An infinite color analogue of Rado's theorem

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Abstract

Let R be a subring of the complex numbers and $\mathfrak{a}$ be a cardinal. A system $\mathcal{L}$ of linear homogeneous equations with coefficients in R is called $\mathfrak{a}$-regular over R if, for every $\mathfrak{a}$-coloring of the nonzero elements of R, there is a monochromatic solution to $\mathcal{L}$ in distinct variables. In 1943, Rado classified those finite systems of linear homogeneous equations that are $\mathfrak{a}$-regular over R for all positive integers $\mathfrak{a}$. For every infinite cardinal $\mathfrak{a}$, we classify those finite systems of linear homogeneous equations that are $\mathfrak{a}$-regular over R. As a corollary, for every positive integer s, we have $2^{\aleph_0} > \aleph_s$ if and only if the equation $x_0 + sx_1 = x_2 + \cdots + x_{s+2}$ is $\aleph_0$-regular over $\mathbb{R}$. This generalizes the case $s = 1$ due to Erdős.

1 Introduction

One of the first results in Ramsey theory is Schur's theorem [21], which states that for every finite coloring of the positive integers, there is a monochromatic solution to the equation $x + y = z$. In 1927, van der Waerden [24] proved his celebrated theorem that every finite coloring of the positive integers contains arbitrarily long monochromatic arithmetic progressions. These two classical theorems of Schur and van der Waerden were beautifully generalized by Rado in his 1933 thesis [18] and even further in 1943 [19]. For an $m \times n$ matrix $A = (a_{ij})$ with entries in a subring R of the complex numbers, denote by $\mathcal{L} = \mathcal{L}(A)$ the system of linear homogeneous equations

$$\sum_{j=1}^{n} a_{ij}x_j = 0 \qquad \text{for } 1 \leq i \leq m.$$

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Let $\mathfrak{a}$ be a cardinal number. The system $\mathcal{L}$ is called $\mathfrak{a}$-*regular over* R if, for every $\mathfrak{a}$-coloring of the elements of R, there is a monochromatic solution to $\mathcal{L}$ in distinct variables. The system $\mathcal{L}$ is called *regular over* R if it is $\mathfrak{a}$-regular over R for all positive integers $\mathfrak{a}$. The matrix A with column vectors $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n$ is said to satisfy the *columns condition* if there exists a partition $\{1, 2, \ldots, n\} = D_1 \cup \ldots \cup D_p$ such that $\sum_{i \in D_1} \mathbf{c}_i = \mathbf{0}$ and for each $j \in \{2, 3, \ldots, p\}$, $\sum_{i \in D_j} \mathbf{c}_i$ is a linear combination of $\{\mathbf{c}_i : i \in \bigcup_{k=1}^{j-1} D_k\}$. Rado [19] proved that the system $\mathcal{L}(A)$ is regular over R if and only if the matrix A satisfies the columns condition and there is a solution to $\mathcal{L}(A)$ in distinct variables.

Our main result is Theorem 1, which is an infinite color analogue of Rado's theorem. For an infinite cardinal $\mathfrak{a}$ and subring R of the complex numbers, Theorem 1 classifies those finite systems of linear homogeneous equations that are $\mathfrak{a}$-regular over R. Before jumping into the main result, we start by describing some well known $\aleph_0$-colorings of the real numbers that are free of monochromatic solutions in distinct variables to particular systems of linear homogeneous equations.

There is an $\aleph_0$-coloring of the nonzero real numbers without a monochromatic solution to the Schur equation $x + y = z$ in distinct variables. For a real nummber $r > 1$, we first define the coloring $c_r : \mathbb{R}_{>0} \to \mathbb{Z}$ of the positive real numbers by $c_r(x) = \lfloor \log_r x \rfloor$. We have $c_2(x) = i$ if and only if x lies in the interval $[2^i, 2^{i+1})$. It follows that if $c_2(x) = c_2(y) = i$, then $c_2(x + y) = i + 1$. Therefore, c_2 is free of monochromatic solutions to $x + y = z$. If we use $\aleph_0$ more colors to color the negative real numbers, and give 0 any color, then we can extend c_2 to an $\aleph_0$-coloring of the real numbers that is free of monochromatic solutions to $x + y = z$ in distinct variables. If a matrix A satisfies that not all row sums of A are zero, then there is an r such that c_r is free of monochromatic solutions to $\mathcal{L}(A)$ and it follows that $\mathcal{L}(A)$ is not $\aleph_0$-regular over $\mathbb{R}$.

If we assume the axiom of choice, which we do until Section 5, then there are other systems of linear equations that are not $\aleph_0$-regular over $\mathbb{R}$. We note that $\mathbb{R}$ is a vector space over $\mathbb{Q}$. Assuming the axiom of choice, every vector space has a basis. In particular, there is a well-ordered basis $B = \{b_p\}_{p < 2^{\aleph_0}}$ for $\mathbb{R}$ as a $\mathbb{Q}$-vector space. Such a basis B is known as a *Hamel basis*. Therefore, every real number x has a unique representation

$$x = \sum_{j=1}^{k} q_j b_{p_j}$$

with $p_1 < \ldots < p_k$ and $q_j \in \mathbb{Q} \setminus \{0\}$ for $1 \leq j \leq k$. We may view each real number x as a weighted finite subset of B, the weight being the vector $w(x) = (q_1, \ldots, q_k)$ and the subset being $e(x) = \{b_{p_1}, \ldots, b_{p_k}\}$. According to Komjáth [16], Rado proved that there is an $\aleph_0$-coloring of $\mathbb{R}$ that is free of monochromatic 3-term arithmetic progressions. Rado's coloring is defined by assigning each real number x its weight $w(x)$. In 1969, Ceder [4] made the same observation as Rado. Later that year, Ceder [5] showed that there are no linear homogeneous equations in 3 variables that are $\aleph_0$-regular over $\mathbb{R}$. This follows from the general observation that for a vector space V over a countable field F, we can assign to an element $x \in V$ its weight $w(x)$, and this countable coloring is free of monochromatic solutions in distinct variables to every system $\mathcal{L}$ of linear homogeneous equations such that the only solutions to $\mathcal{L}$ with $x_i \in \{0,1\}$ for $i = 1, \ldots, n$ satisfy $x_1 = \cdots = x_n$.

Erdős and Kakutani [11] in 1943 proved that the continuum hypothesis is equivalent to there being a countable coloring of the real numbers such that each monochromatic subset is linearly independent over $\mathbb{Q}$. Erdős followed this by proving that the Sidon equation $x_1 + x_2 = x_3 + x_4$ is $\aleph_0$-regular over $\mathbb{R}$ is equivalent to the negation of the continuum hypothesis. A proof of this result can be found in Davies [6].

Analogous to the columns condition of Rado, we find, for every infinite cardinal $\mathfrak{a}$, necessary and sufficient conditions on the matrix A for the system $\mathcal{L}(A)$ to be $\mathfrak{a}$-regular over R. We first need a few definitions, which we borrow from Komjáth [16].

