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Distributional Properties of Killed Exponential Functionals and Short-time Behavior of Lévy-driven Stochastic Differential Equations

Dissertation

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For the mind does not require filling like a bottle, but rather, like wood, it only requires kindling to create in it an impulse to think independently and an ardent desire for the truth.

Plutarch

Abstract

This thesis covers different properties of solutions to Lévy-driven stochastic differential equations and can be divided into two main parts.

In **Chapter 2**, we consider various distributional properties of killed exponential functionals of Lévy processes. For two independent Lévy processes ξ and η and an exponentially distributed random variable τ with parameter $q > 0$, independent of ξ and η, the killed exponential functional is given by $V_{q,\xi,\eta} := \int_0^\tau \mathrm{e}^{-\xi_{s-}} \,\mathrm{d}\eta_s$. Interpreting the case $q = 0$ as $\tau = \infty$ almost surely, the random variable $V_{q,\xi,\eta}$ is a natural generalization of the exponential functional $\int_0^\infty \mathrm{e}^{-\xi_{s-}} \,\mathrm{d}\eta_s$, the law of which describes the stationary distribution of a generalized Ornstein-Uhlenbeck process. Similar to the case without killing, there are two ways to view the distribution of $V_{q,\xi,\eta}$, leading to two main approaches to studying its properties. In the first part of the chapter, we consider the random variable $V_{q,\xi,\eta}$ as a stopped stochastic integral and, using tools from probability theory and infinitely divisible distributions, the support and continuity of the law of the killed exponential functional are characterized, and many sufficient conditions for absolute continuity are given. As an intermediate step, sufficient conditions for absolute continuity of $\int_0^t \mathrm{e}^{-\xi_{s-}} \mathrm{d}\eta_s$ for fixed $t \geq 0$ as well as for integrals of the form $\int_0^\infty f(s) \,\mathrm{d}\eta_s$ for deterministic functions f are obtained. Applying the same techniques to the case $q = 0$ further yields results on the absolute continuity of the improper integral $\int_0^\infty \mathrm{e}^{-\xi_{s-}} \,\mathrm{d}\eta_s$.

As in the case without killing, it can be shown that the law of the killed exponential functional arises as the stationary distribution of a solution to a stochastic differential equation. Since the solution is closely related to the generalized Ornstein-Uhlenbeck process and, in particular, a Markov process, tools from functional analysis become applicable to study the distribution of $V_{q,\xi,\eta}$. This is the content of the second part of Chapter 2. Here, the infinitesimal generator of the process is calculated and used to derive different distributional equations describing the law of $V_{q,\xi,\eta}$, as well as functional equations for its Lebesgue density in the absolutely continuous case. We then consider different special cases and examples to obtain more explicit information on the law of the killed exponential functional and to illustrate some applications of the equations. As in the first part of the chapter, considering $q = 0$ allows to extend the results to the classical exponential functional $\int_0^\infty \mathrm{e}^{-\xi_{s-}} \,\mathrm{d}\eta_s$.

In **Chapter 3**, we consider solutions to general Lévy-driven stochastic differential equations of the form $\mathrm{d}X_t = \sigma(X_{t-})\mathrm{d}L_t$, $X_0 = x$ where the function σ is twice continuously differentiable and maximal of linear growth, and the driving Lévy process $L = (L_t)_{t\geq 0}$ is either vector or matrix-valued. While the almost sure short-time behavior of Lévy processes is well-known and can be characterized in terms of the generating triplet, there is no complete characterization of the behavior of the process X. Using methods from stochastic calculus, we derive limiting results for stochastic integrals of the from $t^{-p} \int_{0+}^t \sigma(X_{t-})\mathrm{d}L_t$ to show that the behavior of the quantity $t^{-p}(X_t - X_0)$ for $t \downarrow 0$ almost surely mirrors

the behavior of $t^{-p}L_t$. Generalizing t^p to a suitable function $f : [0,\infty) \to \mathbb{R}$ then yields a tool to derive explicit law of the iterated logarithm-type results for the solution from the behavior of the driving Lévy process and also allows to give statements for convergence in probability or in distribution.

Zusammenfassung

Die vorliegende Arbeit thematisiert verschiedene Eigenschaften der Lösungen von Lévy-getriebenen stochatstischen Differentialgleichungen und ist inhaltlich in zwei Teile untergliedert.

In **Kapitel 2** werden Verteilungseigenschaften von gestoppten exponentiellen Funktionalen (”killed exponential functionals”) charakterisiert. Diese sind für zwei unabhängige Lévy-Prozesse ξ und η sowie eine von beiden Prozessen unabhängige, exponentialverteilte Zufallsvariable τ mit Parameter $q > 0$ durch das Integral $V_{q,\xi,\eta} := \int_0^\tau \mathrm{e}^{-\xi_{s-}}\,\mathrm{d}\eta_s$ gegeben. Wird der Fall $q = 0$ als $\tau = \infty$ fast sicher interpretiert, ergibt sich das uneigentliche Integral $\int_0^\infty \mathrm{e}^{-\xi_{s-}}\,\mathrm{d}\eta_s$, dessen Verteilung die stationäre Verteilung eines verallgemeinerten Ornstein-Uhlenbeck-Prozesses beschreibt. Ähnlich wie im Fall $q = 0$ gibt es zwei Möglichkeiten, die Verteilung von $V_{q,\xi,\eta}$ zu betrachten, was zu zwei verschiedenen Ansätzen zur Untersuchung ihrer Eigenschaften führt. Im ersten Teil des Kapitels wird die Zufallsvariable $V_{q,\xi,\eta}$ als gestopptes stochastisches Integral betrachtet. Unter Verwendung von wahrscheinlichkeitstheoretischen Hilfsmitteln ergibt sich so eine vollständige Charakterisierung des Trägers und der Stetigkeit der Verteilung sowie verschiedene hinreichende Bedingungen für Absolutstetigkeit. Als Zwischenschritt werden zudem hinreichende Bedingungen für die Absolutstetigkeit von $\int_0^t \mathrm{e}^{-\xi_{s-}}\mathrm{d}\eta_s$ für feste $t \geq 0$ sowie für Integrale der Form $\int_0^\infty f(s)\,\mathrm{d}\eta_s$ für deterministische Funktionen f hergeleitet. Die Anwendung der Methoden auf den Fall $q = 0$ liefert außerdem neue Ergebnisse für die Absolutstetigkeit des uneigentlichen Integrals $\int_0^\infty \mathrm{e}^{-\xi_{s-}}\,\mathrm{d}\eta_s$.

Ähnlich wie für das uneigentliche Integral $\int_0^\infty \mathrm{e}^{-\xi_{s-}}\,\mathrm{d}\eta_s$ kann gezeigt werden, dass die Verteilung von $V_{q,\xi,\eta}$ als stationäre Verteilung der Lösung einer geeigneten stochastischen Differentialgleichung auftritt. Da diese Lösung eng mit dem verallgemeinerten Ornstein-Uhlenbeck-Prozess verwandt und insbesondere ein Markov-Prozess ist, können Werkzeuge aus der Funktionalanalysis verwendet werden, um die Verteilung von $V_{q,\xi,\eta}$ weiter zu untersuchen. Dies ist der Inhalt des zweiten Teils von Kapitel 2. Die Berechnung des infinitesimalen Generators des Prozesses stellt hier ein wichtiges Hilfsmittel dar, um verschiedene Gleichungen, die die Verteilung von $V_{q,\xi,\eta}$ direkt oder ihre Lebesgue-Dichte im absolutstetigen Fall beschreiben, herzuleiten. Durch die anschließende Betrachtung verschiedener Spezialfälle und Beispiele wird im Anschluss die Anwendung der Gleichungen illustriert. Wie im ersten Teil des Kapitels erlaubt die Wahl $q = 0$ das uneigentliche Integral $\int_0^\infty \mathrm{e}^{-\xi_{s-}}\,\mathrm{d}\eta_s$ in die Formulierung der Ergebnisse mit einzubeziehen.

In **Kapitel 3** werden schließlich Lösungen von allgemeinen Lévy-getriebenen stochastischen Differentialgleichungen der Form $\mathrm{d}X_t = \sigma(X_{t-})\mathrm{d}L_t$, $X_0 = x$ betrachtet, wobei die Funktion σ als zweimal stetig differenzierbar und maximal linear wachsend gewählt wird und der treibende Lévy-Prozess $L = (L_t)_{t\geq 0}$ entweder vektor- oder matrixwertig ist. Während das fast sichere Kurzzeitverhalten von Lévy-Prozessen bekannt ist und auf Basis des charakteristischen Tripletts angegeben werden kann, gibt es keine solche Charak-

terisierung des Verhaltens des Prozesses X. Unter Verwendung von Methoden aus der stochastischen Analysis werden in Kapitel 3 zunächst Konvergenzresultate für stochastische Integrale der Form $t^{-p}\int_{0+}^{t}\sigma(X_{t-})\mathrm{d}L_t$ hergeleitet und gezeigt, dass $t^{-p}(X_t - X_0)$ für $t \downarrow 0$ fast sicher das Verhalten von $t^{-p}L_t$, d.h. des treibenden Lévy-Prozesses, widerspiegelt. Die Verallgemeinerung von t^p auf geeignete Funktionen $f : [0,\infty) \to \mathbb{R}$ liefert außerdem ein Werkzeug, um explizite Resultate im Stil des Gesetzes vom iterierten Logarithmus für die Lösung der stochastischen Differentialgleichung aus dem Verhalten des treibenden Lévy-Prozesses abzuleiten und ermöglicht ebenfalls Aussagen für Konvergenz in Wahrscheinlichkeit und in Verteilung.

Contents

1. Introduction

Lévy processes naturally arise as the continuous-time analog of discrete random walks $Y = (Y_n)_{n\in\mathbb{N}}$ with $Y_n = \sum_{k=1}^n X_k$, where $(X_k)_{k\in\mathbb{N}}$ is a sequence of independent and identically distributed random variables. Due to the variety of examples that fit the definition, from continuous Brownian motion to pure-jump processes such as Gamma or compound Poisson processes, they are used in modeling random phenomena in a wide range of disciplines. On the other hand, Lévy processes exhibit connections to different probabilistic and analytic subfields of mathematics, making the analysis of their properties as well as the quantities defined from them relevant from a theoretical point of view as well.

The topics covered in this thesis are connected to Lévy-driven stochastic differential equations and showcase some of the variety of questions that can be posed in this context. On the one hand, we consider killed exponential functionals of Lévy processes, which, although of interest in their own in the context of Lévy integrals and infinitely divisible distributions, are mainly considered due to their law arising as the stationary distribution of the solution of a specific Lévy-driven stochastic differential equation. Here, the equation is univariate and driven by two independent real-valued Lévy processes, one of which may include killing. On the other hand, we consider the short-time behavior of solutions to general Lévy-driven stochastic differential equations, which includes a multivariate setting and does not require the components of the driving process to be independent.

In the following Section 1.1, we give a brief overview of the key concepts used throughout the thesis, as well as the notation. The main results of Chapters 2 and 3 are then summarized in Section 1.2.

1.1. Preliminaries and Notation

Throughout the thesis, we consider real-valued, $\mathbb{R}^d$-valued or $\mathbb{R}^{d\times n}$-valued stochastic processes defined on a probability space $(\Omega, \mathcal{F}, P)$, where we write "a.s." and "a.e." to abbreviate "almost surely" and "almost every(where)", respectively. In addition, we are given a filtration $(\mathcal{F}_t)_{t\geq 0}$, which is always assumed to satisfy the usual hypotheses (see [50, p. 3]), i.e. $\mathcal{F}_0$ contains all P-nullsets and the filtration is right-continuous.

By the usual convention, the space $\mathbb{R}^d$ is identified with $\mathbb{R}^{d\times 1}$, i.e. the elements are interpreted as column vectors. The transpose of a vector or matrix x is denoted by x^T. Further, $\langle \cdot, \cdot \rangle$, and $\|\cdot\|$ denote the Euclidean scalar product and Euclidean norm on $\mathbb{R}^d$, respectively, unless otherwise specified. For a matrix m, we denote its Frobenius norm by $\|m\|$ and for some calculations, m is vectorized by writing its entries column-wise into a vector m^{vec}. We further note the subsets $\mathbb{R}^* = \mathbb{R}\setminus\{0\}$, $\mathbb{R}_+ = [0,\infty)$ and $\mathbb{R}_- = (-\infty, 0]$, which are mainly used in characterizing the support of killed exponential functionals in

Section 2.1. The topological closure of a set $A \subseteq \mathbb{R}$ is denoted by $\overline{A}$, the indicator function of A by $\mathbb{1}_A$, and the Borel σ-algebras on $\mathbb{R}^d$ and $\mathbb{R}^*$ are denoted by $\mathcal{B}_d$ and $\mathcal{B}_1^*$, respectively.

The law of a random variable Y is denoted by either $\mathcal{L}(Y)$ or P_Y and to specify it we write $\stackrel{d}{=}$ for equality in distribution. The (one-dimensional) Lebesgue measure is denoted by λ and (absolute) continuity, as well as densities, are always assumed to be with respect to λ unless stated otherwise. We say that a random variable Y is (absolutely) continuous if its distribution $\mathcal{L}(Y)$ has this property. The Dirac measure at $x \in \mathbb{R}$ is denoted by δ_x and the image measure of a general measure μ under a mapping g is denoted by $g(\mu)$. When given two measures μ_1, μ_2, their convolution is denoted by $\mu_1 * \mu_2$. We use the same notation for the convolution of a measure μ with an integrable function g in Section 2.4, by which we mean the function with value $(\mu * g)(x) = \int_{\mathbb{R}} g(x-y)\mu(\mathrm{d}y)$ at $x \in \mathbb{R}$.

A stochastic process having right continuous paths with finite left limits is referred to as càdlàg. For any càdlàg process X, we denote by X_{s-} the left-hand limit of X at time $s \in (0, \infty)$ and by $\Delta X_s = X_s - X_{s-}$ its jumps. The process X_{s-} is càglàd, i.e. left-continuous with finite right limits. The space of real-valued càdlàg functions on $[0, \infty)$ is denoted by $D([0,\infty),\mathbb{R})$ and for $f \in D([0,\infty),\mathbb{R})$, the quantities $f(s-)$ and $\Delta f(s)$ are defined analogously. For the analysis in Section 2.4, we further need to introduce some function spaces. The space of continuous functions $\mathbb{R} \to \mathbb{R}$ is denoted by $C(\mathbb{R})$ while the subspaces of bounded functions, functions vanishing at infinity and compactly supported functions are referred to as $C_b(\mathbb{R})$, $C_0(\mathbb{R})$ and $C_c(\mathbb{R})$, respectively. For a number $n \in \mathbb{N}$, we denote by $C^n(\mathbb{R})$ the space of functions $\mathbb{R} \to \mathbb{R}$ that are n times continuously differentiable with $C^\infty(\mathbb{R})$ denoting that the property holds for every n. Functions in the subspaces $C_0^n(\mathbb{R})$ are n times continuously differentiable with the function itself, as well as the first n derivatives vanishing at infinity. The spaces $C_c^n(\mathbb{R})$ are defined analogously and $C_c^\infty(\mathbb{R})$ is also referred to as the space of test functions. A similar notation is used for functions mapping $\mathbb{R}^d$ into $\mathbb{R}$.

1.1.1. Lévy Processes

In this section, the general definition of a Lévy process is given along with some important properties that are used in the later chapters.

Definition 1.1 (Lévy Process). *An $\mathbb{R}^d$-valued stochastic process $L = (L_t)_{t\geq 0}$ is called a **Lévy process** if*

(i) $L_0 = 0$ a.s.,

(ii) the process has independent increments, i.e. for every $n \in \mathbb{N}$ and $0 < t_1 < \cdots < t_n$, the random variables L_0, $L_{t_1} - L_0, \ldots$, $L_{t_n} - L_{t_{n-1}}$ are independent,

(iii) the process has stationary increments, i.e. for any $s < t$ it holds that $L_t - L_s \stackrel{d}{=} L_{t-s}$,

(iv) the process is stochastically continuous, i.e. for any $t \geq 0$ and $\varepsilon > 0$ it holds that $\lim_{s\to t} P(|L_s - L_t| > \varepsilon) = 0$,

(v) the paths of the process are a.s. càdlàg.

*An incresing Lévy process is referred to as a **subordinator**.*

Note that any $\mathbb{R}^{d\times n}$-valued Lévy process can be identified with an $\mathbb{R}^{nd}$-valued one by vectorization. A key property of any Lévy process L is the fact that for every $t \geq 0$, the distribution of L_t is infinitely divisible, i.e. it has an nth convolution root for any $n \in \mathbb{N}$. More precisely, if $t \geq 0$ and $\mu = \mathcal{L}(L_t)$, then, for all $n \in \mathbb{N}$, there exists a probability measure μ_n on $\mathbb{R}^d$ such that

$$\mu = (\mu_n)^n = \underbrace{\mu_n * \cdots * \mu_n}_{n \text{ times}}.$$

The property of infinite divisibility implies a very specific form of the characteristic function which is given by the Lévy-Khintchine formula (see e.g. [58, Thm. 8.1]).

Theorem 1.2 (Lévy-Khintchine Formula). *Let $D := \{x \in \mathbb{R}^d : \|x\| \leq 1\}$ denote the closed unit ball.*

(i) If μ is an infinitely divisible distribution on $\mathbb{R}^d$, then the characteristic function $\widehat{\mu}$ of μ satisfies

$$\widehat{\mu}(z) = \exp\Big(-\frac{1}{2}\langle z, Az\rangle + \mathrm{i}\langle \gamma, z\rangle + \int_{\mathbb{R}^d} \Big(\exp(\mathrm{i}\langle z, s\rangle) - 1 - \mathrm{i}\langle z, s\rangle \mathbb{1}_D(x)\Big)\nu(\mathrm{d}s)\Big), \tag{1.1}$$

for all $z \in \mathbb{R}^d$, where $A \in \mathbb{R}^{d\times d}$ is a symmetric nonnegative definite matrix, ν is a measure on $\mathbb{R}^d$ satisfying $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}^d} \min\{\|x\|^2, 1\}\nu(\mathrm{d}x) < \infty$, and $\gamma \in \mathbb{R}^d$.

(ii) The representation of $\widehat{\mu}$ in (i) by A, ν and γ is unique.

(iii) Conversely, if A is a symmetric nonnegative-definite $d \times d$ matrix, ν is a measure satisfying $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}^d} \min\{\|x\|^2, 1\}\nu(\mathrm{d}x) < \infty$, and $\gamma \in \mathbb{R}^d$, then there exists an infinitely divisible distribution μ whose characteristic function is given by (1.1).

Considering an $\mathbb{R}^d$-valued Lévy process $L = (L_t)_{t\geq 0}$, we obtain that its characteristic function is given by

$$\varphi_L(z) = \mathbb{E}\mathrm{e}^{\mathrm{i}zL_t} = \exp(t\psi_L(z)),\ z \in \mathbb{R}^d,$$

where the ψ_L is of the same form as the exponent on the right-hand side of (1.1). Hence, all information on the distribution of the Lévy process L can be encoded in the triplet (A, ν, γ) corresponding to the infinitely divisible distribution $\mu = \mathcal{L}(L_1)$.

Definition 1.3 (Characteristic Triplet of a Lévy Process). *The triplet (A, ν, γ) in* (1.1) *is called **generating** or **characteristic triplet** of the infinitely divisible distribution μ or, if $\mu = \mathcal{L}(L_1)$, of the Lévy process L. In the latter case, we often write (A_L, ν_L, γ_L) to emphasize the correspondence. The individual components of the characteristic triplet are referred to as the **Gaussian covariance matrix**, the **Lévy measure** and the **location parameter**, respectively. If the Lévy process is real-valued, we replace the 1×1 matrix A_L by $\sigma_L^2 \geq 0$.*

From the form of the characteristic function in (1.1), it readily follows that any Lévy process can be decomposed into an independent sum

$$L_t = W_t + \gamma t + J_t$$

where $W = (W_t)_{t\geq 0}$ is a Brownian motion and $J = (J_t)_{t\geq 0}$ includes the jumps of the process. Further analysis of the jump structure of the paths of L reveals that the Lévy measure ν_L satisfies

$$(\lambda \otimes \nu_L)([s,t] \times B) = \mathbb{E}(\#\{s \leq u \leq t, \Delta L_u \in B\}),\ 0 \leq s < t,\ B \in \mathcal{B}_d,$$

where $\#\{s \leq u \leq t, \Delta L_u \in B\}$ denotes the number of jumps of $(L_u)_{u\geq 0}$ for $u \in [s,t]$ with jump height in B, and is thus directly connected to the height and intensity of the jumps of L. By the Lévy-Itō-decomposition (see e.g. [58, Thm. 19.2] and [58, Thm. 19.3]), it follows that the process J can, therefore, be decomposed further according to the desired jump height of the individual parts. Although an independent sum is obtained whenever the sets representing the jump heights of the individual parts are disjoint, only the decomposition

$$J_t = J_t^{(1)} + J_t^{(2)}$$

with $|\Delta J_t^{(1)}| \leq 1$ and $|\Delta J_t^{(2)}| > 1$ for all $t \geq 0$ is used in the later sections. In this representation, $J^{(1)}$ is a pure-jump zero-mean martingale with Lévy measure $\nu_{J^{(1)}} = \nu_L|_D$ and $J^{(2)}$ is a compound Poisson process with Lévy measure $\nu_{J^{(1)}} = \nu_L|_{\mathbb{R}^d\setminus D}$. Besides the jump structure, many distributional and path properties of a Lévy process can be deduced from its characteristic triplet. A selection is presented in Theorem 1.4 below. The results are given in Theorem 21.9, Example 25.12 and Theorem 19.3 of [58].

Theorem 1.4 (Properties of Lévy processes)**.** *Let L be an $\mathbb{R}^d$-valued Lévy process with characteristic triplet (A_L, ν_L, γ_L).*

(i) Whenever $A_L = 0$ and $\int_D \|x\|\nu_L(\mathrm{d}x) < \infty$, the sample paths of L are a.s. of finite variation, i.e. there is a set Ω_0 with $P(\Omega_0) = 1$ such that

$$v((0,t], L_j(\omega)) = \sup_{\pi} \sum_{k=1}^{n} |(L_j)_{s_{k-1}}(\omega) - (L_j)_{s_k}(\omega)| < \infty$$

for all $j = 1, \dots, d$, $\omega \in \Omega_0$, and $t \geq 0$, where the supremum is taken over all partitions π of $(0,t]$ and $0 = s_0 < s_1 < \cdots < s_n = t$.

(ii) Whenever either $A_L \neq 0$ or $\int_D \|x\|\nu_L(\mathrm{d}x) = \infty$, the sample paths of L are a.s. of infinite variation.

(iii) If $\int_{\mathbb{R}\setminus D} \|x\|\nu_L(\mathrm{d}x) < \infty$, then $\mathbb{E}\|L_1\| < \infty$ and the expected value is given by

$$\mathbb{E}(L_t) = t\Big(\int_{\mathbb{R}\setminus D} x\nu_L(\mathrm{d}x) + \gamma_L\Big).$$

(iv) Whenever $\nu_L = 0$, the paths of L are a.s. continuous.

Properties of ν_L in dimension one are sometimes also given in terms of its tail function, in which case we write $\overline{\Pi}_L^{(+)}(x) = \nu_L((x,\infty))$ and $\overline{\Pi}_L^{(-)}(x) = \nu_L((-\infty,-x))$ for the right and left tail, respectively, as well as $\overline{\Pi}_L(x) = \overline{\Pi}_L^{(+)}(x) + \overline{\Pi}_L^{(-)}(x)$ for $x > 0$. In higher dimensions, the intervals (x,∞) and $(-\infty,-x)$ are replaced by the set $\{y : \|y\| > x\}$ to define $\overline{\Pi}_L$. The setting of Theorem 1.4 (i) gives rise to another important quantity.

Definition 1.5 (Drift)**.** *Whenever the Lévy measure ν_L satisfies $\int_{\|x\|\leq 1}\|x\|\nu_L(\mathrm{d}x) < \infty$, the characteristic exponent of L can be rewritten in the form*

$$\psi_L(z) = -\frac{1}{2}\langle z, A_L z\rangle + \mathrm{i}\langle \gamma_L^0, z\rangle + \int_{\mathbb{R}} (\mathrm{e}^{\mathrm{i}\langle z,x\rangle} - 1)\nu_L(\mathrm{d}x), \quad z \in \mathbb{R}^d,$$

*where $\gamma_L^0 = \gamma_L - \int_D x\nu_L(\mathrm{d}x)$ is called the **drift** of L. Whenever γ_L^0 is well-defined, we use γ_L^0 instead of γ_L to describe the distribution of L_t and give the characteristic triplet as (A_L, ν_L, γ_L^0).*

A further consequence of the stationary and independent increments of a Lévy process is the Markov property. More precisely, Lévy processes are characterized by a transition function that is both temporally and spacially homogenous (see e.g. [58, Thm. 10.5]).

Definition 1.6. *A mapping $P_{s,t}(x, B)$ of $x \in \mathbb{R}^d$ and $B \in \mathcal{B}_d$ with $0 \leq s \leq t < \infty$ is called a **transition function** on $\mathbb{R}^d$ if*

(i) $B \mapsto P_{s,t}(x, B)$ *is a probability measure for any fixed* x*,*

(ii) $x \mapsto P_{s,t}(x, B)$ *is measurable for any fixed* B*,*

(iii) $P_{s,s}(x, B) = \delta_x(B)$ *for* $s \geq 0$*,*

(iv) $P_{s,t}$ *satisfies the Chapman-Kolmogorov identity*

$$\int_{\mathbb{R}^d} P_{s,t}(x, \mathrm{d}y)P_{t,u}(y, B) = P_{s,u}(x, B),\ 0 \leq s \leq t \leq u.$$

*If, in addition, $P_{s+h,t+h}(x, B)$ does not depend on h, then $P_{s,t}$ is called **temporally homogeneous**. A transition function is called **spacially homogenous** or **translation invariant** if it satisfies $P_{s,t}(x, B) = P_{s,t}(0, B - x)$ for any s, t, x, and B, where the set $B-x$ is given by $\{y - x : y \in B\}$.*

The transition function corresponding to an $\mathbb{R}^d$-valued Lévy process L is given by

$$P_{s,t}(x, B) = P(L_t - L_s \in B - x) = \mathcal{L}(L_{t-s})(B - x) = P_{t-s}(x, B),\ 0 \leq s \leq t.$$

From this definition, one readily obtains corresponding linear operators through

$$P_t : C_0 \to C_0,\ (P_t f)(x) = \int_{\mathbb{R}^d} P_t(x, \mathrm{d}y)f(y) = E^x[f(X_t)],\ t \geq 0. \tag{1.2}$$

The family $\{P_t : t \geq 0\}$ is a (strongly continuous) semigroup on $C_0(\mathbb{R}^d)$ with norm $\|P_t\| = 1$, as is e.g. shown in [58, Thm. 31.5]. By considering its infinitesimal generator, one obtains another quantity directly linked to the Lévy process L. The domain of a linear operator $\mathcal{A}$ is denoted by $\mathrm{dom}(\mathcal{A})$ below.

Definition 1.7. *For a strongly continuous semigroup $\{P_t : t \geq 0\}$ of operators on $C_0(\mathbb{R}^d)$, the infinitesimal generator is the linear operator defined by*

$$\mathcal{A}f(x) = \lim_{t\downarrow 0} \frac{1}{t}(P_t f(x) - f(x))$$

on the set of functions $f \in C_0(\mathbb{R}^d)$ for which the limit exists uniformly.

We can now give the infinitesimal generator for the semigroup of operators given in (1.2), which is revisited in Section 2.4. The result can e.g. be found in [58, Thm. 31.5]. Recall that D denotes the closed unit disk in $\mathbb{R}^d$.

Theorem 1.8 (Infinitesimal Generator of a Lévy Process)**.** *Let L be an $\mathbb{R}^d$-valued Lévy process with characteristic triplet (A_L, ν_L, γ_L), the semigroup $\{P_t : t \geq 0\}$ as defined as in (1.2) and let $\mathcal{A}^L$ be its infinitesimal generator. Then $C_0^2(\mathbb{R}^d) \subseteq \operatorname{dom}(L)$ and $\mathcal{A}^L$ acts on $f \in C_0^2$ by*

$$\mathcal{A}^L f(x) = \frac{1}{2}\nabla^T A_L \nabla f(x) + \gamma_L^T \nabla f(x) + \int_{\mathbb{R}^d} \Big(f(x+s) - f(x) - \nabla^T f(x) s \mathbb{1}_D(x) \Big) \nu_L(\mathrm{d}s),$$

where $\nabla = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d})^T$.

1.1.2. Stochastic Integration

Stochastic integration is a key concept used in Chapters 2 and 3. In this section, the main ingredients are briefly summarized, starting with the definition of a semimartingale. A full construction of the stochastic integral is e.g. carried out in the books [45] and [50].

Definition 1.9 (Finite-Variation Process)**.** *Let $f : [0, \infty) \to \mathbb{R}$ and $t \geq 0$. Then the quantity*

$$v((0,t], f) = \sup_{\pi} \sum_{k=1}^{n} |f(s_{k-1}) - f(s_k)|,$$

*where the supremum is taken over all partitions π of $(0,t]$ and $0 = s_0 < s_1 < \cdots < s_n = t$, is called the **total variation of f over** $(0,t]$. Whenever $v((0,t], f) < \infty$ for all $t \geq 0$, we say that f is of **finite variation on compacts**. A real-valued stochastic process Y defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ is called **finite-variation process**, if it is adapted to the filtration, has a.s. càdlàg paths and a.e. path of Y is of finite variation on compacts.*

Definition 1.10 ((Local) Martingale)**.** *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ be a filtered probability space. A stochastic process $M = (M_t)_{t\geq 0}$ is called a **martingale**, if*

(i) M is adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$,

(ii) $\mathbb{E}|M_t| < \infty$ for all $t \geq 0$,

(iii) $\mathbb{E}[M_t|\mathcal{F}_s] = M_s$ P-a.s. for all $s, t \geq 0$ with $s \leq t$.

*The process M is called a **local martingale**, if it is adapted to $(\mathcal{F}_t)_{t\geq 0}$ and there exists a sequence $(\tau_n)_{n\in\mathbb{N}}$ of stopping times with $\lim_{n\to\infty} \tau_n = \infty$ a.s. such that the stopped process M^{τ_n} is a uniformly integrable martingale for any $n \in \mathbb{N}$.*

Definition 1.11 (Semimartingale)**.** *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ be a filtered probability space. A real-valued stochastic process $X = (X_t)_{t\geq 0}$ is called a **semimartingale**, if there exists a decomposition of the form*

$$X_t = X_0 + M_t + Y_t, \ t \geq 0,$$

where $M = (M_t)_{t\geq 0}$ is a local martingale and $Y = (Y_t)_{t\geq 0}$ is a finite-variation process. Further, a vector or matrix-valued stochastic process is called a semimartingale if every component has this property.

The Lévy-Itō decomposition introduced in Section 1.1.1 directly implies that every Lévy process is a semimartingale (see also [50, p. 55]) such that integration with respect to a real-valued Lévy process is well-defined. To extend the terminology to the multivariate setting considered in Chapter 3, we use the following definitions (see e.g. [32] or [50]).

Definition 1.12 (Multivariate Stochastic Integrals)**.** *Let H be an $\mathbb{R}^{n\times d}$-valued adapted càglàd process, X and Y be two semimartingales taking values in $\mathbb{R}^{d\times m}$ and $\mathbb{R}^{m\times n}$, respectively, and $a, b \in \mathbb{R}$ with $a < b$. Then the **multivariate stochastic integral** is defined through its components by*

$$\begin{aligned}\left(\int_{a+}^{b} H_s \mathrm{d}X_s\right)_{i,j} &= \sum_{k=1}^{d} \int_{a+}^{b} (H_{i,k})_{s-} \mathrm{d}(X_{k,j})_s, \;\; i = 1, \dots, n, \;\; j = 1, \dots, m, \\ \left(\int_{a+}^{b} \mathrm{d}Y_s H_s\right)_{i,j} &= \sum_{k=1}^{n} \int_{a+}^{b} (H_{k,j})_{s-} \mathrm{d}(Y_{i,k})_s, \;\; i = 1, \dots, m, \;\; j = 1, \dots, d.\end{aligned}$$

Note that the properties of one type of the multivariate stochastic integral readily carry over to the other by transposition of the matrix-valued semimartingales. Similar to the univariate case, the integral is associative and linear in both integrand and integrator.

Definition 1.13 (Multivariate Quadratic Covariation)**.** *Let X and Y be an $\mathbb{R}^{n\times d}$-valued and an $\mathbb{R}^{d\times m}$-valued semimartingale, respectively and $a, b \in \mathbb{R}$ with $a < b$. Then the **quadratic covariation** of X and Y takes the form*

$$[X, Y]_{a+}^{b} = X(b)Y(b) - X(a)Y(a) - \int_{a+}^{b} X_{s-} \mathrm{d}Y_s - \int_{a+}^{b} \mathrm{d}X_s Y_{s-}$$

and its individual components are given by

$$\left([X, Y]_{a+}^{b}\right)_{i,j} = \sum_{k=1}^{d} [X_{i,k}, Y_{k,j}]_{a+}^{b}, \;\; i = 1, \dots, n, \;\; j = 1, \dots, m.$$

By considering the quadratic covariation component-wise it readily follows that it is bilinear and again a semimartingale. For the compatibility with the multivariate stochastic integral, we further note the following properties (see e.g. [32]).

Theorem 1.14. *Let G and H be two adapted càglàd processes taking values in $\mathbb{R}^{n\times k}$ and $\mathbb{R}^{l\times m}$, respectively, and let X and Y be two matrix-valued semimartingales taking values in $\mathbb{R}^{k\times d}$ and $\mathbb{R}^{d\times l}$, respectively. Then*

$$\left[\int_{0+}^{\cdot} G_s \mathrm{d}X_s, \int_{0+}^{\cdot} \mathrm{d}Y_s H_s\right]_{0+}^{t} = \int_{0+}^{t} \mathrm{d}\left(\int_{0+}^{s} G_r \mathrm{d}([X, Y]_{0+}^{r})\right) H_s$$

and the multivariate integration by parts formula takes the form

$$\int_{(0,t]} X_{s-} \mathrm{d}Y_s = X_t Y_t - X_0 Y_0 - \int_{(0,t]} \mathrm{d}X_s Y_{s-} - [X, Y]_{0+}^{t}.$$

Remark 1.15. In [52], Rajput and Rosinski introduced integration of a deterministic function with respect to a Lévy process based on the spectral decomposition of the integrator. Although the definition of the integral is quite different to the semimartingale integral given above, both integrals agree in particular for bounded measurable functions. If a function $f : [0,t] \to \mathbb{R}$ is integrable in the sense of Rajput and Rosinski, then the distribution of the integral $\int_0^t f(s)\,\mathrm{d}\eta_s$ is infinitely divisible with characteristic exponent $\psi_f(z) = \int_0^t \psi_\eta(f(s)z)\,\mathrm{d}s$, $z \in \mathbb{R}$, and characteristic triplet $(\sigma_f^2, \nu_f, \gamma_f)$ given by

$$\sigma_f^2 = \sigma_\eta^2 \int_0^t f(s)^2\,\mathrm{d}s, \tag{1.3}$$

$$\nu_f(B) = \int_0^t \int_{\mathbb{R}} \mathbb{1}_{B\setminus\{0\}}(f(s)x)\,\nu_\eta(\mathrm{d}x)\,\mathrm{d}s,\ B \in \mathcal{B}_1, \tag{1.4}$$

$$\gamma_f = \int_0^t \left[f(s)\gamma_\eta + \int_{\mathbb{R}} f(s)x \left(\mathbb{1}_{\{|f(s)x|\le 1\}} - \mathbb{1}_{\{|x|\le 1\}} \right) \nu_\eta(\mathrm{d}x) \right] \mathrm{d}s, \tag{1.5}$$

see [52, Prop. 2.6, Thm. 2.7] or [58, Prop. 57.10].

1.1.3. Stochastic Differential Equations

Using the notion of the stochastic integral from the previous Section 1.1.2, we give some key concepts related to the solution of stochastic differential equations that are used throughout Chapters 2 and 3. First, a general definition is presented to introduce the assumptions made throughout the thesis. Unless specified otherwise, the term stochastic differential equation (SDE) refers to an expression of the form

$$\mathrm{d}X_t = \sigma(X_{t-})\mathrm{d}L_t,\ t \ge 0, \tag{1.6}$$

with some starting random variable X_0, where L denotes an $\mathbb{R}^d$-valued semimartingale or Lévy process and $\sigma : \mathbb{R}^n \to \mathbb{R}^{n\times d}$. In the context of this thesis, σ is chosen to satisfy the linear growth condition

$$\|\sigma(x)\|^2 \le c(1 + \|x\|^2)$$

for some $c > 0$ and all $x \in \mathbb{R}^n$ and to be twice continously differentiable. Note that the latter in particular implies that $\sigma(X)$ is again a semimartingale by the Itō formula (see e.g. [50, p. 78]). The differential notation in (1.6) is to be understood as

$$X_t = X_0 + \int_{0+}^t \sigma(X_{t-})\mathrm{d}L_t,\ t \ge 0, \tag{1.7}$$

since the processes involved are usually neither differentiable nor continuous.

Definition 1.16 (Solution to an SDE)**.** *An adapted càdlàg process $X = (X_t)_{t\ge 0}$ is called a* ***(strong) solution*** *to* (1.6) *if it satisfies* (1.7) *for any $t \ge 0$.*

To answer the question of existence and uniqueness, we note the following result (see [50, Thm. 7, p. 259]).

Theorem 1.17. *Let Z be an $\mathbb{R}^d$-valued semimartingale with $Z_0 = 0$ and J be an adapted càdlàg process taking values in $\mathbb{R}^n$. Assume further that $\sigma : \mathbb{R}^n \mapsto \mathbb{R}^{n\times d}$ satisfies the following two conditions for all $i = 1,\dots,d$, $j = 1,\dots,n$ and any $\mathbb{R}^n$-valued, adapted càdlàg processes X, Y.*

(i) For any stopping time τ, $X^{\tau-} = Y^{\tau-}$ implies $\sigma_{i,j}(X)^{\tau-} = \sigma_{i,j}(Y)^{\tau-}$.

(ii) There exists an increasing (finite) process $K = (K_t)_{t>0}$ such that the condition $|\sigma_{i,j}(X)_t - \sigma_{i,j}(Y)_t| \leq K_t \sup_{0<s\leq t} \|X_s - Y_s\|$ holds a.s. for any $t \geq 0$.

Then the stochastic differential equation

$$X_t = J_t + \int_{0+}^t \sigma(X_{s-})\mathrm{d}Z_s$$

has a unique (strong) solution that is also adapted and càdlàg. Moreover, if J is a semimartingale, so is the solution X.

Note that σ being twice continuously differentiable and maximal of linear growth ensures that Theorem 1.17 is applicable such that we always work with a unique strong solution to the SDE (1.6) in the following chapters.

Remark 1.18. If the SDE (1.6) is driven by a real-valued Lévy process or an $\mathbb{R}^d$-valued Lévy process with independent components, the independent increments of the process imply that the solution X, whenever existent, is a (strong) Markov process (see [50, Thm. 32, p. 300]). Under additional assumptions on the Lévy measure of the driving process, the solution to (1.6) can further be shown to be a so-called Lévy type Feller process (see [37]) which is characterized by a representation similar to (1.1), but with a triplet of the form $(A(x), \nu(x), \gamma(x))$ that depends on the initial condition $X_0 = x \in \mathbb{R}^n$, as well as the function σ and the characteristic exponent of the driving Lévy process.

1.1.4. (Killed) Exponential Functionals of Lévy Processes

Lastly, we note two important stochastic differential equations that are referenced throughout the thesis and introduce the killed exponential functional studied in Chapter 2.

Definition 1.19 ((Multivariate) Stochastic Exponential)**.** *Let $L = (L_t)_{t\geq 0}$ be an $\mathbb{R}^{d\times d}$-valued Lévy process or semimartingale and* Id $\in \mathbb{R}^{d\times d}$ *denote the identity matrix. Then the unique (strong) solution $X = (X_t)_{t\geq 0}$ to the SDE*

$$\mathrm{d}X_t = X_{t-}\mathrm{d}L_t,\ t > 0, \quad X_0 = \mathrm{Id}$$

is referred to as ***(left) stochastic exponential of*** L *and denoted by $\overleftarrow{\mathcal{E}}(L)$. Similarly, the unique (strong) solution Y to the SDE*

$$\mathrm{d}Y_t = \mathrm{d}L_t Y_{t-},\ t > 0, \quad Y_0 = \mathrm{Id}$$

is called ***right stochastic exponential of*** L *and denoted by $\overrightarrow{\mathcal{E}}(L)$. Unless specified otherwise, the term "stochastic exponential" refers to the left stochastic exponential and we omit the arrow.*

Note that the solutions X and Y in Definition 1.19 generally do not coincide unless $d = 1$, but can be related to one another through transposition of the matrix-valued semimartingales. In the univariate case, the stochastic exponential can be given explicitly using the

Doléans-Dade formula (see e.g. [50, Thm. 37, p. 84]). In particular, the stochastic exponential of a real-valued Lévy process L with characteristic triplet $(\sigma_L^2, \nu_L, \gamma_L)$ is given by

$$\mathcal{E}(L)_t = \mathrm{e}^{L_t - t\sigma_L^2/2} \prod_{0<s\le t} (1+\Delta L_s)\mathrm{e}^{-\Delta L_s}, \quad t \ge 0. \tag{1.8}$$

Definition 1.20 (Generalized Ornstein-Uhlenbeck Process)**.** *Let ξ and η be two independent real-valued Lévy processes and define another Lévy process U through*

$$\mathcal{E}(U)_t = \mathrm{e}^{-\xi_t}. \tag{1.9}$$

Then the unique (strong) solution to the SDE

$$\mathrm{d}X_t = X_{t-}\,\mathrm{d}U_t + \mathrm{d}\eta_t,\ t \ge 0,$$

with starting random variable X_0 independent of ξ and η is called the ***generalized Ornstein-Uhlenbeck (GOU) process*** *driven by ξ and η.*

Similar to the univariate stochastic exponential, the GOU process $X = (X_t)_{t\ge0}$ can also be given in a closed form (see [44, p.428])

$$X_t = \mathrm{e}^{-\xi_t}\left(\int_0^t \mathrm{e}^{\xi_{s-}}\mathrm{d}\eta_s + X_0\right),\ t \ge 0, \tag{1.10}$$

in which the driving processes are more visible. As the driving Lévy processes are assumed to be independent, the GOU process is a Markov process. Observe that the exponential function on the righ-hand side of (1.9) is strictly positive, such that (1.8) directly implies that U cannot have jumps of size -1 or less, i.e. its Lévy measure ν_U satisfies $\nu_U((-\infty,-1]) = 0$. The absence of jumps of size -1 in U further implies that X is a Feller process, i.e. that the semigroup $(P_t)_{t\ge0}$ of linear operators corresponding to the process satisfies $P_t(C_0(\mathbb{R})) \subseteq C_0(\mathbb{R})$ and $\lim_{t\downarrow0} P_t f = f$ for all $f \in C_0(\mathbb{R})$ with the convergence holding with respect to the supremum norm on $C_0(\mathbb{R})$ (see [5, Thm. 3.1] or [37, Ex. 4.3]). Noting that

$$U_t = -\xi_t + t\sigma_\xi^2/2 + \sum_{0<s\le t}\left(\mathrm{e}^{-\Delta\xi_s} - 1 + \Delta\xi_s\right),\ t \ge 0, \tag{1.11}$$

$$\xi_t = -U_t + t\sigma_U^2/2 - \sum_{0<s\le t}\Big(\ln(1+\Delta U_s) - \Delta U_s\Big),\ t \ge 0, \tag{1.12}$$

it follows that (1.11) defines a bijection from the class of all Lévy processes ξ to the class of Lévy processes U with $\nu_U((-\infty,-1]) = 0$, with its inverse given by (1.12). The characteristic triplet of ξ in terms of the one of U has been derived in [7, Lem 3.4] and is given by

$$\sigma_\xi^2 = \sigma_U^2, \quad \nu_\xi = g(\nu_U),$$
$$\gamma_\xi = -\gamma_U + \sigma_U^2/2 + \int_{(-1,\infty)}\left(x\mathbb{1}_{\{|x|\le1\}} - (\ln(1+x))\mathbb{1}_{\{x\in[\mathrm{e}^{-1}-1,\mathrm{e}-1]\}}\right)\nu_U(\mathrm{d}x),$$

where $g : (-1,\infty) \to \mathbb{R}$ is defined by $g(x) = -\ln(1+x)$. Similarly, the characteristic triplet of U in terms of the one of ξ is expressed by

$$\sigma_U^2 = \sigma_\xi^2, \quad \nu_U = h(\nu_\xi), \quad \gamma_U = -\gamma_\xi - \sigma_\xi^2/2 + \int_{[-\log 2,\infty)} \Big[\Big(\mathrm{e}^{-x} - 1\Big) + x\mathbb{1}_{\{|x|\leq 1\}}\Big]\, \nu_\xi(\mathrm{d}x),$$

where $h : \mathbb{R} \to (-1,\infty)$ is given by $h(x) = \mathrm{e}^{-x} - 1$. It further follows from (1.11) and (1.12) that U is of finite variation if and only if ξ is, in which case the drifts are related by $\gamma_U^0 = -\gamma_\xi^0$. A quantity that is closely related to the generalized Ornstein-Uhlenbeck process is the (possibly killed) exponential functional defined from the driving processes ξ and η.

Definition 1.21 ((Killed) Exponential Functional)**.** *Let ξ and η be two independent Lévy processes and let τ be an exponentially distributed random variable with parameter $q > 0$ that is independent of ξ and η. Then*

$$V_{0,\xi,\eta} = \int_0^\infty \mathrm{e}^{-\xi_{s-}}\mathrm{d}\eta_s$$

*is called **exponential functional** of ξ and η or **exponential functional without killing**, whenever the integral converges a.s. to a finite random variable and*

$$V_{q,\xi,\eta} = \int_0^\tau \mathrm{e}^{-\xi_{s-}}\mathrm{d}\eta_s$$

*is called **killed exponential functional** of ξ and η with parameter q.*

Observe that by interpreting the case $q = 0$ as $\tau = \infty$ a.s., the definition of the killed exponential functional includes the improper integral $V_{0,\xi,\eta}$ and is thus a natural generalization. Using (1.9) from the previous section, the random variable $V_{q,\xi,\eta}$ can also be expressed in terms of the Lévy process U. The law of the exponential functional $V_{0,\xi,\eta}$ is naturally obtained as the invariant probability distribution of the GOU process driven by ξ and η provided the processes are not both deterministic and the stochastic integral $\int_0^t \mathrm{e}^{-\xi_{s-}}\,\mathrm{d}\eta_s$ converges a.s. to a finite limit as $t \to \infty$ (see [41, Thm. 2.1]). Necessary and sufficient conditions for the convergence are given in [27]. We show in Section 2.3 that the law of $V_{q,\xi,\eta}$, too, arises as the stationary distribution of a Markov process.

1.2. Main Results

In this section, we give an overview of the main results included in the thesis. Chapter 2 is based on the preprints [9] and [8], which cover various distributional properties of killed exponential functionals of Lévy processes. The results on the support and the continuity properties (cf. [9]) are collected in Sections 2.1 and 2.2, respectively. Here, the main result of Section 2.1 is Theorem 2.3, which gives a complete characterization of the support of the killed exponential functional for all possible combinations of the driving Lévy processes ξ and η, also showing that the support of the killed exponential functional, as opposed to the case without killing, is not necessarily an interval if η is a compound Poisson process.

In Section 2.2, the main results are Theorem 2.23, which gives different sufficient conditions for absolute contnuity of $V_{q,\xi,\eta}$, and Corollary 2.26, which completely characterizes continuity of the distribution. Further, Theorem 2.29 gives different sufficient conditions for the absolute continuity of the improper integral $V_{0,\xi,\eta}$. Since many of the conditions in Theorems 2.23 and 2.29 are derived using conditioning techniques such as

$$P\left(\int_0^\tau \mathrm{e}^{-\xi_{s-}}\, d\eta_s \in B\right) = \int_0^\infty P\left(\int_0^t \mathrm{e}^{-\xi_s}\, d\eta_s \in B \middle| \tau = t\right)\, P_\tau(\mathrm{d}t)$$

for the killed exponential functional, the conditions for absolute continuity of $\int_{0+}^t \mathrm{e}^{-\xi_{s-}}\mathrm{d}\eta_s$ for $t \geq 0$ (Theorem 2.18), as well as for Lévy integrals of the form $\int_{0+}^t f(s-)\mathrm{d}\eta_s$ for deterministic functions f (Corollary 2.14) shown as intermediate steps may be of independent interest. The preprint [9] is joint work with Anita Behme (TU Dresden), Alexander Lindner (Ulm University) and Victor Rivero (CIMAT, Mexico).

The results on the connection between killed exponential functionals and Markov processes, as well as different distributional equations describing the law of the killed exponential functional directly (cf. [8]) are given in Sections 2.3 and 2.4, respectively. Here, the main results of Section 2.3 are Theorem 2.31, which establishes the law of the killed exponential functional as the stationary distribution of a Markov process, and Theorem 2.34 in which the infinitesimal generator of this Markov process is calculated. They yield the key ingredients for deriving different distributional equations for the law of the killed exponential functional, as well as its Lebesgue density in the absolutely continuous case, in Section 2.4.

Here, the equations derived in the first part of the section are based on the techniques developed in [5] and extend the results obtained by Behme and Lindner for the characteristic function (Corollary 2.37) and the density (Proposition 2.39) to include killing. The later parts of Section 2.4 focus on the approach developed in [38] with the main result being Theorem 2.43, which gives a distributional equation for the law of the killed exponential functional under general conditions, as well as the corollaries adding a moment condition (Corollary 2.46) and a finite variation condition (Corollary 2.47) to the theorem, respectively. Further, Corollary 2.48 characterizes continuity and differentiability of the density in the absolutely continuous case, which is an important tool in giving the law of the killed exponential functional explicitly in special cases, as can be seen in Section 2.4.4. Another important result is the content of Remark 2.50 in which a small oversight in [38] is discussed. The preprint [8] is joint work with Anita Behme (TU Dresden) and Alexander Lindner (Ulm University).

Chapter 3 is based on the preprint [53] and treats the short-time behavior of solutions to Lévy-driven SDEs of the form (1.6). Here, the main results are Lemma 3.1, which establishes the short-time behavior of a semimartingale integral based on the convergence of the integrand, and Theorem 3.10, which yields that, under suitable conditions,

$$\lim_{t\downarrow 0}\left(\frac{X_t - x}{t^p} - \frac{\sigma(X_t)L_t}{t^p}\right) = \lim_{t\downarrow 0}\left(\frac{X_t - x}{t^p} - \frac{\sigma(x)L_t}{t^p}\right) = 0$$

with probability one, showing that the the quantity $t^{-p}(X_t - X_0)$ for $t \downarrow 0$ almost surely mirrors the behavior of $t^{-p}L_t$. Hence, the a.s. short-time behavior of the solution X can be derived directly from the behavior of the driving Lévy process. Related results for SDEs driven by general semimartingales are derived in Propositions 3.3 and 3.5. Further, Corollary 3.12 allows to identify an a.s. cluster set $A_X = C(\{X_t/f(t) : t \downarrow 0\})$ for the solution based on the one of the driving Lévy process for suitable scaling functions $f : [0, \infty) \to \mathbb{R}$.

2. Distributional Properties of Killed Exponential Functionals

This chapter aims to study various distributional properties of the (possibly killed) exponential functional introduced in Definition 1.21. While generalized Ornstein-Uhlenbeck processes and the law of $V_{0,\xi,\eta}$ are well studied in the literature, see e.g. the survey paper [14], or [5], [6], [13], [19], [23], [38], [48], killed exponential functionals have so far mainly been considered when $\eta_t = t$, cf. [46], [48], [49], or [66, Thm. 2]. In this chapter, the support (Section 2.1) and continuity of the law of $V_{q,\xi,\eta}$ are characterized and different sufficient conditions for absolute continuity are given (Section 2.2). These two sections are based on the preprint [9]. We further establish the law of the killed exponential functional as the unique invariant probability distribution of a Markov process and calculate the corresponding infinitesimal generator (Section 2.3), from which different equations describing the law of $V_{q,\xi,\eta}$ can be derived (Section 2.4). More precisely, we give functional equations for the characteristic function and the Lebesgue density in the absolutely continuous case as well as different distributional equations that describe the law of the killed exponential functional directly. The analysis carried out in these sections is based on the preprint [8]. We start the discussion of the distributional properties with two examples in which the law of the killed exponential functional can be given explicitly.

