# A PROOF OF MENGER'S THEOREM

Here is a more detailed version of the proof of Menger's theorem on page 50 of Diestel's book.

First let's clarify some details about "separating." Given two sets of vertices $A$ and $B$ in $G$, a third set of vertices $W$ *separates* $A$ from $B$ if every path from a vertex in $A$ to a vertex in $B$ contains a vertex from $W$.

We say that a path is an $A$-$B$ path if it's first vertex is in $A$, it's last vertex is $B$, and none of its internal vertices is in $A$ or $B$. If every $A$-$B$ path contains a vertex of $W$ then $W$ is separating. (The argument here is similar to that used to show that any two vertices connected by a walk are connected by a path.)

So as a special case, $W = A$ separates $A$ from $B$ since a path that starts in $A$ includes at least that vertex from $A$. Thus the definition is not exactly the same as saying that after removing $W$ there remains a vertex in $A$ and a vertex in $B$ that can no longer be connected.

Let us define $k(G, A, B)$ to be the smallest number of vertices in a set that separates $A$ from $B$.

Since either $A$ or $B$ separates $A$ from $B$,

$$k(G, A, B) \leq \min(|A|, |B|). \tag{1}$$

Another annoying special case is when there are no $A$-$B$ paths. Then any set separates $A$ from $B$. In this case $k(G, A, B) = 0$. I got that wrong in lecture.

An important special case in what follows is when $A \subseteq B$. Then the paths of length zero that begin and end at a vertex in $A$ don't go through any vertices that are not in $A$. So a set cannot separated $A$ from $B$ unless is contains $A$. Therefore

$$A \subseteq B \implies k(G, A, B) = |A|. \tag{2}$$

If $A$ is given and there are two sets $B_1$ and $B_2$, with $B_1 \subseteq B_2$, then any set that separates $A$ from $B_2$ will necessarily separate $A$ from $B_1$. Therefore

$$B_1 \subseteq B_2 \implies k(G, A, B_1) \leq k(G, A, B_2) \tag{3}$$

**Theorem 1.** *Let $G$ be a graph with edge set $E$ and vertex set $V$. Suppose $A$ and $B$ are subsets of $V$ and suppose there is at least one $A$-$B$ path. Then the minimum number of vertices separating $A$ from $B$ equals the maximum number of disjoint $A$-$B$ paths*

We'll prove something a little stronger:

**Lemma 1.** *Let $k = k(G, A, B)$. Suppose $k(G, A, B) = k$. Given fewer than $k$ disjoint $A$-$B$ paths*

$$P_1, P_2, \ldots, P_n,$$

*(so $0 \leq n \leq k - 1$) there will exists $n + 1$ $A$-$B$ paths*

$$Q_1, Q_2, \ldots, Q_{n+1}$$

*such that if $b \in B$ is the endpoint of one of the $P_j$ then $b$ will also be an endpoint of one of the $Q_j$.*

(In the book, he keeps track of the $A$-enpoints, but this is not important in making the proof work.)

*Proof.* We prove this by induction on then number $\beta$ of vertices *not* in $B$, so

$$\beta = |G| - |B|.$$

Our base case is $\beta = 0$. This means that $B = G$. By equation 2 we have

$$k = |A|.$$

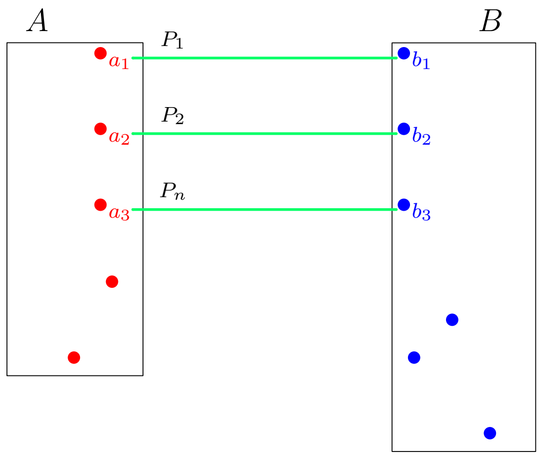
An $A$-$B$ path is just any path of length zero that starts and ends at a vertex $a$ in $A$. Given fewer than $|A|$ disjoint $A$-$B$ paths, we are really looking at fewer than $|A|$ elements of $A$. To this we can add another path of length zero at one of the remaining vertices in $A$ and this gives our longer list of $A$-$B$ paths.

