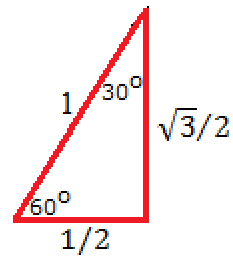
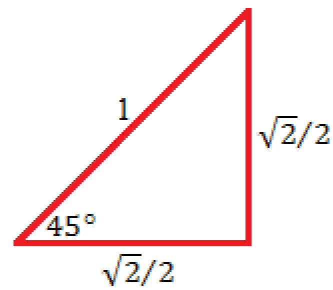
Exact Values of the Sine and Cosine Functions in Increments of 3 degrees

The sine and cosine values for all angle measurements in multiples of 3 degrees can be represented in terms of square-root radicals, and the four common operations of arithmetic. These values can be determined geometrically using three useful right triangles.

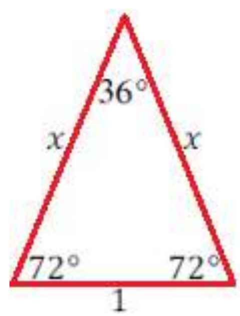
For an equilateral triangle, each of the three interior angles are 60º. Viewing half the triangle gives a 30-60-90 right triangle, with the hypotenuse measuring 1, the short leg $\frac{1}{2}$, and the long leg $\frac{\sqrt{3}}{2}$. The right-triangle trigonometry definitions $\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$ and $\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$, gives $\sin 30° = \frac{1}{2}$, $\cos 30° = \frac{\sqrt{3}}{2}$, $\sin 60° = \frac{\sqrt{3}}{2}$ and $\cos 60° = \frac{1}{2}$.

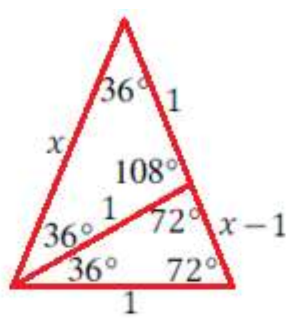


From a square, with interior angles 90º, draw a diagonal from one corner to its opposite corner. This forms a right triangle with two of the square's sides being the legs of the triangle, and the drawn diagonal being the hypotenuse. Let the hypotenuse be 1 unit in length. Then the legs have length $\frac{\sqrt{2}}{2}$ units each. This gives $\sin 45° = \frac{\sqrt{2}}{2}$ and $\cos 45° = \frac{\sqrt{2}}{2}$.



From the isosceles 36-72-72 triangle, let the two long sides be x, and the short side 1. Bisect one of the $72°$ angles, extending the ray to intersect the other side of the triangle. This forms two triangles: an isosceles 36-36-108 triangle whose short sides are both 1, and a smaller isosceles 36-72-72 triangle whose long sides are 1, and short side is $x-1$, as shown in the diagrams below:





The two 36-72-72 triangles are proportional: The ratio of the long sides $x{:}\,1$ is the same proportion as the ratio of the short sides: $1{:}\,(x-1)$. Solve for x by equating two ratios: $\frac{x}{1} = \frac{1}{x-1}$, which then gives $x^2 - x - 1 = 0$. Solving for x using the quadratic formula (and ignoring the negative root) gives $x = \frac{1+\sqrt{5}}{2}$. Splitting the 36-36-108 triangle in half forms a right triangle, a 36-54-90 triangle, with hypotenuse 1, long leg $\frac{x}{2}$ and short leg $\sqrt{1-\left(\frac{x}{2}\right)^2}$. Since $x = \frac{1+\sqrt{5}}{2}$, the long leg of the 36-54-90 triangle is $\frac{1+\sqrt{5}}{4}$ and the short leg is $\frac{\sqrt{10-2\sqrt{5}}}{4}$.

