NOTES ON THE FUNDAMENTAL GROUP

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CONTENTS

1. Introduction to the fundamental group

In this course, we describe the fundamental group, which is an algebraic object we can attach to a geometric space. We will see how this fundamental group can be used to tell us a lot about the geometric properties of the space. Loosely speaking, the fundamental group measures "the number of holes" in a space. For example, the fundamental group of a point or a line or a plane is trivial, while the fundamental group of a circle is $\mathbb{Z}$. Slightly more precisely, the fundamental group of a space X is the space of all loops in X, where we say two loops are equivalent if you can wiggle one to the other.

As a standard application, if two spaces are sufficiently similar, in an appropriate sense to be defined then they will have the same fundamental group. Since the fundamental group is a relatively computable object, this will, right off the bat, give us a way of proving that two spaces are quite different.

Moreover, soon after defining the fundamental group, we will be able to immediately derive a number of interesting consequences. For example, we will prove the Borsuk-Ulam theorem, which implies, among other things, that at any time, there are always some two points on exact opposite sides of the earth, with the same temperature and barometric pressure. We will also use this to show you can always slice a ham sandwich so that there is the same amount of both pieces of bread and ham on each side of the slice (the "Ham Sandwich theorem"). Let's now begin defining the fundamental group.

One excellent source for understanding more about the fundamental group is [Hat02]. Indeed, most of the pictures in this document were copied from [Hat02].

2. Preliminaries: spaces and homotopies

2.1. **Spaces.** As mentioned above, the fundamental group will be a way of assigning a certain group to a given space. So, as a first step, we will introduce spaces and groups:

Definition 2.1. A **space** X is a subset of $\mathbb{R}^n$. A **pointed space** (X, x_0) is a space X together with a point $x_0 \in X$. For (X, x_0) a pointed space, we call x_0 is called the **basepoint** of X.

Warning 2.2. We will often be sloppy about keeping track exactly how a given space X is embedded in $\mathbb{R}^n$.

Remark 2.3 (Unimportant remark)**.** The above definition is "bad" in that it is not natural to embed a give space inside $\mathbb{R}^n$, but rather it is better to consider it as an abstract space in its own right. This makes certain construction easier, since we do not have to keep track of an embedding into $\mathbb{R}^n$. Nevertheless, working with subsets of $\mathbb{R}^n$ is more concrete, and so we will adapt this perspective for most of the course, unless otherwise noted.

For a brief description of a more general notion of space, see Appendix A.

There is a third notion of a space which is perhaps even more correct than that of a topological space: Perhaps you can impress your friends by saying "the category of spaces is the cocompletion of the infinity category point" but, for now, let's just stick to subsets of $\mathbb{R}^n$.

Example 2.4. Here are some examples of spaces we will encounter frequently:

(1) The space $\mathbb{R}^n$, known as **Euclidean** n**-space**. As a special case, we have $\mathbb{R}^0$, which is a point.
(2) The n**-disk**
$$D^n := \{x \in \mathbb{R}^n : |x| \leq 1\}.$$
(3) The n**-sphere**
$$S^n := \left\{x \in \mathbb{R}^{n+1} : |x| = 1\right\}.$$
(4) The **interval** $I := [0, 1] \subset \mathbb{R}$.

2.2. **Maps of spaces.** Now that we've defined our objects of spaces, the next step is to define the maps between the objects.

Definition 2.5. For two spaces X and Y, a map of sets $f : X \to Y$ is **continuous** if for any sequence of points $\{x_i\}_{i=1}^n$ in X converging to $x \in X$, the sequence $\{f(x_i)\}_{i=1}^n$ converges to $f(x) \in Y$. A **continuous**

map of pointed spaces $f : (X, x_0) \to (Y, y_0)$ is a continuous map of spaces $f : X \to Y$ such that $f(x_0) = y_0$.

Loosely, being continuous means that the map should take limits to limits.

Example 2.6. The map

$$\begin{aligned} f\colon I &\to \mathbb{R}^2 \\ x &\mapsto (x, x) \end{aligned}$$

is continuous because for any sequence $x_i \to x$, we have $f(x_i) = (x_i, x_i) \to (x, x) = f(x)$.

Example 2.7 (Non-example)**.** The map

$$\begin{aligned} f\colon I &\to \mathbb{R}^2 \\ x &\mapsto (x, \lceil x \rceil) \end{aligned}$$

is not continuous because the sequence $\frac{1}{n}$ tends to 0, and $f(0) = (0, 0)$, but $f(1/n) = (1/n, 1)$, tends to $(0, 1)$.

Example 2.8. The map $f : [0, 2\pi] \mapsto \mathbb{R}^2$ sending $x \mapsto (\sin x, \cos x)$ is continuous because the functions $\sin x$ and $\cos x$ are continuous. The image is the **unit circle**, which we denote by S^1.

Exercise 2.9. Show that a map

$$\begin{aligned} f\colon S &\to \mathbb{R}^n \\ s &\mapsto (f_1(s), \ldots, f_n(s)) \end{aligned}$$

is continuous if and only if each f_i, viewed as a function $f_i : S \to \mathbb{R}$, is continuous.

Exercise 2.10. Verify f in Example 2.8 is continuous directly from Definition 2.5, using Exercise 2.9 and your favorite definition of sin and cos *Hint:* It may be easier to verify continuity if you choose a well-suited definition.

2.3. **Homotopies and Loops.** We are nearly ready to define the fundamental group. We will define it as the group of all loops, so first we have to say what a loop and a group is.

Definition 2.11. A **path** is a continuous map $f : I \to X$. A **loop** in a pointed space (X, x_0) is a path $f : I \to X$ such that $f(0) = f(1) = x_0$.

Example 2.12. The map $I \to S^1 \subset \mathbb{R}^2$ sending $t \mapsto (\cos 2\pi t, \sin 2\pi t)$ is a loop, where we consider S^1 as the pointed space with basepoint $(1, 0) \in \mathbb{R}^2$.

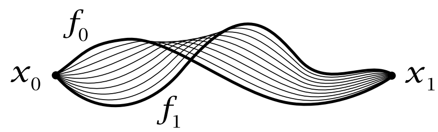

FIGURE 1. A picture of a homotopy between paths f_1 and f_2 from x_0 to x_1

Remark 2.13. It is often convenient to identify a loop $f : I \to X$ with its image $f(I) \subset X$.

Definition 2.14. A **homotopy** of paths on X is a continuous map $f : I \times I' \to X$ with $f(0,t) = f(0,0)$ and $f(1,t) = f(1,0)$ for all t.

A **homotopy** (of loops) on (X, x_0) is a continuous map $f : I \times I' \to X$ with $f(0,t) = f(1,t) = x_0$. Define

$$\begin{aligned} f_t \colon I &\to X \\ s &\mapsto f(s,t). \end{aligned}$$

If $f : I \times I \to X$ is a homotopy, we say f_0 and f_1 are **homotopic** and write $f_0 \sim f_1$. A loop is **nullhomotopic** if it is homotopic to the constant loop (i.e., the loop $f : I \to X$ given by $f(t) = x_0$ for all t).

Remark 2.15. Intuitively, a homotopy is a family of paths interpolating between f_0 and f_1.

Example 2.16. Consider the pointed space $(\mathbb{R}^2, 0)$ (where 0 really denotes the point $(0,0)$)

$$\begin{aligned} f_1 \colon I &\to \mathbb{R}^2 \\ s &\mapsto (1 - \cos 2\pi s, \sin 2\pi s) \end{aligned}$$

and

$$\begin{aligned} f_0 \colon I &\to \mathbb{R}^2 \\ s &\mapsto (0,0)\,. \end{aligned}$$

Note that f_0 and f_1 are homotopic via the homotopy

$$\begin{aligned} f \colon I \times I &\to \mathbb{R}^2 \\ (s,t) &\mapsto (t(1 - \cos 2\pi s), t \sin 2\pi s)\,. \end{aligned}$$

This homotopy linearly interpolates between f_0 and f_1. One can picture this as a circle getting squashed to a point.

Exercise 2.17. Recall that a subset $X \subset \mathbb{R}^n$ is **convex** if for any two points $x, y \in X$, the line segment joining x to y is also contained in X. Generalize Example 2.16 by showing that if f_0, f_1 are two loops in a convex set $X \subset \mathbb{R}^n$ based at the same point x_0, then $f_t(s) = f(s,t) = (1-t)f_0(s) + tf_1(s)$ defines a homotopy between f_0 and f_1.

Example 2.18. As an example of two loops which are not homotopic, consider the pointed space $(\mathbb{R}^2 - \{0\}, (1,0))$ and the two loops $t \mapsto (\cos 2\pi t, \sin 2\pi t)$ and the constant map $t \mapsto (1,0)$ are not homotopic. While this is certainly intuitively believable, it is somewhat tricky to prove. Indeed, we will see this somewhat later; it follows from Theorem 4.1.

We next claim that homotopy defines an equivalence relation on the set of loops in a space. Recall that an equivalence relation S is a relation $\sim$ on a set S that is

(1) reflexive, meaning $x \sim x$
(2) symmetric, meaning $x \sim y \implies y \sim x$
(3) and transitive, meaning $x \sim y$ and $y \sim z \implies x \sim z$.

Remark 2.19 (Unimportant remark)**.** Formally, one should define an equivalence relation as a subset of $S \times S$, but this tends to obfuscate things more than clarify them.

Example 2.20. The relation on the integers $\mathbb{Z}$ defined by "$a \sim b$ if $a - b$ is even" is an equivalence relation. To check the three properties, note that

(1) $a - a = 0$ is even
(2) $a - b = b - a$, so $a - b$ is even if and only if $b - a$ is.
(3) If $a - b$ is even and $b - c$ is even, then $a - c$ is even.

Lemma 2.21. *Let (X, x_0) be a pointed space. Homotopy defines an equivalence relation on the set of loops in (X, x_0).*

Proof. We have to show reflexivity, symmetry, and transitivity. These are probably best understood by drawing pictures. For reflexivity, we have to show any loop is homotopic to itself. That is, for a loop $h : I \to X$ we need a homotopy $f : I \times I \to X$ with $f_0 = h$ and $f_1 = h$. We can simply take $f(s,t) := h(s)$, independent of t. Intuitively, this is just the "constant homotopy." To show symmetry, given a homotopy $f \sim g$, we can "reverse the direction of time" to show $g \sim f$. To show reflexivity, if $f \sim g$ and $g \sim h$, then $f \sim g$ by performing the two homotopies $f \sim g$ and $g \sim h$ at double speed, and composing them.

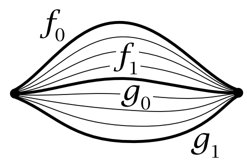

FIGURE 2. A composite of two homotopies

Exercise 2.22. Write out the formulas for symmetry and transitivity to rigorously complete this proof. *Hint:* For symmetry, if $F(s,t)$ is a homotopy between f and g, try $F(s, 1-t)$. For transitivity, if F is a homotopy between f and g and G is a homotopy between g and h, try the function

$$\phi(s,t) := \begin{cases} F(s,2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ G(s,2t-1) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

$\square$

Remark 2.23. Note that the equivalence relation of homotopy partitions the loops in (X, x_0) into equivalence classes called **homotopy classes**.

