) = 0$ for the negative region by Fig.4, and the maximum value exists with $\mu(x)_i = 0$ by Fig.5. Thus, Theorem 7 is also verified by the figures of function $\mu_i(x, R)_P$, $\mu_i(x, R)_N$, $\nu_i(x, R)_P$ and $\nu_i(x, R)_N$.

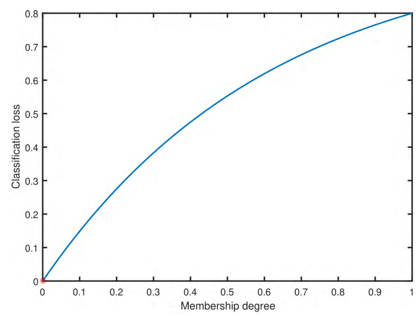

FIGURE 4. Trends of $\mu_i(x, R)_N$ for objects in the negative region.

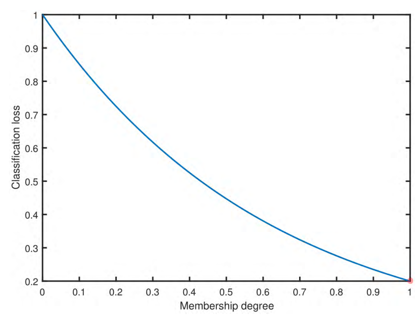

FIGURE 5. Trends of $\nu_i(x, R)_N$ for objects in the negative region.

Theorem 8: Given an intuitionistic fuzzy information table $IS = (U, AT, IV, f)$, a positive region $POS_i(X)$ and a negative region $NEG_i(X)$ in $HIFD_i(II)$, correspondingly, $POS_j(X)$ and $NEG_j(X)$ in $HIFD_j(II)$.

1) For $x \in POS_i(X)$, if $\mu_j(x) \geq \mu_i(x)$ and $\nu_j(x) \leq \nu_i(x)$, then $x \in POS_j(X)$,
2) For $x \in NEG_i(X)$, if $\mu_j(x) \leq \mu_i(x)$ and $\nu_j(x) \geq \nu_i(x)$, then $x \in NEG_j(X)$.

Proof: For any $x \in U$ and $x \in POS_i(X)$, the threshold of the positive region satisfies with $\mu_i(x) \geq (1 - \pi_i(x))\alpha_1$ from the membership degree perspective, thus $\alpha_1 \leq \frac{1}{1+(\nu_i(x)/\mu_i(x))}$. Because $\mu_j(x) \geq \mu_i(x)$ and $\nu_j(x) \leq \nu_i(x)$,

$$\alpha_1 \leq \frac{1}{1 + (\nu_i(x)\big/\mu_i(x))} \leq \frac{1}{1 + (\nu_j(x)\big/\mu_j(x))}.$$

Naturally, $\mu_j(x) \geq (1 - \pi_j(x))\alpha_1$, and $x \in POS_j(X)$ holds. Similarly, for $x \in NEG_i(X)$, $x \in NEG_j(X)$ holds from other perspectives. ■

TABLE 9. A new intuitionistic fuzzy information table.

U	c_1	c_2	c_3
x_1	(0.22, 0.53)	(0.2, 0.53)	(0.17, 0.4)
x_2	(0.23, 0.6)	(0.23, 0.63)	(0.1, 0.5)
x_3	(0.7, 0.2)	(0.8, 0.15)	(0.8, 0.03)
x_4	(0.68, 0.3)	(0.73, 0.26)	(0.75, 0.01)
x_5	(0.6, 0.3)	(0.62, 0.35)	(0.8, 0.2)
x_6	(0.2, 0.6)	(0.18, 0.7)	(0.17, 0.08)
x_7	(0.5, 0.4)	(0.1, 0.7)	(0.28, 0.18)

TABLE 10. The first decision thresholds from the membership degree.