Therefore:

$$\sin 36^\circ = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{\sqrt{10 - 2\sqrt{5}}/4}{1} = \frac{\sqrt{10 - 2\sqrt{5}}}{4},$$

$$\cos 36^\circ = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{(1 + \sqrt{5})/4}{1} = \frac{1 + \sqrt{5}}{4}.$$

The number $\frac{1+\sqrt{5}}{2}$ is called the *golden ratio* and is denoted by ϕ. Thus, we can write

$$\sin 36^\circ = \frac{\sqrt{4 - \phi^2}}{2} \quad \text{and} \quad \cos 36^\circ = \frac{\phi}{2}.$$

The remaining exact representations for angles of multiples of 3° can now be found, using sum-difference and half-angle identities. For example, $\sin 6^\circ = 2 \sin 3^\circ \cos 3^\circ$, and so on. Since different identities may be used, the exact representations may "look" different than other possible representations, but can be shown to be identical in value. The following is a table of all exact values for the sine and cosine of angles of multiples of 3°, up through 45°. All radicals were simplified so that none contained any quotients within them. Whenever appropriate, the equivalent representation in terms of the golden ratio ϕ is given.

All of these values are algebraic, meaning they are the root of some polynomial with integer coefficients. For example, $\sin 90^\circ$ is algebraic. It equals 1, which is the root of the polynomial $x - 1$. Given an algebraic number in the form of radicals and arithmetic operations, one can build a polynomial for which the given number is a root. For example, We know that $\sin 45^\circ = \frac{\sqrt{2}}{2}$, so we set $x = \frac{\sqrt{2}}{2}$, then square both sides and collect terms to one side, we get $x^2 - \frac{1}{2} = 0$. Thus, $2x^2 - 1$ is a polynomial for which $\frac{\sqrt{2}}{2}$ is a root.

Some pairs of these expressions share a common minimal polynomial. For example, $\cos 18^\circ$ and $\sin 36^\circ$ both have the same minimal polynomial, $16x^4 - 20x^2 + 5$.

In the tables that follow, the exact representations are given for angles of multiples of 3 degrees, and the minimal polynomials for each.

Angle: 0º or 0 radians	sin 0º cos 90º	cos 0º sin 90º
Value:	0	1
Minimal Polynomial:	x	$x-1$

Angle: 3º or $\frac{\pi}{60}$ radians	sin 3º cos 87º	cos 3º sin 87º
Decimal Approximation:	0.052335956…	0.998629534…
Representation in radical form:	$\frac{\sqrt{8-\sqrt{3}-\sqrt{15}-\sqrt{10-2\sqrt{5}}}}{4}$	$\frac{\sqrt{8+\sqrt{3}+\sqrt{15}+\sqrt{10-2\sqrt{5}}}}{4}$
Minimal Polynomial:	$65{,}536x^{16}-262{,}144x^{14}+430{,}080x^{12}-372{,}736x^{10}+182{,}784x^{8}-50{,}176x^{6}+7{,}040x^{4}-384x^{2}+1$	

Angle: 6º or $\frac{\pi}{30}$ radians	sin 6º cos 84º	cos 6º sin 84º
Decimal Approximation:	0.104528463…	0.994521895…
Representation in radical form:	$\frac{\sqrt{9-\sqrt{5}-\sqrt{30+6\sqrt{5}}}}{4}$	$\frac{\sqrt{7+\sqrt{5}+\sqrt{30+6\sqrt{5}}}}{4}$
Minimal Polynomial:	$16x^4+8x^3-16x^2-8x+1$	$256x^8-448x^6+224x^4-32x^2+1$

Angle: 9º or $\frac{\pi}{20}$ radians	sin 9º cos 81º	cos 9º sin 81º
Decimal Approximation:	0.156434465…	0.987688341…
Representation in radical form:	$\frac{\sqrt{8-2\sqrt{10+2\sqrt{5}}}}{4}$	$\frac{\sqrt{8+2\sqrt{10+2\sqrt{5}}}}{4}$
Representation in terms of ϕ:	$\frac{\sqrt{2-\sqrt{2+\phi}}}{2}$	$\frac{\sqrt{2+\sqrt{2+\phi}}}{2}$
Minimal Polynomial:	$256x^8-512x^6+304x^4-48x^2+1$	
Minimal Polynomial, ϕ form:	$16x^4-16x^2+2-\phi$	