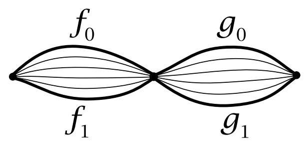

FIGURE 3. If $f_0 \sim f_1$ and $g_0 \sim g_2$ then $f_0 \star g_0 \sim f_1 \star g_1$.

3. THE FUNDAMENTAL GROUP: A DEFINITION AND BASIC PROPERTIES

3.1. **Finally defining the fundamental group.** Finally, we can define the fundamental group. If you are not familiar with the definition of a group, now may be a good time to read the beginning of Appendix B.

Definition 3.1 (Composition of paths)**.** Let $f, g : I \to X$ be two paths. Define the **composition** of f and g, denote $(f \star g) : I \to X$, by

$$(f \star g)(t) := \begin{cases} f(2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ g(2t-1) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Remark 3.2. Note that if f and g are loops, then $f \star g$ will again be a loop.

Remark 3.3. Intuitively, the composition law is just given by following one path, and then the other.

Definition 3.4. The **fundamental group** of (X, x_0), denoted $\pi_1(X, x_0)$, is the group whose underlying set is loops, up to homotopy, (so that two homotopic loops correspond to the same element in the fundamental group) with composition operation given by $[f] \cdot [g] = [f \star g]$ for loops $f, g : I \to X$.

Proposition 3.5. *The fundamental group, $\pi_1(X, x_0)$, is a group.*

Proof. To check it is a group, we have to show there is an identity element, inverses exist, and the group law is associative. First, we construct the identity loop and inverse loop. Then we give an intuitive sketch of why these satisfy the properties of a group. Finally, we leave it as an exercise to complete the proof.

Define the identity e to be the homotopy class of the constant path $f : I \to X$ sending $t \mapsto x_0$. Given a loop $f : I \to X$ define $f^{-1} : I \to X$ by $f^{-1}(t) = f(1-t)$.

Next, we intuitively justify the three axioms in turn. First, the identity axiom makes sense because for any loop f, we have $[f] \cdot e =$

$[f]$ because $[f] \cdot e$ corresponds to going around f at double speed and then staying still, which is homotopic to moving around f at normal speed. Second, the inversion axiom makes sense because if we first go around f and then go backwards, we can linearly move the midpoint of the path backwards along the path until to show it is homotopic to the constant path. Third, the associativity axiom holds because if f, g, h are three loops, then $(f \star g) \star h$ and $f \star (g \star h)$ both result in going around f, g, and h in the same order, albeit at different speeds.

Exercise 3.6. Verify that the above satisfy the axioms of a group as follows

(1) Show that for f a loop, $[f] \cdot e = e \cdot [f] = [f]$.
(2) Show that for f a loop, $[f] \cdot [f^{-1}] = [f^{-1}] \cdot [f] = e$. *Hint:* Let g be the constant loop at x_0. Show that

$$F(s,t) := \begin{cases} f(2st) & \text{if } s \leq \frac{1}{2} \\ f(1-2st) & \text{if } s \geq \frac{1}{2} \end{cases}$$

defines a homotopy between $f \star f^{-1}$ and g.
(3) Show that for f, g, h loops, $([f] \cdot [g]) \cdot [h] = [f] \cdot ([g] \cdot [h])$.

□

Definition 3.7. A space X is **path connected** if there is a path joining any two points (i.e., for all $x, y \in X$ there is some path $f : I \to X$ with $f(0) = x, f(1) = y$). A space is **simply connected** if it is path connected and for all points $x \in X$, $\pi_1(X, x)$.

We next note that the fundamental group of a path connected space does not depend on the choice of basepoint:

Lemma 3.8. *Let X be a path connected space and $x, y \in X$ two points. Then, we have an isomorphism of groups $\pi_1(X, x) \simeq \pi_1(X, y)$.*

Proof. Explicitly, we can construct the isomorphism $\pi_1(X, x) \to \pi_1(X, y)$ as follows. Start by choosing a path η from x to y (meaning $\eta : I \to X$ with $\eta(0) = x, \eta(1) = y$). Then, send a loop γ based at x to the loop $\eta^{-1} \star \gamma \star \eta$, which is a loop based at y.

Exercise 3.9. Verify that if γ and γ' are homotopic then so are $\eta^{-1} \star \gamma \star \eta$ and $\eta^{-1} \star \gamma' \star \eta$, so the above map is a well defined homomorphism of fundamental groups.

Exercise 3.10. Show the above map is an isomorphism by constructing an inverse map sending a loop δ based at y to $\eta \star \delta \star \eta^{-1}$. Verify

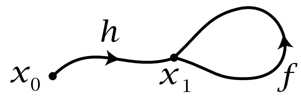

FIGURE 4. A picture of the change of basepoint map by a path h from x_0 to x_1

that this indeed defines an inverse map by checking that the composition of this map with the previous one in both directions is the identity.

$\square$

Remark 3.11. By Lemma 3.8, the fundamental group of a path connected space does not depend on the basepoint, and so it is not particularly important to keep track of the basepoint. For this reason, we will often not be explicit with which basepoint we choose in the remainder of these notes.

Exercise 3.12. Let (X, x_0) be a path connected space. Show that the fundamental group is **abelian**, meaning $[f \star g] = [g \star f]$, if and only if for any $y_0 \in X$ and two paths η_1, η_2 from x_0 to y_0 (meaning $\eta_i(0) = x_0, \eta_i(1) = y_0$) the induced homomorphisms

$$\begin{aligned} \phi_1 : \pi_1(X, x_0) &\to \pi_1(X, y_0) \\ [\gamma] &\mapsto [\eta_1^{-1} \star \gamma \star \eta_1] \end{aligned}$$

and

$$\begin{aligned} \phi_2 : \pi_1(X, x_0) &\to \pi_1(X, y_0) \\ [\gamma] &\mapsto [\eta_2^{-1} \star \gamma \star \eta_2] \end{aligned}$$

are the same homomorphism. *Hint:* Show that $\phi_1 = \phi_2$ if and only if $\phi_1 \phi_2^{-1} = \mathrm{id}$. Show the latter statement holds for all such ϕ_1, ϕ_2 if and only if $a^{-1} b a = b$ for every $a, b \in \pi_1(X, x_0)$.

Definition 3.13. We say a continuous map of spaces $f : X \to Y$ is a **homeomorphism** if there is a map $g : Y \to X$ with $f \circ g = \mathrm{id}$ and $g \circ f = \mathrm{id}$. In this case, we say X is **homeomorphic to** Y and write $X \simeq Y$.

Exercise 3.14. Suppose $f : X \to Y$ is a homeomorphism with $x \mapsto y$. Show that $\pi_1(X, x)$ and $\pi_1(Y, y)$ are isomorphic. *Hint:* Define a homomorphism $\pi_1(f) : \pi_1(X, x) \to \pi_1(Y, y)$ by sending a path $\gamma : I \to X$ to the path $f \circ \gamma : I \to Y$. Since f is a homeomorphism, f

has an inverse map f^{-1}. Show that the corresponding map $\pi_1(f^{-1}) : \pi_1(Y, y) \to \pi_1(X, x)$ is an inverse to $\pi_1(f)$.

3.2. **Examples of a trivial fundamental group.** In this section, we give several examples of spaces with trivial fundamental group. I.e., examples of simply connected spaces.

Example 3.15. The space $(\mathbb{R}^n, 0)$ has trivial fundamental group. To see this, we have to show every loop is homotopic to the constant loop. But indeed, for any loop $f : I \to \mathbb{R}^n$, the homotopy $F(s,t) = t \cdot f(s)$ defines a homotopy between f and the trivial loop.

Exercise 3.16. Generalize Example 3.15 by showing that for any convex subset $X \subset \mathbb{R}^n$ and $x_0 \in X$ we have $\pi_1(X, x_0)$ is the trivial group. In particular, show that D^n has trivial fundamental group. *Hint:* Use Exercise 2.17.

3.3. **Yo I heard you like groups...** We now briefly explore what happens when your space is also a group (such as the space S^1 where you can add two points of S^1 by adding their angles). We will see that this forces the fundamental group to be abelian. This subsection is somewhat peripheral to the discussion, and can safely be skipped on a first reading.

Definition 3.17. A **group space** is a space G with a continuous multiplication map $m : G \times G \to G$ and a continuous inversion map $i : G \to G$ making the underlying set of G into a group.

Theorem 3.18 (Eckmann-Hilton)**.** *Let G be a group space and $e \in G$ be the identity point. Then $\pi_1(G, e)$ is abelian.*

Proof. To show $\pi_1(G, x)$ is abelian, we will show that for any two loops $\gamma, \delta : I \to G$ we have $\gamma \star \delta \sim \delta \star \gamma$. Indeed, for this, we construct a homotopy between the two loops above.

Exercise 3.19. Let $h : I \to G$ denote the constant loop sending $t \mapsto e$. Verify that the homotopy

$$\begin{aligned} F \colon I \times I' &\to G \\ (s,t) &\mapsto m((\gamma \star h)(\max(0, s - t/2)), (h \star \delta)(\min(1, s + t/2)) \end{aligned}$$

defines a homotopy between $\gamma \star \delta$ and $\delta \star \gamma$.

$\square$

Exercise 3.20. Show that $\pi_1(S^1, x_0)$ is abelian. (Later, in Theorem 4.1, we will see it is $\mathbb{Z}$.)

Feel free to skip the following exercise if you have not seen determinants.

Exercise 3.21 (Assuming knowledge of determinants)**.** Let GL_n denote the space of $n \times n$ invertible matrices, viewed as a subspace of $\mathbb{R}^{n^2}$ by sending a matrix $A = (a_{ij})$ to the point in $\mathbb{R}^{n^2}$ whose n^2 coordinates are given by the n^2 entries of A (explicitly, the coordinate in place $n(i-1)+j$ is a_{ij}).

(1) Show that GL_n is an open subset of $\mathbb{R}^{n^2}$. *Hint:* The complement is where the determinant vanishes.
(2) Show that matrix multiplication makes GL_n into a group space. *Hint:* For multiplication and inversion, write out the explicit formula. For this you will need to use the explicit formula for the inverse of a matrix, given by Cramer's rule. It may be helpful to first try the cases $n = 1$ and $n = 2$.
(3) Show that $\pi_1(GL_n, \mathrm{id})$ is abelian.

Remark 3.22. Although we have not yet seen many examples of fundamental groups, it is in general, quite common from the fundamental group to be nonabelian. For example, this is true for the figure 8, (i.e., two circles meeting at a point) see Example 6.6.

4. The fundamental group of the circle

4.1. **Statement of the main result.** So far, we have only seen examples of spaces with trivial fundamental group. If the fundamental group were always trivial, it would not yield any mathematical information. Fortunately, this is not the case. In fact, the prototypical example of a space with nontrivial fundamental group is the circle:

Theorem 4.1. *Let $x_0 \in S^1$ be a point and let $f : I \to S^1$ be the loop going once counterclockwise around the circle at constant speed. Then, the homomorphism $\phi : \mathbb{Z} \to \pi_1(S^1, x_0)$ sending $1 \mapsto [f]$ is an isomorphism.*

We will prove Theorem 4.1 in subsection 4.3 (assuming a particular result on lifting covers), and then we will give a full proof later in subsection C.4. At a certain level the two proofs are really the same proof, though the latter depends on the machinery of universal covers.

4.2. **Applications.** Having setup the theory of the fundamental group, we are now prepared to reap some cool applications.