Definition: Let A be an $m \times n$ matrix with column vectors $\mathbf{c}_1, \ldots, \mathbf{c}_n$.

(1) We call a partition $\mathcal{P} = \{D_1, \ldots, D_l\}$ of the set $[n] = \{1, \ldots, n\}$ *balanced for* A if $\sum_{j \in D_k} \mathbf{c}_j = \mathbf{0}$ holds for every $k \in [l]$.
(2) A collection $\{\mathcal{P}_1, \ldots, \mathcal{P}_s\}$ of balanced partitions for A is called *separative* if for all distinct $u, v \in [n]$, there is a balanced partition $\mathcal{P}_j$ with u and v in different sets in $\mathcal{P}_j$.
(3) We call the system $\mathcal{L}(A)$ *separable* if there exists a separative collection of balanced partitions for A.
(4) For separable $\mathcal{L}(A)$, the *separation number* $s(\mathcal{L}(A))$ is the least positive integer s such that there exists a separative collection of s balanced partitions for A.

We suppose for the rest of the paper that $\mathfrak{a}$ is an infinite cardinal. The successor cardinal of $\mathfrak{a}$ is denoted by $\mathfrak{a}^+$, and the s^{th} successor cardinal of $\mathfrak{a}$ is denoted by $\mathfrak{a}^{+s}$.

The following theorem is our main result.

Theorem 1 *For an infinite cardinal $\mathfrak{a}$ and subring R of the complex numbers, a finite system $\mathcal{L}$ of linear homogeneous equations with coefficients in R is $\mathfrak{a}$-regular over R if and only if $\mathcal{L}$ is separable and $|R| \geq \mathfrak{a}^{+s(\mathcal{L})}$.*

We also have the following result, which is similar in character to results of Komjáth [17] and Schmerl [20].

Theorem 2 *For an infinite cardinal $\mathfrak{a}$, commutative ring R, and subfield $F \subset R$ with $|F| \leq \mathfrak{a}$, there is an $\mathfrak{a}$-coloring of R that is free of monochromatic solutions in distinct variables to all finite systems $\mathcal{L}$ of linear homogeneous equations such that the coefficients of $\mathcal{L}$ are in F and $\mathcal{L}$ is not $\mathfrak{a}$-regular over R.*

For each positive integer s, Komjáth [16] gave an example of a system of linear homogeneous equations that is $\aleph_0$-regular over $\mathbb{R}$ if and only if $2^{\aleph_0} > \aleph_s$. Likewise, for each positive integer s, Corollary 1 gives an example of a single linear homogeneous equation that is $\aleph_0$-regular over $\mathbb{R}$ if and only if $2^{\aleph_0} > \aleph_s$. Corollary 1 follows from Theorem 1 since the separation number for Equation (1) is $s+1$. Corollary 1 generalizes the case $s = 1$ that Erdős solved.

Corollary 1 *For every positive integer s, the linear homogeneous equation*

$$x_1 + sx_2 = x_3 + \cdots + x_{s+3} \tag{1}$$

is $\aleph_0$*-regular over* $\mathbb{R}$ *if and only if* $2^{\aleph_0} > \aleph_s$.

For a system $\mathcal{L}$ of linear homogeneous equations with rational coefficients, Komjáth [16] defines $\lambda(\mathcal{L})$ to be the least cardinal $\mathfrak{b}$ such that if V is a rational vector space of dimension $\mathfrak{b}$, then every $\aleph_0$-coloring of V has a monochromatic solution to $\mathcal{L}$ in distinct variables. If no such cardinal $\mathfrak{b}$ exists, set $\lambda(\mathcal{L}) = \infty$. We note that the dimension and cardinality of an uncountable rational vector space are equal. Komjáth proved that if $\lambda(\mathcal{L}) \leq 2^{\aleph_0}$, then $\mathcal{L}$ is separable. He also proved that if $\mathcal{L}$ is separable, then $\lambda(\mathcal{L}) \leq \aleph_{s(\mathcal{L})}$, where $s(\mathcal{L})$ is the separation number of $\mathcal{L}$. The following theorem demonstrates that Komjáth's upper bound on $\lambda(\mathcal{L})$ is tight.

Theorem 3 *If* $\aleph_s \leq 2^{\aleph_0}$*, then a finite system* $\mathcal{L}$ *of linear homogeneous equations with rational coefficients satisfies* $\lambda(\mathcal{L}) = \aleph_s$ *if and only if* $\mathcal{L}$ *is separable and* $s = s(\mathcal{L})$.

We deduce Theorem 1 and Theorem 3 from results we prove on hypergraph partition relations. These results are discussed in the next section, and proved in Section 4. The deduction of Theorem 1 from results on hypergraph partition relations is established in Section 3. Up until Section 5, we assume the axiom of choice. In Section 5, we investigate the problem of $\aleph_0$-regularity over $\mathbb{R}$ without the axiom of choice.

2 Hypergraph Partition Relations

A *hypergraph* $H = (V, E)$ consists of a set V and a collection E of subsets of V. The elements of V are called *vertices* and the elements of E are called *edges*. A hypergraph H is called k*-uniform* if every edge contains exactly k vertices. The Erdős–Rado arrow notation $\mathfrak{b} \to (\mathfrak{d})^k_{\mathfrak{a}}$ means that for every $\mathfrak{a}$-coloring of the subsets of cardinality k of a set of cardinality $\mathfrak{b}$, there is a monochromatic complete k-uniform hypergraph on $\mathfrak{d}$ vertices. We write $\mathfrak{b} \not\to (\mathfrak{d})^k_{\mathfrak{a}}$ if $\mathfrak{b} \to (\mathfrak{d})^k_{\mathfrak{a}}$ does not hold. Ramsey's theorem [18] can be written using the Erdős–Rado arrow notation: we have

$$\aleph_0 \to (\aleph_0)^k_r$$

for all positive integers r and k.

Erdős [9], [12] in 1942 proved if $\mathfrak{a}$ is an infinite cardinal, then

$$(2^{\mathfrak{a}})^+ \to (\mathfrak{a}^+)^2_{\mathfrak{a}}, \tag{2}$$

and

$$2^{\mathfrak{a}} \not\to (3)^2_{\mathfrak{a}}. \tag{3}$$

In 1943, Erdős and Kakutani [11] proved that if $\mathfrak{a}$ is an infinite cardinal, then there is an $\mathfrak{a}$-coloring of the edges of the complete graph on $\mathfrak{a}^+$ vertices without any monochromatic cycles, but every $\mathfrak{a}$-coloring of the edges of the complete graph on $\mathfrak{a}^{+2}$ vertices contains a monochromatic cycle. For $\mathcal{G}$ a nonempty family of k-uniform hypergraphs and cardinals $\mathfrak{a}$ and $\mathfrak{b}$, the partition relation $\mathfrak{b} \to (\mathcal{G})^k_{\mathfrak{a}}$ is said to hold if, for every $\mathfrak{a}$-coloring of the edges of the complete k-uniform hypergraph on $\mathfrak{b}$ vertices, there is a monochromatic copy of a hypergraph $G \in \mathcal{G}$. We write $\mathfrak{b} \not\to (\mathcal{G})^k_{\mathfrak{a}}$ if $\mathfrak{b} \to (\mathcal{G})^k_{\mathfrak{a}}$ does not hold. Letting $\mathcal{C}$ denote the family of cycles, we can restate the Erdős-Kakutani result as $\mathfrak{b} \to (\mathcal{C})^2_{\mathfrak{a}}$ if and only if $\mathfrak{b} \geq \mathfrak{a}^{+2}$.

