Knaster-Kuratowski-Mazurkiewicz (KKM) Theorem

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Recall that a fixed point of a map $f : X \to X$, defined on a set X, is an element $x \in X$ such that $f(x) = x$. A celebrated theorem of L.E.J. Brouwer stated that if X is a nonempty closed, convex subset of $\mathbb{R}$ (e.g. X is the convex hull of a finite number of points) and $f : X \to X$ is a contnuous map, then f has a fixed point in X. The goal of this note is to use Brouwer's fixed-point theorem to obtain a result for set-valued maps known as the Knaster-Kuratowski-Mazurkiewicz theorem. This theorem turns out to be quite useful in the branch of mathematical economics known as game theory for nonconstructively showing the existence of so-called 'equilibria'.

Suppose $X \subset \mathbb{R}^n$ is nonempty and $M : X \to 2^X$ such that $M(x)$ is a closed nonempty subset of X, for each $x \in X$. We use the notation co to denote the convex hull of a set.

Theorem 1. *(KKM) If* $\operatorname{co}(F) \subset \bigcup_{x\in F} M(x)$ *for every finite subset* $F \subset X$*, then* $\bigcap_{x\in F} M(x) \neq \emptyset$*. Moreover, by the finite intersection property,* $\bigcap_{x\in X} M(x) \neq \emptyset$*, if* X *is compact.*

Proof. We prove the proposition by contradiction. Suppose there exists a nonempty finite subset $F \subset X$ such that $\bigcap_{x\in F} M(x) = \emptyset$. Define a map $\Phi : \operatorname{co}(F) \to \mathbb{R}^n$ by

$$y \mapsto \frac{\sum_{x\in F} d_{M(x)}(y)x}{\sum_{x\in F} d_{M(x)}(y)}, \qquad \forall y \in \operatorname{co}(F)$$

where $d_{M(x)}(y) = \inf_{z\in M(x)} \|y - x\|$ (the distance is the standard Euclidean norm). Note that this map is well-defined: since $y \in \operatorname{co}(F) \subset M(x)$, for some $x \in F$, and $\bigcap_{x\in F} M(x) = \emptyset$, we have that $d_{M(x')}(y) > 0$ for some $x' \in F$.

If we can show that Φ is a continuous self-map of $\operatorname{co}(F)$ into $\operatorname{co}(F)$, then we can apply the Brouwer fixed-point theorem to Φ to obtain a fixed point $z \in \operatorname{co}(F)$. If we define $G := \{x \in F : z \notin M(x)\}$, then

$$z = \frac{\sum_{x\in F} d_{M(x)}(z)x}{\sum_{x\in F} d_{M(x)}(z)} = \frac{\sum_{\substack{x\in F\\ z\notin M(x)}} d_{M(x)}(z)x}{\sum_{\substack{x\in F\\ z\notin M(x)}} d_{M(x)}(z)} = \frac{\sum_{x\in G} d_{M(x)}(z)x}{\sum_{x\in G} d_{M(x)}(z)}$$

implies that $z \in \operatorname{co}(G)$. But then since $z \in \operatorname{co}(G) \subset \bigcup_{x\in G} M(x)$ by hypothesis, we have that $z \in M(x)$, for some $x \in G$, which contradicts the definition of G.

Since F is finite, it is clear that Φ is a self-map. To see that Φ is continuous, it suffices to show $d_{M(x)}(\cdot)$ is continuous, then the result follows since the composition of continuous functions is again continuous. If, for $\epsilon > 0$ given, $\|y - y'\| < \epsilon$, then

$$d_{M(x)}(y) \leq \|y - z\| \leq \|y - y'\| + \|y' - z\| < \epsilon + \|y' - z\|$$

Taking the infimum of the RHS, we obtain that $d_{M(x)}(y') > d_{M(x)}(y) - \epsilon$. By the same argument, we see that $d_{M(x)}(y) > d_{M(x)}(y') - \epsilon$, which completes the proof. □