To start, we prove the Brouwer fixed point theorem. Before proving it, we need to prove that the fundamental group is functorial. That is, we need to show that a continuous map of spaces induces a homomorphism of fundamental groups.

Proposition 4.2. *Let $f : (X, x_0) \to (Y, y_0), g : (Y, y_0) \to (Z, z_0)$ be two continuous maps of pointed spaces. Then,*

(1) f induces a homomorphism $\pi_1(f) : \pi_1(X, x_0) \to \pi_1(Y, y_0)$.
(2) The homomorphism $\pi_1(g) \circ \pi_1(f) : \pi_1(X, x_0) \to \pi_1(Z, z_0)$ is equal to the homomorphism $\pi_1(g \circ f) : \pi_1(X, x_0) \to \pi_1(Z, z_0)$.
(3) For $\mathrm{id} : (X, x_0) \to (X, x_0)$ the identity map, we have $\pi_1(\mathrm{id}) : \pi_1(X, x_0) \to \pi_1(X, x_0)$ is the identity map of groups.

Proof. We prove the three parts in turn. For the first, we define the map by sending a loop $\gamma : I \to X$ to the loop $f \circ \gamma : I \to Y$.

Exercise 4.3. Verify that homotopic loops map to homotopic loops under composition with f, and hence the above construction defines a well-defined map of fundamental groups.

For the second point, using the above definition, we note that for a path $\gamma : I \to X$ we have

$$\begin{aligned}\pi_1(g) \circ \pi_1(f)([\gamma]) &= \pi_1(g)([f \circ \gamma]) \\ &= [g \circ (f \circ \gamma)] \\ &= [(g \circ f) \circ \gamma] \\ &= \pi_1(g \circ f)[\gamma],\end{aligned}$$

as we wished to check.

For the third part, it suffices to note that $\pi_1(\mathrm{id})$ sends any loop to itself, since $\mathrm{id} \circ \gamma = \gamma$. Therefore, it induces the identity morphism on fundamental groups. $\square$

Using this functoriality, we can provide a quick proof of the Brouwer fixed point theorem:

Theorem 4.4 (Brouwer fixed point theorem)**.** *Any continuous map* $h : D^2 \to D^2$ *has a fixed point. That is, there is some* $x \in D^2$ *with* $h(x) = x$.

Proof. Suppose, for the sake of contradiction, that h has no fixed point. Define a continuous map $r : D^2 \to S^1$ as follows. Start with a point $x \in D^2$ and its image $h(x)$. Draw the ray from $h(x)$ to x, and define $r(x)$ to be the intersection of the continuation of this ray with S^1. This is a well defined map because no point is fixed, and two ordered points determine a unique ray. Further, the map r is continuous because if we move x a little bit, $h(x)$ also moves only a little bit and so the ray from $h(x)$ only moves a little bit which means $r(x)$ only moves a little bit.

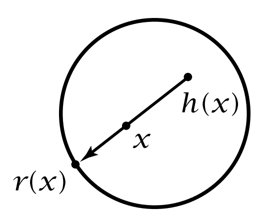

FIGURE 5. A picture of the map r

Observe from the construction that if $x \in S^1$ then $r(x) = x$ because the ray from $f(x)$ to x meets S^1 at x. Let $i : S^1 \to D^2$ denote the inclusion of S^1 as the boundary of D^2. It follows that we have the commuting diagram

$$\begin{tikzcd} S^1 \arrow[rr, "i"] \arrow[dr, "\mathrm{id}"'] & & D^2 \arrow[dl, "r"] \\ & S^1 & \end{tikzcd} \tag{4.1}$$

where the composite map $S^1 \to S^1$ is the identity. This diagram commuting just means that $\mathrm{id} = r \circ i$. Taking fundamental groups, we obtain

$$\begin{tikzcd} \pi_1(S^1, x) \arrow[rr, "\pi_1(i)"] \arrow[dr, "\mathrm{id}"'] & & \pi_1(D^2, x) \arrow[dl, "\pi_1(r)"] \\ & \pi_1(S^1, x) & \end{tikzcd} \tag{4.2}$$

using Proposition 4.2 to obtain this diagram commutes and to deduce that the left map is the identity. But plugging in the groups above, we obtain maps

$$\begin{tikzcd} \mathbb{Z} \arrow[rr, "\pi_1(i)"] \arrow[dr, "\mathrm{id}"'] & & \{e\} \arrow[dl, "\pi_1(r)"] \\ & \mathbb{Z} & \end{tikzcd} \tag{4.3}$$

(where $\{e\}$ denotes the trivial group). So, we obtain that $\mathrm{id} : \mathbb{Z} \to \mathbb{Z}$ can be realized as the composition $\mathbb{Z} \to \{e\} \to \mathbb{Z}$, which is impossible, since the latter composition must send everything to 0, while the identity map $\mathrm{id} : \mathbb{Z} \to \mathbb{Z}$ is surjective. $\square$

Exercise 4.5. Show that any continuous map $f : D^2 \to D^2$ with $f(x) = x$ for all $x \in S^1$ (i.e., all x with $|x| = 1$) is surjective. *Hint:* Use an argument similar to that of Theorem 4.4.

Next, we can provide a neat proof of the fundamental theorem of algebra.

Proposition 4.6. *Every polynomial $f(z) : \mathbb{C} \to \mathbb{C}$ of degree at least 1 has a root.*

Proof. Suppose $f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$ has no root. We may assume $a_n \neq 0$ and divide through by a_n so that $f(z) = z^n +$

$\cdots + z_0$. Then, f defines a continuous map $f : \mathbb{C} \to \mathbb{C} - \{0\}$. Composing f with the continuous map

$$\begin{aligned} p \colon \mathbb{C} - \{0\} &\to S^1 \\ x &\mapsto x/|x| \end{aligned}$$

we obtain a continuous map $g := p \circ f : \mathbb{C} \to S^1$. Now let S^1_r denote the circle of radius r inside $\mathbb{C}$. Restricting g to S^1_r, we obtain a continuous map $g : S^1_r \to S^1$, which is a map between two copies of S^1. On the one hand, we can see that for large r, this sends the loop going once around S^1_r to the loop going n times around S^1 because the leading term of the polynomial dominates, meaning that the map looks like $z \mapsto z^n$.

Exercise 4.7. Spell out the details above, making precise why g sends the generator of $\pi_1(S^1_r, x)$ to n times the generator in $\pi_1(S^1, g(x))$.

On the other hand, as in the proof of Theorem 4.4, the composite

$$S^1_r \longrightarrow \mathbb{C} \longrightarrow S^1 \tag{4.4}$$

induces a morphism on fundamental groups

$$\pi_1(S^1_r, x) \longrightarrow \pi_1(\mathbb{C}, x) \longrightarrow \pi_1(S^1, g(x)), \tag{4.5}$$

which can be identified with the homomorphisms

$$\mathbb{Z} \longrightarrow \{e\} \longrightarrow \mathbb{Z}. \tag{4.6}$$

So, the composite homomorphism must send everything to 0. Hence, we obtain that $n = 0$, and so the polynomial f can only have no roots if $n = 0$, as desired. □

Next, we reach one of the neatest results we'll see in this course, the Borsuk-Ulam theorem.

Theorem 4.8 (Borsuk-Ulam). *Any continuous map $f : S^2 \to \mathbb{R}^2$ has some point x so that $f(x) = f(-x)$.*

Remark 4.9. One "real-world application" of this is that, at any time, there are some two points on opposite sides of the earth with the same temperature and barometric pressure. To see this, just view the surface of the earth as S^2 and let f be the map to $\mathbb{R}^2$ sending a point to the pair (temperature at the point , pressure at the point).

Exercise 4.10. Show the 1-dimensional version of the Borsuk-Ulam theorem. That is, show that any continuous map $f : S^1 \to \mathbb{R}$ has some point with $f(x) = f(-x)$. *Hint:* Apply the intermediate value theorem to $f(x) - f(-x)$.

Proof. Suppose there is no point with $f(x) = f(-x)$. Define the function $g : S^2 \to \mathbb{R}^2 - \{0\}$ sending $x \mapsto f(x) - f(-x)$. Composing g with the continuous map

$$\begin{aligned} p\colon \mathbb{R}^2 - \{0\} &\to S^1 \\ x &\mapsto x/|x| \end{aligned}$$

we obtain a continuous map $h := p \circ g : S^2 \to S^1$. Let $i : S^1 \to S^2$ be the inclusion of the equator. Observe that $-h(x) = h(-x)$ because the same is true of g. That is,

$$g(-x) = f(-x) - f(x) = -(f(x) - f(-x)) = -g(x).$$

Exercise 4.11. Show rigorously that $-(i \circ h)(x) = (i \circ h)(-x)$ implies that the loop passing once around S^1 is sent under $i \circ h$ to a loop passing an odd number of times around S^1. In particular, the resulting homomorphism $\pi_1(S^1, x) \to \pi_1(S^1, h(x))$ is not the constant map.

We now obtain a contradiction, because we also have that the homomorphism $\pi_1(S^1, x) \to \pi_1(S^1, h(x))$ is the constant map. Indeed,

$$S^1 \xrightarrow{\ i\ } S^2 \xrightarrow{\ h\ } S^1 \tag{4.7}$$

induces a morphism on fundamental groups

$$\pi_1(S^1, x) \longrightarrow \pi_1(S^2, x) \xrightarrow{\ h\ } \pi_1(S^1, h(x)). \tag{4.8}$$

We can see that the generator of $\pi_1(S^1, x)$ is mapped to the loop passing once around the equator, which is nullhomotopic in S^2. It follows that the map $\pi_1(i)$ is the constant map sending every element of $\pi_1(S^1, x)$ to the trivial element of $\pi_1(S^2, x)$ (as going n times around the equator in S^2 is also nullhomotopic). It follows that the composite map $h \circ i$ must send everything to $0 \in \mathbb{Z} \simeq \pi_1(S^1, h(x))$. $\square$

Next, we deduce some nice consequences of the Borsuk-Ulam theorem.

Corollary 4.12. *Suppose A, B, C are three closed subsets of S^2 covering S^2. Then, there is some point x so that one of these sets contains both x and $-x$.*

Proof. Define functions $d_B : x \mapsto d(x, B)$ and $d_C : x \mapsto d(x, C)$ where d is the distance function; for a set S, $d(x, S)$ is the smallest real number bigger than or equal to $d(x, s)$ for every $s \in S$. The functions d_B and d_C are well defined because B and C are closed. Then, define the continuous map

$$\begin{aligned} f \colon S^2 &\to \mathbb{R}^2 \\ x &\mapsto (d_B(x), d_C(x)). \end{aligned}$$

By the Borsuk-Ulam theorem, there is some point x with $f(x) = f(-x)$. We will show x and $-x$ lie in the same set. Indeed, if $d_B(x) = d_B(-x) = 0$, this means x and $-x$ lie in B. Similarly, if $d_C(x) = d_C(-x) = 0$, then x and $-x$ lie in C. Finally, if $d_B(x) \neq 0$ and $d_C(x) \neq 0$, then x does not lie in either B or C, so x lies in A. By similar reasoning $-x$ also lies in A. □

Corollary 4.13 (Ham Sandwich theorem)**.** *Let A_1, A_2, A_3 be three closed and bounded sets in $\mathbb{R}^3$. Then there is a 2-plane $P \subset \mathbb{R}^3$ that cuts each of the three sets A_i into two sets of equal volume.*

Remark 4.14. This is called the ham sandwich theorem because you may pretend that A_1 and A_3 two pieces of bread, and A_2 is a piece of ham. The theorem says you can always cut the two pieces of bread and the piece of ham exactly in half, with a slice by a single plane.