Theorem 4 below classifies $\mathfrak{b} \to (\mathcal{G})^2_{\mathfrak{a}}$ for every nonempty family $\mathcal{G}$ of finite graphs and cardinals $\mathfrak{a}$ and $\mathfrak{b}$ with $\mathfrak{a}$ infinite. A star S_n is a graph on $n+1$ vertices with one vertex of degree n and the other n vertices of degree 1. We call a graph a *galaxy* if its connected components are stars.

Theorem 4 *Let $\mathfrak{a}$ and $\mathfrak{b}$ be cardinals with $\mathfrak{a}$ infinite. For a nonempty family $\mathcal{G}$ of finite graphs, the partition relation $\mathfrak{b} \to (\mathcal{G})^2_{\mathfrak{a}}$ holds if and only if (1), (2), (3), or (4) below are true.*

(1) There exists $G = (V, E) \in \mathcal{G}$ with $|E| \leq 1$ and $|V| \leq \mathfrak{b}$.
(2) $\mathfrak{b} = \mathfrak{a}^+$ and there exists a galaxy $G \in \mathcal{G}$.
(3) $\mathfrak{b} > \mathfrak{a}^+$ and there exists a bipartite graph $G \in \mathcal{G}$.
(4) $\mathfrak{b} > 2^{\mathfrak{a}}$.

The forward directions of *(3)* and *(4)* of Theorem 4 follow from the Erdős-Hajnal partition relation (4) below with $k = 2$ and the Erdős partition relation (2), respectively.

The main new result in Theorem 4 is that if $\mathfrak{a}$ is an infinite cardinal, then there is an $\mathfrak{a}$-coloring of the edges of the complete graph on $\mathfrak{a}^+$ vertices such that the only connected monochromatic subgraphs are stars, which strengthens the Erdős-Kakutani result. We give a proof of this result in Section 4 as a warmup to the proof of Theorem 5.

Having fully answered the problem for graphs, we now turn our attention to the general problem for hypergraphs, which requires some terminology. A k-uniform hypergraph $H = (V, E)$ is called *k-partite* if there exists a partition $V = V_1 \cup \ldots \cup V_k$ of the vertex set V such that every edge of H contains exactly one vertex in each V_i. Call H *partite* if H is k-partite for some positive integer k.

We call a hypergraph $H = (V, E)$ an *s-hybrid* if there is a positive integer k and a partition $V = V_1 \cup \ldots \cup V_k$ such that every edge of H contains exactly one vertex in each V_i and, if $v_i \in V_i$ for $1 \leq i \leq s$, then at most one edge of E contains the set $\{v_1, \ldots, v_s\}$. We note that a graph is a galaxy if and only if it is a 1-hybrid. Also, any k-partite graph is vacuously an s-hybrid for $s \geq k$.

A polarized partition theorem due to Erdős and Hajnal [10], [16], [20], states that if $\mathcal{G}$ contains a finite k-partite hypergraph, then

$$\mathfrak{a}^{+k} \to (\mathcal{G})^k_{\mathfrak{a}}. \tag{4}$$

For a nonempty family $\mathcal{G}$ of hypergraphs and for cardinals $\mathfrak{a}$ and $\mathfrak{b}$, the partition relation $\mathfrak{b} \to (\mathcal{G})^{<\omega}_{\mathfrak{a}}$ is said to hold if for every $\mathfrak{a}$-coloring of the finite subsets of a set of cardinality $\mathfrak{b}$, there is a monochromatic copy of a hypergraph $G \in \mathcal{G}$.

Our main result on hypergraph partition relations is Theorem 5 below.

Theorem 5 *Let $\mathfrak{a}$ be an infinite cardinal and s a positive integer such that $\mathfrak{a}^{+s} \leq 2^{\mathfrak{a}}$. For a nonempty family $\mathcal{G}$ of finite hypergraphs that does not contain an s-hybrid, we have*

$$\mathfrak{a}^{+s} \not\to (\mathcal{G})^{<\omega}_{\mathfrak{a}}.$$

3 Reducing Partition Regularity to Hypergraph Partition Relations

In order to make the reduction from partition regularity to hypergraph partition relations, we next define, for a $m \times n$ matrix A, a family $\mathcal{H}(A)$ of finite hypergraphs each with n edges.

Definition: For a $m \times n$ matrix A with column vectors $\mathbf{c}_1, \ldots, \mathbf{c}_n$, let $\mathcal{H}(A)$ be the family of finite hypergraphs such that $H = (W, E)$ is an element of $\mathcal{H}(A)$ with $E = \{e_1, \ldots, e_n\}$ if and only if $\sum_{w \in e_d} \mathbf{c}_d = \mathbf{0}$ holds for every vertex $w \in W$.

Lemma 1, Corollary 2, and Lemma 2 demonstrate a strong connection between separability of $\mathcal{L}(A)$ and the existence of a partite hypergraph in $\mathcal{H}(A)$.

Lemma 1 *A system $\mathcal{L}(A)$ of linear homogeneous equations is separable if and only if there is a partite hypergraph $H \in \mathcal{H}(A)$.*

Proof: We prove a stronger result then the claim of the lemma. We construct a bijection between separative collections of (not necessarily distinct) balanced partitions for A and partite hypergraphs in $\mathcal{H}(A)$. Let $\mathcal{C} = \{\mathcal{P}_1, \ldots, \mathcal{P}_s\}$ be a separative collection of partitions of $[n]$ that are balanced for A, with $\mathcal{P}_i = \{D_{i1}, \ldots, D_{il_i}\}$ for $1 \leq i \leq s$. We associate with each set D_{ij} a vertex w_{ij}. Let $W_i = \{w_{ij} : j \in [l_i]\}$ and $W = W_1 \cup \ldots \cup W_s$. Let $E = \{e_1, \ldots, e_n\}$ with $e_d = \{w_{ij} : d \in D_{ij}\}$ for $d \in [n]$. Since $\mathcal{C}$ is a collection of partitions of $[n]$, then for each $d \in [n]$ and $i \in [s]$ there is exactly one $j \in [l_i]$ such that $d \in D_{ij}$ (which is equivalent to $w_{ij} \in e_d$). Therefore, each edge e_d contains exactly one element from each W_i and $H = (W, E)$ is s-partite with partition $W = W_1 \cup \ldots \cup W_s$. Since each partition $\mathcal{P}_i$ is balanced, then $\sum_{w_{ij} \in e_d} \mathbf{c}_d = \mathbf{0}$ for each vertex w_{ij}. Since $\mathcal{C}$ is separative, then the sets e_{d_1} and e_{d_2} are distinct for $1 \leq d_1 < d_2 \leq n$. Hence, the s-partite hypergraph H is an element of $\mathcal{H}(A)$.