Angle: 12º or $\frac{\pi}{15}$ radians	sin 12º cos 78º	cos 12º sin 78º
Decimal Approximation:	0.207911690…	0.978147600…
Representation in radical form:	$\frac{\sqrt{7-\sqrt{5}-\sqrt{30-6\sqrt{5}}}}{4}$	$\frac{-1+\sqrt{5}+\sqrt{30+6\sqrt{5}}}{8}$
Minimal Polynomial:	$256x^8-448x^6+224x^4-32x^2+1$	$16x^4+8x^3-16x^2-8x+1$

Angle: 15º or $\frac{\pi}{12}$ radians	sin 15º cos 75º	cos 15º sin 75º
Decimal Approximation:	0.258819045…	0.965925826…
Representation in radical form:	$\frac{\sqrt{6}-\sqrt{2}}{4}$	$\frac{\sqrt{6}+\sqrt{2}}{4}$
Minimal Polynomial:	$16x^4 - 16x^2 + 1$	

Angle: 18º or $\frac{\pi}{10}$ radians	sin 18º cos 72º	cos 18º sin 72º
Decimal Approximation:	0.309016994…	0.951056516…
Representation in radical form:	$\frac{\sqrt{5}-1}{4}$	$\frac{\sqrt{10+2\sqrt{5}}}{4}$
Representation in terms of $\boldsymbol{\phi}$:	$\frac{\phi - 1}{2}$	$\frac{\sqrt{2+\phi}}{2}$
Minimal Polynomial:	$4x^2 + 2x - 1$	$16x^4 - 20x^2 + 5$

Angle: 21º or $\frac{7\pi}{60}$ radians	sin 21º cos 69º	cos 21º sin 69º
Decimal Approximation:	0.358367949…	0.933580426…
Representation in radical form:	$\frac{\sqrt{8+\sqrt{3}-\sqrt{15}-\sqrt{10+2\sqrt{5}}}}{4}$	$\frac{\sqrt{8-\sqrt{3}+\sqrt{15}+\sqrt{10+2\sqrt{5}}}}{4}$
Minimal Polynomial:	$65{,}536x^{16}-262{,}144x^{14}+430{,}080x^{12}-372{,}736x^{10}+182{,}784x^{8}-50{,}176x^{6}+7{,}040x^{4}-384x^{2}+1$	

Angle: 24º or $\frac{2\pi}{15}$ radians	sin 24º cos 66º	cos 24º sin 66º
Decimal Approximation:	0.406736643…	0.913545457…
Representation in radical form:	$\frac{\sqrt{7+\sqrt{5}-\sqrt{30+6\sqrt{5}}}}{4}$	$\frac{1+\sqrt{5}+\sqrt{30-6\sqrt{5}}}{8}$
Minimal Polynomial:	$256x^{8}-448x^{6}+224x^{4}-32x^{2}+1$	$16x^{4}-8x^{3}-16x^{2}+8x+1$

Angle: 27º or $\frac{3\pi}{20}$ radians	sin 27º cos 63º	cos 27º sin 63º
Decimal Approximation:	0.453990499…	0.891006524…
Representation in radical form:	$\frac{\sqrt{8-2\sqrt{10-2\sqrt{5}}}}{4}$	$\frac{\sqrt{8+2\sqrt{10-2\sqrt{5}}}}{4}$
Representation in terms of ϕ:	$\frac{\sqrt{2-\sqrt{3-\phi}}}{2}$	$\frac{\sqrt{2+\sqrt{3-\phi}}}{2}$
Minimal Polynomial:	$256x^8-512x^6+304x^4-48x^2+1$	
Minimal Polynomial, ϕ form:	$16x^4-16x^2+1+\phi$	

Angle: 30º or $\frac{\pi}{6}$ radians	sin 30º cos 60º	cos 30º sin 60º
Decimal Approximation:	0.5	0.866025403…
Representation in radical form:	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
Minimal Polynomial:	$2x-1$	$4x^2-3$