U	DA_1	$\alpha_{11}(x)$	$\beta_{11}(x)$
x_1	(0.22, 0.53)	0.50	0.23
x_2	(0.23, 0.6)	0.55	0.25
x_3	(0.7, 0.2)	0.59	0.28
x_4	(0.68, 0.3)	0.65	0.3
x_5	(0.6, 0.3)	0.60	0.28
x_6	(0.2, 0.6)	0.53	0.24
x_7	(0.5, 0.4)	0.60	0.28

TABLE 11. The second decision thresholds from the membership degree.

U	DA_2	$\alpha_{12}(x)$	$\beta_{12}(x)$
x_1	(0.21, 0.51)	0.48	0.22
x_2	(0.23, 0.61)	0.56	0.26
x_3	(0.75, 0.17)	0.61	0.29
x_4	(0.71, 0.28)	0.65	0.31
x_5	(0.61, 0.32)	0.62	0.29
x_6	(0.19, 0.65)	0.55	0.26
x_7	(0.34, 0.52)	0.57	0.27

Example 3: Given an intuitionistic fuzzy information in Table 9 including 7 objects and 3 attributes. Thus, Theorem 8 is discussed from the membership degree perspective as an example. For convenience, the example from the non-membership degree is not shown in this paper.

Tables 10 and 11 show the thresholds of the first and the second decisions where the first decision IFN $DA_1 = c_1$. Furthermore, the second and the third decision IFNs DA_2 and DA_3 are calculated by Definition 8, respectively. From the membership degree perspective, objects x_3, x_4 and x_5 are in the positive region at the first step because $\mu_1(x_3) \geq 0.59$, $\mu_1(x_4) \geq 0.65$ and $\mu_1(x_5) \geq 0.60$ (Table 10).

In addition, according to Tables 11 and 12, the $\mu_1(x_3) \leq \mu_2(x_3)$ and $\nu_2(x_3) \leq \nu_1(x_3)$ which satisfy the condition of 1) of Theorem 8. Observably, x_3 is in the positive region at the second step. At the third step, x_3 is also in the positive region, and satisfies with $\mu_2(x_3) \leq \mu_3(x_3)$ and $\nu_3(x_3) \leq \nu_2(x_3)$. Therefore, 1) of Theorem 8 has been verified with 3 successive decisions of x_3. Object x_4 can reflect 1) of Theorem 8, however, x_5 in the positive region at the first step is not in the same region who satisfies with $\mu_1(x_5) \leq \mu_2(x_5)$ but $\nu_2(x_5) \geq \nu_1(x_5)$ at the second step.

Furthermore, to verify 2) of Theorem 8,

1) at the first step, x_2 and x_6 are in the negative region with $\mu_1(x_2) \leq 0.25$ and $\mu_1(x_6) \leq 0.24$,
2) at the second step, x_2 and x_6 who respectively satisfying with $\mu_1(x_2) \geq \mu_2(x_2)$, $\nu_2(x_2) \geq \nu_1(x_2)$, $\mu_1(x_6) \geq \mu_2(x_6)$ and $\nu_2(x_6) \geq \nu_1(x_6)$ are still classified into the negative region (Fig.11),
3) at the third step, x_1 and x_2 are in the negative region with $\mu_3(x_1) \leq 0.23$ and $\mu_3(x_2) \leq 0.25$, however, x_6 is not in the negative region because $\mu_2(x_6) \leq \mu_3(x_2)$ but $\nu_3(x_1) \geq \nu_2(x_1)$.

TABLE 12. The third decision thresholds from the membership degree.

U	DA_3	$\alpha_{13}(x)$	$\beta_{13}(x)$
x_1	(0.20, 0.54)	0.49	0.23
x_2	(0.19, 0.63)	0.54	0.25
x_3	(0.77, 0.16)	0.61	0.29
x_4	(0. 72, 0.14)	0.57	0.27
x_5	(0.68, 0.34)	0.68	0.32
x_6	(0.18, 0.36)	0.36	0.17
x_7	(0.32, 0.38)	0.46	0.22

Theorem 8 discusses the relationships between decision regions and decision IFNs. If decision IFNs satisfy the prerequisites of Theorem 8, the decision regions are confirmed directly from the membership and non-membership degree perspectives.