We show that the mapping we just described from separative collections of (not necessarily distinct) balanced partitions for A and partite hypergraphs in $\mathcal{H}(A)$ is a bijection by exhibiting its inverse. So suppose $H = (W, E) \in \mathcal{H}(A)$ is an s-partite hypergraph with s-partition $W = W_1 \cup \ldots \cup W_s$. Let $W_i = \{w_{i1}, \ldots, w_{il_i}\}$ for $i \in [s]$ and $E = \{e_1, \ldots, e_n\}$. For each vertex w_{ij}, define the subset $D_{ij} = \{d : e_d \in w_{ij}\}$ of $[n]$. Notice that $\sum_{d \in D_{ij}} \mathbf{c}_d = \mathbf{0}$ follows from the defintion of $\mathcal{H}(A)$. Therefore, $\mathcal{P}_i := D_{i1} \cup \ldots \cup D_{il_i}$ is a balanced partition for A. Since the edges $e_d \in E$ are distinct subsets of W, then the collection $\mathcal{C} := \{\mathcal{P}_1, \ldots, \mathcal{P}_s\}$ of balanced partitions for A is separative. $\square$

Notice that the bijection constructed in the proof of Lemma 1 maps a separative collection of s balanced partitions for A to an s-partite hypergraph in $\mathcal{H}(A)$. We therefore have the following corollary.

Corollary 2 *If $\mathcal{L}$ is a separable system of linear homogeneous equations, then there is an $s(\mathcal{L})$-partite hypergraph $H \in \mathcal{H}(A)$.*

For separable $\mathcal{L}(A)$, the following lemma combined with the previous corollary implies that the minimum s such that there is an s-partite hypergraph in $\mathcal{H}(A)$ and the minimum s' such that there is an s'-hybrid in $\mathcal{H}(A)$ satisfy $s = s' = s(\mathcal{L}(A))$.

Lemma 2 *If $\mathcal{L}(A)$ is a separable system of linear homogeneous equations, then there is no s-hybrid $H \in \mathcal{H}(A)$ with $s < s(\mathcal{L}(A))$.*

Proof: Suppose for contradiction that there is a positive integer $s < s(\mathcal{L}(A))$ and a k-partite s-hybrid $H = (W, E) \in \mathcal{H}(A)$ with k-partition $W = W_1 \cup \ldots \cup W_k$ such that for every set $\{w_1, \ldots, w_s\}$ with $w_i \in W_i$, there is at most one edge that contains $\{w_1, \ldots, w_s\}$. To each subset W_i there is an associated partition $\mathcal{P}_i$ of $[|E|]$ that is balanced for A as in the proof of Lemma 1. Since $s < s(\mathcal{L}(A))$, then the collection $\mathcal{C} = \{\mathcal{P}_1, \ldots, \mathcal{P}_s\}$ of balanced partitions for A is not separative. Hence, there are edges e_{d_1} and e_{d_2} with $d_1 \neq d_2$ such that $e_{d_1} \cap (\bigcup_{i=1}^{s} W_i) = e_{d_2} \cap (\bigcup_{i=1}^{s} W_i)$, contradicting the assumption that H is an s-hybrid. $\square$

For a subset S of a vector space V, we call a system $\mathcal{L}$ of linear homogeneous equations with coefficients in F $\mathfrak{a}$*-regular over* S if, for every $\mathfrak{a}$-coloring of the nonzero elements of S, there is a monochromatic solution to $\mathcal{L}$ in distinct variables.

Let $\langle \mathcal{A}, \prec \rangle$ be a well-ordered set of cardinality $\mathfrak{a}$ and $\mathcal{P}(\mathcal{A})$ denote the set of subsets of $\mathcal{A}$. Let $\phi : V \to \mathcal{P}(\mathcal{A})$ be an injective function. For distinct subsets X and Y of $\mathcal{A}$, let $\delta(X, Y)$ be the least element of $\mathcal{A}$ such that exactly one of X or Y contains $\delta(X, Y)$.

Let B be a basis for a vector space V. For a finite subset $e \subset B$, let $s(e) = \sum_{b \in e} b$. Let $S(B) = \{s(e) : e \text{ is a finite subset of } B\}$.

Theorem 6 *Let V be a vector space over a field F, $\mathfrak{a}$ be an infinite cardinal such that $2^{\mathfrak{a}} \geq |V| > \mathfrak{a} \geq |F|$, and A be a finite matrix with entries in F. The system $\mathcal{L}(A)$ of linear homogeneous equations is $\mathfrak{a}$-regular over V if and only if $|V| \to (\mathcal{H}(A))^{<\omega}_{\mathfrak{a}}$.*

Proof: Let $B = \{b_p : p < |V|\}$ be a basis for V. We have $|V| = |S(B)| = |B|$ since V is an uncountable vector space over a field of lesser cardinality. Notice that $H = (V, E)$ with $E = \{e_1, \ldots, e_n\}$ is a subhypergraph of $[B]^{<\omega}$ that is isomorphic to an element of $\mathcal{H}(A)$ if and only if $\mathbf{x} = (s(e_1), \ldots, s(e_n))$ is a solution to $\mathcal{L}(A)$ in distinct variables. Therefore, $|B| \to (\mathcal{H}(A))^{<\omega}_{\mathfrak{a}}$ is equivalent to $\mathcal{L}(A)$ being $\mathfrak{a}$-regular over $S(B)$. Since $|B| = |V|$ and $S(B) \subset V$, then $|V| \to (\mathcal{H}(A))^{<\omega}_{\mathfrak{a}}$ implies that $\mathcal{L}(A)$ is $\mathfrak{a}$-regular over V.

Our next task is to prove the harder direction: if $\mathcal{L}(A)$ is $\mathfrak{a}$-regular over V, then $|V| \to (H(A))^{<\omega}_{\mathfrak{a}}$. We prove this by proving the contrapositive. So suppose that there is an $\mathfrak{a}$-coloring c of the finite subsets of B that realizes $|V| \not\to (H(A))^{<\omega}_{\mathfrak{a}}$. We then present an $\mathfrak{a}$-coloring Γ_c of the nonzero elements of V that is free of monochromatic solutions to $\mathcal{L}(A)$ in distinct variables.