Example 2.1. [66, Thm. 2] Let $q > 0$, $\eta_t = t$ and $\xi_t = 2B_t + bt$, $t \geq 0$, for some standard Brownian motion $(B_t)_{t\geq 0}$ and $b \in \mathbb{R}$, then

$$V_{q,\xi,\eta} \stackrel{d}{=} \frac{B_{1,\beta}}{2G_\alpha},$$

where $B_{1,\beta} \sim \mathrm{Beta}(1,\beta)$ and $G_\alpha \sim \Gamma(\alpha, 1)$ are independent, and

$$\alpha = \frac{\gamma + b}{2}, \beta = \frac{\gamma - b}{2}, \gamma = \sqrt{2q + b^2}.$$

Example 2.2. [46, Sect. 2] Fix $\alpha \in (0,1)$, set $q = \Gamma(1-\alpha)^{-1}$, $\eta_t = t$ and let ξ_t, $t \geq 0$, be a drift-free subordinator with Lévy measure

$$\nu_\xi(\mathrm{d}x) = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{e}^{-x/\alpha}}{(1-\mathrm{e}^{-x/\alpha})^{\alpha+1}} \mathbb{1}_{(0,\infty)}(x)\mathrm{d}x.$$

Then $V_{q,\xi,\eta}$ has a Mittag-Leffler distribution with parameter α, i.e. its Laplace transform is a Mittag-Leffler function

$$\mathbb{E}[\mathrm{e}^{-tV_{q,\xi,\eta}}] = E_\alpha(-t) = \sum_{k\geq 0} \frac{(-t)^k}{\Gamma(1+\alpha k)},$$

and the distribution of $V_{q,\xi,\eta}$ has a Lebesgue density f_{ML} given by

$$f_{ML}(s) = \frac{1}{\pi\alpha} \sum_{k\geq 0} \frac{(-1)^{k+1}}{k!} \Gamma(\alpha k + 1) s^{k-1} \sin(\pi\alpha k), \quad s > 0.$$

Another important special case of the killed exponential functional is when $\xi_t = 0$, in which case $V_{q,0,\eta} = \eta_\tau$, which can be interpreted as the Lévy process η subordinated by a gamma process with parameters 1 and $q > 0$, evaluated at time 1. The law of $V_{q,0,\eta}$ is then q times the potential measure of η, cf. [58, Def. 30.9]. This and other examples are considered in more detail in Section 2.4.4. Some additional remarks on the results in this chapter and possible extensions are collected in Appendix A.

2.1. Support

In this section, we give the support of the distribution of the killed exponential functional $V_{q,\xi,\eta} = \int_0^\tau e^{-\xi_{s-}} d\eta_s$ for independent Lévy processes ξ and η and an independent exponentially distributed random variable τ with parameter $q \in (0, \infty)$. In [6, Thm. 1] the support of the exponential functional $V_{0,\xi,\eta}$ without killing was completely characterized and, in particular, it was shown that it is always an interval. This is no longer true when considering $q \in (0, \infty)$, as can be seen from the results in this section. We will give a general characterization of the support of $V_{q,\xi,\eta}$ in Theorem 2.3 and then study the case where both ξ and η are pure compound Poisson processes more closely in Proposition 2.10. Note that in case of a deterministic process η, the support of a possibly killed exponential functional has been characterized in [49, Thm. 2.4(2)] in terms of the Wiener-Hopf factorization of ξ, showing that it is always an interval. The special case $q = 0$ (corresponding to $\tau = \infty$) of [49, Thm. 2.4(2)] was already proven by different means in [6]. Both results are included in the following theorem. We will, however, present an alternative proof that does not use the Wiener-Hopf factorization. Observe that Theorem 2.3 covers all possible combinations of ξ and η and, therefore, provides a complete characterization of the support of the exponential functional in the killed case. Below, we refer to a Lévy process as spectrally positive if $\nu_L((-\infty, 0)) = 0$, and spectrally negative if $\nu_L((0, \infty)) = 0$.

Theorem 2.3 (Support of killed exponential functionals). *Consider the killed exponential functional $V := V_{q,\xi,\eta} = \int_0^\tau e^{-\xi_{s-}} d\eta_s$ for $q > 0$ and two independent Lévy processes ξ and η with characteristics $(\sigma_\xi^2, \nu_\xi, \gamma_\xi)$ and $(\sigma_\eta^2, \nu_\eta, \gamma_\eta)$, respectively. As in the introduction we denote by γ_ξ^0, γ_η^0 the drift of ξ or η whenever it exists.*

(i) If $\eta \equiv 0$, then $\operatorname{supp}(V) = \{0\}$.

(ii) Assume that η is deterministic with $\gamma_\eta^0 > 0$, then

$$\operatorname{supp}(V) = \begin{cases} [0, \frac{\gamma_\eta^0}{\gamma_\xi^0}], & \text{if } \xi \text{ is a subordinator with } \gamma_\xi^0 > 0, \\ [0, \infty), & \text{otherwise,} \end{cases}$$

and if $\gamma_\eta^0 < 0$, *then*

$$\operatorname{supp}(V) = \begin{cases} [\frac{\gamma_\eta^0}{\gamma_\xi^0}, 0], & \text{if } \xi \text{ is a subordinator with } \gamma_\xi^0 > 0, \\ (-\infty, 0], & \text{otherwise.} \end{cases}$$

(iii) Assume that one of the following cases holds

(a) η *is of infinite variation,*

(b) η *is of finite variation with* $0 \in \operatorname{supp}(\nu_\eta)$ *and* $\nu_\eta((-\infty,0)) > 0$, $\nu_\eta((0,\infty)) > 0$,

(c) η *is of finite variation with* $\gamma_\eta^0 \neq 0$ *and* $\nu_\eta((-\infty,0)) > 0$, $\nu_\eta((0,\infty)) > 0$,

then $\operatorname{supp}(V) = \mathbb{R}$.

(iv) Assume that η *is non-deterministic, of finite variation and spectrally positive/negative, as well as* $0 \in \operatorname{supp}(\nu_\eta)$ *or* $\gamma_\eta^0 \neq 0$, *then*

$$\operatorname{supp}(V) = \begin{cases} [0, \infty), & \text{if } \eta \text{ is a subordinator,} \\ (-\infty, 0], & \text{if } -\eta \text{ is a subordinator.} \end{cases}$$

If under these assumptions neither η *nor* $-\eta$ *is a subordinator, we have that*

$$\operatorname{supp}(V) = \begin{cases} (-\infty, \frac{\gamma_\eta^0}{\gamma_\xi^0}], & \text{if } \gamma_\eta^0 > 0 \text{ and } \xi \text{ is a subordinator with } \gamma_\xi^0 > 0, \\ [\frac{\gamma_\eta^0}{\gamma_\xi^0}, \infty), & \text{if } \gamma_\eta^0 < 0 \text{ and } \xi \text{ is a subordinator with } \gamma_\xi^0 > 0, \\ \mathbb{R}, & \text{otherwise.} \end{cases}$$

(v) Assume that η *is a compound Poisson process with* $0 \notin \operatorname{supp}(\nu_\eta)$, *then*

$$\operatorname{supp}(V) = \overline{\left\{ \sum_{j=1}^{n} \left(\prod_{k=1}^{j} a_k \right) b_j, \ a_k > 0, \ln a_k \in \Xi, b_j \in \operatorname{supp}(\nu_\eta), n \in \mathbb{N}_0 \right\}}, \tag{2.1}$$

where

$$\Xi = \operatorname{supp}(-\xi_T) \quad \text{with} \quad T \sim \operatorname{Exp}(\nu_\eta(\mathbb{R})), \ \text{independent of } \xi. \tag{2.2}$$

In particular, if $-\xi$ *is a subordinator with* $0 \in \operatorname{supp}(\nu_\xi)$ *or nonzero drift, we have that*

$$\operatorname{supp}(V) = \begin{cases} \{0\} \cup [\inf \operatorname{supp}(\nu_\eta), \infty), & \text{if } \eta \text{ is a subordinator,} \\ (-\infty, \sup \operatorname{supp}(\nu_\eta)] \cup \{0\}, & \text{if } -\eta \text{ is a subordinator,} \\ \mathbb{R}, & \text{otherwise,} \end{cases}$$

and if $\xi \equiv 0$, *then* $\operatorname{supp}(V) = \overline{\left\{ \sum_{j=1}^{n} b_j : b_j \in \operatorname{supp}(\nu_\eta), n \in \mathbb{N}_0 \right\}}$.
In the remaining cases, except when ξ *is a compound Poisson process with* $0 \notin \operatorname{supp}(\nu_\xi)$

and only negative jumps, (2.1) *simplifies to*

$$\operatorname{supp}(V) = \begin{cases} [0,\infty), & \text{if } \eta \text{ is a subordinator,} \\ (-\infty,0], & \text{if } -\eta \text{ is a subordinator,} \\ \mathbb{R}, & \text{otherwise.} \end{cases}$$

In the proof of Theorem 2.3, we will frequently make use of the following three lemmas.

Lemma 2.4. *Let ξ and η be two independent Lévy processes such that $V_{0,\xi,\eta} := \int_0^\infty \mathrm{e}^{-\xi_{s-}} d\eta_s$ exists a.s. Let $q > 0$ be arbitrary. Then*

$$\operatorname{supp}(V_{0,\xi,\eta}) \subseteq \operatorname{supp}(V_{q,\xi,\eta}).$$

Proof. This is clear as for any $q > 0$ the random variable τ may become arbitrarily large. $\square$

Lemma 2.5. *Let ξ and η be two independent Lévy processes with characteristics $(\sigma_\xi^2, \nu_\xi, \gamma_\xi)$ and $(\sigma_\eta^2, \nu_\eta, \gamma_\eta)$, respectively, $q > 0$ fixed and let B_ξ, $B_\eta \subset \mathbb{R}$ be two Borel sets that are bounded away from zero. Define $\widehat{\nu}_\xi := \nu_\xi|_{B_\xi^c}$ and $\widehat{\nu}_\eta := \nu_\eta|_{B_\eta^c}$ as the restrictions of ν_ξ and ν_η on $\mathbb{R} \setminus B_\xi$ and $\mathbb{R} \setminus B_\eta$, respectively, and denote by $\widehat{\xi}$ and $\widehat{\eta}$ the independent Lévy processes*

$$\widehat{\xi}_t = \xi_t - \sum_{\substack{0<s\le t \\ \Delta\xi_s \in B_\xi}} \Delta\xi_s, \quad \widehat{\eta}_t = \eta_t - \sum_{\substack{0<s\le t \\ \Delta\eta_s \in B_\eta}} \Delta\eta_s$$

with Lévy measures $\widehat{\nu}_\xi$ and $\widehat{\nu}_\eta$, respectively. Then

$$\operatorname{supp}(V_{q,\widehat{\xi},\widehat{\eta}}) \subseteq \operatorname{supp}(V_{q,\xi,\eta}).$$

Proof. With positive probability we have $\{\xi_t, \eta_t, 0 \le t \le \tau\} = \{\widehat{\xi}_t, \widehat{\eta}_t, 0 \le t \le \tau\}$ for any realization of τ. Hence the claim follows. $\square$

Recall from Remark 1.15 that for a Lévy process η with characteristic triplet $(\sigma_\eta^2, \nu_\eta, \gamma_\eta)$ and a function $f : [0,t] \to \mathbb{R}$ which is integrable in the sense of Rajput and Rosinski [52, p. 460], the random variable $\int_0^t f(s)\,\mathrm{d}\eta_s$ is infinitely divisible with characteristic triplet $(\sigma_f^2, \nu_f, \gamma_f)$ given by (1.3) – (1.5). From this we derive the following:

Lemma 2.6. *Let η be a Lévy process with characteristic triplet $(\sigma_\eta^2, \nu_\eta, \gamma_\eta)$ and $f : [0,t] \to \mathbb{R}$ be bounded and Borel measurable such that f is not Lebesgue almost everywhere equal to zero. Denote the characteristic triplet of $\int_0^t f(s)\,\mathrm{d}\eta_s$ by $(\sigma_f^2, \nu_f, \gamma_f)$. Then the following are true:*

(i) The Lévy process with characteristic triplet $(\sigma_f^2, \nu_f, \gamma_f)$ is of finite variation if and only if η is of finite variation. In that case, the corresponding drifts γ_f^0 and γ_η^0 are related by $\gamma_f^0 = \gamma_\eta^0 \int_0^t f(s)\mathrm{d}s$.

(ii) If $0 \in \operatorname{supp}(\nu_\eta)$, then $0 \in \operatorname{supp}(\nu_f)$.

(iii) ν_f is infinite if and only if ν_η is infinite.

If, additionally, f is strictly positive on $[0,t]$, then

(iv) $\nu_f((0,\infty)) > 0$ *if and only if* $\nu_\eta((0,\infty)) > 0$, *and similarly for* $(-\infty, 0)$.

Proof. Parts (ii), (iii) and (iv) follow directly from (1.4). For the proof of (i), by measure theoretic induction it follows from (1.4) that

$$\int_{\mathbb{R}} g(x)\,\nu_f(\mathrm{d}x) = \int_0^t \int_{\mathbb{R}} g(f(s)x)\,\nu_\eta(\mathrm{d}x)\,\mathrm{d}s \tag{2.3}$$

for any Borel measurable function $g : \mathbb{R} \to [0,\infty)$ satisfying $g(0) = 0$, and similarly for any Borel measurable function $g : \mathbb{R} \to \mathbb{R}$ with $g(0) = 0$ for which the integrals exist. Applying (2.3) to $g(x) = |x|\mathbb{1}_{[-1,1]}(x)$ then gives

$$\int_{-1}^1 |y|\nu_f(\mathrm{d}y) = \int_0^t |f(s)| \int_{-1/|f(s)|}^{1/|f(s)|} |x|\,\nu_\eta(\mathrm{d}x)\,\mathrm{d}s,$$

from which, in combination with (1.3), the "only if" part readily follows. For the converse, observe that the right-hand side of the above equation can be bounded by

$$\int_0^t |f(s)|\mathrm{d}s \int_{-1}^1 |x|\,\nu_\eta(\mathrm{d}x) + \int_0^t |f(s)|\mathbb{1}_{\{|f(s)|<1\}} \frac{1}{|f(s)|} \nu_\eta(\mathbb{R} \setminus [-1,1])\,\mathrm{d}s,$$

which is finite if η is of finite variation since we assumed boundedness of f. Finally, the expression for the drift γ_f^0 follows from (1.5), (2.3) and the fact that $\gamma_f^0 = \gamma_f - \int_{-1}^1 y\,\nu_f(\mathrm{d}y)$ and similarly for γ_η^0. Observe that (i) could have similarly been derived by a direct application of Theorem 2.10 and Equation (2.16) in [57]. □

We can now prove Theorem 2.3.

Proof of Theorem 2.3. (i) In the case $\eta \equiv 0$, the integrator induces the zero measure, yielding $V = 0$ and thus $\operatorname{supp}(V) = \{0\}$.

(iii) (a,b) First, assume that η is of infinite variation. By conditioning on τ and ξ, as mentioned in the preliminaries, we find

$$P(V \in B) = \int_0^\infty \int_{D([0,\infty),\mathbb{R})} P\Big(\int_0^t \mathrm{e}^{-f(s-)}\,d\eta_s \in B\Big)\,P_\xi(\mathrm{d}f)P_\tau(\mathrm{d}t), \tag{2.4}$$

for all $B \in \mathcal{B}_1$. It follows from Lemma 2.6 that $\int_0^t \mathrm{e}^{-f(s-)}\mathrm{d}\eta_s$ is infinitely divisible with $\int_{-1}^1 |x|\nu_f(\mathrm{d}x) = \infty$ or $\sigma_f^2 > 0$. Thus, [58, Thm. 24.10(i)] implies $P(\int_0^t \mathrm{e}^{-f(s-)}\mathrm{d}\eta_s \in B) > 0$ for all open $B \subseteq \mathbb{R}$. Together with (2.4), this yields $\operatorname{supp}(V) = \mathbb{R}$.

If η is of finite variation with $0 \in \operatorname{supp}(\nu_\eta)$ and $\nu_\eta(\mathbb{R}_+), \nu_\eta(\mathbb{R}_-) > 0$, a similar argument is applicable. Conditioning as in (2.4), we find by Lemma 2.6 that in this case $0 \in \operatorname{supp}(\nu_f)$, as well as $\nu_f(\mathbb{R}_+), \nu_f(\mathbb{R}_-) > 0$. Therefore, $P(\int_0^t \mathrm{e}^{-f(s-)}\mathrm{d}\eta_s \in B) > 0$ for all open $B \subseteq \mathbb{R}$ by [58, Thm. 24.10(ii)] and thus $\operatorname{supp}(V) = \mathbb{R}$ as claimed.

(iv) when $0 \in \operatorname{supp}(\nu_\eta)$:
Let η be as in (iv) and assume additionally that 0 is in the support of ν_η. Assume further that $\operatorname{supp}(\nu_\eta) \subseteq [0,\infty)$, the case $\operatorname{supp}(\nu_\eta) \subseteq (-\infty, 0]$ following by symmetry. Conditioning as in (2.4), Lemma 2.6 yields that $\int_0^t \mathrm{e}^{-f(s-)}\mathrm{d}\eta_s$ (more precisely, the Lévy process

corresponding to it) is of finite variation with $0 \in \operatorname{supp}(\nu_f)$, $\nu_f(\mathbb{R}_+) > 0 = \nu_f(\mathbb{R}_-)$ and drift $\gamma_\eta^0 \int_0^t \mathrm{e}^{-f(s-)}\,\mathrm{d}s$. By [58, Thm. 24.10(iii)] we find that

$$\operatorname{supp}\Big(\int_0^t \mathrm{e}^{-f(s-)}\mathrm{d}\eta_s\Big) = \Big[\gamma_\eta^0 \int_0^t \mathrm{e}^{-f(s-)}\mathrm{d}s, \infty\Big). \tag{2.5}$$

As zero is included in the support of τ, Equation (2.4) shows that $\operatorname{supp}(V) \supseteq [0,\infty)$. Since $V \geq 0$ a.s. when η is a subordinator, we also get the reverse inequality, so that $\operatorname{supp}(V) = [0,\infty)$ when η is a subordinator.

If η is not a subordinator, by assumption we necessarily have $\gamma_\eta^0 < 0$. If then ξ is a subordinator with strictly positive drift γ_ξ^0, then $1/\gamma_\xi^0 \in \operatorname{supp}(\int_0^\infty \mathrm{e}^{-\xi_{s-}}\,\mathrm{d}s)$ by [6, Lem. 1] and (2.4) and (2.5) show $\operatorname{supp}(V) \supseteq [\gamma_\eta^0/\gamma_\xi^0, \infty)$, as τ may become arbitrary large. On the other hand,

$$V_{q,\xi,\eta} \geq \gamma_\eta^0 \int_0^\tau \mathrm{e}^{-\xi_{s-}}\,\mathrm{d}s \geq \gamma_\eta^0 \int_0^\infty \mathrm{e}^{-\gamma_\xi^0 s}\,\mathrm{d}s = \gamma_\eta^0/\gamma_\xi^0, \tag{2.6}$$

showing the converse inequality when ξ is a subordinator with strictly positive drift.

Finally, assume that η is not a subordinator (hence $\gamma_\eta^0 < 0$) and ξ is not a subordinator with strictly positive drift. If ξ does not drift to infinity a.s., then $\int_0^\infty \mathrm{e}^{-\xi_{s-}}\,\mathrm{d}s = +\infty$ by [27, Thm. 2], and if ξ drifts to infinity a.s., then $\int_0^\infty \mathrm{e}^{-\xi_{s-}}\,\mathrm{d}s$ is finite but unbounded by [6, Lem. 1]. As τ may become arbitrarily large, we conclude in both cases from (2.4) and (2.5) that $\operatorname{supp}(V) = \mathbb{R}$ in this case.

(ii), (iii) (c), (iv) when $\gamma_\eta^0 \neq 0$:
Let η be of finite variation with non-zero drift γ_η^0. By the cases already proved we can additionally assume that $0 \notin \operatorname{supp}(\nu_\eta)$, in particular $\nu_\eta(\mathbb{R}) < \infty$. By symmetry, we can further assume without loss of generality that $\gamma_\eta^0 > 0$. We distinguish in the following whether ξ is a subordinator with strictly positive drift or not.

Case 1: Assume ξ is a subordinator with strictly positive drift γ_ξ^0.
If $\nu_\eta(\mathbb{R}_+) = 0$, then $V_{q,\xi,\eta} \leq \gamma_\eta^0/\gamma_\xi^0$ by the same estimates that lead to (2.6), so that $\operatorname{supp}(V) \subseteq (-\infty, \gamma_\eta^0/\gamma_\xi^0]$ in this case. If additionally $\nu_\eta(\mathbb{R}_-) > 0$, choose a constant $K > 0$ such that $\nu_\eta([-K,0)) > 0$ and construct $\widehat{\eta}$ from η by subtracting the jumps that are less than $-K$ of η. Then $V_{0,\xi,\widehat{\eta}}$ exists by [27, Thm. 2] and from [6, Thm. 1(iii)] together with Lemmas 2.4 and 2.5 we obtain

$$\operatorname{supp}(V_{q,\xi,\eta}) \supseteq \operatorname{supp}(V_{q,\xi,\widehat{\eta}}) \supseteq \operatorname{supp}(V_{0,\xi,\widehat{\eta}}) = (-\infty, \gamma_\eta^0/\gamma_\xi^0],$$

which together with the estimate above gives the desired $\operatorname{supp}(V) = (-\infty, \gamma_\eta^0/\gamma_\xi^0]$, so that parts of (iv) are proved.

If $\nu_\eta(\mathbb{R}_+) = \nu_\eta(\mathbb{R}_-) = 0$ (as in (ii)), then η is deterministic, $V_{0,\xi,\eta}$ converges and, unless also ξ is deterministic,

$$\operatorname{supp}(V_{q,\xi,\eta}) \supseteq \operatorname{supp}(V_{0,\xi,\eta}) = [0, \gamma_\eta^0/\gamma_\xi^0]$$

by [6, Lem. 1] and Lemma 2.4, so that together with the previous upper bound we obtain $\operatorname{supp}(V) = [0, \gamma_\eta^0/\gamma_\xi^0]$. If also $\xi_t = \gamma_\xi^0 t$ is deterministic, then $\gamma_\eta^0 \int_0^\tau \mathrm{e}^{-\gamma_\xi^0 s}\,\mathrm{d}s = (1 - \mathrm{e}^{-\tau\gamma_\xi^0})\gamma_\eta^0/\gamma_\xi^0$, giving the same support also in this case.

If $\nu_\eta(\mathbb{R}_+) > 0 = \nu_\eta(\mathbb{R}_-)$ (as in parts of (iv)), then η is a subordinator and hence $V \geq 0$. Choose $K > 0$ such that $\nu_\eta((0, K]) > 0$ and construct $\widehat{\eta}$ from η by subtracting the jumps from η that are greater than K. Again, $V_{0,\xi,\widehat{\eta}}$ exists and its support is $[0, \infty)$ by [6, Thm. 1(ii)] unless ξ is deterministic, so that $\mathrm{supp}(V) = [0, \infty)$ in this case. If $\xi_t = \gamma_\xi^0 t$ is deterministic, then $\mathrm{supp}(V) \supseteq \mathrm{supp}(V_{0,\xi,\eta}) = [\gamma_\eta^0/\gamma_\xi^0, \infty)$ by [6, Thm. 1(ii)]. On the other hand, the assumption $0 \notin \mathrm{supp}(\nu_\eta)$ implies $\nu_\eta(\mathbb{R}) < \infty$, so that with positive probability, η does not jump before time τ. This implies $\mathrm{supp}(V_{q,\gamma_\xi^0 t,\eta}) \supseteq \mathrm{supp}(V_{q,\gamma_\xi^0 t,\gamma_\eta^0 t}) = [0, \gamma_\eta^0/\gamma_\xi^0]$, so that altogether, $\mathrm{supp}(V) = [0, \infty)$ also when ξ is deterministic.

If $\nu_\eta(\mathbb{R}_+), \nu_\eta(\mathbb{R}_-) > 0$ (as in (iii) (c)), choose $K > 0$ such that $\nu_\eta([-K, 0)), \nu_\eta([0, K]) > 0$ and construct $\widehat{\eta}$ from η by subtracting the jumps of absolute size greater than K. Then $\mathrm{supp}(V_{0,\xi,\widehat{\eta}}) = \mathbb{R}$ by [6, Thm. 1(i)] and $\mathrm{supp}(V) = \mathbb{R}$ follows as before using Lemmas 2.4 and 2.5.

Case 2: Assume ξ is not a subordinator with strictly positive drift.
If ξ_t does not drift a.s. to ∞, then $\int_0^\infty \mathrm{e}^{-\xi_{s-}}\, \mathrm{d}s = \infty$ a.s. (cf. [27, Thm. 2]), and if ξ_t drifts a.s. to ∞, then $\int_0^\infty \mathrm{e}^{-\xi_{s-}}\, \mathrm{d}s$ is finite a.s., but has unbounded support by [6, Lem. 1]. In both cases, for any $C > 0$ there is some $t(C) \geq 0$ such that $\int_0^{t(C)} \mathrm{e}^{-\xi_{s-}}\, \mathrm{d}s \geq C$ on a set Ω_C with positive probability. Since $t \mapsto \int_0^t \mathrm{e}^{-\xi_{s-}}\, \mathrm{d}s$ is pathwise continuous and increasing, on the set Ω_C it takes all values in $[0, C]$ when t runs through $[0, t(C)]$. Since τ is independent of (ξ, η) and has a strictly positive density on $[0, \infty)$, it follows that $\mathrm{supp}(\int_0^\tau \mathrm{e}^{-\xi_{s-}}\, \mathrm{d}s) \supset [0, C]$ for each $C > 0$ and hence $\mathrm{supp}(\int_0^\tau \mathrm{e}^{-\xi_{s-}}\, \mathrm{d}s) = [0, \infty)$. This finishes the proof of (ii).

If $\nu_\eta(\mathbb{R}_-) > 0 = \nu_\eta(\mathbb{R}_+)$ (as in parts of (iv)), choose $a < 0$ such that $\nu_\eta((2a, a)) > 0$. Choose $\varepsilon \in (0, 1)$ such that $\sup_{s \in [0,\varepsilon]} |\xi_s| \leq 1$ has positive probability (possible since ξ has càdlàg paths). Then for each $n \in \mathbb{N}$, the probability that $\sup_{s \in [0,\varepsilon]} |\xi_s| \leq 1$ and that simultaneously η has exactly n jumps of size in $(2a, a)$ on $(0, \varepsilon)$ and no other jumps in this interval is strictly positive. On the corresponding set we have

$$2na\mathrm{e} + \varepsilon\gamma_\eta^0/\mathrm{e} \leq \int_0^\varepsilon \mathrm{e}^{-\xi_{s-}}\, \mathrm{d}\eta_s \leq na\mathrm{e}^{-1} + \mathrm{e}\gamma_\eta^0 \varepsilon.$$

In particular, by choosing n sufficiently large, we see that for any given $L < 0$, the probability that $\int_0^\varepsilon \mathrm{e}^{-\xi_{s-}}\, \mathrm{d}\eta_s < L$ is strictly positive. Using as before that $\int_\varepsilon^\infty \mathrm{e}^{-\xi_{s-}}\, \mathrm{d}s$ is unbounded or a.s. equal to $+\infty$, together with the fact that the probability of η having no jumps in the interval $[\varepsilon, \tau]$ is strictly positive, an application of the intermediate value theorem as before shows that $\mathrm{supp}(V_{q,\xi,\eta}) = \mathbb{R}$ in this case.

Finally, if $\nu_\eta(\mathbb{R}_+) > 0 = \nu_\eta(\mathbb{R}_-)$ (as in parts of (iv)) or $\nu(\mathbb{R}_+), \nu(\mathbb{R}_-) > 0$ (as in (iii) (c)), construct $\widehat{\eta}$ from η by omitting the positive jumps of η. Then $\mathrm{supp}(V_{q,\xi,\eta}) \supseteq \mathrm{supp}(V_{q,\xi,\widehat{\eta}})$ by Lemma 2.5 and the claim follows from the cases above.

(v) Assume that η is a compound Poisson process with $0 \notin \mathrm{supp}(\nu_\eta)$. Let $T_1, T_2, \ldots$ denote the jump times of η and $(M_t)_{t \geq 0}$ denote the underlying Poisson process, such that $\eta_t = \sum_{j=1}^{M_t} B_j$ with the i.i.d. random variables $B_1, B_2, \ldots$ following the jump distribution of η. Consider now the random variables $X_k = \xi_{T_k} - \xi_{T_{k-1}}$ for $k \in \mathbb{N}$, setting $T_0 = 0$. Then $X_1, X_2, \ldots$ are i.i.d. and, by the strong Markov property, have the same distribution as ξ_{T_1}. Note that T_1 is exponentially distributed with parameter $\nu_\eta(\mathbb{R})$. With probability one it now holds that

$$\int_0^\tau \mathrm{e}^{-\xi_{s-}}\mathrm{d}\eta_s = \int_0^\tau \mathrm{e}^{-\xi_s}\mathrm{d}\eta_s = \sum_{j=1}^{M_\tau} \mathrm{e}^{-\xi_{T_j}} B_j = \sum_{j=1}^{M_\tau} \mathrm{e}^{-\sum_{k=1}^{j} X_k} B_j = \sum_{j=1}^{M_\tau} \Big(\prod_{k=1}^{j} \mathrm{e}^{-X_k} \Big) B_j.$$

Considering the i.i.d. random variables $A_k = e^{-X_k}$, $k \in \mathbb{N}$, we can thus write

$$V = \sum_{j=1}^{M_\tau} \Big(\prod_{k=1}^{j} A_k \Big) B_j \text{ a.s.}$$

where the random variables $A_1, A_2, \dots$ and $B_1, B_2, \dots$ are mutually independent. Conditioning on $\{M_\tau = n\}$ now leads to (2.1), where $A_k \overset{d}{=} \mathrm{e}^{-\xi_{T_1}}$ implies (2.2).

Finally, it remains to deduce the explicit form of the set depending on the process ξ. Writing $\xi_{T_1} = \int_0^{T_1} \mathrm{e}^0 \mathrm{d}\xi_s$, we see from the cases (ii)–(iv) already proved that, provided ξ is not a compound Poisson process with $0 \notin \operatorname{supp}(\nu_\xi)$, then $\operatorname{supp}(-\xi_{T_1}) = \{0\}$ if ξ is the zero-process, $\operatorname{supp}(-\xi_{T_1}) = (-\infty, 0]$ if ξ is a non-zero subordinator, $\operatorname{supp}(-\xi_{T_1}) = [0, \infty)$ if $-\xi$ is a non-zero subordinator, and $\operatorname{supp}(-\xi_{T_1}) = \mathbb{R}$ otherwise.

If $\operatorname{supp}(\xi_{T_1}) = \mathbb{R}$, it follows immediately from (2.1) that $\operatorname{supp}(V) = [0, \infty)$, $(-\infty, 0]$ or $\mathbb{R}$ whenever η is a subordinator, the negative of a subordinator, or has two-sided jumps, respectively. Similarly, the simplification when $\xi \equiv 0$ is immediate. Whenever $-\xi$ is a subordinator with $0 \in \operatorname{supp}(\nu_\xi)$ or non-zero drift, we clearly have $\operatorname{supp}(\mathrm{e}^{-\xi_{T_1}}) = [1, \infty)$. The corresponding formulas for $\operatorname{supp}(V)$ whenever η or $-\eta$ is a subordinator then follow immediately from (2.1). If η has jumps of both signs, define $z_+ = \inf(\operatorname{supp}(\nu_\eta) \cap \mathbb{R}_+) > 0$ and $z_- = \sup(\operatorname{supp}(\nu_\eta) \cap \mathbb{R}_-) < 0$. For a given number $x \in \mathbb{R}$, choose $n \in \mathbb{N}$ such that $x - (n-1)z_- \geq z_+$, as well as

$$b_1 = \dots = b_{n-1} = z_-, \; a_1 = \dots = a_{n-1} = 1, \; b_n = z_+, \; a_n = \frac{x - (n-1)z_-}{z_+}. \tag{2.7}$$

Then $b_1, \dots, b_n \in \operatorname{supp}(\nu_\eta)$, $a_n \geq 1$ and $\ln(a_1), \dots, \ln(a_n) \in \Xi$ such that we obtain $\sum_{j=1}^n \Big(\prod_{k=1}^j a_k \Big) b_j = x$, i.e. $x \in \operatorname{supp}(V)$ by (2.1) and thus $\operatorname{supp}(V) = \mathbb{R}$ as claimed.

Finally, assume that ξ is a non-zero subordinator or that it is a compound Poisson process with $\nu_\xi(\mathbb{R}_+) > 0$. Construct $\widehat{\eta}$ from η by deleting all jumps whose absolute size is greater than some constant K but such that $\widehat{\eta}$ still has positive and/or negative jumps if η does, and construct $\widehat{\xi}$ from ξ by deleting all its negative jumps. Then $V_{0,\widehat{\xi},\widehat{\eta}}$ exists by [27, Thm. 2], and by [6, Thm. 1] its support is $[0, \infty)$, $(-\infty, 0]$ or $\mathbb{R}$, respectively, depending if η or $-\eta$ or neither of them is a subordinator. The claim then follows from Lemmas 2.4 and 2.5 using $\operatorname{supp}(V_{q,\xi,\eta}) \supseteq \operatorname{supp}(V_{q,\widehat{\xi},\widehat{\eta}}) \supseteq \operatorname{supp}(V_{0,\widehat{\xi},\widehat{\eta}})$. $\square$

Remark 2.7. An alternative proof of some of the cases considered in Theorem 2.3 can be based on a general result regarding the support of solutions of certain random fixed point equations. Namely, as shown in [17, Thm. 2.5.5(1)], if the random variable Z is a solution to the random fixed point equation $Z \overset{d}{=} AZ + B$ for two real-valued random variables A, B, where Z is independent of (A, B), and A and B are such that $P(Ax + B = x) < 1$ for every $x \in \mathbb{R}$, $A \geq 0$ a.s., $P(0 < A < 1) > 0$ and $P(A > 1) > 0$, then $\operatorname{supp}(X)$ is either a half-line or $\mathbb{R}$. Using Theorem 2.31 in

Section 2.3, this result can be applied in our situation whenever neither ξ nor $-\xi$ are subordinators, in which case it is then enough to determine the left and right endpoints of the support.

After obtaining the general result for the support of the killed exponential functional in Theorem 2.3, we will now study the case where both ξ and η are compound Poisson processes more closely. Consider the following two motivating examples.

Example 2.8. Let M and N be two independent Poisson processes. Then $\int_0^t 2^{M_{s-}}\, dN_s$, $t>0$, and $\int_0^\tau 2^{M_{s-}}\, dN_s$ have support $\mathbb{N}_0$. More generally, we have for $a>1$ that

$$\operatorname{supp}\Big(\int_0^\tau a^{M_{s-}}\, dN_s\Big) = \Big\{\sum_{k=0}^N n_k a^k,\ \ N, n_k \in \mathbb{N}_0\Big\},$$

which is neither an interval nor the union of an interval and $\{0\}$.

Example 2.9. Let η be a Poisson process and let ξ be a compound Poisson process whose Lévy measure is supported on the set $-S=-1-C$, where C denotes the classical middle third Cantor set. Thus, both $\operatorname{supp}(\nu_\eta)=\{1\}$ and $\operatorname{supp}(\nu_\xi)$ are bounded away from zero and do not contain an interval. Further, $\nu_\xi(\mathbb{R}_+)=0$. By [18, Cor. 3.4], we have that $C+C=[0,2]$ such that in particular $[2,4]\subseteq \Xi$. By iteration we further see that $\Xi=\{0\}\cup S\cup[2,\infty)$ from which we derive by (2.1) that

$$\operatorname{supp}(V) = \overline{\left\{\sum_{j=1}^n c_j,\ c_j\in\{1\}\cup \mathrm{e}^S\cup[\mathrm{e}^2,\infty), n\in\mathbb{N}_0\right\}}$$

as η always jumps by 1. In particular $\operatorname{supp}(V)$ contains the unbounded interval $[\mathrm{e}^2,\infty)$.

The following proposition collects sufficient conditions for $\operatorname{supp}(V)$ to contain an unbounded interval. In particular, we see that if $\operatorname{supp}(\nu_\xi)$ or $\operatorname{supp}(\nu_\eta)$ contains an interval, then $\operatorname{supp}(V)$ contains an unbounded interval. Recalling the results of Theorem 2.3, it suffices to consider the case when both ξ and η are compound Poisson processes with $0\notin\operatorname{supp}(\nu_\xi)$ and $0\notin\operatorname{supp}(\nu_\eta)$, respectively, as well as $\nu_\xi(\mathbb{R}_+)=0$. Denote by $\lfloor x\rfloor=\max\{z\in\mathbb{Z}: z\le x\}$ the floor function of $x\in\mathbb{R}$.

Proposition 2.10. *Consider the killed exponential functional* $V:=V_{q,\xi,\eta}=\int_0^\tau \mathrm{e}^{-\xi_{s-}}\,\mathrm{d}\eta_s$ *for* $q>0$ *and two independent compound Poisson processes* ξ *and* η *with Lévy measures* ν_ξ *and* ν_η*, respectively, such that* $0\notin\operatorname{supp}(\nu_\eta)$*,* $0\notin\operatorname{supp}(\nu_\xi)$*, and* $\nu_\xi(\mathbb{R}_+)=0$*. Recall that*

$$\operatorname{supp}(\eta_\tau) = \overline{\left\{\sum_{j=1}^n b_j : b_j\in\operatorname{supp}(\nu_\eta), n\in\mathbb{N}_0\right\}}$$

from Theorem 2.3 (v).

(i) Assume there are $\beta<\alpha<0$ *such that* $[\beta,\alpha]\subseteq\operatorname{supp}(\nu_\xi)$ *and set* $k:=\lfloor\frac{\alpha}{\beta-\alpha}\rfloor+1$*, then*

$$\operatorname{supp}(V)\supseteq\begin{cases}\operatorname{supp}(\eta_\tau)\cup[\mathrm{e}^{-k\alpha}\inf\operatorname{supp}(\nu_\eta),\infty), & \textit{if } \eta \textit{ is a subordinator},\\ \operatorname{supp}(\eta_\tau)\cup(-\infty,\mathrm{e}^{-k\alpha}\sup\operatorname{supp}(\nu_\eta)], & \textit{if } -\eta \textit{ is a subordinator},\\ \mathbb{R} & \textit{otherwise.}\end{cases}$$

(ii) *Assume η is a subordinator and there are $0 < \alpha < \beta$ such that $[\alpha, \beta] \subseteq \operatorname{supp}(\nu_\eta)$. Set $k := \lfloor \frac{\alpha}{\beta - \alpha} \rfloor + 1$, then*

$$\operatorname{supp}(V) \supseteq \begin{cases} \{0\} \cup [\alpha, \infty), & \textit{if } \ln(\beta) - \ln(\alpha) \geq -\sup \operatorname{supp}(\nu_\xi), \\ \{0\} \cup \bigcup_{\ell=1}^{k-1} [\ell\alpha, \ell\beta] \cup [k\alpha, \infty), & \textit{otherwise.} \end{cases}$$

(iii) *Assume $-\eta$ is a subordinator and there are $\beta < \alpha < 0$ such that $[\beta, \alpha] \subseteq \operatorname{supp}(\nu_\eta)$. Set $k := \lfloor \frac{\alpha}{\beta - \alpha} \rfloor + 1$, then*

$$\operatorname{supp}(V) \supseteq \begin{cases} (-\infty, \alpha] \cup \{0\}, & \textit{if } \ln(-\beta) - \ln(-\alpha) \geq -\sup \operatorname{supp}(\nu_\xi), \\ (-\infty, k\alpha] \cup \bigcup_{\ell=1}^{k-1} [\ell\beta, \ell\alpha] \cup \{0\}, & \textit{otherwise.} \end{cases}$$

(iv) *Assume $\nu_\eta(\mathbb{R}_-) \neq 0 \neq \nu_\eta(\mathbb{R}_+)$, and there are $0 < \alpha < \beta$ such that $[\alpha, \beta] \subseteq \operatorname{supp}(\nu_\eta)$ or $\beta < \alpha < 0$ such that $[\beta, \alpha] \subseteq \operatorname{supp}(\nu_\eta)$. Then*

$$\operatorname{supp}(V) = \mathbb{R}.$$

(v) *Assume $\nu_\eta(\mathbb{R}_-) \neq 0 \neq \nu_\eta(\mathbb{R}_+)$ and that there are numbers $z_1 < 0$, $z_2 > 0$ in $\operatorname{supp}(\nu_\eta)$ such that $\frac{z_2}{z_1}$ is irrational. Then*

$$\operatorname{supp}(V) = \mathbb{R}.$$

Proof. (i) First, let $\operatorname{supp}(\nu_\eta) \subseteq \mathbb{R}_+$, i.e. η is a subordinator. As, by assumption, the probability that the killing occurs before the first jump of ξ is positive, we have $\operatorname{supp}(\eta_\tau) \subset \operatorname{supp}(V)$. Recalling the structure of $\operatorname{supp}(V)$ from Theorem 2.3 (v), ξ being a compound Poisson process implies that

$$\Xi = \overline{\left\{ \sum_{j=1}^{N} -x_j, x_j \in \operatorname{supp}(\nu_\xi), N \in \mathbb{N}_0 \right\}} \subseteq \mathbb{R}_+.$$

As, by assumption, $[\beta, \alpha] \subseteq \operatorname{supp}(\nu_\xi)$, it follows directly that $[-\alpha, -\beta] \subseteq \Xi$ here, as well as $[-n\alpha, -n\beta] \subseteq \Xi$ for $n \in \mathbb{N}$. In particular, if we have $n \geq k = \lfloor \frac{\alpha}{\beta - \alpha} \rfloor + 1$, it follows that $-(n+1)\alpha < -n\beta$, implying that individual intervals intersect and that, therefore, $[-k\alpha, \infty) \subseteq \Xi$. Choosing $n = 1$, $b_1 = \inf \operatorname{supp}(\nu_\eta)$ and letting a_1 run through $[-k\alpha, \infty)$, we find from (2.1) that $[\mathrm{e}^{-k\alpha} \inf \operatorname{supp}(\nu_\eta), \infty) \subseteq \operatorname{supp}(V)$. If $-\eta$ is a subordinator instead, the claim follows by symmetry.

If η has jumps of both signs, denote $z_+ = \inf(\operatorname{supp}(\nu_\eta) \cap \mathbb{R}_+)$ and $z_- = \sup(\operatorname{supp}(\nu_\eta) \cap \mathbb{R}_-)$ as in the proof of Theorem 2.3 and, for a given $x \in \mathbb{R}$, choose $a_1, \ldots, a_n, b_1, \ldots, b_n$ as in (2.7). Then $\ln(a_1), \ldots, \ln(a_{n-1}) \in \Xi$ and also $\ln(a_n) \in \Xi$ if $n \in \mathbb{N}$ is chosen sufficiently large. From (2.1) we can thus conclude that $x \in \operatorname{supp}(V)$ and, therefore, $\operatorname{supp}(V) = \mathbb{R}$ as claimed.

(ii) Note that the probability that the killing occurs before the first jump of ξ is positive and so is the probability that, additionally, η jumps n times for some $n \in \mathbb{N}$ in the time interval $[0,\tau]$. As $\mathrm{supp}(\nu_\eta)$ contains the interval $[\alpha,\beta]$, it follows that

$$\begin{aligned}\mathrm{supp}(V) &\supseteq \Big\{ \sum_{j=1}^n b_j \Big| n \in \mathbb{N}_0, b_j \in \mathrm{supp}(\nu_\eta) \Big\} \\ &\supseteq \Big\{ \sum_{j=1}^n b_j \Big| n \in \mathbb{N}_0, b_j \in [\alpha,\beta] \Big\} = \{0\} \cup \Big(\bigcup_{\ell=1}^{k-1} [\ell\alpha, \ell\beta] \Big) \cup [k\alpha, \infty),\end{aligned}$$

where we set again $k = \lfloor \frac{\alpha}{\beta-\alpha} \rfloor + 1$. To show that the interval contained in $\mathrm{supp}(V)$ is considerably larger whenever $\ln(\beta) - \ln(\alpha) \geq -\sup \mathrm{supp}(\nu_\xi) =: -y_-$ is satisfied, choose $n = 1$, as well as $a_1 = \mathrm{e}^{-my_-}$ for $m \in \mathbb{N}$ and $b_1 \in [\alpha,\beta]$. From (2.1) it now follows directly that $\{\mathrm{e}^{-my_-}[\alpha,\beta],\ m \in \mathbb{N}\} \subseteq \mathrm{supp}(V)$. Since $\ln(\beta) - \ln(\alpha) \geq -y_-$, we find that $\mathrm{e}^{-my_-}\alpha \leq \mathrm{e}^{-(m-1)y_-}\beta$ such that the intervals $\mathrm{e}^{-(m-1)y_-}[\alpha,\beta]$ and $\mathrm{e}^{-my_-}[\alpha,\beta]$ overlap. In particular, we obtain that $[\alpha,\infty) \cap \mathrm{supp}(V) = [\alpha,\infty)$, yielding the claim. Part (iii) follows by symmetry.

(v) Without loss of generality, assume that $z_1 = -1$ such that $z_2 > 0$ is irrational and let $x \bmod 1$ denote the quantity $x - \lfloor x \rfloor$ for a real number x. Then $z_2 \bmod 1$ is also irrational and the orbit of z_2 under the corresponding rotation in the circle group $([0,1),+)$, where $x_1 + x_2 = (x_1 + x_2) \bmod 1$, is dense (see e.g. [33, Prop. 1.3.3]), i.e.

$$\overline{\{nz_2 \bmod 1,\ n \in \mathbb{N}_0\}} = [0,1].$$

Recalling that the killing may occur before the first jump of the process ξ, it follows from (2.1) that

$$\mathrm{supp}(V) \supseteq \overline{\Big\{ \sum_{j=1}^n b_n \Big| n \in \mathbb{N}_0, b_n \in \mathrm{supp}(\nu_\eta) \Big\}} \supseteq \overline{\{n_1 z_1 + n_2 z_2 | n_1, n_2 \in \mathbb{N}_0\}}.$$

Observe that $z_1 = -1$ implies $z_2 \bmod 1 = z_2 + n_0 z_1$ for some $n_0 \in \mathbb{N}_0$, yielding that the set on the right-hand side must include the interval $[0,1]$. Further, it includes all translations of $[0,1]$ by numbers $z = \tilde{n}_1 z_1 + \tilde{n}_2 z_2$ with fixed $\tilde{n}_1, \tilde{n}_2 \in \mathbb{N}_0$, as can be seen from specifying the first $\tilde{n}_1 + \tilde{n}_2$ jumps. Thus, it follows that $\mathrm{supp}(V) = \mathbb{R}$.

As the existence of $z_1 < 0$ and $z_2 > 0$ with $\frac{z_2}{z_1} \in \mathbb{R} \backslash \mathbb{Q}$ is guaranteed if $\nu_\eta(\mathbb{R}_-) \neq 0 \neq \nu_\eta(\mathbb{R}_+)$ and either $\mathrm{supp}(\nu_\eta) \cap \mathbb{R}_+$ or $\mathrm{supp}(\nu_\eta) \cap \mathbb{R}_-$ contains an interval, Part (iv) follows immeadiately from the preceding argument. □

Example 2.11. Let ξ be a Poisson process and let η be a compound Poison process with Lévy measure $\nu_\eta = \delta_{-1} + \delta_{\sqrt{2}}$, where δ_x denotes the Dirac measure supported at $x \in \mathbb{R}$. Thus, both $\mathrm{supp}(\nu_\xi) = \{1\}$ and $\mathrm{supp}(\nu_\eta) = \{-1, \sqrt{2}\}$ are bounded away from zero and do not contain an interval. However, $\frac{\sqrt{2}}{-1} \in \mathbb{R} \setminus \mathbb{Q}$ and thus $\mathrm{supp}(V) = \mathbb{R}$ by Part (v) of Proposition 2.10.

2.2. Continuity Properties

The following section is concerned with continuity properties of $V_{q,\xi,\eta}$ and the absolute continuity of $V_{0,\xi,\eta}$. Since many of the results are derived by conditioning on the paths of ξ, we first treat the case of integrals of the form $\int_0^\infty f(s)\,\mathrm{d}\eta_s$ with a deterministic function f in Section 2.2.1 below. Here, we characterize continuity of its law and give various sufficient conditions for absolute continuity. In Section 2.2.2, we use some of the results to study the law of $\int_0^t \mathrm{e}^{-\xi_{s-}}\,\mathrm{d}\eta_s$ for fixed $t \geq 0$ arising from conditioning $V_{q,\xi,\eta}$ on $\tau = t$, which can be thought of as deterministic killing. The results are then applied to $V_{q,\xi,\eta}$ in Section 2.2.3, yielding a characterization of the continuity of its law, as well as various sufficient conditions for absolute continuity. Further, sufficient conditions for absolute continuity of the law of $V_{0,\xi,\eta}$ are given in Section 2.2.4. We start with some remarks on the conditioned integrals considered.

Since throughout this chapter we always restrict to independent Lévy processes ξ and η, the stochastic (semimartingale) integral $\int_0^t \mathrm{e}^{-\xi_{s-}}\,\mathrm{d}\eta_s$ given the path $\xi = f$ coincides with the semimartingale integral $\int_0^t \mathrm{e}^{-f(s-)}\,\mathrm{d}\eta_s$ of the deterministic function f with respect to η, which follows from the definition of the semimartingale integral as in Protter [50, Sect. II.4], e.g. when realising ξ and η on a product space. The semimartingale integral $\int_0^t \mathrm{e}^{-f(s-)}\,\mathrm{d}\eta_s$ then agrees with the corresponding stochastic integral in the sense of Rajput and Rosinski [52, p. 460] as both are limits in probability of integrals of simple functions, for which the corresponding integrals trivially agree.

Let η be a one-dimensional Lévy process with characteristic triplet $(\sigma_\eta^2, \nu_\eta, \gamma_\eta)$ and $f : [0,\infty) \to \mathbb{R}$ a deterministic Borel measurable function. We say that f is locally integrable, or more precisely, locally integrable with respect to the independently scattered random measure induced by η, if $f\mathbb{1}_{[0,t]}$ is integrable with respect to η in the sense of Rajput and Rosinski (cf. [52, p. 460], or also Sato [58, Def. 57.8]) for every $t \in (0,\infty)$. The corresponding integral over $[0,t]$ is denoted by $\int_0^t f(s)\,\mathrm{d}\eta_s$. Since $t \mapsto \int_0^t f(s)\,\mathrm{d}\eta_s$ defines an additive process, a version of the integral exists that has càdlàg paths (e.g. [58, Thm. 11.5]), and we shall always assume that such a version is chosen. We say that the improper integral $\int_0^\infty f(s)\,\mathrm{d}\eta_s$ exists, if f is locally integrable and $\int_0^t f(s)\,\mathrm{d}\eta_s$ converges in probability to a finite random variable as $t \to \infty$, equivalently by the independent increments property and the càdlàg paths, if it converges a.s. to a finite random variable as $t \to \infty$ and we denote the limit by $\int_0^\infty f(s)\,\mathrm{d}\eta_s$. A characterization of functions for which the improper integral $\int_0^\infty f(s)\,\mathrm{d}\eta_s$ exists can be found in Sato [56, Prop. 5.5] or [58, Prop. 57.13]. In particular, every locally bounded measurable function is locally integrable, and for functions with bounded support, integrability as defined in Rajput and Rosinski [52, p. 460] is equivalent to the existence of the improper integral.

2.2.1. Absolute Continuity of $\int_0^\infty f(s)\,\mathrm{d}\eta_s$

Recall that for two independent Lévy processes ξ and η, the integral $\int_0^t \mathrm{e}^{-\xi_{s-}}\,\mathrm{d}\eta_s$ conditional on a path $\xi = f$ is equal to the integral $\int_0^t \mathrm{e}^{-f(s-)}\,\mathrm{d}\eta_s$. Hence, the improper integral $V_{0,\xi,\eta} = \int_0^\infty \mathrm{e}^{-\xi_{s-}}\,\mathrm{d}\eta_s$ conditional on $\xi = f$ is equal to $\int_0^\infty \mathrm{e}^{-f(s-)}\,\mathrm{d}\eta_s$, such that the results in this section yield an important tool in studying the law of the exponential functional without killing. Whenever the improper integral $\int_0^\infty f(s)\,\mathrm{d}\eta_s$ exists, its distribution is infinitely divisible with characteristic triplet $(\sigma_f^2, \nu_f, \gamma_f)$, where $\sigma_f^2 = \sigma_\eta^2 \int_0^\infty f(s)^2\,\mathrm{d}s$

and ν_f is given by (1.4) with $t = \infty$, i.e.

$$\nu_f(B) = \int_0^\infty \int_{\mathbb{R}} \mathbb{1}_{B\setminus\{0\}}(f(s)x)\, \nu_\eta(\mathrm{d}x)\, \mathrm{d}s, \quad B \in \mathcal{B}_1. \tag{2.8}$$

Further, the characteristic exponent Ψ_f of $\int_0^\infty f(s)\, \mathrm{d}\eta_s$ is given by

$$\Psi_f(z) = \lim_{t\to\infty} \int_0^t \Psi_\eta(f(s)z)\, \mathrm{d}s, \tag{2.9}$$

cf. [58, Prop. 57.13]. We can hence apply continuity results for infinitely divisible distributions starting with a simple result characterizing continuity of $\int_0^\infty f(s)\, \mathrm{d}\eta_s$, i.e. when the distribution has no atoms.

Proposition 2.12 (Continuity of $\int f(s)\, \mathrm{d}\eta_s$)**.** *Let $\eta = (\eta_t)_{t\geq 0}$ be a one-dimensional Lévy process with characteristic triplet $(\sigma_\eta^2, \nu_\eta, \gamma_\eta)$ and $f : [0,\infty) \to \mathbb{R}$ be a deterministic Borel measurable function such that the improper integral $\int_0^\infty f(s)\, \mathrm{d}\eta_s$ exists and such that $f \neq 0$ on a set of positive Lebesgue measure. Then $\int_0^\infty f(s)\, \mathrm{d}\eta_s$ is continuous (i.e. has no atoms) if and only if*

$$\sigma_\eta^2 > 0, \quad \text{or} \quad \lambda(\{s \in [0,\infty) : f(s) \neq 0\}) \cdot \nu_\eta(\mathbb{R}) = \infty.$$

Proof. By [58, Thm. 27.4], $\int_0^\infty f(s)\, \mathrm{d}\eta_s$ is continuous if and only if $\sigma_f^2 > 0$ or $\nu_f(\mathbb{R}) = \infty$. Since f is not Lebesgue almost everywhere equal to zero, the claim follows by observing that $\sigma_f^2 = \sigma_\eta^2 \int_0^\infty f(s)^2\, \mathrm{d}s$ and $\nu_f(\mathbb{R}) = \lambda(\{s \in [0,\infty) : f(s) \neq 0\})\, \nu_\eta(\mathbb{R})$ by (2.8). ☐

In the following proposition, we collect some known results ensuring absolute continuity of infinitely divisible distributions. Here, we denote by $\Re(z)$ the real part of a complex number z.