Angle: 33º or $\frac{11\pi}{60}$ radians	sin 33º cos 57º	cos 33º sin 57º
Decimal Approximation:	0.54463903…	0.838670567…
Representation in radical form:	$\frac{\sqrt{8-\sqrt{3}-\sqrt{15}+\sqrt{10-2\sqrt{5}}}}{4}$	$\frac{\sqrt{8+\sqrt{3}+\sqrt{15}-\sqrt{10-2\sqrt{5}}}}{4}$
Minimal Polynomial:	$65{,}536x^{16}-262{,}144x^{14}+430{,}080x^{12}-372{,}736x^{10}+182{,}784x^{8}-50{,}176x^{6}+7{,}040x^{4}-384x^{2}+1$	

Comment: the polynomial

$$65{,}536x^{16}-262{,}144x^{14}+430{,}080x^{12}-372{,}736x^{10}+182{,}784x^{8}-50{,}176x^{6}+7{,}040x^{4}-384x^{2}+1$$

Has roots ± sin 3º, ± cos 3º, ± sin 21º, ± cos 21º, ± sin 33º, ± cos 33º, ± sin 39º and ± cos 39º. Thus, the above polynomial can be factored as

$$65{,}536(x^2-\sin^2 3°)(x^2-\cos^2 3°)(x^2-\sin^2 21°)(x^2-\cos^2 21°)(x^2-\sin^2 33°)(x^2-\cos^2 33°)(x^2-\sin^2 39°)(x^2-\cos^2 39°).$$

Angle: 36º or $\frac{\pi}{5}$ radians	sin 36º cos 54º	cos 36º sin 54º
Decimal Approximation:	0.587785252…	0.809016994…
Representation in radical form:	$\frac{\sqrt{10-2\sqrt{5}}}{4}$	$\frac{1+\sqrt{5}}{4}$
Representation in terms of ϕ:	$\frac{\sqrt{3-\phi}}{2}$	$\frac{\phi}{2}$
Minimal Polynomial:	$16x^4-20x^2+5$	$4x^2-2x-1$

Angle: 39º or $\frac{13\pi}{60}$ radians	sin 39º cos 51º	cos 39º sin 51º
Decimal Approximation:	0.62932039…	0.777145961…
Representation in radical form:	$\frac{\sqrt{8-\sqrt{3}+\sqrt{15}-\sqrt{10+2\sqrt{5}}}}{4}$	$\frac{\sqrt{8+\sqrt{3}-\sqrt{15}+\sqrt{10+2\sqrt{5}}}}{4}$
Minimal Polynomial:	$65{,}536x^{16}-262{,}144x^{14}+430{,}080x^{12}-372{,}736x^{10}+182{,}784x^{8}-50{,}176x^{6}+7{,}040x^{4}-384x^{2}+1$	

Angle: 42º or $\frac{7\pi}{30}$ radians	sin 42º cos 48º	cos 42º sin 48º
Decimal Approximation:	0.669130606…	0.743144825…
Representation in radical form:	$\frac{1-\sqrt{5}+\sqrt{30+6\sqrt{5}}}{8}$	$\frac{\sqrt{7-\sqrt{5}+\sqrt{30-6\sqrt{5}}}}{4}$
Minimal Polynomial:	$16x^4-8x^3-16x^2+8x+1$	$256x^8-448x^6+224x^4-32x^2+1$

Angle: 45º or $\frac{\pi}{4}$ radians	sin 45°	cos 45°
Decimal Approximation:	0.707106781…	0.707106781…
Representation in radical form:	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
Minimal Polynomial:	$2x^2-1$	$2x^2-1$

Angles That Are Not Multiples of 3º.

Consider the sine triple-angle identity:

$$\sin(3\theta) = -4\sin^3\theta + 3\sin\theta.$$

Letting $\theta = 10°$, we have

$$\sin 30° = -4\sin^3 10° + 3\sin 10°, \qquad \text{or} \qquad \tfrac{1}{2} = -4\sin^3 10° + 3\sin 10°.$$

Multiplying by 2, we have

$$1 = -8\sin^3 10° + 6\sin 10°.$$

Thus, the polynomial $8x^3 - 6x + 1 = 0$ has $\sin 10°$ as a root, so that $\sin 10°$ is algebraic. Equivalently, the minimal polynomial for $\sin 10°$ is $8x^3 - 6x + 1$.