Let $\mathcal{A}$ be a set of cardinality $\mathfrak{a}$ and $\mathcal{P}(\mathcal{A})$ be the set of subsets of $\mathcal{A}$. In Section 4.2, we define the linear ordering $\prec^*$ of $\mathcal{P}(\mathcal{A})$ and the $\mathfrak{a}$-coloring C_1 of $[\mathcal{P}(\mathcal{A})]^{<\omega}$. By Theorem 7(1), for every color D of C_1, there are disjoint subsets $P_1, \ldots, P_k$ of $\mathcal{P}(A)$ such that every edge of color D contains exactly one vertex in each P_i and every edge $T = \{t_1, \ldots, t_k\} \subset \mathcal{P}(A)$ with $t_k \prec^* \ldots \prec^* t_1$ satisfies $t_j \in P_j$ for $j \in [k]$.

Since B is a basis for V, then for each $x \in V$, there is a unique representation $x = \sum_{i=1}^{k} f_i b_{p_i}$ with $p_1 < \ldots < p_k$. We let $w(x) = (f_1, \ldots, f_k)$ and $e(x) = \{b_{p_1}, \ldots, b_{p_k}\} \subset B$.

Let $\phi : [B]^{<\omega} \to \mathcal{P}(\mathcal{A})$ be an injective function. For an element $x = \sum_{h=1}^{k} f_h b_{p_h}$ of V, define $\sigma(x)$ to be the permutation of $[k]$ such that

$$\phi(b_{p_{\sigma(1)}}) \prec^* \cdots \prec^* \phi(b_{p_{\sigma(k)}}).$$

We define the $\mathfrak{a}$-coloring Γ_c of the nonzero elements of V by

$$\Gamma_c(x) = (C_1(\phi(e(x))), \sigma(x), w(x), c(e(x))).$$

Suppose for contradiction that $\mathbf{x} = (x_1, \ldots, x_n)$ is a monochromatic solution to $\mathcal{L}(A)$ in distinct variables by coloring Γ_c. Let (D, σ, w, d) be the color of the monochromatic solution to $\mathcal{L}(A)$. Let $w = (f_1, \ldots, f_k)$ be the weight that each of the x_j's share. Let $x_j = \sum_{h=1}^{k} f_h b_{p_h,j}$ be the unique basis representation for x_j. Define the finite hypergraph $H(\mathbf{x}) = (V(\mathbf{x}), E(\mathbf{x})) \in \mathcal{H}(A)$ with vertex set $V(\mathbf{x}) \subset B$ defined by $b_j \in V(\mathbf{x})$ if $\pi_j(x_p) \neq 0$ for at least one $p \in \{1, \ldots, n\}$ and edge set $E(\mathbf{x}) = \{e(x_1), \ldots, e(x_n)\}$.

By coloring C_1 and σ, there are disjoint subsets $\{B_1, \ldots, B_k\}$ of B such that every edge $e = \{b_{p_1}, \ldots, b_{p_k}\}$ of color D in the coloring C_1 with $p_1 < \ldots < p_k$ satisfies $b_{p_h} \in B_h$ for $1 \leq h \leq k$.

Therefore, if $b \in B_h$, then the coefficient of b in x_j is f_h if $b \in e(x_j)$ and 0 if $b \notin e(x_j)$. Since $\mathbf{x}$ is a solution to $\mathcal{L}(A)$, then for each $b \in B_h$, we have

$$\mathbf{0} = \sum_{b \in e(x_j)} \mathbf{c}_j f_h = f_h \sum_{b \in e(x_j)} \mathbf{c}_j.$$

Hence, $\sum_{b \in e(x_j)} \mathbf{c}_j = \mathbf{0}$ for all $b \in B$. Therefore the hypergraph $H(\mathbf{x})$, which is colored monochromatic by coloring c, is an element of $\mathcal{H}(A)$. This contradicts that the coloring c is free of monochromatic hypergraphs in $\mathcal{H}(A)$, and completes the proof. $\square$

We now give a proof of Theorem 1, assuming hypergraph partition relations proved in the next section.

Proof of Theorem 1:

Let R be a subring of the complex numbers and $A\mathbf{x} = \mathbf{0}$ be a system of linear equations over R. Let F be the subfield of $\mathbb{C}$ generated by the entries of A, so $|F| = \aleph_0$. Let $V \subset \mathbb{C}$ be the smallest vector space over F containing R, so V contains a basis $B \subset R$ and $|V| = |B| = |R|$. Since $S(B) \subset R$, then as noted in the first paragraph of the proof of Theorem 6, we have $|R| \to (\mathcal{H}(A))^{<\omega}_{\mathfrak{a}}$ is equivalent to $\mathcal{L}(A)$ being $\mathfrak{a}$-regular over $S(B)$. Moreover, by Theorem 6, $|R| \to (\mathcal{H}(A))^{<\omega}_{\mathfrak{a}}$ is equivalent to $\mathcal{L}(A)$ being $\mathfrak{a}$-regular over V. Since $B \subset R \subset V$, then $|R| \to (\mathcal{H}(A))^{<\omega}_{\mathfrak{a}}$ is equivalent to $\mathcal{L}(A)$ being $\mathfrak{a}$-regular over R. By Lemma 1, $\mathcal{L}(A)$ is separable if and only if $\mathcal{H}(A)$ contains a partite hypergraph. By Corollary 2 and Lemma 2, if $\mathcal{L}(A)$ is separable, then the smallest s such that $\mathcal{H}(A)$ contains an s-hybrid is $s(\mathcal{L})$ and $\mathcal{H}(A)$ contains a $s(\mathcal{L})$-partite hypergraph. We complete the proof of Theorem 1 by putting these results together with the hypergraph partition relation (4) and Theorem 5. $\square$

Similar to the proof of Theorem 1, Theorem 2 and Theorem 3 follow in a straightforward manner. We leave the proofs out for brevity.

4 Graph and Hypergraph Colorings

In Subsection 4.1, we present an $\mathfrak{a}$-coloring of the edges of the complete graph on $\mathfrak{a}^+$ vertices such that the only monochromatic subgraphs are galaxies. In Subsection 4.2, we generalize this result, demonstrating that if s is a positive integer satisfying $\mathfrak{a}^{+s} \leq 2^{\mathfrak{a}}$, then there is an $\mathfrak{a}$-coloring of the finite subset of a set of cardinality $\mathfrak{a}^{+s}$ such that the only monochromatic subhypergraphs are s-hybrids. We first need some basic definitions from set theory.

Let $\mathcal{A}$ be a set of cardinalty $\mathfrak{a}$, and define the power set $\mathcal{P}(\mathcal{A})$ as the set of subsets of $\mathcal{A}$. The power set $\mathcal{P}(\mathcal{A})$ has cardinality $2^{\mathfrak{a}}$. Let S be a subset of $\mathcal{P}(\mathcal{A})$ of cardinality $\mathfrak{a}^{+s}$, where s is a positive integer.