Proof. Define a continuous map $f : S^2 \to \mathbb{R}^2$ as follows. For any unit vector $\vec{v}$, consider $\vec{v}$ as a point of S^2. There is some plane Q_v normal to v which cuts A_1 into two pieces of equal volume. Let Q_v^+ be those points in $\mathbb{R}^3$ in the direction of $\vec{v}$ from Q and let Q^- be those points of $\mathbb{R}^3$ in the direction of $-\vec{v}$ from Q. Define $f(\vec{v}) = (f_2(\vec{v}), f_3(\vec{v}))$, where $f_i(\vec{v}) = \mathrm{Vol}(A_i \cap Q_v^+) - \mathrm{Vol}(A_i \cap Q_v^-)$. Here, $\mathrm{Vol}(S)$ denotes the volume of the set S.

Exercise 4.15. Verify that the function f is continuous.

Applying the Borsuk-Ulam theorem, there is some x with $f(x) = f(-x)$. However, we may also note that $f(-x) = -f(x)$ by construction. Therefore, $f(x) = f(-x) = 0$. It follows that the plane Q_x cuts A_1 into two equal pieces also satisfies $\mathrm{Vol}(A_i \cap Q_v^+) = \mathrm{Vol}(A_i \cap Q_v^-)$ for $i = 2, 3$. This means it also cuts A_2 and A_3 into pieces of equal volume, as desired. □

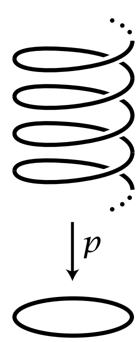

FIGURE 6. The map $\mathbb{R} \to S^1$.

4.3. **Computing the fundamental group of the circle.** Here, we now give a hand-on proof that $\pi_1(S^1, x_0) = \mathbb{Z}$, Theorem 4.1. Later, in subsection C.4, we give a more conceptual proof that depends on much more machinery. We recommend you do not look at that later proof, unless you are particularly keen.

Define

$$\begin{aligned} p \colon \mathbb{R} &\to S^1 \\ t &\mapsto (\cos 2\pi t, \sin 2\pi t)\,. \end{aligned}$$

For now, we prove Theorem 4.1 assuming Lemma C.14, which implies that any loop in S^1 can be lifted uniquely to a path in $\mathbb{R}$ starting at 0, and any homotopy of loops in S^1 can be lifted to a homotopy of their respective lifts in $\mathbb{R}$.

Proof of Theorem 4.1 assuming Lemma C.14. First, let us show the map ϕ of Theorem 4.1 is surjective. Note that the map p sends the path from 0 to n in $\mathbb{R}$ to the loop wrapping n times around S^1. To show the map is surjective, we just need to show that every loop in S^1 is homotopic to a path in $\mathbb{R}$ traveling at constant speed from 0 to n for some n. To see this, using Lemma C.14, we can lift a loop $f : I \to S^1$ uniquely to a loop $\widetilde{f} : I \to \mathbb{R}$ sending $0 \mapsto 0$. Since $p^{-1}(x_0) = \mathbb{Z}$, because $(p \circ \widetilde{f})(1) = x_0$ it follows that $\widetilde{f}(1) \in \mathbb{Z}$. Then, since $\mathbb{R}$ is convex, we can construct a linear homotopy taking $\widetilde{f}$ to the constant path from 0 to $\widetilde{f}(1)$. That is, we obtain a continuous map $F : I \times I' \to \mathbb{R}$ restricting to $\widetilde{f}$ on $I \times \{0\}$ and to the desired path from 0 to $\widetilde{f}(1)$ on $I \times \{1\}$. Then, the map $p \circ F : I \times I' \to S^1$ produces the desired homotopy, showing that ϕ is surjective.

For injectivity, we only need show that two loops wrapping a different number of times around S^1 cannot be homotopic. If they were homotopic, by Lemma C.14, a homotopy between these loops would lift to a homotopy between their lifts to $\mathbb{R}$. But, their lifts have different endpoints in $\mathbb{R}$. This is a contradiction because the lift of the homotopy must fix the endpoint, as the homotopy fixes the endpoint. $\square$

5. Further computations with homotopy groups

We shall next investigate further properties of homotopy groups, such has how the behave under products and homotopy groups of spheres. As an application, we will show $\mathbb{R}^n$ is not homeomorphic to $\mathbb{R}^2$ for any $n \neq 2$.

5.1. **Products.**

Definition 5.1. For $X \subset \mathbb{R}^m$ and $Y \subset \mathbb{R}^n$ the **Cartesian product**, is

$$X \times Y := \{(x, y) : x \in X, y \in Y\}.$$

Exercise 5.2. Let X, Y be two spaces. Choose $x_0 \in X, y_0 \in Y$.

(1) Show that $\pi_1(X, x_0) \times \pi_1(Y, y_0) \simeq \pi_1(X \times Y, (x_0, y_0))$. *Hint:* Given a loop f in X and a loop g in Y, obtain a loop in $X \times Y$ by viewing f as a loop in $X \times \{y_0\}$ and g as a loop in $\{x_0\} \times Y$ and taking $f \star g$ as a loop in $X \times Y$. To show this map is surjective, if you have a loop h in $X \times Y$, one can write this as a function $s \mapsto h_s(x, y) = (f_s(x), g_s(y))$. Show that h_s is homotopic to the composition of the loops $s \mapsto (f_s(x), y_0)$ and $s \mapsto (x_0, g_s(y))$ (where composition is taken in the fundamental group via the $\star$ operation of Definition 3.4).
(2) Compute $\pi_1(S^1 \times S^1, x_0)$. ($S^1 \times S^1$ is known as the torus.)

Exercise 5.3. Let $L \subset \mathbb{R}^3$ be a line (say the line $x = y = 0$ with coordinates x, y, z for $\mathbb{R}^3$). Let $p \in \mathbb{R}^3 - L$. Compute $\pi_1(\mathbb{R}^3 - L, p)$.

Exercise 5.4 (Tricky exercise)**.** Give $\mathbb{R}^4$ coordinates x, y, z, w and let $P_1 \subset \mathbb{R}^4$ be the 2-plane defined by $x = y = 0$ and let $P_2 \subset \mathbb{R}^4$ be the two-plane defined by $z = w = 0$. Let $p \in \mathbb{R}^4 - P_1 - P_2$. Compute $\pi_1(\mathbb{R}^4 - P_1 - P_2, p)$. *Hint:* Write $\mathbb{R}^4 - P_1 - P_2$ as a product.

5.2. **Homotopy groups of spheres.**

Example 5.5. Here is an example of a non-convex set with trivial fundamental group. Take $S^2 \subset \mathbb{R}^3$ the 2-sphere. We claim that for any $x \in S^2$, $\pi_1(S^2, x)$ is the trivial group. To see this, we only need show that any loop is nullhomotopic. First, without loss of generality, by rotating the sphere, we may assume x is the south pole. Let y denote the north pole of S^2. Start with a loop $f : I \to S^2$. If the loop f does not pass through the north pole, you can construct a homotopy between f and the constant loop, by identifying $S^2 - y \simeq \mathbb{R}^2$. If it does pass through the north pole, then you can first deform the loop slightly to miss the north pole, and then perform the linear homotopy following that. We now flush out the details of the above in the following difficult exercise. Note that another proof, which is easier provided one assumes the machinery of van Kampen's theorem, is given in Example 6.13.

Exercise 5.6 (Difficult exercise, assuming Appendix A)**.** Flush out the details of the above sketch as follows:

(1) Show that the map $\mathbb{R}^2 \to S^2 - y$ via the **stereographic projection** sending
$$(a, b) \mapsto \frac{1}{a^2+b^2+1}\left(2a, 2b, a^2+b^2-1\right).$$
is a homeomorphism (see Definition 3.13). *Hint:* As an inverse take the map $S^2 - y \to \mathbb{R}^2$ sending
$$(a, b, c) \mapsto \frac{1}{1-c}(a, b).$$
(2) Show that any path not passing through y is homotopic to the constant path.
(3) Show that any path passing through y is homotopic to a path not passing through y as follows:
  (a) Choose such a path $f : I \to S^2$ passing through y. Choose a small open ball U around y. Use Exercise A.8 to show $f^{-1}(U)$ is an open set of I and $f^{-1}(x)$ is a closed set of I.
  (b) Use Proposition A.18 to show that $f^{-1}(x)$ is compact subset of $f^{-1}(U)$.
  (c) Using that $f^{-1}(U)$ is a possibly infinite collection of intervals and $f^{-1}(x)$ is a compact subset, show there is a finite collection of intervals $I_1 := (a_1, b_1), \ldots, I_n := (a_n, b_n)$ such that $f(I_n) \subset U$ and so that $f(I \setminus \cup_{i=1}^n I_i)$ does not intersect U.
  (d) Perform linear homotopies on each $[a_i, b_i]$ to avoid y and show that each f is homotopic to a path not meeting y.
(4) Conclude that S^2 is simply connected.
(5) Generalize this same argument to show that for all $n \geq 2$, $\pi_1(S^n, x)$ is the trivial group.
(6) Do you see what goes wrong with this argument in the case $n = 1$? (As we have seen, $\pi_1(S^1, x) \simeq \mathbb{Z}$.)

5.3. **An application to $\mathbb{R}^n$.** We can now apply the above to show in Proposition 5.8 that $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$ for $n > 2$.

Lemma 5.7. *Let* $x \in \mathbb{R}^n - \{0\}$. *Then,*

$$\pi_1(\mathbb{R}^n - 0, x) = \begin{cases} \mathbb{Z} & \text{if } n = 2 \\ 0 & \text{if } n > 2 \end{cases}$$

Proof. Note that $\mathbb{R}^n - \{0\}$ is homeomorphic to $\mathbb{R} \times S^{n-1}$. Therefore, using Exercise 3.14 and Exercise 5.2 we see

$$\begin{aligned} \pi_1(\mathbb{R}^n - \{0\}, x) &\simeq \pi_1(\mathbb{R}^1 \times S^{n-1}, (p, q)) \\ &\simeq \pi_1(\mathbb{R}, p) \times \pi_1(S^{n-1}, q) \end{aligned}$$

for some $p \in \mathbb{R}, q \in S^{n-1}$. Therefore, since $\pi_1(\mathbb{R}, p) = 0$, we obtain

$$\pi_1(\mathbb{R}^n - 0, x) = \pi_1(S^{n-1}, x').$$

The claim then follows from our computation of the fundamental groups of S^1 (by Theorem 4.1) and S^n for $n \geq 2$ (in Example 5.5). □

Proposition 5.8. *For* $n > 2$, $\mathbb{R}^2$ *is not homeomorphic to* $\mathbb{R}^n$.