Generally, the relationships of decisions based on H3WDIF-II have been discussed by considering the impact of IFN cost parameters and IFN attribute values in this section. With respect to Theorems 6, 7 and 8, H3WDIF-II can be refined in a certain for dealing with uncertain problem under dynamic intuitionistic fuzzy environments, which results in a simplifier decision process and a more efficient work in practical.

V. EXPERIMENT ANALYSIS

In this section, several simulation experiments have been designed to verify the validity of theorems based on H3WDIF-I and H3WDIF-II, respectively. These experiments are under the environments of 4GB RAM, 2.4GHz CPU and windows 10 system. Besides, the program language is MATLAB. Subsection A shows experimental results based on H3WDIF-I, and Subsection B exhibits the results based on H3WDIF-II.

A. EXPERIMENTS OF H3WDIF-I

To illustrate Theorems 1, 3 and 5 (Theorems 2 and 4 can be verified similarly.), experiments with an UCI dataset (Bank including 4521 objects, 14 condition attributes and 1 decision attribute) are given from the membership degree perspective. As follows, the procedures of H3WDIF-I are presented.

1) Input the original dataset and intuitionistic fuzzy cost parameter matrix.
2) Select the attribute subset U_i ($i = 1, 2, \cdots, |AT|$) as the current granularity level.
3) Select the attribute subset U_{i+1}, and make three-way decisions with IFN cost parameters from the membership degree perspective.
4) Repeat 2) and 3) until $i = |AT|$.
5) Search for x in $POS_i(X)$, and obtain $P_i(X|[x])$ and $\mu_i([x], R)_P$ where i from 1 to $|AT|$ at the i-th step.

Tables 13 and 14 show the conditional probabilities of classification and classification losses at steps 4-9 of H3WDIF-I

TABLE 13. Change rules of classification losses with information table from the 4-th step to 6-th step.

	Step 4		Step 5		Step 6	
	$P_i(X\|[x])$	$\mu_i([x],R)_P$	$P_i(X\|[x])$	$\mu_i([x],R)_P$	$P_i(X\|[x])$	$\mu_i([x],R)_P$
$[x_1]$	0.8251	0.2453	0.8125	0.2605	0.7959	0.28
$[x_2]$	0.8420	0.2246	0.8762	0.1807	0.8182	0.2537
$[x_3]$	0.8420	0.2246	0.8227	0.2483	0.7685	0.311
$[x_4]$	0.8420	0.2246	0.8378	0.2297	0.8730	0.1848
$[x_5]$	0.8208	0.2505	0.8203	0.2511	0.7857	0.2917

TABLE 14. Change rules of classification losses with information table from the 4-th step to 6-th step.

	Step 7		Step 8		Step 9	
	$P_i(X\|[x])$	$\mu_i([x],R)_P$	$P_i(X\|[x])$	$\mu_i([x],R)_P$	$P_i(X\|[x])$	$\mu_i([x],R)_P$
$[x_1]$	0.7692	0.3102	0.6786	0.4039	-	-
$[x_2]$	0.8261	0.2441	0.7241	0.3585	0.6319	0.4473
$[x_3]$	0.7717	0.3074	0.6780	0.4045	0.625	0.4531
$[x_4]$	0.8868	0.1666	0.8667	0.1931	0.8	0.2752
$[x_5]$	0.7833	0.2944	0.6905	0.3924	0.6897	0.3932