Lemma 2.13 (Absolute continuity of infinitely divisible distributions)**.** *Let μ be an infinitely divisible distribution with characteristic triplet (σ^2, ν, γ) and characteristic exponent Ψ.*

(i) If Kallenberg's condition ([31, pp. 794–795])

$$\lim_{\varepsilon\downarrow 0} \varepsilon^{-2} |\ln \varepsilon|^{-1} \left(\sigma^2 + \int_{-\varepsilon}^{\varepsilon} x^2\, \nu(\mathrm{d}x)\right) = \infty, \tag{2.10}$$

or more generally the Hartman-Wintner condition ([29, pp. 287–288])

$$\lim_{|z|\to\infty} \frac{-\Re(\Psi(z))}{\ln(1+|z|)} = \infty \tag{2.11}$$

is satisfied, then μ is absolutely continuous with infinitely often differentiable density with all derivatives vanishing at infinity.

(ii) If

$$\lim_{\varepsilon\downarrow 0} \varepsilon^{-2} |\ln \varepsilon|^{-1} \left(\sigma^2 + \int_{-\varepsilon}^{\varepsilon} x^2\, \nu(\mathrm{d}x)\right) > 1/4, \tag{2.12}$$

or more generally

$$\liminf_{|z|\to\infty} \frac{-\Re(\Psi(z))}{\ln(1+|z|)} > 1/2, \tag{2.13}$$

then μ is absolutely continuous with square integrable density ([10, Cor. 3.6]).

(iii) If the absolutely continuous part ν_{ac} in the Lebesgue decomposition of ν is infinite, then μ is absolutely continuous ([58, Thm. 27.7]).

Kallenberg's condition is classical and his proof ([31, pp. 794–795]) shows that it implies the Hartman-Wintner condition (2.11), which in turn implies that the Fourier transform $\widehat{\mu}$ of μ satisfies $\int_{\mathbb{R}} |x|^k \, |\widehat{\mu}(x)| \, \mathrm{d}x < \infty$ for all $k \in \mathbb{N}$, giving (i). It has been noted by several authors that if the right-hand sides of (2.12) and (2.13) are replaced by > 1 (e.g. [20, p. 853], [36, p. 127] for (2.12) and Hartman and Wintner [29, pp. 794-795] themselves for (2.13)), then μ has a continuous and bounded density vanishing at infinity. The fact that the constant 1 can even be replaced by 1/4 and 1/2 (as done in (2.12) and (2.13) in (ii)) to ensure absolute continuity has been shown by Berger [10, Cor. 3.6]. In fact, Berger's proof shows that (2.12) implies (2.13), which in turn implies square integrability of the Fourier transform of μ, thus giving absolute continuity of μ with square integrable density (cf. [34, Thm. 11.6.1]). Part (iii) is an easy consequence of Sato [58, Thm. 27.7], by observing that the convolution of an absolutely continuous distribution with another distribution is again absolutely continuous.

An interesting feature of integrals of the form $\int_0^\infty f(s) \, \mathrm{d}\eta_s$ is that their distribution is often smoother than the original distribution. A well known example is $\int_0^\infty \mathrm{e}^{-as} \, \mathrm{d}\eta_s$ whenever $a > 0$ and η is a non-deterministic Lévy process such that the integral converges, which always gives a self-decomposable and hence absolutely continuous distribution. That this phenomenon can happen also with functions with compact support was exemplified by Nourdin and Simon [47, Thm. A], who showed that $\int_0^t \mathrm{e}^{-as} \, \mathrm{d}\eta_s$ will always be absolutely continuous whenever $a \neq 0$ and η is such that $\sigma_\eta^2 > 0$ or $\nu_\eta(\mathbb{R}) = \infty$. See also Bodnarchuk and Kulik [16, Prop. 2, Thm. 1], who even characterized when $\int_0^t \mathrm{e}^{-as} \, \mathrm{d}\eta_s$ has a bounded density for all $t > 0$. Berger [10, Lem. 4.1] showed that when η has infinite Lévy measure and $f : [0,t] \to \mathbb{R}$ is a C^1-diffeomorphism onto its range, then $\int_0^t f(s) \, \mathrm{d}\eta_s$ is absolutely continuous. Part (iii) below, which essentially is Remark 4.2 in Berger [10], generalizes this result. For the reader's convenience, we include his short proof.

Corollary 2.14 (Absolute continuity of $\int f(s) \, \mathrm{d}\eta_s$)**.** *Let $\eta = (\eta_t)_{t\geq 0}$ be a one-dimensional Lévy process with characteristic triplet $(\sigma_\eta^2, \nu_\eta, \gamma_\eta)$. Let $f : [0,\infty) \to \mathbb{R}$ be a deterministic Borel measurable function such that the improper integral $\int_0^\infty f(s) \, \mathrm{d}\eta_s$ exists. Assume that $f \neq 0$ on a set of positive Lebesgue measure. Denote by $\nu_\eta = \nu_{\eta,\mathrm{ac}} + \nu_{\eta,\mathrm{sing}}$ the Lebesgue decomposition of ν_η in absolutely continuous part $\nu_{\eta,\mathrm{ac}}$ and singular part $\nu_{\eta,\mathrm{sing}}$. Then each of the following conditions implies that $\int_0^\infty f(s) \, \mathrm{d}\eta_s$ is absolutely continuous with respect to Lebesgue measure:*

(i) The characteristic triplet of η satisfies

$$\lambda(\{s \in [0,\infty) : f(s) \neq 0\}) \cdot \liminf_{\varepsilon \downarrow 0} \varepsilon^{-2} |\ln \varepsilon|^{-1} \left(\sigma_\eta^2 + \int_{-\varepsilon}^{\varepsilon} x^2 \, \nu_\eta(\mathrm{d}x) \right) > \frac{1}{4}, \tag{2.14}$$

or more generally, the characteristic exponent Ψ_η satisfies

$$\lambda(\{s \in [0,\infty) : f(s) \neq 0\}) \cdot \liminf_{|z|\to\infty} \frac{-\Re(\Psi_\eta(z))}{\ln(1+|z|)} > \frac{1}{2}. \tag{2.15}$$

(ii) The absolutely continuous part $\nu_{\eta,\mathrm{ac}}$ of ν_η satisfies

$$\lambda(\{s \in [0,\infty) : f(s) \neq 0\}) \cdot \nu_{\eta,\mathrm{ac}}(\mathbb{R}) = \infty.$$

(iii) Preimages of Lebesgue nullsets $B \in \mathcal{B}_1^*$ under the mapping f are again Lebesgue nullsets (i.e. $\lambda(f^{-1}(B)) = 0$ for every $B \in \mathcal{B}_1^*$ with $\lambda(B) = 0$), and

$$\lambda(\{s \in [0,\infty) : f(s) \neq 0\}) \cdot \nu_\eta(\mathbb{R}) = \infty.$$

(iv) There is $b > 0$ such that η_b is absolutely continuous and the function f is constant and different from zero on an interval of length b.

The condition that preimages of Lebesgue nullsets under a mapping are again Lebesgue nullsets is called the Lusin(N^{-1}) condition in the literature. The first condition in Corollary 2.14 (iii) therefore means that $f_{|f^{-1}(\mathbb{R}^*)} : f^{-1}(\mathbb{R}^*) \to \mathbb{R}^*$ satisfies the Lusin(N^{-1}) condition. In probability theory, this is usually expressed in terms of absolute continuity of occupation measures. Namely, defining the measure ϱ_f on $(\mathbb{R}^*, \mathcal{B}_1^*)$ by

$$\varrho_f(B) = \int_0^\infty \mathbb{1}_B(f(t))\,\mathrm{d}t = \lambda(f^{-1}(B)) = (f_{|f^{-1}(\mathbb{R}^*)}(\lambda))(B), \quad B \in \mathcal{B}_1^*,$$

this is equivalent to saying that ϱ_f is absolutely continuous. Observe that the Lusin(N^{-1})-condition is trivially satisfied if there is a countable decomposition of $[0,\infty)$ into intervals of the form $[a_i, b_i)$ such that each $f_{|(a_i,b_i)}$ is a C^1-diffeomorphism onto its range.

Proof of Corollary 2.14. (i) That (2.14) implies (2.15) follows by standard arguments: if $\sigma_\eta^2 > 0$, then this is clear, and if $\sigma_\eta^2 = 0$ use $\sin(x) \geq 2x/\pi$ for $x \in [0,\pi/2]$ so that $1 - \cos(u) = 2(\sin(u/2))^2 \geq 2u^2/\pi^2$ for $|u| \leq \pi$ and hence

$$\frac{-\Re(\Psi_\eta(z))}{\ln(1+|z|)} = \frac{\int_{\mathbb{R}}(1-\cos(xz))\,\nu_\eta(\mathrm{d}x)}{\ln(1+|z|)} \geq 2\frac{(z/\pi)^2 \int_{-\pi/z}^{\pi/z} u^2\,\nu_\eta(\mathrm{d}u)}{\ln(|z|/\pi)} \frac{\ln(|z|/\pi)}{\ln(1+|z|)}$$

for $|z| \geq \pi$. Now assume (2.15). Since $\mathrm{e}^{\Re\Psi_\eta(z)} = \left|\mathrm{e}^{\Psi_\eta(z)}\right| \leq 1$ we have $\Re\Psi_\eta(z) \leq 0$ for each $z \in \mathbb{R}$. Let $T > 0$ be arbitrary. An application of Fatou's lemma and (2.9) then shows

$$\begin{aligned}
\liminf_{|z|\to\infty} \frac{-\Re(\Psi_f(z))}{\ln(1+|z|)} &\geq \liminf_{|z|\to\infty} \int_0^T \frac{-\Re(\Psi_\eta(f(t)z))}{\ln(1+|z|)}\,\mathrm{d}t \\
&\geq \int_0^T \liminf_{|z|\to\infty} \frac{-\Re(\Psi_\eta(f(t)z))}{\ln(1+|z|)}\,\mathrm{d}t \\
&= \int_0^T \mathbb{1}_{\{f(t)\neq 0\}}\,\mathrm{d}t \cdot \liminf_{|z|\to\infty} \frac{-\Re(\Psi_\eta(z))}{\ln(1+|z|)}.
\end{aligned}$$

Letting $T \to \infty$ we conclude

$$\liminf_{|z|\to\infty} \frac{-\Re(\Psi_f(z))}{\ln(1+|z|)} \geq \lambda(\{s \in [0,\infty) : f(s) \neq 0\}) \cdot \liminf_{|z|\to\infty} \frac{-\Re(\Psi_\eta(z))}{\ln(1+|z|)} > \frac{1}{2}$$

by (2.15). Hence Ψ_f satisfies (2.13) showing that $\int_0^\infty f(s)\,\mathrm{d}\eta_s$ has a square integrable density.

(ii) Denoting

$$\begin{aligned} \nu_{f,1}(B) &:= \int_0^\infty \int_{\mathbb{R}} \mathbb{1}_{B\setminus\{0\}}(f(t)x)\,\nu_{\eta,\mathrm{ac}}(\mathrm{d}x)\,\mathrm{d}t \quad \text{and} \\ \nu_{f,2}(B) &:= \int_0^\infty \int_{\mathbb{R}} \mathbb{1}_{B\setminus\{0\}}(f(t)x)\,\nu_{\eta,\mathrm{sing}}(\mathrm{d}x)\,\mathrm{d}t \end{aligned}$$

for $B \in \mathcal{B}_1$ we have $\nu_f = \nu_{f,1} + \nu_{f,2}$ by (2.8). Now if $B \in \mathcal{B}_1$ has Lebesgue measure zero, then so has $B/f(s)$ for $f(s) \neq 0$, and it follows

$$\int_{\mathbb{R}} \mathbb{1}_{B\setminus\{0\}}(f(s)x)\nu_{\eta,\mathrm{ac}}(\mathrm{d}x) = \begin{cases} \nu_{\eta,\mathrm{ac}}(B/f(s)) = 0, & \text{if } f(s) \neq 0, \\ \int_{\mathbb{R}} \mathbb{1}_{B\setminus\{0\}}(0)\,\nu_{\eta,\mathrm{ac}}(\mathrm{d}x) = 0, & \text{if } f(s) = 0, \end{cases}$$

showing that $\nu_{f,1}$ is absolutely continuous. Since

$$\nu_{f,1}(\mathbb{R}) = \lambda(\{s \in [0,\infty) : f(s) \neq 0\})\,\nu_{\eta,\mathrm{ac}}(\mathbb{R}) = \infty$$

by (2.8) and assumption, it follows from Lemma 2.13 (iii) that each infinitely divisible distribution with Lévy measure $\nu_{f,1}$ is absolutely continuous. Since $\nu_f = \nu_{f,1} + \nu_{f,2}$, each such distribution (with Gaussian variance zero) is a convolution factor of $\mathcal{L}(\int_0^\infty f(s)\,\mathrm{d}\eta_s)$, showing that also $\int_0^\infty f(s)\,\mathrm{d}\eta_s$ is absolutely continuous.

(iii) For each $B \in \mathcal{B}_1^*$ with $\lambda(B) = 0$, and each $x \neq 0$, the set B/x is also a Lebesgue nullset and hence so is $f^{-1}(B/x)$ by the stated condition. Using Fubini's theorem and (2.8) we then obtain that

$$\nu_f(B) = \int_{\mathbb{R}} \int_0^\infty \mathbb{1}_{B\setminus\{0\}}(f(t)x)\,\mathrm{d}t\,\nu_\eta(\mathrm{d}x) = \int_{\mathbb{R}} \lambda(f^{-1}(B/x))\,\nu_\eta(\mathrm{d}x) = 0,$$

showing that ν_f is absolutely continuous. Using $\nu_f(\mathbb{R}) = \infty$ by assumption the claim then follows again from Lemma 2.13 (iii).

(iv) Let $f(x) = c \neq 0$ for $x \in (a, a+b)$. Then

$$\int_0^\infty f(s)\,\mathrm{d}\eta_s = \int_0^a f(s)\,\mathrm{d}\eta_s + c(\eta_{a+b} - \eta_a) + \int_b^\infty f(s)\,\mathrm{d}\eta_s.$$

The result then follows by observing that $c(\eta_{a+b} - \eta_a) \stackrel{d}{=} c\eta_b$ is absolutely continuous by assumption and independent from $\int_0^a f(s)\,\mathrm{d}\eta_s$ and $\int_{a+b}^\infty f(s)\,\mathrm{d}\eta_s$. $\square$

We are also interested in continuity and absolute continuity of randomly stopped functionals such as $\int_0^R f(s)\,\mathrm{d}\eta_s$, where R is an independent time taking values in $(0,\infty)$. Observe that since we have chosen a càdlàg version of $(\int_0^t f(s)\,\mathrm{d}\eta_s)_{t\geq 0}$, $\int_0^R f(s)\,\mathrm{d}\eta_s$ is a random variable.

Corollary 2.15 (Absolute continuity of $\int_0^R f(s)\,\mathrm{d}\eta_s$). *Let $\eta = (\eta_t)_{t\geq 0}$ be a one-dimensional Lévy process with characteristic triplet $(\sigma_\eta^2, \nu_\eta, \gamma_\eta)$ and denote by $\nu_\eta = \nu_{\eta,\mathrm{ac}} + \nu_{\eta,\mathrm{sing}}$ the Lebesgue decomposition of ν_η. Let $f : [0,\infty) \to \mathbb{R}$ be a Borel measurable deterministic function that is locally integrable with respect to η with $\lambda(\{s \in [0,t] : f(s) \neq 0\}) > 0$ for all $t > 0$. Let R be a random variable with values in $(0,\infty)$ that is independent of η and consider the random variable $\int_0^R f(s)\,\mathrm{d}\eta_s$. Then each of the following conditions implies that $\int_0^R f(s)\,\mathrm{d}\eta_s$ is absolutely continuous:*

(i) The characteristic triplet of η satisfies Kallenberg's condition (2.10)*, or more generally, the characteristic exponent Ψ_η satisfies the Hartman–Wintner condition* (2.11)*.*

(ii) $\nu_{\eta,\mathrm{ac}}(\mathbb{R}) = \infty$.

(iii) Preimages of Lebesgue nullsets $B \in \mathcal{B}_1^*$ under the mapping f are again Lebesgue nullsets and $\nu_\eta(\mathbb{R}) = \infty$.

(iv) The function f is constant and different from zero in a neighbourhood of zero and η_t is absolutely continuous for each $t > 0$.

(v) η is of finite variation with non-zero drift, f is Lebesgue almost everywhere different from zero, and R is absolutely continuous with respect to Lebesgue measure.

Proof. Under each of the conditions (i) – (iv), it follows from Corollary 2.14 that $\int_0^t f(s)\,\mathrm{d}\eta_s$ is absolutely continuous for each $t > 0$. Conditioning on $R = t$ then gives for each $B \in \mathcal{B}_1$ with $\lambda(B) = 0$ that

$$P\left(\int_0^R f(s)\,\mathrm{d}\eta_s \in B\right) = \int_0^\infty P\left(\int_0^t f(s)\,\mathrm{d}\eta_s \in B\right) P_R(\mathrm{d}t) = 0,$$

showing absolute continuity of $\int_0^R f(s)\,\mathrm{d}\eta_s$.

Now assume condition (v) and denote by γ_η^0 the drift of η. By interpreting the integral as a pathwise Lebesgue–Stieltjes integral, we can condition on the paths $(\eta_t)_{t\geq 0} = (g(t))_{t\geq 0}$. Since $f \neq 0$ Lebesgue almost everywhere and since the paths of η are of the form $g(t) = \gamma_\eta^0 t + \sum_{0<s\leq t} \Delta g(s)$ with $\lambda(s \in (0,t] : \Delta g(s) \neq 0) = 0$, the functions

$$H_g : (0,\infty) \to \mathbb{R}, \quad t \mapsto \int_0^t f(s)\,\mathrm{d}g(s) = \gamma_\eta^0 \int_0^t f(s)\,\mathrm{d}s + \sum_{0<s\leq t} f(s)\Delta g(s)$$

are Lebesgue almost everywhere differentiable with derivatives $f(t)\gamma_\eta^0 \neq 0$. Theorem 4.2 in Davydov et al. [22] shows that the image measure $H_g(\lambda)$ is absolutely continuous, for P_η almost every path g of η. For a Borel set B with $\lambda(B) = 0$ we then have $\lambda(H_g^{-1}(B)) = 0$ and by absolute continuity of R that $P(H_g(R) \in B) = P(R \in H_g^{-1}(B)) = 0$. Absolute continuity of $\int_0^R f(t)\,\mathrm{d}\eta_t$ then follows from

$$\begin{aligned} P\left(\int_0^R f(s)\,\mathrm{d}\eta_s \in B\right) &= \int_{D([0,\infty),\mathbb{R})} P\left(\int_0^R f(s)\,\mathrm{d}\eta_s \in B \,\middle|\, \eta = g\right) P_\eta(\mathrm{d}g) \\ &= \int_{D([0,\infty),\mathbb{R})} P(H_g(R) \in B)\, P_\eta(\mathrm{d}g) = 0 \end{aligned}$$

for all $B \in \mathcal{B}_1$ with $\lambda(B) = 0$. ☐

2.2.2. Conditions for (Absolute) Continuity of $\int_0^t \mathrm{e}^{-\xi_{s-}}\, \mathrm{d}\eta_s$

In this section, we give sufficient conditions for absolute continuity of $V_{\xi,\eta}(t) := \int_0^t \mathrm{e}^{-\xi_{s-}}\, \mathrm{d}\eta_s$ for fixed $t > 0$ and characterize continuity of its law. Recall that throughout Chapter 2, ξ and η denote independent Lévy processes with characteristic triplets $(\sigma_\xi^2, \nu_\xi, \gamma_\xi)$ and $(\sigma_\eta^2, \nu_\eta, \gamma_\eta)$, and characteristic exponents Ψ_ξ and Ψ_η, respectively. Further, we denote the Lebesgue decompositions of ν_ξ and ν_η by

$$\nu_\xi = \nu_{\xi,\mathrm{ac}} + \nu_{\xi,\mathrm{sing}} \quad \text{and} \quad \nu_\eta = \nu_{\eta,\mathrm{ac}} + \nu_{\eta,\mathrm{sing}},$$

respectively, where "ac" marks the absolutely continuous and "sing" the singular part. In this context, the integral $V_{\xi,\eta}(t)$ can be thought of as exponential functional subject to deterministic killing at $t > 0$, but also occurs when conditioning $V_{q,\xi,\eta}$ on $\tau = t$, yielding an important tool for studying the continuity properties of the killed exponential functional in Section 2.2.3. We start with an example which is due to Lifshits [39].

Example 2.16. Let ξ be a Brownian motion with variance $\sigma_\xi^2 > 0$ and drift $\gamma_\xi^0 \in \mathbb{R}$. Then $\int_0^t \mathrm{e}^{-\xi_s}\, \mathrm{d}s$ is absolutely continuous for each $t > 0$. This is stated in Problem 9.1 of [22, p. 53], but can also be deduced from Theorem 1 in [39], which shows that random variables of the form $\int_0^t g(\xi_s)\, \mathrm{d}s$ are absolutely continuous, provided g is locally Lipschitz with derivative g' that is non-zero and continuous on some set of full Lebesgue measure, and the autocovariance function $k(s,s') = \operatorname{Cov}(\xi_s, \xi_{s'})$ satisfies the non-degeneracy condition $\int_0^t \int_0^t k(s,s') h(s) h(s')\, \mathrm{d}s\, \mathrm{d}s' > 0$ for any Lebesgue integrable function $h : [0,t] \to \mathbb{R}$ that is not almost everywhere equal to zero. The condition on g is satisfied for $g(x) = \mathrm{e}^{-x}$, and the non-degeneracy condition is equivalent to the fact that $\int_0^t \xi_s h(s)\, \mathrm{d}s$ is not constant for each integrable h that is not almost everywhere equal to zero. To see that the latter condition is satisfied, we can assume without loss of generality that $\sigma_\xi^2 = 1$ and $\gamma_\xi^0 = 0$. If then $\int_0^t \xi_s h(s)\, \mathrm{d}s$ is constant for some integrable function h, it must necessarily be equal to its expectation which is zero. Denoting $H(s) = \int_0^s h(u)\, \mathrm{d}u$ and using partial integration we conclude

$$0 = \int_0^t \xi_s\, \mathrm{d}H(s) = \xi_t H(t) - \int_0^t H(s)\, \mathrm{d}\xi_s = \int_0^t (H(t) - H(s))\, \mathrm{d}\xi_s,$$

where we used that the quadratic covariation of H and ξ is zero (H being of finite variation). Taking the variance of this we see that $\int_0^t (H(t) - H(s))^2\, \mathrm{d}s = 0$, showing that H is constant, which in turn implies that h is almost everywhere equal to zero. Hence the non-degeneracy condition is satisfied and [39, Thm. 1] gives absolute continuity of $\int_0^t \mathrm{e}^{-\xi_s}\, \mathrm{d}s$.

Although we will not be able to characterize absolute continuity of $V_{\xi,\eta}(t)$ completely, we will give various sufficient conditions that cover many cases of interest. A key condition below will be that

$$\int_{\mathbb{R}} \Re\left(\frac{1}{1 - \Psi_\xi(z)}\right) \mathrm{d}z < \infty, \tag{2.16}$$

where again $\Re(z)$ denotes the real part of a complex number z. To see the importance of (2.16) in connection with Corollary 2.14 (iii), for each path f of ξ, define the occupation measure $\varrho_{f,t}$ on $\mathcal{B}_1$ by

$$\varrho_{f,t}(B) = \int_0^t \mathbb{1}_B(f(s))\, \mathrm{d}s, \quad B \in \mathcal{B}_1.$$

As shown by Hawkes [30] (cf. Bertoin [11, Thm. V.1]), condition (2.16) is equivalent to the fact that $P_\xi(f)$-a.s., the occupation measure $\varrho_{f,t}$ is absolutely continuous. This in turn is equivalent to saying that $P_\xi(f)$-a.s., preimages of Lebesgue nullsets under the mapping $f : [0,t] \to \mathbb{R}$ are again Lebesgue nullsets, i.e. that $f : [0,t] \to \mathbb{R}$ satisfies the Lusin(N^{-1})-condition. A further equivalent condition can be expressed in terms of potential measures: For each $p \in [0,\infty)$, the p-potential measure W_ξ^p of ξ is defined by

$$W_\xi^p(B) = \int_0^\infty \mathrm{e}^{-pt} P(\xi_t \in B)\,\mathrm{d}t = E\left(\int_0^\infty \mathrm{e}^{-pt} \mathbb{1}_B(\xi_t)\,\mathrm{d}t\right), \quad B \in \mathcal{B}_1, \tag{2.17}$$

e.g. [58, Def. 30.9]; in other literature such as [11] this appears also under the name of resolvent kernel when $p > 0$. Condition (2.16) is then equivalent to the fact that W_ξ^p has a bounded Lebesgue density for some, equivalently all, $p > 0$, cf. [11, Thm. II.16] or [58, Thm. 43.3, Rem. 43.6]. Finally, condition (2.16) is further equivalent to the fact that single points are not essentially polar under ξ, equivalently that the p-capacity $C^p(\{0\})$ of ξ is strictly positive for some, equivalently all, $p > 0$, see [11, Sect. II.3] or [58, Def. 41.14, 42.6] for the definitions of essentially polar sets and the p-capacity and [11, Thm. II.16] or [58, Prop. 43.2, Thm. 43.3] for the corresponding results. We collect some known examples when condition (2.16) is satisfied.

Example 2.17 (Sufficient conditions for (2.16))**.** Let ξ be a Lévy process. If ξ is of finite variation, then (2.16) holds if and only if the drift γ_ξ^0 of ξ is different from zero ([11, Cor. II.20 (ii)], [58, Thm. 43.13]). Condition (2.16) also holds if $\sigma_\xi^2 > 0$ ([58, Thm. 43.21 Case 6]) or more generally if ξ is α-stable with index $\alpha \in (1,2]$ ([58, Ex. 43.22]). A non-deterministic 1-stable process ξ satisfies (2.16) if and only if it is not strictly 1-stable, cf. [58, Ex. 43.7]. Condition (2.16) is further satisfied, when

$$\int_0^1 x\,\nu_\xi(\mathrm{d}x) < \infty = \int_{-1}^0 |x|\,\nu_\xi(\mathrm{d}x) \quad \text{or} \quad \int_{-1}^0 |x|\,\nu_\xi(\mathrm{d}x) < \infty = \int_0^1 x\nu_\xi(\mathrm{d}x), \tag{2.18}$$

cf. [58, Thm. 43.24].

We can now give sufficient conditions for $V_{\xi,\eta}(t) = \int_0^t \mathrm{e}^{-\xi_{s-}}\,\mathrm{d}\eta_s$ to be absolutely continuous.

Theorem 2.18 (Sufficient conditions for absolute continuity of $V_{\xi,\eta}(t)$)**.** *Let ξ and η be as above. Let $t \in (0,\infty)$ be fixed and assume that one of the following conditions is satisfied:*

(i) The characteristic triplet of η satisfies

$$\liminf_{\varepsilon\downarrow 0} \varepsilon^{-2}|\ln\varepsilon|^{-1}\left(\sigma_\eta^2 + \int_{-\varepsilon}^{\varepsilon} x^2\,\nu_\eta(dx)\right) > \frac{1}{4t}, \tag{2.19}$$

or more generally

$$\liminf_{|z|\to\infty} \frac{-\Re(\Psi_\eta(z))}{\ln(1+|z|)} > \frac{1}{2t}. \tag{2.20}$$

In particular, this is satisfied when $\sigma_\eta^2 > 0$.

(ii) The absolutely continuous part of ν_η is infinite: $\nu_{\eta,\mathrm{ac}}(\mathbb{R}) = +\infty$.

(iii) The characteristic exponent Ψ_ξ of ξ satisfies Condition (2.16) *and $\nu_\eta(\mathbb{R}) = \infty$ (recall from Example 2.17 that* (2.16) *is in particular satisfied when $\sigma_\xi^2 > 0$, when ξ is of finite variation with non-zero drift, or when ξ satisfies* (2.18)*).*

(iv) $\nu_{\xi,\mathrm{ac}}(\mathbb{R}) = \nu_\eta(\mathbb{R}) = \infty$.

(v) η *is of finite variation with non-zero drift and* ξ *is such that* $\sigma_\xi^2 > 0$ *or* $\nu_\xi(\mathbb{R}) = \infty$.

(vi) ξ *is a compound Poisson process and* η_s *is absolutely continuous for all* $s > 0$.

Then $\int_0^t \mathrm{e}^{-\xi_{s-}}\, d\eta_s$ *is absolutely continuous.*

Proof. (i) – (iii), (vi): Conditioning on the paths $\xi = f$, we obtain for any Borel set B with $\lambda(B) = 0$ that

$$P\left(\int_0^t \mathrm{e}^{-\xi_{s-}}\, d\eta_s \in B\right) = \int_{D([0,\infty),\mathbb{R})} P\left(\int_0^t \mathrm{e}^{-f(s-)}\, d\eta_s \in B\right) P_\xi(df). \tag{2.21}$$

Hence absolute continuity of $V_{\xi,\eta}(t)$ will follow if one can show that $P_\xi(f)$-a.s., $\int_0^t \mathrm{e}^{-f(s-)}\, \mathrm{d}\eta_s$ is absolutely continuous. But since $\lambda(\{s \in [0,t] : \mathrm{e}^{-f(s-)} \neq 0\}) = t$, this follows from Corollary 2.14 (i)–(iv) and the discussion preceeding Example 2.17 regarding the Condition (2.16) for (iii). For (vi), observe that each path of a compound Poisson process is constant in a neighbourhood of zero.

(v) Assume first that $\nu_\xi(\mathbb{R}) = \infty$. The proof is similar to the proof given in [13, Thm. 3.9b)], but we give the argument here since that theorem is not directly applicable to our situation. For given $\varepsilon > 0$, denote the time of the i'th jump of ξ with absolute jump size greater than ε by $T_i(\varepsilon)$, $i \in \mathbb{N}$. Define the process ξ' by $\xi'_s = \xi_s - \sum_{0<u\le s, |\Delta\xi_u|>\varepsilon} \Delta\xi_u$. Then η, ξ', $(T_i(\varepsilon))_{i\in\mathbb{N}}$ and $(\Delta\xi_{T_i(\varepsilon)})_{i\in\mathbb{N}}$ are all independent by the Lévy-Itô decomposition. Now condition first on the set $\{T_2(\varepsilon) \le t\}$ and then on all quantities present apart the time $T_1(\varepsilon)$ of the first jump of ξ (i.e. condition on $\eta = g$, $\xi' = f$, $T_i(\varepsilon) = t_i$, $i \ge 2$ and $\Delta\xi_{T_i} = y_i$, $i \in \mathbb{N}$). Conditional on this set and these quantities, we have

$$\int_0^t \mathrm{e}^{-\xi_{s-}}\, \mathrm{d}\eta_s = \int_0^{T_1(\varepsilon)} \mathrm{e}^{-f(s-)} \mathrm{d}g(s) + \int_{T_1(\varepsilon)+}^{T_2(\varepsilon)} \mathrm{e}^{-y_1 - f(s-)}\, \mathrm{d}g(s) + \int_{T_2(\varepsilon)+}^{t} \mathrm{e}^{-h(s-)}\, \mathrm{d}g(s),$$

where the function h corresponds (on $\{s > T_2(\varepsilon)\}$) to the path of $(\xi_s)_{s>T_2(\varepsilon)}$, which is known under this conditioning. Then $\int_{T_2(\varepsilon)+}^t \mathrm{e}^{-h(s-)}\, \mathrm{d}g(s)$ is constant and since g is of the form $g(s) = \gamma_\eta^0 s + \sum_{0<u\le s} \Delta g(u)$, where γ_η^0 denotes the drift of η, the function

$$H : (0, T_2(\varepsilon)] \ni u \mapsto \int_0^u \mathrm{e}^{-f(s-)}\, \mathrm{d}g(s) + \int_{u+}^{T_2(\varepsilon)} \mathrm{e}^{-y_1 - f(s-)}\, \mathrm{d}g(s)$$

is Lebesgue almost everywhere differentiable in u with derivative $\mathrm{e}^{-f(u-)}\gamma_\eta^0 - \mathrm{e}^{-y_1 - f(u-)}\gamma_\eta^0$. Since $y_1 \neq 0 \neq \gamma_\eta^0$, this derivative is Lebesgue almost everywhere different from 0. By Theorem 4.2 in Davydov et al. [22], the image measure under H of the Lebesgue measure on $(0, T_2(\varepsilon)]$ is absolutely continuous, hence so is $H(T_1(\varepsilon))$ since $T_1(\varepsilon)$ is uniformly distributed on $(0, T_2(\varepsilon))$ (e.g. [58, Prop. 3.4]). But this shows that

$$P\left(\int_0^t \mathrm{e}^{-\xi_{s-}}\, \mathrm{d}\eta_s \in B \middle| T_2(\varepsilon) \le t, \eta = g, \xi' = f, T_i = t_i\, (i \ge 2),\, \Delta\xi_{T_i} = y_i\, (i \in \mathbb{N})\right) = 0$$

for all Borel sets B with $\lambda(B) = 0$. Integrating all conditions out apart from $\{T_2(\varepsilon) \le t\}$, we conclude that $P(\int_0^t \mathrm{e}^{-\xi_{s-}}\, \mathrm{d}\eta_s \in B | T_2(\varepsilon) \le t) = 0$. Letting $\varepsilon \downarrow 0$ and observing that $P(T_2(\varepsilon) \leq t) \to 1$ as $\varepsilon \downarrow 0$ as a consequence of $\nu_\xi(\mathbb{R}) = +\infty$, we conclude $P(\int_0^t \mathrm{e}^{-\xi_{s-}}\,\mathrm{d}\eta_s \in B) = 0$ and hence absolute continuity of $V_{\xi,\eta}(t)$.

Now assume that $\nu_\xi(\mathbb{R}) < \infty$ and $\sigma_\xi^2 > 0$. Since ξ satisfies Condition (2.16), by (iii) we can additionally assume that $\nu_\eta(\mathbb{R}) < \infty$. Denote by M the time of the last jump of ξ or η before time t, i.e. the last time such that neither ξ nor η jumps in $(M, t]$ (if no jump occurs, then $M = 0$). Then $M < t$ a.s., and conditional on $M = m$, $(\xi_s)_{s\in[0,m]} = (f(s))_{s\in[0,m]}$ and $(\eta_s)_{s\in[0,m]} = (g(s))_{s\in[0,m]}$, we have

$$\int_0^t \mathrm{e}^{-\xi_{s-}}\,\mathrm{d}\eta_s = \int_0^m \mathrm{e}^{-f(s-)}\,\mathrm{d}g(s) + \gamma_\eta^0 \mathrm{e}^{-f(m)} \int_{m+}^t \mathrm{e}^{-(\xi_{s-}-\xi_m)}\,\mathrm{d}s.$$

But the first term is constant and $(\xi_{s-} - \xi_m)_{s\in(m,t]}$ is a Brownian motion with drift γ_ξ^0 under this conditioning, hence the second term is absolutely continuous by Example 2.16. Hence, $V_{\xi,\eta}(t)$ is absolutely continuous under this conditioning, and integrating the conditions out we see that $V_{\xi,\eta}(t)$ is absolutely continuous.

(iv) Choose a set $D \in \mathcal{B}_1$ with $\nu_{\xi,\mathrm{sing}}(D) = \nu_{\xi,\mathrm{ac}}(\mathbb{R} \setminus D) = 0$. For each $\varepsilon \in (0,1)$, denote by R_ε the time of the first jump of ξ with jump size in $D \cap ((-1,-\varepsilon) \cup (\varepsilon, 1))$, and by Y_ε its jump size. On the set $\{R_\varepsilon < t\}$ we can write

$$\int_0^t \mathrm{e}^{-\xi_{s-}}\,\mathrm{d}\eta_s \;=\; \int_0^{R_\varepsilon} \mathrm{e}^{-\xi_{s-}}\,\mathrm{d}\eta_s + \mathrm{e}^{-Y_\varepsilon}\left(\mathrm{e}^{-\xi_{R_\varepsilon-}} \int_{R_\varepsilon+}^t \mathrm{e}^{-(\xi_{s-}-\xi_{R_\varepsilon})}\,\mathrm{d}\eta_s\right).$$

Observe that $\mathrm{e}^{-Y_\varepsilon}$ is independent from $(\int_0^{R_\varepsilon} \mathrm{e}^{-\xi_{s-}}\,\mathrm{d}\eta_s, \mathrm{e}^{-\xi_{R_\varepsilon-}} \int_{R_\varepsilon+}^t \mathrm{e}^{-(\xi_{s-}-\xi_{R_\varepsilon})}\,\mathrm{d}\eta_s)$. Further, conditioning on $\xi = f$ and R_ε, we see from Proposition 2.12 that $\int_{R_\varepsilon+}^t \mathrm{e}^{-(f(s-)-f(R_\varepsilon))}\,\mathrm{d}\eta_s$ has no atoms, i.e.

$$P\left(\int_{R_\varepsilon+}^t \mathrm{e}^{-(f(s-)-f(R_\varepsilon))}\,\mathrm{d}\eta_s = b \middle| R_\varepsilon = r\right) = 0$$

for all $b \in \mathbb{R}$ and $r \in (0,t)$. Integrating out the condition we see similarly to (2.21) that also $\int_{R_\varepsilon+}^t \mathrm{e}^{-(\xi_{s-}-\xi_{R_\varepsilon})}\,\mathrm{d}\eta_s$ has no atoms when conditioned on the set $\{R_\varepsilon < t\}$. In particular, $\mathrm{e}^{-\xi_{R_\varepsilon-}} \int_{R_\varepsilon+}^t \mathrm{e}^{-(\xi_{s-}-\xi_{R_\varepsilon})}\,\mathrm{d}\eta_s \neq 0$ a.s. on $\{R_\varepsilon < t\}$. Conditioning on $(\int_0^{R_\varepsilon} \mathrm{e}^{-\xi_{s-}}\,\mathrm{d}\eta_s,$ $\mathrm{e}^{-\xi_{R_\varepsilon-}} \int_{R_\varepsilon+}^t \mathrm{e}^{-(\xi_{s-}-\xi_{R_\varepsilon})}\,\mathrm{d}\eta_s) = (h_1, h_2)$ and observing that $\mathrm{e}^{-Y_\varepsilon}$ is absolutely continuous, we see that $h_1 + \mathrm{e}^{-Y_\varepsilon} h_2$ is absolutely continuous. Integrating out h_1 and h_2, it follows that $\int_0^t \mathrm{e}^{-\xi_{s-}}\,\mathrm{d}\eta_s$ is absolutely continuous on $\{R_\varepsilon < t\}$ (similar to the proof of (v)). Letting $\varepsilon \downarrow 0$ it follows that $\int_0^t \mathrm{e}^{-\xi_{s-}}\, d\eta_s$ is absolutely continuous since $P(\{R_\varepsilon < t\}) \to 1$, again similar to the proof of (v). $\square$

When $\sigma_\xi^2 > 0$, we obtain in particular:

Corollary 2.19. *Let η and ξ be two independent Lévy processes such that $\sigma_\xi^2 > 0$. Then $V_{\xi,\eta}(t)$ is absolutely continuous if and only if η is neither the zero process nor a compound Poisson process.*

Proof. If η is a compound Poisson process, the probability that η does not jump before time t is positive, and on this set $V_{\xi,\eta}(t) = 0$, hence $V_{\xi,\eta}(t)$ has an atom at zero and hence is not absolutely continuous. Similarly, if η is the zero process, then $V_{\xi,\eta}(t) = 0$. Otherwise, η is of finite variation with non-zero drift, or satisfies $\nu_\eta(\mathbb{R}) = \infty$ or $\sigma_\eta^2 > 0$. It then follows from Theorem 2.18 (v),(iii),(i) that $V_{\xi,\eta}(t)$ is absolutely continuous. $\square$

Remark 2.20. Conditions (i)-(iii) and (vi) of Theorem 2.18 ensure absolute continuity for functionals of the form $\int_0^t g(\xi_{s-})\,\mathrm{d}\eta_s$ for more general functions g than the exponential function. To be more precise, let $g : [0,t] \to \mathbb{R}$ be continuous, such that $s \mapsto g(\xi_{s-})$ is càglàd.
(i) Assume that $g(0) \neq 0$ and that Condition (vi) of Theorem 2.18 is satisfied. Then $\int_0^t g(\xi_{s-})\,\mathrm{d}\eta_s$ is absolutely continuous by the same proof.
(ii) Assume that one of the conditions (i) - (iii) of Theorem 2.18 is satisfied, denote by N the zero set of g, and assume further that

$$\int_0^t P(\xi_s \in N)\,\mathrm{d}s = 0. \tag{2.22}$$

Using Fubini's theorem, we conclude that $E\int_0^t \mathbb{1}_N(\xi_s)\,\mathrm{d}s = 0$, showing that

$$\lambda(\{s \in [0,t] : \xi_s \in N\}) = \lambda(\{s \in [0,t] : g(\xi_s) = 0\}) = 0$$

almost surely. The result then follows by the same proof as in Theorem 2.18 (i)–(iii) and observing that $\int_0^t g(\xi_{s-})\,\mathrm{d}\eta_s$ and $\int_0^t g(\xi_s)\,\mathrm{d}\eta_s$ are a.s. equal.
When N is countable, sufficient conditions for (2.22) are that $\sigma_\xi^2 > 0$ or $\nu_\xi(\mathbb{R}) = \infty$ (by [58, Thm. 27.4]), or that ξ is of finite variation with non-zero drift since then (2.16) is satisfied which gives absolute continuity of the potential measure W_ξ^1 and hence of W_ξ^0. Another sufficient condition obviously is that $g \neq 0$ on $[0,t]$.

We can now characterize continuity of $V_{\xi,\eta}(t)$.

Corollary 2.21 (Continuity of $V_{\xi,\eta}(t)$)**.** *Let ξ and η be two independent Lévy processes and let $t > 0$. Then $V_{\xi,\eta}(t)$ has atoms (i.e. is not continuous) if and only if η is the zero process, or η is a compound Poisson process, or*

$$\sigma_\eta^2 = \sigma_\xi^2 = 0, \quad \nu_\eta(\mathbb{R}) < \infty \quad \textit{and} \quad \nu_\xi(\mathbb{R}) < \infty. \tag{2.23}$$

Proof. As seen in the proof of Corollary 2.19, if η is the zero process or a compound Poisson process, then $V_{\xi,\eta}(t)$ has an atom at zero and hence is not continuous. If (2.23) holds but η is neither a compound Poisson process nor the zero process, then η has drift $\gamma_\eta^0 \neq 0$ and ξ is of finite variation and finite jump activity with drift $\gamma_\xi^0 \in \mathbb{R}$. The probability that both η and ξ do not jump before time t is positive, and on this set we have $V_{\xi,\eta}(t) = \gamma_\eta^0 \int_0^t \mathrm{e}^{-\gamma_\xi^0 s}\,\mathrm{d}s$, so that $V_{\xi,\eta}(t)$ has an atom.

Now assume that neither (2.23) is satisfied nor that η is a compound Poisson process nor the zero process. If $\sigma_\eta^2 > 0$ or $\nu_\eta(\mathbb{R}) = \infty$, conditioning on $\xi = f$ we see that $\int_0^t \mathrm{e}^{-f(s-)}\,\mathrm{d}\eta_s$ is continuous by Proposition 2.12. Hence

$$P(V_{\xi,\eta}(t) = b) = \int_{D([0,\infty),\mathbb{R})} P(V_{\xi,\eta}(t) = b | \xi = f)\, P_\xi(\mathrm{d}f) = 0$$

for each $b \in \mathbb{R}$, so that $V_{\xi,\eta}(t)$ is continuous. If $\nu_\eta(\mathbb{R}) < \infty$ and $\sigma_\eta^2 = 0$, we must have $\gamma_\eta^0 \neq 0$ since η is neither a compound Poisson process nor the zero process. Since (2.23) is violated, necessarily $\sigma_\xi^2 > 0$ or $\nu_\xi(\mathbb{R}) = \infty$. Then $V_{\xi,\eta}(t)$ is absolutely continuous and hence continuous by Theorem 2.18 (v). $\square$

Remark 2.22. (i) It is clear that a pure types theorem does not hold for the law of $\int_0^t \mathrm{e}^{-\xi_{s-}}\, \mathrm{d}\eta_s$. The simplest counterexample is when $\xi = 0$ and η is a compound Poisson process with absolutely continuous jump distribution. Then $V_{\xi,\eta}(t) = \eta_t$ whose distribution has an atom at zero, but restricted to $\mathbb{R}^*$ has a density. Similar examples can be constructed when both ξ and η are compound Poisson processes.
(ii) An example when $\int_0^t \mathrm{e}^{-\xi_{s-}}\, \mathrm{d}\eta_s$ is continuous singular is easily constructed by choosing $\xi = 0$ and for η a process for which η_t is continuous singular, examples of which are given in [58, Thms. 27.19, 27.23].

2.2.3. Conditions for (Absolute) Continuity of $\int_0^\tau \mathrm{e}^{-\xi_{s-}}\, \mathrm{d}\eta_s$

In this section we obtain sufficient conditions (Theorem 2.23) for absolute continuity and a characterization of continuity of $V_{q,\xi,\eta} = \int_0^\tau \mathrm{e}^{-\xi_{s-}}\, \mathrm{d}\eta_s$ (Corollary 2.26). In part (vii) of Theorem 2.23, we need the so called ACP-condition for η. Recall the p-potential measure from Equation (2.17). We say that η satisfies the ACP condition if

$$\text{the potential measure } W_\eta^p \text{ is absolutely continuous for some } p \in [0, \infty). \tag{2.24}$$

This is equivalent to saying that W_η^p is absolutely continuous for all $p \in [0, \infty)$, cf. [58, Rem. 41.12]. If $p \in (0, \infty)$ and T is an exponentially distributed time with parameter p, independent of η, then by conditioning on $T = t$ we have

$$P(\eta_T \in B) = \int_0^\infty P(\eta_t \in B)\, P_T(\mathrm{dt}) = p \int_0^\infty \mathrm{e}^{-pt} P(\eta_t \in B)\, \mathrm{d}t = p W_\eta^p(B), \quad B \in \mathcal{B}_1.$$

Hence (2.24) means nothing else than that η_T is absolutely continuous for any (equivalently: some) exponentially distributed independent time T with parameter in $(0, \infty)$. Obviously, (2.24) is satisfied when η_t is absolutely continuous for each $t > 0$, but the converse is not true in general, see e.g. [58, Rem. 41.13]. Recall also from Section 2.2.2 that $\int_{\mathbb{R}} \Re\big(\frac{1}{1-\Psi_\eta(z)}\big)\, \mathrm{d}z < \infty$ if and only if W_η^1 has a bounded density, so that (2.16) for η implies (2.24) for η. In particular, η satisfies (2.24) if η is of finite variation with non-zero drift, if $\sigma_\eta^2 > 0$, or if η satisfies (2.18) (with ν_η replacing ν_ξ). Examples exist when η satisfies (2.24) but not (2.16), see [58, Thm. 43.21]; one such example is when η is a non-deterministic strictly α-stable process of index $\alpha \in (0, 1)$.

Theorem 2.23 (Sufficient conditions for absolute continuity of $V_{q,\xi,\eta}$)**.** *Let ξ, η and τ as above with $q \in (0, \infty)$. Suppose that one of the following conditions is satisfied:*

(i) The characteristic triplet of η satisfies Kallenberg's condition (2.10)*, or more generally the Hartman–Wintner condition* (2.11)*. In particular, this is satisfied when $\sigma_\eta^2 > 0$.*

(ii) The absolutely continuous part of ν_η is infinite: $\nu_{\eta,\mathrm{ac}}(\mathbb{R}) = \infty$.

(iii) The characteristic exponent Ψ_ξ of ξ satisfies Condition (2.16) *and $\nu_\eta(\mathbb{R}) = \infty$ (recall from Example 2.17 that* (2.16) *is in particular satisfied when $\sigma_\xi^2 > 0$, when ξ is finite variation with non-zero drift, or when ξ satisfies* (2.18)*).*

(iv) $\nu_{\xi,\mathrm{ac}}(\mathbb{R}) = \nu_\eta(\mathbb{R}) = \infty$.

(v) η is of finite variation with non-zero drift.

(vi) ξ is a compound Poisson process and η satisfies the ACP condition (2.24).

Then $V_{q,\xi,\eta} = \int_0^\tau \mathrm{e}^{-\xi_{s-}}\, d\eta_s$ is absolutely continuous.

Proof. (i) – (iv): By Theorem 2.18, any of the given conditions (i) to (iv) implies absolute continuity of $V_{\xi,\eta}(t) = \int_0^t \mathrm{e}^{-\xi_{s-}}\, d\eta_s$ for all $t > 0$. Let $B \in \mathcal{B}_1$ be a Lebesgue-null set. Conditioning on $\tau = t$ we obtain

$$P\left(\int_0^\tau \mathrm{e}^{-\xi_{s-}}\, d\eta_s \in B\right) = \int_0^\infty P\left(\int_0^t \mathrm{e}^{-\xi_s}\, d\eta_s \in B \,\middle|\, \tau = t\right) P_\tau(\mathrm{d}t) = \int_0^\infty 0\, P_\tau(\mathrm{d}t) = 0,$$

showing that $\int_0^\tau e^{-\xi_{s-}}\, d\eta_s$ is absolutely continuous.

(v) If η is of finite variation with non-zero drift, we can condition on the path $\xi = f$. By Corollary 2.15 (v), $\int_0^\tau \mathrm{e}^{-f(s-)} \mathrm{d}\eta_s$ will be absolutely continuous for each path f. Integrating the condition out we see as in (2.21) that $\int_0^\tau \mathrm{e}^{-\xi_{s-}}\, \mathrm{d}\eta_s$ is absolutely continuous.

(vi) Denote by T the time of the first jump of ξ. This is exponentially distributed with parameter $\nu_\xi(\mathbb{R}) \in (0, \infty)$ and independent from τ. Then, conditional on $\{\tau > T\}$, the random variables $\tau - T$ and T are conditionally independent and (conditionally) exponentially distributed with parameters q and $q + \nu_\xi(\mathbb{R})$, respectively. Conditional on $\{\tau > T\}$ we have

$$V_{q,\xi,\eta} = \int_0^T \mathrm{d}\eta_s + \int_{T+}^\tau \mathrm{e}^{-\xi_{s-}}\, \mathrm{d}\eta_s = \eta_T + \int_{T+}^{T+(\tau-T)} \mathrm{e}^{-\xi_{s-}}\, \mathrm{d}\eta_s.$$

Since the two summands are conditionally independent and the first is absolutely continuous by (2.24) as T is conditionally exponentially distributed with parameter $q + \nu_\xi(\mathbb{R})$, the distribution of $V_{q,\xi,\eta}$ conditional on $\{\tau > T\}$ is absolutely continuous. On the other hand, conditional on $\{\tau < T\}$ we have $V_{q,\xi,\eta} = \eta_\tau$ with τ being conditionally exponentially distributed with parameter $q + \nu_\xi(\mathbb{R})$, so that (2.24) gives absolute continuity of $V_{q,\xi,\eta}$ also under this conditioning. Adding up the two cases shows absolute continuity of $V_{q,\xi,\eta}$. □

When η is deterministic but not the zero-process, Pardo et al. [48, Thm. 2.1] showed that $V_{q,\xi,\eta}$ has a density and they also obtained various properties of it. Observe that the existence of the density in this case can also be seen from Theorem 2.23 (v). We also remark that similar to Remark 2.20, many of the results of Theorem 2.23 can be extended to functionals of the form $\int_0^\tau g(\xi_{s-})\, \mathrm{d}\eta_s$ for sufficiently nice functions g. For example, if $g : [0, \infty) \to \mathbb{R}$ is continuous, ξ satisfies $\int_0^\infty P(\xi_s \in N)\, \mathrm{d}s = 0$ for the zero set N of g, and one of the conditions (i) – (iii) of Theorem 2.23 is satisfied, then also $\int_0^\tau g(\xi_{s-})\, \mathrm{d}\eta_s$ will be absolutely continuous by the same proof. Similar to Corollary 2.19 we now obtain:

Corollary 2.24. *Assume that $\sigma_\xi^2 > 0$ and $q \in (0, \infty)$. Then $V_{q,\xi,\eta}$ is absolutely continuous if and only if η is neither a compound Poisson process nor the zero process.*

Proof. If η is a compound Poisson process, then the probability that η does not jump before time τ is positive, hence $V_{q,\xi,\eta}$ has an atom at zero and is hence not continuous, and similarly when η is the zero process. If η is neither a compound Poisson process nor the zero process, it is of finite variation with non-zero drift, or satisfies $\nu_\eta(\mathbb{R}) = \infty$ or $\sigma_\eta^2 > 0$, in which case $V_{q,\xi,\eta}$ is absolutely continuous by Theorem 2.23 (v),(iii), or (i). □

When ξ is a compound Poisson process, it is easy to show that the sufficient Condition (vi) of Theorem 2.23 is actually also necessary:

Corollary 2.25. *Let ξ be a compound Poisson process and $q \in (0,\infty)$. Then $V_{q,\xi,\eta}$ is absolutely continuous if and only if η satisfies the ACP condition* (2.24)*.*

Proof. Sufficiency of the ACP condition follows from Theorem 2.23 (vi). For the converse, assume that $V_{q,\xi,\eta}$ is absolutely continuous and denote by T the time of the first jump of ξ. Then also conditional on $\{\tau < T\}$, $V_{q,\xi,\eta} = \eta_\tau$ is absolutely continuous. But τ is conditional exponentially distributed with parameter $q + \nu_\xi(\mathbb{R})$, showing that η satisfies the ACP condition. $\square$

Continuity of $V_{q,\xi,\eta}$ can be characterized as follows:

Corollary 2.26 (Continuity of $V_{q,\xi,\eta}$)**.** *Let ξ and η be as above and let $q \in (0,\infty)$. Then $V_{q,\xi,\eta}$ is continuous if and only if η is neither a compound Poisson process nor the zero process. If η is a compound Poisson process or the zero process, then $V_{q,\xi,\eta}$ has an atom at 0.*

Proof. We have already seen in the proof of Corollary 2.24 that if η is a compound Poisson process or the zero process, then $V_{q,\xi,\eta}$ has an atom at 0 and is hence not continuous. If η is neither a compound Poisson process nor the zero process and does not satisfy Condition (2.23), then $P(V_{\xi,\eta}(t) = b) = 0$ for any $t > 0$ and $b \in \mathbb{R}$ by Corollary 2.21. Continuity of $V_{q,\xi,\eta}$ then follows by conditioning on $\tau = t$ via

$$P(V_{q,\xi,\eta} = b) = \int_0^\infty P\left(V_{\xi,\eta}(t) = b \middle| \tau = t\right) P_\tau(\mathrm{d}t) = \int_0^\infty 0\, P_\tau(\mathrm{d}t) = 0.$$

Finally, if η is neither a compound Poisson process nor the zero process but satisfies Condition (2.23), then it must be of finite variation with non-zero drift, so that $V_{q,\xi,\eta}$ is absolutely continuous by Theorem 2.23 (v). $\square$

Example 2.27. If η is a compound Poisson process, then $V_{q,\xi,\eta}$ has an atom at zero and hence trivially cannot be absolutely continuous. But its distribution restricted to $\mathbb{R}^*$ can be absolutely continuous, as we show now. Denote the time of the first jump of η by R. Conditional on $\{\tau > R\}$, we can write

$$V_{q,\xi,\eta} \;=\; \mathrm{e}^{-\xi_R}\,\Delta\eta_R + \int_{R+}^{\tau} \mathrm{e}^{-\xi_{s-}}\,\mathrm{d}\eta_s = \mathrm{e}^{-\xi_R}\left(\Delta\eta_R + \int_{R+}^{R+(\tau-R)} \mathrm{e}^{-(\xi_{s-}-\xi_R)}\,\mathrm{d}\eta_s\right),$$

with $\mathrm{e}^{-\xi_R}$, $\Delta\eta_R$ and $\int_{R+}^{R+(\tau-R)} \mathrm{e}^{-(\xi_{s-}-\xi_R)}\,\mathrm{d}\eta_s$ being conditionally independent, similar to the proof of Theorem 2.23 (vi). If now ξ satisfies the ACP condition, or the jump distribution of η is absolutely continuous, we conclude that $V_{q,\xi,\eta}$ conditional on $\{\tau > R\}$ is absolutely continuous. In particular, the law of $V_{q,\xi,\eta}$ is not of pure type.