Proof. Suppose they were homeomorphic by a map $\phi : \mathbb{R}^2 \to \mathbb{R}^n$. Say $\phi(0) = p$. Then, it would follow that $\mathbb{R}^2 - \{0\}$ is homeomorphic to $\mathbb{R}^n - p$. But, homeomorphic spaces have isomorphic fundamental groups by Exercise 3.14. So, by Lemma 5.7, $\mathbb{R}^2 - \{0\}$ is not homeomorphic to $\mathbb{R}^n - p$. □

Exercise 5.9. Show also that $\mathbb{R}^1$ is not homeomorphic to $\mathbb{R}^n$ for any $n \geq 1$ by removing a point from both. *Hint:* Show that $\mathbb{R}^1 - \{0\}$ is not path connected, while $\mathbb{R}^n - p$ for any point p is path connected.

6. Van Kampen's theorem

Having seen some nice applications of $\pi_1(S^1, x_0) = \mathbb{Z}$, we would like to be able to compute fundamental groups. The key tool for this is known as van Kampen's theorem, which lets you compute the fundamental group of a union of spaces X_i in terms of the fundamental groups of the X_i and their overlaps. In order to state van Kampen's theorem, we first need to introduce a certain product operation on groups, known as the free product.

Definition 6.1. Let $G_1, G_2, \ldots, G_n$ be groups. The **free product** of $G_1, \ldots, G_n$, denoted $G_1 * G_2$ is the group whose elements are **words** of the form $g_1 g_2 \cdots g_m$ so that each each each g_i in some $G_{j(i)}$, modulo the equivalence relation that if g_i and g_{i+1} both lie in the same G_j, and $g_i \cdot g_{i+1} = g'$ then the word

$$g_1 g_2 \cdots g_{i-1} g_i g_{i+1} g_{i+2} \cdots g_m$$

is identified with the word

$$g_1 g_2 \cdots g_{i-1} g' g_{i+2} \cdots g_m.$$

Multiplication of words is given by concatenation, the inverse of $g_1 \cdots g_m$ is $g_m^{-1} \cdots g_1^{-1}$, and the identity is the empty word.

Exercise 6.2. Verify that the free product of two groups is indeed a group.

Example 6.3. What does the free product $\mathbb{Z} * \mathbb{Z}$ look like? Write the first copy of $\mathbb{Z}$ multiplicatively as a^n for $n \in \mathbb{Z}$ and write the second copy as b^m for $m \in \mathbb{Z}$. Typical elements of $\mathbb{Z} * \mathbb{Z}$ might look like a^2 or b^{-13} or ab^3a^2 or $b^{-6}abab^3$. We also can write elements in the form $b^3b^2a^3a^{-5}a^2b$ which can be simplified to

$$b^3b^2a^3a^{-5}a^2b = b^5a^{-2}a^2b = b^5a^0b = b^5b = b^6.$$

Exercise 6.4 (Universal property of free groups)**.** Show that for groups $G_1 \ldots, G_n$ and group homomorphisms $f_i : G_i \to H$, there exists a unique homomorphism $f : G_1 * \cdots * G_n \to H$ so that the composition $G_i \xrightarrow{\phi_i} G_1 * \cdots * G_n \xrightarrow{f} H$ is equal to f_i, where $\phi_i : G_i \to G_1 * \cdots * G_n$ sends an element g to the length one word consisting of g.

To state van Kampen's theorem, we will also need the notion of a minimal normal subgroup generated by a subgroup. If you are not familiar with normal subgroups, see subsection B.1 and more specifically Definition B.21.

Theorem 6.5 (van Kampen's theorem). *Suppose $X = \cup_{i=1}^{n} A_\alpha$ where each A_α contains a given basepoint $x_0 \in X$ so that each A_α is path connected and each $A_\alpha \cap A_\beta$ is path connected. We have homomorphisms $\pi_1(A_\alpha, x_0) \to \pi_1(X, x_0)$ induced by the inclusions $A_\alpha \to X$ and homomorphisms $i_{\alpha\beta} : \pi_1(A_\alpha \cap A_\beta, x_0) \to \pi_1(A_\alpha, x_0)$ induced by the inclusions $A_\alpha \cap A_\beta \to X$.*

(1) The homomorphism $\Phi : \pi_1(A_1, x_0) * \cdots * \pi_1(A_n, x_0) \to \pi_1(X, x_0)$ is surjective.
(2) If further each $A_\alpha \cap A_\beta \cap A_\gamma$ is path connected, then the kernel of Φ is the minimal normal subgroup N generated by all elements of the form $i_{\alpha\beta}(\omega) i_{\beta\alpha}(\omega)^{-1}$ for $\omega \in \pi_1(A_\alpha \cap A_\beta, x_0)$ so Φ induces an isomorphism $\pi_1(X, x_0) \simeq \pi_1(A_1, x_0) * \cdots * \pi_1(A_n, x_0)/N$.

The idea for showing surjectivity is that if you have any loop in the total space, you can break it up as a sequence of loops, each one passing only through a single one of the spaces. To determine the kernel, it is not too difficult to check that the kernel contains N because going around one loop and then passing around the same loop in the reverse direction is homotopic to the identity. To conclude one then has to do a careful, fidgety argument to show this is precisely the kernel, which we omit. See [Hat02, Section 1.2, Theorem 1.20] for a proof.

6.1. **Computing examples of fundamental groups with van Kampen's theorem.**

Example 6.6. Let's start by computing the fundamental group of a figure 8. This can also be thought of as the fundamental group of two circles joined a point. Call this space X and let x_0 be the point where the two circles meet. To apply van Kampen's theorem, we need to cover X by two open sets. Take as our first open set A_1 an open set containing the first circle, and extending slightly into the second circle and as our second open set A_2, take an open set containing the second circle, and extending slightly into the first circle. Note, we are not allowed to just take the respective circles themselves, as they are not open. Then, we may note $\pi_1(A_i, x_0) \simeq \pi_1(S^1, x_0) \simeq \mathbb{Z}$ (this can either be seen directly, or using the machinery of deformation retracts, to be introduced following this example). Further, $A_1 \cap A_2$ has trivial fundamental group (the intersection looks like the letter "x"). By van Kampen's theorem the map $\Phi : \pi_1(A_1, x_0) * \pi_1(A_2, x_0) \to \pi_1(X, x_0)$ is both surjective and has trivial kernel, so it is an isomorphism. We

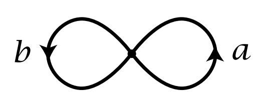

FIGURE 7. A picture of the figure 8 with one generator of the fundamental group given by a and the other given by b

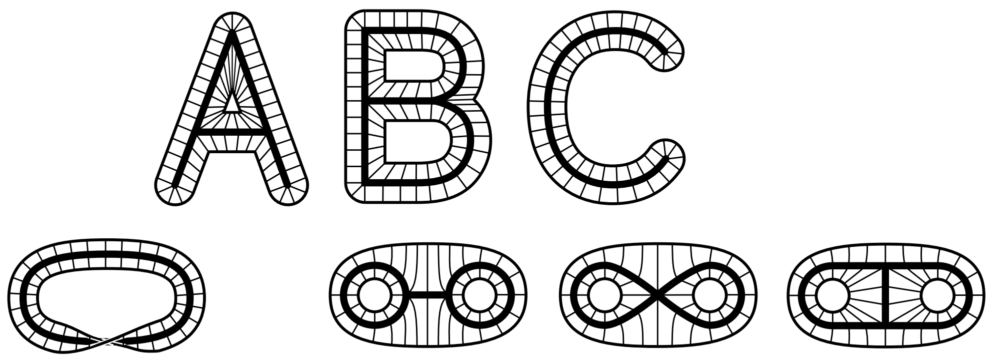

FIGURE 8. Some examples of deformation retracts

conclude that

$$\pi_1(X, x_0) \simeq \pi_1(A_1, x_0) * \pi_1(A_2, x_0) \simeq \mathbb{Z} * \mathbb{Z}.$$

As we saw in Example 6.6 to deal with the condition from van Kampen's that the sets A_i are open, we had to identify the fundamental group of a circle with that of an open set slightly bigger than the circle. In general, you are free to slightly thicken things in this way without changing the fundamental group, as is explained by the notion of a deformation retract, which we now define.

Definition 6.7. Let X be a space and $Y \subset X$ a subspace. A **deformation retract** of X onto Y is a continuous map $f : X \times I \to X$ such that

(1) $f(x, 0) = x$,
(2) $f(x, 1) \in Y$, and
(3) for all $y \in Y$, $f(y, t) = y$.

We say Y is a deformation retract of X if such a map exists.

The main feature of deformation retracts is that they do not change the fundamental group:

Lemma 6.8. *Let* $Y \subset X$. *Suppose* Y *is a deformation retract of* X *and let* $g : Y \to X$ *denote the inclusion. Then, for* $x \in Y$, *the map* $\pi_1(g) : \pi_1(Y, x) \to \pi_1(X, x)$ *is an isomorphism.*

Proof. Recall the map $\pi_1(g)$ sends a loop $\delta : I \to Y$ to the loop $g \circ \delta : I \to X$. Let $f : X \times I' \to X$ be the deformation retract of X onto Y. We define an inverse map $\pi_1(X, x) \to \pi_1(Y, x)$ by sending a loop $\gamma : I \to X$ to the loop $\gamma' : I \to Y$ given by $\gamma'(s) = f(\gamma(s), 1)$.

Exercise 6.9. Complete the proof as follows:

(1) Verify that $g \circ \gamma'$ is homotopic to γ (viewed as loops $I \to X$). *Hint:* Show that the composite $I \times I' \xrightarrow{\gamma, \mathrm{id}} X \times I' \xrightarrow{f} X$ defines a homotopy between $g \circ \gamma$ and γ'.
(2) Show that the map $\pi_1(X, x) \to \pi_1(Y, x)$ just constructed is a two sided inverse to $\pi_1(g)$.
(3) Conclude that $\pi_1(g)$ is an isomorphism.

□

Definition 6.10. We say a space X is **contractible** if there is some point $p \in X$ so that p is a deformation retract of X.

Remark 6.11. Intuitively, a contractible space is one you can contract to a point.

Using the trick of deformation retracts, we can now compute many more examples of fundamental groups.

Exercise 6.12. Let X_n be a "bouquet of n circles" (so that the $n = 2$ case was Example 6.6) which consists of n circles meeting a common point x. Using the same technique as in Example 6.6, show that $\pi_1(X_n, x) \simeq \underbrace{\mathbb{Z} * \mathbb{Z} * \cdots * \mathbb{Z}}_{n}$, a free product of n copies of $\mathbb{Z}$.

Next, we can compute $\pi_1(S^n, x_0)$ in a much easier fashion than we did in Example 5.5.

Example 6.13. We will now show that $\pi_1(S^2, x) = 0$. Choose x on the equator and write $S^2 = A_1 \cup A_2$ with A_1 and open set containing all points with z-coordinate more than $-1/2$ and A_2 the open set with all coordinates containing z coordinates less than $1/2$. Note that $A_1 \cap A_2$ is path connected (it is an annulus), and hence, by van Kampen's theorem, the fundamental group of S^2 is generated by the fundamental groups of A_1 and A_2. But both A_1 and A_2 are contractible, and so the have trivial fundamental group. It follows that S^2 has trivial fundamental group.