Let $<$ denote the well-ordering of the ordinals. For a well-ordered set $\langle R, \prec\rangle$ and $a \in R$, define the *segment* $R_a = \{b \in R : b \prec a\}$. The *cardinality* $|R|$ of a set R is the least ordinal α such that there exists a bijection $\phi : S \to \alpha$. For a set R, let $<_R$ denote a well-ordering of R that is order-isomorphic to the well-ordering $<$ of $|R|$, that is, there is a bijection $\phi : R \to |R|$ such that for all $x, y \in R$, we have $x <_R y$ if and only if $\phi(x) < \phi(y)$. Notice that for a well-ordered set $\langle R, <_R\rangle$ and $y \in R$, the inequality $|R_y| < |R|$ holds. For $y \in S$, let $\phi_y : S_y \to |S_y|$ be a bijection.

For a cardinal κ and set S, define

$$[S]^\kappa = \{T \subset S : |T| = \kappa\} \text{ and } [S]^{<\kappa} = \{T \subset S : |T| < \kappa\}.$$

For the well-ordered set $\langle \mathcal{A}, <_{\mathcal{A}}\rangle$ and distinct subsets $X, Y \subset \mathcal{A}$, let

$$\delta(X, Y) = \min\{a \in \mathcal{A} : \text{exactly one of } X \text{ and } Y \text{ contains } a\}$$

and

$$X \prec^* Y \text{ if } \delta(X, Y) \in Y.$$

The linear ordering $\prec^*$ is called the *lexicographic ordering* of the power set $\mathcal{P}(\mathcal{A})$.

4.1 Coloring with Galaxies

We assume in this subsection that $s = 1$, so that $|S| = \mathfrak{a}^+$. We start by defining three colorings c_1, c_2, and c_3 of $[S]^2$ below. For $x = \{x_0, x_1\} \in [S]^2$ with $x_0 <_S x_1$, let

$$c_1(x) = \delta(x_0, x_1) \tag{5}$$

$$c_2(x) = \begin{cases} 0 & \text{if } x_0 \prec^* x_1 \\ 1 & \text{if } x_1 \prec^* x_0 \end{cases} \tag{6}$$

$$c_3(x) = \phi_{x_1}(x_0) \tag{7}$$

Notice that the set of colors for coloring c_1 is $\mathcal{A}$, the set of colors for coloring c_2 is $\{0,1\}$, and the set of colors for c_3 is the set of ordinals less than $\mathfrak{a}$. Hence, the product coloring $c = c_1 \times c_2 \times c_3$ given by $c(x) = (c_1(x), c_2(x), c_3(x))$ is an $\mathfrak{a}$-coloring of $[S]^2$. We now show that the only monochromatic subgraphs in the coloring c are galaxies. Suppose G is a monochromatic subgraph of color $(\delta, \epsilon, \alpha)$ in the coloring c. Define the partition $S = S_0 \cup S_1$ by $S_0 = \{y : y \in S \text{ and } \delta \notin y\}$ and $S_1 = \{y : y \in S \text{ and } \delta \in y\}$. Notice that every edge $\{x_0, x_1\}$ that is monochromatic of color δ in coloring c_1 with $x_0 \prec^* x_1$ satisfies $x_0 \in S_0$ and $x_1 \in S_1$. Hence, G is bipartite with bipartition $\{S_0, S_1\}$. By coloring c_2, every vertex of $S_{1-\epsilon}$ is larger than its neighbors in G by the well-ordering $<_S$. By the coloring c_3, a vertex y is adjacent in G to at most one other vertex y' satisfying $y' <_S y$. Therefore, every vertex of $S_{1-\epsilon}$ has degree at most 1 in G, and so G must be a galaxy.

4.2 Coloring with s-hybrids

Our first task is to define three colorings, C_1, C_2, and C_3. The coloring C_1 colors all the finite subsets of $\mathcal{P}(\mathcal{A})$, but sometimes (it will be clear by context) we consider the coloring C_1 with its domain restricted to the finite subsets of S. The colorings C_2 and C_3 only color the finite subsets of S. We will prove that all the monochromatic subhypergraphs in the coloring C_1 are partite. Each of the colorings C_1, C_2, C_3 use at most $\mathfrak{a}$ colors, and we will prove that the only monochromatic subhypergraphs in the $\mathfrak{a}$-coloring $C = C_1 \times C_2 \times C_3$ of the finite subsets of S are s-hybrids.

Let $T = \{t_1, \ldots, t_k\}$ be a finite subset of $P(\mathcal{A})$ listed in decreasing lexicographic order: $t_k \prec^* \ldots \prec^* t_1$. Let $C_1(\emptyset) = 0$ and otherwise let $C_1(T)$ be the $|T| \times |T|$ matrix $C_1(T) = (\delta_{ij})$ such that $\delta_{ij} = \delta(t_i, t_j)$ if $i \neq j$ and $\delta_{ij} = 0$ if $i = j$. Notice that the colors of C_1 are 0 or finite matrices whose entries are 0 or elements of $\mathcal{A}$. Therefore, C_1 is an $\mathfrak{a}$-coloring.

If T is a subset of S, then we next assign color $C_2(T)$ and $C_3(T)$ to T. Let $C_2(T) = C_3(T) = 0$ if $k \leq s$. To define $C_2(T)$ and $C_3(T)$ when $k > s$, we first recursively define a listing $T(1)$, …, $T(k)$ of the elements of T and a family

$$S \supset S(T,1) \supset \ldots \supset S(T,k)$$

of subsets of S. Let $T(1)$ be the largest element of T by well ordering $<_S$ and $S(T,1)$ denote the segment $S_{T(1)}$. Once $T(j-1)$ and $S(T,j-1)$ have been defined, let $T(j)$ denote the largest element of $T \setminus \{T(1), \ldots, T(j-1)\}$ in the well ordering $<_{S(T,j-1)}$ of $S(T,j-1)$ and let $S(T,j)$ be the segment $S(T,j-1)_{T(j)}$.

Let $C_2(T)$ be the permutation π of $[k]$ such that $t_{\pi(i)} = T(i)$ for $i \in [k]$. Since the colors of C_2 are 0 or finite permutations, then C_2 is an $\aleph_0$-coloring.

Notice that

$$|S(T,k)| < \ldots < |S(T,1)| < |S|,$$

so the set $S(T,s)$ has cardinality at most $\mathfrak{a}$.

Let $C_3(T)$ be the image of the set $\{T(s+1), \ldots, T(k)\}$ by a bijection $\phi_{T(1),\ldots,T(s)} : [S(T,s)]^\omega \to |S(T,s)|$. Notice that the image of $\phi_{T(1),\ldots,T(s)}$ consists of ordinals less than $\mathfrak{a}$, so C_3 is an $\mathfrak{a}$-coloring. Also, if T and T' are distinct finite subsets of S each with at least s elements and $T(i) = T'(i)$ for $i \in [s]$, then $C_3(T) \neq C_3(T')$ since $\phi_{T(1),\ldots,T(s)}$ maps T and T' to distinct ordinals less than $\mathfrak{a}$.