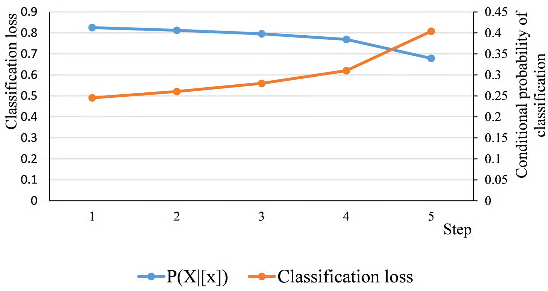

FIGURE 6. Classification losses of x_1.

in a multi-granularity space. Let $[x_1]$-$[x_5]$ represent the corresponding equivalence classes, thus the experimental results of 5 equivalence classes are provided in Tables 13 and 14. As an example, 0.8251 and 0.2453 in the first line of Table 13 denote the conditional probability of classification and classification loss of $[x_1]$ at the 4-th step, respectively (The meanings of Tables 15 and 16 are similar to Tables 14 and 15).

Tables 13 and 14 reveal that classification losses of equivalence classes decrease with increasing of conditional probability and increase with its decreasing in the positive regions conversely in a multi-granularity space as $[x_3]$. In addition, the equivalence classes $[x_1]$, $[x_2]$, $[x_4]$ and $[x_5]$ in the positive regions also intuitively reflect these two change rules (Figs.6-9). These change rules of classification losses match to Theorem 1. Therefore, Theorem 1 is verified by an experiment with UCI dataset.

For effective validations, a consistent dataset has been transformed from Bank. Tables 15 and 16 show that classification losses of all equivalence classes equate to 0 eventually except for $[x_7]$ and $[x_{10}]$. The reason of the difference is that there are 14 attributes (including 13 conditional attributes and 1 decision attribute) in Bank. However, the results of 6 steps (steps 2-7) are only exhibited in Tables 15 and 16, which do not include decisions in the optimal granularity level for $[x_7]$ and $[x_{10}]$. Generally, classification losses will globally equate to 0 at one step. Accordingly, the validation of Theorem 3 is analyzed with a consistent dataset.

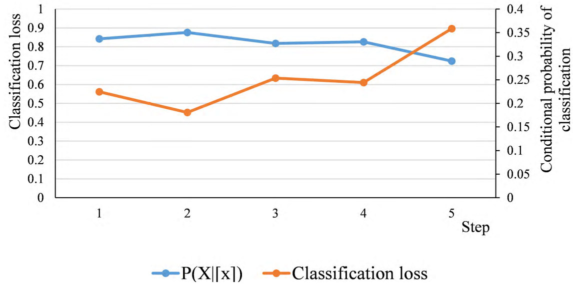

FIGURE 7. Classification losses of x_2.

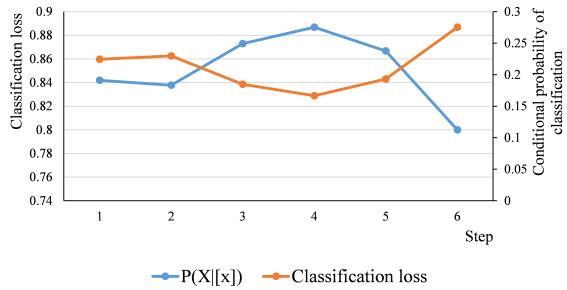

FIGURE 8. Classification losses of x_4.

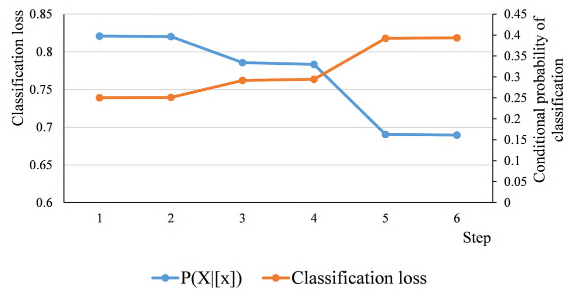

FIGURE 9. Classification losses of x_5.

TABLE 15. Change rules of classification losses with consistent decision table from the 2-th step to 4-th step.