We end this section with an example where $V_{q,\xi,\eta}$ is continuous but not absolutely continuous.

Example 2.28. Let $0 < \alpha < 1$, c be an integer such that $c > 1/(1-\alpha)$, let $a_n = 2^{-c^n}$ for $n \in \mathbb{N}$ and define the Lévy measure ν_η by $\nu_\eta := \sum_{n=1}^\infty a_n^{-\alpha}\delta_{a_n}$. Then ν_η is infinite with $\int_{|x|\le 1} |x|\, \nu_\eta(dx) < \infty$. Let η be the subordinator with Lévy measure ν_η and drift 0. According to the final part of Example 41.23 in Sato [58], the potential measure W_η^q of η is continuous singular for any $q > 0$. Now let τ be an exponentially distributed random variable with parameter $q > 0$ and let ξ be the zero process. Then $V_{q,\xi,\eta} = \eta_\tau$ which has the same distribution as qW_η^q as seen in the discussion preceeding Theorem 2.23 and is hence continuous singular.

Further, to obtain an example with non-deterministic integrand, let ξ' be a compound Poisson process and denote by T the time of its first jump. With positive probability, ξ' does not jump before time τ, i.e. $\{T > \tau\}$ has positive probability, and on this set we have $V_{q,\xi',\eta} = \eta_\tau$. Since conditionally on $\{T > \tau\}$, τ is exponentially distributed with parameter $q + \nu_\xi(\mathbb{R})$, also the conditional distribution of η_τ given $\{T > \tau\}$ is continuous singular. We conclude that $V_{q,\xi',\eta}$ has a non-trivial singular part.

2.2.4. Conditions for Absolute Continuity of $\int_0^\infty \mathrm{e}^{-\xi_{s-}}\, \mathrm{d}\eta_s$

Finally, we consider the exponential functional without killing. Let ξ and η be two independent Lévy processes such that $V_{0,\xi,\eta} := \int_0^\infty \mathrm{e}^{-\xi_{s-}}\, \mathrm{d}\eta_s$ converges a.s. and η is not the zero process. A characterization when the integral converges in terms of the characteristic triplet of ξ and η is given by Erickson and Maller [27, Thm. 2]. In particular, ξ has to drift to ∞ a.s., which implies that it is transient.

Although much more attention has been paid to $V_{0,\xi,\eta}$ rather than $V_{q,\xi,\eta}$ when $q > 0$, not too many sufficient conditions for absolute continuity of $V_{0,\xi,\eta}$ are known. Bertoin et al. [13, Thm. 3.9 (a)] show that if η is of finite variation with non-zero drift and $\nu_\xi(\mathbb{R}) > 0$, then $V_{0,\xi,\eta}$ will be absolutely continuous. They also characterize continuity of $V_{0,\xi,\eta}$ and show that it is always continuous unless both ξ and η are deterministic, cf. [13, Thm. 2.2]. Kuznetsov et al. [38, Cor. 2.5] find that $V_{0,\xi,\eta}$ has a density whenever $\sigma_\eta^2 + \sigma_\xi^2 > 0$ and η and ξ both have finite expectation. Also, it is known that when ξ is spectrally negative, then $V_{0,\xi,\eta}$ is self-decomposable ([13, Rem. (i) in Sect. 2]), and hence absolutely continuous unless it is constant (e.g. [58, Ex. 27.8]), i.e. unless both ξ and η are deterministic.

It is also known that the law of $V_{0,\xi,\eta}$ is of pure type, and even that it is either degenerate, continuous singular, or absolutely continuous, e.g. [7, Sect. 5]. In [42] the law of $\int_0^\infty \mathrm{e}^{-(\ln c)N_{t-}}\, \mathrm{d}\eta_t$ is studied when η and N are two independent Poisson processes and $c > 1$. It is shown that the distribution in that case may be continuous singular or absolutely continuous, depending intrinsically on algebraic properties of c and the ratio of the rates of the two Poisson processes N and η (cf. [42, Thms. 3.1, 3.2]). The question of whether $V_{0,\xi,\eta}$ will always be absolutely continuous for general Lévy processes ξ and η that have both infinite Lévy measure (or only one of them) is still open. Still, in the following theorem, we collect various sufficient conditions for absolute continuity of $V_{0,\xi,\eta}$, many of which are new.

Theorem 2.29 (Sufficient conditions for absolute continuity of $V_{0,\xi,\eta}$)**.** *Let ξ and η be independent such that η is not the zero process and such that $V_{0,\xi,\eta} = \lim_{t\to\infty} \int_0^t \mathrm{e}^{-\xi_{s-}}\, \mathrm{d}\eta_s$ converges almost surely. Suppose that one of the following conditions is satisfied:*

(i) The characteristic triplet of η satisfies Condition (2.19) *with $t := \infty$, or more generally, Condition* (2.20) *with $t := \infty$, i.e. the right-hand sides of these inequalities are replaced by zero. In particular, the conditions are satisfied when $\sigma_\eta^2 > 0$.*

(ii) *The absolutely continuous part of* ν_η *is non-trivial:* $\nu_{\eta,\mathrm{ac}}(\mathbb{R}) > 0$.

(iii) *The characteristic exponent* Ψ_ξ *of* ξ *satisfies Condition* (2.16) *(recall from Example 2.17 that* (2.16) *is in particular satisfied when* $\sigma_\xi^2 > 0$*, when* ξ *is finite variation with non-zero drift, or when* ξ *satisfies* (2.18)*), and at least one of* ξ *and* η *is non-deterministic.*

(iv) *The absolutely continuous part of* ν_ξ *is non-trivial:* $\nu_{\xi,\mathrm{ac}}(\mathbb{R}) > 0$.

(v) η *is of finite variation with non-zero drift, and at least one of* ξ *and* η *is non-deterministic.*

(vi) ξ *is a compound Poisson process and* η *satisfies the ACP condition* (2.24).

(vii) η *is a compound Poisson process and* ξ *satisfies the ACP condition* (2.24).

(viii) ξ *is spectrally negative, and at least one of* ξ *and* η *is non-deterministic.*

Then $V_{0,\xi,\eta} = \int_0^\infty \mathrm{e}^{-\xi_{s-}}\, d\eta_s$ *is absolutely continuous.*

Proof. (viii) As mentioned before, $V_{0,\xi,\eta}$ is self-decomposable when ξ is spectrally negative. Since additionally it is not constant a.s. if additionally at least one of ξ and η is non-deterministic by [13, Thm. 2.2], it is absolutely continuous in this case ([58, Ex. 27.8]).

(v) Let η be of finite variation with non-zero drift. Since $\int_0^\infty \mathrm{e}^{-\xi_{s-}}\, \mathrm{d}\eta_s$ converges, ξ must be transient. It then follows from [13, Thm. 3.9(a)] that $V_{0,\eta,\xi}$ is absolutely continuous when $\nu_\xi(\mathbb{R}) > 0$. If $\nu_\xi(\mathbb{R}) = 0$, then ξ is spectrally negative, and absolute continuity of $V_{0,\xi,\eta}$ follows from (viii).

(i) – (iii): As long as $\nu_\eta(\mathbb{R}) > 0$ in (iii), this follows in complete analogy to the corresponding proof of Theorem 2.18 (i)–(iii) by conditioning on the paths $\xi = f$ and observing that $\lambda(\{s \in [0,\infty) : \mathrm{e}^{-f(s-)} \neq 0\}) = \infty$. If $\nu_\eta(\mathbb{R}) = 0$ in (iii), either $\sigma_\eta^2 > 0$ or η is deterministic and hence of finite variation. The claim then follows from the previously shown cases (i) and (v).

(iv) Choose a set $D \in \mathcal{B}_1$ that is bounded away from zero with $\nu_{\xi,\mathrm{ac}}(D) > 0$ and $\nu_{\xi,\mathrm{sing}}(D) = 0$. Denote by R the time of the first jump of ξ with jumping size in D and by Y its jump size (which has a density by assumption). Observe that R is a stopping time with respect to the augmented filtration. Writing

$$\int_0^\infty \mathrm{e}^{-\xi_{s-}}\, \mathrm{d}\eta_s = \int_0^R \mathrm{e}^{-\xi_{s-}}\, \mathrm{d}\eta_s + \mathrm{e}^{-Y} \left(\mathrm{e}^{-\xi_{R-}} \int_{R+}^\infty \mathrm{e}^{-(\xi_{s-}-\xi_R)}\, \mathrm{d}\eta_s \right),$$

observing that Y is independent from $(\int_0^R \mathrm{e}^{-\xi_{s-}}\, \mathrm{d}\eta_s,\ \mathrm{e}^{-\xi_{R-}} \int_{R+}^\infty \mathrm{e}^{-(\xi_{s-}-\xi_R)}\, \mathrm{d}\eta_s)$ and that $\int_{R+}^\infty \mathrm{e}^{-(\xi_{s-}-\xi_R)}\, \mathrm{d}\eta_s \overset{d}{=} \int_0^\infty \mathrm{e}^{-\xi_{s-}}\, \mathrm{d}\eta_s$ is continuous (hence $\neq 0$ a.s.) by Theorem 2.2 in [13], it follows as in the proof of Theorem 2.18 (iv) that $\int_0^\infty \mathrm{e}^{-\xi_{s-}}\, \mathrm{d}\eta_s$ is absolutely continuous.

(vi) Denote by T the time of the first jump of ξ, which is exponentially distributed with parameter $\nu_\xi(\mathbb{R}) \in (0,\infty)$. Then

$$V_{0,\xi,\eta} = \int_0^T \mathrm{d}\eta_s + \int_{T+}^\infty \mathrm{e}^{-\xi_{s-}}\, \mathrm{d}\eta_s = \eta_T + \mathrm{e}^{-\xi_T} \int_{T+}^\infty \mathrm{e}^{-(\xi_{s-}-\xi_T)}\, \mathrm{d}\eta_s$$

with η_T, $\xi_T = \Delta\xi_T$ and $\int_{T+}^\infty \mathrm{e}^{-(\xi_{s-}-\xi_T)}\,\mathrm{d}\eta_s$ being independent by the strong Markov property. Since η_T is absolutely continuous by the ACP condition, absolute continuity of $V_{0,\xi,\eta}$ follows.

(vii) Denote by R_i the time of the i'th jump of η and by Z_i its jump size. Since ξ and η do not jump together a.s., we have

$$V_{0,\xi,\eta} = \int_0^\infty \mathrm{e}^{-\xi_{s-}}\,\mathrm{d}\eta_s = \sum_{i=1}^\infty \mathrm{e}^{-\xi_{R_i-}} Z_i \overset{a.s.}{=} \mathrm{e}^{-\xi_{R_1}} \left(\sum_{i=1}^\infty \mathrm{e}^{-(\xi_{R_i}-\xi_{R_1})} Z_i \right).$$

But ξ_{R_1} and $\sum_{i=1}^\infty \mathrm{e}^{-(\xi_{R_i}-\xi_{R_1})} Z_i$ are independent by the strong Markov property, and since $V_{0,\xi,\eta}$ is different from zero a.s. (as it has no atom as η is non-deterministic), also $\sum_{i=1}^\infty \mathrm{e}^{-(\xi_{R_i}-\xi_{R_1})} Z_i$ is different from zero almost surely. The claim then follows by observing that $\mathrm{e}^{-\xi_{R_1}}$ is absolutely continuous by the (ACP) condition, since R_1 is exponentially distributed with parameter $\nu_\eta(\mathbb{R}) \in (0,\infty)$. □

Observe that part (viii) above is already covered by parts (i), (iii) and (v), for if ξ is spectrally negative and drifts to infinity, then it is either of finite variation with strictly positive drift, or it is of infinite variation with $\nu_\xi((0,\infty)) = 0$, so that in both cases Ψ_ξ satisfies Condition (2.16) and (viii) follows from (iii) when $\nu_\eta(\mathbb{R}) > 0$. When $\nu_\eta(\mathbb{R}) = 0$, (viii) follows from (i) and (v).

The following result generalizes Corollary 2.5 of Kuznetsov et al. [38] in the sense that it shows that the assumption in [38] that both ξ and η have finite expectation can be omitted for the existence of a density.

Corollary 2.30. *Let ξ and η be independent Lévy processes such that $V_{0,\xi,\eta}$ converges a.s. and such that η is not the zero process. Suppose that $\sigma_\eta^2 + \sigma_\xi^2 > 0$. Then $V_{0,\xi,\eta}$ is absolutely continuous.*

Proof. If $\sigma_\eta^2 > 0$, this follows from Theorem 2.29 (i). If $\sigma_\xi^2 > 0 = \sigma_\eta^2$, then $\nu_\eta(\mathbb{R}) > 0$ or η is deterministic but non-zero. The claim then follows from parts (iii) and (v) of Theorem 2.29, respectively. □

A result similar to Corollary 2.25 does not hold when $q = 0$. This follows by observing that $\int_0^\infty \mathrm{e}^{-\ln(c)N_{t-}}\,\mathrm{d}\eta_t$ can be absolutely continuous for suitable constants $c > 1$ and Poisson processes N and η by [42, Thm. 3.2]; obviously, a Poisson process does not satisfy the ACP condition.

Let us finally mention that similar to Remark 2.20, some of the results of Theorem 2.29 can be easily extended to functionals of the form $\int_0^\infty g(\xi_{s-})\,\mathrm{d}\eta_s$, assuming the convergence of the integral. In particular, when $g : [0,\infty) \to \mathbb{R}$ is continuous with zero set N, if the integral converges, if $\int_0^\infty P(\xi_s \in N)\,\mathrm{d}s = 0$ and if one of the conditions (i)–(iii) of Theorem 2.29 is satisfied, then $\int_0^\infty g(\xi_{s-})\,\mathrm{d}\eta_s$ will be absolutely continuous. Sufficient conditions for $\int_0^\infty P(\xi_s \in N)\,\mathrm{d}s = 0$ have been discussed in Remark 2.20.

2.3. Connection to Markov Processes

As noted in Section 1.1.4 of the introduction, the exponential functional $V_{0,\xi,\eta}$, provided the improper integral converges, describes the stationary distribution of a generalized Ornstein-Uhlenbeck process. This section aims to show that the law of the killed exponential functional $V_{q,\xi,\eta}$, too, arises as the stationary distribution of a Markov process. By calculating the infinitesimal generator of the process, we obtain a key tool for the analysis in Section 2.4.

Theorem 2.31. *Let ξ and η be two independent Lévy processes and $q \in (0,\infty)$. Define the Lévy process U by (1.11) such that $\mathcal{E}(U) = e^{-\xi}$ and let N be a Poisson process with parameter q, which is independent of U. Define*

$$\widetilde{U} := U - N.$$

Then $\widetilde{U}$ is a Lévy process with Lévy measure $\nu_{\widetilde{U}}$ such that $\nu_{\widetilde{U}}(\{-1\}) = q$. Further

$$V_{q,\xi,\eta} \overset{d}{=} \int_0^\infty \mathcal{E}(\widetilde{U})_{s-}\,\mathrm{d}\eta_s \tag{2.25}$$

and $\mathcal{L}(V_{q,\xi,\eta})$ is the unique invariant probability measure of the Markov process $\widetilde{V} = (\widetilde{V}_t)_{t\geq 0}$ satisfying the stochastic differential equation

$$\mathrm{d}\widetilde{V}_t = \widetilde{V}_{t-}\,\mathrm{d}\widetilde{U}_t + \mathrm{d}\eta_t, \quad t \geq 0, \tag{2.26}$$

with starting random variable $\widetilde{V}_0$ independent of $\widetilde{U}$ and η. The solution of (2.26) is given by

$$\widetilde{V}_t = \mathrm{e}^{-\xi_t}\mathbb{1}_{\{N(t)=0\}}\widetilde{V}_0 + \mathrm{e}^{-\xi_t}\int_{T(t)+}^t \mathrm{e}^{\xi_{s-}}\,\mathrm{d}\eta_s, \tag{2.27}$$

where $T(t)$ denotes the time of the last jump of N before t, with the convention that $T(t) = 0$ if no jump of N occurs before time t. In particular, if Z is a random variable, independent of (ξ,η,N) and with the same distribution as $V_{q,\xi,\eta}$, then Z satisfies the random fixed point equation

$$Z \overset{d}{=} \mathrm{e}^{-\xi_t}\mathbb{1}_{\{N(t)=0\}}Z + \mathrm{e}^{-\xi_t}\int_{T(t)+}^t \mathrm{e}^{\xi_{s-}}\,\mathrm{d}\eta_s$$

for each $t > 0$.

Proof. Observe that $\Delta\widetilde{U}_t = -1$ if and only if $\Delta N_t = 1$, i.e. N counts the jumps of size -1 of $\widetilde{U}$, and that the Lévy measure $\nu_{\widetilde{U}}$ of $\widetilde{U}$ is concentrated on $[-1,\infty)$ with $\nu_{\widetilde{U}}(\{-1\}) = q$. Since $\mathcal{E}(\widetilde{U})_t = 0$ whenever $N_t \geq 1$ and $\mathcal{E}(\widetilde{U})_t = \mathcal{E}(U)_t = \mathrm{e}^{-\xi_t}$ on $\{N_t = 0\}$, and since the time of the first jump of N is exponentially distributed with parameter q, we obtain Equation (2.25). By [7, Thm. 2.2] and [7, Prop. 3.2], the differential equation $\mathrm{d}\widetilde{V}_t = \widetilde{V}_{t-}\mathrm{d}\widetilde{U}_t + \mathrm{d}\eta_t$ has a strictly stationary solution, unique in distribution, and the corresponding marginal distribution is given by $\mathcal{E}(U)_\tau \int_0^\tau \mathcal{E}(U)_{s-}^{-1}\,\mathrm{d}\eta_s$, where τ is Exp(q)-distributed, independent from (U,η). Further, the process $(\widetilde{V}_t)_{t\geq 0}$ defined in (2.26) is a time-homogeneous Markov process (cf. [7, Lem. 3.3]), so that the marginal strictly stationary distribution is the unique invariant probability measure of the Markov process. Since $\mathcal{E}(U)_t \int_{(0,t]} \mathcal{E}(U)_{s-}^{-1}\,d\eta_s \overset{d}{=} \int_0^t \mathcal{E}(U)_{s-}\,d\eta_s$ for every fixed $t > 0$ by [7, Lem. 3.1], we

obtain that
$$\mathcal{E}(U)_\tau \int_0^\tau \mathcal{E}(U)_{s-}^{-1}\, d\eta_s \overset{d}{=} \int_0^\tau \mathcal{E}(U)_{s-}\, d\eta_s \overset{d}{=} V_{q,\xi,\eta}$$
by conditioning on τ. This shows that $\mathcal{L}(V_{q,\xi,\eta})$ is the unique invariant probability measure of the Markov process $\widetilde{V}$.

To see the specific form (2.27) of the solution of (2.26), denote for $0 \le s \le t$
$$\mathcal{E}(\widetilde{U})_{s,t} = \exp\Big((\widetilde{U}_t - \widetilde{U}_s) - \sigma_{\widetilde{U}}^2 (t-s)/2\Big) \prod_{s<u\le t} (1+\Delta\widetilde{U}_u) e^{-\Delta\widetilde{U}_u}.$$

By [7, Prop. 3.2, Eq. (2.7)], the solution $\widetilde{V} = (\widetilde{V}_t)_{t\ge 0}$ of (2.26) is given by
$$\begin{aligned}\widetilde{V}_t \;&=\; \mathcal{E}(\widetilde{U})_t \left(\widetilde{V}_0 + \int_{0+}^t \left[\mathcal{E}(\widetilde{U})_{s-}\right]^{-1} \mathrm{d}\eta_s\right) \mathbb{1}_{\{N(t)=0\}} \\ &\quad + \mathcal{E}(\widetilde{U})_{T(t),t} \int_{T(t)+}^t \left[\mathcal{E}(\widetilde{U})_{T(t),s-}\right]^{-1} \mathrm{d}\eta_s\, \mathbb{1}_{\{N(t)\ge 1\}}.\end{aligned}$$

Since $\widetilde{U}_s - \widetilde{U}_{T(t)} = U_s - U_{T(t)}$ for $s \in (T(t),t)$ we see from the Doléans-Dade formula that
$$\mathcal{E}(\widetilde{U})_{T(t),s} = \mathrm{e}^{-(\xi_s - \xi_{T(t)})} \quad \text{for} \quad s \in (T(t),t).$$

Hence, the above can be rewritten (a.s. for fixed t) as
$$\widetilde{V}_t = \mathrm{e}^{-\xi_t} \widetilde{V}_0 \mathbb{1}_{\{N(t)=0\}} + \mathrm{e}^{-(\xi_t - \xi_{T(t)})} \int_{T(t)+}^t \mathrm{e}^{\xi_{s-} - \xi_{T(t)}}\, \mathrm{d}\eta_s.$$

Since ξ and $T(t)$ are independent of η, we can pull out $\mathrm{e}^{-\xi_{T(t)}}$ from the last integral leading to (2.27). The desired fixed point equation is now immediate, since $\mathcal{L}(V_{q,\xi,\eta})$ is the invariant probability measure of $\widetilde{V}$ with independent starting value $\widetilde{V}_0$. $\square$

Remark 2.32. In the setting of Theorem 2.31, define the killed Lévy process $\widetilde{\xi}$ with killing rate q and cemetery ∞ by
$$\widetilde{\xi}_t = \begin{cases} \xi_t, & t < \tau, \\ +\infty, & t \ge \tau, \end{cases}$$
where τ is Exp(q)-distributed, independent of (ξ, η). Then
$$V_{q,\xi,\eta} = \int_0^\infty \mathrm{e}^{-\widetilde{\xi}_{s-}}\, \mathrm{d}\eta_s,$$
so that $V_{q,\xi,\eta}$ can be seen as an exponential functional with respect to $\widetilde{\xi}$ and η. The killed Lévy process $\widetilde{\xi}$ and $\widetilde{U}$ are related by $\mathcal{E}(\widetilde{U}) = \mathrm{e}^{-\widetilde{\xi}}$.

Remark 2.33. Since in the situation of Theorem 2.31, the characteristic triplet of $-N$ is given by $(0, q\delta_{-1}, -q)$, where δ_{-1} denotes the Dirac measure at -1, the characteristic triplet of $\widetilde{U}$ can be expressed in terms of the characteristic triplet of U via
$$\sigma_{\widetilde{U}}^2 = \sigma_U^2, \quad \nu_{\widetilde{U}} = \nu_U + q\delta_{-1}, \quad \gamma_{\widetilde{U}} = \gamma_U - q.$$

Additionally, if U is of finite variation, then so is $\widetilde{U}$, and the drifts of $\widetilde{U}$ and U are equal. The key difference between the describing stochastic differential equations for the exponential functional without killing and the killed exponential functional can then be seen in the Lévy measure of U and $\widetilde{U}$, respectively, since $\nu_U((-\infty,-1])=0$ while $\nu_{\widetilde{U}}(\{-1\})=q$.

Recall from Section 1.1.1 that the infinitesimal generator $\mathcal{A}^{\widetilde{V}}$ of $(\widetilde{V}_t)_{t\geq 0}$ is the linear operator defined by

$$\mathcal{A}^{\widetilde{V}}f(x)=\lim_{t\to 0}\frac{\mathbb{E}^x[f(\widetilde{V}_t^x)]-f(x)}{t},\quad x\in\mathbb{R},$$

on the set of functions $f\in C_b(\mathbb{R})$ for which this limit exists uniformly in x. Here, $\widetilde{V}_t^x$ denotes the solution of (2.26) with initial value $\widetilde{V}_0^x=x$ and $\mathbb{E}^x$ denotes the corresponding expectation. As a starting point of the analysis, consider the generalized Ornstein–Uhlenbeck process, i.e. the Markov process given by

$$V_t^x=x+\int_0^t V_{s-}^x\,\mathrm{d}U_t+\eta_t,$$

which is the solution of the differential equation $dV_t=V_{t-}\,dU_t+d\eta_t$ with starting random variable $V_0^x=x$. As shown in [5, Thm. 3.1, Cor. 3.2, Cor. 3.3], $(V_t^x)_{t\geq 0}$ is a (rich) Feller process and the domain of its infinitesimal generator $\mathcal{A}^V$ contains the space

$$C_{0,pl}^2(\mathbb{R}):=\left\{f\in C_0^2(\mathbb{R}):\lim_{|x|\to\infty}\Big(|xf'(x)|+|x^2f''(x)|\Big)=0\right\},$$

where the added subscript *pl* refers to the power law decay of the derivatives, on which $\mathcal{A}^V$ acts by

$$\begin{aligned}\mathcal{A}^Vf(x)&=\mathcal{A}^\eta f(x)-f'(x)x\gamma_\xi+\frac{1}{2}(f''(x)x^2+f'(x)x)\sigma_\xi^2\\&\quad+\int_{\mathbb{R}}\Big(f(x\mathrm{e}^{-y})-f(x)+f'(x)xy\mathbb{1}_{\{|y|\leq 1\}}\Big)\,\nu_\xi(\mathrm{d}y)\\&=\mathcal{A}^\eta f(x)+xf'(x)\gamma_U+\frac{1}{2}x^2f''(x)\sigma_U^2\\&\quad+\int_{\mathbb{R}}\Big(f(x+xy)-f(x)-xyf'(x)\mathbb{1}_{\{|y|\leq 1\}}\Big)\nu_U(\mathrm{d}y),\end{aligned}\tag{2.28}$$

where $\mathcal{A}^\eta$ denotes the infinitesimal generator of the Lévy process η, which is by Theorem 1.8 of the introduction given by

$$\mathcal{A}^\eta f(x)=\gamma_\eta f'(x)+\frac{1}{2}\sigma_\eta^2f''(x)+\int_{\mathbb{R}}\Big(f(x+y)-f(x)-f'(x)y\mathbf{1}_{|y|\leq 1}\Big)\,\nu_\eta(\mathrm{d}y)$$

for $f\in C_{0,pl}^2(\mathbb{R})$. From this we can derive the generator of $\widetilde{V}$ as follows.

Theorem 2.34. *Let $q\in[0,\infty)$, $(\widetilde{V}_t)_{t\geq 0}$ as defined in (2.26) and assume that $V_{0,\xi,\eta}$ converges a.s. whenever $q=0$ is considered. Then the set $C_{0,pl}^2(\mathbb{R})$ is contained in $\mathrm{dom}(\mathcal{A}^{\widetilde{V}})$, and for $f\in C_{0,pl}^2(\mathbb{R})$ we have*

$$\mathcal{A}^{\widetilde{V}}f(x)=\mathcal{A}^Vf(x)+q\,(f(0)-f(x)),$$

with $\mathcal{A}^V$ *as given in* (2.28).

Proof. The case $q=0$ was shown in [5]. Let $q>0$ and $f \in C^2_{0,pl}(\mathbb{R})$. Then for each $t>0$,

$$\begin{aligned}\frac{\mathbb{E}^x[f(\widetilde{V}_t^x)]-f(x)}{t} &= \frac{\mathbb{E}^x[f(\widetilde{V}_t^x)|N_t=0]\mathbb{P}(N_t=0)-f(x)}{t}\\ &\quad+\mathbb{E}^x[f(\widetilde{V}_t^x)|N_t=1]\frac{\mathbb{P}(N_t=1)}{t}\\ &\quad+\mathbb{E}^x[f(\widetilde{V}_t^x)|N_t\geq 2]\frac{\mathbb{P}(N_t\geq 2)}{t}. \end{aligned}\tag{2.29}$$

Since f is bounded and $\mathbb{P}(N_t \geq 2) = o(t)$ as $t \to 0$, the last term tends to 0, uniformly in $x \in \mathbb{R}$, as $t \to 0$. Denote the time of the last jump of N before t by $T(t)$. Then

$$\widetilde{V}_t = \mathrm{e}^{-\xi_t}\left(x+\int_0^t \mathrm{e}^{\xi_{s-}}\,\mathrm{d}\eta_s\right)\mathbb{1}_{\{N(t)=0\}} + \left(\mathrm{e}^{-(\xi_t-\xi_{T(t)})}\int_{(T(t),t]}\mathrm{e}^{\xi_s-\xi_{T(t)}}\,\mathrm{d}\eta_s\right)\mathbb{1}_{\{N(t)\geq 1\}}$$

by [7, Prop. 3.2]. Since $\mathbb{P}(N_t=1) = qt\mathrm{e}^{-qt}$, we conclude that

$$\lim_{t\to 0}\mathbb{E}^x[f(\widetilde{V}_t^x)|N_t=1]\frac{\mathbb{P}(N_t=1)}{t} = f(0)q,$$

uniformly in x. Finally, since $\mathbb{P}(N_t=0)=\mathrm{e}^{-qt}$ we can write

$$\frac{\mathbb{E}^x[f(\widetilde{V}_t^x|N_t=0)]P(N_t=0)-f(x)}{t} = \frac{\mathbb{E}^x[f(V_t^x)]-f(x)}{t} + \frac{\mathrm{e}^{-qt}-1}{t}\mathbb{E}^x[f(V_t^x)].$$

The first of these terms converges uniformly in x to $\mathcal{A}^V f(x)$ as $t \to 0$, and the second uniformly to $-qf(x)$ since $\mathbb{E}^x[f(V_t^x)]$ converges uniformly to $f(x)$ since $(V_t^x)_{t\geq 0}$ is a Feller process ([5, Thm. 3.1]). Together with (2.29) this gives the claim. □

Remark 2.35. (i) Alternatively, the above theorem could be shown following the proof of Theorem 3.1 and Corollary 3.2 in [5] and replacing U by $\widetilde{U}$ to allow for jumps of size -1. Observing that the characteristics γ_U and $\gamma_{\widetilde{U}}$ differ by $q = -\int_{\{-1\}} y\nu_{\widetilde{U}}(dy)$ then leads to the same result.
(ii) Aside from the expression in Theorem 2.34, the operator $A^{\widetilde{V}}$ can also be given in terms of the characteristics of $\widetilde{U}$. As $\nu_{\widetilde{U}}(\{-1\}) = q$ we have for$f \in C^2_{0,pl}(\mathbb{R})$ that

$$\begin{aligned}\mathcal{A}^{\widetilde{V}}f(x) &= \mathcal{A}^\eta f(x) + xf'(x)\gamma_{\widetilde{U}} + \frac{1}{2}x^2 f''(x)\sigma^2_{\widetilde{U}}\\ &\quad+\int_{\mathbb{R}}\Big(f(x+xy)-f(x)-xyf'(x)\mathbb{1}_{[-1,1]}(y)\Big)\nu_{\widetilde{U}}(\mathrm{d}y)\end{aligned}\tag{2.30}$$

which is (2.28) with U replaced by $\widetilde{U}$.

The key to deriving the equations describing $\mathcal{L}(V_{q,\xi,\eta})$ in the following sections lies in the fact that the law of the killed exponential functional is the unique invariant probability law of the Markov process in (2.26) and thus the equation

$$\int_{\mathbb{R}} \mathcal{A}^{\widetilde{V}} f(x)\,\mathcal{L}(V_{q,\xi,\eta})(\mathrm{d}x) = 0 \tag{2.31}$$

holds for every function f in the domain of the operator $\mathcal{A}^{\widetilde{V}}$ (see e.g. [40, Thm. 3.37]; although the proof is given for Feller processes only, one can see from the argument given that this must hold true also for invariant probability measures of general Markov processes). In view of Theorem 2.34, this is in particular satisfied for $f \in C^2_{0,pl}(\mathbb{R})$. We also note the following special case as a key tool for Section 2.4.2.

Corollary 2.36. *The space* $C_c^\infty(\mathbb{R})$ *is a subset of* $\mathrm{dom}(\mathcal{A}^{\widetilde{V}})$ *and* (2.31) *holds for every test function* f*.*

2.4. Distributional Equations for Killed Exponential Functionals

2.4.1. Equations Derived by Fourier and Laplace Methods

In this section, we use the infinitesimal generator obtained in Theorem 2.34 to derive distributional equations for the law of the killed exponential functional, as well as a functional equation to describe its density, using the method developed in [5] for the case without killing. Throughout the analysis, we set $\mathcal{L}(V_{q,\xi,\eta}) = \mu$ and denote its characteristic function by $\varphi_{q,\xi,\eta}$. The following conclusion now follows in complete analogy to Theorem 4.1 and Corollary 4.3 in [5], using Lemma 4.2 of [5]. For convenience, the following corollary is given in the characteristics of the original Lévy process ξ, as well as in the characteristics of $\widetilde{U}$ with $\nu_{\widetilde{U}}(\{-1\}) = q$.

Corollary 2.37. *Let $q \geq 0$ and assume that the exponential functional converges a.s. whenever $q = 0$ is considered. Further, let $h \in C_c^\infty(\mathbb{R})$ such that $h(x) = 1$ for $|x| \leq 1$ and $h(x) = 0$ for $|x| \geq 2$. Set $h_n(x) := h(x/n)$ and $f_{u,n}(x) = \mathrm{e}^{iux} h_n(x)$ for $u \in \mathbb{R}$, $n \in \mathbb{N}$, and $x \in \mathbb{R}$. Then*

$$\begin{aligned}
\psi_\eta(u)\varphi_{V_{q,\xi,\eta}}(u) &= q(\varphi_{V_{q,\xi,\eta}}(u) - 1) \\
&\quad + \lim_{n\to\infty} \Bigg(\gamma_\xi \int_{\mathbb{R}} x f'_{u,n}(x)\, \mu(\mathrm{d}x) - \frac{\sigma_\xi^2}{2} \int_{\mathbb{R}} (x^2 f''_{u,n}(x) + x f'_{u,n}(x))\, \mu(\mathrm{d}x) \\
&\quad - \int_{\mathbb{R}} \int_{\mathbb{R}} (f_{u,n}(x\mathrm{e}^{-y}) - f_{u,n}(x) + xy f'_{u,n}(x) \mathbb{1}_{\{|y|\leq 1\}})\, \nu_\xi(\mathrm{d}y)\, \mu(\mathrm{d}x) \Bigg) \\
&= - \lim_{n\to\infty} \Bigg(\gamma_{\widetilde{U}} \int_{\mathbb{R}} x f'_{u,n}(x)\, \mu(\mathrm{d}x) + \frac{\sigma_{\widetilde{U}}^2}{2} \int_{\mathbb{R}} x^2 f''_{u,n}(x)\, \mu(\mathrm{d}x) \\
&\quad + \int_{\mathbb{R}} \int_{[-1,\infty)} (f_{u,n}(x + xy) - f_{u,n}(x) - xy f'_{u,n}(x) \mathbb{1}_{\{|y|\leq 1\}})\, \nu_{\widetilde{U}}(\mathrm{d}y)\, \mu(\mathrm{d}x) \Bigg)
\end{aligned} \tag{2.32}$$

for all $u \in \mathbb{R}$. If additionally $\mathbb{E}V_{q,\xi,\eta}^2 < \infty$, then, for all $u \in \mathbb{R}$,

$$\begin{aligned}
\psi_\eta(u)\varphi_{V_{q,\xi,\eta}}(u) &= q(\varphi_{V_{q,\xi,\eta}}(u) - 1) + \gamma_\xi u \varphi'_{V_{q,\xi,\eta}}(u) - \frac{\sigma_\xi^2}{2} \left(u^2 \varphi''_{V_{q,\xi,\eta}}(u) + u \varphi'_{V_{q,\xi,\eta}}(u) \right) \\
&\quad - \int_{\mathbb{R}} \left(\varphi_{V_{q,\xi,\eta}}(u\mathrm{e}^{-y}) - \varphi_{V_{q,\xi,\eta}}(u) + uy \varphi'_{V_{q,\xi,\eta}}(u) \mathbf{1}_{|y|\leq 1} \right) \nu_\xi(\mathrm{d}y) \\
&= -\gamma_{\widetilde{U}} u \varphi'_{V_{q,\xi,\eta}}(u) - \frac{\sigma_{\widetilde{U}}^2}{2} u^2 \varphi''_{V_{q,\xi,\eta}}(u) \\
&\quad - \int_{[-1,\infty)} \left(\varphi_{V_{q,\xi,\eta}}(u + uy) - \varphi_{V_{q,\xi,\eta}}(u) - uy \varphi'_{V_{q,\xi,\eta}}(u) \mathbb{1}_{\{|y|\leq 1\}} \right) \nu_{\widetilde{U}}(\mathrm{d}y) \\
&= -\mathbb{E}\left[e^{iuV_{q,\xi\eta}} \psi_{\widetilde{U}}(uV_{q,\xi\eta}) \right].
\end{aligned} \tag{2.33}$$

Remark 2.38. Observe that the integral with respect to $\nu_{\widetilde{U}}$ does not vanish even if ξ (and hence U) is a Brownian motion with drift due to the added point mass at -1.

Equation (2.33) can be solved in special cases, some of which are discussed in Section 2.4.4. Note that it has been shown in [2, Thm. 3.1], that the precondition $\mathbb{E}V_{q,\xi,\eta}^2 < \infty$ is fulfilled if

$$\mathbb{E}[U_1^2] < \infty, \quad \mathbb{E}[\eta_1^2] < \infty, \quad 2\mathbb{E}[U_1] + \mathrm{Var}(U_1) < q, \tag{2.34}$$

and $\lim_{t\to\infty} \mathcal{E}(\widetilde{U})_t = 0$ a.s., the latter obviously being satisfied whenever $q > 0$. If η is a subordinator, an equation similar to (2.33) also holds for the Laplace transforms without any moment condition. Let $\mathbb{L}_Y(u)$ denote the Laplace transform of the law of a random variable Y, e.g. $\mathbb{L}_{V_{q,\xi,\eta}}(u) = \mathbb{E}[\mathrm{e}^{-uV_{q,\xi,\eta}}]$, $u \geq 0$. Similar to Remark 4.5 in [5] we obtain

$$\begin{aligned}\Big(\ln \mathbb{L}_{\eta_1}(u)\Big)\mathbb{L}_{V_{q,\xi,\eta}}(u) =& q(\mathbb{L}_{V_{q,\xi,\eta}}(u) - 1) - \gamma_\xi u\mathbb{E}[V_{q,\xi,\eta}\mathrm{e}^{-uV_{q,\xi,\eta}}] \\ &- \frac{\sigma_\xi^2}{2}\Big(u^2\mathbb{E}[V_{q,\xi,\eta}^2\mathrm{e}^{-uV_{q,\xi,\eta}}] - u\mathbb{E}[V_{q,\xi,\eta}\mathrm{e}^{-uV_{q,\xi,\eta}}]\Big) \\ &- \int_{\mathbb{R}}\Big(\mathbb{L}_{V_{q,\xi,\eta}}(u\mathrm{e}^{-y}) - \mathbb{L}_{V_{q,\xi,\eta}}(u) - uy\mathbb{E}[V_{q,\xi,\eta}\mathrm{e}^{-uV_{q,\xi,\eta}}]\mathbf{1}_{|y|\leq 1}\Big)\,\nu_\xi(\mathrm{d}y),\end{aligned}$$

for $u > 0$, rearranging which yields

$$\begin{aligned}\frac{\ln \mathbb{L}_{\eta_1}(u)}{u}\mathbb{L}_{V_{q,\xi,\eta}}(u) =& q\frac{\mathbb{L}_{V_{q,\xi,\eta}}(u) - 1}{u} + \Big(\gamma_\xi - \frac{\sigma_\xi^2}{2}\Big)\mathbb{L}'_{V_{q,\xi,\eta}}(u) - \frac{\sigma_\xi^2}{2}u\mathbb{L}''_{V_{q,\xi,\eta}}(u) \\ &- \int_{\mathbb{R}}\left(\frac{\mathbb{L}_{V_{q,\xi,\eta}}(u\mathrm{e}^{-y})}{u} - \frac{\mathbb{L}_{V_{q,\xi,\eta}}(u)}{u} + y\mathbb{L}'_{V_{q,\xi,\eta}}(u)\mathbf{1}_{|y|\leq 1}\right)\,\nu_\xi(\mathrm{d}y).\end{aligned} \tag{2.35}$$

Restricting the jump part of ξ to be of finite variation, (2.35) reduces to

$$\begin{aligned}\frac{\ln \mathbb{L}_{\eta_1}(u)}{u}\mathbb{L}_{V_{q,\xi,\eta}}(u) =& q\frac{\mathbb{L}_{V_{q,\xi,\eta}}(u) - 1}{u} + \Big(\gamma_\xi^0 - \frac{\sigma_\xi^2}{2}\Big)\mathbb{L}'_{V_{q,\xi,\eta}}(u) - \frac{\sigma_\xi^2}{2}u\mathbb{L}''_{V_{q,\xi,\eta}}(u) \\ &- \int_{\mathbb{R}}\left(\frac{\mathbb{L}_{V_{q,\xi,\eta}}(u\mathrm{e}^{-y})}{u} - \frac{\mathbb{L}_{V_{q,\xi,\eta}}(u)}{u}\right)\,\nu_\xi(\mathrm{d}y)\end{aligned}$$

and we can derive a functional equation for the density of $V_{q,\xi,\eta}$ in the absolutely continuous case by Laplace inversion. The proof is in complete analogy to the proof for the case $q = 0$ given in Theorem 2.1 in [3] and hence omitted. For $q \geq 0$ we obtain the following result.

Proposition 2.39. *Assume that the jump part of ξ is of finite variation and η is a subordinator, i.e. $\ln \mathbb{L}_{\eta_1}(u) = -\gamma_\eta^0 u - \int_{(0,\infty)}(1 - \mathrm{e}^{-uy})\nu_\eta(\mathrm{d}y)$ for $u \geq 0$. Further assume that $\mathcal{L}(V_{q,\xi,\eta}) = \mu$ is absolutely continuous with density f_μ and, whenever $\sigma_\xi^2 \neq 0$, the function $z \mapsto z^2 f_\mu(z)$ is absolutely continuous on $[0, z]$ for all $z > 0$. Then $f_\mu(z)$ fulfills for λ-a.e. $z > 0$*

$$\begin{aligned}&\gamma_\eta^0 f_\mu(z) - \Big(\gamma_\xi^0 + \frac{\sigma_\xi^2}{2}\Big)z f_\mu(z) - \frac{\sigma_\xi^2}{2}z^2 f'_\mu(z) - q\int_z^\infty f_\mu(s)\mathrm{d}s \\ &\quad = \int_z^\infty \nu_\xi((\ln\tfrac{s}{z}, \infty))f_\mu(s)\mathrm{d}s - \int_0^z \Big(\nu_\xi((-\infty, \ln\tfrac{s}{z})) + \nu_\eta((z - s, \infty))\Big)f_\mu(s)\mathrm{d}s.\end{aligned} \tag{2.36}$$

Recall that various sufficient conditions for absolute continuity of μ were given in Theorems 2.23 and 2.29. Nevertheless, there are cases where Proposition 2.39 is not applicable, e.g. if η is not a subordinator, if $\int_{-1}^1 |x|\nu_\xi(\mathrm{d}x) = \infty$, or if μ is not absolutely continuous. In the next section, we derive a general equation for the law of $V_{q,\xi,\eta}$ without a priori assumptions from which Proposition 2.39 is reobtained as a special case (see Remark 2.49). The proof given in this section, however, is comparably shorter and less technical.

Remark 2.40. Observe that we obtain the functional equation given in (2.3) of [48] in the special case of $\eta_t = t$ and ξ being a subordinator, as $V_{q,\xi,t}$ is always absolutely continuous by [49].

2.4.2. Equations Derived by Schwartz Theory of Distributions

In this section, we give distributional equations for the law of the killed exponential functional using Schwartz theory of distributions, where we follow a similar approach as used in [38, Thm. 2.2] for the exponential functional without killing. While studying the method, we found a small oversight in the proof of said theorem which results in the distributional equation not being applicable in all claimed cases. This is discussed in Remark 2.50. However, we also found that the method works when killing is included and that the moment condition $\mathbb{E}|\xi_1|, \mathbb{E}|\eta_1| < \infty$ of [38] is not needed to arrive at the desired conclusion in both cases. Compared to Section 2.4.1, we now rely more on technical auxiliary results. As a consequence, many a priori assumptions needed in the previous section can be dropped. The main result of this section is Theorem 2.43, which establishes a connection between the characteristic triplets of the processes η and $\widetilde{U}$, and the law of the corresponding killed exponential functional $V_{q,\xi,\eta}$. From this, we directly obtain a functional equation for the density in the absolutely continuous case as well as, similar to [38, Cor. 2.3], a criterion for absolute continuity and continuity or smoothness of the density that extends the one given in Corollary 2.30 in Section 2.2 for the exponential functional without killing to the case $q > 0$. Further, we discuss different special cases. Recall that the process U is constructed from ξ via $\mathrm{e}^{-\xi} = \mathcal{E}(U)$ and that $\widetilde{U}$ is obtained from adding a point mass of $q > 0$ at -1 to the Lévy measure of U. To alleviate some of the notation, we characterize the functions involved in Theorem 2.43 in the following lemma.

Lemma 2.41. *Let ξ, η be two independent Lévy processes such that η is not the zero process and $q \geq 0$. Further, define the functions $B_\eta, B_{\widetilde{U}}, S_\eta, S_{\widetilde{U}}$ by*

$$B_\eta : \mathbb{R} \to \mathbb{R}, \quad B_\eta(z) = \begin{cases} -\nu_\eta(-\infty, \min\{z,-1\}), & \text{if } z < 0, \\ 0, & \text{if } z = 0, \\ \nu_\eta((\max\{z,1\},\infty)), & \text{if } z > 0, \end{cases} \tag{2.37}$$

$$B_{\widetilde{U}} : [1,\infty) \to [0,\infty), \quad B_{\widetilde{U}}(z) = \begin{cases} 0, & \text{if } z = 1, \\ \nu_{\widetilde{U}}((\max\{z-1,1\},\infty)), & \text{if } z > 1, \end{cases} \tag{2.38}$$

$$S_\eta : \mathbb{R} \to [0,\infty), \quad S_\eta(z) = \begin{cases} \int_{-\infty}^{z} (z-y)\nu_\eta|_{[-1,1]}(\mathrm{d}y), & \text{if } z < 0, \\ 0, & \text{if } z = 0, \\ \int_z^\infty (y-z)\nu_\eta|_{[-1,1]}(\mathrm{d}y), & \text{if } z > 0, \end{cases} \tag{2.39}$$

$$S_{\widetilde{U}} : [0,\infty) \to [0,\infty), \quad S_{\widetilde{U}}(z) = \begin{cases} \int_{-\infty}^{z-1} (z-1-y)\nu_{\widetilde{U}}\big|_{[-1,1]}(\mathrm{d}y), & \text{if } z \in [0,1), \\ 0, & \text{if } z = 1, \\ \int_{z-1}^\infty (y-z+1)\nu_{\widetilde{U}}\big|_{[-1,1]}(\mathrm{d}y), & \text{if } z > 1. \end{cases} \tag{2.40}$$

Then both $B_{\widetilde{U}}$ *and* B_η *are bounded and hence locally integrable with respect to* λ *and both* S_η *and* $z \mapsto S_{\widetilde{U}}(z+1)$, $z \in \mathbb{R}$ *are integrable with respect to* λ*. In particular, the convolution* $B_\eta * \mu$ *is defined everywhere and bounded and the convolution* $S_\eta * \mu$ *is defined everywhere, is* λ*-a.e. finite and integrable. Further, the functions* $z \mapsto \int_{0+}^z (B_\eta * \mu)(x)\mathrm{d}x$ *and* $z \mapsto \int_{0+}^z \int_{0+}^t B_{\widetilde{U}}(\frac{t}{x})\mu(\mathrm{d}x)\mathrm{d}t$ *are locally integrable with respect to* λ*.*

Proof. First, note that $|B_\eta(z)| \le \nu_\eta(\mathbb{R} \setminus [-1,1]) < \infty$ and $|B_{\widetilde{U}}(z)| \le \nu_{\widetilde{U}}([1,\infty)) < \infty$ implies that B_η and $B_{\widetilde{U}}$ are bounded, respectively. For S_η, an application of Fubini's theorem yields for $z > 0$ that

$$\begin{aligned}\int_{0+}^\infty |S_\eta(t)|\mathrm{d}t &\le \int_{0+}^\infty \int_z^\infty |y - z|\nu_\eta|_{[-1,1]}(\mathrm{d}y)\mathrm{d}z \\ &= \int_{0+}^\infty \int_{0+}^y |y-z|\mathrm{d}z\nu_\eta|_{[-1,1]}(\mathrm{d}y) \\ &= \int_{0+}^\infty \frac{y^2}{2}\nu_\eta|_{[-1,1]}(\mathrm{d}y) < \infty,\end{aligned}$$

and similarly for $z < 0$, showing that S_η is indeed integrable. The same argument applies to $z \mapsto S_{\widetilde{U}}(z+1)$. The remaining assertions now follow from standard results on the convolution of bounded or measurable functions and finite measures. □

The term involving $S_{\widetilde{U}}$ in the distributional equation (2.41) below is considered in the following lemma.

Lemma 2.42. *Let* $q \ge 0$ *and* $S_{\widetilde{U}}$ *as defined in* (2.40)*. Then*

$$\varrho(\mathrm{d}z) = \Big(\mathbb{1}_{\{z>0\}} \int_0^\infty x S_{\widetilde{U}}(\tfrac{z}{x})\mu(\mathrm{d}x) + \mathbb{1}_{\{z<0\}} \int_{-\infty}^0 |x| S_{\widetilde{U}}(\tfrac{z}{x})\mu(\mathrm{d}x)\Big)\mathrm{d}z,$$

defines a locally finite measure on $\mathcal{B}_1(\mathbb{R})$*.*

Proof. Let $B \subset \mathbb{R}$ be compact. We first consider $B \in [0,\infty)$, i.e. $B \subseteq [0,R]$ for sufficiently large $R \in \mathbb{R}$. As $S_{\widetilde{U}}$ is nonnegative by definition, we obtain

$$\int_B \varrho(\mathrm{d}z) \le \int_{0+}^R \int_0^{z-} x S_{\widetilde{U}}(\tfrac{z}{x})\mu(\mathrm{d}x)\mathrm{d}z + \int_{0+}^R \int_{z+}^\infty x S_{\widetilde{U}}(\tfrac{z}{x})\mu(\mathrm{d}x)\mathrm{d}z,$$

in which we can insert the cases given in (2.40). Applying Fubini's theorem now yields

$$\begin{aligned}\int_{0+}^R \int_0^{z-} x S_{\widetilde{U}}(\tfrac{z}{x})\mu(\mathrm{d}x)\mathrm{d}z &\le \int_{0+}^1 \int_{0+}^R \int_x^{x+xy} (xy - z + x)\mathrm{d}z\mu(\mathrm{d}x)\nu_{\widetilde{U}}|_{[-1,1]}(\mathrm{d}y) \\ &\le \frac{R^2}{2}\int_{0+}^1 y^2 \nu_{\widetilde{U}}(\mathrm{d}y) < \infty\end{aligned}$$

for the first term. For the second term, write

$$\int_{0+}^R \int_{z+}^\infty x S_{\widetilde{U}}(\tfrac{z}{x})\mu(\mathrm{d}x)\mathrm{d}z = \int_{-1}^{0-} \int_{0+}^\infty \int_{\min\{x(1+y),R\}}^{\min\{x,R\}} (z - x - xy)\mathrm{d}z\mu(\mathrm{d}x)\nu_{\widetilde{U}}(\mathrm{d}y).$$

Whenever y is bounded away from zero, e.g. considering $y \in [-1,-1/2]$, the inner integral can again be estimated by $\int_0^R z\mathrm{d}z = R^2/2$, thus yielding finiteness of the triple integral as before. For $y \in (-1/2, 0)$, observe that the inner integral vanishes whenever $x > 2R$ and if $x \leq 2R$, it can be bounded by $\int_{x(1+y)}^{x}(z - x - xy)\mathrm{d}z = x^2y^2/2 \leq 2y^2R^2$. Thus, the triple integral is also finite in the last case. Since the same arguments apply for $B \subset (-\infty, 0]$, it follows that ϱ is locally finite. $\square$

The following theorem is the main result of this section. As before, we set $\mathcal{L}(V_{q,\xi,\eta}) = \mu$.