For the following theorem, we use the colorings C_1, C_2, and C_3 that we just defined.

Theorem 7 *Assume $\mathfrak{a}$ is an infinite cardinal and s is a positive integer such that $\mathfrak{a}^{+s} \leq 2^{\mathfrak{a}}$. Let $\mathcal{A}$ be a set of cardinality $\mathfrak{a}$, and S be a subset of the power set $\mathcal{P}(\mathcal{A})$ with cardinality $\mathfrak{a}^{+s}$.*

(1) In the $\mathfrak{a}$-coloring C_1 of $[\mathcal{P}(\mathcal{A})]^{<\omega}$, the monochromatic subhypergraphs are partite. Moreover, for every monochromatic subhypergraph $H = (V, E)$ in the coloring C_1 of $[\mathcal{P}(\mathcal{A})]^{<\omega}$, there are disjoint subsets $P_1, \ldots, P_k$ of $\mathcal{P}(\mathcal{A})$ such that every edge of H contains exactly one vertex in each P_i and every edge $T = \{t_1, \ldots, t_k\} \in E$ with $t_k \prec^* \ldots \prec^* t_1$ satisfies $t_j \in P_j$ for $j \in [k]$.

(2) There is an $\mathfrak{a}$-coloring C of $[S]^{<\omega}$ such that every monochromatic subhypergraph is a s-hybrid.

Proof:

(1) Every monochromatic subhypergraph in the coloring C_1 is uniform since the empty set has color 0, and every k-set with $k \geq 1$ is a $k \times k$ matrix. For $k = 0$ or 1, every k-uniform hypergraph is trivially k-partite. Let $k \geq 2$ and $\Delta = (\delta_{ij})$ be one of the $k \times k$ matrices that is one of the colors of C_1. For $j \in [k]$, define the subset P_j of $\mathcal{P}(\mathcal{A})$ by $x \in P_j$ if and only if $\delta_{ij} \notin x$ for $i < j$ and $\delta_{jh} \in x$ for $j < h \leq k$. For $i, j \in [k]$ with $i < j$, every element of P_i contains δ_{ij} and no element of P_j contains δ_{ij}. Hence, the sets $P_1, \ldots, P_k$ are pairwise disjoint. From the definition of the coloring C_1, every edge in a monochromatic subhypergraph of color Δ has a vertex in each P_j for each $j \in [k]$. Hence, a monochromatic subhypergraph in the $\mathfrak{a}$-coloring C_1 is partite. Moreover, if $T = \{t_1, \ldots, t_k\}$ with $t_k \prec^* \ldots \prec^* t_1$ is an edge of color Δ, then $t_j \in P_j$ for $j \in [k]$.

(2) Let C be the product coloring $C = C_1 \times C_2 \times C_3$, where C_1, C_2, and C_3 are as previously defined, and the domain of C_1 is restricted to the finite subsets of S. Since each C_i for $i \in \{1, 2, 3\}$ uses at most $\mathfrak{a}$ colors, then the product coloring C uses at most $\mathfrak{a}$ colors.

By the coloring C_1, every monochromatic subhypergraph is k-partite for some positive integer k. If $k \leq s$, then a k-partite hyerpgraph is a s-hybrid. So we may assume $k > s$.

By the colorings C_1 and C_2, for every color $D = (\Delta, \pi, \alpha)$ of C, there is a positive integer k and pairwise disjoint subsets $S_1, \ldots, S_k$ of S such that every finite subset T of S that is colored D by C satisfies $|T| = k$ and $T(i) \in S_i$ for $i \in [k]$. By the coloring C_3, every s-tuple $(v_1, \ldots, v_s)$ with $v_i \in S_i$ satisfies that at most one edge e of color D satisfies $\{v_1, \ldots, v_s\} \subset e$. Hence, the only monochromatic subhypergraphs are s-hybrids. □

For a hypergraph H and cardinal $\mathfrak{b}$, the hypergraph $\mathfrak{b}H$ consists of $\mathfrak{b}$ disjoint copies of H.

Lemma 3 *Let $\mathfrak{a}$ and $\mathfrak{b}$ be infinite cardinals such that $\mathfrak{a} < \mathfrak{b}$ and $\mathfrak{b}$ is a regular cardinal. If H is a k-uniform hypergraph such that $\mathfrak{b} \to (\{H\})^k_{\mathfrak{a}}$, then $\mathfrak{b} \to (\{\mathfrak{b}H\})^k_{\mathfrak{a}}$.*

Proof: We have $\mathfrak{b} = \mathfrak{b} \times \mathfrak{b}$, so every $\mathfrak{a}$-coloring of the edges of the complete k-uniform hypergraph has at least $\mathfrak{b}$ monochromatic copies of H. Since $\mathfrak{b} > \mathfrak{a}$ and $\mathfrak{b}$ is regular, then there are $\mathfrak{b}$ monochromatic copies of H all of the same color. Hence, we have $\mathfrak{b} \to (\mathfrak{b}H)^k_{\mathfrak{a}}$. $\square$

For a hypergraph $H = (V, E)$ and vertex $v \in V$, define the neighborhood hypergraph $N(H, v) = (V', E')$ by

$$V' = \{w : w \in V \setminus \{v\} \text{ and there is an edge } e \in E \text{ such that } \{v, w\} \subset e\}$$

and

$$E' = \{e \setminus \{v\} : v \in e \in E\}.$$

We define the infinite k-uniform hypergraph $H(k, \mathfrak{a})$ recursively. The 1-uniform hypergraph $H(1, \mathfrak{a}) = (V, E)$ has $\mathfrak{a}^+$ vertices and $E = \{\{v\} : v \in V\}$. For integer $k \geq 2$, define the k-uniform hypergraph $H(k, \mathfrak{a})$ which consists of $\mathfrak{a}^+$ disjoint copies of the k-uniform hypergraph which consists of a root vertex v whose neighborhood hypergraph is $H(k-1, \mathfrak{a})$.

Lemma 4 *Let k be a positive integer and $\mathfrak{a}$ an infinite cardinal. For every $\mathfrak{a}$-coloring of the finite subsets of a set of cardinality $\mathfrak{a}^+$ there is a monochromatic copy of $H(k, \mathfrak{a})$.*

Proof: The proof is by induction on k. For $k = 1$, the result follows immediately from the transfinite pigeonhole principle. The induction hypothesis is that the lemma is true for k. Let v be any of the $\mathfrak{a}^+$ vertices, and consider the edges of size $k+1$ containing v. By the induction hypothesis, there is a monochromatic hypergraph where every edge contains v and the neighborhood hypergraph of v is isomorphic to $H(k, \mathfrak{a})$. Combining this with Lemma 3, for every $\mathfrak{a}$-coloring of the finite subsets of a set of cardinality $\mathfrak{a}^+$ there is a monochromatic copy of $H(k+1, \mathfrak{a})$. By induction on k , we have verified the lemma. $\square$

Every 1-hybrid k-uniform graph on at most $\mathfrak{a}^+$ vertices is a subhypergraph of $H(k, \mathfrak{a})$. Hence, Corollary 3 follows immediately from Lemma 4 and Theorem 7.