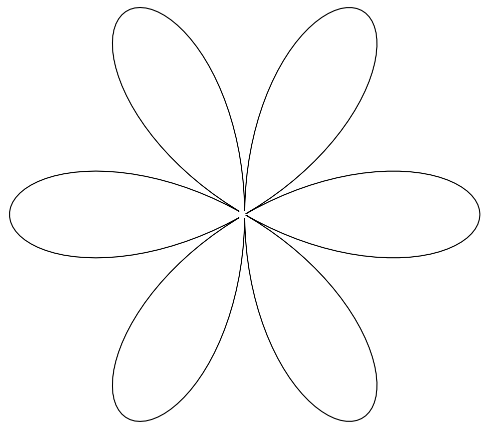

FIGURE 9. A bouquet of 6 circles.

Exercise 6.14. Using the same technique as in Example 6.13, show that $\pi_1(S^n, x) = 0$ for every $n \geq 2$.

Exercise 6.15. In this exercise, we compute the fundamental group of $\mathbb{R}^2 - \{p_1, \ldots, p_n\}$, for $p_1, \ldots, p_n$ n distinct points in $\mathbb{R}^2$.

(1) Show that there is a deformation retract from $\mathbb{R}^2 - \{p\}$ onto S^1.
(2) More generally, show that $\mathbb{R}^2 - \{p_1, \ldots, p_n\}$ deformation retracts onto a bouquet of n circles, as defined in Exercise 6.12.
(3) Conclude using Exercise 6.12 that

$$\pi_1(\mathbb{R}^2 - \{p_1, \ldots, p_n\}, x) \simeq \underbrace{\mathbb{Z} * \mathbb{Z} * \cdots * \mathbb{Z}}_{n}.$$

(Puzzle!) Suppose you have two nails at the same height on a wall and a picture hung on a string. Can you wind the string around the nails in such a way that the picture falls down if either nail is removed but does not fall when both nails remain in place? *Possible hint:* It's possible to solve this without fundamental groups, but fundamental groups can give an idea of where to look. What does this problem have to do with $\pi_1(\mathbb{R}^2 - \{p_1, p_2\}, x)$?

Exercise 6.16. Compute the fundamental group $X := \mathbb{R}^3 - \{L_1, \ldots, L_n\}$, where L_i are lines passing through the origin. *Hint:* Show that X deformation retracts onto the complement of $2n$ points in the sphere and use Exercise 6.15.

6.2. **The inverse problem.** One natural question is the following:

Question 6.17. Given any group G, is there a space X with $\pi_1(X, x) = G$?.

Perhaps surprisingly, the answer is yes, and furthermore, one can build such a space with only circles and disks!

Proposition 6.18. *Given any group* G*, there exists a space* X *with* $\pi_1(X, x) = G$.

We will now indicate the idea of the proof of this. For a complete proof, see [Hat02, Section 1.2, Corollary 1.28]. Let's say you wanted to construct a space with fundamental group $\mathbb{Z} * \mathbb{Z}/(aba^{-1}b^{-1})$, with a the generator of the first copy of $\mathbb{Z}$ and b the generator of the second copy, written multiplicatively. Here's how we can construct the space X with that fundamental group.

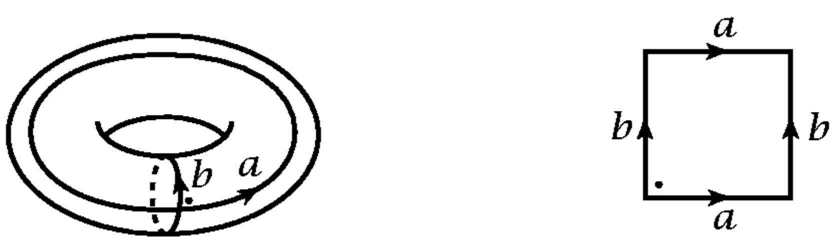

FIGURE 10. A picture of how to construct the a space with fundamental group $\mathbb{Z} * \mathbb{Z}/(aba^{-1}b^{-1})$

Start with a square and label the two vertical edges b and the two horizontal edges a. Start at the lower left hand corner. For each of a or b, we orient the edge in the clockwise direction and for each of a^{-1} or b^{-1} orient the edge in the counterclockwise direction. We now glue the edge a to the edge a^{-1} (they should be identified as they correspond to the same generator). Since they were both oriented rightward, we have obtained a cylinder. Next, glue the two edges b and b^{-1} together. Since they were both oriented upward, we have now obtained a torus. Indeed, we saw that a torus has fundamental group $\mathbb{Z} \times \mathbb{Z}$ in Exercise 5.2 and $\mathbb{Z} * \mathbb{Z}/(aba^{-1}b^{-1}) \simeq \mathbb{Z} \times \mathbb{Z}$, so things worked out in this case. In general, we can construct a space with one edge for each generator of the group, and one disk glued on for each relation. We can then use van Kampen's theorem to verify the resulting space has the correct fundamental group. The details are a bit technical, so again, see [Hat02, Section 1.2, Corollary 1.28] for a full proof.

Appendix A. Topological spaces

In this appendix, we give a brief introduction to some basic properties topological spaces.

Definition A.1. A **topological space** X is a set (also called X) together with a collection of subsets $\{U_i\}_{i \in I}$ called **open sets** with $U_i \subset X$ satisfying the following conditions

(1) The empty set $\emptyset \subset X$ is open.
(2) The set X is open.
(3) An arbitrary union of open sets is open.
(4) The intersection of two open sets is open.

The collection of open subsets is called a **topology** for X.

Example A.2. One example of a topological space is $\mathbb{R}^n$. There are many ways to topologize this. The **Euclidean topology** for $\mathbb{R}^n$ is the topology where the open sets are unions of open balls centered at points.

Exercise A.3. Verify this the Euclidean topology is indeed a topology.

Exercise A.4. Let X be a set and define the **indiscrete topology** on X to be the topology where the only opens are X and $\emptyset$. Define the **discrete topology** to be the topology where any subset of X is open. Show that the discrete and indiscrete topologies are indeed topologies.

Definition A.5. If $\iota : X \subset Y$ with X a set and Y a topological space, then the **induced topology on** X (with respect to ι) is the topology on X whose open sets are those of the form $X \cap U$ for $U \subset Y$ open.

Exercise A.6. Verify that the induced topology is indeed a topology.

Definition A.7. A continuous map of topological spaces $f : X \to Y$ is a map of underlying sets such that for any open $U \subset Y$, the preimage $f^{-1}(U) \subset X$ is open.

Exercise A.8. Let $\mathbb{R}^m$ and $\mathbb{R}^n$ be topologized with the Euclidean topologies and let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be topologized with the induced topologies. Show that a map of sets $f : X \to Y$ is continuous in the sense of Definition A.7 if and only if it is continuous in the sense of Definition 2.5.

Definition A.9. For X and Y two topological spaces, define the **product topology** by taking the topology whose open sets are unions of sets of the form $U \times V$ for $U \subset X$ open and $V \subset Y$ open.

Exercise A.10. Verify that the product topology is a topology. *Hint:* You need to show that an intersection of two sets of the form $(V_1 \times U_1) \cap (V_2 \times U_2)$, for $U_i \subset X, V_i \subset Y$, can be written as a union of open sets of the form $W_i \times Z_i$ for $U_i \subset X, Z_i \subset Y$.

A.1. **Compactness.**

Definition A.11. A space is **compact** if every open cover has a finite subcover.

Lemma A.12. *The interval* $I = [0, 1] \subset \mathbb{R}$ *is compact with the subspace topology induced from the Euclidean topology on* $\mathbb{R}$.

Proof. To see this, let $\{U_i\}$ be a hypothetical open cover with no finite subcover. We want to reach a contradiction. Divide the interval into two pieces $I = [0, 1/2] \cup [1/2, 1]$. If U_i has no finite subcover, then either $U_i \cap [0, 1/2]$ has no finite subcover or $U_i \cap [1/2, 1]$ has no finite subcover. Let I_1 be the subinterval with no finite subcover. Then, split I_1 into two intervals and repeat the process to obtain some I_2 of half the length. Continue this process to obtain an infinite chain $I \supset I_1 \supset I_2 \supset \cdots$.

Exercise A.13. Verify there exists a unique point p in $\cap_{i=1}^{\infty} I_i$. *Hint:* Take the limit of the left endpoints and show this converges.

Since $\{U_i\}$ forms an open cover there is some U_i containing p. Since U_i is open, it contains an open ball around p, so some there is some j with $I_j \subset U_i$. But, we assumed that I_j has no finite subcover coming from U_i, a contradiction. □

Next, we aim to show that products of intervals are compact. For this we need the following:

Lemma A.14. *If* X *and* Y *are two compact topological spaces, then* $X \times Y$ *with the product topology (see Definition A.9) is compact.*

Proof. Let $\{U_i\}_{i \in I}$ be an open cover of $X \times Y$. For any point $p \in X \times Y$, by the definition of the product topology on $X \times Y$, there is some $W \times Z \subset U_i$ with $W \subset X, Z \subset Y$ open. For our purposes, we may replace each U_i in our open cover by a union of sets of the form $W^i_j \times Z^i_j$, each contained in U_i whose union covers U_i. Note that if we can find a finite open cover consisting of such sets which are products, then choosing U_i containing these respective sets will yield a finite subcover of the U_i. Hence, by the above, we may assume all U_i are of the form $W_i \times Z_i$.

Next, note that for any $x \in X$, by compactness of Y, there is a finite collection of $J_x \subset I$ with $\{W_j \times Z_j\}_{j \in J_x}$ covering $\{x\} \times Y$. For each such point x, we note that the open set $W_x := \cap_{j \in J} W_j$ (open because J is finite) defines an open set of X containing x. Further, the finitely many members of this set $W_j \times Z_j$ cover $W_x \times Y$. Applying this same procedure to each $x \in X$, by compactness of X, we can find finitely many such points, call them $x_1, \ldots, x_n$ so that the resulting W_x form an open cover of x. Then, the set $\cup_{i=1}^n J_{x_i}$ determines a finite collection of open sets in our cover which cover $X \times Y$. □

Exercise A.15. Prove that products of intervals are compact, as follows.

(1) Use Lemma A.14 and Lemma A.12 conclude that $I \times I$ is compact.
(2) More generally, by induction on n, prove I^n (meaning a product of n copies of I) is compact.

Our next goal is to prove that a closed and bounded subset of $\mathbb{R}^n$ is compact. This is also known as the Heine-Borel theorem.

Exercise A.16. Show that if X is homeomorphic to Y, then X is compact if and only if Y is compact.

Exercise A.17. Show that a closed subset X of a compact set Y (giving X the induced subspace topology) is compact. *Hint:* Take an open cover U_i of X. By definition of the subspace topology, this is a collection of open sets V_i in Y, so that $V_i \cap X = U_i$. Note that $Y \setminus X$ is an open set, and so the V_i, together with $Y \setminus X$ form an open cover of Y. By compactness, obtain a finite subcover, and deduce that there was a finite subcover of the U_i covering X.

Proposition A.18 (Heine-Borel)**.** *A closed bounded subset of* $\mathbb{R}^n$ *(with the Euclidean topology) is compact.*

Proof. Start with some closed and bounded subset $X \subset \mathbb{R}^n$.

Exercise A.19. By scaling X, show that X is homeomorphic to a subset $X' \subset I^n$.

Since X is homeomorphic to a subset $X' \subset I^n$, by Exercise A.16, it suffices to show that X' is compact. By Exercise A.15, we know that I^n is compact. By Exercise A.17, it follows that X' is compact, as desired. □

Finally, we briefly state the notion of a basis.