	Step 2		Step 3		Step 4	
	$P(X_i\|[x])$	$\mu_i([x],R)_P$	$P_i(X\|[x])$	$\mu_i([x],R)_P$	$P_i(X\|[x])$	$\mu_i([x],R)_P$
$[x_6]$	0.9545	0.0705	0.9167	0.1255	0.875	0.1882
$[x_7]$	0.9545	0.0705	0.9167	0.1255	0.875	0.1882
$[x_8]$	0.8916	0.1601	0.8085	0.2653	0.7391	0.3429
$[x_9]$	0.95	0.0773	0.9375	0.0957	1	0
$[x_{10}]$	0.8632	0.1926	0.75	0.3313	0.6944	0.3885

TABLE 16. Change rules of classification losses with consistent decision table from the 5-th step to 7-th step.

	Step 5		Step 6		Step 7	
	$P_i(X\|[x])$	$\mu_i([x],R)_P$	$P_i(X\|[x])$	$\mu_i([x],R)_P$	$P_i(X\|[x])$	$\mu_i([x],R)_P$
$[x_6]$	0.8333	0.2353	0.6667	0.4152	1	0
$[x_7]$	0.8333	0.2353	0.6667	0.4152	0	0.8
$[x_8]$	0.625	0.4531	0.6923	0.3906	1	0
$[x_9]$	1	0	1	0	1	0
$[x_{10}]$	0.6522	0.4287	0.7333	0.349	0	0.8

Theorem 5 discusses a trend of boundary region with consistent information, and the number of object in the boundary region is shown in Table 17. Evidently, it illustrates that

TABLE 17. Number of objects in the boundary region based on 5 consistent decision tables.

Dataset	Size	Conditional attribute	Number of objects $BND(X)$ at the last step
Adult	30725	11	0
Bank	4521	14	0
Audit	773	17	0
Breast tissue	106	9	0
Banknote authentication	1372	4	0

TABLE 18. Change rules of classification losses with intuitionistic fuzzy information table from the 1-th step to 3-th step.

	Step 1 $\mu_1(x)$ $\nu_1(x)$ $\mu_1(x,R)_P$	Step 2 $\mu_2(x)$ $\nu_2(x)$ $\mu_2(x,R)_P$	Step 3 $\mu_3(x)$ $\nu_3(x)$ $\mu_3(x,R)_P$
x_1	0.9727 0.0215 0.0048	0.7305 0.0260 0.0058	0.6895 0.1888 0.0413
x_2	0.9635 0.0287 0.0064	0.7239 0.0348 0.0077	0.6833 0.1961 0.0428
x_3	0.973 0.0213 0.0047	0.7307 0.0258 0.0057	0.6897 0.1886 0.0412
x_4	0.9748 0.0199 0.0044	0.732 0.024 0.0053	0.6909 0.1871 0.0409
x_5	0.9848 0.012 0.0027	0.7391 0.0145 0.0035	0.6977 0.1792 0.0392

TABLE 19. Change rules of classification losses with intuitionistic fuzzy information table from the 4-th step to 6-th step.

	Step 4 $\mu_4(x)$ $\nu_4(x)$ $\mu_4(x,R)_P$	Step 5 $\mu_5(x)$ $\nu_5(x)$ $\mu_5(x,R)_P$	Step 6 $\mu_6(x)$ $\nu_6(x)$ $\mu_6(x,R)_P$
x_1	0.5962 0.2050 0.0447	0.5930 0.2093 0.0456	0.6460 0.3200 0.0689
x_2	0.5909 0.2122 0.0462	0.5877 0.2164 0.0471	0.6402 0.3261 0.0702
x_3	0.5964 0.2049 0.0447	0.5931 0.2091 0.0456	0.6462 0.3198 0.0689
x_4	0.5974 0.2034 0.0444	0.5942 0.2077 0.0453	0.6473 0.3186 0.0686
x_5	0.6033 0.1957 0.0427	0.6000 0.2000 0.0436	0.6536 0.3120 0.0672