Now we asssume the lemma is true for all $\beta < \beta_0$, where $\beta_0 \geq 1$. We now attempt to prove the lemma for $\beta = \beta_0$.

Suppose we are given $G$, $A$ and $B$ with $k(G, A, B) = k$ and where there are $\beta$ vertices in $G$ that are not in $B$. Suppose further that we are given $P_1, \ldots, P_n$ disjoint $A$-$B$ paths with $n < k$.

Let the set of endpoints of the $P_j$ in $A$ be $a_j$ and the endpoint in $B$ be $b_j$. We will us a line indicate a path with an unknown number of internal vertices. Since we might have paths of length zero, it is possible that the two endpoints drawm are really the same vertex. In the drawings we will assume $n = 3$.

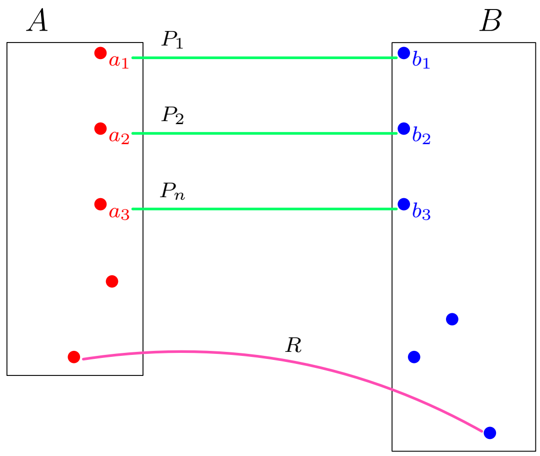
Here are $P_1, \ldots, P_n$, in green:




Since $n < k$, the set $\{b_1, \ldots, b_n\}$ does not separate $A$ from $B$. Therefore there is an $A$-$B$ path $R$ that does end at or go through any of the $b_j$. If we are luck, this path does not contain any of the vertices from the other paths. In that case, we are done, with

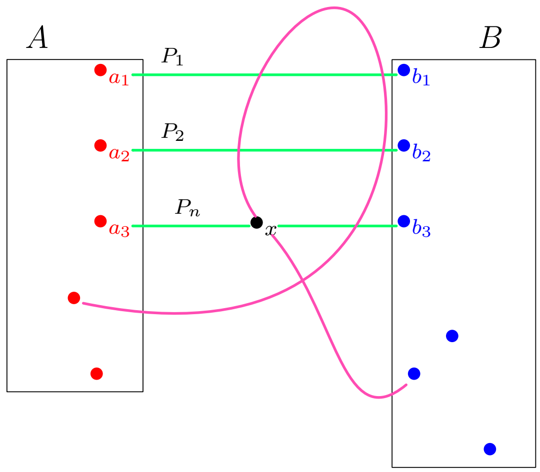
$$Q_1 = P_1, \ldots, Q_n = P_n \text{and} Q_{n+1} = R,$$

as shown here

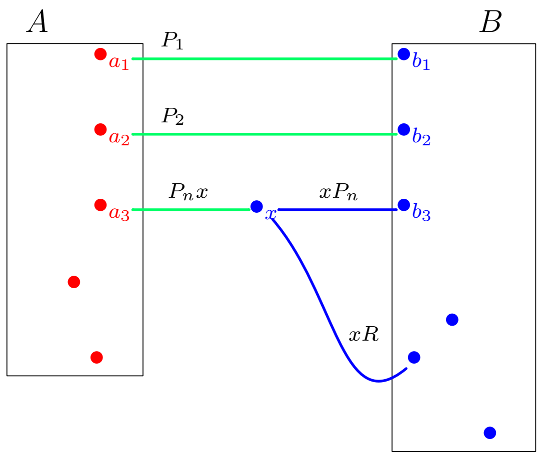



If this is not the case, let $x$ denote the vertex that is the last one on the path $R$ that is also on one of the paths $P_j$. We can reindex the $P_j$,

$a_j$ and $b_j$ so that $x$ is on the path $P_n$.