Suppose C is a nonempty, compact, and convex subset of $\mathbb{R}$ and $f : C \times C \to \mathbb{R}$ is a function which is quasiconcave in x and lower semicontinuous in y. Wan use the KKM theorem to prove the Ky Fan minimax inequality, which states that

$$\inf_{y\in C} \sup_{x\in C} f(x, y) \leq \sup_{x\in C} f(x, x)$$

Proof. For each $x \in C$ fixed, the level set $\{y \in C : f(x, y) \leq f(z, z)\}$ is closed by hypothesis that f is l.s.c. in the second variable. Being the closed subset of C, it is also compact. For $x \in C$, we define the nonempty closed set $M(x)$ to be the level set

$$M(x) := \left\{ y \in C : f(x, y) \leq \sup_{z\in C} f(z, z) \right\}$$

Let $F = \{x_1, \cdots, x_n\}$ be a finite subset of C. I claim that $\operatorname{co}(F) \subset \bigcup_{x\in F} M(x)$. Let $\lambda_1 x_1 + \cdots + \lambda_n x_n$ be an arbitrary convex combination in $\operatorname{co}(F)$. Since f is quasiconcave in the first coordinate, we have the inequalities

$$\sup_{z\in C} f(z, z) \geq f(\lambda_1 x_1 + \cdots + \lambda_n x_n, \lambda_1 x_1 + \cdots + \lambda_n x_n) \geq \min_{1\leq j\leq n} f(x_j, \lambda_1 x_1 + \cdots + \lambda_n x_n)$$

Therefore $\lambda_1 x_1 + \cdots + \lambda_n x_n \in M(x_j)$, for some j.

By the KKM theorem, $\bigcap_{x\in C} M(x) \neq \emptyset$, which implies that there exists $y \in C$ such that

$$f(x,y) \leq \sup_{z\in C} f(z,z), \qquad \forall x \in C$$

Taking the supremum over $x \in C$ in the fiirst coordinate, we conclude that $\sup_{x\in C} f(x,y) \leq \sup_{z\in C} f(z,z)$. Taking the infimum over $y \in C$ on the LHS, we obtain the Ky Fan minimax inequality. □

We can use the Ky Fan minimax inequality to prove the existence of a Nash equilibrium for nonempty compact, convex set strategy sets $C_k \subset \mathbb{R}^n$, for $1 \leq k \leq m$.

Proposition 2. *Let the C_k be as above $C = C_1 \times \cdots \times C_m$ and let $f_1, \cdots, f_m : C \to \mathbb{R}$ be continuous functions such that the function*

$$x_k \in C_k \mapsto f_k(y_1, \cdots, x_k, \cdots, y_m)$$

is convex on C_k, for all $y_i \in C_i$, $i \neq k$, fixed. Then there exists an element $c = (c_1, \cdots, c_m) \in C$ such that

$$f_k(c) \leq f_k(c_1, \cdots, x_k, \cdots, c_m), \qquad \forall x_k \in C_k, \forall 1 \leq k \leq m$$

Proof. Consider the function

$$f(x,y) = \sum_{k=1}^{m} \left[f_k(y) - f_k(y_1, \cdots, x_k, \cdots, y_m)\right], \qquad \forall x, y \in C$$

Observe that f is continuous on $C \times C$, being the composition of continuous functions. I claim that f is quasiconcave in the first coordinate and lower semicontinuous in the second variable. Indeed, in fact, f is a convex function of x, for y fixed, by our hypotheses on the f_k. Since each f_k is continuous on C, it follows that f is continuous in y, for x fixed. We can apply the Ky Fan minimax inequality to f to obtain

$$\inf_{y\in C} \sup_{y\in C} f(x,y) \leq \sup_{x\in C} f(x,x) = 0$$

Since the supremum of l.s.c. functions is again l.s.c. and l.s.c. functions attain their infimum on compact subsets, we see that there exists $c \in C$ such that

$$\sup_{x\in C} f(x,c) = \inf_{y\in C} \sup_{x\in C} f(x,y)$$

Observe that since $f_k(y) - f_k(y_1, \cdots, x_k, y_m) = 0$ for $x_k = y_k$, where $y = (y_1, \cdots, y_m) \in C$ is fixed, we see that

$$\sup_{x\in C} f(x,y) = \sum_{k=1}^{m} \sup_{x\in C} \left[f_k(y) - f(y_1, \cdots, x_k, \cdots, y_m)\right] = \sum_{k=1}^{m} f_k(y) - \inf_{x\in C} f(y_1, \cdots, x_k, \cdots, y_m) \geq 0,$$

and each term on the RHS is nonnegative. Hence, $\inf_{y\in C} \sup_{x\in C} f(x,y) = 0$ and since

$$f_k(c) - \inf_{x\in C} f(c_1, \cdots, x_k, \cdots, x_m) = 0 \Longrightarrow f_k(c) \leq f(c_1, \cdots, x_k, \cdots, c_m)\ \forall x_k \in C_k$$

and for all $k \in \{1, \cdots, m\}$. □