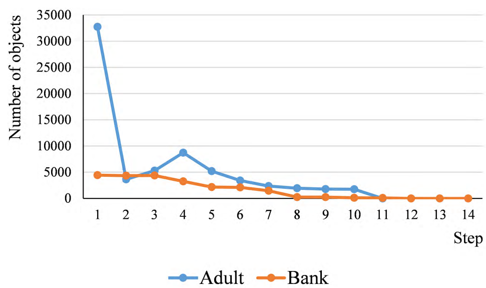

FIGURE 10. Number of objects in the boundary region.

boundary region exists no objects at the last step. Furthermore, the changes of object numbers in the boundary region are also shown based on H3WDIF-I in multi-granularity spaces (Figs.10 and 11). Figs.10 and 11 exhibit the trends of 5 datasets that boundary regions become empty sets in the process of decisions with finer granules, and these results satisfy with Theorem 5.

B. EXPERIMENTS OF H3WDIF-II

In order to verify Theorem 6, two experiments are given with 100 objects and 6 attributes from the membership degree perspective based on H3WDIF-II. As follows, the procedures of experiment are shown.

1) Input the original data, important degrees of attributes and cost parameter matrix.
2) Select the attribute subset U_i ($i = 1, 2, \cdots, |AT|$) as the current granularity level.

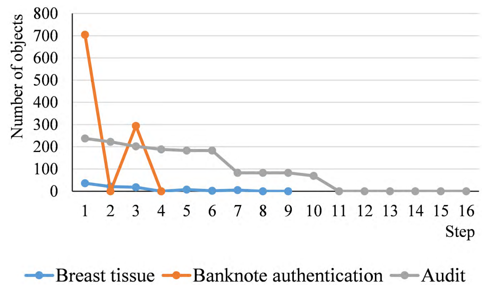

FIGURE 11. Number of objects in the boundary region.

3) Select the attribute subset U_{i+1}, and calculate decision IFNs $DA_{i+1}(x)$ for each object x.
4) Execute 3WDIF-II from the membership degree perspective for each x.
5) Repeat 2) to 4) until $i = |AT|$.
6) Search for x in $POS_i(X)$, and obtain $\mu_i(x)$, $\nu_i(x)$ and $\mu_i(x, R)_P$ where i from 1 to $|AT|$.

Both Tables 18 and 19 show classification losses and decision IFNs of 5 objects in different granularity levels. For example, the 0.5962, 0.2050 and 0.0447 are the membership degree, non-membership degree and classification loss of x_1, respectively. For specific details, x_1 is classified into the positive region at the first and the second steps. Meanwhile, classification loss of x_1 is increasing with the increment of the membership degrees. Therefore, x_1-x_5 are in accord with 1) of Theorem 6.

VI. CONCLUSIONS

Currently, with the rapid development of age, information results in a lot of uncertain problems. Intuitionistic fuzzy set is described by the membership and non-membership degrees, and it is an important mathematic tool for dealing with uncertainty problems. Three-way decision theory divides a universe into three regions involving three actions, which is in accordance with human cognitive habits for complex problem solving. To excavate more effective information under dynamic intuitionistic fuzzy environments, a H3WDIF-I model is established in multi-granularity spaces with IFN cost parameters and fixed attribute values. To analyze the relationships of decisions in two successive granularity levels, the change rules are discussed from the viewpoint of classification losses based on H3WDIF-I. In addition, a H3WDIF-II is also presented to solve problems with both IFN cost parameters and IFN attribute values. To better obtain inner relationships, a H3WDIF-II model is also discussed in multi-granularity spaces. According to change rules of classification losses, there is non-monotonicity between classification losses and the conditional probability of classification based on two proposed models, respectively. Globally, the trends of classification losses and the division relationships of three regions are proven in a multi-granularity space. In the future, a new three-way decision model may be established with hierarchical structures to solve the multiple hierarchy problems in the intuitionistic fuzzy information systems.

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