Theorem 2.43. *Let ξ, η be two independent Lévy processes such that η is not the zero process and $q \geq 0$ such that $V_{0,\xi,\eta}$ converges a.s. whenever $q = 0$ is considered. Further, let the functions $B_\eta, B_{\widetilde{U}}, S_\eta, S_{\widetilde{U}}$ be as in Lemma 2.41. Then there exists a constant $K \in \mathbb{R}$ such that*

$$\begin{aligned}
K\mathrm{d}z = &\Big(\frac{1}{2}\sigma_\eta^2 + \frac{1}{2}z^2\sigma_{\widetilde{U}}^2\Big)\mu(\mathrm{d}z) + (S_\eta * \mu)(z)\mathrm{d}z \\
&+ \Big(\mathbb{1}_{\{z>0\}}\int_0^\infty xS_{\widetilde{U}}(\tfrac{z}{x})\mu(\mathrm{d}x) + \mathbb{1}_{\{z<0\}}\int_{-\infty}^0 |x|S_{\widetilde{U}}(\tfrac{z}{x})\mu(\mathrm{d}x)\Big)\mathrm{d}z \\
&- \int_{0+}^z \Big(\gamma_\eta + x\gamma_{\widetilde{U}}\Big)\mu(\mathrm{d}x)\mathrm{d}z - \int_{0+}^z (B_\eta * \mu)(x)\mathrm{d}x\mathrm{d}z \\
&- \int_{0+}^z \int_{0+}^t B_{\widetilde{U}}(\tfrac{t}{x})\mu(\mathrm{d}x)\mathrm{d}t\mathrm{d}z.
\end{aligned} \tag{2.41}$$

The proof of Theorem 2.43 is based on the proof of Theorem 2.2 in [38] and the individual steps are carried out in Section 2.4.3 below. We sketch the argument briefly. First, taking $f \in C_c^\infty(\mathbb{R})$, the explicit form of $\mathcal{A}^{\widetilde{V}}f(x)$ is inserted into (2.31), allowing to rewrite the left-hand side to the form

$$\int_{\mathbb{R}} A^{\widetilde{V}}f(x)\mu(\mathrm{d}x) = \int_{\mathbb{R}} f''(z)G_1(\mathrm{d}z) + \int_{\mathbb{R}} f'(z)G_2(\mathrm{d}z)$$

for suitable G_1 and G_2. We can then use partial integration to rewrite the above integrals to all include the same function, namely f'', yielding the form

$$\int_{\mathbb{R}} f''(z)G_1(\mathrm{d}z) + \int_{\mathbb{R}} f'(z)G_2(\mathrm{d}z) = \int_{\mathbb{R}} f''(z)G(\mathrm{d}z),$$

where G can be identified with a distribution in the sense of Schwartz. Using (2.31) and the definition of the distributional derivative, it follows that this distribution satisfies $G'' = 0$. By solving this ordinary differential equation (ODE) over the distribution space, one can find an equivalent expression for G. Identifying the remaining constants then yields (2.41). Alternatively, one can derive a distributional equation for the law of the killed exponential functional from the corresponding equation in the case without killing. This approach is further discussed in Appendix A.1.

Whenever μ is absolutely continuous with respect to the Lebesgue measure, (2.41) directly yields a functional equation for the density. Recall that various sufficient conditions for absolute continuity were given in Theorems 2.23 and 2.29 in Section 2.2. Note in particular that whenever μ is continuous, the existence of a density is equivalent to the existence of a density of $\mu|_{\mathbb{R}\setminus\{0\}}$. If η is not the zero process, μ is continuous if and only if $q > 0$ and η is not a compound Poisson process (see Corollary 2.26) or if $q = 0$ and ξ and η are not simultaneously deterministic (cf. [13, Thm. 2.2]). In the case that $q > 0$ and η is a compound Poisson process, it is $\mu(\{0\}) > 0$ such that the measure cannot be absolutely continuous, however, it is still possible for $\mu|_{\mathbb{R}\setminus\{0\}}$ to have a density (see Corollary 2.45 below). We thus formulate the following result in the slightly more general setting that only $\mu|_{\mathbb{R}\setminus\{0\}}$ has a density. The proof is immediate from Theorem 2.43 and is, therefore, omitted.

Corollary 2.44. *Under the conditions of Theorem 2.43, assume that $\mu|_{\mathbb{R}\setminus\{0\}}$ has a density f_μ with respect to the Lebesue measure. Then there exists a constant $K \in \mathbb{R}$ such that*

$$\begin{aligned}
&\Big(\frac{1}{2}\sigma_\eta^2 + \frac{1}{2}z^2\sigma_{\widetilde{U}}^2\Big)f_\mu(z) + (S_\eta * f_\mu)(z) + S_\eta(z)\mu(\{0\}) \\
&\quad + \mathbb{1}_{\{z>0\}}\int_0^\infty xS_{\widetilde{U}}(\tfrac{z}{x})f_\mu(x)\mathrm{d}x + \mathbb{1}_{\{z<0\}}\int_{-\infty}^0 |x|S_{\widetilde{U}}(\tfrac{z}{x})f_\mu(x)\mathrm{d}x \\
&= K + \int_0^z \Big(\gamma_\eta + x\gamma_{\widetilde{U}}\Big)f_\mu(x)\mathrm{d}x - \mathbb{1}_{\{z<0\}}\gamma_\eta\mu(\{0\}) \\
&\quad + \int_0^z (B_\eta * f_\mu)(x)\mathrm{d}x + \int_0^z\int_0^t B_{\widetilde{U}}(\tfrac{t}{x})f_\mu(x)\mathrm{d}x\mathrm{d}t
\end{aligned} \tag{2.42}$$

for λ-a.e. $z \in \mathbb{R}$.

It was shown in [38, Cor. 2.5] that the law of the exponential functional $V_{0,\xi,\eta}$ admits a continuous density on $\mathbb{R}\setminus\{0\}$ if $\sigma_\xi^2+\sigma_\eta^2 > 0$, as well as $\mathbb{E}|\xi_1| < \infty$, $\mathbb{E}|\eta_1| < \infty$ and $\mathbb{E}\xi_1 < 0$. The following corollary generalizes this to general $q \geq 0$. As in Theorem 2.43, we do not require a moment condition. The proof is given in Section 2.4.3. Observe that $\sigma_{\widetilde{U}}^2 = \sigma_\xi^2$.

Corollary 2.45. *In addition to the assumptions of Theorem 2.43, let $\sigma_{\widetilde{U}}{}^2 + \sigma_\eta^2 > 0$.*

(i) If $\sigma_\eta^2 > 0$, then μ has a continuous density f_μ on $\mathbb{R}$.

(ii) If $\sigma_{\widetilde{U}}^2 > 0$, then $\mu|_{\mathbb{R}\setminus\{0\}}$ has a continuous density f_μ on $\mathbb{R}\setminus\{0\}$.

(iii) In both cases, there exist constants $M_1, M_2 > 0$ such that for all $z \neq 0$ it holds

$$(\sigma_\eta^2 + z^2\sigma_{\widetilde{U}}^2)f_\mu(z) \leq M_1 + M_2|z|.$$

Note that the above results, in particular (2.41) and (2.42), are derived under very weak assumptions. Thus, the equations can be simplified further whenever more properties of the processes ξ and η are known. We discuss some special cases in the following corollaries, the proofs of which are also given in Section 2.4.3.

Corollary 2.46 (Finite First Moments)**.** *Under the assumptions of Theorem 2.43 let further $\mathbb{E}|\eta_1| < \infty$ and $\mathbb{E}|\widetilde{U}_1| < \infty$. Denote the expectation of η_1 and $\widetilde{U}_1$ by γ_η^1 and $\gamma_{\widetilde{U}}^1$, respectively, and define the functions*

$$S_\eta^{FM} : \mathbb{R} \to [0,\infty), \quad S_\eta^{FM}(z) = \begin{cases} \int_{-\infty}^z (z-y)\nu_\eta(\mathrm{d}y), & \text{if } z < 0, \\ 0, & \text{if } z = 0, \\ \int_z^\infty (y-z)\nu_\eta(\mathrm{d}y), & \text{if } z > 0, \end{cases}$$

$$S_{\widetilde{U}}^{FM} : [0,\infty) \to [0,\infty), \quad S_{\widetilde{U}}^{FM}(z) = \begin{cases} \int_{-\infty}^{z-1} (z-1-y)\nu_{\widetilde{U}}(\mathrm{d}y), & \text{if } z \in [0,1), \\ 0, & \text{if } z = 1, \\ \int_{z-1}^\infty (y-z+1)\nu_{\widetilde{U}}(\mathrm{d}y), & \text{if } z > 1. \end{cases}$$

Then the following hold true:

(i) *There exists a constant $K \in \mathbb{R}$ such that*

$$\begin{aligned} K\mathrm{d}z = &\Big(\frac{1}{2}\sigma_\eta^2 + \frac{1}{2}z^2\sigma_{\widetilde{U}}^2\Big)\mu(\mathrm{d}z) + (S_\eta^{FM} * \mu)(z)\mathrm{d}z \\ &+ \Big(\mathbb{1}_{\{z>0\}} \int_0^\infty x S_{\widetilde{U}}^{FM}(\tfrac{z}{x})\mu(\mathrm{d}x) + \mathbb{1}_{\{z<0\}} \int_{-\infty}^0 |x| S_{\widetilde{U}}^{FM}(\tfrac{z}{x})\mu(\mathrm{d}x)\Big)\mathrm{d}z \\ &- \int_{0+}^z \Big(\gamma_\eta^1 + x\gamma_{\widetilde{U}}^1\Big)\mu(\mathrm{d}x)\mathrm{d}z, \end{aligned} \tag{2.43}$$

where the right-hand side of the equation defines a locally finite measure on $\mathcal{B}_1(\mathbb{R})$.

(ii) *If additionally $\mathbb{E}|\mathcal{E}(U)_1| < \mathrm{e}^q$ or, equivalently, $\gamma_{\widetilde{U}}^1 < 0$, then $\mathbb{E}|V_{q,\xi,\eta}| = \int |x|\mu(\mathrm{d}x) < \infty$ and the constant K in (2.43) takes the form*

$$K = -\int_{0+}^\infty \Big(\gamma_\eta^1 + x\gamma_{\widetilde{U}}^1\Big)\mu(\mathrm{d}x) = \int_{-\infty}^0 \Big(\gamma_\eta^1 + x\gamma_{\widetilde{U}}^1\Big)\mu(\mathrm{d}x).$$

Moreover, if additionally $\sigma_\eta^2 + \sigma_{\widetilde{U}}^2 > 0$, then the density f_μ of $\mu|_{\mathbb{R}\setminus\{0\}}$ is bounded.

Corollary 2.47 (Finite Variation)**.** *Under the assumptions of Theorem 2.43 let further η and $\widetilde{U}$ be of finite variation, i.e. $\sigma_\eta^2 = \sigma_{\widetilde{U}}^2 = 0$ and $\int_{[-1,1]} |x|\nu_\eta(\mathrm{d}x), \int_{[-1,1]} |x|\nu_U(\mathrm{d}x) < \infty$. Denote by γ_η^0 and $\gamma_{\widetilde{U}}^0$ the drifts of η and $\widetilde{U}$, respectively, and define the functions*

$$B_\eta^{FV} : \mathbb{R} \to \mathbb{R}, \quad B_\eta^{FV}(z) = \begin{cases} -\nu_\eta((-\infty, z)), & \text{if } z < 0, \\ 0, & \text{if } z = 0, \\ \nu_\eta((z,\infty)), & \text{if } z > 0, \end{cases} \tag{2.44}$$

$$B_{\widetilde{U}}^{FV} : [0,\infty) \to \mathbb{R}, \quad B_{\widetilde{U}}^{FV}(z) = \begin{cases} -\nu_{\widetilde{U}}((-\infty, z-1)), & \text{if } z \in [0,1), \\ 0, & \text{if } z = 1, \\ \nu_{\widetilde{U}}((z-1,\infty)), & \text{if } z > 1. \end{cases} \tag{2.45}$$

Then the equation

$$\begin{aligned} 0 = &\Big(\gamma_\eta^0 + z\gamma_{\widetilde{U}}^0\Big)\mu(\mathrm{d}z) + (B_\eta^{FV} * \mu)(z)\mathrm{d}z \\ &+ \Big(\mathbb{1}_{\{z>0\}} \int_0^\infty B_{\widetilde{U}}^{FV}(\tfrac{z}{x})\mu(\mathrm{d}x) - \mathbb{1}_{\{z<0\}} \int_{-\infty}^0 B_{\widetilde{U}}^{FV}(\tfrac{z}{x})\mu(\mathrm{d}x)\Big)\mathrm{d}z \end{aligned} \tag{2.46}$$

holds and the quantities on the right-hand side define a locally finite measure.

Assuming finite variation only of the jump parts of the processes η and $\widetilde{U}$, Corollary 2.45 can be extended to differentiability.

Corollary 2.48 (Differentiable Density)**.** *Under the assumptions of Theorem 2.43, let the jump parts of $\widetilde{U}$ and η be of finite variation. Further, let $\sigma_\eta^2 + \sigma_{\widetilde{U}}^2 > 0$ and γ_η^0, $\gamma_{\widetilde{U}}^0$, B_η^{FV} and $B_{\widetilde{U}}^{FV}$ as in Corollary 2.47.*

(i) *If $\sigma_\eta^2 > 0$, then the density f_μ of μ is continuously differentiable on $\mathbb{R} \setminus \{0\}$.*

(ii) If $\sigma^2_{\widetilde{U}} > 0 = \sigma^2_\eta$ and $q = 0$, or η is not a compound Poisson process, then μ has a density f_μ on $\mathbb{R}$ which is continuously differentiable on $\mathbb{R} \setminus \{0\}$.

(iii) The density f_μ satisfies the equation

$$\begin{aligned}\Big(\frac{1}{2}\sigma^2_\eta + \frac{1}{2}z^2\sigma^2_{\widetilde{U}}\Big)f'_\mu(z) + z\sigma^2_{\widetilde{U}}f_\mu(z) - \Big(\gamma^0_\eta + z\gamma^0_{\widetilde{U}}\Big)f_\mu(z) - B^{FV}_\eta(z)\mu(\{0\}) \\ = (B^{FV}_\eta * f_\mu)(z) + \mathbb{1}_{\{z>0\}}\int_0^\infty B^{FV}_{\widetilde{U}}(\tfrac{z}{x})f_\mu(x)\mathrm{d}x - \mathbb{1}_{\{z<0\}}\int_{-\infty}^0 B^{FV}_{\widetilde{U}}(\tfrac{z}{x})f_\mu(x)\mathrm{d}x,\end{aligned} \tag{2.47}$$

which under the conditions (i) and (ii) is valid for all $z \in \mathbb{R} \setminus \{0\}$ and still holds λ-a.e. whenever $\sigma^2_{\widetilde{U}} > 0 = \sigma^2_\eta$, but the additional assumptions of (ii) are not satisfied. In the latter case, f_μ is λ-a.e. differentiable.

Observe that $\mu(\{0\}) = 0$ when $\sigma^2_\eta > 0$, or $\sigma^2_\eta + \sigma^2_{\widetilde{U}} > 0$ and $q = 0$, or $q > 0$, $\sigma^2_{\widetilde{U}} > 0$ and η is neither a compound Poisson process nor the zero process.

Remark 2.49. Using the relation between the characteristic triplets of ξ, U, and $\widetilde{U}$ established in Section 2.3 and Theorem 2.31, as well as the fact that $\gamma^0_{\widetilde{U}} = \gamma^0_U = -\gamma^0_\xi + \frac{1}{2}\sigma^2_\xi$ whenever $\int_{[-1,1]}|y|\nu_\xi(\mathrm{d}y) < \infty$, one finds that Proposition 2.39 is a special case of Corollaries 2.47 and 2.48. In particular, Equation (2.36) is reobtained from (2.46) and (2.47) for $z > 0$. If η is a subordinator, similar formulas are obtained from (2.46) and (2.47) for $z < 0$, which are readily seen to be satisfied by $f_\mu(z) = 0$ for $z < 0$. Note, however, that neither of the corollaries requires ξ or η to be a subordinator.

Remark 2.50. While studying the method, we found that the distributional equation given in (2.3) of [38] for the case $q = 0$ and $\mathbb{E}|\xi_1|, \mathbb{E}|\eta_1| < \infty$ does not hold in general. The cause of this lies in equation (2.6) of the paper, where it is stated that for functions $f \in C_c^\infty((0,\infty))$ the left-hand side of the equation $\int_\mathbb{R} A^V f(x)\mu(\mathrm{d}x) = 0$ simplifies to an integral over the positive real line, i.e. $\int_0^\infty \mathcal{A}^V f(x)\mu(\mathrm{d}x) = 0$. Evaluating the generator for such a function f leads to

$$\begin{aligned}\int_\mathbb{R} \mathcal{A}^V f(x)\mu(\mathrm{d}x) = &\int_0^\infty \mathcal{A}^V f(x)\mu(\mathrm{d}x) \\ &+ \int_{-\infty}^0 \frac{\sigma^2_\eta}{2}f''(x) + (\gamma_\eta - x\gamma_\xi)f'(x) + \frac{\sigma^2_\xi}{2}(x^2 f''(x) + xf'(x))\mu(\mathrm{d}x) \\ &+ \int_{-\infty}^0\int_{-\infty}^0 \Big(f(x+y) - f(x) - yf'(x)\mathbb{1}_{\{|y|\le 1\}}\Big)\nu_\eta(\mathrm{d}y)\mu(\mathrm{d}x) \\ &+ \int_{-\infty}^0\int_0^\infty \Big(f(x+y) - f(x) - yf'(x)\mathbb{1}_{\{|y|\le 1\}}\Big)\nu_\eta(\mathrm{d}y)\mu(\mathrm{d}x) \\ &+ \int_{-\infty}^0\int_\mathbb{R} \Big(f(x\mathrm{e}^{-y}) - f(x) + f'(x)xy\mathbb{1}_{\{|y|\le 1\}}\Big)\nu_\xi(\mathrm{d}y)\mu(\mathrm{d}x).\end{aligned} \tag{2.48}$$

Observe that the second, third and last term are zero as $f(x) = 0$ for $x \le 0$. However, the fourth term may not, e.g. in the case $\xi_t = t$ and η being a pure-jump process with Lévy measure $\nu_\eta = \delta_2 + \delta_{-2}$. For this example, one can construct a nonnegative test function supported on the interval $[\frac{1}{2}, \frac{3}{2}]$ for which the term in question is nonzero. Nevertheless, whenever η is a subordinator, and thus $V_{0,\xi,\eta} \ge 0$ a.s., or η does not have any positive jumps, the fourth term of (2.48) vanishes such that all conclusions drawn from equation (2.6) in [38], in particular the distributional equation (2.3), remain valid. Otherwise, the equation does not necessarily hold, as can be seen from the following example. Let $\xi_t = t$ and η be a pure-jump process with the Lévy measure given by $\nu_\eta(\mathrm{d}x) = \mathrm{e}^{-|x|}\mathrm{d}x$. We can derive the distribution of $V_{0,\xi,\eta}$ explicitly from [28, Thm. 2.1(f)], yielding that the exponential functional has the same distribution as the difference of two independent Exp(1)-distributed random variables, i.e. a Laplace distribution with parameters 0 and 1. As μ is known, one can readily check that Equation (2.3) of [38] does not hold for this example. The tail function and the integrated tails for $x > 0$ are given by

$$\nu_\eta((x,\infty)) = \int_x^\infty \mathrm{e}^{-t}\mathrm{d}t = \mathrm{e}^{-x}, \quad \overline{\overline{\Pi}}_\eta^{(+)}(x) := \int_x^\infty \nu_\eta((t,\infty))\mathrm{d}t = \mathrm{e}^{-x},$$

and similarly

$$\overline{\overline{\Pi}}_\eta^{(-)}(x) := \int_x^\infty \nu_\eta((-\infty,-t))\mathrm{d}t = \mathrm{e}^{-x}, \quad x > 0.$$

Therefore, Equation (2.3) in [38] reads

$$\begin{aligned} &-\int_v^\infty \mu(\mathrm{d}x)\mathrm{d}v + \Big(\frac{1}{v}\int_0^v \mathrm{e}^{-(v-x)}\mu(\mathrm{d}x)\Big)\mathrm{d}v + \Big(\frac{1}{v}\int_v^\infty \mathrm{e}^{-(x-v)}\mu(\mathrm{d}x)\Big)\mathrm{d}v \\ &-\int_v^\infty \frac{1}{w^2}\Big(\int_0^w \mathrm{e}^{-(w-x)}\mu(\mathrm{d}x) + \int_w^\infty \mathrm{e}^{-(x-w)}\mu(\mathrm{d}x)\Big)\mathrm{d}w\mathrm{d}v = 0, \quad v > 0 \end{aligned} \tag{2.49}$$

for this specific example. Note that due to the choice of the processes the remaining parameters (in the notation of [38]) are given by $b_\xi = -1$, $\sigma_\xi^2 = \sigma_\eta^2 = 0$ and $\overline{\overline{\Pi}}_\xi^{(+)} = \overline{\overline{\Pi}}_\xi^{(-)} = 0$. Inserting $\mu(\mathrm{d}x) = \frac{1}{2}\mathrm{e}^{-|x|}\mathrm{d}x$ into the left-hand side of (2.49), we obtain

$$\begin{aligned} &-\frac{\mathrm{e}^{-v}}{2}\mathrm{d}v + \frac{\mathrm{e}^{-v}}{2}\mathrm{d}v + \frac{\mathrm{e}^{-v}}{4v}\mathrm{d}v - \frac{1}{4}\Big(\int_v^\infty \frac{2\mathrm{e}^{-w}}{w}\mathrm{d}w + \frac{\mathrm{e}^{-v}}{v} - \int_v^\infty \frac{\mathrm{e}^{-w}}{w}\mathrm{d}w\Big)\mathrm{d}v \\ &= \Big(-\frac{1}{4}\int_v^\infty \frac{\mathrm{e}^{-w}}{w}\mathrm{d}w\Big)\mathrm{d}v, \end{aligned}$$

which is not the zero measure, contradicting (2.49). However, one can check that the equation given in (2.46) is satisfied for this example. Observing that $U_t = \widetilde{U}_t = -t$ here and verifying that η and $\widetilde{U}$ are of finite variation, Corollary 2.47 is applicable. Hence, it holds

$$0 = -z\mu(\mathrm{d}z) + (B_\eta^{FV} * \mu)(z)\mathrm{d}z, \tag{2.50}$$

by (2.46), where B_η^{FV} can be calculated explicitly as

$$B_\eta^{FV}(z) = \begin{Bmatrix} \mathrm{e}^{-z}, & z > 0 \\ -\mathrm{e}^{z}, & z < 0 \end{Bmatrix} = \operatorname{sign}(z)\mathrm{e}^{-|z|}.$$

Therefore, (2.50) now reads

$$z\mu(\mathrm{d}z) = \Big(\int_{-\infty}^z \mathrm{e}^{-z+s}\mu(\mathrm{d}s) - \int_z^\infty \mathrm{e}^{z-s}\mu(\mathrm{d}s)\Big)\mathrm{d}z$$

and it is readily checked that the equation indeed holds for $\mu(\mathrm{d}x) = \frac{1}{2}\mathrm{e}^{-|x|}\mathrm{d}x$. Instead of using the results from [28], one could also solve (2.50) directly. Since $B_\eta^{FV} * \mu$ is integrable, so is $z\mu(\mathrm{d}z)$ by (2.50), such that taking Fourier transforms leads to

$$-i\varphi'_{V_{0,\xi,\eta}}(x) = -\frac{2x}{x^2+1}\varphi_{V_{0,\xi,\eta}}(x).$$

This yields $\varphi_{V_{0,\xi,\eta}}(x) = (x^2+1)^{-1}$, from which the exact distribution of $V_{0,\xi,\eta}$ is readily obtained by Fourier inversion. Alternatively, one could also observe that $\mathbb{E}V^2_{0,\xi,\eta} < \infty$ by (2.34) and find the distribution of $V_{0,\xi,\eta}$ from solving (2.33), or equivalently (4.8) in [5], for the given characteristics and performing a Fourier inversion of the solution.

2.4.3. Proofs for Section 2.4.2

The proof of Theorem 2.43 consists of several steps which are shown as separate lemmas. First, the left-hand side of (2.31) is rewritten to a suitable form.

Lemma 2.51. *Under the assumptions of Theorem 2.43 we have for every $f \in C_c^\infty(\mathbb{R})$ that*

$$\int_{\mathbb{R}} A^{\widetilde{V}} f(x)\mu(\mathrm{d}x) = \int_{\mathbb{R}} f''(z)G_1(\mathrm{d}z) + \int_{\mathbb{R}} f'(z)G_2(\mathrm{d}z),$$

where the individual contributions are given by

$$\begin{aligned} G_1(\mathrm{d}z) &= \Big(\frac{1}{2}\sigma_\eta^2 + \frac{1}{2}z^2\sigma_{\widetilde{U}}^2\Big)\mu(\mathrm{d}z) + (S_\eta * \mu)(z)\mathrm{d}z \\ &\quad + \Big(\mathbb{1}_{\{z>0\}}\int_0^\infty xS_{\widetilde{U}}(\tfrac{z}{x})\mu(\mathrm{d}x) + \mathbb{1}_{\{z<0\}}\int_{-\infty}^0 |x|S_{\widetilde{U}}(\tfrac{z}{x})\mu(\mathrm{d}x)\Big)\mathrm{d}z \\ G_2(\mathrm{d}z) &= \Big(\gamma_\eta + z\gamma_{\widetilde{U}}\Big)\mu(\mathrm{d}z) + (B_\eta * \mu)(z)\mathrm{d}z \\ &\quad + \int_{0+}^z B_{\widetilde{U}}(\tfrac{z}{x})\mu(\mathrm{d}x)\mathrm{d}z \end{aligned}$$

with the functions B_η, $B_{\widetilde{U}}$, S_η, and $S_{\widetilde{U}}$ given as in Equations (2.37) *to* (2.40)*.*

Proof. By linearity, we can split $\mathcal{A}^{\widetilde{V}}f(x)$ according to (2.30) and rewrite the corresponding integrals separately. Firstly, for the terms originating from the Gaussian and drift components it follows that

$$\begin{aligned} &\int_{\mathbb{R}}\Big(\frac{1}{2}\sigma_\eta^2 f''(x) + \gamma_\eta f'(x) + \frac{1}{2}x^2 f''(x)\sigma_{\widetilde{U}}^2 + xf'(x)\gamma_{\widetilde{U}}\Big)\mu(\mathrm{d}x) \\ &\quad = \int_{\mathbb{R}} f''(x)\Big(\frac{1}{2}\sigma_\eta^2 + \frac{1}{2}x^2\sigma_{\widetilde{U}}^2\Big)\mu(\mathrm{d}x) + \int_{\mathbb{R}} f'(x)\Big(\gamma_\eta + x\gamma_{\widetilde{U}}\Big)\mu(\mathrm{d}x) \end{aligned}$$

such that their contributions to G_1 and G_2 are readily identified. For the terms corresponding to the jump parts of the processes, the integrals with respect to the Lévy measure are split according to the value of the indicator function in the integrand. Starting with the contribution of the big jumps of η, we find for $y > 1$ that

$$\begin{aligned} \int_{\mathbb{R}}\int_{1+}^\infty \Big(f(x+y) - f(x)\Big)\nu_\eta(\mathrm{d}y)\mu(\mathrm{d}x) &= \int_{\mathbb{R}}\int_{1+}^\infty\int_x^{x+y} f'(t)\mathrm{d}t\nu_\eta(\mathrm{d}y)\mu(\mathrm{d}x) \\ &= \int_{\mathbb{R}} f'(t)\int_{-\infty}^t \nu_\eta\Big((\max\{t-x,1\},\infty)\Big)\mu(\mathrm{d}x)\mathrm{d}t, \end{aligned}$$

where interchanging the order of integration is allowed due to the compact support of f' and the involved measures being finite. A similar calculation applies if $y < -1$. Using the function B_η defined in (2.37), the term reads as

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}\setminus[-1,1]} \Big(f(x+y)-f(x)\Big)\nu_\eta(\mathrm{d}y)\mu(\mathrm{d}x) &= \int_{\mathbb{R}} f'(t)\int_{\mathbb{R}} B_\eta(t-x)\mu(\mathrm{d}x)\mathrm{d}t\\ &= \int_{\mathbb{R}} f'(t)(B_\eta * \mu)(t)\mathrm{d}t.\end{aligned}$$

The big jumps of $\widetilde{U}$ are treated in the same way, although the result cannot be interpreted as a linear convolution here. For $x > 0$ it follows that

$$\int_{0+}^{\infty}\int_{1+}^{\infty} \Big(f(x+xy)-f(x)\Big)\nu_{\widetilde{U}}(\mathrm{d}y)\mu(\mathrm{d}x) = \int_0^\infty f'(t)\int_{0+}^{t} \nu_{\widetilde{U}}\Big((\max\{\tfrac{t}{x}-1,1\},\infty)\Big)\mu(\mathrm{d}x)\mathrm{d}t,$$

and the calculation for $x < 0$ is again similar. Using the function $B_{\widetilde{U}}$ introduced in (2.38) now yields the desired form as

$$\int_{\mathbb{R}}\int_{1+}^{\infty} \Big(f(x+xy)-f(x)\Big)\nu_{\widetilde{U}}(\mathrm{d}y)\mu(\mathrm{d}x) = \int_{\mathbb{R}} f'(t)\int_{0+}^{t} B_{\widetilde{U}}(\tfrac{t}{x})\mu(\mathrm{d}x)\mathrm{d}t.$$

Note that the argument of $B_{\widetilde{U}}$ is always greater or equal to one due to t and x being of the same sign with $|x| \leq |t|$. The approach to the terms corresponding to the small jumps of η and $\widetilde{U}$, respectively, is similar. However, we obtain a contribution to G_1 instead of G_2 here. For η, using the Taylor formula, this leads to

$$\begin{aligned}&\int_{\mathbb{R}}\int_{[-1,1]} \Big(f(x+y)-f(x)-yf'(x)\Big)\nu_\eta(\mathrm{d}y)\mu(\mathrm{d}x)\\ &\quad= \int_{\mathbb{R}}\int_{[-1,1]}\int_x^{x+y} f''(t)(x+y-t)\mathrm{d}t\nu_\eta(\mathrm{d}y)\mu(\mathrm{d}x).\end{aligned}$$

A direct computation similar to Lemma 2.41 shows, since $|f''|$ is compactly supported and thus bounded by a constant, that

$$\Big|\int_x^{x+y} |f''(t)(x+y-t)|\mathrm{d}t\Big| \leq \frac{Cy^2}{2}.$$

Thus, Fubini's theorem is applicable and we find for $y > 0$ that

$$\begin{aligned}&\int_{\mathbb{R}}\int_{(0,1]} \Big(f(x+y)-f(x)-yf'(x)\Big)\nu_\eta(\mathrm{d}y)\mu(\mathrm{d}x)\\ &\quad= \int_{\mathbb{R}} f''(t)\int_{-\infty}^{t}\int_{t-x}^{\infty} \Big(y-(t-x)\Big)\nu_\eta\Big|_{[-1,1]}(\mathrm{d}y)\mu(\mathrm{d}x)\mathrm{d}t,\end{aligned}$$

with a similar calculation holding for $y < 0$. Adding both terms, one obtains

$$\int_{\mathbb{R}}\int_{[-1,1]} \Big(f(x+y)-f(x)-yf'(x)\Big)\nu_\eta(\mathrm{d}y)\mu(\mathrm{d}x) = \int_{\mathbb{R}} f''(t)(S_\eta * \mu)(t)\mathrm{d}t,$$

where the function S_η is taken from (2.39). For the last term involving the small jumps of $\widetilde{U}$, it follows similarly that

$$\begin{aligned}&\int_{\mathbb{R}}\int_{[-1,1]}\Big(f(x+xy)-f(x)-xyf'(x)\Big)\nu_{\widetilde{U}}(\mathrm{d}y)\mu(\mathrm{d}x)\\&\quad=\int_{\mathbb{R}}\int_{[-1,1]}\int_{x}^{x+xy}f''(t)(x+xy-t)\mathrm{d}t\nu_{\widetilde{U}}(\mathrm{d}y)\mu(\mathrm{d}x).\end{aligned}$$

As f has compact support, there is some $R>0$ such that $\operatorname{supp}(f)\subseteq[-R,R]$. Let now $x,y>0$ and denote the set $\operatorname{supp}(f)\cap[x,x+xy]$ by $M=M_{x,y}$. As $M=\emptyset$ if $x>R$ and $|f''|$ is bounded by some constant C, it follows that

$$\int_{0+}^{1}\int_{0}^{\infty}\int_{x}^{x+xy}|f''(t)(x+xy-t)|\mathrm{d}t\mu(\mathrm{d}x)\nu_{\widetilde{U}}(\mathrm{d}y)\leq\frac{CR^2}{2}\int_{(0,1]}y^2\nu_{\widetilde{U}}(\mathrm{d}y)<\infty$$

with a similar calculation as in Lemma 2.42. When considering $x<0$, R is replaced by $-R$. If $y<0$ and $x>0$, we can split the interval $[-1,0)$ at some intermediate point y_0, say $y_0=\frac{1}{2}$, and estimate the respective integrals separately. For $y\in(-\frac{1}{2},0)$, observe that $M=\operatorname{supp}(f)\cap[x+xy,x]=\emptyset$ if $x>2R$ as $x+xy>\frac{x}{2}$ for the given values of y. Thus, we can use similar estimates as for $y>0$. For $y\in[-1,-\frac{1}{2}]$, note that $M\subseteq[0,R]$ and that, therefore, $x(1+y)\in M$ only if $x(1+y)\leq R$. This yields

$$C\int_{[-1,-\frac{1}{2}]}\int_{0}^{\infty}\int_{M}(t-x-xy)\mathrm{d}t\mu(\mathrm{d}x)\nu_{\widetilde{U}}(\mathrm{d}y)\leq\frac{CR^2}{2}\nu_{\widetilde{U}}([-1,-\tfrac{1}{2}])<\infty,$$

due to $[-1,-\frac{1}{2}]$ being bounded away from zero. Again, similar arguments are applicable for negative values of x, i.e. when $y<0$ and $x<0$ yielding integrability in the last case. Interchanging the order of integration and rewriting the term to include $S_{\widetilde{U}}$ as defined in (2.40) now leads to

$$\begin{aligned}&\int_{0}^{\infty}\int_{0}^{1}\int_{x}^{x+xy}f''(t)(x+xy-t)\mathrm{d}t\nu_{\widetilde{U}}(\mathrm{d}y)\mu(\mathrm{d}x)\\&\qquad+\int_{0}^{\infty}\int_{-1}^{0}\int_{x+xy}^{x}f''(t)(t-x-xy)\mathrm{d}t\nu_{\widetilde{U}}(\mathrm{d}y)\mu(\mathrm{d}x)\\&\quad=\int_{0}^{\infty}f''(t)\int_{0}^{\infty}xS_{\widetilde{U}}(\tfrac{t}{x})\mu(\mathrm{d}x)\mathrm{d}t.\end{aligned}$$

As the remaining two terms yield a similar result with the opposite sign, the complete term can be rewritten as

$$\begin{aligned}&\int_{\mathbb{R}}\int_{[-1,1]}\Big(f(x+xy)-f(x)-xyf'(x)\Big)\nu_{\widetilde{U}}(\mathrm{d}y)\mu(\mathrm{d}x)\\&\quad=\int_{\mathbb{R}}f''(t)\Big(\mathbb{1}_{\{t>0\}}\int_{0}^{\infty}xS_{\widetilde{U}}(\tfrac{t}{x})\mu(\mathrm{d}x)-\mathbb{1}_{\{t<0\}}\int_{-\infty}^{0}xS_{\widetilde{U}}(\tfrac{t}{x})\mu(\mathrm{d}x)\Big)\mathrm{d}t.\end{aligned}$$

Summing up the individual contributions now yields G_1 and G_2 as claimed. □

Lemma 2.52. *Under the assumptions of Theorem 2.43 we have for all $f\in C_c^\infty(\mathbb{R})$ that*

$$\int_{\mathbb{R}}f''(z)G_1(\mathrm{d}z)+\int_{\mathbb{R}}f'(z)G_2(\mathrm{d}z)=\int_{\mathbb{R}}f''(z)G(\mathrm{d}z),$$

where G can be identified with a distribution in the sense of Schwartz and is given by

$$\begin{aligned}
G(\mathrm{d}z) &= \Big(\frac{1}{2}\sigma_\eta^2 + \frac{1}{2}z^2\sigma_{\widetilde{U}}^2\Big)\mu(\mathrm{d}z) + (S_\eta * \mu)(z)\mathrm{d}z \\
&\quad + \Big(\mathbb{1}_{\{z>0\}}\int_0^\infty x S_{\widetilde{U}}(\tfrac{z}{x})\mu(\mathrm{d}x) + \mathbb{1}_{\{z<0\}}\int_{-\infty}^0 |x| S_{\widetilde{U}}(\tfrac{z}{x})\mu(\mathrm{d}x)\Big)\mathrm{d}z \\
&\quad - \int_{0+}^z \Big(\gamma_\eta + x\gamma_{\widetilde{U}}\Big)\mu(\mathrm{d}x)\mathrm{d}z - \int_{0+}^z (B_\eta * \mu)(x)\mathrm{d}x\mathrm{d}z \\
&\quad - \int_{0+}^z\int_{0+}^t B_{\widetilde{U}}(\tfrac{t}{x})\mu(\mathrm{d}x)\mathrm{d}t\mathrm{d}z,
\end{aligned}$$

with the functions S_η, $S_{\widetilde{U}}$, B_η and $B_{\widetilde{U}}$ as defined in Equations (2.37) *to* (2.40)*.*

Proof. The term involving $G_1(\mathrm{d}z)$ in Lemma 2.51 is already of the desired form, and, by Lemmas 2.41 and 2.42, it follows that $G_1(\mathrm{d}z)$ yields finite values when evaluated over compact subsets of $\mathbb{R}$. For the terms included in G_2, observe that $z \mapsto \int_{0+}^z G_2(\mathrm{d}w)$ is càdlàg on $\mathbb{R}$ and locally of bounded variation. Partial integration then shows

$$\begin{aligned}
\int_{\mathbb{R}} f'(z)G_2(\mathrm{d}z) &= -\int_{\mathbb{R}} f''(z)\int_{0+}^z G_2(\mathrm{d}w)\mathrm{d}z \\
&= -\int_{\mathbb{R}} f''(z)\Big(\int_{0+}^z \Big(\gamma_\eta + x\gamma_{\widetilde{U}}\Big)\mu(\mathrm{d}x) + \int_{0+}^z (B_\eta * \mu)(x)\mathrm{d}x \\
&\qquad + \int_{0+}^z\int_{0+}^t B_{\widetilde{U}}(\tfrac{t}{x})\mu(\mathrm{d}x)\mathrm{d}t\mathrm{d}z\Big)\mathrm{d}z.
\end{aligned}$$

This contribution to G also yields finite values when evaluated over compact subsets of $\mathbb{R}$ by Lemma 2.41. Summing up the terms, we find that G is of the claimed form and locally finite, which allows to interpret the measure as a distribution in the sense of Schwartz. □

The following lemma now allows us to identify the distribution G through solving an ordinary differential equation.

Lemma 2.53. *The distribution $G(\mathrm{d}z)$ in Lemma 2.52 is of the form $C_1 z\mathrm{d}z + C_2\mathrm{d}z$ for some constants $C_1, C_2 \in \mathbb{R}$.*

Proof. By Equation (2.31) and Lemma 2.52 it holds for all $f \in C_c^\infty(\mathbb{R})$ that

$$\int_{\mathbb{R}} A^{\widetilde{V}} f(x)\mu(\mathrm{d}x) = \int_{\mathbb{R}} f''(z)G(\mathrm{d}z) = \langle f'', G\rangle = 0,$$

where $\langle\cdot,\cdot\rangle$ denotes dual pairing. From the definition of the distributional derivative it now follows that

$$\langle f'', G\rangle = -\langle f', G'\rangle = \langle f, G''\rangle = 0.$$

As the above holds for all test functions f and $\mathbb{R}$ is an open set, we can conclude that G'' must be the zero distribution. Using results on the antiderivative of distributions, e.g. from [25, Thm. 4.3], we find that the solution is given by $G(\mathrm{d}z) = C_1 z\mathrm{d}z + C_2\mathrm{d}z$ and is unique up to the choice of constants. □

Lastly, we note the following lemma to identify one of the constants.

Lemma 2.54. *The distribution $G(\mathrm{d}z)$ in Lemma 2.52 satisfies*

$$\frac{1}{\ln(t)}\int_1^t \frac{1}{z^2}G(\mathrm{d}z) \to 0,\ t\to\infty. \tag{2.51}$$

Proof. Using linearity, we can once more treat every summand of G separately. First, we find for the contribution of the Gaussian parts of η and $\widetilde{U}$ that

$$\frac{1}{\ln(t)}\int_1^t \Big(\frac{1}{z^2}\frac{1}{2}\sigma_\eta^2+\frac{1}{2}\sigma_{\widetilde{U}}^2\Big)\mu(\mathrm{d}x) \leq \frac{1}{\ln(t)}\Big(\frac{1}{2}\sigma_\eta^2+\frac{1}{2}\sigma_{\widetilde{U}}^2\Big)\mu([1,t]) \to 0,\ t\to\infty,$$

yielding the desired value of the limit as μ is a finite measure. For the contribution of the drift first observe that

$$\lim_{z\to\infty}\frac{1}{z}\int_0^z x\mu(\mathrm{d}x)=0,$$

i.e. for every $\varepsilon>0$ we can find a value R_ε such that $\frac{1}{z}\int_0^z x\mu(\mathrm{d}x)\leq\varepsilon$ if $z>R_\varepsilon$. This yields

$$\Big|\int_1^t\frac{1}{z^2}\int_0^z \gamma_{\widetilde{U}}x\mu(\mathrm{d}x)\mathrm{d}z\Big| \leq |\gamma_{\widetilde{U}}|\Big(\int_1^{R_\varepsilon}\frac{1}{z^2}\int_0^z x\mu(\mathrm{d}x)\mathrm{d}z+\varepsilon\int_{R_\varepsilon}^t\frac{1}{z}\mathrm{d}z\Big)$$

which implies that

$$0\leq \limsup_{t\to\infty}\Big|\frac{1}{\ln(t)}\int_1^t\frac{1}{z^2}\int_0^z\Big(\gamma_\eta+x\gamma_{\widetilde{U}}\Big)\mu(\mathrm{d}x)\Big|\leq|\gamma_{\widetilde{U}}|\varepsilon.$$

Since the above statement holds for every $\varepsilon>0$, we can conclude that the limit is zero. For the contribution of the small jumps of η, recall that $S_\eta*\mu$ is integrable with respect to λ by Lemma 2.41. Therefore, we find that

$$0\leq\lim_{t\to\infty}\frac{1}{\ln(t)}\int_1^t\frac{1}{z^2}(S_\eta*\mu)(z)\mathrm{d}z\leq\lim_{t\to\infty}\frac{1}{\ln(t)}\int_{\mathbb{R}}(S_\eta*\mu)(z)\mathrm{d}z=0.$$

For the summand involving $S_{\widetilde{U}}$, splitting up the inner integral leads to

$$\int_1^t\frac{1}{z^2}\int_0^\infty xS_{\widetilde{U}}(\tfrac{z}{x})\mu(\mathrm{d}x)\mathrm{d}z=\int_1^t\frac{1}{z^2}\int_{\frac{z}{2}}^{2z}xS_{\widetilde{U}}(\tfrac{z}{x})\mu(\mathrm{d}x)\mathrm{d}z+\int_1^t\frac{1}{z^2}\int_{2z}^\infty xS_{\widetilde{U}}(\tfrac{z}{x})\mu(\mathrm{d}x)\mathrm{d}z,$$

as $x<\frac{z}{2}$ implies that $\frac{z}{x}-1>1$ and, therefore, we have $S_{\widetilde{U}}(\frac{z}{x})=0$ in this case. Since $S_{\widetilde{U}}(\frac{z}{x})$ is nonnegative by (2.40), a direct calculation leads to

$$\begin{aligned}\int_{2z}^\infty xS_{\widetilde{U}}(\tfrac{z}{x})\mu(\mathrm{d}x)&=\int_{2z}^\infty\int_{-1}^{\frac{z}{x}-1}\Big(z-x(y+1)\Big)\nu_{\widetilde{U}}(\mathrm{d}y)\mu(\mathrm{d}x)\\&=\int_{-1}^{-\frac{1}{2}}\int_{2z}^{\frac{z}{y+1}}\Big(z-x(y+1)\Big)\mu(\mathrm{d}x)\nu_{\widetilde{U}}(\mathrm{d}y)\\&\leq\int_{-1}^{-\frac{1}{2}}\Big(z-2z(y+1)\Big)\int_{2z}^{\frac{z}{y+1}}\mu(\mathrm{d}x)\nu_{\widetilde{U}}(\mathrm{d}y)\\&\leq z\mu([2z,\infty))\nu_{\widetilde{U}}([-1,-\tfrac{1}{2}]),\end{aligned}$$

which, since $\mu([2z,\infty)) \to 0$ as $z \to \infty$, implies that the term is in $o(z)$. We can thus apply the same reasoning as for the contribution of the drift terms and conclude that

$$\lim_{t\to\infty} \frac{1}{\ln(t)} \int_1^t \frac{1}{z^2} \int_{2z}^{\infty} x S_{\widetilde{U}}(\tfrac{z}{x})\mu(\mathrm{d}x)\mathrm{d}z = 0.$$

If $z/2 \le x \le 2z$, consider

$$\begin{aligned}
\int_1^t \int_{\frac{z}{2}}^{2z} \frac{x}{z^2} S_{\widetilde{U}}(\tfrac{z}{x})\mu(\mathrm{d}x)\mathrm{d}z &= \int_{\frac{1}{2}}^{2t} \int_{\max\{\frac{x}{2},1\}}^{\min\{t,2x\}} \frac{x}{z^2} S_{\widetilde{U}}(\tfrac{z}{x})\mathrm{d}z\mu(\mathrm{d}x) \\
&\le \int_{\frac{1}{2}}^{2t} \int_{\frac{x}{2}}^{x(1-\varepsilon)} \frac{x}{z^2} S_{\widetilde{U}}(\tfrac{z}{x})\mathrm{d}z\mu(\mathrm{d}x) + \int_{\frac{1}{2}}^{2t} \int_{x(1-\varepsilon)}^{x(1+\varepsilon)} \frac{x}{z^2} S_{\widetilde{U}}(\tfrac{z}{x})\mathrm{d}z\mu(\mathrm{d}x) \\
&\quad + \int_{\frac{1}{2}}^{2t} \int_{x(1+\varepsilon)}^{2x} \frac{x}{z^2} S_{\widetilde{U}}(\tfrac{z}{x})\mathrm{d}z\mu(\mathrm{d}x)
\end{aligned}$$

for some $\varepsilon \in (0,1/2)$. In the case that the values of z are bounded away from $z = x$, we can use (2.40) to estimate $S_{\widetilde{U}}(\frac{z}{x})$ by a constant $C_\varepsilon > 0$, implying that

$$\int_{x(1+\varepsilon)}^{2x} \frac{1}{z^2} S_{\widetilde{U}}(\tfrac{z}{x})\mathrm{d}z \le C_\varepsilon \int_{x(1+\varepsilon)}^{2x} \frac{1}{z^2}\mathrm{d}z = C_\varepsilon\Big(\frac{1}{x(1+\varepsilon)} - \frac{1}{2x}\Big)$$

with a similar estimate also holding for $z \in [x/2, x(1-\varepsilon)]$. If the values of z are close to the singularity at $z = x$, we find that

$$\begin{aligned}
\int_{x(1-\varepsilon)}^{x(1+\varepsilon)} \frac{1}{z^2} S_{\widetilde{U}}(\tfrac{z}{x})\mathrm{d}z &\le \frac{1}{x^2(1-\varepsilon)^2} \int_{x(1-\varepsilon)}^{x} S_{\widetilde{U}}(\tfrac{z}{x})\mathrm{d}z + \frac{1}{x^2}\int_x^{x(1+\varepsilon)} S_{\widetilde{U}}(\tfrac{z}{x})\mathrm{d}z \\
&= \frac{1}{x(1-\varepsilon)^2}\int_{-\varepsilon}^{0} S_{\widetilde{U}}(t+1)\mathrm{d}t + \frac{1}{x}\int_0^{\varepsilon} S_{\widetilde{U}}(t+1)\mathrm{d}t \qquad (2.52)
\end{aligned}$$

by a suitable substitution. Note that both integrals are finite by the definition of $\nu_{\widetilde{U}}$, since

$$\begin{aligned}
\int_0^{\varepsilon} S_{\widetilde{U}}(t+1)\mathrm{d}t &= \int_0^{\varepsilon}\int_t^1 (y-t)\nu_{\widetilde{U}}(\mathrm{d}y)\mathrm{d}t \\
&= \int_0^1 \int_0^{\min\{\varepsilon,y\}} (y-t)\mathrm{d}t\nu_{\widetilde{U}}(\mathrm{d}y) \le \int_0^1 y^2 \nu_{\widetilde{U}}(\mathrm{d}y), \qquad (2.53)
\end{aligned}$$

and a similar estimate holds for $t \in [-\varepsilon, 0)$. Therefore, one obtains an estimate in terms of $\frac{1}{x}$ in all cases, implying

$$\begin{aligned}
0 &\le \lim_{t\to\infty} \frac{1}{\ln(t)} \int_1^t \frac{1}{z^2}\int_{\frac{z}{2}}^{2z} x S_{\widetilde{U}}(\tfrac{z}{x})\mu(\mathrm{d}x)\mathrm{d}z \le \lim_{t\to\infty}\frac{1}{\ln(t)}\int_{\frac{1}{2}}^{t} x\frac{\widetilde{C}}{x}\mu(\mathrm{d}x) \\
&= \widetilde{C}\lim_{t\to\infty}\frac{1}{\ln(t)}\mu([\tfrac{1}{2},t]) = 0
\end{aligned}$$

with a suitable constant $\widetilde{C}$. For the term corresponding to the big jumps of η, observe that the function B_η is bounded and satisfies $\lim_{|z|\to\infty} B_\eta(z) = 0$ by (2.37). This implies that also $(B_\eta * \mu)(z) \to 0$, as can be seen by partitioning the domain of integration of the convolution with respect to the values of the function B_η. Thus, we also find that

$$\lim_{|z|\to\infty} \frac{1}{z}\int_0^z (B_\eta * \mu)(x)\mathrm{d}x = 0,$$

i.e. the corresponding summand in G is in $o(z)$, which, in combination with the above arguments is enough to conclude that this term also does not contribute to the limit in (2.51). Similarly, we observe that also $B_{\widetilde{U}}$ is bounded and satisfies $\lim_{|t|\to\infty} B_{\widetilde{U}}(t) = 0$ as $\nu_{\widetilde{U}}((1,\infty)) < \infty$, which, together with μ being a finite measure implies that

$$\lim_{z\to\infty} \int_0^z B_{\widetilde{U}}(\tfrac{z}{x})\mu(\mathrm{d}x) = 0$$

by dominated convergence. Therefore, the corresponding antiderivative appearing in G is in $o(z)$ as desired. □

Using the above lemmas, we can now prove Theorem 2.43.