Definition A.20. A **basis** for a topology on X is a collection $\mathcal{U}$ of open sets $U_i \subset X$ so that every open $U \subset X$ can be written as a union $U = \cup_i U_i$ for each $U_i \in \mathcal{U}$.

APPENDIX B. GROUP THEORY

In this appendix, we develop introductory group theory, which we will use elsewhere in the notes in the course of our investigation of fundamental groups.

Definition B.1. A group G is a set G together with a multiplication operation $\cdot : G \times G \to G$ and an **identity** $e \in G$ satisfying the following properties

Identity For every $g \in G$, we have $e \cdot g = g \cdot e = g$.

Associativity For $g, h, k \in G$, we have $(g \cdot h) \cdot k = g \cdot (h \cdot k)$.

Inverses For every $g \in G$, there is an **inverse** denoted $g^{-1} \in G$ so that $g \cdot g^{-1} = g^{-1} \cdot g = e$.

Remark B.2. We will often omit the multiplication operation $\cdot$ from the notation of a group when it is understood from context.

Exercise B.3. Verify, directly from the definition that every group has a unique identity element. Show that for any $g \in G$, g has a unique inverse, and so the name g^{-1} is justified.

Definition B.4. A **homomorphism** of groups $f : G \to H$ is a map of sets such that $f(e_G) = e_H$ and $f(g \cdot_G g') = f(g) \cdot_H f(g')$, where the subscripts denote the identity and multiplication in the corresponding group.

Definition B.5. A group homomorphism $f : G \to H$ is **injective** if $f(g) = f(g') \implies g = g'$. It is **surjective** if for every $h \in H$ there is some $g \in G$ with $f(g) = h$. It is **bijective** (also known as an **isomorphism**) if it is both injective and surjective. If $f : G \to H$ is bijective, we write $G \simeq H$.

Exercise B.6. Show that a group homomorphism $f : G \to H$ is injective if and only if $f^{-1}(e_H) = e_G$.

Exercise B.7. Show that a group homomorphism $f : G \to H$ is bijective if and only if there is a group homomorphism $f^{-1} : H \to G$ so that $f^{-1} \circ f = \mathrm{id}_G, f \circ f^{-1} = \mathrm{id}_H$. *Hint:* Show that a map is bijective if and only if there is a unique element of G mapping to any given element of H. Use this to define an inverse map.

Definition B.8. The **kernel** of a group homomorphism $f : G \to H$ is the set of elements $g \in G$ with $f(g) = e_H$.

Definition B.9. A **subgroup** H of G is a subset $H \subset G$ so that

(1) $e_G \in H$.

(2) For any $g \in H$ we also have $g^{-1} \in H$.
(3) If $g, g' \in H$ then $g \cdot_G g'$ is also in H.

Exercise B.10. Show that the kernel of a group homomorphism is a subgroup.

Definition B.11. A group is **abelian** if for all $a, b \in G$, we have $a \cdot b = b \cdot a$.

Example B.12 (Non-example)**.** The group of permutations of three elements is not abelian because if you first switch elements 1 and 2, and then switch elements 2 and 3, this is not the same as first switching 2 and 3 and then switching 1 and 2. In the first case, you end up sending $1 \mapsto 3 \mapsto 2 \mapsto 1$ while in the second case you end up sending $1 \mapsto 2 \mapsto 3 \mapsto 1$.

B.1. **Normal subgroups and quotients.**

Definition B.13. A subgroup $H \subset G$ is **normal** if for all $g \in G$ and $h \in H$ we have $ghg^{-1} \in H$.

Definition B.14. Let $H \subset G$ be a subgroup. Construct the **quotient** G/H as the set of all elements $g \in G$ modulo the equivalence relation $g_1 \sim g_2$ if there is some $h \in H$ with $g_1 = g_2 h$. The equivalence class of g is called the **coset of** g and the coset is notated gH.

Exercise B.15. Verify that the relation $\sim$ as defined in Definition B.14 is indeed an equivalence relation.

Exercise B.16. Show that if $H \subset G$ is normal then G/H can be given the structure of a group by $gH \cdot g'H = (gg')H$.

Exercise B.17 (Universal property of quotients)**.** For $H \subset G$ a subgroup, show that for any group homomorphism $f : G \to G'$ with $f(H) = e_{G'}$ there is a unique map of sets $\overline{f} : G/H \to G'$ so that the composition of $G \to G/H$ (given by $g \mapsto gH$) with $\overline{f}$ is equal to f. Show that $\overline{f}$ is a group homomorphism when H is normal (the normality hypothesis is only so that G/H can be given the structure of a group, using Exercise B.16).

Definition B.18. Let $S \subset G$ be a subset (which is not necessarily a subgroup). The **subgroup generated by** S is the intersection of all subgroups of G containing S.

Remark B.19. Intuitively, you can obtain the subgroup generated by a set by just throwing in all inverses and then throwing in all repeated products of such elements.

Exercise B.20. Show that the intersection of any collection of subgroups of a group is again a subgroup. Conclude that the subgroup generated by a set is indeed a subgroup.

Definition B.21. Let $H \subset G$. The **minimal normal subgroup** generated by H is the subgroup generated by

$$\left\{ g \in G : \text{ there exists } k \in G, h \in H \text{ with } g = khk^{-1} \right\}.$$

Exercise B.22 (Tricky Exercise)**.** Let $H \subset G$. Show that the minimal normal subgroup is indeed a normal subgroup. *Hint:* To show it is a subgroup, use Exercise B.20. To show it is normal (which is the same as saying it is invariant under conjugation; the conjugates of an element h are those of the form ghg^{-1} and a set S is invariant under conjugation if for all $s \in S$ $gsg^{-1} \in S$), use that the generating set is invariant under conjugation.

APPENDIX C. UNIVERSAL COVERS

Warning C.1. The following section is written at a level above the rest of these notes. I recommend you do not read this section, it is merely here for completeness.

We now construct universal covers. As an application, we will compute the fundamental group of S^1.

Definition C.2. A topological space X is **path connected** if for any two points $x, y \in X$, there is a path $f : I \to X$ with $f(0) = x, f(1) = y$. A topological space X is **connected** if there is no way to write $X = X_1 \cup X_2$ with $X_1, X_2 \subset X$ both open sets with $X_i \neq \emptyset$ for $i = 1, 2$. A topological space X is **locally path connected** if there is a basis of open sets which are path connected. A topological space X is **semi-locally path connected** if for every point $x \in X$, there is an open set U_x with the image of the map $\pi_1(U_x, x) \to \pi_1(X, x)$ equal to 1.

Exercise C.3. Show that any path connected space is connected.

The universal cover is a special type of covering space, which we now introduce.

Definition C.4. A **covering space** Y of a space X is a topological space Y with a continuous map $f : Y \to X$ such that around every $p \in X$ there is some open set $U_p \ni p$ with $f^{-1}(U_p)$ homeomorphic to a disjoint union of copies of U_p, each of which is mapped homeomorphically to U_p under f. A **pointed covering space** (Y, y_0) of a space (X, x_0) is a covering space $f : Y \to X$ with $f(y_0) = x_0$.

Definition C.5. Let X be a topological space. A **universal cover** is a pointed covering space $f : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ so that for any pointed covering space $g : (Y, y_0) \to (X, x_0)$ there is a unique continuous map $h : (\widetilde{X}, \widetilde{x}_0) \to (Y, y_0)$ making $(\widetilde{X}, x_0)$ into a covering space of (Y, y_0) so that $f = g \circ h$, i.e.,

(C.1)
$$\begin{tikzcd} \widetilde{X} \arrow[rr, "h"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & X & \end{tikzcd}$$

commutes.

Exercise C.6. Show that if $(\widetilde{X}, \widetilde{x}_0)$ is a universal cover, then it is unique up to isomorphism. That is, if $(\widetilde{X}', \widetilde{x}_0')$ is any other universal cover, there is a unique pointed covering map $(\widetilde{X}, \widetilde{x}_0) \to (\widetilde{X}', \widetilde{x}_0')$ which is an isomorphism. *Hint:* Use Definition C.5 to construct unique maps $\widetilde{X} \to \widetilde{X}'$ and $\widetilde{X}' \to \widetilde{X}$ and use Definition C.5 again to show that the composition of these maps is the identity, using that the identity is one such map.

It would be nice to know whether universal covers exist. Fortunately, they do, in the following pleasant situation, as we now construct.

Definition C.7 (Construction)**.** Let X be a path connected, semi-locally simply connected, locally path connected space. Choose some $x_0 \in X$. We construct a universal cover $\widetilde{X}$ explicitly as

$$\widetilde{X} := \{[\gamma] : \gamma : I \to X \text{ is a path with } \gamma(0) = x_0\}.$$

We next define a topology on $\widetilde{X}$. Let $\mathcal{U}$ denote the collection of path connected open sets U with some $x \in U$ so that $\pi_1(U, x) \to \pi_1(X, x)$.

Exercise C.8. Show that the map $\pi_1(U, x) \to \pi_1(X, x)$ is independent of the choice of point $x \in U$ and $\mathcal{U}$ forms a basis for the topology on X.

We define a topology on $\widetilde{X}$ by declaring that all open sets are unions of sets of the following form: For every point $p \in X$, let $U \in \mathcal{U}$ and define $U_{[\gamma]}$ be the set of points $[\xi] \in \widetilde{X}$ with $\xi = \eta \star \gamma$ (where $\star$ denotes composition of paths, see Definition 3.4) for $\eta : I \to U$ a path so that $\eta(0) = \gamma(1)$.

Exercise C.9. In this exercise, we check that $U_{[\gamma]}$ form a base for a topology on $\widetilde{X}$.

(1) Verify that the intersection of two sets of the form $U_{[\gamma]}$ is a union of such sets.
(2) Use this to show that unions of sets of the form $U_{[\gamma]}$ define a topology on $\widetilde{X}$.
(3) Show, via the definition, that $U_{[\gamma]}$ form a base for this topology.

By Exercise C.9, we can define a topology on $\widetilde{X}$ with $U_{[\gamma]}$ as a basis. We have a map $\widetilde{X} \to X$ sending $[\gamma] \mapsto \gamma(1)$.

We would like to show the construction of Definition C.5 is indeed a universal cover. We begin by showing it is a covering space.

Lemma C.10. *Let X be a path connected, semi-locally simply connected, locally path connected space. The construction $\widetilde{X}$ of Definition C.5 is a covering space of X via the map $p : \widetilde{X} \to X$ sending $[\gamma] \mapsto \gamma(1)$.*

Proof. This is proven in the following several exercises.

Exercise C.11. Prove Lemma C.10 in the following steps.

(1) Show that the map $p : \widetilde{X} \to X$ is a homeomorphism when restricted to a map $p|_{U_{[\gamma]}} : U_{[\gamma]} \to U$. *Hint:* Show this restriction is a homeomorphism by showing it induces a bijection between open sets.
(2) Show that $p : \widetilde{X} \to X$ is continuous, using that it is locally a homeomorphism, as shown in (1).
(3) Show that $p : \widetilde{X} \to X$ is a covering space, again using (1).