We have no need for the part of $R$ before $x$. We do need $xR$, the part of $R$ from $x$ on, which we show in blue. We also need $xP_n$, the part of $P_n$ after $x$, which we also show in blue. Finally, we need $P_nx$, the part of $P_n$ before $x$, which we show in green.




Let $B'$ equal all the vertices in $B$ together with all the vertices on the blue paths $xR$ and $xP_n$. Since $B \subseteq B'$ we know by equation 3 that

$$n < k \leq k(B, A, B')$$

Therefore we can apply the induction hypothesis to the strictly larger set $B'$ and the $n$ paths

$$P_1, \ldots, P_{n-1}, P_nx.$$

These have endpoints

$$b_1, \ldots, b_{n-1}, x$$

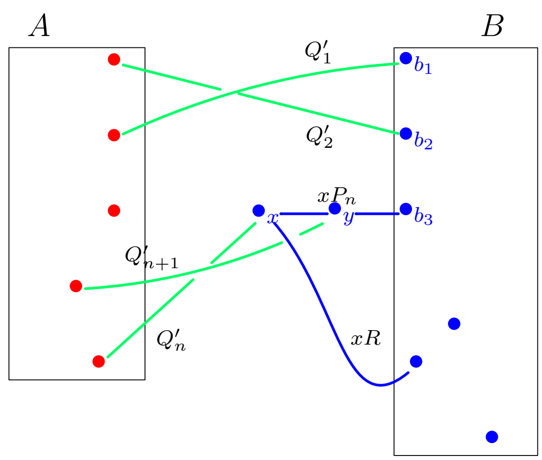
We conclude that there are disjoint $A$-$B'$ paths

$$Q'_1, \ldots, Q'_{n+1}$$

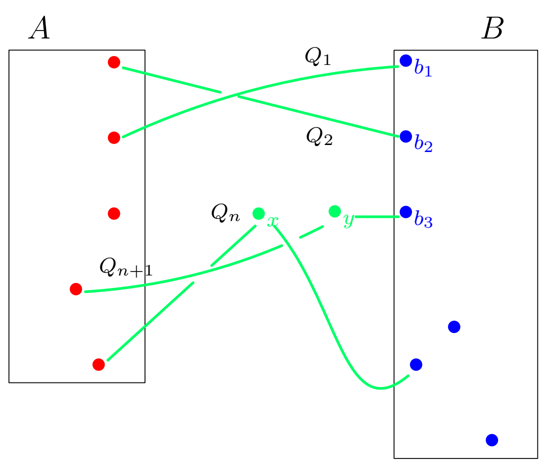
whose endpoints are $\{b_1, \ldots, b_{n-1}, x, y\}$ where all we know about $y$ is that is it in $B'$ and is not equal to $b_1, \ldots, b_{n-1}$ or $x$. We can reindex the $Q'_j$ so that the $B$-endpoint of $Q'_j$ is $b_j$ for $j < n$, the $B$-endpoint of $Q'_n$ is $x$ and the $B$-endpoint of $Q'_{n+1}$ is $y$. We have no idea which elements in $A$ are the other endpoints.

Since $B'$ contains vertices from $B$, from $xP_n$ and $xR$, there are three cases to consider:

Case 1 – $y$ is on $xP_n$ : Recall that $y$ cannot equal $x$. Here is the picture:



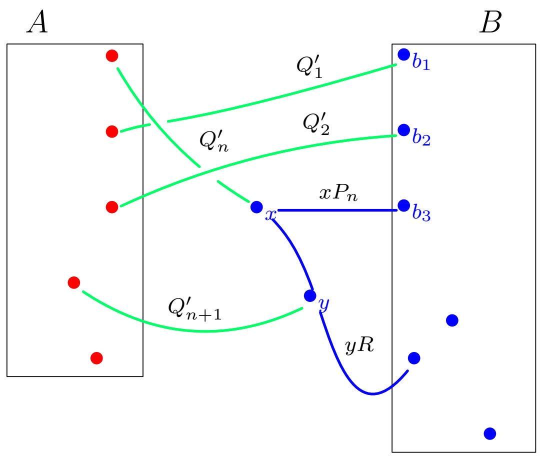

Extend $Q'_n$ with $xR$ to create $Q_n$ and extend $Q'_n$ with $yP_n$ :