Corollary 3 *For every infinite cardinal $\mathfrak{a}$ and family $\mathcal{G}$ of hypergraphs, we have $\mathfrak{a}^+ \to (\mathcal{G})^{<\omega}_{\mathfrak{a}}$ if and only if $\mathcal{G}$ contains a 1-hybrid hypergraph on at most $\mathfrak{a}^+$ vertices.*

We next prove Erdős and Hajnal's polarized partition relation (4). A proof of this result can also be found in the papers of Schmerl [20] and Komjáth [16]. The complete k-partite hypergraph $P(k; n) = (V, E)$ is defined by $V = V_1 \cup \ldots \cup V_k$ with $|V_i| = n$ for $1 \leq i \leq k$, and $(v_1, \ldots, v_k) \in E$ if $v_i \in V_i$ for $1 \leq i \leq k$, and there are no other edges. Notice that every k-partite hypergraph on n vertices is a subhypergraph of $P(k; n)$.

Lemma 5 *Let k and n be positive integers and $\mathfrak{a}$ an infinite cardinal. Every $\mathfrak{a}$-coloring of the edges of the complete k-uniform hypergraph on $\mathfrak{a}^{+k}$ vertices contains a monochromatic $P(k; n)$.*

Proof: The proof is by induction on k. For $k = 1$, this result follows immediately from the transfinite pigeonhole principle. The induction hypothesis is that the lemma is true for k. Consider an $\mathfrak{a}$-coloring of the complete $(k+1)$-uniform hypergraph $K^{(k+1)}_{\mathfrak{a}^{+(k+1)}}$ on $\mathfrak{a}^{+(k+1)}$ vertices. Partition the $\mathfrak{a}^{+(k+1)}$ vertices into two sets, X and Y, such that $|X| = \mathfrak{a}^{+(k+1)}$ and $|Y| = \mathfrak{a}^{+k}$. For each $x \in X$, consider the edges of $K^{(k+1)}_{\mathfrak{a}^{+(k+1)}}$ that include x and k vertices from Y. By the induction hypothesis, there is a monochromatic $(k+1)$-hypergraph such that each edge contains x and the neighborhood of x is a copy of $P(k;n)$ with vertices in Y. Make a pigeonhole for each copy of $P(k;n)$ with vertices in Y and color of the $\mathfrak{a}$ colors. There are $\mathfrak{a}^{+k}$ such pigeonholes. Place a vertex $x \in X$ in a pigeonhole if the neighborhood of x in the color of the pigeonhole contains the copy of $P(k;n)$ of the pigeonhole. Since there are $\mathfrak{a}^{+(k+1)}$ such vertices $x \in X$ and only $\mathfrak{a}^{+k}$ pigeonholes, then there are n vertices in one pigeonhole. These n vertices, along with the vertices of the copy of $P(k;n)$ of the pigeonhole, are the vertices of a monochromatic $P(k+1;n)$ in the color of the pigeonhole. By induction on k, we have verified the lemma. □

5 Regularity without the axiom of choice

In this section we study infinite color regularity over $\mathbb{R}$ without the axiom of choice. Finite color regularity over $\mathbb{R}$ without the axiom of choice was studied by Fox and Radoičić [13] and also by Alexeev, Fox, and Graham [1].

We first define the axioms we will be using. In 1942 Bernays [3] formulated the axiom known as the principle of dependent choice.

Definition: Principle of dependent choices If E is a binary relation on a nonempty set A and for every $a \in A$ there exists $b \in A$ with aEb, then there exists a sequence $a_1, a_2, \ldots, a_n, \ldots$ such that $a_n E a_{n+1}$ for every $n < \omega$.

The principle of dependent choice is usually denoted by DC. The axiom of choice implies DC, but not conversely (Theorem 8.2 in [15]). As usual, ZF is short for Zermelo-Fraenkel system of axioms, and ZFC is short for Zermelo-Fraenkel system of axioms with the axiom of choice.

Definition: Axiom LM Every set of real numbers is Lebesgue measurable.

Axiom LM is not consistent with ZFC. However, In 1970, assuming the existence of an inaccessible cardinal, Solovay proved the following consistency result.

Theorem 8 (Solovay, [23]) *The system of axioms ZF + DC + LM is consistent.*

We call a system

$$\sum_{i=1}^{n} a_{ij} x_i = 0 \qquad \text{for } 1 \le j \le m$$

of linear homogeneous equations *homothetic* if $\sum_{i=1}^{n} a_{ij} = 0$ for $1 \le j \le m$. Rado [19] first proved that if $\mathcal{L}$ is a system of linear homogeneous equations that is not homothetic, then there is a countable coloring of the real numbers without a monochromatic solution to $\mathcal{L}$ in distinct variables. The following theorem classifies those systems of linear homogeneous equations that are $\aleph_0$-regular in ZF+DC+LM.

Theorem 9 *In ZF+DC+LM, a system $\mathcal{L}$ of homogeneous linear equations is $\aleph_0$-regular over $\mathbb{R}$ if and only if $\mathcal{L}$ is homothetic and there is a solution to $\mathcal{L}$ in distinct variables.*

This classification is considerably different from the classification in ZFC given by Theorem 1. For example, $x_1 + x_2 = 2x_3$ is $\aleph_0$-regular over $\mathbb{R}$ in ZF+DC+LM but not $\aleph_0$-

regular over $\mathbb{R}$ in ZFC. The proof of Theorem 9 follows from a result of Ceder [5].

If $S \subset \mathbb{R}^n$, then a *homothetic copy* of S is a set $aS+b = \{as+b : s \in S\}$ where $a, b \in \mathbb{R}$ and $a \neq 0$. Notice that a system $\mathcal{L}$ of linear homogeneous equations is homothetic if and only if for every solution $(x_1, \ldots, x_n)$ of $\mathcal{L}$, we have $(ax_1+b, \ldots, ax_n+b)$ is also a solution of $\mathcal{L}$ for all a and b in $\mathbb{R}$. Hence the solution space of a system $\mathcal{L}$ of linear equations is closed under taking homothetic copies if and only $\mathcal{L}$ is homothetic. The last ingredient of the proof of Theorem 9 is the following theorem of Ceder [5].

Theorem 10 (Ceder 1969) *If S is a finite subset of $\mathbb{R}^n$, then every countable coloring of $\mathbb{R}^n$ with each color class Lebesgue measurable contains a monochromatic homothetic copy of S.*

Since the set of solutions to a linear homogeneous system of equations is closed by homothetic copies, then Theorem 9 follows from Theorem 10.

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