□

Proposition C.12. *Let X be a path connected, semi-locally simply connected, locally path connected space. Then, the construction $p : \widetilde{X} \to X$ of Definition C.7 is simply connected.*

Proof. Choose $\widetilde{x}_0 \in \widetilde{X}$ mapping to $x \in X$ under the projection p of Lemma C.10. We want to show $\pi_1(\widetilde{X}, \widetilde{x}_0)$ is the trivial group. The proof will proceed in two steps

(1) First, we will show the image of the map $\pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x_0)$ is trivial.
(2) Second, we will show the map $\pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x_0)$ is injective

Once we prove these two facts, the result will follow.

C.1. **Step 1: Trivial image.** We check that the image of $\pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x_0)$ is trivial. We need to check that any loop $f : I \to \widetilde{X}$ in $\widetilde{X}$ ending at $\widetilde{x}_0$ maps to a nullhomotopic loop in X.

Exercise C.13. Show that the composition $p \circ f : I \to X$, is a loop in X whose homotopy class if $f(1)$

Since f is a loop ending at $\widetilde{x}_0$, which represents the homotopy class of the constant loop at x, by definition of $\widetilde{X}$, it follows that $p \circ f$ is nullhomotopic.

C.2. **Step 2: Injectivity.** To show the map is injective, we need to show that any loop $h : I \to \widetilde{X}$ based at $\widetilde{x}_0$ which maps to a contractible loop in X is contractible in $\widetilde{X}_0$. If we could lift the homotopy $I \times I' \to X$ to a homotopy $I \times I' \to \widetilde{X}$ between h and the trivial loop, we would conclude h is nullhomotopic, so the map is injective.

Lemma C.14. *Let* $p : Y \to X$ *be any covering space with* $y_0 \mapsto x_0$*. Any homotopy* $f : I \times I' \to X$ *based at* x_0 *(meaning* $f(0,t) = f(1,t) = x_0$*) lifts to a unique homotopy* $\widetilde{f} : I \times J \to Y$ *for* $I = [0,1]$ *and* $J = [r,s]$ *with* $[r,s]$ *any interval (including possibly* $r = s$*). By lifts, we mean* $p \circ \widetilde{f} = f$*, i.e., there is a commutative diagram*

(C.2)
$$\begin{tikzcd} I \times J \arrow[rr, "\widetilde{f}"] \arrow[dr, "f"'] & & Y \arrow[dl, "p"] \\ & X. & \end{tikzcd}$$

Let us see why this will finish the proof of injectivity. By Lemma C.14, we can lift the homotopy $I \times I' \to X$ to a homotopy $I \times I' \to Y$. It suffices to check that this is indeed a homotopy between h and the constant loop. For this we will use the uniqueness claim of Lemma C.14. First, we claim that this lift $\widetilde{f}$ satisfies $\widetilde{f}(s,0) = h(s)$. Indeed, this follows by taking J to be a point (the interval $[0,0]$) and noting that h is a lift of $f_0 : I \times \{0\} \to X$. Similarly, the fact that $\widetilde{f}(s,1) = \widetilde{x}_0$ follows from the uniqueness part of Lemma C.14 and fact that the constant loop at $\widetilde{x}_0$ is a lift of the constant loop at x.

To complete the injectivity, we now prove Lemma C.14.

Proof of Lemma C.14. To start, if $J = [r,s]$ with $r > s$, we can rescale and translate so that $J = [0,1]$. The case $r = s$ is covered at the end of the proof in Exercise C.22. To prove the lemma, choose a sufficiently fine open cover U_i of $I \times J$ so that the preimage under p of $f(U_i)$ is a disjoint union of copies of $f(U_i)$. This is possible by the definition of covering space, first choosing open sets $V_i \subset X$ where this holds, and then taking their preimages in $I \times J$.

Exercise C.15. For all U_i show that the possible lifts $U_i \to Y$ of $f : U_i \to X$ are in bijection with connected components (i.e., maximal connected subsets) of $p^{-1}(f(U_i))$, using that the map p is a homeomorphism from each connected component of the preimage of $f(U_i)$ onto U_i.

Exercise C.16. Show that we may assume the U_i are connected, by replacing U_i by its connected components. Further, show that we can assume the U_i are boxes of the form $\left(\frac{a-1}{n}, \frac{a+1}{n}\right), \left(\frac{b-1}{n}, \frac{b+1}{n}\right)$, for $0 \leq a, b \leq n$, interpreted suitably in the case a or b is equal to 0 or n *Hint:* For this, use compactness of $I \times J$, as shown in Exercise A.15. Relabel these U_i as $U_{a,b}$.

Exercise C.17. Verify that for any topological space V with an open cover U_i, to define a map $f : V \to X$ it suffices to define a collection of maps $f_i : U_i \to X$ so that f_i and f_j agree when restricted to $U_i \cap U_j$.

Exercise C.18. Show that for each fixed a we can construct a lift $\cup_{i=0}^{n-2} U_{a,i}$. *Hint:* Use Exercise C.15 and Exercise C.17.

Exercise C.19. Show that there is a unique lift $\widetilde{f}_0 : \cup_{i=0}^{n} U_{0,i} \to Y$ such that $\widetilde{f}_0(0, 0) = \widetilde{x}_0$. *Hint:* Use Exercise C.15.

Exercise C.20. Assume that we have constructed a lift $\widetilde{f}_{a-1} : \cup_{i=0}^{n} U_{a-1,i} \to Y$. Show that there is a unique lift $\widetilde{f}_a : \cup_{i=0}^{n} U_{a,i} \to Y$ agreeing with $\widetilde{f}_{a-1}$ on $(\cup_{i=0}^{n} U_{a,i}) \cap (\cup_{i=0}^{n} U_{a-1,i})$.

Exercise C.21. Conclude the proof in the case $r < s$ by induction on a.

This concludes the case $J = [r, s]$ with $r < s$. It only remains to deal with the case $r = s$.

Exercise C.22. Finally, prove the case that $J = [r, r]$ in a fashion similar to the above. *Hint:* This is much easier than the above case, since we only have to worry about the I coordinate. Essentially, it is the same as Exercise C.19.

□

This completes the verification of injectivity, and hence completes the proof of Proposition C.12. □

We still have not yet shown $\widetilde{X}$ is a universal cover! We use Lemma C.14 (proved in the course of Proposition C.12) to do this now.

Theorem C.23. *Let X be a path connected, semi-locally simply connected, locally path connected space. Then the construction $(\widetilde{X}, \widetilde{x}_0)$ of Definition C.5 is a universal cover of (X, x_0).*

Proof. We need to show that that for any covering space $f : (Y, y_0) \to (X, x_0)$ there is a unique lift $\widetilde{p} : (\widetilde{X}, \widetilde{x}_0) \to (Y, y_0)$ of $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$. Recall that a point $z \in \widetilde{X}$ is a homotopy class of a path $f : I \to X$. By Lemma C.14, there is a unique lift of f, call it $\widetilde{f} : I \to Y$ with $f(0) = y_0$. Define $\widetilde{p}([z]) := \widetilde{f}(1)$.

Exercise C.24. Show that $\widetilde{p}$ is well defined by verifying that two homotopic paths of X lift to the same path in Y. *Hint:* Use Lemma C.14.

Exercise C.25. Verify that the map $\widetilde{p}$ is continuous. *Hint:* Show that in fact $\widetilde{p} : \widetilde{X} \to Y$ makes $\widetilde{X}$ into a covering space using that $Y \to X$ is a covering space via the definition of the topology on $\widetilde{X}$.

So, we have now constructed the desired map. It remains to show it is the unique map of *pointed spaces* $(\widetilde{X}, \widetilde{x}_0) \to (Y, y_0)$ commuting with the projection to (X, x_0).

Exercise C.26. Verify that $\widetilde{p}$ is the unique map of pointed spaces $(\widetilde{X}, \widetilde{x}_0) \to (Y, y_0)$ commuting with the projection to (X, x_0). *Hint:* Use the uniqueness part of Lemma C.14.

□

C.3. **Properties of universal covers.** We can use the universal cover to deduce the fundamental group of a space.

Corollary C.27. *Let X be a path connected, semi-locally simply connected, locally path connected space. Let $p : \widetilde{X} \to X$ be the universal cover. Then for $x \in X$, we have a bijection of sets $\phi : p^{-1}(x) \simeq \pi_1(X, x)$, where ϕ sends a point of $\widetilde{X}$, to the corresponding homotopy class of a path in X via the definition of $\widetilde{X}$ as in Definition C.5.*

Proof. First, note that ϕ is well defined because every point of $p^{-1}(x)$ corresponds to a path in X starting and ending at x, which means that path is a loop. We check ϕ is injective and surjective. To show it is surjective, we need to show that every homotopy class of a loop is represented by a point in $p^{-1}(x)$. We know every class is represented by some point of $\widetilde{X}$. However, the only points of $\widetilde{X}$ which correspond to loops based at x (and not just paths) are those points in $p^{-1}(x)$, by definition of $\widetilde{X}$.

It remains to check injectivity. That is, we only need so show that $\widetilde{x}_0$ is the only point mapping to the nullhomotopy class in X. However, this too follows by definition of $\widetilde{X}$, since its points correspond to homotopy classes of loops in X and so $\widetilde{x}_0$ is the only point of $\widetilde{X}$ mapping to the nullhomotopic class. □

Corollary C.28. *Let X be a path connected, semi-locally simply connected, locally path connected space. Suppose that Y is simply connected. Then Y is the universal cover of X.*

Proof. To start, we claim that the identity map $\widetilde{Y} \to Y$ is a universal covering.

Exercise C.29. Show that a simply connected, semi-locally simply connected, locally path connected space Z is its own universal cover. *Hint:* Show that the construction of Definition C.5 outputs Z again. To show this, demonstrate that if the fiber over one point consists of a single point, then the fiber over any point consists of a single point. Show the fiber over a basepoint of z consists of a single point using that Z is simply connected. Show that a map from a covering space of Z where every fiber has degree 1 is an isomorphism.

Exercise C.30. Show that $\widetilde{X}$ is a universal cover of Y, using the definition Definition C.5.

Exercise C.31. Conclude that $\widetilde{X}$ and $\widetilde{Y} = Y$ are both universal covers of Y, hence show that $Y \to X$ is a universal cover. *Hint:* Use Exercise C.6.

□

C.4. **Fundamental group of the circle.** We are finally ready to compute the fundamental group of the circle, proving Theorem 4.1.

Proof of Theorem 4.1. Consider the map $p : \mathbb{R} \to S^1$ sending $t \mapsto (\cos 2\pi t, \sin 2\pi t)$.

Exercise C.32. Verify $p : \mathbb{R} \to S^1$ is a covering space.

We claim that in fact $\mathbb{R} \to S^1$ is the universal covering space. Indeed, this follows from Corollary C.28 because $\pi_1(\mathbb{R}, 0) = 0$, as shown in Example 3.15, and one can easily verify that $\mathbb{R}$ satisfies the hypotheses of Corollary C.28.

From Corollary C.27, it follows that $f^{-1}(0)$ is in bijection with the fundamental group $\pi_1(S^1, x_0)$. However, $f^{-1}(0)$ is precisely the integer points of $\mathbb{R}$. Under the map p, the integer n precisely maps to the loop winding n times around S^1. This shows that all homotopy classes are represented by such loops.

Exercise C.33. Verify the resulting map $\mathbb{Z} \to \pi_1(S^1, x_0)$ which we have shown is a bijection also respects the group structures on the two groups, and hence defines an isomorphism.

□

References

[Hat02] Allen Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002.