The cosine triple-angle identity is similar:

$$\cos(3\theta) = 4\cos^3\theta - 3\cos\theta.$$

Letting $\theta = 40°$, we have

$$\cos 120° = 4\cos^3 40° - 3\cos 40°, \qquad \text{or} \qquad -\tfrac{1}{2} = 4\cos^3 40° - 3\cos 40°.$$

Multiplying by –2, we have

$$1 = -8\cos^3 40° + 6\cos 40°.$$

This means that $\cos 40°$ is also a root of the polynomial $8x^3 - 6x + 1$.

Now, letting $\theta = 20°$, we have the following identity:

$$\cos 60° = 4\cos^3 20° - 3\cos 20°, \qquad \text{or} \qquad \tfrac{1}{2} = 4\cos^3 20° - 3\cos 20°.$$

Thus, $\cos 20°$ is a root of $8x^3 - 6x + 1$. This means that the polynomial $8x^3 - 6x + 1$ can be factored as

$$8x^3 - 6x + 1 = 8(x - \sin 10°)(x - \cos 40°)(x + \cos 20°).$$

A similar approach shows that $\cos 10°$, $\sin 40°$ and $\sin 20°$ are all roots of

$$64x^6 - 96x^4 + 36x^2 - 3,$$

and that this polynomial can be factored as

$$64x^6 - 96x^4 + 36x^2 - 3 = 64(x^2 - \sin^2 40°)(x^2 - \cos^2 10°)(x^2 - \sin^2 20°).$$

Another interesting case involves the angles 5°, 25° and 35°. The polynomial

$$4{,}096x^{12} - 12{,}288x^{10} + 13{,}824x^{8} - 7{,}168x^{6} + 1{,}680x^{4} - 144x^{2} + 1$$

has $\pm\sin 5°$, $\pm\cos 5°$, $\pm\sin 25°$, $\pm\cos 25°$, $\pm\sin 25°$ and $\pm\cos 35°$ as roots. This polynomial can be factored as

$$4{,}096(x^2 - \sin^2 5°)(x^2 - \cos^2 5°)(x^2 - \sin^2 25°)(x^2 - \cos^2 25°)(x^2 - \sin^2 35°)(x^2 - \cos^2 35°).$$

In this manner, we have taken care of all angles that are multiples of 5º, but not of 3º.

What About the Sine of 1°?

Using the sine triple-angle identity, we can try to brute force a minimal polynomial that has sin 1º as a root. We start with

$$\sin 3° = -4\sin^3 1° + 3\sin 1°.$$

Replacing sin 3º with its exact radical representation gives

$$\frac{\sqrt{8 - \sqrt{3} - \sqrt{15} - \sqrt{10 - 2\sqrt{5}}}}{4} = -4\sin^3 1° + 3\sin 1°.$$

We can go a few more steps. Multiply through by 4, and let $x = \sin 1º$:

$$\sqrt{8 - \sqrt{3} - \sqrt{15} - \sqrt{10 - 2\sqrt{5}}} = -16x^3 + 12x.$$

Square both sides and subtract 8:

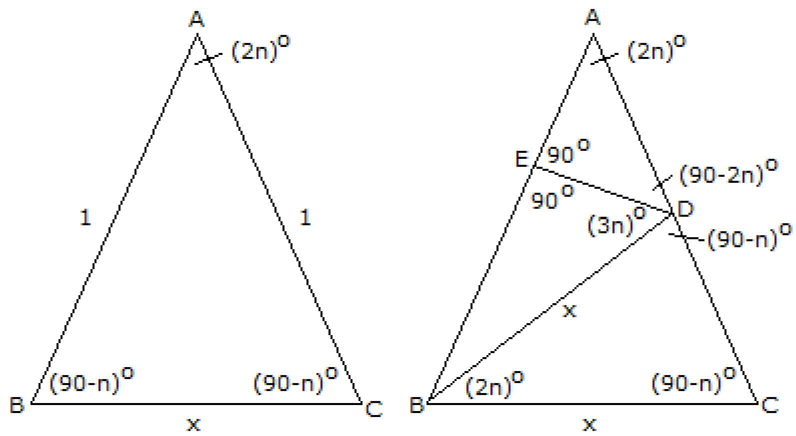
$$-\sqrt{3} - \sqrt{15} - \sqrt{10 - 2\sqrt{5}} = 256x^6 - 384x^4 + 144x^2 - 8.$$