Proof of Theorem 2.43. Starting from (2.31), we first rewrite the left-hand side according to Lemmas 2.51 and 2.52 to arrive at

$$\int_{\mathbb{R}} A^{\widetilde{V}} f(x)\mu(\mathrm{d}x) = \int_{\mathbb{R}} f''(z)G(\mathrm{d}z) = 0.$$

Recall that this equation holds for every $f \in C_c^\infty(\mathbb{R})$ by Corollary 2.36. Using the results from Lemma 2.53, we find that G'' equals the zero distribution and thus $G = C_1 z\mathrm{d}z + C_2\mathrm{d}z$ for some constants $C_1, C_2 \in \mathbb{R}$. Identifying the equivalent expressions for G from Lemmas 2.52 and 2.53 now yields

$$\begin{aligned} C_1 z\mathrm{d}z + C_2\mathrm{d}z =& \Big(\frac{1}{2}\sigma_\eta^2 + \frac{1}{2}z^2\sigma_{\widetilde{U}}^2\Big)\mu(\mathrm{d}z) + (S_\eta * \mu)(z)\mathrm{d}z \\ &+ \Big(\mathbb{1}_{\{z>0\}}\int_0^\infty xS_{\widetilde{U}}(\tfrac{z}{x})\mu(\mathrm{d}x) + \mathbb{1}_{\{z<0\}}\int_{-\infty}^0 |x|S_{\widetilde{U}}(\tfrac{z}{x})\mu(\mathrm{d}x)\Big)\mathrm{d}z \\ &- \int_{0+}^z \big(\gamma_\eta + x\gamma_{\widetilde{U}}\big)\mu(\mathrm{d}x)\mathrm{d}z - \int_{0+}^z (B_\eta * \mu)(x)\mathrm{d}x\mathrm{d}z \\ &- \int_{0+}^z\int_{0+}^t B_{\widetilde{U}}(\tfrac{t}{x})\mu(\mathrm{d}x)\mathrm{d}t\mathrm{d}z. \end{aligned} \tag{2.54}$$

In order to arrive at (2.41), the values of C_1 and C_2 have to be identified. To determine C_1, observe that

$$\frac{1}{\ln(t)}\int_1^t \frac{1}{z^2}(C_1 z + C_2)\mathrm{d}z = C_1 + C_2\frac{1-\frac{1}{t}}{\ln(t)} \to C_1,\ t\to\infty,$$

such that we can give its value by applying the above transformations to both sides of (2.54) and letting $t\to\infty$. From Lemma 2.54, this limit is equal to zero. Renaming $K = C_2$, we arrive at (2.41). □

Proof of Corollary 2.45. (i) and (ii), Existence: From the form of (2.41), we see that

$$\Big(\frac{1}{2}\sigma_\eta^2 + \frac{1}{2}z^2\sigma_{\widetilde{U}}^2\Big)\mu(\mathrm{d}z) = H(z)\mathrm{d}z$$

for some locally integrable function H. It follows that μ has a density f_μ on $\mathbb{R}$ whenever $\sigma_\eta^2 > 0$ and that $\mu|_{\mathbb{R}\setminus\{0\}}$ has a density f_μ on $\mathbb{R}\setminus\{0\}$ whenever $\sigma_{\widetilde{U}}^2 > 0 = \sigma_\eta^2$. In both cases, Corollary 2.44 implies that f_μ must satisfy (2.42) for λ-a.e. $z \in \mathbb{R}$.

(iii) Since $S_\eta \geq 0$ and $S_{\widetilde{U}} \geq 0$, the term $(\frac{1}{2}\sigma_\eta^2 + \frac{1}{2}z^2\sigma_{\widetilde{U}}^2)f_\mu$ can be bounded by the right-hand side of (2.42). Observe that all quantities in this bound, apart from $\mathbb{1}_{\{z<0\}}\gamma_\eta\mu(\{0\})$ if $\gamma_\eta\mu(\{0\}) \neq 0$, are continuous functions in z. In particular, the right-hand side of (2.42) is locally bounded in $z \in \mathbb{R}$, continuous on $\mathbb{R}\setminus\{0\}$ and, whenever $\mu(\{0\}) = 0$ (which is in particular satisfied if $\sigma_\eta^2 > 0$), also continuous on $\mathbb{R}$. Observing further that $B_\eta * f_\mu$ is integrable and $B_{\widetilde{U}}$ is bounded by definition (cf. Lemma 2.41), we see that the right-hand side of (2.42) can be bounded by $M_1 + M_2|z|$ for $z \in \mathbb{R}$ and suitable constants $M_1, M_2 \geq 0$, yielding the desired bound

$$\Big(\frac{1}{2}\sigma_\eta^2 + \frac{1}{2}z^2\sigma_{\widetilde{U}}^2\Big)f_\mu(z) \leq M_1 + M_2|z|, \quad \forall z \neq 0.$$

(ii), Continuity: Let $\sigma_{\widetilde{U}}^2 > 0$ and $\sigma_\eta^2 \geq 0$. As the right-hand side of (2.42) is continuous on $\mathbb{R}\setminus\{0\}$, it suffices to show that S_η, $S_\eta * f_\mu$, as well as $z \mapsto \mathbb{1}_{\{z>0\}}\int_0^\infty S_{\widetilde{U}}(\frac{z}{x})f_\mu(x)\mathrm{d}x$ and $z \mapsto \mathbb{1}_{\{z<0\}}\int_{-\infty}^0 |x|S_{\widetilde{U}}(\frac{z}{x})f_\mu(x)\mathrm{d}x$ are continuous on $\mathbb{R}\setminus\{0\}$. Write

$$S_\eta(z) = \int_\varepsilon^\infty (y-z)\mathbb{1}_{[z,\infty)}(y)\nu_\eta|_{[-1,1]}(\mathrm{d}y)$$

for $z > \varepsilon > 0$, and observe that the function $z \mapsto (y-z)\mathbb{1}_{[z,\infty)}(y)$ is continuous at $z_0 > \varepsilon$ for all values of y. Thus, an application of Lebesgue's dominated convergence theorem yields that S_η is continuous at $z_0 > \varepsilon$. Since $\varepsilon > 0$ was arbitrary and we can apply a similar argument for $z_0 < 0$, it follows that S_η is continuous on $\mathbb{R}\setminus\{0\}$. To show that $S_\eta * f_\mu$ is continuous at $z_0 > 0$, let $\varepsilon \in (0, z_0)$ as well as $\delta \in (0,1)$ and decompose

$$S_\eta(z) = S_\eta^{\delta,1}(z) + S_\eta^{\delta,2}(z), \tag{2.55}$$

where the functions on the right-hand side are defined similar to (2.39) with $\nu_\eta|_{[-1,1]}$ replaced by $\nu_\eta|_{[-\delta,\delta]}$ or $\nu_\eta|_{[-1,1]\setminus[-\delta,\delta]}$ for $S_\eta^{\delta,1}$ or $S_\eta^{\delta,2}$, respectively. Then $S_\eta^{\delta,1}$ and $S_\eta^{\delta,2}$ are continuous on $\mathbb{R}\setminus\{0\}$ and $S_\eta^{\delta,2}$ is bounded by $\nu_\eta([-1,1]\setminus[-\delta,\delta]) < \infty$ by definition. The latter implies that $S_\eta^{\delta,2} * f_\mu$ is continuous on $\mathbb{R}$ for every $\delta \in (0,1)$ (see e.g. [61, Thm. 14.8]). For the treatment of $S_\eta^{\delta,1} * f_\mu$, recall from Lemma 2.41 that S_η is integrable with respect to λ. Since $S_\eta^{\delta,1}$ converges (point-wise) to zero as $\delta \downarrow 0$ and $S_\eta^{\delta,1} \leq S_\eta$, it follows that

$$\lim_{\delta\downarrow 0}\int_{-\frac{z_0}{4}}^{\frac{z_0}{4}} S_\eta^{\delta,1}(z)\mathrm{d}z = 0$$

by dominated convergence. By part (iii) of the corollary, we can bound f_μ by a constant $M_3 > 0$ on $[z_0/4, 7z_0/4]$. For $z \in (z_0/2, 3z_0/2)$ and $0 < \delta < z_0/4$ we have $S_\eta^{\delta,1} = 0$ for $|y| > z_0/4$ and hence

$$\Big(S_\eta^{\delta,1} * f_\mu\Big)(z) = \int_{-\frac{z_0}{4}}^{\frac{z_0}{4}} f_\mu(z-x)S_\eta^{\delta,1}(x)\mathrm{d}x \leq M_3\int_{-\frac{z_0}{4}}^{\frac{z_0}{4}} S_\eta^{\delta,1}(x)\mathrm{d}x.$$

Choosing δ small enough, the above estimate on the right-hand side becomes arbitrarily small. Together with the previously established continuity of $S_\eta^{\delta,2}$ and (2.55), this shows continuity of $S_\eta * f_\mu$ at $z_0 > 0$. Applying a similar argument for $z_0 < 0$, we can conclude that $S_\eta * f_\mu$ is continuous on $\mathbb{R} \setminus \{0\}$.

It remains to consider the terms involving $S_{\widetilde{U}}$. First, we establish continuity of the mapping $z \mapsto \int_0^\infty x S_{\widetilde{U}}(\frac{z}{x}) f_\mu(x) \mathrm{d}x$ in $z_0 > 0$. As for (2.55), let $\delta \in (0,1)$ and decompose

$$S_{\widetilde{U}}(z) = S_{\widetilde{U}}^{\delta,1}(z) + S_{\widetilde{U}}^{\delta,2}(z), \tag{2.56}$$

where the quantities $S_{\widetilde{U}}^{\delta,1}$ and $S_{\widetilde{U}}^{\delta,2}$ are defined similar to (2.40) with $\nu_{\widetilde{U}}|_{[-1,1]}$ replaced by $\nu_{\widetilde{U}}|_{[-\delta,\delta]}$ or $\nu_{\widetilde{U}}|_{[-1,1]\setminus[-\delta,\delta]}$, respectively. As in the treatment of S_η, a bound for $S_{\widetilde{U}}^{\delta,2}$ is readily obtained from the definition since

$$S_{\widetilde{U}}^{\delta,2}(z) \le \int_{-1}^{z-1} (z-1-(-1)) \nu_{\widetilde{U}}|_{[-1,1]\setminus[-\delta,\delta]}(\mathrm{d}y) \le z \nu_{\widetilde{U}}([-1,1] \setminus [-\delta,\delta])$$

for $z \in [0,1]$ and, setting $M_4^\delta = \nu_{\widetilde{U}}([-1,1] \setminus [-\delta,\delta])$,

$$S_{\widetilde{U}}^{\delta,2}(z) \le \int_0^1 y \nu_{\widetilde{U}}|_{[-1,1]\setminus[-\delta,\delta]}(\mathrm{d}y) \le M_4^\delta$$

for $z > 1$. Further, $S_{\widetilde{U}}^{\delta,2}$ is continuous on $[0,\infty) \setminus \{1\}$, as can be seen from applying a similar argument as for $S_\eta^{\delta,2}$. Writing

$$\int_0^\infty x S_{\widetilde{U}}^{\delta,2}(\tfrac{z}{x}) f_\mu(x) \mathrm{d}x = \int_0^\infty x \mathbb{1}_{(0,z)}(x) S_{\widetilde{U}}^{\delta,2}(\tfrac{z}{x}) f_\mu(x) \mathrm{d}x + \int_0^\infty x \mathbb{1}_{(z,\infty)}(x) S_{\widetilde{U}}^{\delta,2}(\tfrac{z}{x}) f_\mu(x) \mathrm{d}x,$$

the integrand can be bounded by $(x\mathbb{1}_{(0,z)}(x) M_4^\delta + z \mathbb{1}_{(z,\infty)}(x) M_4^\delta) f_\mu(x)$ such that the continuity of the mapping $z \mapsto \int_0^\infty x S_{\widetilde{U}}^{\delta,2}(\frac{z}{x}) f_\mu(x) \mathrm{d}x$ in $z_0 > 0$ follows by dominated convergence. Since we can apply a similar argument for the continuity in $z_0 < 0$ of the corresponding function on the negative real numbers and we have for $z_0 = 0$ that

$$\lim_{z \downarrow 0} \int_0^\infty x S_{\widetilde{U}}^{\delta,2}(\tfrac{z}{x}) f_\mu(x) \mathrm{d}x = \int_0^\infty x S_{\widetilde{U}}^{\delta,2}(0) f_\mu(x) \mathrm{d}x = 0,$$

it follows that the mapping

$$z \mapsto \mathbb{1}_{\{z>0\}} \int_0^\infty x S_{\widetilde{U}}^{\delta,2}(\tfrac{z}{x}) f_\mu(x) \mathrm{d}x + \mathbb{1}_{\{z<0\}} \int_{-\infty}^0 |x| S_{\widetilde{U}}^{\delta,2}(\tfrac{z}{x}) f_\mu(x) \mathrm{d}x$$

is continuous on $\mathbb{R}$. Therefore, it only remains to consider the term involving $S_{\widetilde{U}}^{\delta,1}$. In order to do so, observe that the support of $S_{\widetilde{U}}^{\delta,1}$ is contained in the interval $[1-\delta, 1+\delta]$, that $S_{\widetilde{U}}^{\delta,1} \le S_{\widetilde{U}}$ by definition and that, as a consequence of the integrability of $S_{\widetilde{U}}$ (cf. Lemma 2.41) we have that

$$0 \le \lim_{\delta \downarrow 0} \int_{\mathbb{R}} S_{\widetilde{U}}^{\delta,1}(x) \mathrm{d}x \le \lim_{\delta \downarrow 0} \int_{1-\delta}^{1+\delta} S_{\widetilde{U}}(x) \mathrm{d}x = 0.$$

Using the substitution $v = z/x$ for $z > 0$, it follows that

$$\int_0^\infty x S_{\widetilde{U}}^{\delta,1}(\tfrac{z}{x}) f_\mu(x) \mathrm{d}x = \int_{\frac{z}{1+\delta}}^{\frac{z}{1-\delta}} x S_{\widetilde{U}}^{\delta,1}(\tfrac{z}{x}) f_\mu(x) \mathrm{d}x = \int_{1-\delta}^{1+\delta} \frac{z^2}{v^3} S_{\widetilde{U}}^{\delta,1}(v) f_\mu(\tfrac{z}{v}) \mathrm{d}v.$$

Since f_μ is locally bounded on $\mathbb{R} \setminus \{0\}$ by part (iii) of the corollary, the above quantity becomes arbitrarily small for sufficiently small $\delta > 0$ when $z \in (z_0/2, 3z_0/2)$ for $z_0 > 0$. Together with (2.56) and the already established continuity of the terms involving $S_\eta^{\delta,2}$, it follows that the mapping

$$z \mapsto \mathbb{1}_{\{z>0\}} \int_0^\infty x S_{\widetilde{U}}(\tfrac{z}{x}) f_\mu(x) \mathrm{d}x + \mathbb{1}_{\{z<0\}} \int_{-\infty}^0 |x| S_{\widetilde{U}}(\tfrac{z}{x}) f_\mu(x) \mathrm{d}x$$

is continuous on $\mathbb{R}\setminus\{0\}$. The desired continuity of f_μ on $\mathbb{R}\setminus\{0\}$ hence follows from (2.42).

(i), Continuity: Now assume that $\sigma_\eta^2 > 0$. As μ has a density on $\mathbb{R}$, it follows that $\mu(\{0\}) = 0$ and using the same argument as in the proof of part (ii), it is sufficient to show that the mappings $z \mapsto (S_\eta * f_\mu)(z)$ and

$$z \mapsto \mathbb{1}_{\{z>0\}} \int_0^\infty x S_{\widetilde{U}}(\tfrac{z}{x}) f_\mu(x) \mathrm{d}x + \mathbb{1}_{\{z<0\}} \int_{-\infty}^0 |x| S_{\widetilde{U}}(\tfrac{z}{x}) f_\mu(x) \mathrm{d}x$$

are continuous at $z = 0$. Observe that by (iii), f_μ is not only locally bounded on $\mathbb{R} \setminus \{0\}$, but on $\mathbb{R}$ whenever $\sigma_\eta^2 > 0$, such that we can use the methods from part (ii) also for $z = 0$ in this case. Note that f_μ is in particular bounded on $[-1, 1]$, and that

$$\lim_{\delta \downarrow 0} \int_{-1}^1 S_\eta^{\delta,1}(z) \mathrm{d}z = \lim_{\delta \downarrow 0} \int_{-1}^1 S_{\widetilde{U}}^{\delta,1}(z) \mathrm{d}z = 0.$$

The terms including $S_{\widetilde{U}}^{\delta,1}$ and $S_\eta^{\delta,1}$ thus become arbitrarily small in a neighborhood of zero. Since the terms involving $S_{\widetilde{U}}^{\delta,2}$ and $S_\eta^{\delta,2}$ are again continuous, we find that f_μ is also continuous at $z = 0$. This finishes the proof. $\square$

Remark 2.55. It seems tempting to iterate the proof of Corollary 2.45 to obtain further smoothness properties of f_μ. Such an argument would require being able to show that $f_\mu \in C(\mathbb{R})$ implies $S_\eta * f_\mu \in C^1(\mathbb{R})$ or at least $S_\eta * f_\mu \in C^1(\mathbb{R} \setminus \{0\})$, as well as similar statements for the other quantities on the right-hand side of (2.42). This claim is, however, not true in general. A counterexample is given by $\nu_\eta(\mathrm{d}x) = x^{-5/2} \mathbb{1}_{(0,1)}(x) \mathrm{d}x$ and $f \in C_c(\mathbb{R})$ being a density that satisfies $f(x) = c((x-2)^{1/3} + 2)$ for $x \in [2, 3]$ and $f(x) = c(2 - (2-x)^{1/3})$ for $x \in [1, 2]$, where $c > 0$ is a suitable norming constant. Since $S_\eta(x) \sim 4/3 x^{-1/2}$ as $x \downarrow 0$ and $f'(x) = \frac{c}{3}|x-2|^{-2/3}$ for $x \in [1, 3]$, an application of Fatou's lemma shows that

$$\liminf_{x \downarrow 2} \frac{(S_\eta * f)(x) - (S_\eta * f)(2)}{x - 2} = \infty$$

such that $S_\eta * f$ is not differentiable in $x = 2$. Hence, an easy iterative argument seems not to be possible in the general case considered in Theorem 2.43 and Corollary 2.45. Observe, however, that Corollary 2.48 gives conditions for $f_\mu \in C^1(\mathbb{R} \setminus \{0\})$ and Example 2.56 below gives a concrete example when $\sigma_\eta^2 > 0$ and $f_\mu \in C^1(\mathbb{R} \setminus \{0\}) \setminus C^1(\mathbb{R})$. Furthermore, note that restricting the characteristics of the Lévy processes involved may yield much stronger smoothness properties than the general case, e.g. if $\eta_t = t$, where the density of the law of the killed exponential functional with $q \geq 0$ is infinitely often differentiable on $\mathbb{R} \setminus \{0\}$ for most choices of ξ (see [49, Thm. 2.4(3)]).

Proof of Corollary 2.46. (i) Observe first that $S_\eta^b = S_\eta^{FM} - S_\eta$ and $S_{\widetilde{U}}^b = S_{\widetilde{U}}^{FM} - S_{\widetilde{U}}$ are bounded functions vanishing at infinity and that $S_{\widetilde{U}}^b = 0$ for $z \leq 1$. Thus, the convolution $S_\eta^{FM} * \mu$ can be written as a sum of an integrable and a bounded function and hence is locally integrable with respect to λ. The analog of Lemma 2.42 involves the measure ϱ^{FM}, which takes the form

$$\varrho^{FM}(\mathrm{d}z) = \varrho(\mathrm{d}z) + \Big(\mathbb{1}_{\{z>0\}} \int_0^z S_{\widetilde{U}}^b(\tfrac{z}{x})\mu(\mathrm{d}x) - \mathbb{1}_{\{z<0\}} \int_z^0 x S_{\widetilde{U}}(\tfrac{z}{x})\mu(\mathrm{d}x)\Big)\mathrm{d}z,$$

from which it is visible that the right-hand side of (2.43) defines a locally finite measure. As the moment condition implies that both $\int_{\mathbb{R}\setminus[-1,1]} |x|\nu_\eta(\mathrm{d}y)$ and $\int_{(1,\infty)} |x|\nu_{\widetilde{U}}(\mathrm{d}y)$ are finite, it follows that

$$\begin{aligned} &\int_{\mathbb{R}} \Big(f(x+y) - f(x) - yf'(x)\mathbb{1}_{|y|\leq 1}(y)\Big)\nu_\eta(\mathrm{d}y) + \gamma_\eta f'(x) \\ &\quad = \int_{\mathbb{R}} \Big(f(x+y) - f(x) - yf'(x)\Big)\nu_\eta(\mathrm{d}y) + \gamma_\eta^1 f'(x) \end{aligned} \tag{2.57}$$

with a similar relation also holding true for $\widetilde{U}$. One now follows the proofs of Lemmas 2.51 and 2.52, i.e. considers the integral with respect to μ, shows that Fubini's Theorem is applicable for the terms involving multiple integrals and thus recovers a similar distribution $G^{FM}(\mathrm{d}z)$. To show e.g. that

$$\int_{1+}^\infty \int_0^\infty \int_x^{x+xy} |f''(t)(x+xy-t)|\mathrm{d}t\mu(\mathrm{d}x)\nu_{\widetilde{U}}(\mathrm{d}y) < \infty$$

for $f \in C_c^\infty(\mathbb{R})$ with $\mathrm{supp}(f) \subset [-R, R]$, observe that the inner integral can be estimated by $RC(x + xy) \leq R^2C(1 + y)$ for a suitable constant $C \geq 0$ such that $|f''(t)| \leq C$ for $t \in [-R, R]$. Together with the moment condition, this yields that the triple integral, too, is finite. Following the remaining steps of the proof of Theorem 2.43, we also obtain that

$$\lim_{t\to\infty} \frac{1}{\ln(t)} \int_1^z \frac{1}{z^2}\Big(S_\eta^b(z) + \int_0^z x S_{\widetilde{U}}^b(\tfrac{z}{x})\mu(\mathrm{d}x)\Big)\mathrm{d}z = 0,$$

which yields Equation (2.43) as claimed.

(ii) Since $\mathbb{E}|\mathcal{E}(U)_1| = \mathbb{E}\mathcal{E}(U)_1 = \mathrm{e}^{\mathbb{E}U_1}$ (see [2, Prop. 3.1]) and $\gamma_{\widetilde{U}}^1 = \mathbb{E}\widetilde{U}_1 = \mathbb{E}U_1 - q$ by definition, the condition $\mathbb{E}|\mathcal{E}(U)_1| < \mathrm{e}^q$ is equivalent to $\gamma_{\widetilde{U}}^1 < 0$. Further, $\mathbb{E}|\mathcal{E}(U)_1| < \mathrm{e}^q$ implies $\int_{\mathbb{R}} |x|\mu(\mathrm{d}x) < \infty$, as is shown in [2, Thm. 3.1]. Let $G^{FM}(\mathrm{d}z)$ denote the right-hand side of (2.43). To determine the value of the constant, we use a similar approach as in Lemma 2.54, showing that

$$\lim_{t\to\infty} t\int_t^\infty \frac{1}{z^2} G^{FM}(\mathrm{d}z) = -\int_{0+}^\infty \Big(\gamma_\eta^1 + x\gamma_{\widetilde{U}}^1\Big)\mu(\mathrm{d}x). \tag{2.58}$$

To see that (2.58) holds, observe first that $\lim_{t\to\infty} t\mu((t,\infty)) = 0$ as a consequence of $t\mu((t,\infty)) \leq \int_{(t,\infty)} |x|\mu(\mathrm{d}x) < \infty$. This implies

$$\lim_{t\to\infty} t\int_t^\infty \Big(\frac{\sigma_\eta^2}{2z^2} + \frac{\sigma_{\widetilde{U}}^2}{2}\Big)\mu(\mathrm{d}z) = 0.$$

Further, note that $S_\eta^{FM} * \mu = S_\eta * \mu + S_\eta^b * \mu$. Here, $S_\eta * \mu$ is integrable with respect to λ, from which we obtain

$$\lim_{t\to\infty} t \int_t^\infty \frac{1}{z^2}\Big(S_\eta * \mu\Big)(z)\mathrm{d}z \leq \lim_{t\to\infty} \int_t^\infty (S_\eta * \mu)(z)\mathrm{d}z = 0,$$

and $S_\eta^b * \mu$ is bounded with $\lim_{t\to\infty} S_\eta^b * \mu(z) = 0$. Hence, the integral involving S_η^{FM} also vanishes. Next, observe that $S_{\widetilde{U}}$ is bounded on $[0,\infty) \setminus [1/2, 2]$ and that $\lim_{z\to\infty} S_{\widetilde{U}}(z) = 0$. Since $\int |x|\mu(\mathrm{d}x) < \infty$, an application of Lebesgue's dominated convergence theorem yields

$$\lim_{z\to\infty} \int_{(0,\frac{z}{2})\cup(2z,\infty)} x S_{\widetilde{U}}^{FM}(\tfrac{z}{x})\mu(\mathrm{d}x) = 0$$

and hence

$$\lim_{t\to\infty} t \int_t^\infty \frac{1}{z^2} \int_{(0,\frac{z}{2})\cup(2z,\infty)} x S_{\widetilde{U}}^{FM}(\tfrac{z}{x})\mu(\mathrm{d}x)\mathrm{d}z = 0.$$

For $z/2 \leq x \leq z$ we find, similar to (2.52) and (2.53), that for some constant $C > 0$

$$t\int_t^\infty \frac{1}{z^2}\int_{\frac{z}{2}}^{2z} x S_{\widetilde{U}}^{FM}(\tfrac{z}{x})\mu(\mathrm{d}x)\mathrm{d}z \leq t\int_{\frac{t}{2}}^\infty x \int_{\frac{x}{2}}^\infty \frac{1}{z^2} S_{\widetilde{U}}^{FM}(\tfrac{z}{x})\mathrm{d}z\mu(\mathrm{d}x) \leq t\int_{\frac{t}{2}}^\infty x\frac{C}{x}\mu(\mathrm{d}x),$$

where the right-hand side converges to zero as $t \to \infty$. Lastly, observe that

$$\lim_{t\to\infty} t\int_t^\infty \frac{1}{z^2}\int_{0+}^z \Big(\gamma_\eta^1 + x\gamma_{\widetilde{U}}^1\Big)\mu(\mathrm{d}x)\mathrm{d}z = \int_{0+}^\infty \Big(\gamma_\eta^1 + x\gamma_{\widetilde{U}}^1\Big)\mu(\mathrm{d}x),$$

which yields the value of K and thus finishes the proof of (2.58). That the constant can also be written as $\int_{-\infty}^0 \Big(\gamma_\eta^1 + x\gamma_{\widetilde{U}}^1\Big)\mu(\mathrm{d}x)$ follows by a similar argument considering $|t|\int_{-\infty}^t z^{-2}G^{FM}(\mathrm{d}z)$ for $t \to -\infty$, or alternatively from $\mathbb{E}(V_{q,\xi,\eta})\mathbb{E}(\widetilde{U}_1) = -\mathbb{E}(\eta_1)$ (cf. [2, Thm. 3.3a]). Now assume that $\sigma_\eta^2 + \sigma_{\widetilde{U}}^2 > 0$. By Corollary 2.45, $\mu|_{\mathbb{R}\setminus\{0\}}$ has a density f_μ. Note, however, that rearranging the terms in (2.43) and using the positivity of S_η^{FM} and $S_{\widetilde{U}}^{FM}$ as in the proof of Corollary 2.45 leads to

$$\Big(\frac{1}{2}\sigma_\eta^2 + \frac{1}{2}z^2\sigma_{\widetilde{U}}^2\Big)f_\mu(z) \leq K + \int_{0+}^z \Big(\gamma_\eta^1 + \gamma_{\widetilde{U}}^1\Big)\mu(\mathrm{d}x) \leq |K| + |\gamma_\eta^1| + |\gamma_{\widetilde{U}}^1|\int_{\mathbb{R}} |x|\mu(\mathrm{d}x) < \infty$$

here due to the moment condition. Thus, f_μ is bounded. □

Proof of Corollary 2.47. Observe first that $B_\eta^I = B_\eta^{FV} - B_\eta$ and $B_{\widetilde{U}}^I = B_{\widetilde{U}}^{FV} - B_{\widetilde{U}}$ are integrable with respect to λ due to the finite variation condition. In particular, we have that $B_\eta^{FV} * \mu = B_\eta * \mu + B_\eta^I * \mu$ is the sum of a bounded and an integrable function and hence locally integrable. In order to obtain the analog of Lemma 2.41 in the finite variation case, one needs to show that the mapping $z \mapsto \int_{0+}^\infty B_{\widetilde{U}}(\frac{z}{x})\mu(\mathrm{d}x)$ is locally integrable with respect to λ. Since $B_{\widetilde{U}}$ is bounded, it is sufficient to consider $B_{\widetilde{U}}^I$ for which we find

$$\begin{aligned}\int_{0+}^R \int_0^{z-} B_{\widetilde{U}}^I(\tfrac{z}{x})\mu(\mathrm{d}x)\mathrm{d}z &= \int_{0+}^R\int_{x+}^R\int_{\frac{z}{x}-1}^1 \nu_{\widetilde{U}}(\mathrm{d}y)\mathrm{d}z\mu(\mathrm{d}x)\\ &\leq \int_{0+}^R\int_{0+}^1\int_x^{x+xy}\mathrm{d}z\nu_{\widetilde{U}}(\mathrm{d}y)\mu(\mathrm{d}x) \leq R\mu([0,R])\int_{0+}^1 y\nu_{\widetilde{U}}(\mathrm{d}y),\end{aligned}$$

showing that the triple integral is finite due to the finite variation condition. The other quantities can be estimated similarly. Thus, the right-hand side of (2.46) defines a locally finite measure, which we denote by $G^{FV}(\mathrm{d}z)$. With the jump part of $\widetilde{U}$ and η being of finite variation, it follows that $\int_{[-1,1]} |y|\nu_\eta(\mathrm{d}y)$ and $\int_{[-1,1]} |y|\nu_{\widetilde{U}}(\mathrm{d}y)$ are finite, in particular

$$\begin{aligned}\int_{\mathbb{R}} \Big(f(x+y) - f(x) - yf'(x)\mathbb{1}_{|y|\le 1}(y)\Big)\nu_\eta(\mathrm{d}y) + \gamma_\eta f'(x) \\ = \int_{\mathbb{R}} \Big(f(x+y) - f(x)\Big)\nu_\eta(\mathrm{d}y) + \gamma_\eta^0 f'(x) \qquad (2.59)\end{aligned}$$

with a similar relation also holding true for $\widetilde{U}$. As in the proof of Corollary 2.46, there is no need to split the integrals with respect to Lévy measures when rewriting as in Lemma 2.51 such that the jumps of η and $\widetilde{U}$, respectively, only yield a single term. Since also $\sigma_\eta^2 = \sigma_{\widetilde{U}}^2 = 0$ by assumption, all terms can be rewritten to only include f' and there is no need to consider antiderivatives when following the argument of Lemma 2.52. This implies that the distribution G^{FV} obtained satisfies $\int_{\mathbb{R}} f'(z)G^{FV}(\mathrm{d}z) = 0$ for all $f \in C_c^\infty(\mathbb{R})$, hence $(G^{FV})' = 0$, giving the form $G^{FV}(\mathrm{d}z) = C\mathrm{d}z$ for a single real constant C. We have thus obtained an equivalent to (2.41) in the finite variation case. In order for the constant to vanish, we need, similar to (2.51), that

$$C = \lim_{t\to\infty} \frac{1}{\ln(t)} \int_1^t \frac{1}{z} G^{FV}(\mathrm{d}z) = 0. \qquad (2.60)$$

Using the results obtained in the proof of Lemma 2.54, one can directly conclude that the drift term, as well as the terms involving B_η and $B_{\widetilde{U}}$ as defined in (2.37) and (2.38), respectively, satisfy the desired asymptotics, leaving only B_η^I and $B_{\widetilde{U}}^I$ to be considered. Since $B_\eta^I * \mu$ is integrable with respect to λ, it readily follows that

$$0 \le \lim_{t\to\infty} \frac{1}{\ln(t)} \int_1^t \frac{1}{z}\Big(B_\eta^I * \mu\Big)(z)\mathrm{d}z \le \lim_{t\to\infty} \frac{1}{\ln(t)} \int_1^\infty \frac{1}{z}\Big(B_\eta^I * \mu\Big)(z)\mathrm{d}z = 0$$

Further, treating $B_{\widetilde{U}}^I$ similar to $S_{\widetilde{U}}$ in Lemma 2.54, we find for $x > 2z$ that

$$\begin{aligned}\int_{2z}^\infty B_{\widetilde{U}}^I(\tfrac{z}{x})\mu(\mathrm{d}x) = \int_{2z}^\infty \int_{-1}^{\frac{z}{x}-1} \nu_{\widetilde{U}}(\mathrm{d}y)\mu(\mathrm{d}x) = \int_{-1}^{-\frac{1}{2}} \int_{2z}^{\frac{z}{y+1}} \mu(\mathrm{d}x)\nu_{\widetilde{U}}(\mathrm{d}y) \\ \le \nu_{\widetilde{U}}\Big([-1,-\tfrac{1}{2}]\Big)\mu\Big([2z,\infty]\Big),\end{aligned}$$

which converges to zero as $z \to \infty$, and for $x \le 2z$ that

$$\int_1^t \frac{1}{z} \int_{\frac{z}{2}}^{2z} B_{\widetilde{U}}^I(\tfrac{z}{x})\mu(\mathrm{d}x)\mathrm{d}z = \int_{\frac{1}{2}}^t \int_{\max\{\frac{x}{2},1\}}^{\min\{t,2x\}} \frac{1}{z} B_{\widetilde{U}}^I(\tfrac{z}{x})\mathrm{d}z\mu(\mathrm{d}x).$$

Here, note that $B_{\widetilde{U}}^{FV}$ can be estimated by a constant if the argument is bounded away from the singularity at $z = x$ and that a suitable substitution implies

$$\int_{x(1-\varepsilon)}^{x(1+\varepsilon)} \frac{1}{z} B_{\widetilde{U}}^I(\tfrac{z}{x})\mathrm{d}z \le \Big(\frac{1}{1-\varepsilon}\int_{-\varepsilon}^0 \nu_{\widetilde{U}}\Big([-1,s)\Big)\mathrm{d}s + \int_0^\varepsilon \nu_{\widetilde{U}}\Big((s,\infty)\Big)\mathrm{d}s\Big)\frac{1}{x}.$$

As $\widetilde{U}$ is of finite variation by assumption, it follows that the above term is finite and thus

$$\lim_{t\to\infty} \frac{1}{\ln(t)} \int_1^t \frac{1}{z} \int_0^\infty B^I_{\widetilde{U}}(\tfrac{z}{x})\mu(\mathrm{d}x)\mathrm{d}z = 0,$$

yielding (2.60) as claimed. □

Proof of Corollary 2.48. With the jump parts of the processes η and $\widetilde{U}$ being of finite variation, we can follow the proof of Corollary 2.47 and rewrite their contribution to the distribution G in terms of B^{FV}_η and $B^{FV}_{\widetilde{U}}$. However, as $\sigma^2_\eta + \sigma^2_{\widetilde{U}} > 0$, an argument similar to Lemmas 2.52 to 2.54 is needed to find the equivalent to (2.41) for the case considered. Note in particular that the desired asymptotics for B^{FV}_η and $B^{FV}_{\widetilde{U}}$ follow directly from (2.60) in the proof of Corollary 2.47. Since μ has density f_μ on $\mathbb{R}\setminus\{0\}$ by Corollary 2.45, we also find an equivalent to Equation (2.42) which is given for a suitable constant $K \in \mathbb{R}$ by

$$\begin{aligned}\Big(\frac{1}{2}\sigma^2_\eta + \frac{1}{2}z^2\sigma^2_{\widetilde{U}}\Big)f_\mu(z) = -K &+ \int_{0+}^z \Big(\gamma^0_\eta + x\gamma^0_{\widetilde{U}}\Big)f_\mu(x)\mathrm{d}x + \mathbb{1}_{\{z<0\}}\gamma^0_\eta\mu(\{0\})\\ &+ \int_{0+}^z (B^{FV}_\eta * f_\mu)(x)\mathrm{d}x + \int_{0+}^z B^{FV}_\eta(x)\mathrm{d}x\mu(\{0\}) \qquad (2.61)\\ &+ \int_{0+}^z \Big(\mathbb{1}_{\{t>0\}}\int_0^\infty B^{FV}_{\widetilde{U}}(\tfrac{t}{x})f_\mu(x)\mathrm{d}x - \mathbb{1}_{\{t<0\}}\int_{-\infty}^0 B^{FV}_{\widetilde{U}}(\tfrac{t}{x})f_\mu(x)\mathrm{d}x\Big)\mathrm{d}t.\end{aligned}$$

Here, the terms involving γ^0_η, $\gamma^0_{\widetilde{U}}$, B^{FV}_η and $B^{FV}_{\widetilde{U}}$ are locally integrable with respect to the Lebesgue measure as a result of calculations similar to Lemma 2.41. This implies that the respective integrals are differentiable λ-a.e. (see e.g. [21, Thm. 6.3.6]) such that the right-hand side of (2.61) and thus $(\frac{1}{2}\sigma^2_\eta + \frac{1}{2}z^2\sigma^2_{\widetilde{U}})f_\mu(z)$ is differentiable λ-a.e., implying that this must hold for f_μ as well. Equation (2.47) now follows by differentiation.

Further, observe that $f_\mu \in C^0(\mathbb{R}\setminus\{0\})$ by Corollary 2.45. Hence, differentiability of f_μ follows by showing that the terms on the right-hand side of (2.61) are in $C^1(\mathbb{R}\setminus\{0\})$, or equivalently, by the fundamental theorem of calculus, that the functions that are integrated over $(0,z]$ are continuous on $\mathbb{R}\setminus\{0\}$. As this is trivially satisfied for the mapping $x \mapsto (\gamma^0_\eta + x\gamma^0_{\widetilde{U}})f_\mu(x)$ and the assumptions of both (i) and (ii) imply that $\mu(\{0\}) = 0$ by Corollary 2.26, it remains to consider the terms involving B^{FV}_η and $B^{FV}_{\widetilde{U}}$. Similar to the treatment of S_η in Corollary 2.45, let $x_0 > 0$, choose $0 < \varepsilon < x_0/2$ and define B^ε_η by replacing ν_η by $\nu_\eta|_{\mathbb{R}\setminus[-\varepsilon,\varepsilon]}$ in (2.44). Then B^ε_η is bounded and continuous at all but countably many points such that $x \mapsto (B^\varepsilon_\eta * f_\mu)(x) = \int_{\mathbb{R}} B^\varepsilon_\eta(x-t)f_\mu(t)\mathrm{d}t$ is continuous at x_0 by dominated convergence. Next, observe that the remainder $B^{FV}_\eta - B^\varepsilon_\eta$ is only supported on a subset of $[-\varepsilon,\varepsilon]$ and integrable due to the finite variation condition. Therefore, the mapping

$$x \mapsto \Big(\Big(B^{FV}_\eta - B^\varepsilon_\eta\Big) * f_\mu\Big)(x) = \int_{-\varepsilon}^{\varepsilon} f_\mu(x-t)\Big(B^{FV}_\eta - B^\varepsilon_\eta\Big)(t)\mathrm{d}t$$

is continuous by another application of Lebesgue's dominated convergence theorem as $f_\mu(x-t)$ is uniformly bounded in $x \in [x_0-\varepsilon, x_0+\varepsilon]$ and $t \in [-\varepsilon,\varepsilon]$. Applying a similar argument for $x_0 < 0$, it follows that $x \mapsto (B^{FV}_\eta * f_\mu)(x)$ is continuous on $\mathbb{R}\setminus\{0\}$ as desired. The terms involving $B^{FV}_{\widetilde{U}}$ can be treated similarly to the ones involving $S_{\widetilde{U}}$ in Corollary 2.45. First, decompose

$$\int_0^\infty B^{FV}_{\widetilde{U}}(\tfrac{t}{x})f_\mu(x)\mathrm{d}x = \int_{(0,\frac{t}{2})\cup(\frac{3}{2}t,\infty)} B^{FV}_{\widetilde{U}}(\tfrac{t}{x})f_\mu(x)\mathrm{d}x + \int_{\frac{t}{2}}^{\frac{3}{2}t} B^{FV}_{\widetilde{U}}(\tfrac{t}{x})f_\mu(x)\mathrm{d}x \tag{2.62}$$

and fix $t_0 > 0$. Observe that $B^{FV}_{\widetilde{U}}$ is bounded on $\mathbb{R}\setminus[2/3,2]$ and that the mapping $t \mapsto \mathbb{1}_{(0,\frac{t}{2})\cup(\frac{3}{2}t,\infty)}(x)B^{FV}_{\widetilde{U}}(\frac{t}{x})$ is continuous at t_0 for all but countably many $x > 0$. Therefore, continuity of the mapping $t \mapsto \int_{\mathbb{R}} \mathbb{1}_{(0,\frac{t}{2})\cup(\frac{3}{2}t,\infty)}(x)B^{FV}_{\widetilde{U}}(\frac{t}{x})f_\mu(x)\mathrm{d}x$ in $t_0 > 0$ follows by dominated convergence. Further, the second term on the right-hand side of (2.62) can be rewritten using a suitable substitution, yielding

$$\begin{aligned}\int_{\frac{t}{2}}^{\frac{3}{2}t} B^{FV}_{\widetilde{U}}(\tfrac{t}{x})f_\mu(x)\mathrm{d}x &= \int_{\frac{t}{2}}^{t} \nu_{\widetilde{U}}((\tfrac{t}{x}-1,\infty))f_\mu(x)\mathrm{d}x - \int_t^{\frac{3}{2}t} \nu_{\widetilde{U}}((-\infty,\tfrac{t}{x}-1))f_\mu(x)\mathrm{d}x\\ &= \int_0^1 \nu_{\widetilde{U}}((w,\infty))f_\mu(\tfrac{t}{w+1})\frac{t}{(w+1)^2}\mathrm{d}w\\ &\quad - \int_{-\frac{1}{3}}^0 \nu_{\widetilde{U}}((-\infty,w))f_\mu(\tfrac{t}{w+1})\frac{t}{(w+1)^2}\mathrm{d}w,\end{aligned}$$

where choosing $0 < \varepsilon < t_0/2$ implies that $f_\mu(t/(w+1))$ can be uniformly bounded in $t \in [t_0-\varepsilon, t_0+\varepsilon]$ and $w \in [-1/3, 1]$. Since

$$\int_0^1 \nu_{\widetilde{U}}((w,\infty))\frac{1}{(w+1)^2}\mathrm{d}w + \int_{-\frac{1}{3}}^0 \nu_{\widetilde{U}}((-\infty,w))\frac{1}{(w+1)^2}\mathrm{d}w < \infty,$$

the right-hand side of (2.62) is continuous at $t_0 > 0$ by dominated convergence. Since a similar argument can be applied for $t_0 < 0$, the term is continuous on $\mathbb{R}\setminus\{0\}$. Therefore, the right-hand side of (2.61) and hence the left-hand side of (2.61) are in $C^1(\mathbb{R}\setminus\{0\})$, which shows that $f_\mu \in C^1(\mathbb{R}\setminus\{0\})$. In particular, Equation (2.47) holds for all $z \in \mathbb{R}\setminus\{0\}$ if the assumptions of part (i) or (ii) of the corollary are satisfied. □

2.4.4. Applications and Examples

In this section, we consider various applications of the equations in Sections 2.4.1 and 2.4.2, respectively, deriving explicit information on the law of the killed exponential functional in special cases. The first example is concerned with the special case $\xi \equiv 0$, which is the Lévy process η subordinated by a gamma process with parameters 1 and $q > 0$, evaluated at time 1. The law of $V_{q,0,\eta}$ is q times the potential measure of η, cf. [58, Def. 30.9].

Example 2.56. Let $q > 0$ and $\xi \equiv 0$, i.e. $V_{q,0,\eta} = \eta_\tau$. Since $\sigma_\xi = \gamma_\xi = 0$ and ν_ξ is the zero measure, the limit term in (2.32) vanishes, yielding

$$\psi_\eta(u)\varphi_{V_{q,\xi,\eta}}(u) = q(\varphi_{V_{q,\xi,\eta}}(u) - 1)$$

such that we recover the known formula for the characteristic function of the potential measure from [58, Prop. 37.4]. One can also use the results in Section 2.4.2 to give a distributional equation for $\mu = \mathcal{L}(\eta_\tau)$, and hence for the potential measure, by observing that the characteristics of $\widetilde{U}$ are given by $(0, q\delta_{-1}, -q)$ whenever $\xi \equiv 0$. For example, if η

is of finite variation, one obtains from (2.46) that

$$\gamma_\eta^0 \mu(\mathrm{d}z) + \Big(B_\eta^{FV} * \mu\Big)(z)\mathrm{d}z + q\Big(\mathbb{1}_{\{z<0\}}\mu((-\infty, z]) - \mathbb{1}_{\{z>0\}}\mu([z,\infty))\Big)\mathrm{d}z = 0,$$

and if $\sigma_\eta^2 > 0$, but the jump part of η is still of finite variation, it follows from Corollaries 2.45 and 2.48 that μ has a density $f_\mu \in C^0(\mathbb{R}) \cap C^1(\mathbb{R} \setminus \{0\})$ that satisfies

$$\frac{1}{2}\sigma_\eta^2 f_\mu'(z) - \gamma_\eta^0 f_\mu(z) = \Big(B_\eta^{FV} * f_\mu\Big)(z) + q\Big(\mathbb{1}_{\{z<0\}}\int_{-\infty}^z f_\mu(x)\mathrm{d}x - \mathbb{1}_{\{z>0\}}\int_z^\infty f_\mu(x)\mathrm{d}x\Big).$$

In the special case of η being a standard Brownian motion, we have $\sigma_\eta^2 = 1$, $\gamma_\eta^0 = 0$ and B_η^{FV}=0 and one readily checks that the solution of the differential equation is given by

$$f_\mu(z) = \sqrt{\frac{q}{2}}\mathrm{e}^{-\sqrt{2q}|z|} = q\Big(\frac{1}{\sqrt{2q}}\mathrm{e}^{-\sqrt{2q}|z|}\Big),$$

which is q times the potential density given in [58, Ex. 30.11].

The following example collects some cases in which the solution of Equation (2.33) can be given explicitly.

Example 2.57. (i) Assume that $q > 0$ and that $(\xi_t)_{t\geq 0}$ is deterministic and not the zero process, i.e. $\xi_t = \gamma_\xi t$ with $\gamma_\xi \neq 0$. We need to assure $-2\gamma_\xi < q$ and $\mathbb{E}\eta_1^2 < \infty$ in order to have (2.34). Under this assumption, setting $\varphi := \varphi_{V_{q,\xi,\eta}}$, Equation (2.33) reduces to

$$\gamma_\xi u\varphi'(u) + (q - \psi_\eta(u))\varphi(u) = q.$$

For any $u > c > 0$ the solution to this inhomogeneous first-order ODE is given by

$$\varphi(u) = \exp\Big(\int_c^u \frac{\psi_\eta(s) - q}{\gamma_\xi s}\mathrm{d}s\Big)\Big[\varphi(c) + \int_c^u \frac{q}{\gamma_\xi t}\exp\Big(-\int_c^t \frac{\psi_\eta(s) - q}{\gamma_\xi s}\mathrm{d}s\Big)\mathrm{d}t\Big]. \tag{2.63}$$

Now assume that $\gamma_\xi > 0$ and that $\psi_\eta(s) \sim \alpha s^\beta$ near zero for some $\alpha \in \mathbb{C}\setminus\{0\}$ and $\beta > 0$. Then $\int_0^u \frac{\psi_\eta(s)}{\gamma_\xi s}\mathrm{d}s$ exists and letting $c \downarrow 0$ in (2.63) leads to

$$\varphi(u) = \exp\Big(\int_0^u \frac{\psi_\eta(s)}{\gamma_\xi s}\mathrm{d}s\Big)u^{-q/\gamma_\xi}\int_0^u \frac{q}{\gamma_\xi}t^{q/\gamma_\xi - 1}\exp\Big(-\int_0^t \frac{\psi_\eta(s)}{\gamma_\xi s}\mathrm{d}s\Big)\mathrm{d}t \tag{2.64}$$

for $u > 0$. In the trivial case of $\psi_\eta(u) = iu$ where

$$V_{q,\xi,\eta} = \int_0^\tau \mathrm{e}^{-\gamma_\xi t}\mathrm{d}t = \frac{1}{\gamma_\xi}(1 - \mathrm{e}^{-\gamma_\xi \tau}), \tag{2.65}$$

Equation (2.64) simplifies to

$$\varphi(u) = \exp\Big(\frac{i}{\gamma_\xi}u\Big)u^{-q/\gamma_\xi}\int_0^u \frac{q}{\gamma_\xi}t^{q/\gamma_\xi - 1}\exp\Big(-\frac{it}{\gamma_\xi}\Big)\mathrm{d}t$$

for $u > 0$, which in the special case $q = \gamma_\xi$ can be further simplified to

$$\varphi(u) = \frac{qi}{u}\Big(1 - \exp\Big(\frac{u}{q}i\Big)\Big), \quad u > 0, \tag{2.66}$$

as characteristic function of (2.65) with $\gamma_\xi = q$. Observe from the explicit form of φ that the characteristic function has zeroes such that the law of the killed exponential functional cannot be infinitely divisible in this case. If we assume η to be a Brownian motion without drift instead, i.e. $\psi_\eta(u) = -\frac{\sigma_\eta^2}{2}u^2$, then (2.34) holds, and whenever $\gamma_\xi > 0$, Equation (2.64) reduces to

$$\varphi(u) = \exp\Big(- \frac{\sigma_\eta^2}{\gamma_\xi}u^2\Big)u^{-q/\gamma_\xi}\int_0^u \frac{q}{\gamma_\xi}t^{q/\gamma_\xi - 1}\exp\Big(\frac{\sigma_\eta^2}{\gamma_\xi}t^2\Big)\mathrm{d}t.$$

This can be further simplified for various values of q/γ_ξ, e.g. if $q = 2\gamma_\xi$, we have

$$\varphi(u) = \frac{q}{2\sigma_\eta^2}u^{-2}\Big(1 - \exp\Big(- \frac{2\sigma_\eta^2 u^2}{q}\Big)\Big), \quad u > 0,$$

and if $q = 4\gamma_\xi$ we have

$$\varphi(u) = u^{-4}\frac{q^2}{8\sigma_\eta^4}\Big[\Big(\frac{4\sigma_\eta^2 u^2}{q} - 1\Big) + \exp\Big(- \frac{4u^2\sigma_\eta^2}{q}\Big)\Big], \quad u > 0,$$

as characteristic function of the killed exponential functional. Finally, if we assume η to be a compound Poisson process with intensity k and exponentially distributed jumps with parameter $a > 0$ such that $\psi_\eta(u) = k\frac{iu}{a-iu}$, then we derive from (2.64) for $u > 0$ that

$$\begin{aligned}\varphi(u) &= (u+ia)^{-k/\gamma_\xi}u^{-q/\gamma_\xi}\int_0^u \frac{q}{\gamma_\xi}t^{q/\gamma_\xi - 1}(t+ia)^{k/\gamma_\xi}\mathrm{d}t \\ &= \Big(\frac{a-iu}{a}\Big)^{-k/\gamma_\xi} {}_2F_1\Big(\frac{q}{\gamma_\xi}, -\frac{k}{\gamma_\xi}; 1 + \frac{q}{\gamma_\xi}; \frac{iu}{a}\Big),\end{aligned}$$

where ${}_2F_1(\cdot,\cdot;\cdot;\cdot)$ denotes the hypergeometric function (see e.g. Formulas 15.3.1 and 15.1.1 in [1]) and z^{-k/γ_ξ} for $z \in \mathbb{C}$ is interpreted as $\exp(-\frac{k}{\gamma_\xi}\log(z))$ with log denoting the principal branch of the complex logarithm. Setting formally $q = 0$ in the above expression, we obtain $\varphi(u) = (\frac{a-iu}{a})^{-k/\gamma_\xi}$, which is the characteristic function of the Gamma$(k/\gamma_\xi, a)$-distribution. Indeed, $V_{0,\xi,\eta}$ is Gamma$(k/\gamma_\xi, a)$-distributed, cf. [28, Thm. 2.1(f)]. Note that we can also obtain this fact from setting $q = 0$ and considering $c \downarrow 0$ in (2.63).
(ii) Let $(\xi_t)_{t\geq 0}$ be a Brownian motion with drift γ_ξ. In order to have (2.34), we need to assume that $2(\sigma_\xi^2 - \gamma_\xi) < q$ and $\mathbb{E}\eta_1^2 < \infty$. Under this assumption, Equation (2.33) leads to the following inhomogeneous second-order ODE (again setting $\varphi := \varphi_{V_{q,\xi,\eta}}$)

$$\frac{\sigma_\xi^2}{2}u^2\varphi''(u) + \Big(\frac{\sigma_\xi^2}{2} - \gamma_\xi\Big)u\varphi'(u) + (\psi_\eta(u) - q)\varphi(u) = -q,$$

which for $\gamma_\xi = \frac{\sigma_\xi^2}{2} < \frac{q}{2}$ reduces to

$$\frac{\sigma_\xi^2}{2} u^2 \varphi''(u) + (\psi_\eta(u) - q)\varphi(u) = -q.$$

In particular, assuming $(\eta_t)_{t\geq 0}$ to be a Brownian motion without drift, the resulting ODE

$$u^2 \varphi''(u) - \Big(\frac{\sigma_\eta^2}{\sigma_\xi^2} u^2 + \frac{2q}{\sigma_\xi^2}\Big)\varphi(u) = -\frac{2q}{\sigma_\xi^2}, \tag{2.67}$$

is a Bessel-type equation. Using the substitution $\varphi_{hom}(u) = \sqrt{u}\, g_{hom}\big(\frac{\sigma_\eta}{\sigma_\xi} u\big)$ for $u > 0$, it is easily checked that a function φ_{hom} satisfies the homogeneous equation corresponding to (2.67) on $(0, \infty)$ if and only if g_{hom} satisfies the homogeneous modified Bessel equation

$$v^2 g''(v) + v g'(v) - \Big(v^2 + \frac{2q}{\sigma_\xi^2} + \frac{1}{4}\Big) g(v) = 0$$

for $v \in (0, \infty)$. Denoting $\alpha := (2q/\sigma_\xi^2 + 1/4)^{1/2} > 0$, two linear independent solutions of this modified Bessel equation are given by the modified Bessel functions I_α and K_α of first and second kind, respectively (cf. [65, pp. 77-78]), hence the general solution of the homogeneous equation corresponding to (2.67) is given by

$$\varphi_{hom}(u) = c_1 \sqrt{u} I_\alpha\Big(\frac{\sigma_\eta}{\sigma_\xi} u\Big) + c_2 \sqrt{u} K_\alpha\Big(\frac{\sigma_\eta}{\sigma_\xi} u\Big), \quad u > 0$$

with complex constants c_1, c_2. Whenever $3\sigma_\xi^2 = q$, one easily verifies that a particular solution of (2.67) is given by

$$\varphi_{part}(u) = 2\frac{q}{\sigma_\eta^2} u^{-2} = 6\frac{\sigma_\xi^2}{\sigma_\eta^2} u^{-2},$$

and hence in this case

$$\varphi(u) = 6\frac{\sigma_\xi^2}{\sigma_\eta^2} u^{-2} + c_1 \sqrt{u} I_{5/2}\Big(\frac{\sigma_\eta}{\sigma_\xi} u\Big) + c_2 \sqrt{u} K_{5/2}\Big(\frac{\sigma_\eta}{\sigma_\xi} u\Big), \quad u > 0.$$

Observe that

$$K_{5/2}(z) = \sqrt{\frac{\pi}{2z}} \mathrm{e}^{-z} \Big(1 + \frac{3}{z} + \frac{3}{z^2}\Big)$$

and that $I_{5/2}(z) \sim \mathrm{e}^z / \sqrt{2\pi z}$ as $z \to \infty$ (cf. [65, p. 80 Eqs. (10),(12)]). Since φ is bounded as a characteristic function, we obtain $c_1 = 0$ when letting $u \to \infty$, and using $\lim_{u \downarrow 0} \varphi(u) = 1$ we obtain $c_2 = -\sqrt{8\sigma_\eta/(\pi \sigma_\xi)}$. Altogether we obtain

$$\varphi(u) = 6\frac{\sigma_\xi^2}{\sigma_\eta^2} u^{-2} - 2\mathrm{e}^{-\sigma_\eta u/\sigma_\xi} \Big(1 + \frac{3\sigma_\xi}{\sigma_\eta u} + \frac{3\sigma_\xi^2}{\sigma_\eta^2 u^2}\Big)$$

for $u > 0$ whenever $\gamma_\xi = \sigma_\xi^2/2 = q/6 > 0$ and η is a Brownian motion without drift and variance σ_η^2. Replacing u by $|u|$ in the right-hand side the above formula also holds for $u \in \mathbb{R} \setminus \{0\}$ by symmetry of φ.