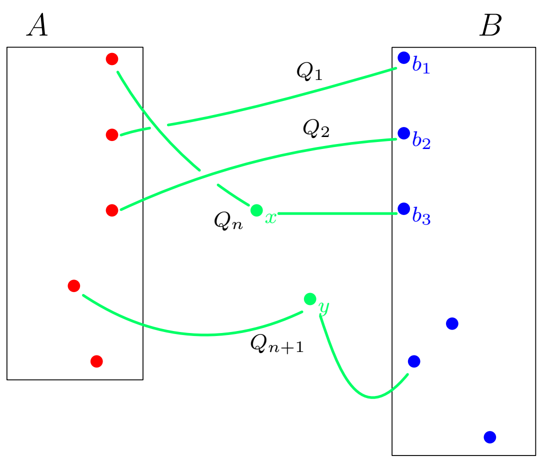

The desired new disjoint paths are

$$Q_1 = Q'_1, \quad \ldots, \quad Q_{n-1} = Q'_{n-1}, \quad Q_n = Q'_n \circ xR, \quad Q_{n+1} = Q'_{n+1} \circ yP_n$$

Case 2 – $y$ is on $xR$ : Recall that $y$ cannot equal $x$. Here is the picture:




This time, concatenate $Q'_n$ with $xP_n$ and concatenate $Q'_{n+1}$ with $yR$ : $x$. Here is the picture:



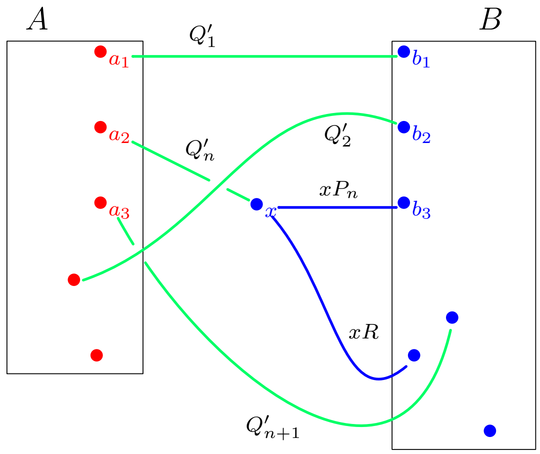

The desired new disjoint paths are

$$Q_1 = Q'_1, \quad \ldots, \quad Q_{n-1} = Q'_{n-1}, \quad Q_n = Q'_n \circ xP_n, \quad Q_{n+1} = Q'_{n+1} \circ yR$$

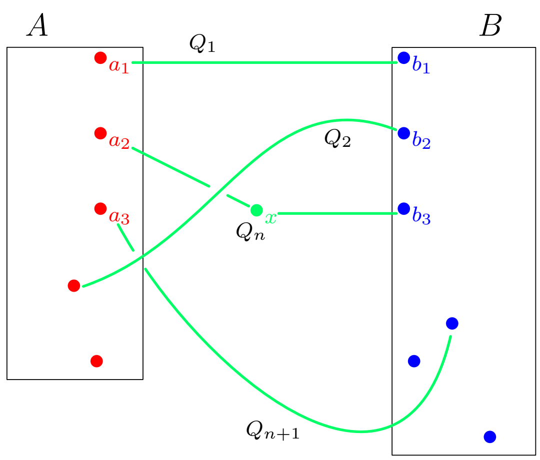
Case 2 – $y$ is not on $xR$ or $xP_n$. This means that $y$ is in $B$ and $y$ does not equal $b_n$, the $B$-endpoint of $xP_n$. When we applied the induction hypothetis we were guaranteed that $y$ would not equal $b_1, \ldots, b_{n-1}$ so in fact

$$y \neq b_j \quad (\text{for all} j, 1 \leq j \leq n)$$

Here is the picture:




This time we can use $Q'_{n+1}$ as it is, and we extend $Q'_n$ by $xP_n$, as shown here:




The desired new disjoint paths are

$$Q_1 = Q'_1, \quad \ldots, \quad Q_{n-1} = Q'_{n-1}, \quad Q_n = Q'_n \circ xP_n, \quad Q_{n+1} = Q'_{n+1}$$

□