Square both sides again and simplifying gives

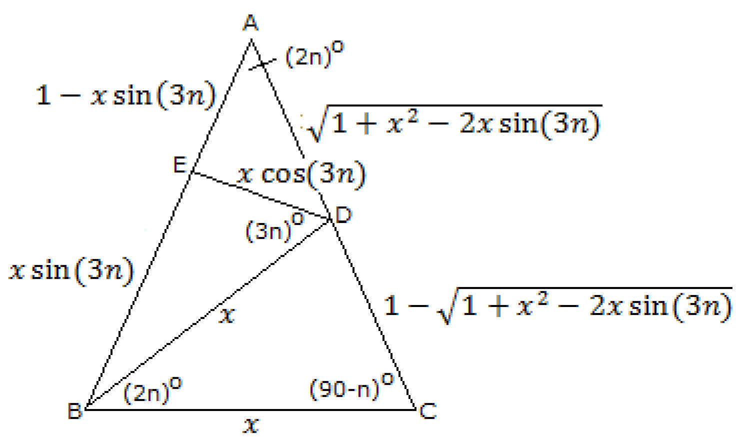
$$\sqrt{5} + \sqrt{30 + 6\sqrt{5}} = 16{,}384x^{12} - 49{,}152x^{10} + 55{,}296x^{8} - 28{,}672x^{6} + 6{,}720x^{4} - 576x^{2} + 9.$$

Unfortunately, this is going nowhere fast, even though it should eventually work. The problem is that the next squaring step does not appreciably simplify the left side into fewer radicals, yet expands the right side into a 24th-power polynomial. You get the idea.

The following construction results in a general way to illustrate the value of $\sin n°$. Start with an isosceles triangle ΔABC with two sides of length 1, and the remaining side of length x. Let the angle opposite x be $(2n)°$.



In the above diagram, continue the construction as follows: Draw segment BD such that its length is also x, then draw segment DE such that it meets segment AB at a right angle. We can now label the various lengths as follows: $|DE| = x\cos(3n)$ and $|BE| = x\sin(3n)$, so therefore, $|AE| = 1 - x\sin(3n)$. The Pythagorean formula gives the length of $|AD| = \sqrt{1 + x^2 - 2x\sin(3n)}$. In turn, the length $|CD| = 1 - \sqrt{1 + x^2 - 2x\sin(3n)}$. This is shown in the following figure:



Now, drop a perpendicular from A to segment BC, and also a perpendicular from B to segment CD. In doing so, we have split the angle measurement $(2n)°$ into $n°$. Importantly, note that triangles ΔABC and ΔBCD are proportional. We can now define $\sin n°$ in two ways using the "opposite over hypotenuse" construction for right angles:

$$\sin n° = \frac{x}{2} \text{ and } \sin n° = \frac{1 - \sqrt{1 + x^2 - 2x\sin(3n)}}{2x}.$$

Relating the two expressions, we have:

$$\frac{1 - \sqrt{1 + x^2 - 2x\sin(3n)}}{2x} = \frac{x}{2}.$$

This simplifies to $x^2 = 1 - \sqrt{1 + x^2 - 2x\sin(3n)}$. After squaring away the radical, the equation becomes

$$x^4 - 3x^2 + 2x\sin(3n) = 0$$

Since $x = 0$ produces a trivial case, we ignore it and divide out by x:

$$x^3 - 3x + 2\sin(3n) = 0$$

This cubic polynomial has three roots. Let a be the positive root of this polynomial that is closest to 0. Therefore, $\sin(n°) = \frac{a}{2}$, or $a = 2\sin(n°)$. For example, to find $\sin 1°$, let $n = 1$ and we get have $x^3 - 3x + 2\sin(3°) = 0$. A calculator shows that $a = 0.0349048$ ... is a root of this polynomial. Therefore, $\sin 1° = \frac{0.0349048}{2} = 0.0174524$, which is also confirmed via calculator.

Prepared by Scott Surgent (surgent@asu.edu) Please report errors to me if you see one. Updated Nov-2018
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