The next example illustrates some of the results of Section 2.4.2.

Example 2.58. Let $q \geq 0$ and assume that both ξ and η are Brownian motions with or without drift, i.e. $\eta_t = \sigma_\eta B_t + \gamma_\eta^0 t$ and $\xi_t = \sigma_\xi W_t + \gamma_\xi^0 t$ where $(B_t)_{t\geq 0}$ and $(W_t)_{t\geq 0}$ denote two independent standard Brownian motions, $\gamma_\eta^0, \gamma_\xi^0 \in \mathbb{R}$, $\sigma_\eta^2 + \sigma_\xi^2 > 0$ and η is not the zero process. It follows from Corollaries 2.45 and 2.48 that the law of the killed exponential functional is absolutely continuous and that the density f_μ is continously differentiable on $\mathbb{R} \setminus \{0\}$. Observing that the characteristics of the process $\widetilde{U}$ are given by $(\sigma_\xi^2, q\delta_{-1}, \gamma_{\widetilde{U}})$ with drift $\gamma_{\widetilde{U}}^0 = -\gamma_\xi^0 + \sigma_\xi^2/2$, we find from (2.47) that f_μ satisfies

$$\frac{1}{2}\Big(\sigma_\eta^2 + z^2\sigma_{\widetilde{U}}^2\Big) f_\mu'(z) - \Big(\gamma_\eta^0 + z(\gamma_{\widetilde{U}}^0 - \sigma_{\widetilde{U}}^2)\Big) f_\mu(z) \\ + q\mathbb{1}_{\{z>0\}} \int_z^\infty f_\mu(x)\mathrm{d}x - q\mathbb{1}_{\{z<0\}} \int_{-\infty}^z f_\mu(x)\mathrm{d}x = 0 \tag{2.68}$$

for $z \neq 0$. Note that the integral terms vanish whenever $q = 0$ such that (2.68) reduces to an ordinary differential equation. In this case, we obtain

$$\frac{f_\mu'(z)}{f_\mu(z)} = \frac{\gamma_\eta^0 + (\gamma_{\widetilde{U}}^0 - \sigma_{\widetilde{U}}^2)z}{\frac{1}{2}\sigma_\eta^2 + \frac{1}{2}\sigma_{\widetilde{U}}^2 z^2}$$

for $z \neq 0$, from which the explicit solution can be derived by logarithmic integration. Assuming that $\sigma_\eta^2, \sigma_\xi^2 \neq 0$, it follows that

$$f_\mu(z) = C\Big(\sigma_\eta^2 + z^2\sigma_{\widetilde{U}}^2\Big)^{-1+\gamma_{\widetilde{U}}^0/\sigma_{\widetilde{U}}^2} \exp\left(\frac{2\gamma_\eta^0}{\sigma_\eta\sigma_{\widetilde{U}}} \arctan\left(\frac{\sigma_{\widetilde{U}}}{\sigma_\eta} z\right)\right)$$

where $C > 0$ is a norming constant. Note that even though the equation is solved for $z > 0$ and $z < 0$ separately, the continuity of f_μ implies that the same norming constant can be used on both sides. In particular, the result obtained for f_μ above coincides with the density of the exponential functional given in [28, Thm. 2.1(d)]. Let now $q \neq 0$. In this case, the integro-differential equation (2.68) yields an ordinary differential equation for the distribution function $F_\mu(z) = \int_{-\infty}^z f_\mu(x)\mathrm{d}x$ of the killed exponential functional, which is given by

$$\begin{aligned} &\frac{1}{2}\Big(\sigma_\eta^2 + z^2\sigma_{\widetilde{U}}^2\Big) F_\mu''(z) - \Big(\gamma_\eta^0 + z(\gamma_{\widetilde{U}}^0 - \sigma_{\widetilde{U}}^2)\Big) F_\mu'(z) - qF_\mu(z) = -q, \quad z > 0, \\ &\frac{1}{2}\Big(\sigma_\eta^2 + z^2\sigma_{\widetilde{U}}^2\Big) F_\mu''(z) - \Big(\gamma_\eta^0 + z(\gamma_{\widetilde{U}}^0 - \sigma_{\widetilde{U}}^2)\Big) F_\mu'(z) - qF_\mu(z) = 0, \quad z < 0. \end{aligned}$$

Exemplarily, we choose $q = 2$, $\sigma_\xi^2 = 4$, $\gamma_\eta^0 = 1$ and $\sigma_\eta^2 = \gamma_\xi^0 = 0$. In this case it is $V_{q,\xi,\eta} \geq 0$ a.s. due to η being a deterministic subordinator. Hence, (2.68) reduces to

$$2z^2 f_\mu'(z) + (2z - 1)f_\mu(z) + 2\int_z^\infty f_\mu \mathrm{d}x = 0, \quad z > 0$$

and we find that the tail function $T_\mu(z) = 1 - F_\mu(z)$ satisfies

$$2z^2 T''_\mu(z) + (2z-1)T'_\mu(z) - 2T_\mu(z) = 0, \quad z > 0. \tag{2.69}$$

The general solution of (2.69) is given by

$$T_\mu(z) = c_1 z \mathrm{e}^{-1/(2z)} + c_2(2z-1) = z(c_1\mathrm{e}^{-1/(2z)} + 2c_2) - c_2$$

and it is readily checked that the constants must satisfy $c_1 = 2$ and $c_2 = -1$ in order to obtain a tail function that satisfies $\lim_{z\downarrow 0} T_\mu(z) = 1$ and $\lim_{z\to\infty} T_\mu(z) = 0$. Deriving $T_\mu(z) = 1 - 2z(1-\exp(-\frac{1}{2z}))$, it follows that the density is given by

$$f_\mu(z) = -T'_\mu(z) = 2 - \Big(\frac{1}{z} + 2\Big)\exp\Big(-\frac{1}{2z}\Big)$$

for $z > 0$. Observe in particular that $V_{q,\xi,\eta} \overset{d}{=} Z_1/2Z_2$, where Z_1 is uniformly distributed on $[0,1]$ and $Z_2 \overset{d}{=} \mathrm{Exp}(1)$ is independent of Z_1. This coincides with the results from [66, Thm. 2] given in Example 2.1 at the beginning of the section.

Observe that, so far, both processes ξ and η were assumed to be continuous in the examples considered. We conclude this section by discussing two examples in which ξ, and hence U, is a pure-jump process.

Example 2.59. Let ξ be a Poisson process with intensity $c > 0$ and $\eta_t = \sigma_\eta B_t$, where $\sigma^2_\eta > 0$ and $(B_t)_{t\geq 0}$ is a standard Brownian motion. Using the connection between ξ and U established in Section 1.1.4, it is readily checked that $\sigma^2_{\widetilde{U}} = 0$, $\nu_{\widetilde{U}} = c\delta_{\mathrm{e}^{-1}-1} + q\delta_{-1}$, as well as $\gamma^0_{\widetilde{U}} = -\gamma^0_\xi = 0$, and it follows that

$$B^{FV}_{\widetilde{U}} = \begin{cases} -q, & \text{if } z \in (0, \frac{1}{\mathrm{e}}], \\ -(c+q), & \text{if } z \in (\frac{1}{e}, 1), \\ 0, & \text{if } z \geq 1 \text{ or } z = 0, \end{cases}$$

for the function $B^{FV}_{\widetilde{U}}$ as defined in Corollary 2.47. By Corollaries 2.45 and 2.48, μ has a density $f_\mu \in C^0(\mathbb{R}) \cap C^1(\mathbb{R}\setminus\{0\})$ that satisfies

$$\begin{aligned} \frac{\sigma^2_\eta}{2} f'_\mu(z) = &-\mathbb{1}_{\{z>0\}}\Big(q\int_z^\infty f_\mu(x)\mathrm{d}x + c\int_z^{\mathrm{e}z} f_\mu(x)\mathrm{d}x\Big) \\ &+ \mathbb{1}_{\{z<0\}}\Big(q\int_{-\infty}^z f_\mu(x)\mathrm{d}x + c\int_{\mathrm{e}z}^z f_\mu(x)\mathrm{d}x\Big) \end{aligned} \tag{2.70}$$

for $z \neq 0$. Observe that in particular $\mu(\{0\}) = 0$ as a consequence of $\sigma^2_\eta > 0$. Since the right-hand side of (2.70) is differentiable, so is the left-hand side, such that we obtain $f_\mu \in C^2(\mathbb{R}\setminus\{0\})$, as well as

$$\frac{\sigma^2_\eta}{2} f''_\mu(z) = qf_\mu(z) + c\Big(f_\mu(z) - f_\mu(\mathrm{e}z)\Big)$$

for $z \neq 0$ by differentiating (2.70).

Example 2.60. Assume now that ξ is a compound Poisson process with Lévy measure $\nu_\xi(\mathrm{d}x) = \mathrm{e}^{-x}\mathbb{1}_{(0,\infty)}\mathrm{d}x$, $\eta_t = \sigma_\eta B_t + \gamma_\eta t$, where $\sigma_\eta^2 > 0$ and $(B_t)_{t\geq 0}$ again denotes a standard Brownian motion, as well as $q > 0$. As in Example 2.59, it follows from Corollaries 2.45 and 2.48 that μ is absolutely continuous with density $f_\mu \in C^0(\mathbb{R}) \cap C^1(\mathbb{R} \setminus \{0\})$. Using the relation between ν_ξ and $\nu_{\widetilde{U}}$, we can give the function $B_{\widetilde{U}}^{FV}$ as

$$B_{\widetilde{U}}^{FV}(z) = \begin{cases} 0, & z > 1, \\ -(z+q), & z \in [0,1), \end{cases}$$

such that Equation (2.47) reads

$$\frac{1}{2}\sigma_\eta^2 f_\mu'(z) = \gamma_\eta f_\mu(z) - \mathbb{1}_{\{z>0\}} \int_z^\infty \left(\frac{z}{x} + q\right) f_\mu(x)\mathrm{d}x + \mathbb{1}_{\{z<0\}} \int_{-\infty}^z \left(\frac{z}{x} + q\right) f_\mu(x)\mathrm{d}x.$$

Since $f_\mu \in C^0(\mathbb{R}) \cap C^1(\mathbb{R} \setminus \{0\})$, the integral terms are differentiable in $z \neq 0$ and it follows that $f_\mu' \in C^1(\mathbb{R} \setminus \{0\})$. Thus, $f_\mu \in C^2(\mathbb{R} \setminus \{0\})$ and differentiating the equation leads to

$$\begin{aligned} \frac{1}{2}\sigma_\eta^2 f_\mu''(z) = \gamma_\eta f_\mu'(z) &- \mathbb{1}_{\{z>0\}}\left(\int_z^\infty \frac{1}{x} f_\mu(x)\mathrm{d}x - (1+q) f_\mu(z)\right) \\ &+ \mathbb{1}_{\{z<0\}}\left(\int_{-\infty}^z \frac{1}{x} f_\mu(x)\mathrm{d}x + (1+q) f_\mu(z)\right) \end{aligned}$$

for $z \neq 0$. This shows that $f_\mu'' \in C^1(\mathbb{R}\setminus\{0\})$ and hence $f_\mu \in C^3(\mathbb{R}\setminus\{0\})$. Differentiating the equation once more finally eliminates the integrals and leads to the third-order linear ODE

$$\frac{1}{2}\sigma_\eta^2 f_\mu'''(z) = \gamma_\eta f_\mu''(z) + (1+q) f_\mu'(z) + \frac{1}{z} f_\mu(z),$$

which is satisfied for all $z \neq 0$.

3. Short-time Behavior of Solutions to Lévy-driven SDEs

The aim of this chapter is a characterization of the a.s. short-time behavior of the stochastic process $X = (X_t)_{t\geq 0}$ which is the solution of an SDE of the form

$$\mathrm{d}X_t = \sigma(X_{t-})\mathrm{d}L_t,\ X_0 = x \in \mathbb{R}^n, \tag{3.1}$$

where $L = (L_t)_{t\geq 0}$ is an $\mathbb{R}^d$-valued Lévy process and the function $\sigma : \mathbb{R}^n \to \mathbb{R}^{n\times d}$ is twice continuously differentiable and maximal of linear growth. We describe the short-time behavior of X by comparing it to the behavior of suitable functions based on the analysis given in the preprint [53].

For real-valued Lévy processes, results by Shtatland [64] and Rogozin [54] characterize the almost sure convergence of the quotient L_t/t for $t \downarrow 0$ in terms of the total variation of the paths of the process, which was generalized to determining the behavior of the quotient for arbitrary positive powers of t in [15], [51] and [12] from the characteristic triplet. The exact scaling function f for law of the iterated logarithm-type (LIL-type) results of the form $\limsup_{t\downarrow 0} L_t/f(t) = c$ a.s. for a deterministic constant c was determined by Khinchine for Lévy processes that include a Gaussian component (see e.g. [58, Prop. 47.11]) and in e.g. [59] and [60] for more general types of Lévy processes. A multivariate counterpart to these LIL-type results was derived in the recent paper [26], showing that the short-time behavior of the driving process in (1.6) is already well-understood. For the solution X, the situation becomes more difficult. It was shown in [62] and [37] that X is, under suitable conditions, a so-called Lévy-type Feller process, i.e. the characteristic function of X_t can be expressed using a characteristic triplet similar to the driving Lévy process with the triplet $(A(x), \nu(x), \gamma(x))$ additionally depending on the initial condition $x \in \mathbb{R}^n$ and the function σ. The short and long-time behavior of such Feller processes can be characterized in terms of power-law functions using a generalization of Blumenthal-Getoor indices (see [63]), where the symbol of the process plays the role of the characteristic exponent. Using similar methods, an explicit short-time LIL in one dimension was derived in [35].

The definition of a Lévy-type Feller process suggests that one can think of X as "locally Lévy" and, since the short-time behavior of the process is determined by the path behavior in an arbitrarily small neighborhood of zero, the process X thus should directly mirror the short-time behavior of the driving Lévy process. We confirm this hypothesis in terms of power-law functions in Proposition 3.3 and Theorem 3.10 below by showing that the almost sure finiteness of $\lim_{t\downarrow 0} t^{-p}L_t$ implies the almost sure convergence of the quantity $t^{-p}(X_t - X_0)$ and that similar results hold for $\limsup_{t\downarrow 0} t^{-p}(X_t - X_0)$ and $\liminf_{t\downarrow 0} t^{-p}(X_t - X_0)$ with probability one whenever $\lim_{t\downarrow 0} t^{-p/2}L_t$ exists almost surely. Using knowledge on the form of the scaling function for the driving Lévy process, the limit theorems can be generalized to suitable functions $f : [0, \infty) \to \mathbb{R}$ to derive explicit LIL-type results for the solution of (3.1). As another application, we will also briefly study convergence in distribution and in probability, showing that results on the short-time behavior of the driving process translate here as well. Note that the results given below partially overlap with characterizations obtained from other approaches such as the generalization of Blumenthal-Getoor indices for Lévy-type Feller processes discussed e.g. in [63] while also covering new cases such as a.s. limits for $t \downarrow 0$. Whenever possible, we work with general semimartingales and include converse results to reobtain the limiting behavior of the driving process from the solution. As a first step, we give a lemma that characterizes the a.s. short-time behavior of a stochastic integral when the behavior of the integrand is known.

Lemma 3.1. *Let $X = (X_t)_{t\geq 0}$ be a real-valued semimartingale, $p > 0$ and $\varphi = (\varphi_t)_{t\geq 0}$ an adapted càglàd process such that $\lim_{t\downarrow 0} t^{-p}\varphi_t$ exists and is finite with probability one. Then*

$$\frac{1}{t^p}\int_{0+}^{t} \varphi_s \mathrm{d}X_s \to 0 \text{ a.s. for } t \downarrow 0. \tag{3.2}$$

Proof. Define the process ψ (ω-wise) by

$$\psi_t := \begin{cases} t^{-p}\varphi_t, \ t > 0, \\ \lim_{s\downarrow 0} s^{-p}\varphi_s, \ t = 0, \end{cases}$$

possibly setting $\psi_0(\omega) = 0$ on the null set where the limit does not exist. By definition, ψ is càglàd, and, as $\lim_{t\downarrow} t^{-p}\varphi_t$ exists a.s. in $\mathbb{R}$, $\mathcal{F}_0$ contains all null sets by assumption and the filtration is right-continuous, ψ_0 is $\mathcal{F}_0$-measurable. Therefore, ψ is also adapted. This implies that the semimartingale

$$Y_t := \int_{0+}^{t} \psi_s \mathrm{d}X_s$$

is indeed well-defined, allowing to rewrite the process considered in (3.2) using the associativity of the stochastic integral. This leads to

$$\int_{0+}^{t} \varphi_s \mathrm{d}X_s = \int_{0+}^{t} s^p\psi_s \mathrm{d}X_s = \int_{0+}^{t} s^p \mathrm{d}Y_s,$$

which implies

$$\frac{1}{t^p}\int_{0+}^{t} \varphi_s \mathrm{d}X_s = \frac{1}{t^p}\left(t^pY_t - \int_{0+}^{t} Y_s \mathrm{d}(s^p)\right) = Y_t - \frac{1}{t^p}\int_{0+}^{t} Y_s \mathrm{d}(s^p)$$

by partial integration. As Y is a semimartingale which has a.s. càdlàg paths additionally satisfying $Y_0 = 0$ by definition, we have $\lim_{t\downarrow 0} Y_t = 0$ with probability one. The term remaining on the right-hand side is a path-by-path Lebesgue-Stieltjes integral. Note that, as $p > 0$, the integrator is increasing, thus implying the monotonicity of the corresponding integral. This leads to

$$\inf_{0<s\leq t} Y_s \leq \frac{1}{t^p}\int_{0+}^{t} Y_s \mathrm{d}(s^p) \leq \sup_{0<s\leq t} Y_s.$$

Recalling $\lim_{t\downarrow 0} Y_t = 0$ a.s., we can conclude that the above terms vanish with probability one as $t \downarrow 0$, which yields the claim. □

Remark 3.2. (i) By defining the process $\psi_s = (\psi_{i,j})_{i,j}$ component-wise and considering Y_t as either $(Y_t)_{i,j} = \sum_{k=1}^d \int_{0+}^t (\psi_s)_{i,k} \mathrm{d}(X_s)_{k,j}$ or $(Y_t)_{i,j} = \sum_{k=1}^d \int_{0+}^t (\psi_s)_{k,j} \mathrm{d}(X_s)_{i,k}$, the lemma naturally extends to multivariate stochastic integrals with ψ and X being $\mathbb{R}^{n\times d}$-valued and $\mathbb{R}^{d\times m}$-valued semimartingales, respectively.
(ii) The function t^p in the denominator may be replaced by an arbitrary continuous function $f : [0,\infty) \to \mathbb{R}$ that is increasing and satisfies $f(0) = 0$ and $f(t) > 0$ for all $t > 0$.

Lemma 3.1 is the key tool to deriving a.s. short-time limiting results for the solution of a stochastic differential equation.

Proposition 3.3. *Let L be an $\mathbb{R}^d$-valued semimartingale satisfying $L_0 = 0$, $v \in \mathbb{R}^d$, $p > 0$, and $\sigma : \mathbb{R}^d \to \mathbb{R}^{n\times d}$ twice continuously differentiable and maximal of linear growth. Define $X = (X_t)_{t\geq 0}$ as the solution of* (3.1). *Then*

$$\lim_{t\downarrow 0} \frac{L_t}{t^p} = v \text{ a.s.} \Rightarrow \lim_{t\downarrow 0} \frac{X_t - x}{t^p} = \sigma(x)v \text{ a.s.}$$

Proof. Let $\lim_{t\downarrow 0} t^{-p} L_t = v$ with probability one. By definition, X satisfies the equation

$$X_t = x + \int_{0+}^t \sigma(X_{s-})\mathrm{d}L_s.$$

Applying partial integration to the individual components yields

$$\begin{aligned}\left(\frac{X_t - x}{t^p}\right)_i &= \frac{1}{t^p} \sum_{k=0}^d \int_{0+}^t \sigma_{i,k}(X_{s-})\mathrm{d}(L_k)_s \\ &= \frac{1}{t^p} \sum_{k=0}^d \Big(\sigma_{i,k}(X_t)(L_k)_t - \sigma_{i,k}(x)(L_k)_0 - \int_{0+}^t (L_k)_{s-}\mathrm{d}\sigma_{i,k}(X_s) \\ &\quad - \big[\sigma_{i,k}(X), L_k\big]_t\Big) \end{aligned} \tag{3.3}$$

As $t^{-p}L_t \to v$ a.s. by assumption and $X_t \to x = X_0$ a.s. by definition of X, the first term on the right-hand side of (3.3) converges a.s. to the desired limit as $t \downarrow 0$. Thus, the claim follows if we can show that the remaining terms vanish when the limit is considered. Since $L_0 = 0$ a.s., this is true for the second term and, as $\sigma(X)$ is again a semimartingale, Lemma 3.1 is applicable for the third term of (3.3), showing that it converges a.s. to zero. Since σ is twice continuously differentiable, applying Itō's formula for X in the quadratic covariation appearing in the last term yields

$$\begin{aligned}\big[\sigma_{i,k}(X), L_k\big]_t = \Big[&\sigma_{i,k}(x) + \sum_{j=1}^n \int_{0+}^{\cdot} \frac{\partial \sigma_{i,k}}{\partial x_j}(X_{s-})\mathrm{d}(X_j)_s \\ &+ \frac{1}{2}\sum_{j_1,j_2=1}^n \int_{0+}^{\cdot} \frac{\partial^2 \sigma_{i,k}}{\partial x_{j_1}\partial x_{j_2}}(X_{s-})\mathrm{d}\big[X_{j_1}, X_{j_2}\big]_s^c \\ &+ \sum_{0<s\leq \cdot} \Big(\sigma_{i,k}(X_s) - \sigma_{i,k}(X_{s-}) - \sum_{j=1}^n \frac{\partial \sigma_{i,k}}{\partial x_j}(X_{s-})\Delta(X_j)_s\Big), L_k\Big]_t. \end{aligned} \tag{3.4}$$

By linearity of the quadratic covariation, the right-hand side of (3.4) is split into seperate terms that can be treated individually. Further, using the associativity of the stochastic integral and the fact that continuous finite variation terms do not contribute to the quadratic covariation, it follows that many of the terms vanish, leaving

$$\Big[\sigma_{i,k}(X), L\Big]_t = \sum_{j=1}^{n} \int_{0+}^{t} \frac{\partial \sigma_{i,k}}{\partial x_j}(X_{s-})\mathrm{d}[X_j, L_k]_s \tag{3.5}$$
$$+ \Big[\sum_{0<s\leq \cdot} \Big(\sigma_{i,k}(X_s) - \sigma_{i,k}(X_{s-}) - \sum_{j=1}^{n} \frac{\partial \sigma_{i,k}}{\partial x_j}(X_{s-})\Delta(X_j)_s \Big), L_k \Big]_t .$$

For the first term observe that the quadratic variation process is of finite variation, such that the integral is given by a path-by-path Lebesgue-Stieltjes integral. By the definition of X, it follows that

$$[X_j, L_k]_t = \sum_{l=1}^{d} \int_{0+}^{t} \sigma_{j,l}(X_{s-})\mathrm{d}[L_l, L_k]_s.$$

Denoting integration with respect to the total variation measure of a process Y as $\mathrm{d}TV_Y$, the individual integrals can be estimated by

$$\Big| \int_{0+}^{t} \sigma_{j,l}(X_{s-})\mathrm{d}[L_l, L_k]_s \Big| \leq \int_{0+}^{t} \big|\sigma_{j,l}(X_{s-})\big| \mathrm{d}TV_{[L_l,L_k]}(s)$$
$$\leq \Big(\int_{0+}^{t} \big|\sigma_{j,l}(X_{s-})\big| \mathrm{d}[L_l, L_l]_s \Big)^{\frac{1}{2}} \Big(\int_{0+}^{t} \big|\sigma_{j,l}(X_{s-})\big| \mathrm{d}[L_k, L_k]_s \Big)^{\frac{1}{2}}$$
$$\leq \sup_{0<s\leq t} \big|\sigma_{j,l}(X_{s-})\big| \sqrt{[L_l, L_l]_t}\sqrt{[L_k, L_k]_t},$$

using the Kunita-Watanabe inequality (see e.g. [50, Th. II.25]) and the fact that the resulting integrals have increasing integrators. Further, the above estimates also show that the total variation of $\int_{0+}^{t} \sigma_{j,l}(X_{s-})\mathrm{d}[L_l, L_k]$ satisfies this estimate. For the quadratic variation terms note that since $(L_0)_{k,l} = 0$ a.s. and

$$[L_k, L_k]_t = (L_k)_t^2 - 2\int_{0+}^{t} (L_k)_{s-}\mathrm{d}(L_k)_s,$$

it follows from the assumption and the one-dimensional version of Lemma 3.1 that

$$\lim_{t\downarrow 0} \frac{1}{t^p}[L_k, L_k]_t = \lim_{t\downarrow 0} \frac{1}{t^p}\sqrt{[L_l, L_l]_t}\sqrt{[L_k, L_k]_t} = 0$$

with probability one. Thus,

$$0 \leq \limsup_{t\downarrow 0} \Big| \frac{1}{t^p} \int_{0+}^{t} \sigma_{j,l}(X_{s-})\mathrm{d}[L_l, L_k]_s \Big| = 0 \text{ a.s.},$$

and a similar estimate holds true for the total variation of $\int_{0+}^{t} \sigma_{j,l}(X_{s-})\mathrm{d}[L_l, L_k]_s$. Denoting the total variation process of Y at t by $TV(Y)_t$, we obtain the bound

$$\frac{1}{t^p}\Big| \sum_{j=1}^{n} \int_{0+}^{t} \frac{\partial \sigma_{i,k}}{\partial x_j}(X_{s-})\mathrm{d}[X_j, L_k]_s \Big| \leq \sum_{j=1}^{n} \sup_{0<s\leq t} \Big| \frac{\partial \sigma_{i,k}}{\partial x_j}(X_{s-}) \Big| \frac{1}{t^p} TV\Big([X_j, L_k]\Big)_t$$

for the first term on the right-hand side of (3.5), showing that it vanishes a.s. when the limit $t \downarrow 0$ is considered. Lastly, denote

$$\Big[\sum_{0<s\le t}\Big(\sigma_{i,k}(X_s)-\sigma_{i,k}(X_{s-})-\sum_{j=1}^{n}\frac{\partial\sigma_{i,k}}{\partial x_j}(X_{s-})\Delta(X_j)_s\Big),L_k\Big]_t =: [J,L_k]_t$$

for the jump term remaining in (3.5). Using the Kunita-Watanabe inequality and recalling that $[L_k, L_k] = o(t^p)$ by the previous estimate, it remains to consider the quadratic variation of the process J. Evaluating

$$[J,J]_t=\sum_{0<s\le t}(\Delta J_s)^2=\sum_{0<s\le t}\Big(\sigma_{i,k}(X_s)-\sigma_{i,k}(X_{s-})-\sum_{j=1}^{n}\frac{\partial\sigma_{i,k}}{\partial x_j}(X_{s-})\Delta(X_j)_s\Big)^2,$$

and noting that

$$\sup_{0<s\le t}\left|\frac{\partial^2\sigma_{i,k}}{\partial x_{j_1}\partial x_{j_2}}(X_{s-})\right|<\infty$$

for all $j_1, j_2 = 1,\dots,n$ and a fixed $t \ge 0$ as $\sigma \in C^2$ and X is a càdlàg process, we can conclude that

$$[J,J]_t\le C\sum_{0<s\le t}\|\Delta X_s\|^2=C\sum_{0<s\le t}\|\sigma(X_{s-})\Delta L_s\|^2\le C'\sum_{0<s\le t}\|\Delta L_s\|^2\le C'\sum_{k=1}^{d}[L_k,L_k]_t$$

for some finite (random) constants C, C'. This shows that both terms in (3.5) are indeed $o(t^p)$ and do not contribute when the limit $t \downarrow 0$ in (3.3) is considered. Hence, the limit is equal to $\sigma(X_0)v$ a.s., which is the claim. □

Remark 3.4. (i) Observe that $\lim_{t\downarrow 0} t^{-p}L_t = v$ implies $[L, L] = o(t^p)$ here. Whenever L is a Lévy process, the same assumption yields $[L, L]_t = o(t^{2p})$ (see Lemma 3.9 below).
(ii) Similar to Lemma 3.1, one can replace t^p by any other continuous, increasing function $f : [0,\infty) \to \mathbb{R}$ that satisfies $f(0) = 0$ and $f(t) > 0$ for all $t > 0$.
(iii) Since the short-time behavior of the process is determined by its behavior in an arbitrarily small neighborhood of zero, Proposition 3.3 and many of the results below are also applicable when the solution of the SDE is only well-defined on some interval $[0, \varepsilon]$ with $\varepsilon > 0$. Thus, the linear growth condition can be omitted if one replaces t by $\min\{t, \varepsilon\}$ in the calculations.

Whenever one can assure that $\sigma(X_{s-})$ is invertible, the implication in Proposition 3.3 is indeed an equivalence. This yields the following counterpart to [63, Thm. 4.4] for almost sure limits at zero.

Proposition 3.5. *Let L be an $\mathbb{R}^d$-valued semimartingale, $v \in \mathbb{R}^d$ and $p > 0$ and X the solution to* (3.1)*. Let further $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ be twice continuously differentiable, maximal of linear growth and such that $\sigma(X_{t-})$ has a.s. full rank for $t \ge 0$, where we set $X_{0-} = x$. Then*

$$\lim_{t\downarrow 0}\frac{L_t}{t^p}=v \text{ a.s.} \Leftrightarrow \lim_{t\downarrow 0}\frac{X_t-x}{t^p}=\sigma(x)v \text{ a.s.} \tag{3.6}$$

Proof. As Proposition 3.3 yields the first implication, let $\lim_{t\downarrow 0} t^{-p}(X_t - x) = \sigma(X_0)v$ with probability one. Using that $\sigma(X_{s-})$ has a.s. full rank, we can recover L from X via

$$L_t = \int_{0+}^{t} \Big(\sigma(X_{s-})\Big)^{-1} \mathrm{d}X_s.$$

Since $\lim_{t\downarrow 0} X_t = x$ a.s., it is $\|\sigma(X_t) - \sigma(x)\| < 1$ a.s. for sufficiently small $t > 0$. This implies

$$\begin{aligned}\sigma(x)\sigma(X_t)^{-1} &= \Big(\mathrm{Id} - (\mathrm{Id} - \sigma(X_t)\sigma(x)^{-1})\Big)^{-1} = \sum_{k=0}^{\infty}\Big(\mathrm{Id} - \sigma(X_t)\sigma(x)^{-1}\Big)^k \\ &= \mathrm{Id} + \Big(\mathrm{Id} - \sigma(X_t)\sigma(x)^{-1}\Big) + R_t,\end{aligned}$$

where the Neumann series converges a.s. in norm. Observe that we have by Taylor's formula

$$\frac{1}{t^p}(\sigma(X_t) - \sigma(x))_{i,j} = \frac{1}{t^p}\sum_{k=1}^{n} \frac{\partial \sigma_{i,j}}{\partial x_k}(x)(X_t - x)_{i,j} + \frac{1}{t^p} r_{i,j}(t)$$

where the remainder term satisfies $r_{i,j}(t) = O((X_t - x)^2) = o(t^p)$. Thus,

$$\lim_{t\downarrow 0} \frac{1}{t^p}\big(\mathrm{Id} - \sigma(X_t)\sigma(x)^{-1}\Big) = \lim_{t\downarrow 0} \frac{1}{t^p}\big(\sigma(x) - \sigma(X_t)\Big)\sigma(x)^{-1}$$

exists a.s. from which it follows that also $R_t = o(t^p)$ with probability one. Hence,

$$\begin{aligned}\sigma(x)\frac{1}{t^p}L_t &= \frac{1}{t^p}\int_{0+}^{t}\Big(\sigma(x)(\sigma(X_{s-}))^{-1} - \mathrm{Id}\Big)\mathrm{d}X_s + \frac{1}{t^p}\int_{0+}^{t} \mathrm{Id}\ \mathrm{d}X_s \\ &= \frac{1}{t^p}\int_{0+}^{t}\Big(\sigma(x)(\sigma(X_{s-}))^{-1} - \mathrm{Id}\Big)\mathrm{d}X_s + t^{-p}\Big(X_t - x\Big).\end{aligned}$$

and we find that the limit for $t \downarrow 0$ exists a.s. and is equal to $\sigma(X_0)v$ by Lemma 3.1 and the assumption. This yields the claim since $\sigma(X_0)$ has full rank with probability one. □

Proposition 3.5 is in particular applicable for the stochastic exponential by vectorization of the matrix-valued stochastic processes. Here, the condition $\det(\mathrm{Id} + \Delta L_s) \neq 0$ for all $s \geq 0$ ensures that the inverse $\mathcal{E}(L)^{-1}$ is well defined (see e.g. [32]).

Corollary 3.6. *Let L be an $\mathbb{R}^{d\times d}$-valued semimartingale satisfying* $\det(\mathrm{Id} + \Delta L_s) \neq 0$ *for all $s \geq 0$, $v \in \mathbb{R}^{d\times d}$ and $p > 0$. Then*

$$\lim_{t\downarrow 0} \frac{L_t}{t^p} = v \text{ a.s.} \Leftrightarrow \lim_{t\downarrow 0} \frac{\mathcal{E}(L) - \mathrm{Id}}{t^p} = v \text{ a.s.} \tag{3.7}$$

Remark 3.7. In the case that L is a Lévy process, the a.s. limit v appearing for $p = 1$ in Corollary 3.6 is the drift of L. A result by Shtatland and Rogozin (see [64] and [54]) directly links the existence of this limit to the process having sample paths of bounded variation. Since the stochastic exponential $\mathcal{E}(L)$ has paths of bounded variation iff this holds true for the paths of L, a similar connection can be made for $\mathcal{E}(L)$. Denote by BV the set of stochastic processes having sample paths of bounded variation, then Corollary 3.6

implies

$$\lim_{t\downarrow 0}\frac{\mathcal{E}(L)-\mathrm{Id}}{t}\text{ exists a.s.}\Leftrightarrow \lim_{t\downarrow 0}\frac{L_t}{t}\text{ exists a.s.}\Leftrightarrow L\in BV\Leftrightarrow \mathcal{E}(L)\in BV.$$

Considering Proposition 3.3 in the context of Lévy processes yields the following result.

Proposition 3.8. *Let L be an $\mathbb{R}^{d\times m}$-valued Lévy process satisfying $\lim_{t\downarrow 0} t^{-p}L_t = v$ a.s. for some $v\in\mathbb{R}^{d\times m}$, $p>0$ and let $X=(X_t)_{t\geq 0}$ be an $\mathbb{R}^{n\times d}$-valued semimartingale. Then*

$$\frac{1}{t^p}\int_{0+}^t X_{s-}\mathrm{d}L_s \to X_0 v \text{ a.s.},\ t\downarrow 0.$$

If additionally $\lim_{t\downarrow 0} t^{-p}X_t = w$ for some matrix $w\in\mathbb{R}^{n\times d}$ such that $wv=0$, then

$$\frac{1}{t^{2p}}\int_{0+}^t X_{s-}\mathrm{d}L_s \to 0 \text{ a.s.},\ t\downarrow 0.$$

Note that the above proposition holds in particular when $X_t=\sigma(L_t)$ for a suitable function σ, but the dependence on the driving process is not needed to conclude the convergence. This is due to the following property of the quadratic variation of a Lévy process.

Lemma 3.9. *Let L be given as in Proposition 3.8, then, a.s., $[L,L]_t = o(t^{2p})$ as $t\downarrow 0$.*

Proof. Using the Kunita-Watanabe inequality to estimate the individual components in the multivariate case, we may restrict the argument to $d=1$. Here, the quadratic variation $[L,L]_t$ of L is a Lévy process of bounded variation given by

$$[L,L]_t = \sigma^2 t + \sum_{0<s\leq t}(\Delta L_s)^2,$$

with the constant σ being the variance of the Gaussian part of L (if present). In the case that $p<1/2$, applying Khintchine's LIL (see e.g. [58, Prop. 47.11]) implies that $\lim_{t\downarrow 0} t^{-p}L_t = 0$ a.s. holds for any Lévy process. Since we also have $2p<1$, Theorem 1 in [64] yields, regardless of the value of σ^2, that

$$\frac{1}{t^{2p}}[L,L]_t = \frac{1}{t}[L,L]_t\cdot t^{1-2p}\to 0 \text{ a.s.},\ t\downarrow 0.$$

In the case that $p=1/2$, Khinchine's LIL yields $\limsup_{t\downarrow 0} L_t/\sqrt{t}=\infty$ a.s. if the Gaussian part of L is nonzero. As the limit exists and is finite by assumption, the process L must satisfy $\sigma=0$. This implies that the quadratic variation process has no drift, so $[L,L]_t=o(t)$ a.s. by [64, Thm. 1]. In the case that $p>1/2$, consider L with its drift (if present) subtracted from the process. This neither changes the structure of the quadratic variation nor the assumption on the a.s. convergence, but ensures that [12, Thm. 2.1] is applicable. Note that whenever $p>1$ and L is of finite variation with non-zero drift, we have $\lim_{t\downarrow 0} t^{-p}|L_t|=\infty$ by [54], showing that this case is excluded by the assumption. The a.s. existence of $\lim_{t\downarrow 0} t^{-p}L_t$ further implies that the Lévy measure ν_L of L satisfies (cf. [12])

$$\int_{[-1,1]}|x|^{1/p}\nu_L(\mathrm{d}x)<\infty.$$

Noting that $\Delta[L,L]_t = f(\Delta L_t)$ for $f(x) = x^2$, it follows that $\nu_{[L,L]}(B) = \nu_L(f^{-1}(B))$ for all sets $B \subseteq [-1,1]$. As we can now treat $\nu_{[L,L]}$ as an image measure, it is

$$\int_{[-1,1]} |x|^{1/2p} \nu_{[L,L]}(\mathrm{d}x) = \int_{[0,1]} |x|^{1/2p} \nu_{[L,L]}(\mathrm{d}x) = \int_{[-1,1]} |x|^{1/p} \nu_L(\mathrm{d}x) < \infty.$$

Thus, the quadratic variation satisfies the same integral condition with $2p$ instead of p. As $[L,L]_t$ is a bounded variation Lévy process without drift, part (i) of [12, Thm. 2.1] yields the claim in the last case. □

Proof of Proposition 3.8. First, let X be a general semimartingale. Without loss of generality, we can assume $X_0 = 0$ a.s., since

$$\frac{1}{t^p} \int_{0+}^t X_{s-} \mathrm{d}L_s = \frac{1}{t^p} \int_{0+}^t (X_{s-} - X_0) \mathrm{d}L_s + \frac{1}{t^p} X_0 L_t$$

and $t^{-p} X_0 L_t \to X_0 v$ a.s. for $t \downarrow 0$ by assumption. As X is a semimartingale, we have

$$\frac{1}{t^p} \int_{0+}^t X_{s-} \mathrm{d}L_s = \frac{1}{t^p} X_t L_t - \frac{1}{t^p} X_0 L_0 - \frac{1}{t^p} \int_{0+}^t \mathrm{d}X_s L_{s-} - \frac{1}{t^p} [X,L]_{0+}^t$$

by partial integration. Applying $X_0 = 0$ for the first two summands and Lemma 3.1 for the integral on the right-hand side, it follows that the terms vanish with probability one as $t \downarrow 0$. For the covariation, we have

$$\left| \left(\frac{1}{t^p} [X,L]_t \right)_{i,j} \right| \leq \sum_{k=1}^d \left| \frac{1}{t^p} [X_{i,k}, L_{k,j}]_t \right| \leq \sum_{k=1}^d \frac{1}{t^p} \sqrt{[X_{i,k}, X_{i,k}]_t} \sqrt{[L_{k,j}, L_{k,j}]_t}$$

by the Kunita-Watanabe inequality. As each component $L_{k,j}$ of L is again a Lévy process satisfying $\lim_{t\downarrow 0} t^{-p} (L_{k,j})_t = v_{k,j}$ a.s., one can conclude that $[L_{k,j}, L_{k,j}]_t = o(t^{2p})$ by Lemma 3.9. Therefore, it follows that $t^{-p}[X,L]_t \to 0$ a.s. for $t \downarrow 0$, yielding the first part of the proposition. Assume next that additionally $\lim_{t\downarrow 0} t^{-p} X_t = w$ for some $w \in \mathbb{R}^{n\times d}$ with $wv = 0$. One can argue similar to the proof of Lemma 3.1 and define an adapted stochastic process ψ (ω-wise) by

$$\psi_t := \begin{cases} t^{-p} X_t, \ t > 0, \\ \lim_{s\downarrow 0} s^{-p} X_s, \ t = 0, \end{cases}$$

possibly setting $\psi_0(\omega) = 0$ on the null set where the limit does not exist. Using the associativity of the stochastic integral, rewrite

$$\int_{0+}^t X_{s-} \mathrm{d}L_s = \int_{0+}^t s^p \psi_{s-} \mathrm{d}L_s = \int_{0+}^t s^p \mathrm{d}Y_s,$$

where $Y_t = \int_{0+}^t \psi_{s-} \mathrm{d}L_s$ is a well-defined semimartingale, and use integration by parts to obtain

$$\frac{1}{t^{2p}} \int_{0+}^t X_{s-} \mathrm{d}L_s = \frac{1}{t^{2p}} \left(t^p Y_t - \int_{0+}^t Y_s \mathrm{d}(s^p) \right) = \frac{1}{t^p} Y_t - \frac{1}{t^{2p}} \int_{0+}^t Y_s \mathrm{d}(s^p). \tag{3.8}$$

By the first part of the proposition, it is

$$\lim_{t \downarrow 0} \frac{1}{t^p} Y_t = \lim_{t \downarrow 0} \frac{1}{t^p} \int_{0+}^t \psi_{s-} \mathrm{d}L_s = \psi_0 v = wv = 0$$

with probability one, while the path-wise Lebesgue-Stieltjes integral can be estimated by

$$\frac{1}{t^p} \inf_{0<s\leq t} (Y_s)_{i,j} \leq \frac{1}{t^{2p}} \int_{0+}^t (Y_s)_{i,j} \mathrm{d}(s^p) \leq \frac{1}{t^p} \sup_{0<s\leq t} (Y_s)_{i,j}, \; i = 1, \dots, m, \; j = 1, \dots, n,$$

due to the integrand being an increasing function. As $wv = 0$, both bounds converge to zero with probability one, hence

$$\lim_{t \downarrow 0} \frac{1}{t^{2p}} \int_{0+}^t Y_s \mathrm{d}(s^p) = 0$$

almost surely. Thus, the limit for $t \downarrow 0$ of (3.8) exists with probability one and is equal to zero. □

The above proposition is in particular applicable for solutions of Lévy-driven SDEs. An inspection of the proof of Proposition 3.3 shows that, a.s.,

$$[\sigma_{i,k}(X), L_k]_t = O([L_1, L_1]_t + \dots + [L_d, L_d]_t).$$

Since $[L_k, L_k]_t = o(t^{2p})$ for any $k = 1, \dots, d$ by Lemma 3.9, it follows that

$$[\sigma(X), L]_t = o(t^{2p})$$

with probability one whenever L is a Lévy process satisfying $\lim_{t\downarrow 0} t^{-p} L_t = 0$ a.s. and X is the solution of (3.1). We use this fact to consider the almost sure lim sup and lim inf behavior of the quotient $t^{-p}(X_t - x)$ including the divergent case. Note that the condition $\lim_{t\downarrow 0} t^{-p/2} L_t = 0$ a.s. is satisfied whenever $p/2 > 1/2$ and $\int_0^1 x^{2/p} \nu_L(\mathrm{d}x) < \infty$ by [12, Thm. 2.1].

Theorem 3.10. *Let L be an $\mathbb{R}^d$-valued Lévy process such that $\lim_{t\downarrow 0} t^{-p/2} L_t = 0$ a.s. for some $p > 0$. Further, let $\sigma : \mathbb{R}^n \to \mathbb{R}^{n\times d}$ be twice continuously differentiable and maximal of linear growth and define $X = (X_t)_{t\geq 0}$ as the solution of the SDE* (3.1)*. Then, a.s.,*

$$\lim_{t \downarrow 0} \left(\frac{X_t - x}{t^p} - \frac{\sigma(X_t) L_t}{t^p} \right) = \lim_{t \downarrow 0} \left(\frac{X_t - x}{t^p} - \frac{\sigma(x) L_t}{t^p} \right) = 0. \tag{3.9}$$

In particular, if $\sigma(x)$ has rank d, we have

$$\lim_{t \downarrow 0} \frac{\|L_t\|}{t^p} = \infty \text{ a.s.} \;\Rightarrow\; \lim_{t \downarrow 0} \frac{\|X_t - x\|}{t^p} = \infty \text{ a.s.} \tag{3.10}$$

Proof. Similar to the proof of Proposition 3.3, we use integration by parts and rewrite

$$\frac{X_t - x}{t^p} - \frac{\sigma(X_t) L_t}{t^p} = -\frac{1}{t^p} \int_{0+}^t \mathrm{d}\sigma(X_s) L_{s-} - \frac{1}{t^p} [\sigma(X), L]_t. \tag{3.11}$$

The claim follows by showing that the desired limiting behavior for the right-hand side.

For the term involving the quadratic variation, this is immediate from the previous calculations. Hence, it remains to study the behavior of the integral. Using the Itō formula once more yields

$$\begin{aligned}\left(\int_{0+}^{t} \mathrm{d}\sigma(X_s)L_{s-}\right)_i &= \sum_{k=1}^{d}\int_{0+}^{t}(L_k)_{s-}\mathrm{d}\sigma_{i,k}(X_s)\\ &= \sum_{k=1}^{d}\int_{0+}^{t}(L_k)_{s-}\mathrm{d}\Big(\sigma_{i,k}(x) + \sum_{j=1}^{n}\int_{0+}^{s}\frac{\partial\sigma_{i,k}}{\partial x_j}(X_{r-})\mathrm{d}(X_j)_r\\ &\quad + \frac{1}{2}\sum_{j_1,j_2=1}^{n}\int_{0+}^{s}\frac{\partial^2\sigma_{i,k}}{\partial x_{j_1}\partial x_{j_2}}(X_{r-})\mathrm{d}\big[X_{j_1},X_{j_2}\big]^c_r + (J_{i,k})_s\Big),\end{aligned}$$

where the jump term is again denoted by J and the component of σ included in it is carried as a subscipt. Observe that by associativity of the stochastic integral, it follows that

$$\begin{aligned}&\int_{0+}^{t}(L_k)_{s-}\mathrm{d}\Big(\int_{0+}^{s}\frac{\partial^2\sigma_{i,k}}{\partial x_{j_1}\partial x_{j_2}}(X_{r-})\mathrm{d}\big[X_{j_1},X_{j_2}\big]^c_r\Big)\\ &= \sum_{l_1=1}^{n}\sum_{l_2=1}^{n}\int_{0+}^{t}(L_k)_{s-}\frac{\partial^2\sigma_{i,k}}{\partial x_{j_1}\partial x_{j_2}}(X_{s-})\sigma_{j_1,l_1}(X_{s-})\sigma_{j_2,l_2}(X_{s-})\mathrm{d}[L_{l_1},L_{l_2}]^c_s\\ &=: \sum_{l_1=1}^{n}\sum_{l_2=1}^{n}\int_{0+}^{t}(L_k)_{s-}M_{s-}\mathrm{d}[L_{l_1},L_{l_2}]^c_s,\end{aligned}$$

which is a sum of pathwise Lebesgue-Stieltjes integrals. Thus,

$$\begin{aligned}&\frac{1}{t^p}\Big|\int_{0+}^{t}(L_k)_{s-}\mathrm{d}\Big(\int_{0+}^{s}\frac{\partial^2\sigma_{i,k}}{\partial x_{j_1}\partial x_{j_2}}(X_{r-})\mathrm{d}\big[X_{j_1},X_{j_2}\big]^c_r\Big)\Big|\\ &\le \frac{1}{t^p}\sum_{l_1=1}^{d}\sum_{l_2=1}^{d}\sup_{0<s\le t}\Big|(L_k)_{s-}M_{s-}\Big|\sqrt{[L_{l_1},L_{l_1}]_t}\sqrt{[L_{l_2},L_{l_2}]_t}.\end{aligned}$$

As σ is in particular C^2, the supremum on the right-hand side is bounded and we conclude that the bound obtained converges to zero with probability one by Lemma 3.9. For the jump term we have

$$\frac{1}{t^p}\Big|\int_{0+}^{t}(L_k)_{s-}\mathrm{d}(J_{i,k})_s\Big| \le \frac{1}{t^p}\sum_{0<s\le t}|(L_k)_{s-}|\cdot|\Delta(J_{i,k})_s|$$

by definition. However, since $\sigma\in C^2(\mathbb{R}^d)$, it follows from Taylor's formula that

$$|\Delta(J_{i,k})_s| = \Big|\sigma_{i,k}(X_s)-\sigma_{i,k}(X_{s-})-\sum_{j=1}^{n}\frac{\partial\sigma_{i,k}}{\partial x_j}(X_{s-})\Delta(X_j)_s\Big| \le C\|\Delta X_s\|^2 \le C'\|\Delta L_s\|^2$$

for some finite (random) constants $C, C'\ge 0$ such that

$$\frac{1}{t^p}\Big|\int_{0+}^{t}(L_k)_{s-}\mathrm{d}(J_{i,k})_s\Big| \le \frac{1}{t^p}C'\sup_{0<s\le t}|(L_k)_s|\sum_{j=1}^{d}[L_j,L_j]_t,$$

which also converges a.s. to zero as $t \downarrow 0$. For the last term, observe first that

$$\int_{0+}^{t} (L_k)_{s-} \mathrm{d}\Big(\int_{0+}^{s} \frac{\partial \sigma_{i,k}}{\partial x_j}(X_{r-}) \mathrm{d}(X_j)_r \Big) = \sum_{l=1}^{d} \int_{0+}^{t} (L_k)_{s-} \frac{\partial \sigma_{i,k}}{\partial x_j}(X_{s-}) \sigma_{j,l}(X_{s-}) \mathrm{d}(L_l)_s$$

by the associativity of the stochastic integral. Including the summation over k and j, this can be rewritten as

$$\sum_{k=1}^{d}\sum_{l=1}^{d} \int_{0+}^{t} (L_k)_{s-} \Big(\sum_{j=1}^{d} \frac{\partial \sigma_{i,k}}{\partial x_j}(X_{s-}) \sigma_{j,l}(X_{s-}) \Big) \mathrm{d}(L_l)_s = \sum_{k=1}^{d}\sum_{l=1}^{d} \int_{0+}^{t} (L_k)_{s-} (M_{i,k,l})_{s-} \mathrm{d}(L_l)_s,$$

where we note that $\sup_{0<s\le t} |(M_{i,k,l})_s|$ is bounded for any fixed small $t \ge 0$ and continuous at zero due to the continuity of σ and its derivatives. Since $\lim_{t\downarrow 0} t^{-p/2} L_t = 0$ with probability one, it follows that, a.s., $\lim_{t\downarrow 0} t^{-p/2} (L_k)_t (M_{i,k,l})_t$ exists. Thus, the second part of Proposition 3.8 is applicable and one can conclude that the integral also converges to zero with probability one. Since $\lim_{t\downarrow 0} t^{-p/2} L_t = 0$ with probability one, we have

$$\begin{aligned} 0 &\le \left\| \frac{\sigma(X_t) L_t}{t^p} - \frac{\sigma(x) L_t}{t^p} \right\| \le \left\| \frac{\sigma(X_t) - \sigma(x)}{t^{p/2}} \right\| \cdot \left\| \frac{L_t}{t^{p/2}} \right\| \\ &\le \sum_{j=1}^{n} \sup_{0<s\le t} \left\| \frac{\partial \sigma}{\partial x_j}(X_t) \right\| \cdot \left\| \frac{X_t - x}{t^{p/2}} \right\| \cdot \left\| \frac{L_t}{t^{p/2}} \right\|. \end{aligned} \tag{3.12}$$

As $t \downarrow 0$, the first term converges with probablity one by the assumptions on σ and Proposition 3.3 is applicable for the second one. Using that $\lim_{t\downarrow 0} t^{-p/2} L_t = 0$ a.s., the right-hand side of (3.12) converges to zero with probability one as $t \downarrow 0$. If $\sigma(x)$ has rank d, Equation (3.10) follows immediately from the convergence result in (3.9). □

Theorem 3.10 allows to characterize the a.s. short-time behavior of the solution to a Lévy-driven SDE in terms of power law functions. In order to derive precise LIL-type results, we now turn to more general functions. Note that, whenever the driving Lévy process has a Gaussian component, its a.s. short-time behavior is determined by Khintchine's LIL (see e.g. [58, Prop. 47.11]). Hence, Lemma 3.9 readily generalizes to continuous increasing functions $f : [0, \infty) \to \mathbb{R}$ with $f(0) = 0$ and $f(t) > 0$ for all $t > 0$, as any function f such that $\lim_{t\downarrow 0} L_t/f(t)$ exists in $\mathbb{R}$ must satisfy $\sqrt{2t \ln(\ln(1/t))}/f(t) \to 0$ and it follows

$$\lim_{t\downarrow 0} \frac{[L,L]_t}{(f(t))^2} = \lim_{t\downarrow 0} \Big(\frac{[L,L]_t}{t} \frac{t}{2t\ln(\ln(1/t))} \frac{2t\ln(\ln(1/t))}{(f(t))^2} \Big) = 0 \text{ a.s.} \tag{3.13}$$

by [64, Thm. 1]. Thus, $[L,L]_t = o(f(t)^2)$ and we can replace the function $t^{p/2}$ for some $p > 0$ in Theorem 3.10 by f in this case and obtain a precise short-time behavior for the solutions of stochastic differential equations that include a diffusion part. In the case that L does not include a Gaussian component, $[L,L]$ is a finite variation process without drift satisfying $\lim_{t\downarrow 0} t^{-1}[L,L]_t = 0$ a.s. by [64, Thm. 1], such that an argument similar to (3.13) is still applicable if f decays sufficiently fast as $t \downarrow 0$. For the general case, we combine Theorem 3.10 with the precise information on possible scaling functions derived in [26]. Note that the conditions of Corollary 3.11 below immediately follow from Khinthchine's LIL whenever $h \equiv 1$, as the process does not include a Gaussian part by assumption.

Corollary 3.11. *Let L be a purely non-Gaussian $\mathbb{R}^d$-valued Lévy process and $f:[0,\infty)\to\mathbb{R}$ be of the form $f(t)=\sqrt{t\ln(\ln(1/t))h(1/t)^{-1}}$, where $h:[0,\infty)\to[0,\infty)$ is a continuous and non-decreasing slowly varying function, such that the set of cluster points of $L_t/f(t)$ as $t\downarrow 0$ is bounded with probability one. Further, let $\sigma:\mathbb{R}^d\to\mathbb{R}^{n\times d}$ be twice continuously differentiable and maximal of linear growth and define $X=(X_t)_{t\geq 0}$ as the solution of* (3.1)*. Then, a.s.,*

$$\lim_{t\downarrow 0}\left(\frac{X_t-x}{f(t)}-\frac{\sigma(X_t)L_t}{f(t)}\right)=\lim_{t\downarrow 0}\left(\frac{X_t-x}{f(t)}-\frac{\sigma(x)L_t}{f(t)}\right)=0. \tag{3.14}$$

In particular, if $\sigma(x)$ has rank d, we have

$$\lim_{t\downarrow 0}\frac{\|L_t\|}{f(t)}=\infty\text{ a.s.}\ \Rightarrow\ \lim_{t\downarrow 0}\frac{\|X_t-x\|}{f(t)}=\infty\text{ a.s.}$$

Proof. As the scaling function is of the form $f(t)=t^{1/2}\ell(1/t)$ with a slowly varying function ℓ by assumption, the a.s. boundedness of the cluster points of $L_t/f(t)$ in $\mathbb{R}^d$ implies that, for all $\varepsilon\in(0,1/2)$,

$$\lim_{t\downarrow 0}\frac{L_t}{t^{(1/2-\varepsilon)}}=\lim_{t\downarrow 0}\frac{L_t}{f(t)}\cdot\ell(1/t)t^{\varepsilon}=0$$

with probability one. Thus, Theorem 3.10 is applicable with $p/2=1/2-\varepsilon$, yielding

$$\lim_{t\downarrow 0}\left(\frac{X_t-x}{t^{1-2\varepsilon}}-\frac{\sigma(x)L_t}{t^{1-2\varepsilon}}\right)=0$$

almost surely. Using the explicit form of f and choosing $\varepsilon\in(0,1/4)$, it follows that

$$\lim_{t\downarrow 0}\left(\frac{X_t-x-\sigma(x)L_t}{f(t)}\right)=\lim_{t\downarrow 0}\left(\frac{X_t-x-\sigma(x)L_t}{t^{1-2\varepsilon}}\cdot\frac{t^{1-2\varepsilon}}{f(t)}\right)=0$$

with probability one, which is (3.14), and the remaining claims follow in analogy to the proof of Theorem 3.10. □

The above results show that the almost sure short-time LIL-type behavior of the driving Lévy process directly translates to the solution of the stochastic differential equation (3.1). We also note the following statement for the conversion of the corresponding cluster set.

Corollary 3.12. *Under the assumptions of Corollary 3.11 let* $\limsup_{t\downarrow 0}\|L_t\|/f(t)$ *be bounded with probability one. Then there exists an a.s. cluster set $A_X=C(\{X_t/f(t):t\downarrow 0\})$ for the solution X of* (3.1) *which is obtained from the cluster set $A_L=C(\{L_t/f(t):t\downarrow 0\})$ of the driving Lévy process L via $A_X=\sigma(x)A_L$.*

Corollary 3.12 implies in particular that A_X shares the properties of A_L derived in [26, Thm. 2.4] and that there is a one-to-one correspondence between the cluster sets whenever $\sigma(x)$ has rank d. As $\sigma(x)=\mathrm{Id}$ for the stochastic exponential, we have $A_X=A_L$ for this example, mirroring the statement of Corollary 3.6.

Lastly, we use Theorem 3.10 to show that one can also translate more general limiting results at zero from the driving Lévy process to the solution of (3.1). Here, convergence in distribution and convergence in probability are denoted by $\xrightarrow{\mathcal{D}}$ and $\xrightarrow{P}$, respectively. As the short-time behavior of Brownian motion is well-known, L is taken to be a purely non-Gaussian Lévy process and we further assume the drift of L, whenever existent, to be equal to zero. Note that sufficient conditions for the attraction of a Lévy process to normality are e.g. given in [24, Thm. 2.5]. Thus, the conditions of Corollary 3.13 below are readily checked from the characteristic triplet of L and e.g. satisfied for a Lévy process with a symmetric Lévy measure such as $\nu_L(\mathrm{d}x) = \exp(-|x|)\mathbb{1}_{[-1,1]}(x)\mathrm{d}x$.

Corollary 3.13. *Let $d = 1$, L as specified above and assume that there is a continuous increasing function $f : [0,\infty) \to [0,\infty)$ such that $f(t)^{-1}L_t \xrightarrow{\mathcal{D}} Y$ as $t \downarrow 0$, where the random variable Y follows a non-degenerate stable law with index $\alpha \in (0,2]$. Let further $\sigma : \mathbb{R} \to \mathbb{R}$ be twice continuously differentiable and maximal of linear growth and define $X = (X_t)_{t\geq 0}$ as the solution of* (3.1) *such that the initial condition $x \in \mathbb{R}^n$ satisfies $\sigma(x) \neq 0$. Then*

$$\frac{X_t - x}{f(t)} \xrightarrow{\mathcal{D}} \sigma(x)Y. \tag{3.15}$$

Whenever f is regularly varying with index $a \in (0,1/2]$ at zero, (3.15) *also holds if the random variable Y is a.s. constant.*

Proof. If $\alpha = 2$, i.e. Y is normally disributed, the convergence of $L_t/f(t)$ implies that

$$\lim_{t\downarrow 0} t\overline{\Pi}_L^{(\#)}(xf(t)) = 0 \tag{3.16}$$

for all $x > 0$ and $\# \in \{+,-\}$ by [43, Prop. 4.1]. Choosing $x = 1$, note that the condition (3.16) is not sufficient to imply the integrability of $\overline{\Pi}_L^{(\#)}(f(t))$ over $[0,1]$. However, since the distribution of Y is non-degenerate, the scaling function f is regularly varying with index $1/2$ at zero (see [24, Thm. 2.5]) such that also

$$\lim_{t\downarrow 0} t\overline{\Pi}^{\#} L(t^{1/2-\varepsilon}) = 0.$$

This yields the estimate

$$\overline{\Pi}_L^{\#}(t^{(1/2-\varepsilon)k}) \leq \frac{C_t}{t^k} \tag{3.17}$$

where C_t is bounded as $t \downarrow 0$ and the function is thus integrable over $[0,1]$ for $0 \leq k < 1$. By assumption, L does not have a Gaussian component and the drift of the process is equal to zero whenever it is defined. Hence,

$$\int_{0+}^{t} \overline{\Pi}^{\#}(t^{(1/2-\varepsilon)k})\mathrm{d}t < \infty$$

for both $\# = +$ and $\# = -$ and thus $\lim_{t\downarrow 0} t^{-(1/2-\varepsilon)k} L_t = 0$ a.s. by [12, Thm. 2.1]. Applying Theorem 3.10, we obtain

$$\lim_{t\downarrow 0} \left(\frac{X_t - x}{t^{k-2\varepsilon k}} - \frac{\sigma(x)L_t}{t^{k-2\varepsilon k}} \right) = 0$$

with probability one. It now follows for $k - 2\varepsilon k > 1/2$ that

$$\lim_{t \downarrow 0} \left(\frac{X_t - x - \sigma(x) L_t}{f(t)} \right) = \lim_{t \downarrow 0} \left(\frac{X_t - x - \sigma(x) L_t}{t^{k-2\varepsilon k}} \cdot \frac{t^{k-2\varepsilon k}}{f(t)} \right) = 0$$

almost surely, which yields the desired convergence of $f(t)^{-1}(X_t - x)$. If Y follows a nondegenerate stable law with index $\alpha \in (0, 2)$, the right-hand side of (3.16) is to be replaced by the tail function $\overline{\Pi}_Y^{\#}(x)$ (see [43, Prop. 4.1]) and it follows from the proof of [43, Thm. 2.3] that the scaling function f is regularly varying with index $1/\alpha$ at zero in this case. Thus, we can derive a bound similar to (3.17) and argue as before. Noting that [43, Prop. 4.1] does not require the law of the limiting random variable to be non-degenerate, the argument is also applicable if Y is a.s. constant and f is regularly varying with index $a \in (0, 1/2]$ at zero. □

One can also give a result for convergence in probability. The conditions can be checked directly from the characteristic triplet of the driving Lévy process using [24, Thm. 2.2] and are e.g. satisfied for finite variation Lévy processes. As the limiting random variable is a.s. constant, the proof is immediate from Corollary 3.13.

Corollary 3.14. *Let $d = 1$, L as above and assume that there is a continuous increasing function $f : [0, \infty) \to [0, \infty)$ that is regularly varying with index $a \in (0, 1/2]$ at zero such that $f(t)^{-1} L_t \xrightarrow{P} v$ for some finite value $v \in \mathbb{R}$ as $t \downarrow 0$. Let further $\sigma : \mathbb{R} \to \mathbb{R}$ be twice continuously differentiable and maximal of linear growth and define $X = (X_t)_{t \geq 0}$ as the solution of* (1.6)*. Then*

$$\frac{X_t - x}{f(t)} \xrightarrow{P} \sigma(x) v.$$

A. Some Additional Remarks on Chapter 2

In this section, we collect some additional remarks on the contents of Chapter 2 regarding the methods used, as well as possible extensions, that were not included in [9] and [8], but might be of interest either in their own or for further investigations. Recall that ξ and η denote independent Lévy processes with characteristic triplets $(\sigma_\xi^2, \nu_\xi, \gamma_\xi)$ and $(\sigma_\eta^2, \nu_\eta, \gamma_\eta)$, and characteristic exponents Ψ_ξ and Ψ_η, respectively. Further, τ denotes an exponentially distibuted random variable with parameter $q \geq 0$ that is independent of ξ and η and $\tau = \infty$ a.s. whenever $q = 0$.

A.1. An Alternative Approach to Corollary 2.45

First, we give an approach to deriving an analog of [38, Cor. 2.4] for $V_{q,\xi,\eta}$ directly from the result in the case without killing. To keep the argument below more transparent, the Lévy measures ν_ξ and ν_η are assumed to be supported only on $[-1, 1]$ to account for the moment condition in [38] and we exclude the case of η being equal to zero or a compound Poisson process to avoid additional terms including point masses (cf. Corollary 2.26). Recall that the infinitesimal generator $A^{\widetilde{V}}$ (cf. Theorem 2.34) is given by

$$A^{\widetilde{V}} = A^V + q(f(0) - f(x)),$$

where A^V denotes the infinitesimal generator of a generalized Ornstein-Uhlenbeck process driven by ξ and η. Denoting again $\mu = \mathcal{L}(V_{q,\xi,\eta})$, it follows that

$$\int_{\mathbb{R}} A^{\widetilde{V}} f(x)\mu(\mathrm{d}x) = \int_{\mathbb{R}} A^V f(x)\mu(\mathrm{d}x) + \int_{\mathbb{R}} q(f(0) - f(x))\mu(\mathrm{d}x) = 0 \tag{A.1}$$

holds for all functions f in the domain of the operator, in particular for $f \in C_c^\infty(\mathbb{R})$. We use this decomposition to show a special case of Corollary 2.45. Note that, although η is assumed to be a subordinator below, the same approach can also be applied in other cases, as long as the equation and the asymptotics of the individual terms are known in the case without killing.

Proposition A.1. *Let ξ, η be two independent Lévy processes such that η is a subordinator that is neither a compound Poisson process nor the zero process and the Lévy measures ν_ξ and ν_η are only supported on $[-1, 1]$. Let further $q > 0$. If $\sigma_\xi^2 > 0$, then μ has a density with respect to the Lebesgue measure on $\mathbb{R}$.*

Proof. Since μ is continuous by assumption and η is a subordinator, it is enough to consider functions $f \in C_c^\infty((0, \infty))$. With the moment condition $\mathbb{E}|\xi_1|, \mathbb{E}|\eta_1| < \infty$ in [38] satisfied due to neither ξ nor η possessing big jumps, we can follow the proof of [38, Thm. 2.3] and rewrite the term involving A^V in (A.1). Note, however, that the process ξ considered here refers to the negative of the exponent in the paper and that some quantities need to be modified slightly. It follows that

$$\int_0^\infty A^V f(x)\mu(\mathrm{d}x) = \int_0^\infty g'(z)\nu(\mathrm{d}z), \tag{A.2}$$

where $f \in C_c^\infty((0,\infty))$, $g(z) = zf'(z)$ and ν is given for $z > 0$ similar to Equation (2.3) in [38] by

$$\begin{aligned}\nu(\mathrm{d}z) = &-\gamma_\xi \int_z^\infty \mu(\mathrm{d}x)\mathrm{d}z + \frac{\sigma_\xi^2}{2} z\mu(\mathrm{d}z) \\ &+ \int_{(z,ez]} \overline{\overline{\Pi}}_\xi^{(+)}\Big(\ln\frac{x}{z}\Big)\mu(\mathrm{d}x)\mathrm{d}z + \int_{[\frac{z}{e},z)} \overline{\overline{\Pi}}_\xi^{(-)}\Big(\ln\frac{z}{x}\Big)\mu(\mathrm{d}x)\mathrm{d}z + \gamma_\eta \int_z^\infty \frac{\mu(\mathrm{d}x)}{x}\mathrm{d}z \\ &+ \frac{1}{z}\int_{(0\wedge(z-1),z)} \overline{\overline{\Pi}}_\eta^{(+)}(z-x)\mu(\mathrm{d}x)\mathrm{d}z - \int_z^\infty \frac{1}{t^2}\int_{(0\wedge(t-1),t)} \overline{\overline{\Pi}}_\eta^{(+)}(t-x)\mu(\mathrm{d}x)\mathrm{d}t\mathrm{d}z.\end{aligned} \tag{A.3}$$

As ν yields finite values when evaluated on compact subsets of $(0,\infty)$, it can be interpreted as a distribution in the sense of Schwartz and the right-hand side of (A.2) can be rewritten as $\langle g', \nu\rangle$ to emphasize the dual pairing. Due to the assumptions made in the proposition, Lemma 2.4 in [38] also holds for (A.3) such that its limiting behavior is characterized by

$$\lim_{z\to\infty} \frac{1}{z}|\nu|(a,z) = 0, \quad a > 0$$

and consequently,

$$\lim_{z\to\infty} \frac{1}{h(z)}|\nu|(a,z) = 0, \quad a > 0 \tag{A.4}$$

holds for all functions $h(z)$ such that $(h(z))^{-1} = \mathcal{O}(z^{-1})$. This condition is later used similarly to the lemma in [38] to determine the constants appearing in the solution.

For the term in (A.1) that constitutes the contribution of the killing we have

$$\int_0^\infty q\cdot(f(0) - f(x))\mu(\mathrm{d}x) = \int_0^\infty f(x)(-q)\mu(\mathrm{d}x)$$

setting $f(0) = 0$ due to the choice $f \in C_c^\infty((0,\infty))$. Again, the measure on the right-hand side is interpreted as a distribution in the sense of Schwartz and the term can be rewritten as $\langle f, -q\mu\rangle$. Using (A.1) and the definition of the distributional derivative, this leads to the equation

$$\langle g', \nu\rangle + \langle f, -q\mu\rangle = \langle f, (x\nu')'\rangle + \langle f, -q\mu\rangle = \langle f, (x\nu')' - q\mu\rangle = 0$$

which holds for all $f \in C_c^\infty((0,\infty))$. This implies that the distribution $(x\nu')' - q\mu$ has empty support in $(0,\infty)$, which can only be the case if it is the zero distribution. Thus,

$$(x\nu')' - q\mu = 0 \quad\Longleftrightarrow\quad (x\nu')' = q\mu \tag{A.5}$$

such that ν is determined, similarly to G in the proof of Theorem 2.43, by solving an ordinary differential equation (ODE). Note, however, that the the ODE derived in this approach is inhomogenous as opposed to the homogenous equation $G'' = 0$ arising in Section 2.4.3. As every distribution has a well-defined antiderivative (see e.g. [25, Thm. 4.3]), we can obtain $x\nu'$ from (A.5) as an antiderivative of $q\mu$. It can be given explicitly as

$$f \mapsto \int_0^\infty f(t) q\mu((0,t])\mathrm{d}t,$$

e.g. by using a similar argument as in Example 6.14 of [55]. Since this antiderivative is only unique up to a constant, we find that

$$x\nu'(\mathrm{d}x) = q\mu((0,x])\mathrm{d}x + C_1\mathrm{d}x \quad \Longleftrightarrow \quad \nu'(\mathrm{d}x) = \frac{q\mu((0,x])\mathrm{d}x}{x} + \frac{C_1\mathrm{d}x}{x} \tag{A.6}$$

for some constant $C_1 \in \mathbb{R}$. In the particular case considered, the function $x \mapsto \mu((0,x])$ is continuous due to μ not having atoms. The antiderivative ν of ν' can now be determined similarly to the solution of (A.5) by superposition of the individual solutions for the terms on the right-hand side of (A.6). As μ is continuous, $g(x) = x^{-1}q\mu((0,x])$ is also continuous for $x > 0$ Thus, using results from [55] yields that an antiderivative of $x^{-1}q\mu((0,x])\mathrm{d}x$ is given by $\tilde{g}(x)\mathrm{d}x$ with $\tilde{g}$ denoting an antiderivative of g, e.g. $\tilde{g}(x) = \int_1^x t^{-1}q\mu((0,t])\mathrm{d}t$. Note, however, that a similar result would also follow from [55] if $t \mapsto \mu((0,t])$ is only right-continuous. The second term on the right-hand side of (A.6) can be treated similarly and corresponds to the solution of the homogenous equation obtained in [38]. Hence, it follows

$$\nu(\mathrm{d}x) = \int_1^x \frac{1}{t} q\mu((0,t])\mathrm{d}t\mathrm{d}x + C_1\ln(x)\mathrm{d}x + C_2\mathrm{d}x, \tag{A.7}$$

where $C_2 \in \mathbb{R}$ has to be included as the antiderivative of the distribution is only determined up to an additive constant. Note that ν as given above is the same as in (A.3). Since the limiting behavior of ν is known from (A.4), it can now be used to determine the constants C_1 and C_2. To find C_1, observe that the corresponding term will behave like $z\ln(z)$ when evaluated on sets (a,z) for $a > 0$. Choosing $h(z) = z\ln(z)$ and applying (A.4) to the left-hand side of (A.7) leads to

$$0 = \lim_{z\to\infty} \frac{1}{z\ln(z)} \int_a^z \int_1^x \frac{1}{t} q\mu((0,t])\mathrm{d}t\mathrm{d}x + C_1,$$

such that C_1 can be determined by calculating the limit. As the integrand in the first term is continuous by assumption, the double integral can be interpreted as a Riemann integral and is thus twice continuously differentiable. As the same holds for $z\ln(z)$ if $z > 0$, l'Hôspital's rule can be applied to calculate the limit, yielding

$$\begin{aligned}\lim_{z\to\infty} \frac{1}{z\ln(z)} \int_a^z \int_1^x \frac{1}{t} q\mu((0,t])\mathrm{d}t\mathrm{d}x &= \lim_{z\to\infty} \frac{1}{\ln(z) + \frac{1}{z}z} \int_1^z \frac{1}{t} q\mu((0,t])\mathrm{d}t \\ &= \lim_{z\to\infty} z\frac{1}{z} q\mu((0,z]) = q\mu((0,\infty)),\end{aligned}$$

i.e. $C_1 = -q\mu((0,\infty))$. Note that the limit can also be derived without relying on the continuity assumption by giving upper and lower bounds for the integrand. To find C_2, observe that the corresponding term in (A.7) behaves like z when evaluated on (a,z) for $a>0$. Using (A.4) and the explicit value of C_1, this yields

$$0 = q\lim_{z\to\infty}\frac{1}{z}\Big(\int_a^z\int_1^x\frac{1}{t}\mu((0,t])\mathrm{d}t\mathrm{d}x - \mu((0,\infty))\int_a^z\ln(x)\mathrm{d}x\Big) + C_2 \tag{A.8}$$

such that the constant is again determined by calculating the limit. To do so, rewrite the second term as

$$\mu((0,\infty))\int_a^z\ln(x)\mathrm{d}x = \int_a^z\int_1^x\frac{1}{t}\mu((0,\infty))\mathrm{d}t\mathrm{d}x$$

such that the difference in (A.8) is given by

$$\frac{1}{z}\Big(\int_a^z\int_1^x\frac{1}{t}\mu((0,t])\mathrm{d}t\mathrm{d}x - \mu((0,\infty))\int_a^z\ln(x)\mathrm{d}x\Big) = -\frac{1}{z}\int_a^z\int_1^x\frac{1}{t}\mu((t,\infty))\mathrm{d}t\mathrm{d}x.$$

Note that one may replace a by 1 here, as the additional constants obtained from rewriting the term will vanish when the limit is evaluated. For z fixed and finite, Fubini's Theorem can be applied, yielding

$$-\frac{1}{z}\int_a^z\int_1^x\frac{1}{t}\mu((t,\infty))\mathrm{d}t\mathrm{d}x = -\int_1^z\frac{1}{t}\mu((t,\infty))\mathrm{d}t + \frac{1}{z}\int_1^z\mu((t,\infty))\mathrm{d}t.$$

Here, the second term on the right-hand side has already been treated on page 9 of [38] and is shown to vanish when the limit is evaluated. From (A.8) we can see that the limit for $z\to\infty$ must also exist for the other term, such that the constant C_2 is given by

$$C_2 = q\int_1^\infty\frac{1}{t}\mu((t,\infty))\mathrm{d}t.$$

Replacing C_1 and C_2 in (A.7) and simplifying the term now leads to

$$\begin{aligned}\nu(\mathrm{d}z) &= \int_1^z\frac{1}{t}q\mu((0,t])\mathrm{d}t\mathrm{d}z - q\mu((0,\infty))\ln(z)\mathrm{d}z + q\int_1^\infty\frac{1}{t}\mu((t,\infty))\mathrm{d}t\mathrm{d}z\\ &= q\int_z^\infty\frac{1}{t}\mu((t,\infty))\mathrm{d}t\mathrm{d}z.\end{aligned}$$

As ν is the same as in (A.3), we arrive at

$$\begin{aligned}q\int_z^\infty\frac{1}{t}\mu((t,\infty))\mathrm{d}t\mathrm{d}z = &-\gamma_\xi\int_z^\infty\mu(\mathrm{d}x)\mathrm{d}z + \frac{\sigma_\xi^2}{2}z\mu(\mathrm{d}z)\gamma_\eta\int_z^\infty\frac{\mu(\mathrm{d}x)}{x}\mathrm{d}z\\ &+\int_{(z,ez]}\overline{\overline{\Pi}}_\xi^{(+)}\Big(\ln\frac{x}{z}\Big)\mu(\mathrm{d}x)\mathrm{d}z + \int_{[\frac{z}{e},z)}\overline{\overline{\Pi}}_\xi^{(-)}\Big(\ln\frac{z}{x}\Big)\mu(\mathrm{d}x)\mathrm{d}z\\ &+\frac{1}{z}\int_{(0\wedge(z-1),z)}\overline{\overline{\Pi}}_\eta^{(+)}(z-x)\mu(\mathrm{d}x)\mathrm{d}z\\ &-\int_z^\infty\frac{1}{t^2}\int_{(0\wedge(t-1),t)}\overline{\overline{\Pi}}_\eta^{(+)}(t-x)\mu(\mathrm{d}x)\mathrm{d}t\mathrm{d}z.\end{aligned} \tag{A.9}$$

Note that all terms in (A.9) except $(\sigma_\xi^2/2)z\mu(\mathrm{d}z)$ are by definition absolutely continuous with respect to the Lebesgue measure such that the existence of a density is implied. In this case it also follows from (A.9) that the density is bounded on compact subsets of $(0, \infty)$. □

A.2. Self-Decomposability

Aside from the distributional properties studied in Sections 2.1 to 2.4, many more aspects of the law of the killed exponential functional can be considered. One such example is the question of self-decomposability. We call a random variable X self-decomposable, if for any $c \in (0,1)$ there exist random variables X' and Y_c such that $X \overset{d}{=} cX' + Y_c$ with $X' \overset{d}{=} X$, where the summands on the right-hand side are assumed to be independent. Equivalently, this can be expressed through the factorization property $\Phi_X(z) = \Phi_X(cz)\Phi_c(z)$ for all $z \in \mathbb{R}$, where Φ_X and Φ_c denote the characteristic functions of X and Y_c, respectively. Exponential functionals of the form $\int_0^\infty \mathrm{e}^{-as}\mathrm{d}\eta_s$, where $a > 0$ and η is a Lévy process satisfying $\int_{\mathbb{R}\setminus[-\mathrm{e},\mathrm{e}]}(\ln|y|)\nu_\eta(\mathrm{d}y) < \infty$, are classical examples of self-decomposable random variables. It was further shown in [13, Rem. to Thm. 2.2] that the deterministic subordinator $\xi_t = at$ can be replaced by any Lévy process ξ that has no positive jumps and drifts to ∞ a.s. as $t \to \infty$, showing that the law of $V_{0,\xi,\eta}$ is self-decomposable under mild assumptions. If $\eta \equiv 0$, the law of $V_{q,\xi,\eta}$ is the zero measure and hence clearly self-decomposable. However, a simple condition similar to the one given in [13] cannot hold for the killed exponential functional, as can be seen from the examples below.

Example A.2. (i) Let $\eta_t = t$, $\xi_t = at$ and $q = a > 0$. In this case, the characteristic function φ of $V_{q,\xi,\eta}$ was calculated explicitly in (2.66). Note that φ has zeroes, such that the corresponding distribution cannot be self-decomposable as it is not infinitely divisible (cf. [58, Cor. 15.11]). Further, any infinitely divisible distribution has unbounded support unless it is degenerate (cf. [58, Thm. 24.3]), such that the law of the killed exponential functional is never self-decomposable if η is deterministic and ξ is a subordinator with positive drift (see also Section 2.1). If $\xi \equiv 0$, we have $V_{q,0,t} = \int_0^\tau \mathrm{e}^0\mathrm{d}t = \tau$, which is exponentially distributed and hence has a self-decomposable law by [58, Ex. 15.13].
(ii) Whenever η is a compound Poisson process, the law of the killed exponential functional has an atom at zero. Since self-decomposable distributions are absolutely continuous unless degenerate (see [58, Thm. 28.4]), the law of $V_{q,\xi,\eta}$ cannot be self-decomposable regardless of the choice of ξ.
(iii) Let $\xi \equiv 0$ and η be a standard Brownian motion. In this case, $V_{q,\xi,\eta} \overset{d}{=} \eta_\tau$ and the law of the killed exponential functional is absolutely continuous by Corollary 2.45. Since $\mathcal{L}(V_{q,\xi,\eta})$ is equal to q times the potential measure, its density is given by

$$f(x) = q \cdot v_q(x) = q \cdot \frac{1}{\sqrt{2q}} \exp(-|x|/\sqrt{2q})$$

where v_q denotes the density of the corresponding potential measure derived in [58, Ex. 30.11]. Note, however, that f is identified as the density of a two-sided exponential distribution in [58, Ex. 45.4], which is self-decomposable (cf. [58, Ex.15.14]).

A.3. Extension to the Multivariate Case

Throughout Chapter 2, the killed exponential functional has been considered for real-valued Lévy processes ξ and η only. Note, however, that all relevant quantities can be formulated in a multivariate setting as well (see [4]). We give a brief overview of the tools needed to extend the notion of the (killed) exponential functional to the multivariate case as well as the main differences to the univariate setting analyzed in Chapter 2. Let (U, L) be an $\mathbb{R}^{d\times d} \times \mathbb{R}^d$-valued Lévy process with U possibly being independent of L. Then the solution to the SDE

$$\mathrm{d}V_t = \mathrm{d}U_t V_{t-} + \mathrm{d}L_t \tag{A.10}$$

with starting random variable V_0 is called a multivariate generalized Ornstein-Uhlenbeck (mGOU) process. Alternatively, one can assume L to be $\mathbb{R}^{d\times d}$-valued and obtain a matrix-valued generalized Ornstein-Uhlenbeck process that includes the stochastic exponential by setting $L \equiv 0$. In either case, the process $V = (V_t)_{t\geq 0}$ is well-defined as the unique strong solution to the SDE by Theorem 1.17. It was shown in [4, Thm. 3.4] that, similar to (1.10) in the univariate case, the mGOU process can be given explicitly as

$$V_t = \overleftarrow{\mathcal{E}}(X)_t^{-1}\Big(V_0 + \int_{0+}^t \overrightarrow{\mathcal{E}}(X)_{s-}\mathrm{d}Y_s\Big),\ t \geq 0,$$

where $\det(\mathrm{Id} + \Delta X) \neq 0$ for all $t \geq 0$ such that $\overleftarrow{\mathcal{E}}(X)^{-1}$ is well-defined. The Lévy processes X and Y are considered the driving processes of V and play the role of ξ and η, respectively. Further, U and L in (A.10) are obtained through the relations

$$\begin{aligned} \overrightarrow{\mathcal{E}}(U)_t &= \overleftarrow{\mathcal{E}}(X)_t^{-1},\ t \geq 0, \\ L_t &= Y_t + \sum_{0<s\leq t} ((\mathrm{Id} + \Delta X_s)^{-1} - \mathrm{Id})\Delta Y_s - [X, Y]_t^c,\ t \geq 0, \end{aligned}$$

where $[\cdot,\cdot]^c$ denotes the continuous part of the quadratic covariation. Note that U, too, satisfies $\det(\mathrm{Id}+\Delta U_t) \neq 0$ for all $t \geq 0$, matching the condition $\Delta U_t \neq -1$ in the univariate case. Whenever V_0 is independent of (X, Y), the mGOU process is a time-homogenous Markov process which, under suitable conditions, has a unique stationary distribution that is given by the law of the improper integral

$$V_{0,U,L} = \int_{0+}^{\infty} \overleftarrow{\mathcal{E}}(U)_{s-}\mathrm{d}L_s, \tag{A.11}$$

where the limit exists a.s. if we assume e.g.

$$\mathbb{E}[\ln\max\{1, \|U_1\|\}], \mathbb{E}[\ln\max\{1, \|L_1\|\}] < \infty, \text{ and } \mathbb{E}[\ln\|\overleftarrow{\mathcal{E}}(U)_{t_0}\|] < 0 \text{ for some } t_0 > 0$$

in the above setting (cf. [4, Thm. 5.4]). Thus, $V_{0,U,L}$ is the multivariate analog to the exponential functional in the case without killing. Similar to the univariate case, it can be shown that the mGOU process is a Feller process under suitable assumptions on the Lévy measure of U and one can give the infinitesimal generator of the process. We show the first result by vectorization of the SDE (A.10) and evaluation of the necessary and sufficient condition for the solution to be a rich Feller process given in [37, Thm. 1.1] similar to [37, Ex. 4.3]. We call a Feller process rich if the space of test functions lies in the domain of its infinitesimal generator.

Lemma A.3. *Let (U, L) be a Lévy process taking values in $\mathbb{R}^{d\times d} \times \mathbb{R}^d$. Then the mGOU process driven by U and L, i.e. the solution of the stochastic differential equation*

$$\mathrm{d}V_t = \mathrm{d}U_t V_{t-} + \mathrm{d}L_t,\ t > 0, \quad V_0 = v \in \mathbb{R}^d$$

is a rich Feller process if the Lévy measure ν_U of U satisfies

$$\nu_U(\{m \in \mathbb{R}^{d\times d} : \det(m + \mathrm{Id}) = 0\}) = 0. \tag{A.12}$$

Whenever ν_U has atoms in the set $\{m \in \mathbb{R}^{d\times d} : \det(m + \mathrm{Id}) = 0\}$, the solution is not a rich Feller process.

Proof. We compare (A.12) to the necessary and sufficient condition derived in [37]. First, we rewrite the SDE to the form

$$\mathrm{d}V_t = \sigma(V_{t-})\mathrm{d}Z_t \tag{A.13}$$

where $Z = (Z_t)_{t\geq 0}$ is an $\mathbb{R}^{d^2+d}$-valued Lévy process such that

$$Z = \begin{pmatrix} U^{vec} \\ L \end{pmatrix} = (U_{1,1}, U_{2,1}, \ldots, U_{d,1}, U_{2,1}, \ldots, U_{d,d}, L_1, L_2, \ldots, L_d)^T \tag{A.14}$$

and $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times(d^2+d)}$ is given by

$$\sigma((x_1, \ldots, x_d)^T) = \left(x^T \otimes_K \mathrm{Id} \mid \mathrm{Id}\right),$$

where $\otimes_K$ denotes the Kronecker product and $\mathrm{Id} \in \mathbb{R}^{d\times d}$ the identity matrix. Note that by Theorem 1.17, V is a strong solution of (A.13) such that the conditions of [37, Thm. 1.1] are satisfied. Thus, the solution is a rich Feller process if and only if

$$\nu_Z(A_r(x)) = \nu_Z(\{y \in \mathbb{R}^{d^2+d} : \sigma(x)y \in B(-x, r)\}) \overset{\|x\|\to\infty}{\to} 0 \tag{A.15}$$

holds for all $r > 0$, where $B(-x, r) \subset \mathbb{R}^d$ denotes the ball of radius r centered at $-x$. For the given SDE and $y = (y_1, y_2) \in \mathbb{R}^{d^2+d}$ with $y_1 \in \mathbb{R}^{d^2}$ and $y_2 \in \mathbb{R}^d$, the condition reads

$$\begin{aligned} \sigma(x)y \in B(-x, r) &\Leftrightarrow \|\sigma(x)y + x\| < r \\ &\Leftrightarrow \|y_1^{mat}x + y_2 + x\| = \|(y_1^{mat} + \mathrm{Id})x + y_2\| < r \end{aligned}$$

where $y_1^{mat} \in \mathbb{R}^{d\times d}$ is constructed from y_1 by reversing the vectorization. Assume now that $\nu_U(\{y_1 \in \mathbb{R}^{d^2} : \det(y_1^{mat} + \mathrm{Id}) = 0\}) = 0$. Whenever $\det(y_1^{mat} + \mathrm{Id}) \neq 0$, the linear mapping $x \mapsto (y_1^{mat} + \mathrm{Id})x$ is continuous, one-to-one and onto and takes the value $-y_2$ for exactly one $x_0 \in \mathbb{R}^d$. In particular, $x \mapsto \|(y_1^{mat} + \mathrm{Id})x + y_2\|$ is continuous and unbounded for $\|x\| \to \infty$ such that

$$\lim_{\|x\|\to\infty} \mathbb{1}_{A_r(x)\cap\{y=(y_1,y_2)\in\mathbb{R}^{d^2+d}:\det(y_1^{mat}+\mathrm{Id})\neq 0\}} = 0.$$

Since ν_Z is a Lévy measure on $\mathbb{R}^{d^2+d} \setminus \{(0, \ldots, 0)^T\}$ and $A_r(x)$ does not contain a neighborhood of the zero vector if $\|x\|$ is sufficiently large, it follows by dominated convergence that (A.15) is satisfied.

Next, let $m \in \mathbb{R}^{d\times d}$ with $\det(m+\mathrm{Id}) = 0$ and $\nu_U(\{m\}) > 0$. We show necessity by contradiction. Assume that $\nu_Z(A_r(x)) \to 0$ as $\|x\| \to \infty$. Since $\det(m+\mathrm{Id}) = 0$, the mapping $x \mapsto (m+\mathrm{Id})x$ is not one-to-one and onto. In particular, the linear system $(m+\mathrm{Id})x = 0$ has infinitely many solutions and there exists a unit vector $x_0 \in \mathbb{R}^d$ with $\ell(m+\mathrm{Id})x_0 = 0$ for all $\ell \in \mathbb{R}$. Choosing $x = \ell x_0$ yields

$$\|(m+\mathrm{Id})x + y_2\| = \|y_2\|,$$

i.e., $(m^{vec}, y_2) \in A_r(\ell x_0)$ for any $y_2 \in \mathbb{R}^d$ with $\|y\| < r$. Hence,

$$\lim_{\ell\to\infty} \nu_Z(A_r(\ell x_0)) = \nu_Z(\{m\} \times \{y_2 \in \mathbb{R}^d : \|y_2\| < r\}). \tag{A.16}$$

Observe that the right-hand side tends to $\nu_Z(\{m\} \times \mathbb{R}^d) = \nu_U(\{m\}) > 0$ as $r \to \infty$ and is, therefore, positive for large enough r, contradicting $\lim_{\|x\|\to\infty} \nu_Z(A_r(x)) = 0$. □

A similar result holds if we assume L, and hence V, to be matrix-valued. Note, however, that, unlike in one dimension, (A.12) is in general not necessary in the vector or matrix-valued case, as one can construct examples where $\nu_U(\cdot + \mathrm{Id})$ has mass, but no atoms, on the non-invertible matrices, but the condition given in [37] is still satisfied.

Example A.4. Set $d = 2$ and let U and L be two independent Lévy processes such that L is a Brownian motion and U is a compound Poisson process with $\nu_U(\cdot + \mathrm{Id})$ having mass on the non-invertible matrices, but no atoms, such that the jumps of the process are of the form

$$\Delta U_s + \mathrm{Id} = \begin{pmatrix} 1 & a \\ 1 & a \end{pmatrix} = y_1^{mat} + \mathrm{Id}$$

with some $a \in [0,1)$ that is distributed according to the one-dimensional Lebesgue measure λ restricted to $[0,1)$. In particular, (A.12) does not hold, as

$$\nu_U(\{m \in \mathbb{R}^{d\times d} : \det(m+\mathrm{Id}) = 0\}) = \lambda([0,1)) = 1 \neq 0.$$

Observe that for $x = (x_1, x_2)^T$ and $y_2 = (z_1, z_2)^T$ it is

$$\|(y_1^{mat} + \mathrm{Id})x + y_2\| = \sqrt{2(x_1 + a x_2)^2 + \|y_2\|^2 + 2(x_1 z_1 + a x_2 z_1 + x_1 z_2 + a x_2 z_2)}.$$

As L does not jump, we can set $y_2 = 0$, and to check (A.15) it is sufficient to show that the set

$$C_r(x) = \left\{ a \in [0,1) : (x_1 + a x_2)^2 < \frac{r^2}{2} \right\}$$

or, equivalently, the set

$$\widetilde{C}_r(x) = \left\{ a \in [0,1) : \left(1 + 2a \frac{x_1 x_2}{x_1^2 + x_2^2}\right) < \frac{r^2}{2\|x\|^2} \right\}$$

reduces to a Lebesgue null set as $\|x\| \to \infty$. Observe that the function

$$(0,0)^T \neq (x_1,x_2)^T \mapsto 2a\frac{x_1x_2}{x_1^2+x_2^2}$$

only takes values in the interval $[-a,a]$ such that the term on the left-hand side of the inequality defining the elements of $\widetilde{C}_r(x)$ is strictly positive regardless of the choice of the continuous curve $(x_1,x_2)^T = (f_1(t),f_2(t))^T$ with $\lim_{t\to\infty}\|(f_1(t),f_2(t))^T\| = \infty$. Thus, the set reduces to the empty set as $\|x\| \to \infty$. Since, by the choice of U and L, the Lévy measure of the process Z defined in (A.14) is finite, it follows

$$\lim_{\|x\|\to\infty} \nu_Z(A_r(x)) = 0,$$

by dominated convergence, showing that (A.15) may be satisfied even if (A.12) is not.

Next, we calculate the infinitesimal generator $\mathcal{A}^V$ of the mGOU process using results from [37]. Alternatively, $\mathcal{A}^V$ can be derived using the multivariate version of Itō's formula similar to the proof of [5, Thm. 3.1].

Lemma A.5. *Let (U,L) be an $\mathbb{R}^{d\times d}\times\mathbb{R}^d$-valued Lévy processes such that the components U and L are independent. Further, assume that* (A.12) *holds. Then the infinitesimal generator $\mathcal{A}^V$ of the mGOU process V satisfying the SDE*

$$\mathrm{d}V_t = \mathrm{d}U_tV_{t-} + \mathrm{d}L_t,\ t>0,\quad V_0 = v\in\mathbb{R}^d$$

acts on functions $f\in\mathbb{C}_c^\infty(\mathbb{R}^d)$ by

$$\begin{aligned}\mathcal{A}^V f(x) = \mathcal{A}^L f(x) &+ \frac{1}{2}\Big((x^T\otimes_K \mathrm{Id})A_U(x\otimes_K \mathrm{Id})\nabla\Big)^T\nabla f(x)\\ &+\gamma_U^T(x\otimes_K\mathrm{Id})\nabla f(x) \qquad\qquad (\mathrm{A}.17)\\ &+\int_{\mathbb{R}^{d^2}} f(x+s^T(x\otimes_K\mathrm{Id})) - f(x) - s^T(x\otimes_K\mathrm{Id})\nabla f(x)\mathbb{1}_D(s)\nu_U(\mathrm{d}s),\end{aligned}$$

where $\mathcal{A}^L$ is the infinitesimal generator of the Lévy process L as given in Theorem 1.8, D is the unit disk and $\otimes_K$ denotes the Kronecker product. Further, $C_c^2(\mathbb{R}^d)$ and $C_{0,pl}^2(\mathbb{R}^d)$ are contained in the domain of $\mathcal{A}^V$ and (A.17) *also holds for these functions. Here, the space $C_{0,pl}^2(\mathbb{R}^d)$ is given by*

$$C_{0,pl}^2(\mathbb{R}^d) = \Big\{f\in C_0^2(\mathbb{R}^d): \lim_{\|x\|\to\infty}\Big(\sum_{k=1}^d(1+\|x\|)\Big|\frac{\partial f}{\partial x_k}(x)\Big| + \sum_{k,l=1}^d(1+\|x\|)^2\Big|\frac{\partial^2 f}{\partial x_k\partial x_l}(x)\Big|\Big)=0\Big\}.$$

Proof. Whenever (A.12) is satisfied, [37, Thm. 1.1] yields the the infinitesimal generator of V as

$$\mathcal{A}^V f(x) = -\int_{\mathbb{R}^d} e^{ixy}q(x,y)\widehat{f}(y)\mathrm{d}y,\ f\in C_c^\infty(\mathbb{R}^d), x\in\mathbb{R}^d,$$

where $\widehat{f}$ denotes the Fourier transform of the function f and $q(x,y)$ is the so-called symbol of the process V, which can be calculated from the characteristic exponent ψ_Z of Z via $q(x,y) = \psi_Z(\sigma(x)^Ty)$. Thus, $\mathcal{A}^V$ is readily obtained from evaluating the integral. Using linearity and the independence of the processes U and L, we can calculate the contributions of the individual processes separately. The contribution of L is readily identified, as the terms obtained from the characteristic exponent ψ_L of L reproduce the infinitesimal generator of the Lévy process as given in Theorem 1.8, i.e.,

$$\begin{aligned}
&-\int_{\mathbb{R}^d} e^{ixy}\psi_L(y)\widehat{f}(y)\mathrm{d}y = \mathcal{A}^L f(x)\\
&= \frac{1}{2}\nabla^T A_L \nabla f(x) + \gamma_L^T \nabla f(x) + \int_{\mathbb{R}^d} f(x+s) - f(x) - \nabla^T f(x) s \mathbb{1}_D(s)\nu_L(\mathrm{d}s).
\end{aligned}$$

For the contribution of U, the Gaussian, drift and jump part can be treated separately. Starting with the Gaussian part, observe that

$$\begin{aligned}
&-\frac{1}{2}\int_{\mathbb{R}^d} e^{ixy} y^T (x^T \otimes_K \mathrm{Id}) A_U (x \otimes_K \mathrm{Id}) y \widehat{f}(y)\mathrm{d}y\\
&= \frac{1}{2}\Big((x^T \otimes_K \mathrm{Id}) A_U (x \otimes_K \mathrm{Id})\nabla\Big)^T \nabla f(x)
\end{aligned}$$

since every multiplication by some y_i in the integrand yields a partial derivative when the inverse Fourier transform is considered. Note that, since A_U is symmetric, the matrix $(x^T \otimes_K \mathrm{Id}) A_U (x \otimes_K \mathrm{Id})$ is symmetric as well. Further, a similar calculation yields the contribution of the drift part, which is given by

$$-\int_{\mathbb{R}^d} e^{ixy} i\langle \gamma_U, (x \otimes_K \mathrm{Id})y\rangle \widehat{f}(y)\mathrm{d}y = \gamma_U^T (x \otimes_K \mathrm{Id})\nabla f(x).$$

Lastly, we obtain for the integral term

$$\begin{aligned}
&-\int_{\mathbb{R}^d}\int_{\mathbb{R}^{d^2}} e^{ixy}\Big(\exp(i\langle s, (x \otimes_K \mathrm{Id})y\rangle) - 1 - i\langle s, (x \otimes_K \mathrm{Id})y\rangle \mathbb{1}_D(s)\Big)\nu_U(\mathrm{d}s)\widehat{f}(y)\mathrm{d}y\\
&= \int_{\mathbb{R}^{d^2}} f(x + s^T(x \otimes_K \mathrm{Id})) - f(x) - s^T(x \otimes_K \mathrm{Id})\nabla f(x)\mathbb{1}_D(s)\nu_U(\mathrm{d}s),
\end{aligned}$$

by observing that the first term of each integral yields a phase shift. Summing up the individual terms yields $A^V f(x)$ for $f \in C_c^\infty(\mathbb{R}^d)$ as claimed.

It follows from [37, Thm. 1.1] that $C_c^2(\mathbb{R}^d) \subset \mathrm{dom}(G)$ and it is further readily checked that the right-hand side of (A.17) is also well-defined for $f \in C_{0,pl}^2(\mathbb{R}^d)$ by generalizing the bounds obtained in [5] to higher dimensions. Denote the integro-differential operator on the right-hand side of (A.17) by H and observe that

$$\begin{aligned}
\Big|\gamma_U^T (x \otimes_K \mathrm{Id})\nabla f(x) + \gamma_L^T \nabla f(x)\Big| &\le \max_j \gamma_j \sum_{k=1}^d \Big(1 + \sum_{l=1}^d |x_l|\Big)\Big|\frac{\partial f}{\partial x_l}(x)\Big|\\
&\le C \max_j \gamma_j \sum_{k=1}^d (1 + \|x\|)\Big|\frac{\partial f}{\partial x_l}(x)\Big|
\end{aligned}$$

by the equivalence of norms on $\mathbb{R}^{d^2}$ and that

$$\begin{aligned}
&\Big|\frac{1}{2}\Big((x^T \otimes_K \mathrm{Id})A_U(x \otimes_K \mathrm{Id})\nabla\Big)^T \nabla f(x) + \frac{1}{2}\nabla^T A_L \nabla f(x)\Big| \\
&\leq \frac{1}{2}\max_{i,j}|A_{i,j}| \sum_{k,l=1}^{d}\Big(1 + \sum_{m,n=1}^{d}|x_m x_n|\Big)\Big|\frac{\partial^2 f}{\partial x_k \partial x_j}(x)\Big| \\
&\leq \frac{1}{2}\max_{i,j}|A_{i,j}| \sum_{k,l=1}^{d}(1+\|x\|)^2\Big|\frac{\partial^2 f}{\partial x_k \partial x_j}(x)\Big|
\end{aligned}$$

by the Hölder inequality. Denoting $\|f\|_\infty = \sup_{x\in\mathbb{R}^d}|f(x)|$, we further obtain

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^{d^2}} f(x + s^T(x \otimes_K \mathrm{Id})) - f(x) - s^T(x \otimes_K \mathrm{Id})\nabla f(x)\mathbb{1}_D(s)\nu_U(\mathrm{d}s)\Big| \\
&\leq \sum_{k,l=1}^{d}\|x\|^2\Big|\frac{\partial^2 f}{\partial x_k \partial x_l}(x)\Big|\int_D \|s\|^2\nu_U(\mathrm{d}s) + 2\|f\|_\infty \nu_U(\mathbb{R}^{d^2}\setminus D)
\end{aligned}$$

by splitting the integral and applying the Taylor formula for the integrand in the first term and a general estimate for the second one. Similarly,

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^{d^2}}\Big(f(x+s) - f(x) - \nabla^T f(x)s\mathbb{1}_D(s)\Big)\nu_L(\mathrm{d}s)\Big| \\
&\leq \sum_{k,l=1}^{d}\Big|\frac{\partial^2 f}{\partial x_k \partial x_l}(x)\Big|\int_D \|s\|^2\nu_L(\mathrm{d}s) + 2\|f\|_\infty \nu_L(\mathbb{R}^d\setminus D),
\end{aligned}$$

which also yields a finite bound. Hence, $Hf(x)$ is indeed in $C_0(\mathbb{R}^d)$ for the functions f considered, as dominated convergence is applicable for the integrals. We can thus conclude that both $C_c^2(\mathbb{R}^d)$ and $C_{0,pl}^2(\mathbb{R}^d)$ are contained in $\mathrm{dom}(H)$. To see that H extends uniquely to the desired spaces, observe that the above bound for $|Hf(x)|$ can be made uniform in $x \in \mathbb{R}^d$ by taking the supremum and equip $C_c^\infty(\mathbb{R}^d)$, $C_c^2(\mathbb{R}^d)$ or $C_{0,pl}^2(\mathbb{R}^d)$, respectively, with the norm

$$\|f\|_{pl} = \|f\|_\infty + \sum_{k=1}^{d}\Big\|(1+\|x\|)\frac{\partial f}{\partial x_k}\Big\|_\infty + \sum_{k,l=1}^{d}\Big\|(1+\|x\|)^2\frac{\partial^2 f}{\partial x_k \partial x_l}\Big\|_\infty.$$

It now follows that there is a constant $C \geq 0$ such that

$$\|Hf(x)\|_\infty \leq C\|f\|_{pl},$$

if f is taken from any one of the considered function spaces. Since $\mathcal{A}^V$ is a closed operator, H, interpreted as an operator $(C_c^\infty(\mathbb{R}^d), \|.\|_{pl}) \to (C_0(\mathbb{R}^d, \|.\|_\infty)$, is bounded and hence continuous, $C_c^\infty(\mathbb{R}^d)$ is dense in both $C_c^2(\mathbb{R}^d)$ and $C_{0,pl}^2(\mathbb{R}^d)$, and $(C_0(\mathbb{R}^d), \|.\|_\infty)$ is a Banach space, H uniquely extends to the desired spaces. $\square$

Again, a similar result holds if we assume L, and hence V, to be matrix-valued. Observe in particular that the structure of the infinitesimal generator in (A.17) mirrors (2.28) in the one-dimensional case and that, similar to [5, Thm. 3.1, Cor. 3.3], it can also be calculated without the assumption of independence, yielding a more general form. Letting τ denote an exponentially distributed random variable and defining

$$V_{q,U,L} = \int_{0+}^{\tau} \overleftarrow{\mathcal{E}}(U)_{s-} \mathrm{d}L_s \tag{A.18}$$

now gives the multivariate analog for the killed exponential functional and thus the starting point for an analysis similar to Chapter 2 for the multivariate case.

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Erklärung

Hiermit versichere ich, Jana Katharina Reker, dass ich die vorliegende Arbeit selbständig angefertigt habe und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt sowie die wörtlich oder inhaltlich übernommenen Stellen als solche kenntlich gemacht habe. Ich erkläre außerdem, dass diese Arbeit weder im In- noch im Ausland in dieser oder ähnlicher Form in einem anderen Promotionsverfahren vorgelegt wurde.

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